Sparsity in Machine Learning

EN.580.709 - Fall 2019

Supervised Learning

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We only get to measure the *empirical risk*,

$$\hat{R}_S(h) = \frac{1}{n} \sum_i L(h(\mathbf{x}_i), y_i)$$

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$$\min_{\boldsymbol{\beta}} \frac{1}{n} \sum_{i}^{n} (y_i - \boldsymbol{\beta}^T \mathbf{x}_i) = \frac{1}{n} \|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2$$

$$\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix} \in \mathbb{R}^n, \quad \mathbf{X} = \begin{bmatrix} -\mathbf{x}_1^T- \\ -\mathbf{x}_2^T- \\ \vdots \\ -\mathbf{x}_n^T- \end{bmatrix} \in \mathbb{R}^{n \times m}, \quad \boldsymbol{\beta} = \begin{bmatrix} \beta_1 \\ \beta_2 \\ \vdots \\ \beta_n^T \end{bmatrix} \in \mathbb{R}^m$$

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Ridge (Regularized) Regression

$$\min_{\boldsymbol{\beta}} \mathcal{L} = \frac{1}{n}\|\mathbf{y} - \mathbf{X}\boldsymbol{\beta}\|_2^2 + \lambda\|\boldsymbol{\beta}\|_2^2$$

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Polynomial Regression

Let $x \in \mathbb{R}$,

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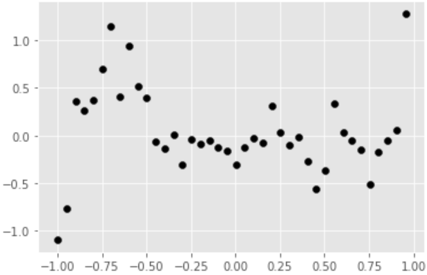

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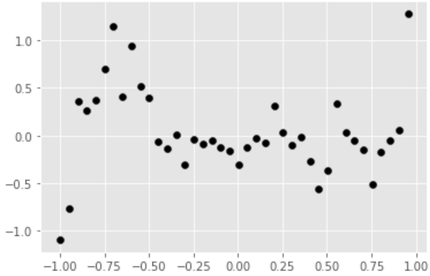
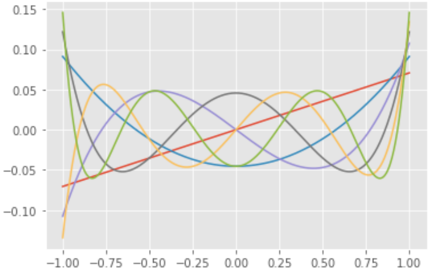


$\phi_j(x), \ 1 \le j \le m$



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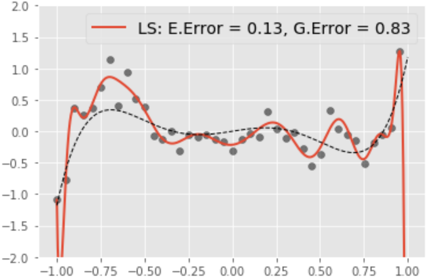

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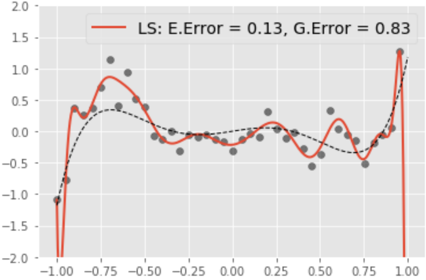

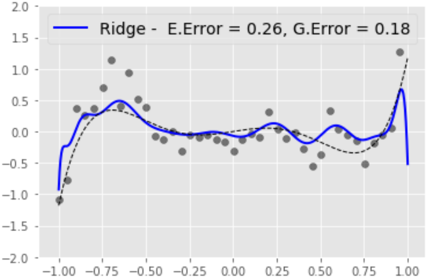

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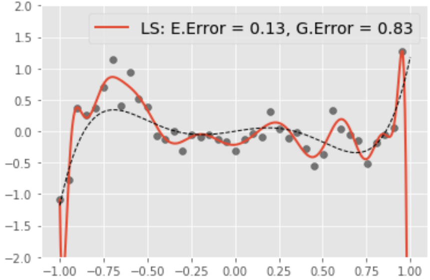

