Def (Generating Set and Span): Consider a vector space $V = (\mathcal{V}, +, \cdot)$ and a set of vectors $\mathcal{A} = \{x_1, \dots x_k\} \subseteq \mathcal{V}$. If every vector $v \in \mathcal{V}$ can be expressed as a linear combination of vectors $x_1, \dots x_k$, $\mathcal{A}$ is called a generating set of V. The set of all linear combinations in $\mathcal{A}$ is called the span of $\mathcal{A}$. If $\mathcal{A}$ spans the vector space, we write $V = \text{span}[\mathcal{A}]$ or $V = \text{span}[x_1, \dots, x_k]$

Def (Basis): Consider a vector space $V = (\mathcal{V}, +, \cdot)$ and $\mathcal{A} \subseteq \mathcal{V}$. A generating set $\mathcal{A}$ of V is called minimal if there exists no smaller set $\tilde{\mathcal{A}} \subseteq \mathcal{A} \subseteq \mathcal{V}$ that spans V. Every linearly independent generating set of V is minimal and is called a basis of V.

$\Rightarrow$ A basis is a minimal generating set and a maximal linearly independent set of vectors.

Let $V = (\mathcal{V}, +, \cdot)$ be a vector space and $\mathcal{B} \subseteq \mathcal{B} \neq \emptyset$. Then, the following statements are equivalent:

- $\mathcal{B}$ is a basis of V.
- $\mathcal{B}$ is a minimally generating set
- $\mathcal{B}$ is a maximally linearly independent set of vectors in V, i.e., adding any other vector to this set will make it linearly dependent.
- Every vector $x \in V$ is a linear combination of vectors from $\mathcal{B}$ and every linear combination is unique: $x = \sum_{i=1}^{k} \lambda_i b_i = \sum_{i}^{k} \psi_i b_i$ and $\lambda_i, \psi_i \in \mathbb{R}, b_i \in \mathcal{B}$ and it follows that $\lambda_i = \psi_i, i = 1 \dots k$

In $\mathbb{R}^3$, the canonical / standard basis is: $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$

Every vector space V possesses a Basis $\mathcal{B}$. There can be many bases of a vector space, but all bases possess the same number of elements, the basis vectors.

The dimension of V is the number of basis vectors of V. $\Rightarrow$ the number of independent directions in the vectorspace.

in the vector space

A basis of a subspace $U = \text{span}[x_1, \ldots x_m] \subseteq \mathbb{R}^n$ can be found by executing the following steps:

1) Write the spanning vectors as columns of a matrix A.
2) Determine the row-echelon form of A.
3) The spanning vectors associated with the pivot columns are a basis of U.

$\begin{bmatrix} 1 & 2 & 3 & -1 \\ 2 & -1 & -4 & 8 \\ -1 & 1 & 7 & -5 \\ -1 & 2 & 5 & -6 \\ -1 & 2 & -3 & 1 \end{bmatrix} \in \mathbb{R}^5$ check if linearly independent: $\sum_{i=1}^{4} \lambda_i x_i = 0$ $\sim \begin{bmatrix} 1 & 2 & 3 & -1 \\ 0 & 1 & 2 & -2 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ x_1, x_2, x_4 linearly independent.

$\lambda_1 x_1 + \lambda_2 x_2 + \lambda_4 x_4 = 0$

$\{x_1, x_2, x_4\}$ is a basis of U

Rank - the number of linearly independent columns of a matrix $A \in \mathbb{R}^{m \times n}$ equals the number of linearly independent rows

Properties of rank:

- $rk(A) = rk(A^T)$ - col rank = row rank
- The columns of $A \in \mathbb{R}^{m \times n}$ span a subspace $U \subseteq \mathbb{R}^m$ with $\dim(U) = rk(A)$
- The rows of $A \in \mathbb{R}^{m \times n}$ span a subspace $W \subseteq \mathbb{R}^n$ with $\dim(W) = rk(A)$. A basis of W can be found by applying Gaussian elimination to A^T.
- For all $A \in \mathbb{R}^{n \times n}$, it holds that A is regular (invertible) iff $rk(A) = n$
- For all $A \in \mathbb{R}^{m \times n}$ and all $b \in \mathbb{R}^m$ it holds that the linear equation system $Ax = b$ can be solved iff $rk(A) = rk(A|b)$
- For $A \in \mathbb{R}^{m \times n}$ the subspace of solutions for $Ax = 0$ possesses dimension $n - rk(A)$. (nullspace)
- A matrix $A \in \mathbb{R}^{m \times n}$ has full rank if its rank equals the largest possible rank for a matrix of the same dimensions. $rk(A) = \min(m, n)$

Suppose A is m by n with $m < n$. (more unknowns than equations). Then there are nonzero solutions to $Ax = 0$.

(special solutions in the nullspace)

There will be at least one free variables.

Vectors $x_1, \ldots x_n$ are lin independent if no combination gives the zero vector (except the zero combination)

if the null space of A is only the zero vector rank $= n$ $N(A) = \{0\}$ - no free variables

dependent if $Ac = 0$ for nonzero. rank $< n$, free variables

Vectors $v_1 \dots v_n$ span a space - The space consists of all combinations of those vectors.

Basis for a space is a sequence of vectors with 2 properties:

1) independent 2) Span the space.