Lin Alg (MIT - Strang) Lecture 21 - Eigenvalues + Eigenvectors

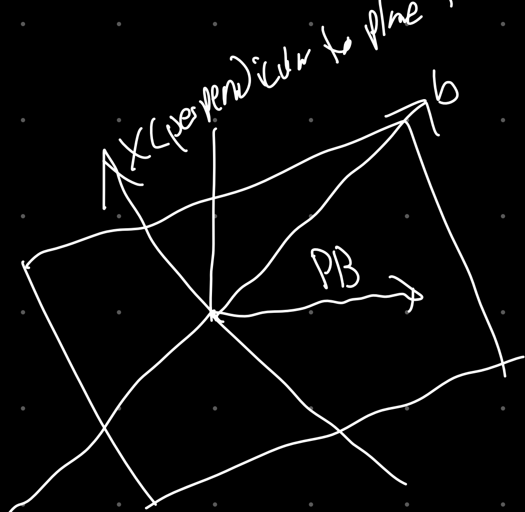
$\det[A-\lambda I]=0$

Ax parallel to x - eigenvectors

$Ax=\lambda x$ ← lambda

If A is singular, $\lambda=0$ is an eigenvalue

What are the x's (eigenvectors and λ's (eigenvalues for a projection matrix?



Any x in the plane will be an eigenvector

$Px=x \quad \lambda=1$

Any x that's perpendicular to the plane

$Px=0x \quad \lambda=0$

eigenvalues: 1, 0

$A=\begin{bmatrix}0&1\\1&0\end{bmatrix}$ $x=\begin{bmatrix}1\\1\end{bmatrix}$ $Ax=\begin{bmatrix}1\\1\end{bmatrix}$ $\lambda=1$ $Ax=x$

$x=\begin{bmatrix}-1\\1\end{bmatrix}$ $Ax=\begin{bmatrix}1\\-1\end{bmatrix}$ $\lambda=-1$ $Ax=-x$

$n\times n$ matrices will have n eigenvalues

The sum of the eigenvalues (λ)'s is equal to the diagonal entries of A. (a11 + a22 + ... + ann)

How to solve $Ax=\lambda x$ with 2 unknowns

Rewrite: $(A-\lambda I)x=0$ Singular ($\det=0$)

$\det(A-\lambda I)=0$ ⇒ find λ first.

$A=\begin{bmatrix}3&1\\1&3\end{bmatrix}$ $\det(A-\lambda I)=\begin{vmatrix}3-\lambda&1\\1&3-\lambda\end{vmatrix}$

$=(3-\lambda)^2-1=\lambda^2-6\lambda+8$

$0=\lambda^2-\underbrace{6}_{\text{trace }(3+3)}\lambda+\underline{8}$ determinant

$=(\lambda-4)(\lambda-2) \quad \lambda_1=4 \quad \lambda_2=2$

$A-4I\begin{bmatrix}-1&1\\1&-1\end{bmatrix}$ $x_1=\begin{bmatrix}1\\1\end{bmatrix}$

$A-2I\begin{bmatrix}1&1\\1&1\end{bmatrix}$ $x=\begin{bmatrix}-1\\1\end{bmatrix}$

$Ax=\lambda x \quad (A+3I)x=\lambda x+3x=(\lambda+3)x$

$Ax=\lambda x$, B has eigenvalues α_1,α_2

$Bx=\alpha x \quad (A+B)x\neq(\lambda+\alpha)x$ unless B is a multiple of the Identity Matrix

eigenvalues are not linear and don't multiply

Example $Q=\begin{bmatrix}0&-1\\1&0\end{bmatrix}$ trace $=0+0=\lambda_1+\lambda_2$

90° rotation

$\det=1=\lambda_1\lambda_2$ $\det(Q-\lambda I)=\begin{bmatrix}-\lambda&-1\\1&-\lambda\end{bmatrix}=\lambda^2+1=0 \quad \lambda_1=i \quad \lambda_2=-i$

i - multiplies to 1, adds to 0. not real.

$A = \begin{bmatrix} 3 & 1 \\ 0 & 3 \end{bmatrix}$ eigenvalues: 3 $\det(A - \lambda I) = (3-\lambda)(3-\lambda)$ $\lambda_1 = 3$ $\lambda_2 = 3$

if a matrix is triangular, the eigenvalues are on the diagonal.

$(A - \lambda I)x = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\begin{bmatrix} x \end{bmatrix} = 0$ $x_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ $x_2 = \begin{bmatrix} \text{one without linear dependence} \end{bmatrix}$

Only 1 eigenvector, 2 eigenvalues

def: Let $A \in \mathbb{R}^{n \times n}$ be a square matrix. Then $\lambda \in \mathbb{R}$ is an eigenvalue of A and $x \in \mathbb{R}^n \setminus \{0\}$ is the corresponding eigenvector of A if $Ax = \lambda x$. $Ax = \lambda x$ is the eigenvalue equation.

