

Part I

高等数学

Chapter 1

函数、极限、连续

1.1 考点解析

1.2 经典例题

例 1. (1) $\lim_{x \rightarrow 0} (\cos x)^{\frac{1}{\ln(1+x^2)}}$

$$\begin{aligned} \text{sol} &= \lim_{x \rightarrow 0} [1 + (\cos x - 1)]^{\frac{1}{\ln(1+x^2)}} \\ &= \exp \lim_{x \rightarrow 0} \frac{\cos x - 1}{\ln(1+x^2)} = \exp \lim_{x \rightarrow 0} \frac{-\frac{1}{2}x^2}{x^2} = e^{-\frac{1}{2}} \end{aligned}$$

(2) $\lim_{x \rightarrow +\infty} (\sqrt{x + \sqrt{x}} - \sqrt{x - \sqrt{x}})$

$$\begin{aligned} \text{sol} &= \lim_{x \rightarrow +\infty} \frac{2\sqrt{x}}{\sqrt{x + \sqrt{x}} + \sqrt{x - \sqrt{x}}} \\ &= \lim_{x \rightarrow +\infty} \frac{2}{\sqrt{1 + \frac{1}{\sqrt{x}}} + \sqrt{1 - \frac{1}{\sqrt{x}}}} = 1 \end{aligned}$$

例 2. (1) $\lim_{x \rightarrow 0} \left[\frac{a}{x} - \left(\frac{1}{x^2} - a^2 \right) \ln(1 + ax) \right]$, a 为常数。

$$\begin{aligned} \text{sol} &= \lim_{x \rightarrow 0} \frac{ax + \ln(1 + ax)}{x^2} + a^2 \lim_{x \rightarrow 0} \ln(1 + ax) \\ &= \lim_{x \rightarrow 0} \frac{a - \frac{a}{1+ax}}{2x} + a^2 \lim_{x \rightarrow 0} ax = \lim_{x \rightarrow 0} \frac{a^2}{2(1 + ax)} + 0 = \frac{1}{2}a^2 \end{aligned}$$

$$\begin{aligned}
(2) \quad & \lim_{x \rightarrow 0} \frac{e^{\sin 2x} - e^{2 \sin x}}{x^3} \\
\text{sol} = & \lim_{x \rightarrow 0} e^{2 \sin x} \lim_{x \rightarrow 0} \frac{e^{\sin 2x - 2 \sin x} - 1}{x^3} \\
= & \lim_{x \rightarrow 0} \frac{\sin 2x + 2 \sin x}{x^3} \\
= & \lim_{x \rightarrow 0} \frac{2 \cos 2x - 2 \cos x}{3x^2} & \text{or} = \lim_{x \rightarrow 0} \frac{2 \sin x (\cos x - 1)}{x^3} \\
= & \lim_{x \rightarrow 0} \frac{-4 \sin 2x + 2 \sin x}{6x} & = \lim_{x \rightarrow 0} -\frac{2 \sin x \frac{1}{2} x^2}{x^3} \\
= & \lim_{x \rightarrow 0} \frac{-8 \sin 2x + 2 \cos x}{6} & = \lim_{x \rightarrow 0} -\frac{\sin x}{x} \\
= & -1 & = -1
\end{aligned}$$

例 3. (1) $\lim_{x \rightarrow 0} \frac{(1+x)^{\frac{1}{x}} - e}{x}$

$$\begin{aligned}
\text{sol} = & \lim_{x \rightarrow 0} \frac{e^{\frac{\ln(1+x)}{x}} - e}{x} \\
= & \lim_{x \rightarrow 0} \frac{e^{\frac{\ln(1+x)}{x} - 1} - 1}{x} e = \lim_{x \rightarrow 0} \frac{\frac{\ln(1+x)}{x} - 1}{x} e \\
= & \lim_{x \rightarrow 0} \frac{\ln(1+x) - x}{x^2} e = \lim_{x \rightarrow 0} \frac{\frac{1}{1+x} - 1}{2x} e \\
= & \lim_{x \rightarrow 0} -\frac{1}{2(1+x)} e = -\frac{1}{2} e
\end{aligned}$$

(2) $\lim_{x \rightarrow 1} \frac{x - x^x}{1 - x + \ln x}$

$$\begin{aligned}
\text{sol} = & \lim_{x \rightarrow 1} \frac{1 - x^{(x-1)}}{1 - x + \ln x} \lim_{x \rightarrow 1} x = \lim_{x \rightarrow 1} \frac{1 - e^{(x-1) \ln x}}{1 - x + \ln x} \\
= & \lim_{x \rightarrow 1} \frac{-(x-1) \ln x}{1 - x + \ln x} = \lim_{x \rightarrow 1} -\frac{\ln x + \frac{x-1}{x}}{-1 + \frac{1}{x}} \\
= & \lim_{x \rightarrow 1} -\frac{\frac{1}{x} + \frac{1}{x^2}}{-\frac{1}{x^2}} = \lim_{x \rightarrow 1} \frac{x+1}{1} = 2
\end{aligned}$$

(3) $\lim_{n \rightarrow \infty} \left(n \tan \frac{1}{n} \right)^{n^2}$

$n \rightarrow +\infty$ 是 $x \rightarrow +\infty$ 的特殊情况, 将 n 换为 x , 有 $\lim_{x \rightarrow +\infty} (x \tan \frac{1}{x})^{x^2}$,
 $x \rightarrow +\infty$ 求极限不方便, 倒代换 $t = \frac{1}{x}$,

$$\text{sol} \xrightarrow{t=\frac{1}{x}} \lim_{t \rightarrow 0^+} \left(\frac{\tan t}{t} \right)^{\frac{1}{t^2}} = \exp \lim_{t \rightarrow 0^+} \frac{\ln(\frac{\tan t}{t})}{t^2}$$

$$\begin{aligned}
&= \exp \lim_{t \rightarrow 0^+} \frac{\ln(\frac{\tan t}{t} - 1 + 1)}{t^2} = \exp \lim_{t \rightarrow 0^+} \frac{\frac{\tan t}{t} - 1}{t^2} \\
&= \exp \lim_{t \rightarrow 0^+} \frac{\tan t - t}{t^3} = \exp \lim_{t \rightarrow 0^+} \frac{\frac{\sin^2 t}{\cos^2 t}}{3t^2} \\
&= \exp \lim_{t \rightarrow 0^+} \frac{t^2}{3t^2} = e^{\frac{1}{3}}
\end{aligned}$$

$$(4) \lim_{x \rightarrow 0^+} \ln x \ln(1-x)$$

$$\begin{aligned}
&\text{sol} \xrightarrow{\ln(1-x) \sim -x} \lim_{x \rightarrow 0^+} -x \ln x \\
&= \lim_{x \rightarrow 0^+} -\frac{\ln x}{\frac{1}{x}} = \lim_{x \rightarrow 0^+} \frac{\frac{1}{x}}{\frac{1}{x^2}} \\
&= \lim_{x \rightarrow 0^+} x = 0
\end{aligned}$$

$$\text{例 4. (1) } \lim_{x \rightarrow +\infty} [x - (1 + e^{-x}) \ln(1 + e^x)]$$

sol : 看 e^x 不顺眼, 换

1°

$$\begin{aligned}
&\xrightarrow{t=e^x \rightarrow \infty} \lim_{t \rightarrow \infty} [\ln t - (1 + t^{-1}) \ln(1 + t)] \\
&= \lim_{t \rightarrow \infty} [\ln t - \ln(1 + t) - \frac{\ln(1 + t)}{t}] \\
&= \lim_{t \rightarrow \infty} [\ln \frac{t}{1 + t} - \frac{\ln(1 + t)}{t}] \\
&= \lim_{t \rightarrow \infty} \ln \frac{1}{\frac{1}{t} + 1} - \lim_{t \rightarrow \infty} \frac{\frac{1}{1+t}}{1} \\
&= 0 - 0 = 0
\end{aligned}$$

2°

$$\begin{aligned}
&\xrightarrow{t=e^{-x} \rightarrow 0^+} \lim_{t \rightarrow 0^+} [-\ln t - (1 + t) \ln(1 + t^{-1})] \\
&= \lim_{t \rightarrow 0^+} [-\ln t - \ln(1 + \frac{1}{t}) - t \ln(1 + \frac{1}{t})] \\
&= \lim_{t \rightarrow 0^+} [-\ln(t + 1)] - \lim_{t \rightarrow 0^+} \frac{\ln(1 + \frac{1}{t})}{\frac{1}{t}} \\
&= \lim_{t \rightarrow 0^+} [-\ln(t + 1)] - \lim_{t \rightarrow 0^+} \frac{\frac{-\frac{1}{t^2}}{1 + \frac{1}{t}}}{-\frac{1}{t^2}} \\
&= 0 - 0 = 0
\end{aligned}$$

3° 或恒等变换

$$\begin{aligned}
 &= \lim_{x \rightarrow +\infty} [x - \ln(1 + e^x)] - \lim_{x \rightarrow +\infty} \frac{\ln(1 + e^x)}{e^x} \\
 &= \lim_{x \rightarrow +\infty} [\ln e^x - \ln(1 + e^x)] - \lim_{x \rightarrow +\infty} \frac{\ln(1 + e^x)}{e^x} \\
 &= \lim_{x \rightarrow +\infty} \ln \frac{1 + e^x}{e^x} - \lim_{x \rightarrow +\infty} \frac{\frac{e^x}{1+e^x}}{e^x} \\
 &= 0 - 0 = 0
 \end{aligned}$$

$$(2) \lim_{x \rightarrow 0} \left(\frac{2+e^{\frac{1}{x}}}{1+e^{\frac{4}{x}}} + \frac{\sin x}{|x|} \right)$$

遇绝对值，分左右极限。

sol

$$\lim_{x \rightarrow 0^+} \left(\frac{2 + e^{\frac{1}{x}}}{1 + e^{\frac{4}{x}}} + \frac{\sin x}{|x|} \right) = \lim_{x \rightarrow 0^+} \left(\frac{2e^{-\frac{4}{x} + e^{-\frac{3}{x}}}}{e^{-\frac{4}{x} + 1}} + 1 \right) = 1$$

$$\lim_{x \rightarrow 0^-} \left(\frac{2 + e^{\frac{1}{x}}}{1 + e^{\frac{4}{x}}} + \frac{\sin x}{|x|} \right) = 1$$

\therefore 原式 = 1

$$(3) \lim_{x \rightarrow 0} \frac{3 \sin x + x^2 \cos \frac{1}{x}}{(1 + \cos x) \ln(1 + x)}$$

$$\begin{aligned}
 \text{sol} &= \lim_{x \rightarrow 0} \frac{3 \sin x + x^2 \cos \frac{1}{x}}{2x} = \lim_{x \rightarrow 0} \frac{3 \sin x}{2x} + \lim_{x \rightarrow 0} \frac{x \cos \frac{1}{x}}{2} \\
 &= \frac{3}{2}
 \end{aligned}$$

$$\text{例 5. (1) } \lim_{x \rightarrow +\infty} \left(\frac{\pi}{2} - \arctan x \right)^{\frac{1}{\ln x}}$$

$$\text{sol} = \exp \lim_{x \rightarrow +\infty} \frac{\ln(\frac{\pi}{2} - \arctan x)}{\ln x} = \exp \lim_{x \rightarrow +\infty} \frac{-x}{(1+x^2)(\frac{\pi}{2} - \arctan x)}$$

1°

$$= \exp \lim_{x \rightarrow +\infty} \frac{-\frac{1}{x}}{(\frac{1}{x^2} + 1)(\frac{\pi}{2} - \arctan x)}$$

$$= \exp \lim_{x \rightarrow +\infty} \frac{-\frac{x}{1+x^2}}{\frac{\pi}{2} - \arctan x}$$

$$= \exp \lim_{x \rightarrow +\infty} \frac{-\frac{1-x^2}{(1+x^2)^2}}{-\frac{1}{1+x^2}}$$

$$= \exp \lim_{x \rightarrow +\infty} \frac{1-x^2}{1+x^2} = e^{-1}$$

2°

$$\begin{aligned} &= \exp \lim_{x \rightarrow +\infty} \frac{-\frac{1}{x}}{(1+\frac{1}{x^2})(\frac{\pi}{x} - \arctan x)} \\ &= \exp \lim_{x \rightarrow +\infty} \frac{-\frac{1}{x}}{\frac{\pi}{2} - \arctan x} \\ &= \exp \lim_{x \rightarrow +\infty} \frac{\frac{1}{x^2}}{-\frac{1}{1+x^2}} = e^{-1} \end{aligned}$$