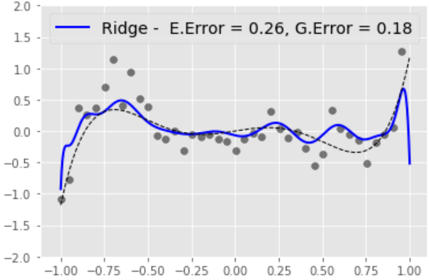

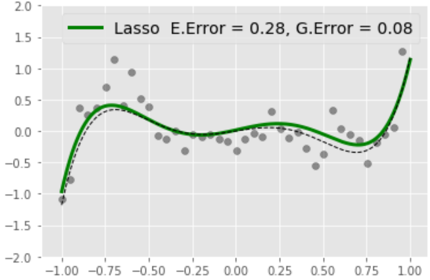

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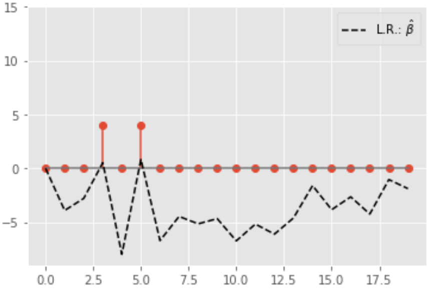

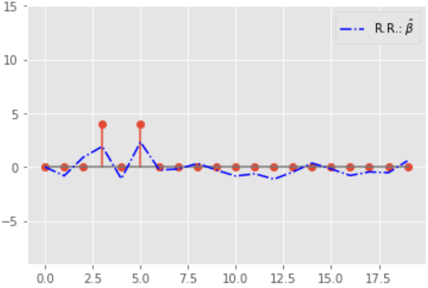

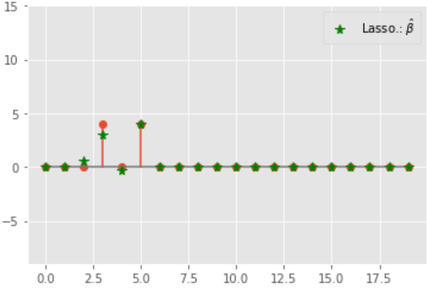

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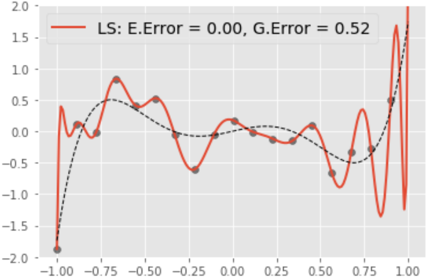

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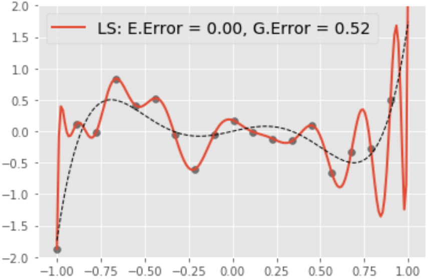

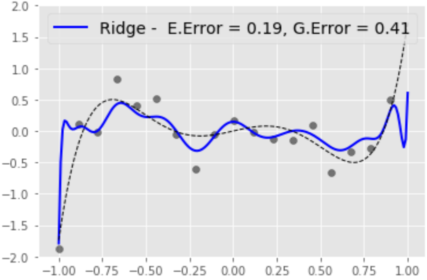

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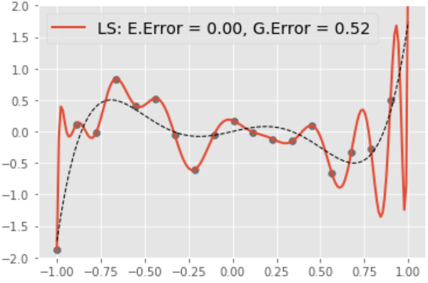

