Elimination < success / failure

Back - Substitution

Elimination Matrices

Matrix Multiplication

MIT Lin Alg Lec 2 - Elimination w/ Matrices (Strang)

$x + 2y + z = 2 \quad r_1$

$3x + 8y + z = 12 \quad r_2$

$4y + z = 2 \quad r_3$

1st pivot

$$\overset{A}{\begin{bmatrix} \boxed{1} & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix}} = \overset{b}{\begin{bmatrix} 2 \\ 12 \\ 2 \end{bmatrix}} \rightarrow$$

$Ax = b$

purpose - to eliminate x from r_2

$$\begin{matrix} & 1 & 2 & 1 \\ 3x & 0 & \boxed{2} & -2 \\ (2,1) & 0 & 4 & 1 \end{matrix} \xrightarrow{(3,2)} \begin{matrix} \boxed{1} & 2 & 1 \\ 0 & \boxed{2} & -2 \\ 0 & 0 & \boxed{5} \end{matrix}$$

U - upper triangular

Goal: get from A to U (upper triangular matrix)

* pivots cannot be zero

To find determinant, multiply the pivots. For prev ex: $1 \cdot 2 \cdot 5 = 10$

How could this operation have failed? - if 1st pivot was 0 → would require a row switch
- if second pivot was 6

if there is a nonzero below a pivot point that is a zero, row switch

Augmented Matrix:

$A \to U$, $b \to c$

$$\begin{bmatrix} \boxed{1} & 2 & 1 & 2 \\ 3 & 8 & 1 & 12 \\ 0 & 4 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 1 & 2 \\ 0 & \boxed{2} & -2 & 6 \\ 0 & 4 & 1 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} \boxed{1} & 2 & 1 & 2 \\ 0 & \boxed{2} & -2 & 6 \\ 0 & 0 & \boxed{5} & -10 \end{bmatrix}$$

A, b → u, c

$Ux = c$

$x + 2y + z = 2$

$2y - 2z = 6$

$5z = -10$

$(2, 1, -2)$

Solve for z: $z = -2$

Solve for y: $2y + 4 = 6$, $y = 1$

Solve for x: $x + 2 - 2 = 2$, $x = 2$

Back Substitution: Solving equations in reverse order via substitution when the system is triangular.

Matrices

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ \ldots \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} =$$

Combination of the columns.

3x col 1

$$\begin{bmatrix} 3 + 8 + 5 \\ \ldots \end{bmatrix} \begin{bmatrix} 16 \\ \ldots \end{bmatrix}$$

$\begin{bmatrix} 0 & 4 & 1 \end{bmatrix} \begin{bmatrix} 5 \end{bmatrix}$ 4x col2 5x col3 $\begin{bmatrix} 12+3+15 \\ 0+16+5 \end{bmatrix} = \begin{bmatrix} 49 \\ 21 \end{bmatrix}$

$$\begin{bmatrix} 1 & 2 & 7 \end{bmatrix} \begin{bmatrix} - & - & - \\ - & - & - \\ - & - & - \end{bmatrix}$$

1x3   3x3

by row   row x matrix = row

column x matrix = column $[4 \; 24 \; 35]$

1x row1 + 2x row2 + 7x row3

$$\begin{bmatrix} 1 & 2 & 1 \\ 3 & 8 & 1 \\ 0 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 4 & 1 \end{bmatrix}$$

Subtract 3x row1 from row 2

$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ → identity matrix

$\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ Elementary Matrix

E_{21} → fixed the 2 1 position. row 2, column 1

"E" is labeled by the row and column it is fixing. row 1st then column.

Steps: Subtract 2x row2 from row3

$$\underbrace{\begin{bmatrix} 1 & 2 & 1 \\ 0 & 2 & -2 \\ 0 & 0 & 5 \end{bmatrix}}_{U} \qquad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} E_{32}$$

$(E_{21} A) E_{32} = U$

multiply the E's first to get the matrix that does everything at once.