3°

$$\begin{aligned} &= \exp \lim_{x \rightarrow +\infty} \frac{-1}{2x(\frac{\pi}{2} - \arctan x) - 1} \\ &\xrightarrow{\text{先求}} \lim_{x \rightarrow +\infty} x(\frac{\pi}{2} - \arctan x) \\ &= \lim_{x \rightarrow +\infty} \frac{\frac{\pi}{2} - \arctan x}{\frac{1}{x}} \\ &= \lim_{x \rightarrow +\infty} \frac{-\frac{1}{1+x^2}}{-\frac{1}{x^2}} = 1 \\ &\therefore \exp \lim_{x \rightarrow +\infty} \frac{-1}{2-1} = e^{-1} \end{aligned}$$

$$(2) \lim_{x \rightarrow 0^+} \frac{1-\cos x}{\sqrt{1-x}-\cos \sqrt{x}}$$

$$\begin{aligned} \text{sol} &= \lim_{x \rightarrow 0^+} \frac{\sqrt{1-x} \cos x}{1-x-\cos^2 \sqrt{x}} \frac{1-\cos^2 x}{1+\cos^2 x} \\ &= \lim_{x \rightarrow 0^+} \frac{\sqrt{1-x} + \cos \sqrt{x}}{1+\cos x} \frac{\sin^2 x}{1-x-\cos^2 \sqrt{x}} \\ &= \lim_{x \rightarrow 0^+} \frac{2}{2} \frac{x^2}{\sin^2 \sqrt{x} - x} \\ &\xrightarrow{t=\sqrt{x}} \lim_{x \rightarrow 0^+} \frac{t^4}{\sin^2 t - t^2} = \lim_{x \rightarrow 0^+} \frac{2t^3}{\sin t \cos t - t} \\ &= \lim_{x \rightarrow 0^+} \frac{6t^2}{\cos^2 t - \sin^2 t - 1} = \lim_{x \rightarrow 0^+} \frac{3t^2}{-\sin^2 t} \\ &= -3 \end{aligned}$$

例 6. 已知 $\lim_{x \rightarrow \infty} [(x^5 + 6x^4 + 2)^\alpha - x] = \beta$, 求 α, β 的值。

$$\begin{aligned} \text{sol: } & \because \lim_{x \rightarrow \infty} [(x^5 + 6x^4 + 2)^\alpha - x] = \beta \\ & \therefore \lim_{x \rightarrow +\infty} [(x^5 + 7x^4 + 2)^\alpha - x] = \lim_{x \rightarrow +\infty} x \left[\frac{(x^5 + 7x^4 + 2)^\alpha}{x} - 1 \right] = 0 \\ & \text{即 } \lim_{x \rightarrow +\infty} \left(\frac{x^5 + 7x^4 + 2}{x^{\frac{1}{\alpha}}} \right)^\alpha = 1 \\ & \therefore \frac{1}{\alpha} = 5 \Rightarrow \alpha = 5 \end{aligned}$$

$$\text{又 } \because \beta = \lim_{x \rightarrow +\infty} (\sqrt[5]{x^5 + 7x^4 + 2} - x)$$

1°

$$\begin{aligned} &= \lim_{x \rightarrow +\infty} x(\sqrt[5]{x^5 + 7x^4 + 2} - 1) \\ &= \lim_{x \rightarrow +\infty} x \frac{1}{5} \left(\frac{7}{x} + \frac{2}{x^5} \right) = \frac{7}{5} \end{aligned}$$

2°

$$\begin{aligned} & \xrightarrow{t=\frac{1}{x} \rightarrow 0^+} \lim_{t \rightarrow 0^+} \left(\sqrt[5]{\frac{1}{t^5} + \frac{7}{t^4} + 2} - \frac{1}{t} \right) \\ &= \lim_{t \rightarrow 0^+} \frac{1}{t} (\sqrt[5]{1 + 7t + 2t^5} - 1) \\ &= \lim_{t \rightarrow 0^+} \frac{1}{t} \cdot \frac{1}{5} (7t + 2t^5) = \frac{7}{5} \end{aligned}$$

例 7. 设 $f(x)$ 在 $(0, +\infty)$ 内可导, $f(x) > 0$, $\lim_{x \rightarrow +\infty} f(x) = 1$ 且 $\lim_{h \rightarrow 0} \left[\frac{f(x+hx)}{f(x)} \right]^{\frac{1}{h}} = e^{\frac{1}{x}}$, 求 $f(x)$ 。

$$\begin{aligned} \text{sol: } & \because \lim_{h \rightarrow 0} \left[\frac{f(x+hx)}{f(x)} \right]^{\frac{1}{h}} = e^{\frac{1}{x}} \\ & \therefore \lim_{h \rightarrow 0} \frac{\ln \frac{f(x+hx)}{f(x)}}{h} = \frac{1}{x} \end{aligned}$$

1° “+1-1” 法

$$\lim_{h \rightarrow 0} \frac{\ln \left(\frac{f(x+hx)-f(x)}{f(x)} + 1 \right)}{h} = \frac{1}{x}$$

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{f(x+hx) - f(x)}{hf(x)} &= \frac{1}{x} \\ \lim_{h \rightarrow 0} \frac{f(x+hx) - f(x)}{hx} &= \frac{1}{x^2} f(x) \\ f'(x) &= \frac{1}{x^2} f(x) \\ \therefore f(x) &= e^{-\frac{1}{x}}\end{aligned}$$

2°

$$\begin{aligned}\lim_{h \rightarrow 0} \frac{\ln f(x+hx) - \ln f(x)}{h} &= \frac{1}{x} \\ \lim_{h \rightarrow 0} \frac{\ln f(x+hx) - \ln f(x)}{hx} \cdot x &= \frac{1}{x} \\ x \cdot [\ln f(x)]' &= \frac{1}{x} \\ \therefore (\ln f(x))' &= \frac{1}{x^2} \\ \ln f(x) &= -\frac{1}{x} \\ f(x) &= e^{-\frac{1}{x}}\end{aligned}$$

3° 泰勒展开法

$$\begin{aligned}f(x) &= f(x_0) + f'(x_0)(x - x_0) + o(x - x_0) \\ \therefore f(x + \Delta x) &= f(x) + f'(x)\Delta x + o(\Delta x) \\ \therefore \ln f(x + hx) &= \ln f(x) + [\ln f(x)]'hx + o(hx) \\ \text{则 } \lim_{h \rightarrow 0} \frac{\ln f(x + hx) - \ln f(x)}{h} &= \frac{1}{x} \\ \lim_{h \rightarrow 0} \frac{[\ln f(x)]'hx + o(hx)}{h} &= \frac{1}{x} \\ [\ln f(x)]' &= \frac{1}{x^2} \\ \therefore f(x) &= e^{-\frac{1}{x}}\end{aligned}$$

例 8. 设 $f(x)$ 在 $x=0$ 处可导, $f(0) \neq 0$, $f'(0) \neq 0$, 当 $h \rightarrow 0$ 时, $af(h) + bf(2h) - f(0)$ 是比 h 更高阶的无穷小, 求 a, b 的值。

$$\text{sol} : \because \lim_{h \rightarrow 0} af(h) + bf(2h) - f(0) = 0$$

并且 $f(x)$ 在 $x=0$ 可导, \rightarrow 连续, 代入

$$af(0) + bf(0) - f(0) = 0$$

$$(a + b - 1)f(0) = 0$$

$$a + b = 1$$

回代

$$\begin{aligned} & \lim_{h \rightarrow 0} \frac{af(h) + bf(2h) - (a+b)f(0)}{h} \\ &= \lim_{h \rightarrow 0} a \frac{f(h) - f(0)}{h} + \lim_{h \rightarrow 0} 2b \frac{f(2h) - f(0)}{2h} \\ &= af'(0) + 2bf'(0) = 0 \\ &\therefore \begin{cases} a + b = 1 \\ a + 2b = 0 \end{cases} \Rightarrow \begin{cases} a = 2 \\ b = -1 \end{cases} \end{aligned}$$

例 9. 设 $f(x)$ 为连续函数, $f(0) \neq 0$, 求极限 $\lim_{x \rightarrow 0} \frac{\int_0^x (x-t)f(t)dt}{x \int_0^x f(t)dt}$ 。

sol :

$$\begin{aligned} & \lim_{x \rightarrow 0} \frac{xf(x) + \int_0^x f(t)dt - xf(x)}{xf(x) + \int_0^x f(t)dt} \\ &= \lim_{x \rightarrow 0} \frac{\int_0^x f(t)dt}{xf(x) + \int_0^x f(t)dt} \end{aligned}$$

1°

$$\begin{aligned} \text{原式} &= \lim_{x \rightarrow 0} \frac{\frac{\int_0^x f(t)dt}{x}}{f(x) + \frac{\int_0^x f(t)dt}{x}} \\ &\text{先计算 } \lim_{x \rightarrow 0} \frac{\int_0^x f(t)dt}{x} \\ &= \lim_{x \rightarrow 0} f(x) \\ &= f(0) \\ \therefore \text{原式} &= \lim_{x \rightarrow 0} \frac{f(0)}{f(x) + f(0)} \\ &= \frac{1}{2} \end{aligned}$$

2° 泰勒展开

$$\text{设 } F(x) = \int_0^x f(t)dt$$

$$\begin{aligned}
F(x) &= F(0) + F'(0)x + o(x) \\
&= 0 + f(x)x + o(x) \\
\therefore \text{原式} &= \lim_{x \rightarrow 0} \frac{F(x)}{xf(x) + F(x)} \\
&= \lim_{x \rightarrow 0} \frac{xf(x) + o(x)}{xf(x) + xf(x) + o(x)} \\
&= \lim_{x \rightarrow 0} \frac{f(x) + \frac{o(x)}{x}}{2f(x) + \frac{o(x)}{x}} \\
&= \frac{1}{2}
\end{aligned}$$

例 10. 当 $x \rightarrow 0$ 时, $1 - \cos x \cos 2x \cos 3x$ 与 ax^n 为等价无穷小, 求常数 a 与 n 的值。

sol : 泰勒展开

$$\begin{aligned}
\cos x &= 1 - \frac{1}{2}x^2 + o(x^2) \\
\therefore 1 - \cos x \cos 2x \cos 3x \\
&= 1 - [1 - \frac{1}{2}x^2 + o(x^2)][1 - \frac{1}{2}(2x)^2 + o(x^2)][1 - \frac{1}{2}(3x)^2 + o(x^2)] \\
&= (\frac{1}{2} + \frac{4}{2} + \frac{9}{2})x^2 + o(x^2) \\
&= 7x^2 + o(x^2) \\
&\therefore \text{与 } ax^n \text{ 等价无穷小} \\
&\therefore n = 2, a = 7
\end{aligned}$$

例 11. 当 $x \rightarrow 0$ 时, $x^2 + \ln(1+x)\ln(1-x)$ 与 ax^n 为等价无穷小, 求常数 a 与 n 的值。

sol : 泰勒展开

$$\begin{aligned}
\ln(1+x) &= x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3) \\
\therefore x^2 + \ln(1+x)\ln(1-x) \\
&= x^2 + [x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3)][-x - \frac{1}{2}(-x)^2 + \frac{1}{3}(-x)^3 - o(x^3)] \\
&= x^2 - [x - \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3)][x + \frac{1}{2}x^2 + \frac{1}{3}x^3 + o(x^3)] \\
&= (-\frac{1}{3} - \frac{1}{3} + \frac{1}{4})x^4 + o(x^4)
\end{aligned}$$

$$\begin{aligned}
&= -\frac{5}{12}x^4 + o(x^4) \\
&\because \text{与 } ax^n \text{ 等价无穷小} \\
&\therefore n = 4, a = -\frac{5}{12}
\end{aligned}$$

例 12. 已知 $\lim_{x \rightarrow 0} \frac{a \tan x + b(1 - \cos x)}{c \ln(1 - 2x) + d(1 - e^{-x^2})} = 2$, ($a^2 + c^2 \neq 0$), 则 ()

$$A. b = 4d \quad B. b = -4d \quad C. a = 4c \quad D. a = -4c$$

sol :