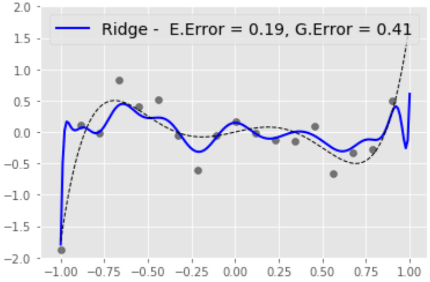

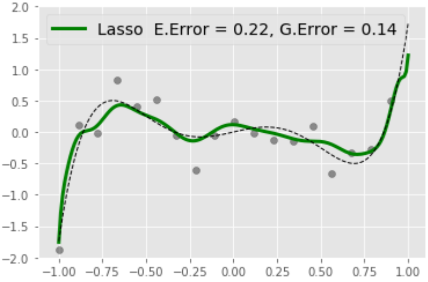

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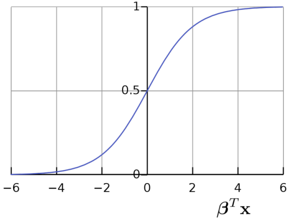

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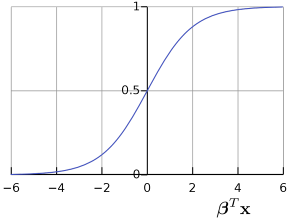

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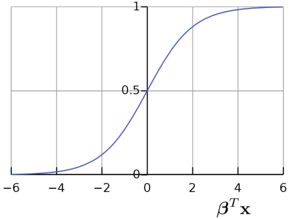

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For log. regression, $\mu(X) = P(Y=1|X=\mathbf{x})$, and $g(u) = \log \frac{u}{1-u}$

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How do we fit $\boldsymbol{\beta}$? Maximize the (log) likelihood $P(\{\mathbf{y}_i\}|\boldsymbol{\beta}, \{\mathbf{x}_i\})$:

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$$\mathcal{L}(\boldsymbol{\beta}) = \prod_{i=1}^{n} P(Y = y_i|\mathbf{x}_i) = \prod_{i=1}^{n} P(Y = 1|\mathbf{x}_i)^{y_i} P(Y = 0|\mathbf{x}_i)^{1-y_i}$$

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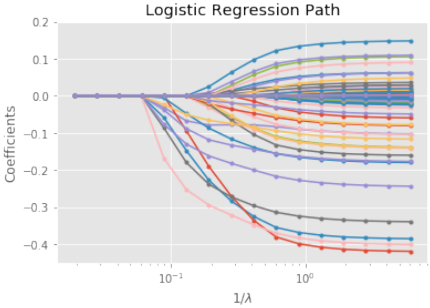

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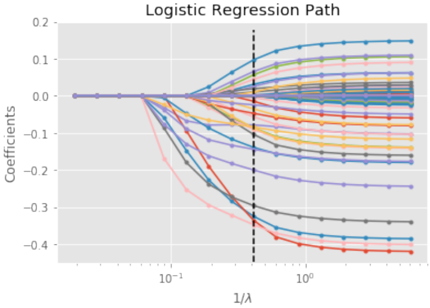

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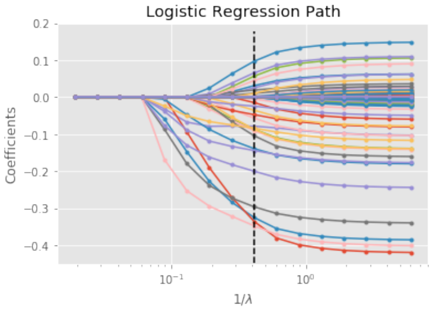

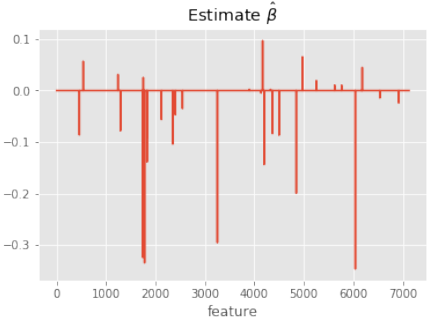

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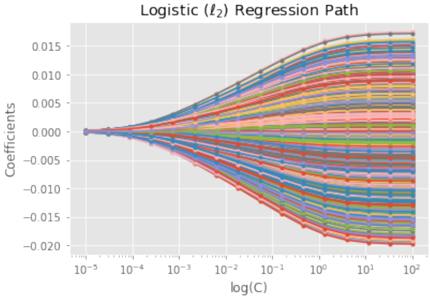

