Friday

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STAT 135 LAB

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Random Variables - a random number

- discrete & continuous ← 2 types
  - ↳ can take only a finite or countably infinite values
  - continuous R.V.: can take on a continuum of values

Mean of an RV

- AKA Expected value (center of mass of prob. function)
- Discrete RV w/ prob funct. $\mathbb{P}(x)$, $\mathbb{E}[X] = \sum_i x_i \mathbb{P}(x_i)$

  provided that $\sum_i |x_i| \mathbb{P}\{x_i\} < \infty$
- Continuous RV w/ pdf $f(x)$,

  $\mathbb{E}[X] = \int_{-\infty}^{\infty} x f(x)\,dx$, provided $\int_{-\infty}^{\infty} x f(x)\,dx < \infty$

Variance

- Indication of how dispersed/spread out the prob. dist. is about its central or expected value.
- If X is an R.V. w/ $\mathbb{E}[X]$, $Var(X) = \mathbb{E}[[X - \mathbb{E}[X]]^2]$

  (provided that $\mathbb{E}[X]$ exists)
- $Var(X) = \mathbb{E}[[X - \mathbb{E}(X)]^2] = \mathbb{E}[X^2 - 2X\mathbb{E}[X] + \mathbb{E}[X]^2] = \mathbb{E}[X^2] - 2\mu^2 + \mu^2$

  $= \mathbb{E}[X^2] - \mu^2$

  $\boxed{Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2}$

Covariance

if X & Y positively associated, as $X > \mu_x$, Y tends to be smaller than μ_y

- Measures 2 RV's variance from ~~their~~ means.
- If X & Y jointly distributed RV's, w/ Expectation μ_x & μ_y,

  $Cov(X,Y) = \mathbb{E}[(X-\mu_x)(Y-\mu_y)]$ (provided that expectation exists)
- Correlation: strength btw 2 vars, $Corr(X,Y) = \frac{Cov(X,Y)}{SD(X)\,SD(Y)}$

$Cov(X,Y) = \mathbb{E}[(X-\mu_x)(Y-\mu_y)] = \mathbb{E}[XY - \mu_y X - \mu_x Y + \mu_x\mu_y]$

$= \mathbb{E}[XY] - \mathbb{E}[X]\mu_y - \mathbb{E}[Y]\mu_x + \mu_x\mu_y = \mathbb{E}[XY] - 2\mathbb{E}[X]\mathbb{E}[Y] + \mathbb{E}[X]\mathbb{E}[Y]$

$= \mathbb{E}[XY] - \mathbb{E}[X]\mathbb{E}[Y]$

$$\begin{aligned} Cov(a+X, Y) &= \mathbb{E}[(a+X-\mathbb{E}[a+X])(Y-\mathbb{E}[Y])] \\ &= \mathbb{E}[(\cancel{a}+X-\cancel{a}+\mathbb{E}[X])(Y-\mathbb{E}[Y])] \\ &= \mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = Cov(X,Y) \end{aligned}$$

$$\begin{aligned} Cov(aX, bY) &= \mathbb{E}[(aX-\mathbb{E}[aX])(bY-\mathbb{E}[bY])] \\ &= \mathbb{E}[a(X-\mathbb{E}[X])\cdot b(Y-\mathbb{E}[Y])] \\ &= ab\,\mathbb{E}[(X-\mathbb{E}[X])(Y-\mathbb{E}[Y])] = ab\,Cov(X,Y) \end{aligned}$$

$$Cov(X, Y+Z) = Cov(X,Y) + Cov(X,Z)$$

$$\begin{aligned} Cov(aW+bX, cY+dZ) &= Cov(aW+bX, cY) + Cov(aW+bX, dZ) \\ &= Cov(aW, cY) + Cov(bX, cY) + Cov(aW, dZ) + Cov(bX, dZ) \\ &= ac\,Cov(W,Y) + bc\,Cov(X,Y) + ad\,Cov(W,Z) + bd\,Cov(X,Z) \end{aligned}$$

$$Cov\left(\sum_i aX_i, \sum_j bY_j\right) = \sum_i \sum_j Cov(X_i, Y_j)$$

correct??