1° 洛必达

$$\lim_{x \rightarrow 0} \frac{a \frac{1}{\cos^2 x} + b \sin x}{c \frac{-2}{1-2x} + d 2x e^{-x^2}} = \frac{a}{-2c} = 2 \Rightarrow a = -4c$$

2°

$$\lim_{x \rightarrow 0} \frac{a \frac{\tan x}{x} + b \frac{\ln(1 - \cos x)}{x}}{c \frac{\ln(1 - 2x)}{x} + d \frac{1 - e^{-x^2}}{x}} = \frac{a}{-2c} = 2 \Rightarrow a = -4c$$

例 13. 设 $f(x)$ 对一切正数 x_1, x_2 有 $f(x_1, x_2) = f(x_1) + f(x_2)$, 且 $f(x)$ 在 $x = 1$ 处连续, 证明 $f(x)$ 在 $(0, +\infty)$ 连续。

sol : 取 $x_1 = x_2 = 1 \Rightarrow f(1) = f(1) + f(1) \Rightarrow f(1) = 0$

$$\lim_{\Delta x \rightarrow 0} f(x + \Delta x) = \lim_{\Delta x \rightarrow 0} f\left[x\left(1 + \frac{\Delta x}{x}\right)\right]$$

$$= \lim_{\Delta x} \left[f(x) + f\left(1 + \frac{\Delta x}{x}\right) \right]$$

$\therefore f(x)$ 在 $x = 1$ 处连续

$$\therefore \lim_{\Delta x \rightarrow 0} f\left(1 + \frac{\Delta x}{x}\right) = f(1) = 0$$

$$\therefore \lim_{\Delta x \rightarrow 0} f(x + \Delta x) = \lim_{\Delta x \rightarrow 0} [f(x) + 0] = f(x)$$

$\therefore f(x)$ 在 $(0, +\infty)$ 连续

例 14. 设 $f(x)$ 在 $[0, +\infty)$ 上可导, $f(0) < 0$, $f'(x) \geq k > 0$, (k 为常数), 证明方程 $f(x) = 0$ 在 $(0, +\infty)$ 上有唯一根。

sol : 构造 $F(x) = f(x) - f(0) - kx$

$$F'(x) = f'(x) - k \geq 0$$

又 $\because F(0) = 0$
 $\therefore F(x) \geq F(0) = 0$
 $\therefore f(x) \geq f(0) + kx$
 \therefore 当 $f(0) + kx_0 = 0 \Rightarrow x_0 = -\frac{f(0)}{k} > 0$ 时 $f(-\frac{f(0)}{k}) \geq 0$
 且 $f(0) < 0, f'(x) \geq k > 0$
 $\therefore f(x) = 0$ 在 $(0, +\infty)$ 上有唯一根

例 15. 设 $f(x)$ 在 $[a, +\infty)$ 上二阶可导, $f(a) < 0, f'(a) > 0$ 当 $x > a$ 时, $f''(x) > 0$, 证明方程 $f(x)$ 在 $(a, +\infty)$ 上存在唯一根。

sol: $\because f''(x) > 0$
 \therefore 当 $x \in (a, +\infty)$ 时 $f'(x) > f'(a)$
 构造 $F(x) = f(x) - f(a) - f'(a)x$
 $F'(x) = f'(x) - f'(a) > 0 \quad (x \in (a, +\infty))$
 $\therefore f(x) - f(a) - f'(a)x > -f'(a)a$
 $f(x) > f(a) + f'(a)x = f'(a)a$
 令 $f(a) + f'(a)x_0 - f'(a)a = 0 \Rightarrow x_0 = a - \frac{f(a)}{f'(a)} > a$
 $\therefore f(x_0) > 0$
 又 $\because f(a) < 0, f'(x) > f'(a) > 0$
 $\therefore f(x) = 0$ 在 $(a, +\infty)$ 上存在唯一根

例 16. 设 $x_1 > 0$, 当 $n \geq 1$ 时, $x_{n+1} = \frac{1}{2}(x_n + \frac{a}{x_n})$, a 为正常数, 证明数列 $\{x_n\}$ 存在极限并求其极限。

sol: $x_{n+1} - x_n = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) - x_n = \frac{a - x_n^2}{2x_n}$
 $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right) \geq \sqrt{a}$
 $\therefore x_{n+1} - x_n \leq 0$
 $\therefore n \geq 2$ 时 $\{x_n\}$ 单调不增, 有下界为 $\sqrt{a} \Rightarrow \lim_{n \rightarrow \infty} x_n$ 存在
 设 $\lim_{n \rightarrow \infty} x_n = A$, 在 $x_{n+1} = \frac{1}{2}\left(x_n + \frac{a}{x_n}\right)$ 两边令 $n \rightarrow \infty$
 则有 $A = \frac{1}{2}\left(A + \frac{a}{A}\right) \Rightarrow A = \sqrt{a}$
 $\therefore \lim_{n \rightarrow \infty} x_n = \sqrt{a}$

例 17. 设 $a > 0$, $x_1 > \sqrt{a}$, 当 $n \geq 1$ 时, $x_{n+1} = \sqrt{a + x_n}$, 证明 $\{x_n\}$ 存在极限并求出其极限。

$$\text{sol} : x_{n+1}^2 = a + x_n$$

$$\therefore x_{n+1}^2 - x_n^2 = x_n - x_{n-1}$$

$$\therefore x_3^2 - x_2^2 = x_2 - x_1 = \sqrt{a + \sqrt{a}} - \sqrt{a} > 0$$

$$\therefore x_3 > x_2$$

$$x_{n+1} = \frac{a}{x_{n+1}} + \frac{x_n}{x_{n+1}} \leq \frac{a}{x_1} + 1 = \sqrt{a} + 1$$

显然 $x_{n+1} > x_n$, $\{x_n\}$ 单调增加, 有上界为 $\sqrt{a} + 1 \Rightarrow \lim_{n \rightarrow \infty} x_n$ 存在

设 $\lim_{n \rightarrow \infty} x_n = A$, 在 $x_{n+1}^2 = a + x_n$ 两边令 $n \rightarrow \infty$

$$\text{则有 } A^2 = a + A \Rightarrow A = \frac{1}{2} \pm \sqrt{\frac{1}{4} + a}$$

$$\text{又 } \because A \geq \sqrt{a} > 0 \quad \therefore A = \frac{1}{2} + \sqrt{\frac{1}{4} + a}$$

$$\therefore \lim_{n \rightarrow \infty} x_n = \frac{1}{2} + \sqrt{\frac{1}{4} + a}$$

例 18. 设 $x_1 > 0$, 当 $n \geq 1$ 时, $x_{n+1} = \sqrt{ax_n}$, 证明 $\{x_n\}$ 存在极限并求出其极限。

$$\begin{aligned} \text{sol} : x_{n+1} - x_n &= \sqrt{ax_n} - x_n \\ &= \sqrt{x_n}(\sqrt{a} - \sqrt{x_n}) \end{aligned}$$

$$\begin{aligned} x_{n+1} - a &= \sqrt{ax_n} - a \\ &= \sqrt{a}(\sqrt{x_n} - \sqrt{a}) \end{aligned}$$

(i) 当 $x_1 < a$ 时

$$\therefore \sqrt{x_1} - \sqrt{a} < 0 \therefore x_{n+1} - x_n > 0, \text{ 且 } x_{n+1} - a < 0 \Rightarrow x_{n+1} < a$$

$$\therefore \{x_n\} \text{ 单调递增, 且上界为 } a \Rightarrow \lim_{n \rightarrow \infty} x_n \text{ 存在}$$

设 $\lim_{n \rightarrow \infty} x_n = A$, 在 $x_{n+1} = \sqrt{ax_n}$ 两边令 $n \rightarrow \infty$

$$\text{则有 } A = \sqrt{aA} \Rightarrow A = a$$

$$\therefore \lim_{n \rightarrow \infty} x_n = a$$

(ii) 当 $x_1 > a$ 时

$$\therefore \sqrt{x_1} - \sqrt{a} > 0 \therefore x_{n+1} - x_n < 0, \text{ 且 } x_{n+1} - a > 0 \Rightarrow x_{n+1} > a$$

$\therefore \{x_n\}$ 单调递减, 且下界为 $a \Rightarrow \lim_{n \rightarrow \infty} x_n$ 存在

设 $\lim_{n \rightarrow \infty} x_n = A$, 在 $x_{n+1} = \sqrt{ax_n}$ 两边令 $n \rightarrow \infty$

则有 $A = \sqrt{aA} \Rightarrow A = a$

$\therefore \lim_{n \rightarrow \infty} x_n = a$

(iii) 当 $x_1 = a$ 时, $x_{n+1} = x_n = a \Rightarrow \lim_{n \rightarrow \infty} x_n = a$

\therefore 综上所述, 数列 $\{x_n\}$ 存在极限, 且 $\lim_{n \rightarrow \infty} x_n = a$

例 19. 设 $x_1 = 2$, 当 $n \geq 1$ 时, $x_{n+1} = 1 + \frac{x_n}{1+x_n}$, 证明 $\{x_n\}$ 存在极限并求出其极限。

$$\text{sol: } x_{n+1} - x_n = 1 + \frac{x_n}{x_{n+1}} - 1 - \frac{x_{n-1}}{1+x_{n-1}}$$

$$= \frac{x_n(1+x_{n-1}) - (1+x_n)x_{n-1}}{(1+x_n)(1+x_{n-1})}$$

$$\therefore x_2 = 1 + \frac{2}{1+2} = 1\frac{2}{3} < x_1$$

$\therefore x_{n+1} - x_n < 0$, 即 $\{x_n\}$ 单调递减

$\therefore x_{n+1} > 1 \quad \therefore$ 有下界为 1 $\Rightarrow \lim_{n \rightarrow \infty} x_n$ 存在

设 $\lim_{n \rightarrow \infty} x_n = A$, 对 $x_{n+1} = 1 + \frac{x_n}{1+x_n}$ 两边令 $n \rightarrow \infty$

$$\therefore A = 1 + \frac{A}{1+A} \Rightarrow A = \frac{1 \pm \sqrt{5}}{2}$$

$$\therefore A > 1 \quad \therefore A = \frac{1 + \sqrt{5}}{2}$$

$$\therefore \lim_{n \rightarrow \infty} x_n = \frac{1 + \sqrt{5}}{2}$$

例 20. 设 $0 < x_1 < 3$, 当 $n \geq 1$ 时, $x_{n+1} = \sqrt{x_n(3-x_n)}$, 证明 $\{x_n\}$ 存在极限并求出其极限。

$$\text{sol: } \frac{x_{n+1}}{x_n} = \frac{\sqrt{x_n(3-x_n)}}{x_n} = \sqrt{\frac{3}{x_n} - 1}$$

$$x_{n+1} = \sqrt{x_n(3-x_n)} \leq \frac{3}{2}$$

$$\text{or } x_{n+1}^2 - \frac{9}{4} = -x_n^2 + 3x_n - \frac{9}{4} = -(x_n - \frac{3}{2})^2 \leq 0$$

$$\therefore \text{当 } n \geq 2 \text{ 时 } \frac{x_{n+1}}{x_n} > 1, \text{ 且 } x_{n+1} \leq \frac{3}{2}$$

即 $\{x_n\}$ 单调不减, 且有上界为 $\frac{3}{2} \Rightarrow \lim_{n \rightarrow \infty} x_n$ 存在

设 $\lim_{n \rightarrow \infty} x_n = A$, 对 $x_{n+1} = \sqrt{x_n(3-x_n)}$ 两边令 $n \rightarrow \infty$

$$\therefore A = \sqrt{A(3-A)} \Rightarrow A = 0 \quad \text{or} \quad A = \frac{3}{2} \quad \because x_2 > 0 \therefore A = \frac{3}{2}$$

$$\therefore \lim_{n \rightarrow \infty} x_n = \frac{3}{2}$$

例 21. 设 $0 < a < 1$, $x_1 = \frac{a}{2}$, 当 $n \geq 1$ 时, $x_{n+1} = \frac{a+x_n^2}{2}$, 证明 $\{x_n\}$ 存在极限并求出其极限。

