Chapter 2: Uniqueness and Uncertainty

We will initially focus on the problem

$$(P_0): \quad \min_x \|x\|_0 \text{ s.t. } Ax = b.$$

We will relax this problem later, e.g. to allow for small deviations.

We are first interested in the question: $\boxed{\text{is there a unique solution to } (P_0)\text{?}}$

[Consider avoiding the 2-ortho case and going directly to the general case.]

Consider for simplicity, first, the concatenation of two orthonormal (unitary) matrices: Ψ, Φ. : $A = [\Psi, \Phi]$, $\Psi^T\Psi = \Phi^T\Phi = I_n$. Consider, e.g., the identity and Fourier matrices. Assume $\exists\, \underline{b} \neq \underline{0}$: $b = \Psi\alpha = \Phi\beta$. What can we say about the sparsity of α and β? In particular, can a signal be sparse in time and frequency domains?

The answer clearly depends on the relation between Ψ and Φ. If they are the same, then both α, β can be arbitrarily sparse. A usefull measure of distance or similarity is the mutual coherence:

$$\mu(\Psi, \Phi) = \max_{1 \le i,j \le n} |\psi_i^T \phi_j|.$$

Consider:

$S = supp(\alpha)$.

$|\beta_j|^2 = |\langle b, \phi_j \rangle|^2$

$= |\langle \Psi_S \alpha_S, \phi_j \rangle|^2$

$= |\langle \alpha_S, \Psi_S^T \phi_j \rangle|^2$

$\le \|\alpha_S\|^2 \|\underbrace{\Psi_S^T \phi_j}\|^2$

$\psi_i^T \phi_j \le \mu$

$\le \|\alpha_S\|^2 \mu^2 |S|$.

One can easily see that for two orthogonal matrices $\frac{1}{\sqrt{n}} \le \mu(A) \le 1$. (To see the lower bound note that each column in $(\Psi^T\Phi)$ – an ortho matrix too – must have a norm of 1. Thus, the minimal maximal entry (if the energy is spread) is $1/\sqrt{n}$, so that $\sum_{i=1}^{n} a_i^2 = 1$.)

With $\mu(\Psi, \Phi)$ we have an uncertainty principle.

<u>Theorem:</u> For an arbitrary pair of orthogonal basis with mut. coherence $\mu(\Psi, \Phi)$ and $\underline{b} \neq \underline{0}$, so that $b = \Psi\alpha = \Phi\beta$,

Then $\|\alpha\|_0 + \|\beta\|_0 \ge \frac{2}{\mu(\Psi, \Phi)}$.

<u>Proof:</u> Note that $\|b\|_2 = \|\alpha\|_2 = \|\beta\|_2$. Let $I \triangleq supp(\alpha)$. Then, $|\beta_j|^2 = |\underline{b}^T \phi_j| = |\underbrace{\sum_{i \in I} \alpha_i \psi_i^T}_{b} \phi_j|^2$

From Cauchy-Schwartz: $|\beta_j|^2 \le \|\alpha\|_2^2 |\sum_{i \in I} \psi_i^T \phi|^2 \le \|b\|_2^2 \mu^2(\Psi, \Phi) \underbrace{|I|}_{\|\alpha\|_0}$.

$\langle x, y \rangle \le \|x\| \|y\|$

Then, $\sum |\beta_j|^2 = \|b\|_2^2 \le \|b\|_2^2 \mu^2(\Psi, \Phi) \|\alpha\|_0 \|\beta\|_0$.

geometric mean, $\Rightarrow \sqrt{\|\alpha\|_0 \|\beta\|_0} \ge \left(\frac{1}{\mu}\right)$ $\Rightarrow \|\alpha\|_0 \|\beta\|_0 \ge \left(\frac{1}{\mu}\right)^2$.

$\sqrt[n]{x_1 \cdot x_2 \cdot x_3 \cdots x_n}$ $\qquad\qquad$ $\frac{x_1+x_2+\dots+x_n}{n}$.

Employing the geometric-~~algebraic~~ arithmetic means relation: $\forall a,b>0$, $\sqrt{ab} \leq \frac{a+b}{2}$, then

$$1 \leq \mu(\Psi,\Phi)\, \tfrac{1}{2}(\|\alpha\|_0 + \|\beta\|_0) \quad \Rightarrow \quad \frac{\|\alpha\|_0+\|\beta\|_0}{2} \geq \sqrt{\|\alpha\|_0 \|\beta\|_0} \geq \frac{1}{\mu}$$

$$\Rightarrow \quad \|\alpha\|_0 + \|\beta\|_0 \geq \frac{2}{\mu(\Psi,\Phi)}.$$

Indeed, if one has $[I, F]$, signal cannot have less than $2\sqrt{n}$ non-zeros in time and in frequency. Thus, if a signal has less than $\sqrt{n}$ entries in one domain, it must be denser in the other.

In turn, this also implies an uncertainty in the cardinality of x: $\underline{b} = [\Psi, \Phi] x$.

Suppose $\exists\, x_1$ and x_2 : $Ax_1 = Ax_2$. Then $x_1 - x_2 = e \in \mathcal{N}(A)$: $Ae = 0.$, $\underline{e} \neq 0$.