$b^m a^n \sum_i^n \sum_j^m Cov(X_i, Y_j)$?.

$$Cov(X,X) = Var(X)$$

$$Var(X) = \mathbb{E}[(X-\mu_X)^2]$$

$$\begin{aligned} Var(X+Y) &= Cov(X+Y, X+Y) = Cov(X+Y, X) + Cov(X+Y, Y) \\ &= Cov(X,X) + Cov(Y,X) + Cov(X,Y) + Cov(Y,Y) \\ &= Var(X) + 2\,Cov(X,Y) + Var(Y) \end{aligned}$$

If $X \perp\!\!\!\perp Y$, $Cov(X,Y) = 0$, and $Var(X+Y) = Var(X) + Var(Y)$

If X_i's are independent, $\Rightarrow Cov(X_i, X_j) = 0$ for $i \neq j$, hence

$$Var\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n Var(X_i)$$

$$Var(aX) = a^2\,Var(X)$$

$$Var(a+bX) = b^2\,Var(X)$$

Variance important for ch 7 & 8

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LAW OF LARGE NUMBERS

→ As $n \to \infty$, $\frac{X_1 + \dots + X_n}{n} = \bar{X}_n \xrightarrow{LLN} \mathbb{E}[X]$

if you have large sample size, sample mean approaches the true mean as $n \to \infty$

WEAK LAW OF LARGE #'s (AKA "LAW OF AVERAGES")

- Let $X_1, X_2, \dots,$ be seq. of i.i.d. RV's, each having finite mean, $\mathbb{E}[X_i] = \mu$, and $Var(X_i) = \sigma^2$, then

For $\forall \varepsilon > 0$: $\left\{ \left[\frac{X_1 + \dots + X_n}{n} - \mu \right] > \varepsilon \right\} \to 0$ as $n \to \infty$

$\iff \mathbb{P}\{ |\bar{X}_n - \mu| > \varepsilon \} \to 0$ as $n \to \infty$

LOW: When sample size (n) is large, probability is nearly 1, that the sample mean is close to the true mean

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STRONG LAW OF LARGE #'s

Let $X_1, X_2, \dots,$ be seq. of i.i.d. RV's, each having a finite mean $\mu = \mathbb{E}[X_i]$. Then $\mathbb{P}\{ \lim_{n \to \infty} \bar{X}_n = \mu \} = 1$

LOW: w/ prob 1, the sample mean converges to the true mean as the sample size tends to ∞

CENTRAL LIMIT THEOREM (CLT)

- Layman's: CLT states that sum of large num of iid has a distribution that is approx. Normal

- More Formal: Let $X_1, X_2, \dots$ be seq of iid RV's having mean μ, and $Var(X_i) = \sigma^2$, and common distribution function F. Then $\lim_{n \to \infty} \mathbb{P}\left\{ \frac{(X_1 + \dots + X_n) - \mu}{\sigma \sqrt{n}} \leq a \right\} = \Phi(a)$, $-\infty < a < \infty$
where $\Phi(x)$ is a CDF for a $N(0,1)$

- Note that $\mathbb{E}[X_1 + \dots + X_n] = \mathbb{E}[\sum_i^n X_i] = n\mu$

$Var(X_1 + \dots + X_n) = Var(\sum_i X_i) = n\sigma^2$

So $Z_n = \frac{\sum_i^n X_i - n\mu}{\sigma \sqrt{n}}$ is standardizing the RV $\sum_i^n X_i$

Binomial Dist

- Sum of iid bernoulli's
- num. successes in n indep. trials, w/ prob. p of success
- Params: n - # trials, $n \in \{1, 2, \ldots$
  p - prob of success, $p \in [0, 1]$
- $\mathbb{P}\{S = k\} = \binom{n}{k} p^k (1-p)^{n-k}$, where $k = 0, 1, \ldots, n$
- $S = X_1 + X_2 + \cdots + X_n$, where X_i = indicator of success on trial i.
- $\mathbb{E}[S] = np$, $Var(S) = npq = np(1-p)$