例 22. 设 $a_0 \geq b_0 > 0$, 当 $n \geq 0$ 时, $a_{n+1} = \frac{a_n+b_n}{2}$, $b_{n+1} = \frac{2a_nb_n}{a_n+b_n}$, 证明数列 $\{a_n\}$ 与 $\{b_n\}$ 的极限均存在, 并求 $\lim_{n \rightarrow \infty} a_n$, $\lim_{n \rightarrow \infty} b_n$ 。

$$\text{sol: } a_{n+1} - a_n = \frac{b_n - a_n}{2} \quad \frac{b_{n+1}}{b_n} = \frac{2a_n}{1_n + b_n}$$

$$\frac{a_{n+1}}{b_{n+1}} = \frac{(a_n + b_n)^2}{4a_nb_n} \geq \frac{(2\sqrt{a_nb_n})^2}{4a_nb_n} = 1$$

$$\therefore a_n \geq b_n$$

$\therefore \{a_n\}$ 为单调不增, $\{b_n\}$ 为单调不减, 且 $a_0 \geq a_n \geq b_n \geq b_0$

$\Rightarrow \lim_{n \rightarrow \infty} a_n, \lim_{n \rightarrow \infty} b_n$ 存在, 设 $\lim_{n \rightarrow \infty} a_n = A, \lim_{n \rightarrow \infty} b_n = B$

则分别对 $a_{n+1} = \frac{a_n + b_n}{2}, b_{n+1} = \frac{2a_nb_n}{a_n + b_n}$ 两边令 $n \rightarrow \infty$

$$\Rightarrow \begin{cases} A = \frac{A+B}{2} \\ B = \frac{2AB}{A+B} \end{cases} \Rightarrow A = B$$

$$\therefore a_{n+1}b_{n+1} = a_nb_n = \cdots = a_0b_0$$

$$\therefore AB = a_0b_0 \Rightarrow A = B = \sqrt{a_0b_0}$$

$$\therefore \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n = \sqrt{a_0b_0}$$

例 23. 设 $x_0 > 0, y_0 > 0$, 当 $n \geq 0$ 时, $x_{n+1} = \sqrt{x_n y_n}$, $y_{n+1} = \frac{x_n + y_n}{2}$, 证明数列 $\{x_n\}$ 与 $\{y_n\}$ 的极限均存在, 且 $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n$ 。

$$\text{sol: } \frac{x_{n+1}}{x_n} = \sqrt{\frac{y_n}{x_n}}, y_{n+1} - y_n = \frac{x_n - y_n}{2}$$

$$\because y_{n+1} = \frac{1}{2}(x_n + y_n) \geq \sqrt{x_n y_n} = x_{n+1}$$

$\therefore \{x_n\}$ 单调不减, $\{y_n\}$ 单调不增

$\because y_0 \geq y_n \geq x_n \geq x_0 \Rightarrow \lim_{n \rightarrow \infty} x_n, \lim_{n \rightarrow \infty} y_n$ 存在

设 $\lim_{n \rightarrow \infty} x_n = X, \lim_{n \rightarrow \infty} y_n = Y$, 则对 $x_{n+1} = \sqrt{x_n y_n}$,

$$y_{n+1} = \frac{x_n + y_n}{2} \text{ 两边分别令 } n \rightarrow \infty$$

$$\Rightarrow \begin{cases} X = \sqrt{XY} \\ Y = \frac{X+Y}{2} \end{cases} \Rightarrow X = Y$$

$$\text{即 } \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n$$

例 24. 设 $x_1 = a, x_2 = b, a < b$, 当 $n \geq 2$ 时, $x_{n+1} = \frac{x_n + x_{n-1}}{2}$, 求

(1) $y_n = x_n - x_{n-1}, n \geq 2$;

(2) $\sum_{k=2}^n y_k$;

(3) $\lim_{n \rightarrow \infty} x_n$.

sol :

(1)

$$x_{n+1} - x_n = \frac{x_n + x_{n+1}}{2} - x_n = -\frac{1}{2}(x_n - x_{n-1})$$

$$\therefore y_{n+1} = -\frac{1}{2}y_n$$

$$y_2 = x_2 - x_1 = b - a \Rightarrow y_n = \left(-\frac{1}{2}\right)^{n-2}(b - a)$$

(2)

$$\begin{aligned} \sum_{k=2}^n y_k &= \sum_{k=2}^n (b - a) \left(-\frac{1}{2}\right)^{k-2} \\ &= (b - a) \frac{1 - \left(-\frac{1}{2}\right)^{n-1}}{1 - \left(-\frac{1}{2}\right)} \\ &= (b - a) \frac{1 - \left(-\frac{1}{2}\right)^{n-1}}{\frac{3}{2}} \end{aligned}$$

(3)

$$x_n = y_n + y_{n-1} + \cdots + y_2 + x_1$$

$$\begin{aligned}
&= \sum_{k=2}^n y_k + x_1 \\
&= (b-a) \frac{1 - (-\frac{1}{2})^{n-1}}{\frac{3}{2}} + x_1 \\
\therefore \lim_{n \rightarrow \infty} x_n &= \lim_{n \rightarrow \infty} (b-a) \frac{1 - (-\frac{1}{2})^{n-1}}{\frac{3}{2}} + a \\
&= \frac{2}{3}(b-a) + a = \frac{1}{3}(2b-a)
\end{aligned}$$

例 25. 设 $x_1 = 1$, $x_2 = 2$, 当 $n \geq 2$ 时, $x_{n+1} = \sqrt{x_n x_{n+1}}$, 求

(1) $y_n = \ln x_n - \ln x_{n-1}$, $n \geq 2$;

(2) $\sum_{k=2}^n y_k$;

(3) $\lim_{n \rightarrow \infty} x_n$.

sol :

(1)

$$\begin{aligned}
\ln x_{n+1} &= \frac{1}{2}(\ln x_n + \ln x_{n-1}) \\
\ln x_{n+1} - \ln x_n &= -\frac{1}{2}(\ln x_n - \ln x_{n-1}) \\
\therefore y_{n+1} &= -\frac{1}{2}y_n \\
y_2 &= \ln 2 \\
\therefore y_n &= \left(-\frac{1}{2}\right)^{n-2} \ln 2
\end{aligned}$$

(2)

$$\sum_{k=2}^n y_k = \sum_{k=2}^n \ln 2 \left(-\frac{1}{2}\right)^{k-2} = \ln 2 \frac{1 - (-\frac{1}{2})^{n-1}}{\frac{3}{2}}$$

(3)

$$\begin{aligned}
\lim_{n \rightarrow \infty} x_n &= \exp \lim_{n \rightarrow \infty} \ln x_n \\
&= \exp \lim_{n \rightarrow \infty} [y_n + y_{n-1} + \cdots + y_2 + \ln x_1]
\end{aligned}$$

$$\begin{aligned}
&= \exp \lim_{n \rightarrow \infty} \left[\sum_{k=2}^n y_k + \ln x_1 \right] \\
&= \exp \lim_{n \rightarrow \infty} \left[\ln 2 \frac{1 - \left(-\frac{1}{2}\right)^{n-1}}{\frac{3}{2}} + \ln 1 \right] \\
&= \exp \left[\ln 2 \cdot \frac{2}{3} + 0 \right] \\
&= \sqrt[3]{4}
\end{aligned}$$

例 26. 设 n 为正整数, 且 $n\pi \leq x < (n+1)\pi$,

(1) 证明 $2n \leq \int_0^x |\sin t| dt < 2(n+1)$;

(2) 求 $\lim_{x \rightarrow +\infty} \frac{\int_0^x |\sin t| dt}{x}$ 。

sol :

(1)

$$\begin{aligned}
\int_0^{n\pi} |\sin t| dt &\leq \int_0^x |\sin t| dt \leq \int_0^{(n+1)\pi} |\sin t| dt \\
&\Downarrow \\
2n &\leq \int_0^x |\sin t| dt \leq 2(n+1)
\end{aligned}$$

(2)

$$\begin{aligned}
\frac{2n}{x} &\leq \frac{\int_0^x |\sin t| dt}{x} \leq \frac{2(n+1)}{x} \\
&\Downarrow \\
\lim_{n \rightarrow \infty} \frac{2n}{(n+1)\pi} &\leq \frac{2n}{x} \leq \frac{\int_0^x |\sin t| dt}{x} \leq \frac{2(n+1)}{x} \leq \lim_{n \rightarrow \infty} \frac{2(n+1)}{n\pi} \\
&\because \lim_{n \rightarrow \infty} \frac{2n}{(n+1)\pi} = \frac{2}{\pi} \\
&\quad \lim_{n \rightarrow \infty} \frac{2(n+1)}{n\pi} = \frac{2}{\pi} \\
&\therefore \lim_{x \rightarrow \infty} \frac{\int_0^x |\sin t| dt}{x} = \frac{2}{\pi}
\end{aligned}$$

例 27. 当 $0 < x < 1$ 时, 证明 $\sin \frac{\pi x}{2} > x$; 又设 $0 < x_1 < 1$, 当 $n \geq 1$ 时, $x_{n+1} = \sin \frac{\pi x_n}{2}$, 证明数列 $\{x_n\}$ 存在极限并求出其极限。sol :

(1)

$$\text{令 } f(x) = \sin \frac{\pi x}{2} - x$$

$$f'(x) = \frac{\pi}{2} \cos \frac{\pi x}{2} - 1$$

$$\text{令 } f'(x_0) = 0 \Rightarrow x_0 = \frac{2}{\pi} \arccos \frac{2}{\pi}$$

$$0 < x < x_0 \text{ 时 } f'(x) > 0 \quad \therefore f(x) > f(0) = 0$$

$$x_0 < x < 1 \text{ 时 } f'(x) > 0 \quad \therefore f(x) > f(0) = 0$$

$$\therefore f(x) > 0 \quad (x \in (0, 1))$$

$$\therefore \text{当 } 0 < x < 1 \text{ 时, } \sin \left[\frac{2}{\pi} x \right] > x$$

(2)

已知 $0 < x < 1$ 设 $0 < x_n < 1$

$$\text{则由 (1) 可知: } 0 < x_n < x_{n+1} = \frac{\pi x_n}{2} < 1$$

$$\therefore 0 < x_{n+1} < 1$$

$$\therefore x_{n+1} > x_n \Rightarrow \{x_n\} \text{ 单调递增} \Rightarrow \lim_{n \rightarrow \infty} \text{ 存在}$$

$$\text{设 } \lim_{n \rightarrow \infty} x_n = A$$

$$\text{则 } A = \sin \frac{\pi A}{2} \quad \text{显然 } A = 1$$

$$\therefore \lim_{n \rightarrow \infty} x_n = 1$$

例 28. 证明方程 $x^n + x^{n+1} + \cdots + x^2 + x = 1$, ($n \geq 2$) 在 $(0, 1)$ 上存在唯一根, 并将此根记为 x_n , 证明数列 $\{x_n\}$ 存在极限并求出其极限。

sol :

(1)

$$\text{令 } f(x) = x^n + x^{n+1} + \cdots + x^2 + x - 1$$

$$f^{[prime]}(x) = nx^{n-1} + (n-1)x^{n-2} + \cdots + 2x + 1$$

$$x \in (0, 1) \text{ 时 } f'(x) > 0 \text{ 显然成立}$$

$$\therefore f(0) = -1 < 0 \quad f(1) = n - 1 > 0$$

$$\therefore f(x) = 0 \text{ 在 } (0, 1) \text{ 上存在有唯一根, 即}$$

$$x^n + x^{n-1} + \cdots + x^2 + x = 1 \quad (n \geq 2) \text{ 在 } (0, 1) \text{ 上存在唯一根}$$

(2)

$$n=2 \text{ 时 } x^2 + x = 1 \Rightarrow x_2$$

$$n=3 \text{ 时 } x^3 + x^2 + x = 1 \Rightarrow x_3$$

$$n \text{ 时 } x_n^n + x_n^{n-1} + \cdots + x_n = 1 \Rightarrow x_n$$

$$n+1 \text{ 时 } x_{n+1}^{n+1} + x_{n+1}^n + \cdots + x_{n+1} = 1 \Rightarrow x_{n+1}$$