$A = U$ what matrix takes me from A to U?

$\leftarrow \underbrace{(E_{32} E_{21})}A = U$

Associative Law of Matrices

Associative Law of Matrices: Given three matrices A, B, and C of the same size, the order in which we multiply does not change the result.

=> mat mult is associative

Permutation

Exchange rows 1 and 2

$$\underbrace{\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}}_{P} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} c & d \\ a & b \end{bmatrix}$$

- exchange the rows of the identity matrix to find P.

take none of that col

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

take one of this one

take one of c1 => they flip. take none of c2

$$\begin{bmatrix} 0 \\ 1 \end{bmatrix} \begin{bmatrix} b \\ d \end{bmatrix}_{c_1} \quad \begin{bmatrix} 1 \\ 0 \end{bmatrix} \begin{bmatrix} a \\ c \end{bmatrix}_{c_2} = \begin{bmatrix} b & a \\ d & c \end{bmatrix}$$

- how to exchange columns of a matrix?

$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} b & a \\ d & c \end{bmatrix}$ inverse? $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$? not invert

$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ inverse

Column operations, multiply on the right. row operations, multiply on the left.

$A \times B \neq B \times A$ Commutative Law does not apply to matrices

Don't think how to get from A to U, but how to get from U to A.

"inverse"

Inverse of a matrix:

$$\underset{E}{\begin{bmatrix} 1 & 0 & 0 \\ -3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$$ find the matrix that undoes the elimination to give the identity.

$r_2 + 3r_1$ $\underset{I}{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}$ $E^{-1}\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Mathematics for Machine Learning 2.2-2.3 (Deisenroth et al.)

The system of linear equations

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 2 & 2 \\ 2 & 0 & 3 & 1 \end{array}\right]$$ has no solution, adding $r_1 + r_2$ yields $2\ 0\ 3 | 5$, which contradicts $r_3: 2\ 0\ 3 | 1$

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 2 & 2 \\ 0 & 1 & 1 & 2 \end{array}\right]$$ r_1 and r_3 give us $x_1 = 1$ $r_1 + r_2 = 2\ 0\ 3 | 5 \Rightarrow x_3 = 1$. Thus, $(1,1,1)$ is the only solution.

$$\left[\begin{array}{ccc|c} 1 & 1 & 1 & 3 \\ 1 & -1 & 2 & 2 \\ 2 & 0 & 3 & 5 \end{array}\right]$$ $1+2=3$, its redundant. Thus we have a 3-dimensional system operating on a 2 dimensional plane. Thus this system has infinitely many solutions.

Def: Matrix: with $m,n \in \mathbb{N}$ a real-valued (m,n) matrix A is a $m \cdot n$ tuple of elements a_{ij}, $i=1 \dots m$, $j=1 \dots n$, which is ordered according to a rectangular scheme consisting of m rows and n columns.

Matrix Addition: $$A + B := \begin{bmatrix} a_{11}+b_{11} & \cdots & a_{1n}+b_{1n} \\ \vdots & & \vdots \\ a_{m1}+b_{m1} & \cdots & a_{mn}+b_{mn} \end{bmatrix} \in \mathbb{R}^{m \times n}$$

Matrix Multiplication

for matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{n \times k}$, the elements c_{ij} of the product $C = AB \in \mathbb{R}^{m \times k}$ are computed as:

$$c_{ij} = \sum_{l=1}^{n} a_{il} b_{lj} \quad i = 1 \dots m \quad j = 1 \dots k$$

This summation means to compute c_{ij}, we multiply the elements of the ith row of A with the jth column of B and sum them. $\Rightarrow$ dot product

$\Rightarrow$ matrices can only be multiplied if their "neighboring" dimensions match.

$$\underbrace{\underset{n\times k}{A}\ \underset{k\times m}{B}}_{\text{must match}} = \underset{n\times m}{C}$$

The product BA is not defined because $n \neq m$. AB is defined because $k = k$

match! product is defined.