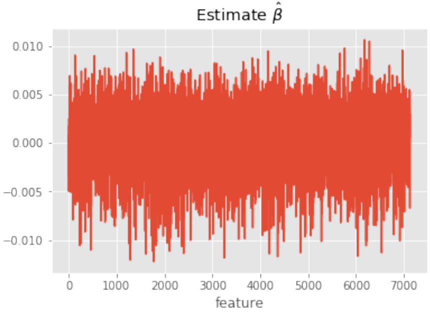

Multi-class Logistic Regression

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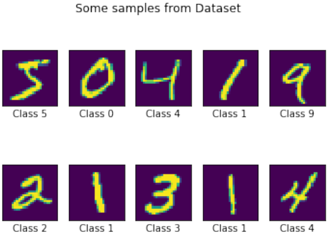
Digits Classification



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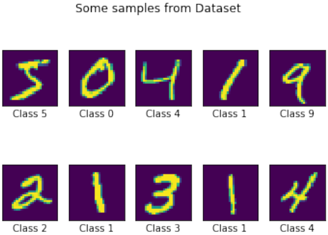

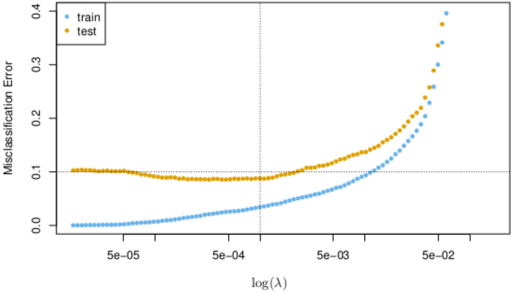

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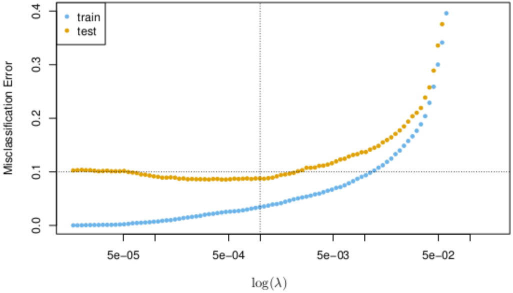

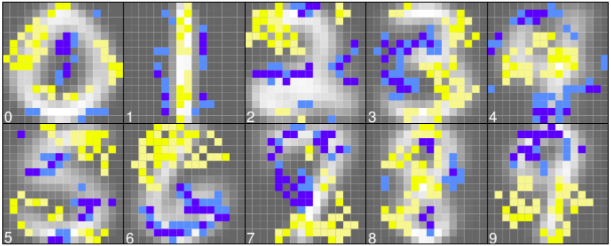

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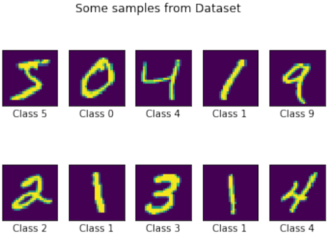

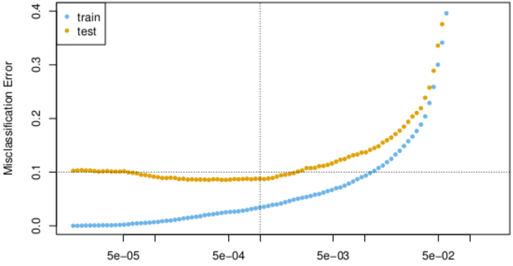

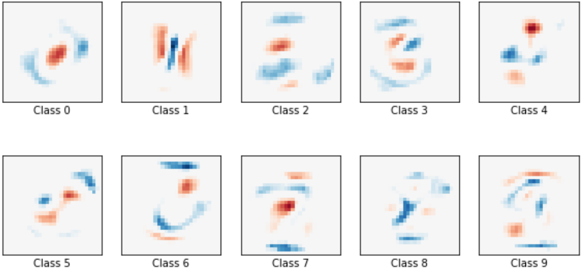

Sparsity in Machine Learning
Part II

EN.580.709 - Fall 2019

Recall Lasso

- $\mathbf{y} \in \mathbb{R}^n$: response, $\mathbf{X} \in \mathbb{R}^{n \times m}$: predictors/features