若 $x_{n+1} > x_n$, 则 $x_{n+1}^{n+1} < 0$ 矛盾

$$\therefore x_{n+1} \leq x_n \text{ 且 } x_n > 0 \Rightarrow \lim_{n \rightarrow \infty} x_n \text{ 存在}$$

$$\lim_{n \rightarrow \infty} x_n^n + x_n^{n-1} + \cdots + x_n^2 + x_n = 1 \Rightarrow \lim_{n \rightarrow \infty} x_n \cdot \frac{1 - x_n^n}{1 - x_n} = 1$$

$$\because 0 < x_n \leq x_2 = \frac{-1 + \sqrt{5}}{2} < 1$$

$$\therefore \lim_{n \rightarrow \infty} x_n^n = 0, \text{ 设 } \lim_{n \rightarrow \infty} x_n = A$$

$$\therefore \lim_{n \rightarrow \infty} x_n \cdot \frac{1 - x_n^n}{1 - x_n} = \lim_{n \rightarrow \infty} x_n \cdot \frac{1}{1 - x_n} = A \cdot \frac{1}{1 - A} = 1$$

$$\Rightarrow A = \frac{1}{2} \quad \therefore \lim_{n \rightarrow \infty} x_n = \frac{1}{2}$$

Chapter 2

导数与微分

2.1 考点解析

2.2 经典例题

例 1. 设 $f(x)$ 在 x_0 处二阶可导, 求 $\lim_{h \rightarrow 0} \frac{f(x_0+h)+f(x_0-h)-2f(x_0)}{h^2}$ 。

sol :

1°

$\because f(x)$ 在 x_0 处二阶可导

$\therefore f(x)$ 一阶导数在 x_0 的某个邻域内有定义

$$\begin{aligned} & \therefore \lim_{h \rightarrow 0} \frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{h^2} \\ &= \lim_{h \rightarrow 0} \frac{f'(x_0+h) - f'(x_0-h)}{2h} \\ &= \frac{1}{2} \left[\lim_{h \rightarrow 0} \frac{f'(x_0+h) - f'(x_0)}{h} + \lim_{h \rightarrow 0} \frac{f'(x_0-h) - f'(x_0)}{-h} \right] \\ &= \frac{1}{2} \cdot 2f''(x_0) = f''(x_0) \end{aligned}$$

2° 泰勒展开

$$\begin{aligned} f(x_0+h) &= f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + o(h^2) \\ \therefore \lim_{h \rightarrow 0} \frac{f(x_0+h) + f(x_0-h) - 2f(x_0)}{h^2} \\ &= \lim_{h \rightarrow 0} \frac{f(x_0) + f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + o(h^2)}{h^2} \end{aligned}$$

$$\begin{aligned}
& + \lim_{h \rightarrow 0} \frac{f(x_0) - f'(x_0)h + \frac{1}{2}f''(x_0)h^2 + o(h^2) - 2f(x_0)}{h^2} \\
& = \lim_{h \rightarrow 0} \frac{f''(x_0)h^2 + o(h^2)}{h^2} \\
& = f''(x_0)
\end{aligned}$$

例 2. 设 $f(0) = 0$, $f'(0) = 1$, $f''(0) = 2$, 求 $\lim_{x \rightarrow 0} \frac{f(x) - x}{x^2}$ 。

sol :

1°

$$\lim_{x \rightarrow 0} \frac{f(x) - x}{x^2} = \lim_{x \rightarrow 0} \frac{f'(x) - 1}{2x} = \lim_{x \rightarrow 0} \frac{f''}{2} = 1$$

2°

$$\begin{aligned}
f(x + 0) &= f(0) + f'(0)x + \frac{1}{2}f''(0)x^2 + o(x^2) = x + x^2 + o(x^2) \\
\lim_{x \rightarrow 0} \frac{f(x) - x}{x^2} &= \lim_{x \rightarrow 0} \frac{x + x^2 + o(x^2) - x}{x^2} = \lim_{x \rightarrow 0} \frac{x^2 + o(x^2)}{x^2} = 1
\end{aligned}$$

例 3. 设 $f(x) = \begin{cases} \frac{\sin x}{x} & x \neq 0 \\ 1 & x = 0 \end{cases}$, 求 $f'(0)$, $f''(0)$ 。

$$\begin{aligned}
\text{sol : } f'(0) &= \lim_{x \rightarrow 0} \frac{\frac{\sin x}{x} - 1}{x} = \lim_{x \rightarrow 0} \frac{\sin x - x}{x^2} = \lim_{x \rightarrow 0} \frac{\cos x - 1}{2x} \\
&= \lim_{x \rightarrow 0} \frac{\frac{1}{2}x^2}{2x} = 0 \quad \text{or} \quad = \lim_{x \rightarrow 0} \frac{-\sin x}{2} = 0 \\
x \neq 0 \text{ 时 } f'(x) &= \frac{x \cos x - \sin x}{x^2} \\
\therefore f''(0) &= \lim_{x \rightarrow 0} \frac{f'(x) - f'(0)}{x} \\
&= \lim_{x \rightarrow 0} \frac{x \cos x - \sin x}{x^3} \\
&= \lim_{x \rightarrow 0} \frac{\cos x - x \sin x - \cos x}{3x^2} \\
&= \lim_{x \rightarrow 0} \frac{-\sin x}{3x} = -\frac{1}{3}
\end{aligned}$$

例 4. 设 $f(x) = \begin{cases} \frac{1}{x} - \frac{1}{e^x - 1} & x \neq 0 \\ \frac{1}{2} & x = 0 \end{cases}$, 试讨论 $f(x)$ 在 $x = 0$ 处的连续性、可导性。

sol :

(1) 连续性

$$\begin{aligned}
 \lim_{x \rightarrow 0} f(x) &= \lim_{x \rightarrow 0} \frac{1}{x} - \frac{1}{e^x - 1} \\
 &= \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x(e^x - 1)} = \lim_{x \rightarrow 0} \frac{e^x - 1 - x}{x^2} = \lim_{x \rightarrow 0} \frac{e^x - 1}{2x} = \lim_{x \rightarrow 0} \frac{e^x}{2} \\
 &= \frac{1}{2} = f(0) \\
 \therefore &\text{连续}
 \end{aligned}$$

(2) 可导性

$$\begin{aligned}
 f'(0) &= \lim_{x \rightarrow 0} \frac{f(x) - f(0)}{x} \\
 &= \lim_{x \rightarrow 0} \frac{\frac{1}{x} - \frac{1}{e^x - 1} - \frac{1}{2}}{x} = \lim_{x \rightarrow 0} \frac{(2-x)(e^x - 1) - 2x}{2x^2(e^x - 1)} \\
 &= \lim_{x \rightarrow 0} \frac{(2-x)e^x - 2 - x}{2x^3} = \lim_{x \rightarrow 0} \frac{-e^x + (2-x)e^x - 1}{6x^2} \\
 &= \lim_{x \rightarrow 0} \frac{-e^x + (1-x)e^x}{12x} = \lim_{x \rightarrow 0} \frac{-xe^x}{12x} = -\frac{1}{12} \\
 \therefore &\text{可导}
 \end{aligned}$$

例 5. $f(x)$ 在 $x = 0$ 处二阶可导, $f(0) = 0$, $g(x) = \begin{cases} \frac{f(x)}{x} & x \neq 0 \\ f'(0) & x = 0 \end{cases}$, 证明 $g'(x)$ 在 $x = 0$ 处连续。

$$\begin{aligned}
 \text{sol : } g'(0) &= \lim_{x \rightarrow 0} \frac{\frac{f(x)}{x} - f'(0)}{x} \\
 &= \lim_{x \rightarrow 0} \frac{f(x) - xf'(0)}{x^2} = \lim_{x \rightarrow 0} \frac{f'(x) - f'(0) - xf''(0)}{2x} = \frac{1}{2}f'' \\
 x \neq 0 \text{ 时 } g'(x) &= \frac{f'(x) - f(x)}{x^2} \\
 \lim_{x \rightarrow 0} g'(x) &= \lim_{x \rightarrow 0} \frac{f'(x)x - f(x)}{x^2} = \lim_{x \rightarrow 0} \frac{f'(x) + xf''(x) - f'(x)}{2x} \\
 &= \lim_{x \rightarrow 0} \frac{f''(x)}{2} = \frac{f''(0)}{2} = \frac{f''(0)}{2} = g'(0) \\
 \therefore g'(x) &\text{在 } x = 0 \text{ 处连续}
 \end{aligned}$$

例 6. 设 $f(x)$ 在 $(-\infty, +\infty)$ 上连续, $\lim_{x \rightarrow 0} \frac{f(x)}{x} = A$, A 为常数, $\varphi(x) =$

$\int_0^1 f(xt)dt$, 试讨论 $\varphi'(x)$ 的连续性。

$$\text{sol} : \because \lim_{x \rightarrow 0} \frac{f(x)}{x} = A \quad \therefore f(0) = 0$$

$$\therefore \varphi(0) = \int_0^1 f(0)dt = 0$$

$$x = 0 \text{ 时, } \varphi(x) \xrightarrow{u=xt} \varphi(x) = \int_0^x f(u) \frac{du}{x} = \frac{1}{x} \int_0^x f(u)du$$

$$\therefore \varphi'(0) = \lim_{x \rightarrow 0} \frac{\frac{1}{x} \int_0^x f(u)du - 0}{x} = \lim_{x \rightarrow 0} \frac{\int_0^x f(u)du}{x^2} = \lim_{x \rightarrow 0} \frac{f(x)}{2x} = \frac{A}{2}$$

$$x \neq 0 \text{ 时, } \varphi'(x) = \frac{f(x)x - \int_0^x f(u)du}{x^2}$$

$$\therefore \lim_{x \rightarrow 0} \varphi'(x) = \lim_{x \rightarrow 0} \frac{f(x)x - \int_0^x f(u)du}{x^2} = \lim_{x \rightarrow 0} \frac{f(x)}{x} - \lim_{x \rightarrow 0} \frac{\int_0^x f(u)du}{x^2}$$

$$= A - \lim_{x \rightarrow 0} \frac{f(x)}{2x} = A - \frac{A}{2} = \frac{A}{2}$$

$$\therefore \varphi(x) \text{ 连续}$$

例 7. 设 $f(x)$ 在 $(-\infty, +\infty)$ 上有定义, 对任意 x, y 有 $f(x+y) = e^x f(y) + e^y f(x)$, 且 $f(x)$ 在 $x=0$ 处可导, $f'(0) = 2$, 证明 $f(x)$ 在任一点处可导, 并求 $f(x)$ 。

$$\text{sol} : \text{令 } x = y = 0 \Rightarrow f(0) = 2f(0) \Rightarrow f(0) = 0$$

$$\therefore f'(x) = \lim_{\Delta x \rightarrow 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} \frac{e^x f(\Delta x) + e^{\Delta x} f(x) - f(x)}{\Delta x}$$

$$= \lim_{\Delta x \rightarrow 0} e^x \frac{f(\Delta x) - f(0)}{\Delta x} + \lim_{\Delta x \rightarrow 0} f(x) \frac{e^{\Delta x} - 1}{\Delta x}$$

$$= e^x \cdot f'(0) + f(x) = 2e^x + f(x)$$

$$\therefore f'(x) - f(x) = 2e^x \Rightarrow f(x) = e^{-\int -1dx} \left[C + \int 2e^x \cdot e^{\int -1dx} dx \right]$$

$$\therefore f(x) = e^x [C + 2x] = 2xe^x + Ce^x$$

$$\therefore f'(0) = 2 \Rightarrow C = 0$$

$$\therefore f(x) = 2xe^x$$

例 8. 设 $f(x)$ 在 $(-\infty, +\infty)$ 上连续, $f(1) = 3$, 对任意 $x > 0, y > 0$ 有 $\int_1^{xy} f(x)dt = x \int_1^y f(t)dt + y \int_1^x f(t)dt$, 求 $f(x)$ 。

sol : 对 y 求导 :

$$xf(xy) = xf(y) + \int_1^x f(t)dt$$

$$\text{令 } y = 1$$

$$xf(x) = 3x + \int_1^x f(t)dt$$

1°

$$\text{令 } F(x) = \int_1^x f(t)dt$$

$$xF'(x) = 3x + F(x) \Rightarrow F'(x) - \frac{1}{x}F(x) = 3$$

$$F(x) = e^{-\int -\frac{1}{x}dx} \left[C + \int 3e^{\int -\frac{1}{x}dx} dx \right]$$