$A = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} \in \mathbb{R}^{2\times3}$, $B = \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} \in \mathbb{R}^{3\times2}$. We obtain:

$$AB = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}\begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix} \in \mathbb{R}^{2\times2} \quad \begin{matrix} 0+2+0 & 2-2+3 \\ 0+2+0 & 6-2+1 \end{matrix} = \begin{bmatrix} 2 & 3 \\ 2 & 5 \end{bmatrix}$$

$$BA = \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 4 & 2 \\ -2 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 0+6 & 4 & 2 \\ -2 & 0 & 2 \\ 3 & 2 & 1 \end{bmatrix}$$

Mat Mult is not commutative, $AB \neq BA$.

Def: Identity Matrix: In $\mathbb{R}^{n\times n}$, we define the identity matrix as the $n\times n$ matrix containing 1 on the diagonal and 0 everywhere else. $I_n = \begin{bmatrix} 1 & \cdots & 0 & \cdots & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & & \vdots & & 0 \\ \vdots & & \vdots & & \vdots \\ 0 & & 0 & & 1 \end{bmatrix}$

Properties of matrixes:

Associativity: $\forall A \in \mathbb{R}^{m\times n}, B \in \mathbb{R}^{n\times p}, C \in \mathbb{R}^{p\times q}: (AB)C = A(BC)$ for any matrices "A" of size "$m\times n$, B of size $n\times p$, and C of size $p\times q$, $(AB)C = A(BC) \Rightarrow$ the product of 2 matrices times a 3rd matrix is the same as the product of switching one of the two matrices in the parentheses with the one outside.

Distributivity: $\forall A, B \in \mathbb{R}^{m\times n}$ $C, D \in \mathbb{R}^{n\times p}: (A+B)C = AC + BC$; $A(C+D) = AC + AD$.

you can distribute scalar matrix A across the sum of $(C+D)$ like so: the product of AC + the product of AD is the same as the product of $A(C+D)$

Multiplication with the identity matrix: $\forall A \in \mathbb{R}^{m\times n}: I_m A = A I_n = A$; $I_m \neq I_n$

Multiplying matrix $A^{m\times n}$ · Identity Matrix I_m ($I_m A$) is equivalent to A · Identity$_n$: AI_n, which is equivalent to A.

Def: Inverse: Consider a square matrix $A \in \mathbb{R}^{n\times n}$. Let matrix $B \in \mathbb{R}^{n\times n}$ have the property that $AB = I_n = BA$. B is called the inverse of A and denoted by A^{-1}.

The inverse of a matrix is the matrix that allows you to revert a matrix back to its original state after performing a transformation.

Not every matrix A has an inverse A^{-1}. If it does, the matrix A is called regular / invertible / unique.

$\Rightarrow$ when the matrix inverse exists, it is unique.

The Matrices $A = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 4 & 5 \\ 6 & 7 & 7 \end{bmatrix}$, $B = \begin{bmatrix} -7 & -7 & 6 \\ 2 & 1 & -1 \\ 4 & 5 & 4 \end{bmatrix}$ are inverse to each other since $AB = I = BA$

$AB = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 4 & 5 \\ 6 & 7 & 7 \end{bmatrix} \begin{bmatrix} -7 & -7 & 6 \\ 2 & 1 & -1 \\ 4 & 5 & 4 \end{bmatrix} \begin{bmatrix} -7 & 4 & 4 \\ -28 & 4 & 25 \\ 36 & -7 & 28 \end{bmatrix}$ $BA = \begin{bmatrix} -7 & -7 & 6 \\ 2 & 1 & -1 \\ 4 & 5 & 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 1 \\ 4 & 4 & 5 \\ 6 & 7 & 7 \end{bmatrix} = \begin{bmatrix} -7 & 4 & 4 \\ -28 & 4 & 25 \\ 36 & -7 & 28 \end{bmatrix}$

$$\begin{bmatrix} -7 & & \\ & 4 & \\ & & 28 \end{bmatrix} = I \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} -7 & & \\ & 4 & \\ & & 28 \end{bmatrix}$$