Poisson

Poisson scatter — # raindrops in a square follows Pois dist.
Poisson process — # arrivals in time

- Discrete RV N_μ where is # arrivals in given time period in Poisson arrival process,
  or # points in a given area in a Poisson Random Scatter,
  where $\mathbb{E}[X] = \mu$.
  $N_\mu \in \mathbb{Z}^+$ $(0, 1, 2 \cdots)$
- Prob. function $\mathbb{P}\{N_\mu = k\} = e^{-\mu} \frac{\mu^k}{k!}$ $(k = 0, 1, 2, \ldots)$
- $\mathbb{E}[N_\mu] = \mu$, $Var(N_\mu) = \mu$

☆ when Daniel took 135, the final was all about Poisson Dist.

Geom Dist.

trials before 1st success

- Discrete R.V., T, w/ possible values that are positive integers
- Parameter: p = success probability
- $T :=$ # trials until 1st success, in independent trials w/ prob. p of success on each trial.
- Prob. function $\mathbb{P}\{T = n\} = (1-p)^{n-1} p$, $(n = 1, 2, \ldots)$
- Other version of geom:
  - Let $F = T - 1$ denote # failures before 1st success.
  - $F \sim geom(p)$ on $\{0, 1, \ldots\}$

- $\mathbb{E}[T] = \frac{1}{p}$
- $Var(T) = \frac{1-p}{p^2}$

- $\mathbb{E}[F] = \mathbb{E}[T-1] = \frac{1}{p} - 1 = \frac{1-p}{p}$
- $Var(F) = Var(T-1) = Var(T)$

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NEGATIVE BINOMIAL → sum of iid geometrics.

- Discrete R.V., F_r
- # failures until r'th success, in Bernoulli trials w/ prob., p, of success on each
- poss. values = $F_r = 0, 1, \ldots$ $F_r \in \mathbb{Z}^+$
- Params: p = success probability
  r = # successes
- Prob. Function: $\mathbb{P}\{F_r = n\} = \binom{n+r-1}{r-1} p^r (1-p)^n$, $(n \in \{0, 1, \ldots\})$
- $\mathbb{E}[F_r] = \frac{r(1-p)}{p}$
- $Var(F_r) = \frac{r(1-p)}{p^2}$

BETA DIST.

- Continuous R.V., X_{rs} $x^r(1-x)^{r-s}$
- Takes on value btw 0 and 1
- Params: $r > 0$, $s > 0$
- Density fn. $\frac{\mathbb{P}\{X_{rs} \in dx\}}{dx} = \frac{\Gamma(r)\Gamma(s)}{\Gamma(r+s)} x^{r-1}(1-x)^{s-1}$ $(0 \le x \le 1)$

$$= \left(\frac{1}{\int_0^1 x^{r-1}(1-x)^{s-1}}\right) \cdot x^{r-1}(1-x)^{s-1}, \text{ where } \Gamma(r) = \int_0^\infty t^{r-1}e^{-t}\,dt$$

- $\mathbb{E}[X_{rs}] = \frac{r}{r+s}$
- $Var(X_{rs}) = \frac{rs}{(r+s)^2(r+s+1)}$
- Sources & Applications:
  - order statistics of uniform variables
  - Bayesian inference for unknown probabilities

EXPONENTIAL DIST'N

- Continuous RV, T with possible values $(0, \infty)$.
- Often T interpreted as lifetime
- Params: λ = rate
- Density fn: $\frac{\mathbb{P}\{T \in dt\}}{dt} = \lambda e^{-\lambda t}$, $t \ge 0$
- CDF: $\mathbb{P}\{T \le t\} = 1 - e^{-\lambda t}$, $t \ge 0$
- $\mathbb{E}[T] = \frac{1}{\lambda}$, $Var(T) = \frac{1}{\lambda^2}$
- T = time until next arrival in Poisson process w/ rate λ

☆ Last semester STAT 135 final was all about Exponential Dist.