$$= x \left[C + \int \frac{3}{x} dx \right]$$

$$= Cx + 3x \ln x$$

$$\therefore f(x) = F'(x) = C + 3 \ln x + 3$$

$$\because f(1) = 3 \quad \therefore C = 0$$

$$\therefore f(x) = 3 \ln x + 3$$

2°

$$\because f(x) = \frac{3x + \int_0^x f(t)dt}{x}$$

$$\therefore f(x) \text{ 可导}$$

$$f(x) + xf'(x) = 3 + f(x) \Rightarrow f'(x) = \frac{3}{x}$$

$$\therefore f(x) = 3 \ln x + C$$

$$\because f(1) = 3 \Rightarrow C = 3$$

$$\therefore f(x) = 3 \ln x + 3$$

例 9. 设 $f(x) = \begin{cases} x^3 \sin \frac{1}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$, 求使得 $f^{(n)}(0)$ 存在的最大 n 。

$$\text{sol: } f'(0) = 0$$

$$\begin{aligned} f''(0) &= \lim_{x \rightarrow 0} \frac{f'(x) - 0}{x} \\ &= \lim_{x \rightarrow 0} \frac{3x^2 \sin \frac{1}{x} - x \cos \frac{1}{x}}{x} \end{aligned}$$

$$= \lim_{x \rightarrow 0} 3x \sin \frac{1}{x} - \cos \frac{1}{x}$$

振荡, 极限不存在

$\therefore f''(0)$ 不存在 $\therefore n = 1$

例 10. 设 $f(x) = 3x^3 + x^2|x|$, 求使得 $f^{(n)}(0)$ 存在的最大 n 。

设 $g(x) = x^2|x|$

$$g(x) = \begin{cases} x^3 & x \geq 0 \\ -x^3 & x < 0 \end{cases} \Rightarrow g'(x) = \begin{cases} 3x^2 & x \geq 0 \\ -3x^2 & x < 0 \end{cases}$$

$$\lim_{x \rightarrow 0} 3x^2 = \lim_{x \rightarrow 0} -3x^2 = 0 \quad \therefore \text{可导}$$

$$g''(x) = \begin{cases} 6x & x \geq 0 \\ -6x & x < 0 \end{cases}$$

$$\lim_{x \rightarrow 0} 6x = \lim_{x \rightarrow 0} -6x = 0 \quad \therefore \text{可导}$$

$$g^{(3)}(x) = \begin{cases} 6 & x \geq 0 \\ -6 & x < 0 \end{cases}$$

$$\lim_{x \rightarrow 0} 6 \neq \lim_{x \rightarrow 0} -6 \quad \therefore \text{不可导}$$

又 $\because 3x^3$ 对 $f(x)$ 的可导阶数无影响

$\therefore n$ 最大为 3

例 11. 设 $y = (2x+3)^3(3x+2)^2$, 求 $y^{(5)}$, $y^{(6)}$ 。

sol : y 最高阶为 x^5

$$\therefore y^{(6)} = 0$$

$$y^{(5)} = 2^2 \cdot 3^2 \cdot 5! = 8 \times 9 \times 120 = 8640$$

例 12. 设 $y = \frac{x^4 - x^3 + 2x^2 - 3x}{x-1}$, 求 $y^{(5)}$ 。

$$\text{sol : } y = \frac{x^3(x-1) + (2x-1)(x-1) - 1}{x-1} = x^3 + 2x - 1 - \frac{1}{x-1}$$

$$\therefore y^{(5)} = -1 \cdot (-1)^5 \cdot 5! \frac{1}{(x-1)^6} = \frac{120}{(x-1)^6}$$

例 13. 设 $y = y(x)$ 由方程 $y = \cos x + xe^y$ 所确定, 求 $y'(0)$, $y''(0)$ 。

$$\text{sol : } y' = -\sin x + e^y + x \cdot e^y \cdot y'$$

$$\begin{aligned} \text{令 } x=0 &\Rightarrow y=1 \quad y'=e^y \Rightarrow y'(0)=e \\ y'' &= -\cos x + e^y \cdot y' + e^y \cdot y' + x[e^y \cdot y']' \\ \text{令 } x=0 &\Rightarrow y'' = -1 + e^y \cdot y' + e^y \cdot y' = -1 + 2e^2 \end{aligned}$$

例 14. 设 $y = y(x)$ 由方程 $2y^3 - 2y^2 + 2xy - x^2 = 1$ 所确定, 求 $f(x)$ 的驻点, 并判别该驻点是否为极值点。

$$\begin{aligned} \text{sol: } 6y^2y' - 4yy' + 2y + 2xy' - 2x &= 0 \\ \text{令 } y' = 0 &\Rightarrow y = x \quad \text{回代} \\ 2x^3 - 2x^2 + 2x^2 - x^2 &= 1 \\ 2x^3 - x^2 - 1 &= 0 \\ (x-1)(2x^2 + x + 1) &= 0 \\ \therefore \text{驻点为 } x &= 1 \\ \therefore 12yy'^2 + 6y^2y'' - 4y'^2 - 4yy'' + 2y' + 2y' + 2xy'' - 2 &= 0 \\ 4y'' = 2 &\Rightarrow y'' = \frac{1}{2} \quad \therefore \text{为极小值点} \end{aligned}$$

例 15. 通过变换 $x = \sin t$ 化简方程 $(1-x^2)\frac{d^2y}{dx^2} - x\frac{dy}{dx} + a^2y = 0$, 并求原方程的通解。

$$\begin{aligned} \text{sol: } \frac{dy}{dt} &= \cos t \frac{dy}{dx} \\ \frac{d^2y}{dt^2} &= -\sin t \frac{dy}{dx} + \cos t \frac{d\frac{dy}{dx}}{dx} \frac{dx}{dt} \\ &= -\sin t \frac{dy}{dx} + \cos^2 t \frac{d^2y}{dx^2} \\ &= -x \frac{dy}{dx} + (1-x^2) \frac{d^2y}{dx^2} \\ \therefore \frac{d^2y}{dt^2} + a^2y &= 0 \end{aligned}$$

(1)

$$\begin{aligned} a=0 \text{ 时 } y'' &= 0 \\ \lambda^2 = 0 &\Rightarrow \lambda_1 = \lambda_2 = 0, \Delta = 0 \\ \therefore y &= (C_1 + C_2 t)e^{0t} = C_1 + C_2 t = C_1 + C_2 \arcsin x \end{aligned}$$

(2)

$$a \neq 0 \text{ 时 } y'' + a^2y = 0$$

$$\begin{aligned}
\lambda^2 + a^2 &= 0 \Rightarrow \lambda_1 = ai, \lambda_2 = -ai, \Delta < 0 \\
\therefore y &= e^{0t}(C_1 \cos at + C_2 \sin at) \\
&= C_1 \cos at + C_2 \sin at \\
&= C_1 \cos(a \cdot \arcsin x) + C_2 \sin(a \cdot \arcsin x)
\end{aligned}$$

例 16. 通过变换 $x = \frac{u}{\cos x}$ 化简方程 $\cos x \frac{d^2 y}{dx^2} - 2 \sin x \frac{dy}{dx} + 3y \cos x = e^x$, 并求原方程的通解。

$$\begin{aligned}
\text{sol : } u &= y \cdot \cos x \\
\therefore \frac{du}{dx} &= \cos x \frac{dy}{dx} - y \sin x \\
\frac{d^2 y}{dx^2} &= \cos x \frac{d^2 y}{dx^2} - \sin x \frac{dy}{dx} - \sin x \frac{dy}{dx} - y \cos x \\
&= \cos x \frac{d^2 y}{dx^2} - 2 \sin x \frac{dy}{dx} - y \cos x \\
\therefore u'' + 4u &= e^x \\
\lambda^2 + 4 &= 0 \Rightarrow \lambda_1 = 2i, \lambda_2 = -2i \\
\therefore \bar{u} &= e^{0x}(C_1 \cos 2x + C_2 \sin 2x) \\
u^* &= Ae^x \\
\text{又 } \therefore u'' + 4u &= e^x \Rightarrow A = \frac{1}{5} \\
\therefore u &= C_1 \cos 2x + C_2 \sin 2x + \frac{1}{5}e^x \\
\therefore y &= \frac{u}{\cos x} = C_1 \frac{\cos 2x}{\cos x} + C_2 \frac{\sin 2x}{\cos x} + \frac{e^x}{5 \cos x} \\
&= C_1 \frac{\cos 2x}{\cos x} + C_2 \sin x + \frac{e^x}{5 \cos x}
\end{aligned}$$

例 17. 设 $f(x) = y(x)$ 满足方程 $y'' + (x + e^{2y})y'^3 = 0$, $y' \neq 0$, 试将该方程化为 $y = y(x)$ 的反函数 $x = x(y)$ 满足的微分方程, 并求原方程的通解。

$$\begin{aligned}
\text{sol : } \frac{dx}{dy} &= \frac{1}{y'} \quad \frac{d^2 x}{dy^2} = \frac{d(\frac{1}{y'})}{dx} \frac{dx}{dy} = -\frac{\frac{y''}{y'^2}}{y'} = -\frac{y''}{y'^3} \\
\frac{d^2 x}{dy^2} - x &= e^{2y} \quad \lambda^2 - 1 = 0 \Rightarrow \lambda = \pm 1 \\
\bar{x} &= C_1 e^y + C_2 e^{-y} \quad x^* = Ae^{2y} \\
\therefore 4Ae^{2y} - Ae^{2y} &= e^{2y} \Rightarrow A = \frac{1}{3} \\
\therefore x &= C_1 e^y + C_2 e^{-y} + \frac{1}{3}e^{2y}
\end{aligned}$$

Chapter 3

导数与微分

3.1 考点解析

3.2 经典例题

例 1. (1) $\int \frac{\ln x}{x} dx$

$$\text{sol: } \int \ln x d \ln x = \frac{1}{2} \ln^2 x + C$$

(2) $\int \frac{1}{x^2} e^{-\frac{1}{x}} dx$

$$\text{sol: } \int e^{-\frac{1}{x}} d\left(-\frac{1}{x}\right) = e^{-\frac{1}{x}} + C$$

(3) $\int \sqrt{\frac{\arcsin x}{1-x^2}} dx$

$$\text{sol: } \int \sqrt{\arcsin x} d \arcsin x = \frac{2}{3} (\arcsin x)^{\frac{3}{2}} + C$$

(4) $\int \frac{\ln \tan x}{\sin x \cos x} dx$

$$\text{sol: } \int \ln \tan x d \ln \tan x = \frac{1}{2} (\ln \tan x)^2$$

例 2. (1) $\int \frac{1+\ln x}{(x \ln x)^2} dx$

$$\text{sol: } \int \frac{1}{(x \ln x)^2} d(x \ln x) = -\frac{1}{(x \ln x)^2} dx$$

$$(2) \int \frac{\ln(1+x) - \ln x}{x(1+x)} dx$$

$$\text{sol} : - \int \ln(1+x) - \ln x d[\ln(1+x) - \ln x] = -\frac{1}{2}[\ln(1+x) - \ln x]^2 + C$$

$$(3) \int \sqrt{\frac{x}{1-x\sqrt{x}}} dx$$

$$\begin{aligned} \text{sol} : &= \int \frac{\sqrt{x}}{\sqrt{1-x\sqrt{x}}} dx \\ &\because (1-x\sqrt{x})' = -\sqrt{x} - \frac{1}{2}x\frac{1}{\sqrt{x}} = -\frac{3}{2}\sqrt{x} \\ &\therefore = \int \frac{1}{\sqrt{1-x\sqrt{x}}} \left(-\frac{2}{3}\right) d(1-x\sqrt{x}) \\ &= -\frac{2}{3} \times 2(1-x\sqrt{x})^{\frac{1}{2}} + C \\ &= -\frac{4}{3}(1-x\sqrt{x})^{\frac{1}{2}} + C \end{aligned}$$

$$(4) \int \frac{1+x}{x(1+xe^x)} dx$$

$$\begin{aligned} \text{sol} : &\because (xe^x)' = e^x + xe^x = e^x(1+x) \\ &\therefore \int \frac{1+x}{x(1+xe^x)} dx \\ &= \int \frac{e^x(1+x)}{xe^x(1+e^x)} dx = \int \frac{1}{xe^x(1+xe^x)} dx e^x \\ &\xrightarrow{u=xe^x} \int \frac{1}{u(1+u)} du = \int \frac{1+u}{u(1+u^2)} du = \int \frac{1+u}{u} d\frac{u}{1+u} \\ &= \ln \left| \frac{u}{1+u} \right| = \ln \left| \frac{xe^x}{1+xe^x} \right| \end{aligned}$$