Def: Transpose: For $A \in \mathbb{R}^{m \times n}$ the matrix $B \in \mathbb{R}^{n \times m}$ with $b_{ij} = a_{ji}$ is called the transpose of A. We write $B = A^T$

$AA^{-1} = I = A^{-1}A$ $\quad (AB)^{-1} = B^{-1}A^{-1}$ $\quad (A+B)^{-1} \neq A^{-1} + B^{-1}$

$(A^T)^T = A$ $\quad (AB)^T = B^T A^T$ $\quad (A+B)^T = A^T + B^T$

Def: Symmetric Matrix: A matrix $A \in \mathbb{R}^{n \times n}$ is symmetric if $A = A^T$

* only $n \times n$ matrices can be symmetric
* the sum of symmetric matrices is also symmetric, but the product of symmetric matrices is not.

multiplication by a scalar: $A \in \mathbb{R}^{m \times n}$ $\lambda \in \mathbb{R}$ $\lambda A = K$, $k_{ij} = \lambda a_{ij}$ => λ scales each element of A.

Associativity: $\lambda(BC) = (\lambda B)C = B(\lambda C) = \lambda(BC)$, $B \in \mathbb{R}^{m \times n}$, $C \in \mathbb{R}^{n \times k}$

=> allows us to move scalars

$(\lambda C)^T = C^T \lambda^T = C^T \lambda = \lambda C^T$; $\lambda = \lambda^T \; \forall \lambda \in \mathbb{R}$

$C = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \lambda = \begin{bmatrix} \lambda & 2\lambda \\ 3\lambda & 4\lambda \end{bmatrix} + \begin{bmatrix} 4 & 24 \\ 34 & 44 \end{bmatrix} = \lambda C + 4C$

Distributivity:

$\lambda(B+C) = \lambda B + \lambda C$, $B, C \in \mathbb{R}^{m \times n}$

$$\begin{aligned} 2x_1 + 3x_2 + 5x_3 &= 1 \\ 4x_1 - 2x_2 - 7x_3 &= 8 \\ 9x_1 + 5x_2 - 3x_3 &= 2 \end{aligned} \Rightarrow \left[\begin{array}{ccc|c} 2 & 3 & 5 & 1 \\ 4 & -2 & -7 & 8 \\ 9 & 5 & -3 & 2 \end{array}\right] = \underbrace{\begin{bmatrix} 2 & 3 & 5 \\ 4 & -2 & -7 \\ 9 & 5 & -3 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}}_{x} = \underbrace{\begin{bmatrix} 1 \\ 8 \\ 2 \end{bmatrix}}_{b}$$

Matrix form → $Ax = b$ product Ax is a linear combination of the columns of A.

$$\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 42 \\ 8 \end{bmatrix}$$

2 equations, 4 unknowns, thus we should expect infinite solutions

$b = 42\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 8\begin{bmatrix} 0 \\ 1 \end{bmatrix}$ $[42, 8, 0, 0]$ → particular solution

To capture all other solutions, we express a third column using the first two columns:

$$\begin{bmatrix} 8 \\ 2 \end{bmatrix} = 8\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 2\begin{bmatrix} 0 \\ 1 \end{bmatrix} \Rightarrow 0 = 8c_1 + 2c_2 - c_3 + 0c_4 \quad (x_1, x_2, x_3, x_4) = (8, 2, -1, 0)$$

any scaling by $\lambda_1 \in \mathbb{R}$ produces the zero vector: $\begin{bmatrix} 1 & 0 & 8 & -4 \\ 0 & 1 & 2 & 12 \end{bmatrix}\left(\lambda \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix}\right) = \lambda(8c_1 + 2c_2 - c_3) = 0$

We can use the 4th column instead for the same result: $\lambda \begin{bmatrix} -4 \\ 12 \\ 0 \\ -1 \end{bmatrix} = \lambda(-4c_1 + 12c_2 - c_4) = 0$

