Stat 135 Spring 2018 MT Problem #2 Solutions

Name: Solutions
SID: Solutions
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Problem 2:

Suppose I am conducting an experiment where X_i represents the waiting time until a component fails, and I have 100 independent observations of X_i, exponential random variables with unknown λ.

(a) Find the MoM estimate of λ.
Proof: The first moment of an exponential is $\frac{1}{\lambda}$. Hence we have

$$\frac{1}{\lambda} = \overline{x} \Rightarrow \hat{\lambda} = \frac{1}{\overline{x}}$$

(b) Find the ML estimate of λ.
Proof: The likelihood function is

$$\mathcal{L}(\lambda \mid x_1, \ldots, x_{100}) = \prod_{i=1}^{100} \lambda e^{-\lambda x_i} = \lambda^{100} e^{-\lambda \sum_{i=1}^{100} x_i}$$

And the log likelihood is

$$\ell(\lambda \mid x_1, \ldots, x_{100}) = 100 \log(\lambda) - \lambda \sum_{i=1}^{100} x_i$$

The first derivative is

$$\frac{d}{d\lambda} \ell(\lambda \mid x_1, \ldots, x_{100}) = \frac{100}{\lambda} - \sum_{i=1}^{100} x_i = 0 \Rightarrow \hat{\lambda} = \frac{1}{\overline{x}}$$

The second derivative is

$$\frac{d^2}{d\lambda^2} \ell(\lambda \mid x_1, \ldots, x_{100}) = -\frac{100}{\lambda^2} < 0$$

Hence, $\hat{\lambda}$ is a maximum.

(c) Find Fisher Information and estimated variance of the ML estimate.
Proof: The Fisher Information is

$$\mathcal{I}_n(\lambda) = -\mathbb{E}\left[\frac{d^2}{d\lambda^2} \ell(\lambda \mid x_1, \ldots, x_{100})\right] = \frac{100}{\lambda^2}$$

Hence the estimated variance is

$$\mathrm{Var}(\hat{\lambda}) = \frac{1}{\mathcal{I}(\lambda)} = \frac{\lambda^2}{100}$$

(d) Suppose that instead of data for all 100 X_i, the data are paired so we have 50 observations of independent gamma$(2,\lambda)$ random variables. (In other words, each of the 50 observations has density $\lambda^2 ye\lambda y$ for $y \geq 0$.) You may write these observations as Y_i for i= $1,\ldots,50$. Find the MLE for λ based on these 50 observations.
Proof: First the likelihood function

$$\mathcal{L}(\lambda \,|\, y_1,\ldots,y_{50}) = \prod_{i=1}^{50} \lambda^2 y_i e^{-\lambda y_i} = \lambda^{100}\left(\prod_{i=1}^{50} y_i\right) e^{-\lambda \sum_{i=1}^{50} y_i}$$

Its log likelihood is

$$\ell(\lambda \,|\, y_1,\ldots,y_{50}) = 100\log(\lambda) + \sum_{i=1}^{50}\log(y_i) - \lambda\sum_{i=1}^{50} y_i$$

Its first derivative is

$$\frac{d}{d\lambda}\ell(\lambda \,|\, y_1,\ldots,y_{50}) = \frac{100}{\lambda} - \sum_{i=1}^{50} y_i = 0 \Rightarrow \hat{\lambda} = \frac{2}{\overline{y}}$$

The second derivative is

$$\frac{d^2}{d\lambda^2}\ell(\lambda \,|\, y_1,\ldots,y_{100}) = -\frac{100}{\lambda^2} < 0$$

Hence $\hat{\lambda}$ is a maximum.

(e) Is the MLE in d) based on a sufficient statistic? Prove your answer.
Proof: From the likelihood function, we have the following factorization with

$$h(y_1,\ldots,y_{50}) = \prod_{i=1}^{50} y_i$$

And

$$g_\lambda(T(y_1,\ldots,y_{50})) = \lambda^{100} e^{-\lambda \sum_{i=1}^{50} y_i}$$

Hence, $T(y_1,\ldots,y_{50}) = \sum_{i=1}^{50} y_i$ so yes the MLE is based on a sufficient statistic

$$\hat{\lambda} = \frac{2}{\frac{1}{50}T(y_1,\ldots,y_n)}$$

Note, we could also have

$$h(y_1,\ldots,y_{50}) \propto c$$

For some constant c and

$$g_\lambda(T(y_1,\ldots,y_{50})) = \lambda^{100}\prod_{i=1}^{50} y_i e^{-\lambda \sum_{i=1}^{50} y_i}$$

So that

$$T(y_1,\ldots,y_{50}) = \left(\prod_{i=1}^{50} y_i, \sum_{i=1}^{50} y_i\right)$$

Our MLE would still be based on this sufficient statistic as follows. Let $P_2 : (x_1,x_2) \mapsto x_2$ be a projection onto the second coordinate. Then

$$\hat{\lambda} = \frac{2}{\frac{1}{50}P_2(T(y_1,\ldots,y_{50}))}$$