Quiz 3 Solution

Stat 135

1. **Problem #1**
   John wants to find whether a dice is fair on getting a six. So he throws a dice indepedently 10 times to test hypothesis that the probability of getting sixes is $\frac{1}{6}$ versus the alternative that the probability is not $\frac{1}{6}$. He counts the number of sixes observed as X and he decides that the test only rejects his hypothesis if $|X - 5| > 4$.

   (a) What is the significance level of the test?
   Solution:

   $$\alpha = P(\text{reject } H_0 |\ H_0 \text{ is true}) = P(X = 10 \text{ or } X = 0 |\ p = \tfrac{1}{6}) = (\tfrac{5}{6})^{10} + (\tfrac{1}{6})^{10}$$

   Rubrics:
   1 point for correct α function
   1 point for correct calculation for rejection region
   1 point for correct answer

   (b) If in fact, later on somebody tells John that the probability of getting sixes is 0.2, what is the power of the test?
   Solution:

   $$\begin{aligned} \text{power} = 1 - \beta = P(\text{reject } H_0 |\ H_1 \text{ is true}) &= P(X = 10 \text{ or } X = 0 |\ p = 0.2) \\ &= (0.2)^{10} + (0.8)^{10} \end{aligned}$$

2. **Problem #2**
Let $X_1, \ldots, X_{20}$ be a sample from a normal distribution with a variance of 20.

(a) Derive a likelihood ratio test of $H_0 : \mu = 0$ and $H_1 : \mu = 1$ with stating rejection region and then obtain the rejection region if the significance level is set to be 0.05.
Solution:

$$\begin{aligned}
\Lambda = \frac{max_{\theta \in \omega_0}[lik(\theta)]}{max_{\theta \in \omega_1}[lik(\theta)]} &= \frac{\frac{1}{(\sigma\sqrt{2\pi})^n} exp(\frac{-1}{2\sigma^2}\sum_{i=1}^{20}(x_i - \mu_0)^2)}{\frac{1}{(\sigma\sqrt{2\pi})^n} exp(\frac{-1}{2\sigma^2}\sum_{i=1}^{20}(x_i - \mu_1)^2)} \\
&= exp(\frac{-1}{40}(\sum_{i=1}^{20}(x_i - \mu_0)^2 - \sum_{i=1}^{20}(x_i - \mu_1)^2)) \\
&= exp(\frac{-1}{40}(\sum_{i=1}^{20} x_i^2 - \sum_{i=1}^{20}(x_i - 1)^2)) \\
&= exp(\frac{-1}{40}(\sum_{i=1}^{20}(2x_i - 1))) = exp(\frac{-1}{40}(40\bar{X} - 20)) \\
&= exp(0.5 - \bar{X})
\end{aligned}$$

So we reject H_0 as Λ is small. So the rejection region is $\bar{X} > c$ so that we reject H_0.
For $\bar{X}$ under H_0, it follows $N(0, \frac{20}{20}) = N(0, 1)$.
Since $\alpha = 0.05$, we have,

$$P(\bar{X} > c|H_0) = 0.05$$
$$P(\frac{\bar{X} - 0}{1} > \frac{c - 0}{1}) = 0.05$$
$$P(Z > c) = 0.05$$

So based on standard normal table, $c \approx 1.645$
So the rejection region is $\bar{X} > 1.645$.

(b) Assume $P(H_0) = P(H_1)$, what is $P(H_0|x)$? (You don't need to consider joint density for this part.)
Solution:

$$\begin{aligned}
P(H_0|x) &= \frac{P(x|H_0) \times P(H_0)}{P(x|H_0) \times P(H_0) + P(x|H_1) \times P(H_1)} \\
&= \frac{\frac{1}{\sigma\sqrt{2\pi}} exp(\frac{-1}{2\sigma^2}(x - \mu_0)^2)}{\frac{1}{\sigma\sqrt{2\pi}} exp(\frac{-1}{2\sigma^2}(x - \mu_0)^2) + \frac{1}{\sigma\sqrt{2\pi}} exp(\frac{-1}{2\sigma^2}(x - \mu_1)^2)} \\
&= \frac{exp(\frac{-1}{40}x^2)}{exp(\frac{-1}{40}x^2) + exp(\frac{-1}{40}x^2) \times exp(\frac{-1}{40}(-2x + 1))} \\
&= \frac{1}{1 + exp(\frac{1}{40}(2x - 1))}
\end{aligned}$$

3. **Problem #3**
A box contains 9 marbles and there are 3 types of marbles: red, blue and green. Tom believes that there is same number of each type of marble. So he draws 9 times with replacement and gets the following results:

Type of Marble	Counts
Red	2
Blue	4
Green	3

Tom wants to determine whether the numbers of marbles are equal based on the observations. Create a hypothesis test using likelihood ratio test for this question and obtain the conclusion for Tom. (Please use likelihood ratio test for full credit; only partial credit will be given for using Pearson's goodness-of-fit test.)
Formula of multinomial distribution:

$$f(x_1, \dots, x_k; n, p_1, \dots, p_k) = \frac{n!}{x_1! \cdots x_k!} p_1^{x_1} \times \cdots \times p_k^{x_k}$$

Solution:
H_0 : The probability is equal. (Or any answer simiar such as having equal number of balls)
H_1 : The probability is not equal. (Or any answer simiar)

$$\Lambda = \frac{\frac{n!}{x_1! \times \cdots \times x_k!} p_1(\hat{\theta})^{x_1} \times \cdots \times p_k(\hat{\theta})^{x_k}}{\frac{n!}{x_1! \times \cdots \times x_k!} \hat{p_1}^{x_1} \times \cdots \times \hat{p_k}^{x_k}} = \prod_{i=1}^{k} \left(\frac{p_i(\hat{\theta})}{\hat{p}_i}\right)^{x_i}$$

So

$$-2log\Lambda = -2n \sum_{i=1}^{k} \hat{p}_i log\left(\frac{p_i(\hat{\theta})}{\hat{p}_i}\right) = 2 \sum_{i=1}^{k} O_i log\left(\frac{O_i}{E_i}\right)$$

After substitution, we get,

$$-2log\Lambda = 2 \times (2 \times log(2/3) + 4 \times log(4/3) + 3 \times log(3/3)) = 0.6796$$

For this question, degree of freedom = 2
So, p-value = $P(\chi_2^2 > 0.68) \approx 0.7$
Since p-value > 0.05, we do not reject H_0.
So the number of each ball seems to be equal.