General Solution – the set of all solutions:

$$\left\{ x \in \mathbb{R}^4 : x = \begin{bmatrix} 42 \\ 8 \\ 0 \\ 0 \end{bmatrix} + \lambda_1 \begin{bmatrix} 8 \\ 2 \\ -1 \\ 0 \end{bmatrix} + \lambda_2 \begin{bmatrix} -4 \\ 12 \\ 0 \\ -1 \end{bmatrix}, \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

1) Find particular solution $Ax = b$
2) Find all solutions to $Ax = 0$
3) Combine the solutions from steps 1+2.

Elementary Transformations:

- exchange of two rows
- multiplication of a row with a constant
- addition of two rows

Definition: Row-Echelon Form: A matrix is in row-echelon form if:

- All rows that contain only zeros are at the bottom of the matrix; correspondingly, all rows that contain at least one nonzero element that are on top of rows that contain only zeroes.
- Looking at nonzero rows only, the first nonzero number from the left (pivot point / leading coefficient) is always strictly to the right of the pivot of the row above it.
  - basic variables correspond to the pivots ⇒ useful for finding a particular solution.
  - free variables are all other variables

Reduced-Row-Echelon Form – An equation system is in reduced row-echelon form if:

- it is in row-echelon form
- every pivot is 1
- the pivot is the only nonzero entry in its column.

Gaussian Elimination – an algorithm that performs elementary transformations to bring a system of linear equations into reduced row-echelon form.

$$A = \begin{bmatrix} \boxed{1} & 3 & 0 & 0 & 3 \\ 0 & 0 & \boxed{1} & 0 & 9 \\ 0 & 0 & 0 & \boxed{1} & -4 \end{bmatrix}$$

We need to express the non-pivot columns as a linear combination of the pivot column.
We express the non-pivot columns in terms of sums and multiples of the pivot columns that are on their left.

C_2 is $3 \times C_1$: $\begin{bmatrix}3\\-1\\0\\0\\0\end{bmatrix}$ $\begin{bmatrix}3\\0\\9\\-4\\-1\end{bmatrix}$: $\left\{ x \in \mathbb{R}^5 : x = \lambda_1 \begin{bmatrix}3\\-1\\0\\0\\0\end{bmatrix} + \lambda_2 \begin{bmatrix}3\\0\\9\\-4\\-1\end{bmatrix}, \lambda_1, \lambda_2 \in \mathbb{R} \right\}$

Minus one trick: $Ax = 0$, $A \in \mathbb{R}^{k \times n}$, $x \in \mathbb{R}^n$.

Assume that A is in reduced row-echelon form without any rows that just contain zeroes

$$A = \begin{bmatrix} 0 \cdots 0 & 1 & * \cdots * & 0 & * \cdots * & \cdots & 0 & \cdots * \\ \vdots \\ 0 \cdots 0 & \cdots & 0 \cdots 0 & \cdots & 0 \cdots & 0 & \cdots 1 \cdots 0 \end{bmatrix}$$

where $*$ can be any arbitrary real number, with the constraints that the first nonzero entry per row must be 1 and all other entries 0. The columns $j_1 \ldots j_k$ with pivots are the standard unit vectors $e_1 \ldots e_k \in \mathbb{R}^k$

We extend to an $n \times n$ matrix by adding $n-k$ rows of form $[0 \ldots 0 \, -1 \, 0 \ldots 0]$ so that the diagonal of Augmented Matrix A contains either a 1 or -1. Thus, the columns of A that contain the -1 as pivots are solutions of the homogenous equation system $Ax = 0$.