例 3. (1) $\int \frac{1}{x(1+x^n)} dx \quad (n \neq 0)$

$$\begin{aligned} \text{sol} : &\because (x^n)' = nx^{n-1} \\ &\therefore = \frac{1}{n} \int \frac{nx^{n-1}}{x^n(1+x^n)} dx = \frac{1}{n} \int \frac{1}{x^n(1+x^n)} dx^n \\ &= \frac{1}{n} \ln \left| \frac{x^n}{1+x^n} \right| + C \end{aligned}$$

$$(2) \int \frac{1}{(2x^2+1)\sqrt{x^2+1}} dx$$

$$\begin{aligned} \text{sol} : \xrightarrow{x=\tan t} & \int \frac{1}{(2\tan^2 t + 1) \cdot \frac{1}{\cos t}} \cdot \frac{1}{\cos^2 t} dt = \int \frac{\sec t}{2\tan^2 t + 1} dt \\ & = \int \frac{\cos t}{1 + \sin^2 t} dt = \arctan(\sin t) + C = \arctan \frac{x}{\sqrt{x^2+1}} + C \end{aligned}$$

$$(3) \int \frac{\ln x}{(1+x^2)^{\frac{3}{2}}} dx$$

$$\begin{aligned} \text{sol} : \xrightarrow{x=\tan t} & \int \ln(\tan t) \cos^3 t \frac{1}{\cos^2 t} dt = \int \ln(\tan t) \cos t dt \\ & = \sin t \ln \tan t - \int \sin t \frac{\cos t}{\sin t} \frac{1}{\cos^2 t} dt \\ & = \frac{x}{\sqrt{x^2+1}} \ln x - \ln |\sec t + \tan t| + C \\ & = \frac{x \ln x}{\sqrt{x^2+1}} - \ln \left| x + \sqrt{x^2+1} \right| + C \end{aligned}$$

$$\text{例 4. (1) } \int \frac{x^2 \arctan x}{1+x^2} dx$$

sol :

1°

$$\begin{aligned} & = \int \frac{x^2 \arctan x}{1+x^2} dx = \int x^2 \arctan x d \arctan x \\ & \xrightarrow{t=\arctan x} \int \tan^2 t \cdot t dt = \int \left(\frac{1}{\cos^2 t} - 1 \right) t dt \\ & = \int (\sec^2 t - 1) t dt = \int t \sec^2 t dt - \int t dt \\ & = t \tan t - \int \tan t dt - \int t dt \\ & = t \tan t + \ln |\cos t| - \frac{1}{2} t^2 + C \\ & = x \arctan x + \ln \frac{1}{\sqrt{x^2+1}} - \frac{1}{2} \arctan^2 x + C \end{aligned}$$

2°

$$\begin{aligned} & = \int \frac{x^2 + 1 - 1}{1+x^2} \arctan x dx \\ & = \int \arctan x dx - \int \frac{1}{1+x^2} \arctan x dx \end{aligned}$$

$$\begin{aligned}
&= x \arctan x - \int x d \arctan x - \int \frac{1}{x^2 + 1} \arctan x dx \\
&= x \arctan x - \int \frac{x}{1 + x^2} dx - \int \arctan x d \arctan x \\
&= x \arctan x - \frac{1}{2} \ln(1 + x^2) - \frac{1}{2} \arctan^2 x + C
\end{aligned}$$

$$(2) \int \frac{1}{x^2(1+x^2)^2} dx$$

$$\begin{aligned}
\text{sol : } &\xrightarrow{x=\tan t} \int \frac{1}{\tan^2 t (\frac{1}{\cos^2 t})^2} \frac{1}{\cos^2 t} dt = \int \frac{\cos^2 t}{\tan^2 t} dt \\
&= \int \frac{\cos^4 t}{\sin^2 t} dt = \int \frac{(1 - \sin^2 t)^2}{\sin^2 t} dt = \int (\frac{1}{\sin^2 t} - 2 + \sin^2 t) dt \\
&= -\cot t - 2t + \int \frac{1}{2}(1 - \cos 2t) dt \\
&= -\cot t - 2t + \frac{1}{4}(2t - \sin 2t) \\
&= -\cot t - \frac{1}{4} \sin 2t - \frac{3}{2}t \\
&= -\frac{1}{x} - \frac{3}{2} \arctan x - \frac{1}{2} \sin t \cos t \\
&= -\frac{1}{x} - \frac{3}{2} \arctan x - \frac{x}{2(x^2 + 1)} + C
\end{aligned}$$

$$(3) \int \frac{1}{x\sqrt{4-x^2}} dx$$

sol :

1o

$$\begin{aligned}
&\xrightarrow{x=2\sin t} \int \frac{1}{2\sin t \cdot 2\cos t} 2\cos t dt = \frac{1}{2} \ln |\csc t + \cot t| + C \\
&= \frac{1}{2} \ln \left| \frac{2 - \sqrt{4-x^2}}{x} \right| + C
\end{aligned}$$

2o

$$\begin{aligned}
&\xrightarrow{x=\frac{1}{t}} \int \frac{1}{\frac{1}{t}\sqrt{4-\frac{1}{t^2}}} \cdot \left(-\frac{1}{t^2}\right) dt = -\int \frac{1}{\sqrt{4-\frac{1}{t^2}}} \frac{1}{t} dt \\
&= -\int \frac{1}{\sqrt{4t^2-1}} dt = -\frac{1}{2} \int \frac{1}{\sqrt{(2t)^2-1}} d2t \\
&= -\frac{1}{2} \ln |2t + \sqrt{4t^2-1}| + C
\end{aligned}$$

$$= -\frac{1}{2} \ln \left| \frac{2}{x} + \sqrt{4 \frac{1}{x^2} - 1} \right| + C = -\frac{1}{2} \ln \left| \frac{2 + \sqrt{4 - x^2}}{x} \right| + C$$

$$(4) \int \frac{1}{x^2 \sqrt{x^2 - x + 1}} dx$$

$$\begin{aligned} \text{sol : } & \xrightarrow{x=\frac{1}{t}} \int \frac{t^2}{\sqrt{\frac{1}{t^2} - \frac{1}{t} - 1}} \cdot \left(-\frac{1}{t^2}\right) dt = - \int \frac{t}{\sqrt{t^2 - t + 1}} dt \\ &= - \int \frac{t}{\sqrt{(t - \frac{1}{2})^2 + \frac{3}{4}}} dt = - \int \frac{(t - \frac{1}{2}) + \frac{1}{2}}{\sqrt{(t - \frac{1}{2})^2 + \frac{3}{4}}} d(t - \frac{1}{2}) \\ & \xrightarrow{u=t-\frac{1}{2}} - \int \frac{u + \frac{1}{2}}{\sqrt{u^2 + \frac{3}{4}}} du \\ &= - \int \frac{1}{2\sqrt{u^2 + \frac{3}{4}}} du = - \int \frac{1}{2\sqrt{u^2 + \frac{3}{4}}} du^2 - \frac{1}{2} \int \frac{1}{\sqrt{u^2 + \frac{3}{4}}} du \\ &= -\sqrt{u^2 + \frac{3}{4}} - \frac{1}{2} \ln \left| u + \sqrt{u^2 + \frac{3}{4}} \right| + C \\ &= -\sqrt{\left(\frac{1}{x} - \frac{1}{2}\right)^2 + \frac{3}{4}} - \frac{1}{2} \ln \left| \frac{1}{x} - \frac{1}{2} + \sqrt{\left(\frac{1}{x} - \frac{1}{2}\right)^2 + \frac{3}{4}} \right| + C \\ &= -\frac{\sqrt{x^2 - x + 1}}{x} - \frac{1}{2} \ln \left| \frac{1}{x} - \frac{1}{2} + \frac{\sqrt{x^2 - x + 1}}{x} \right| + C \end{aligned}$$

$$\text{例 5. (1) } \int \frac{\arcsin x}{x^2 \sqrt{1-x^2}} dx$$

$$\begin{aligned} \text{sol : } & \xrightarrow{x=\sin t} \int \frac{t}{\sin^2 t \cos t} \cos t dt = \int \frac{t}{\sin^2 t} dt \\ &= - \int t d \cot t = -t \cot t + \int \cot t dt \\ &= -\arcsin x \cdot \frac{\sqrt{1-x^2}}{x} + \ln |\sin t| + C \\ &= -\frac{\sqrt{1-x^2}}{x} \arcsin x + \ln |x| + C \end{aligned}$$

$$(2) \int \frac{x e^x}{\sqrt{e^x - 1}} dx$$

$$\text{sol : } \xrightarrow{t=\sqrt{e^x-1}} \int \frac{\ln(t^2+1)(t^2+1)}{t} \cdot \frac{2t}{t^2+1} dt = \int 2 \ln(t^2+1) dt$$

$$\begin{aligned}
&= 2t \ln(t^2 + 1) - 4 \int \frac{t^2}{t^2 + 1} dt \\
&= 2t \ln(t^2 + 1) - 4 \int \frac{t^2 + 1 - 1}{t^2 + 1} dt \\
&= 2t \ln(t^2 + 1) - 4 \int dt + 4 \int \frac{1}{1 + t^2} dt \\
&= 2t \ln(t^2 + 1) - 4t + 4 \arctan t + C \\
&= 2(x - 2)\sqrt{e^x - 1} + 4 \arctan \sqrt{e^x - 1} + C
\end{aligned}$$

$$(3) \int \frac{1}{2 - \sqrt[3]{x+1}} dx$$

$$(4) \int \frac{\sqrt[3]{x}}{x(\sqrt{x} + \sqrt[3]{x})} dx$$

例 6. (1) $\int \frac{1}{\sqrt{2x+1} + \sqrt{x-1}} dx$

$$(2) \int \frac{1}{(1+e^x)^2} dx$$

$$(3) \int \frac{\arcsin \sqrt{x}}{\sqrt{1-x}} dx$$

$$(4) \int \frac{\arcsin x}{\sqrt{1+x}} dx$$

例 7. (1) $\int \frac{\sqrt{x}}{(x-1)^2} dx$

$$(2) \int \frac{\arctan e^x}{e^x} dx$$

$$(3) \int \frac{\arctan e^x}{e^{2x}} dx$$

$$(4) \int \frac{x^2}{(x \sin x + \cos x)^2} dx$$

例 8. (1) $\int \frac{x}{1+\cos x} dx$

$$(2) \int \frac{1+\sin x}{1+\cos x} e^x dx$$

$$(3) \int \frac{e^x}{x} (1 + x \ln x) dx$$

$$(4) \int e^{-|x|} dx$$

例 9. (1) $\int \frac{1}{\sin 2x - 2 \sin x} dx$

$$(2) \int \frac{1}{\sin^2 x + 3} dx$$

$$(3) \int \frac{1}{2+\cos x} dx$$

$$(4) \int \frac{1}{\sin x + \cos x} dx$$

例 10. 设 $F(x)$ 是 $f(x)$ 的一个原函数, $F(x) > 0$, $F(0) = 1$, 当 $x > 0$ 时,
 $f(x)F(x) = \frac{xe^x}{2(1+x)^2}$, 求 $f(x)$ 。

例 11. 已知 $a \neq b$, 求 A, B 的值, 使得 $\int \frac{dx}{(a+b \cos x)^2} = \frac{A \sin x}{a+b \cos x} + B \int \frac{dx}{a+b \cos x}$ 。

Part II

线性代数

Part III

概率论与数理统计

Part IV

习题

Chapter 4

习题

4.1 习题1

1. 设 $f(x) = \begin{cases} 1 & |x| \leq 1 \\ 0 & |x| > 1 \end{cases}$ $g(x) = \begin{cases} 2 - x^2 & |x| \leq 1 \\ 2 & |x| > 1 \end{cases}$, 求 $f(f(x))$, $f(g(x))$, $g(f(x))$, $g(g(x))$ 。