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Elastic-Net

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Example

Let $Z_1, Z_2 \sim N(0, 1)$

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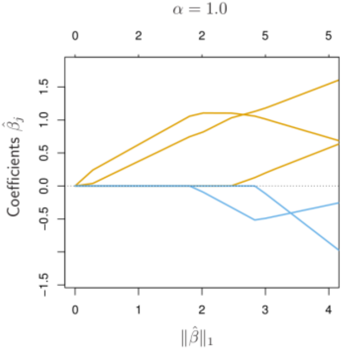

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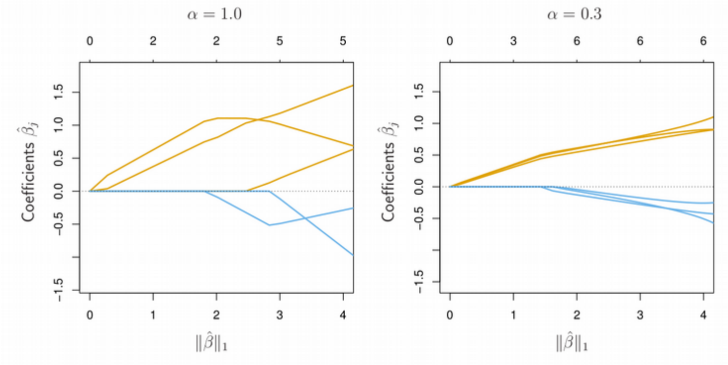

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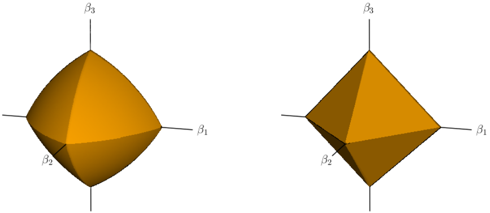

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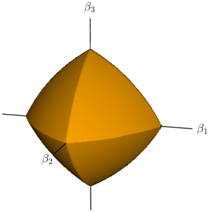

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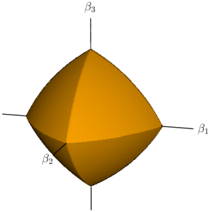

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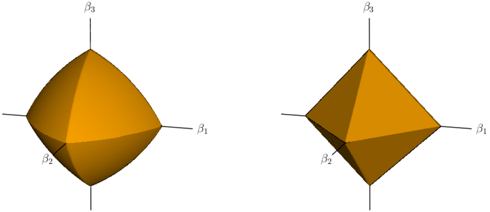

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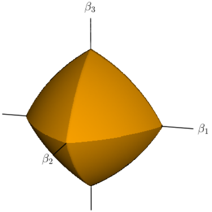

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Leukemia type (AML/ALL) dataset - Golub et al, Science '90

- 38 training samples, 34 test samples, $m = 7129$ genes.
- Record the expression for sample i and gene j

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Leukemia classification example

Method	10-fold CV error	Test error	No. of genes
Golub UR	3/38	4/34	50
SVM RFE	2/38	1/34	31
PLR RFE	2/38	1/34	26
NSC	2/38	2/34	21
Elastic Net	2/38	0/34	45

UR: univariate ranking (Golub et al. 1999)
RFE: recursive feature elimination (Guyon et al. 2002)
SVM: support vector machine (Guyon et al. 2002)
PLR: penalized logistic regression (Zhu and Hastie 2004)
NSC: nearest shrunken centroids (Tibshirani et al. 2002)

[Hui & Hasie, '05]

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- What if there's some natural grouping in the problem, and we want to introduce this as a prior?
- Say, J groups, and each $\boldsymbol{\beta}_j = [\beta_{j,1}, \ldots, \beta_{j,m_j}]$

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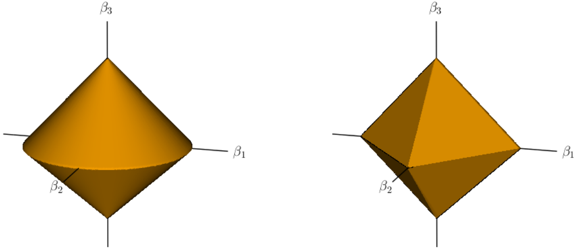

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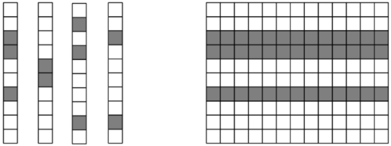

Sparsity vs. joint sparsity

End of this Part

Where are we at?