$\Rightarrow$ These columns form a basis of the solution space $Ax = 0$: the nullspace / kernel

$$\begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & -4 \end{bmatrix} \quad \begin{bmatrix} 1 & 3 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 9 \\ 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & -1 \end{bmatrix} \quad \begin{bmatrix}3\\-1\\0\\0\\0\end{bmatrix} \quad \begin{bmatrix}3\\0\\9\\-4\\-1\end{bmatrix} \quad \left\{ x \in \mathbb{R}^5 : x = \lambda_1 \begin{bmatrix}3\\-1\\0\\0\\0\end{bmatrix} + \lambda_2 \begin{bmatrix}3\\0\\9\\-4\\-1\end{bmatrix}, \lambda_1, \lambda_2 \in \mathbb{R} \right\}$$

Calculating the inverse: To find A^{-1} of $A \in \mathbb{R}^{n \times n}$, we need to find X that satisfies $AX = I_n$

We can use augmented matrix notation $[A \mid I_n] \ldots [I_n \mid A^{-1}]$

If we bring the augmented equation system into reduced row echelon form, we can read out the inverse on the right-hand side:

$$\begin{bmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \left[\begin{array}{cccc|cccc} 1 & 0 & 2 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 & 0 & 1 & 0 \\ 1 & 1 & 1 & 1 & 0 & 0 & 0 & 1 \end{array}\right]$$

use Gaussian Elimination

$$\left[\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & -1 & 2 & -2 & 2 \\ 0 & 1 & 0 & 0 & 1 & -1 & 2 & -2 \\ 0 & 0 & 1 & 0 & 1 & -1 & 1 & -1 \\ 0 & 0 & 0 & 1 & -1 & 0 & -1 & 2 \end{array}\right] A^{-1}$$

Let x_* be a solution of $Ax = b$. The key idea is to set up an iteration of the form:

$$x^{(k+1)} = Cx^{(k)} + d$$

for suitable C and d that reduces the residual error $\|x^{(k+1)} - x_*\|$ in every iteration and converges to x_*.

$$\left[\begin{array}{cc|c} 2 & 1 & 3 \\ 1 & 2 & 4 \end{array}\right] ① \cdot \tfrac{1}{2} \quad \left[\begin{array}{cc|c} 1 & 1/2 & 3/2 \\ 1 & 2 & 4 \end{array}\right] - ① \quad \left[\begin{array}{cc|c} 1 & 1/2 & 3/2 \\ 0 & 1/2 & 5/2 \end{array}\right] \cdot ② \quad \left[\begin{array}{cc|c} 1 & 0 & -1 \\ 0 & 1/2 & 5/2 \end{array}\right] \times 2$$

$\left[\begin{array}{cc|c}1&0&-1\\0&1&5\end{array}\right]$ $\quad -1\begin{bmatrix}2\\1\end{bmatrix}+5\begin{bmatrix}1\\2\end{bmatrix} = \begin{matrix}-2+5\\-1+10\end{matrix} = \begin{bmatrix}3\\9\end{bmatrix}$

$\left[\begin{array}{cc|c}2&1&3\\1&2&4\end{array}\right] \xrightarrow{-1/2\,①} \left[\begin{array}{cc|c}1&1/2&3/2\\0&3/2&5/2\end{array}\right]$

$\left[\begin{array}{cc|c}2&1&3\\1&2&4\end{array}\right] \xrightarrow{-1/2\,①} \left[\begin{array}{cc|c}2&1&3\\0&1&5/2\end{array}\right]$

$y = 5/2$

$2x + 5/2 = 3$

$2x = 1/2$

$x =$

$\frac{4}{3}\begin{bmatrix}2\\1\end{bmatrix} + \frac{3}{4}\begin{bmatrix}1\\2\end{bmatrix} = [$

Mit LinAlg Lecture 3- Multiplication and Inverse Matrices

$A_{m\times n}\; B_{n\times p} = C\,(AB)_{m\times p}$

C_{34} = (row 3 of A) · (column 4 of B)

$A_{31}\cdot B_{14} \quad A_{32}\cdot B_{24} \quad A_{33}\cdot B_{34}\dots = \sum_{k=1}^{n} a_{3k}b_{k4}$

Columns of C are combinations of columns of A.

Rows of C are combinations of Rows of B.