2. $f(x) = \begin{cases} x^2 + x + 1 & x \geq 1 \\ 1 & -1 \leq x < 1 \\ x^2 - x + 1 & x < -1 \end{cases}$, 求 $f(-x)$ 。

3. $f(x) = \begin{cases} 1 & 0 \leq x < 1 \\ -1 & 1 \leq x \leq 2 \end{cases}$, 求 $f(2x)$, $f(x^2)$, $f(x-2)$ 。

4. 作半径为 R 的球的外切正圆锥, 试建立圆锥体积 V 与其高 h 的关系, 何时体积最小。

5. 一矩形内接于半径为 R , 中心角为 2φ ($\varphi < \frac{\pi}{2}$) 的圆扇形中, 矩形的一对对边也平行于扇形中心角的角平分线, 试建立矩形面积 S 与 θ 及 S 与边长 x 的关系, 何时面积最大。

6. 求下列极限

(1) $\lim_{x \rightarrow 0} \frac{\ln(1+x^2)}{\sec x - \cos x}$

(2) $\lim_{x \rightarrow 0} \frac{x - \arcsin x}{(e^x - 1)^3}$

(3) $\lim_{x \rightarrow 0} \left[\frac{1}{\ln(1+x)} - \frac{1}{x} \right]$

(4) $\lim_{x \rightarrow 1} \frac{x-1-x \ln x}{(x-1) \ln x}$

$$(5) \lim_{x \rightarrow +\infty} \left(\frac{2}{\pi} \arctan x \right)^x$$

$$(6) \lim_{x \rightarrow 0^+} \frac{\ln \sin 3x}{\ln \sin 2x}$$

$$(7) \lim_{x \rightarrow 0} \left(\frac{\arcsin x}{x} \right)^{\frac{1}{x^2}}$$

$$(8) \lim_{x \rightarrow 0^+} \left[\frac{\ln x}{(1+x)^2} - \ln \frac{x}{1+x} \right]$$

$$(9) \lim_{x \rightarrow \infty} \frac{e^x - \frac{2}{\pi} x \arctan x}{e^x + x}$$

$$(10) \lim_{x \rightarrow 1} (2-x)^{\tan \frac{\pi x}{2}}$$

$$(11) \lim_{x \rightarrow 0} \left(\frac{\sin x}{x} \right)^{\frac{1}{1-\cos x}}$$

$$(12) \lim_{x \rightarrow +\infty} \left(\frac{\pi}{2} - \arctan x \right)^{\frac{1}{x}}$$

$$(13) \lim_{x \rightarrow 1} \left(\tan \frac{\pi x}{4} \right)^{\tan \frac{\pi x}{2}}$$

$$(14) \lim_{x \rightarrow 0} \frac{x e^{2x} + x e^x - 2 e^{2x} + 2 e^x}{x^3}$$

$$(15) \lim_{x \rightarrow 0^+} (\cot x)^{\frac{1}{\ln x}}$$

$$(16) \lim_{x \rightarrow \infty} x^2 \left(1 - x \sin \frac{1}{x} \right)$$

7. 由条件 $\lim_{x \rightarrow -\infty} (\sqrt{x^2 - 2x + 2} - ax - b) = 0$, 解出 a, b 。

8. 指出下列函数间断点的类型

$$(1) y = \frac{x^2 + x}{|x|(x^2 - 1)}$$

$$(2) y = \frac{x}{\tan x}$$

$$(3) y = \frac{e^{\frac{1}{x}} - 1}{e^{\frac{1}{x}} + 1}$$

$$(4) y = \arctan \frac{1}{x}$$

9. 讨论 $f(x) = \lim_{n \rightarrow \infty} \frac{1 - x^{2n}}{1 + x^{2n}} x$ 的连续性, 若有间断点, 指出其类型。

10. 设函数 $f(x) = \lim_{n \rightarrow \infty} \frac{x^{2n-1} + ax^2 + bx}{x^{2n} + 1}$ 在 $(-\infty, +\infty)$ 上连续, 求 a, b 的值。

11. 设 $f(x)$ 在 $(-\infty, +\infty)$ 内有定义, 且在 $x = 0$ 处连续, 对任意 x_1, x_2 有 $f(x_1 + x_2) = f(x_1) + f(x_2)$ 。证明 $f(x)$ 在 $(-\infty, +\infty)$ 上连续。

12. 设函数 $f(x)$ 在 $[0, 2a] (a > 0)$ 上连续, 且 $f(0) = f(2a)$ 。证明方程 $f(x) = f(x + a)$ 在 $[0, a]$ 上至少有一个根。

13. 直径相同的圆排成 n 行填满了等边三角形, 如图 $n = 3$ 的情况。设 A 为等边三角形的面积, A_n 为所有圆的面积, 求 $\lim_{n \rightarrow \infty} \frac{A_n}{A}$ 。
14. 设数列 $\{x_n\}$, $\{y_n\}$ 满足 $x_n \leq A \leq y_n$, 其中 A 为常数, 且 $\lim_{n \rightarrow \infty} (y_n - x_n) = 0$, 试用夹逼定理证明 $\lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} y_n = A$ 。
15. 试用夹逼定理求下列极限

(1) $\lim_{n \rightarrow \infty} \left(\frac{1}{n^2+n+1} + \frac{2}{n^2+n+1} + \cdots + \frac{n}{n^2+n+1} \right)$

(2) $\lim_{n \rightarrow \infty} (1 + 2^n + 3^n)^{\frac{1}{n}}$

(3) $\lim_{n \rightarrow \infty} \frac{n!}{n^n}$

4.2 习题2

1. 选择题

- (1) 设 $f(x) = \frac{x}{a+e^{bx}}$ 在 $(-\infty, +\infty)$ 内连续, 且 $\lim_{x \rightarrow -\infty} f(x) = 0$, 则 ()。
- (A) $a < 0, b < 0$
(B) $a > 0, b > 0$
(C) $a \leq 0, b > 0$
(D) $a \geq 0, b < 0$
- (2) 设 $f(x)$ 在 $x = a$ 处可导, 则 $|f(x)|$ 在 $x = a$ 处不可导的充分条件是 ()。
- (A) $f(a) = 0$ 且 $f'(a) = 0$
(B) $f(a) = 0$ 且 $f'(a) \neq 0$
(C) $f(a) > 0$ 且 $f'(a) > 0$
(D) $f(a) < 0$ 且 $f'(a) < 0$
- (3) 下列条件与 $f(x)$ 在 x_0 处可导的定义等价的是 ()。
- (A) $\lim_{h \rightarrow 0} \frac{f(x_0+h)-f(x_0-h)}{2h}$ 存在
(B) $\lim_{h \rightarrow 0} \frac{f(x_0+2h)-f(x_0+h)}{h}$ 存在
(C) $\lim_{h \rightarrow 0} \frac{f(x_0)-f(x_0-h)}{h}$ 存在
(D) $\lim_{n \rightarrow 0} n[f(x_0 + \frac{1}{n}) - f(x_0)]$ 存在
- (4) 设 $f(x)$ 可导, $F(x) = f(x)(1 + |\sin x|)$ 。则 $f(0) = 0$ 是 $F(x)$ 在 $x = 0$ 处可导的 ()。
- (A) 充要条件
(B) 充分条件
(C) 必要条件
(D) 无关条件
- (5) 设 $f(x)$ 在 $(-\delta, \delta)$ 内有定义, 且 $|f(x)| \leq x^2$, 则 $x = 0$ 是 $f(x)$ 的 ()。
- (A) 间断点
(B) 连续不可导点
(C) 可导点且 $f'(0) = 0$
(D) 可导点且 $f'(0) \neq 0$
- (6) 设 $f(x) = \begin{cases} \frac{2}{3}x^3 & x \leq 1 \\ x^2 & x > 1 \end{cases}$ 则 $f|_{\text{prime}}$ 在 $x = 1$ 处 ()。
- (A) 左、右导数都存在
(B) 左导数存在, 右导数不存在

- (C) 左导数不存在, 右导数存在
(D) 左、右导数都不存在

2. 设 $f(x) = \begin{cases} x \arctan \frac{1}{x^2} & x \neq 0 \\ 0 & x = 0 \end{cases}$, 则 $f'(x)$ 在 $x = 0$ 处连续
3. 设 $f(x)$ 在 $(-\frac{\pi}{4}, \frac{\pi}{4})$ 上连续, 对任意 x, y 满足 $f(x+y) = \frac{f(x)+f(y)}{1-f(x)f(y)}$, 且 $f(x)$ 在 $x = 0$ 处可导, $f'(0) = 2$.
(1) 用导数定义求 $f'(x)$
(2) 求 $f(x)$
4. $f(x)$ 是周期为 5 的连续函数, 在 $x = 0$ 的某个邻域内满足 $f(1+\sin x) - 3f(1-\sin x) = 8x + \alpha(x)$, $\alpha(x)$ 是当 $x \rightarrow 0$ 时比 x 更高阶的无穷小, 且 $f(x)$ 在 $x = 1$ 处可导, 求曲线 $y = f(x)$ 在点 $(6, f(6))$ 处的切线方程。
5. $f(x)$ 在 $x = 0$ 处满足 $f(0) = 0$ 、 $f'(0) = 0$ 、 $f''(0) = 6$ 求 $\lim_{x \rightarrow 0} \frac{f(\sin^2 x)}{x^4}$ 。
6. $f(x)$ 在 $(-\infty, +\infty)$ 上满足 $f(x+1) = 2f(x)$, 当 $0 \leq x < 1$ 时 $f(x) = x(1-x^2)$ 。证明 $f(x)$ 在 $x = 0$ 处不可导。
7. 设 $f(x) = \begin{cases} \frac{1}{x} - \frac{1}{\ln(1+x)} & x \neq 0, x > -1 \\ -\frac{1}{2} & x = 0 \end{cases}$ 试讨论 $f(x)$ 在 $x = 0$ 处的连续、可导性。
8. 设 $f(x) = \max\{\cos x, |\frac{2}{\pi}x + 1|\}$, 指出 $f(x)$ 不可导的点, 并说明理由。
9. 令 $t = \sqrt{x}$, 将方程 $4x \frac{d^2 y}{dx^2} + 2(1 - \sqrt{x}) \frac{dy}{dx} - 6y = e^{\sqrt[3]{x}}$ 化为 y 对 t 的微分方程, 并求原方程的通解。
10. 设 $f(x) = \begin{cases} \frac{g(x)-e^{-x}}{x} & x \neq 0 \\ 0 & x = 0 \end{cases}$, 其中 $g(x)$ 在 $x = 0$ 处存在二阶导数, 且 $g(0) = 1$ 、 $g'(0) = -1$, 试讨论 f' 在 $x = 0$ 处的连续性。
11. 设 $\rho = \rho(x)$ 是抛物线 $y = \sqrt{x}$ 上任一点 $M(x, y)$ ($x \geq 1$) 处的曲率半径, $s = s(x)$ 是该曲线上介于点 $A(1, 1)$ 与 M 之间的弧长, 计算 $3\rho \frac{d^2 \rho}{ds^2} - (\frac{d\rho}{ds})^2$ 的值。(在直角坐标系下曲率的公式 $K = \frac{|y''|}{(1+y'^2)^{\frac{3}{2}}}$)
12. 从点 $P_1(1, 0)$ 作 x 轴的垂线, 交抛物线 $y = x^2$ 于 $Q_1(1, 1)$, 再从 Q 作抛物线的切线与 x 轴交于 P_2 , 过 P_2 作 x 轴的垂线交抛物线于 Q_2 , 依次重复得 $P_1, Q_1; P_2, Q_2; \dots; P_n, Q_n; \dots$

- (1) 求 $\overline{OP_n}$ 的长度
- (2) 求 $\sum_{n=1}^{\infty} \overline{Q_n P_n}$ 的和
13. $f(x)$ $n+1$ 阶可导, $F(x) = \lim_{t \rightarrow \infty} t^2 [f(x + \frac{\pi}{t}) - f(x)] \sin \frac{x}{t}$, 试求 $F^{(n)}(x)$ 。
14. 试证曲线 $\sqrt{x} + \sqrt{y} = \sqrt{a}$ ($a > 0$) 上任一点的切线在两坐标轴上的截距和为常数。