GAMMA DIST'N

- Continuous R.V., $T_{r,\lambda}$, poss. values $(0, \infty)$
- Params: $r > 0$ (shape)
  $\lambda > 0$ (rate or inverse scale)
- Density fn. $\frac{\mathbb{P}\{T_{r,\lambda} \in dt\}}{dt} = \frac{\lambda^r}{\Gamma(r)} t^{r-1} e^{-\lambda t}, \quad t \geq 0$
- Special cases:
  - Gamma$(1, \lambda)$ is expo(λ)
  - Gamma$(\frac{n}{2}, \frac{1}{2})$ is chi-sq. w/ n deg. of freedom
- Sources:
  - Sum of r indep. exponential variables is gamma
  - Time until r'th arrival in Poisson process w/ rate λ
- Transformations
  - Scaling: if $T \sim$ gamma(r, λ)
    $\lambda T \sim$ gamma$(r, 1)$
  - Sums: For independent $T_i \sim$ gamma(r_i, λ)
    $\sum_i^n T_i \sim$ gamma$(\sum_i^n r_i, \lambda)$

NORMAL DIST'N

- STANDARD NORMAL
  - CONTINUOUS RV, Z, takes on $(-\infty, \infty)$
  - PDF: $\frac{1}{\sqrt{2\pi}} e^{-\frac{1}{2}z^2} = \frac{\mathbb{P}\{Z \in dz\}}{dz}$
  - CDF: $\mathbb{P}\{Z \leq z\} = \Phi(z) = \int_{-\infty}^{z} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx$
  - $\mathbb{E}[Z] = 0$, $Var(Z) = 1$
  - Transformations: -
    - $Z^2 \sim$ gamma$(\frac{1}{2}, \frac{1}{2}) \equiv \chi_1^2$ (chi-sq. w/ 1 deg. of freedom $\equiv$ gamma$(\frac{1}{2}, \frac{1}{2})$)
    - Let $Z_1, Z_2, \ldots, Z_n$ be indep $\sim N(0,1)$
      $Z_1^2 + Z_2^2 + \cdots + Z_n^2 \sim \chi_n^2$ (chi-sq. $\equiv$ gamma$(\frac{n}{2}, \frac{1}{2})$)

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Normal

- R.V., $X = \mu + \sigma Z$, where $Z \sim N(0,1)$
- $\mathbb{E}[X] = \mathbb{E}[\mu + \sigma Z] = \mu + \sigma \mathbb{E}[Z] = \mu$
- $Var(X) = Var(\mu + \sigma Z) = \sigma^2 Var(Z) = \sigma^2$
- PDF: $\dfrac{\mathbb{P}\{X \in dx\}}{dx} = \dfrac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2}$, $(-\infty < x < \infty)$
- Sum of Normals: If $X_i \overset{iid}{\sim} N(\mu_i, \sigma_i^2)$, then
  $\sum_i X_i \sim N\left(\sum_i \mu_i, \sum_i \sigma_i^2\right)$

Chi-Squared Dist'n

- Continuous R.V., U, takes on $(0, \infty)$
- If $Z \sim N(0,1)$, $U = Z^2, \sim \chi^2$
- If $Z_1, Z_2, \ldots, \overset{iid}{\sim} N(0,1)$
  $V = Z_1^2 + Z_2^2 + \cdots + Z_n^2 = \chi^2_n \Rightarrow V$ is gamma$\left(\frac{n}{2}, \frac{1}{2}\right)$
- Density $\dfrac{\left(\frac{1}{2}\right)^{n/2}}{\Gamma\left(\frac{n}{2}\right)} V^{\frac{n}{2}-1} e^{-\frac{V}{2}}$ $\quad V \geq 0$
- <u>Sums</u> if $U \sim \chi^2_n$ and $V \sim \chi^2_m$ and U & V are indep,
  $U + V \sim \chi^2_{n+m}$