$A_{m\times n}\quad B_{n\times p}\quad C_{m\times p}$

Column of A · row of B $\quad m\times 1 \quad 1\times p$

$$\begin{bmatrix}2\\3\\4\end{bmatrix}\begin{bmatrix}1&6\end{bmatrix} = \begin{bmatrix}2&12\\3&18\\4&24\end{bmatrix} \rightarrow \begin{matrix}2&12\\3&18\\4&24\end{matrix}$$

AB = Sum of (Cols of A) · (rows of B)

$$\begin{bmatrix}2&7\\3&8\\4&9\end{bmatrix}\begin{bmatrix}1&6\\0&0\end{bmatrix} = \begin{bmatrix}2\\3\\4\end{bmatrix}\begin{bmatrix}1&6\end{bmatrix} + \begin{bmatrix}7\\8\\9\end{bmatrix}\begin{bmatrix}0&0\end{bmatrix}$$

Block

$$A\left[\begin{array}{c|c}A_1&A_2\\\hline A_3&A_4\end{array}\right] \; B\left[\begin{array}{c|c}B_1&B_2\\\hline B_3&B_4\end{array}\right] = \begin{bmatrix}C_1&C_2\\C_3&C_4\end{bmatrix}$$

$C_1: A_1B_1 + A_2B_3$

$C_2\; A_3B_3 + A_4B_4$

$C_3: B_1A_1 + B_2A_3$

$C_4: B_3A_2 + B_4A_4$

Inverses (Square matrices) $\quad A^{-1}\; A$

If A^{-1} exists, $A^{-1}A = I$ (left inverse) $\quad I = AA^{-1}$ (right inverse)

↑ invertible / nonsingular

Singular case: no inverse

$A = \begin{bmatrix}1&3\\2&6\end{bmatrix}$ I can find a vector x with $Ax = 0 \Rightarrow$ matrix not invertible

$$\begin{bmatrix} 1 & 3 \\ 2 & 4 \end{bmatrix}\begin{bmatrix} 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

↳ non zero vector x

$\Rightarrow$ we can take the vector x to zero, and there is no way for the inverse A^{-1} to recover it.

$$\underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}} \underset{A^{-1}}{\begin{bmatrix} a & c \\ b & d \end{bmatrix}} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$A \times$ col j of A^{-1} = col j of I

Gauss-Jordan (solve 2 problems at once)

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \qquad \left[\begin{array}{cc|cc} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{array}\right] \to \left[\begin{array}{cccc} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{array}\right]$$

$$\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}\begin{bmatrix} c \\ d \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \to \left[\begin{array}{cc|cc} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{array}\right] \qquad \underset{A}{\begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}}\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$\qquad\qquad\qquad\qquad\qquad I \qquad A^{-1}$

$$E[A\ I] = [I\ A^{-1}] \qquad EA = I \text{ tells us } E = A^{-1}$$

Mathematics for Machine Learning 2.4-2.5

def group: Consider a set G and an operation $\otimes : G \times G \to G$ defined on G. Then $G := (G, \otimes)$ is called a group if the following hold:

$\otimes$: Kronecker Product

$$A \otimes B = \begin{pmatrix} aB & bB \\ cB & dB \end{pmatrix}$$

1) closure of G under $\otimes$: $\forall x, y \in G : x \otimes y \in G$
2) Associativity: $\forall x, y, z \in G : (x \otimes y) \otimes z = x \otimes (y \otimes z)$
3) neutral element: $\exists e \in G \ \forall x \in G : x \otimes e = x$ and $e \otimes x = x$
4) Inverse element: $\forall x \in G \ \exists y \in G : x \otimes y = e$ and $y \otimes x = e$, where e is the natural element. we often write x^{-1} to denote the inverse element of x.

The inverse element is defined with respect to the operation $\otimes$ and does not necessarily mean $\frac{1}{x}$

5) Abelian Group: $\forall x, y \in G: x \otimes y = y \otimes x$, then $G = (G, \otimes)$ is an Abelian group (commutative)