The following statements are equivalent:

- λ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$
- There exists an $x \in \mathbb{R}^n \setminus \{0\}$ with $Ax = \lambda x$, or equivalently, $(A - \lambda I_n)x = 0$ can be solved non-trivially ($x \neq 0$)
- $rk(A - \lambda I_n) < n$
- $\det(A - \lambda I_n) = 0$

Def (Colinearity and Codirection): two vectors that point in the same direction are called codirected. Two vectors are colinear if they point in the same or opposite direction

If x is an eigenvector of A associated with eigenvalue λ, then for any $c \in \mathbb{R} \setminus \{0\}$ it holds that cx is an eigenvector of A.

$A(cx) = cAx = c\lambda x = \lambda(cx)$ All vectors collinear to x are also eigenvectors of A.

Theorem: $\lambda \in \mathbb{R}$ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$ iff λ is a root of the characteristic polynomial $p_A(\lambda)$ of A.

def (Algebraic Multiplicity): Let a square matrix A have an eigenvalue λ_i. The algebraic multiplicity of λ_i is the number of times the root appears in the characteristic polynomial.

def (Eigenspace and Eigenspectrum): For $A \in \mathbb{R}^{n \times n}$, the set of all eigenvectors of A is associated with an eigenvalue λ that spans a subspace of $\mathbb{R}^n$ called the eigenspace of A with respect to λ and is denoted by E_λ. The set of all eigenvalues of A is called the eigenspectrum, or just spectrum, of A.

If λ is an eigenvalue of $A \in \mathbb{R}^{n \times n}$, then the corresponding eigenspace E_λ is the solution space of $(A - \lambda I)x = 0$

$I \in \mathbb{R}^{n \times n}$ has the characteristic polynomial $p_I(\lambda) = \det(I - \lambda I) = (1-\lambda)^n = 0$, which has one eigenvalue $\lambda = 1$ that occurs n times. $Ix = \lambda x = 1x$ holds for all vectors $x \in \mathbb{R}^{n \times n} / \{0\}$. The sole eigenspace E_1 of the identity matrix spans n dimensions, and all n standard basis vectors of $\mathbb{R}^n$ are eigenvectors of I.

- A matrix A and its transpose A^T possess the same eigenvalues, but not necessarily the same eigenvectors.
- The eigenspace E_λ is the null space of $A - \lambda I$ since $Ax = \lambda x \iff Ax - \lambda x = 0 \iff (A - \lambda I)x = 0 \iff x \in \ker(A - \lambda I)$
- Symmetric, positive definite matrices always have positive, real eigenvalues.
- eigenvalues are invariant under basis change.

$A = \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix}$ $\quad p_A(\lambda) = \det(A - \lambda I) = \det\left| \begin{bmatrix} 4 & 2 \\ 1 & 3 \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} \right| = \begin{vmatrix} 4-\lambda & 2 \\ 1 & 3-\lambda \end{vmatrix} = (4-\lambda)(3-\lambda) - 2$

$p = 10 - 7\lambda + \lambda^2 = (2-\lambda)(5-\lambda)$ $\quad \lambda_1 = 2 \quad \lambda_2 = 5$

$\lambda = 5 \quad \begin{bmatrix} 4-5 & 2 \\ 1 & 3-5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} -1 & 2 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \quad E_5 = \operatorname{span}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\begin{bmatrix} 4-2 & 2 \\ 1 & 3-2 \end{bmatrix} = \begin{bmatrix} 2 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = 0 \Rightarrow x_2 = -x_1 \begin{bmatrix} 1 \\ -1 \end{bmatrix} \quad E_2 = \operatorname{span}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$

Def (geometric multiplicity): Let λ_i be an eigenvalue of a square matrix A. Then the geometric multiplicity of λ_i is the number of linearly independent eigenvectors associated with λ_i

↳ the dimensionality of the eigenspace spanned by the eigenvectors associated with λ_i

$1 \leq$ geometric multiplicity $\leq$ algebraic multiplicity

Thm 4.12: The eigenvectors $x_1, \dots x_n$ of a matrix $A \in \mathbb{R}^{n \times n}$ with n distinct eigenvalues $\lambda_1, \dots \lambda_n$ are linearly independent.

$\Rightarrow$ eigenvectors of a matrix with n distinct eigenvalues form a basis of $\mathbb{R}^n$.

Def (defective): A square matrix $A \in \mathbb{R}^{n \times n}$ is defective if it possesses fewer than n linearly independent eigenvectors.

a defective matrix cannot have n distinct eigenvalues

If $\operatorname{rank}(A) = n$, then $S := A^T A$ is symmetric, positive definite

Thm 4.15 (Spectral Theorem): If $A \in \mathbb{R}^{n \times n}$ is symmetric, there exists an orthonormal basis of the corresponding vector space V consisting of eigenvectors of A, and each eigenvalue is real.

Thm 4.16: The determinant of a matrix $A \in \mathbb{R}^{n \times n}$ is the product of its eigenvalues

$\det(A) = \prod_{i=1}^{n} \lambda_i$, where $\lambda_i \in \mathbb{C}$ are possibly repeated eigenvalues of A.

Thm 4.17: The trace of a matrix $A \in \mathbb{R}^{n \times n}$ is the sum of its eigenvalues

$\operatorname{tr}(A) = \sum_{i=1}^{n} \lambda_i$, where $\lambda_i \in \mathbb{C}$ are possibly repeated eigenvalues of A.