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$$\min_{\mathbf{x}} \|\mathbf{y} - \mathbf{A}\mathbf{x}\|_2^2 \;\; \text{s.t.} \;\; \|\mathbf{x}\|_0 \leq k$$

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4. Matrix spectral sparsity and robust PCA:

$$\min_{\mathbf{A}} \|\mathbf{X} - \mathbf{A}\| + \lambda \phi(\mathbf{A}) \qquad \longleftarrow$$

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Matrix Spectral Sparsity

EN.580.709 - Fall 2019

Eigenfaces

- Yale B Dataset

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(a) mean face (b) first eigenface (c) second eigenface

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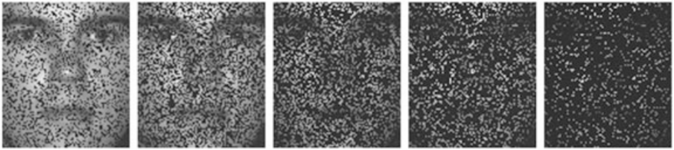
(a) Variation along the first eigenface

Matrix Completion



(a) Face images with (30, 50, 70, 80, 90)% percentage of missing entries

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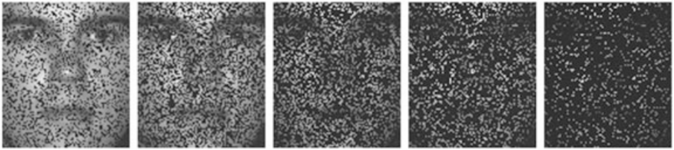

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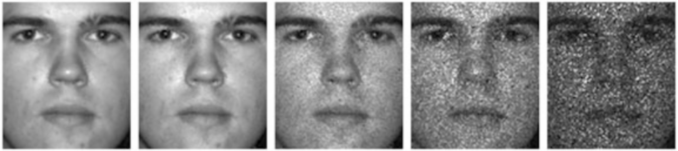
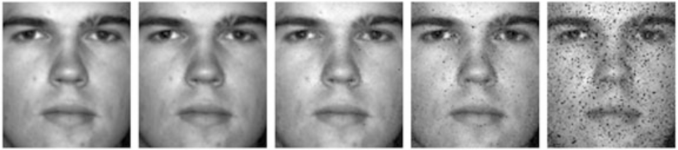
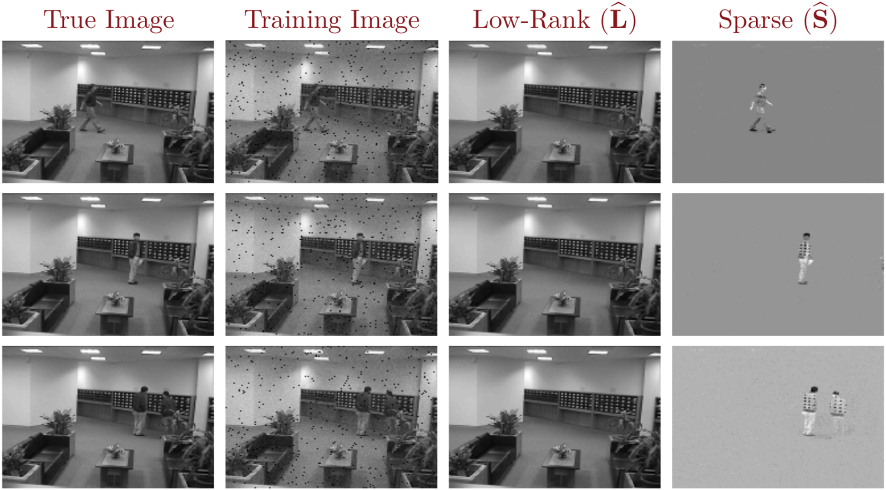


(c) Face images reconstructed by convex optimization with $\tau = 2 \times 10^4$



(d) Face images reconstructed by convex optimization with $\tau = 4 \times 10^5$

Robust PCA


True Image
Training Image
Low-Rank $(\widehat{\mathbf{L}})$
Sparse $(\widehat{\mathbf{S}})$