Note that $Ae = [\Psi, \Phi]\begin{bmatrix} e_\Psi \\ e_\Phi \end{bmatrix} = 0 \Rightarrow \Psi e_\Psi = -\Phi e_\Phi \neq \underline{0}$, since $e \neq 0$.

From the result above, $\|e\|_0 = \|e_\Psi\|_0 + \|e_\Phi\|_0 \geq \frac{2}{\mu(A)}$. Further, $\|e\|_0 \leq \|x_1\|_0 + \|x_2\|_0$,

So $\|x_1\|_0 + \|x_2\|_0 \geq \frac{2}{\mu(A)}$.

Uniqueness 1

If $\exists\, x : Ax = b$, and $\|x\|_0 \leq 1/\mu(A) \Rightarrow$ it's the sparsest!

This uncertainty principle states that two solutions cannot be simultaneously sparse.

$\boxed{\text{Uniqueness in the general case}}$

Eg: $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{matrix} 0 \\ 1 \\ 1 \end{matrix}$

$\tilde{A} = \left[\begin{matrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 2 \\ 0 & 0 & 1 & 0 \end{matrix}\right]$

In the more general case, $A \in \mathbb{R}^{n \times m}$, $n > m$, consider the $\boxed{\text{spark}(A)}$ as the minimal number of linearly dependent columns. This should be contrasted with the rank(A): maximal number of l.i. columns.

The spark gives a simple criterion for uniqueness: note that by definition,

$A\underline{x} = \underline{0} \Rightarrow \|x\|_0 \geq \text{spark}(A)$., because at least spark(A) columns must be linearly combined to obtain the zero vector. Thus, we can have the following:

$\boxed{\text{Theorem}}$ (Uniqueness via Spark).

If a system $Ax = b$ has a solution x: $\|x\|_0 \leq \frac{\text{spark}(A)}{2}$, then it is the sparsest one possible.

proof: Assume $\exists\, y \neq x$, $Ay = b$, an alternative solution. Then $A(x-y) = 0$, and so $\|x\|_0 + \|y\|_0 \geq \|x-y\|_0 \geq \text{spark}(A)$.

Now, if $\|x\|_0 < \frac{\text{spk}(A)}{2} \Rightarrow \|y\|_0 > \frac{\text{spk}(A)}{2}$, and so x is the sparsest possible.

However simple, this is quite cool: even though solving (P_0) is NP-hard, if we are given a candidate solution x we can just check its optimality by counting $\|x\|_0$.

Q: What are bounds for the spark of a matrix? Clearly, $\eta(A) \geq 2$ if we forbid cases of all-zero columns. On the other extreme, $\eta(A) \leq n+1$.

Q2: How many non-zero should be required to guarantee a unique solution if A is Gaussian?

The problem is that, in general, computing $\eta(A)$ is as hard as computing the solution to (P_0), as is still a combinatorial problem. For this reason, one would like other easier way of checking for optimality. One such ways is via the mutual coherence: $\mu(A) = \max_{i \neq j} \frac{|a_i^T a_j|}{\|a_i\|_2 \cdot \|a_j\|_2}$. which characterizes the (maximal) correlation between two atoms. Note that for orthogonal matrices, $\mu = 0$.

If $m > n$ though, $\mu(A) < 1$. In general, for full rank matrices $\mu \geq \sqrt{\frac{m-n}{n(m-1)}} = \Omega\left(\frac{1}{\sqrt{n}}\right)$.

The set of matrices for which this lower bound is tight are called Grassmanian Frames – where the atoms are "as separated as possible". As opposed to the spark, $\mu(A)$ is trivial to compute. Then, the following result is usefull:

<u>Lemma:</u> $\forall A \in \mathbb{R}^{n \times m}$, $\text{spark}(A) \geq 1 + \frac{1}{\mu(A)}$.

<u>proof:</u> Assume without loss of generality that columns are normalized: $\|a_i\|_2 = 1$, as normalization preserves the spark and $\mu(A)$. The Gram matrix $G = A^T A$ satisfies

$$G_{ii} = 1 \quad \forall\ 1 \leq i \leq m, \quad |G_{ij}| \leq \mu(A) \quad \forall\ i \neq j.$$

Consider a leading submatrix of G of size $p \times p$. – Computed as the Gram of p columns for a: $\tilde{G}_p = A_p^T A_p$. Recall Gershgorin's disk theorem, stating that in a square matrix $H \in \mathbb{C}^{n \times n}$, all its eigenvalues must reside in the union of its Gershgorin's disks, defined as the circle centered at H_{ii} and of radius $R_i = \sum_{j \neq i} |a_{ij}|$: the sum of absolute values of its non-diagonal elements per row. Then, if the subGram is diagonally dominant, i.e $G_{ii} = 1 > \sum_{j \neq i} |G_{ij}|$, then $\tilde{G}$ must be positive-definite – why? and so all p columns are linearly independent. This condition implies that $1 > (p-1)\mu(A) \Rightarrow p < 1 + \frac{1}{\mu(A)}$. Thus, $p = 1 + \frac{1}{\mu}$ is the minimal number of columns needed for a linear dependence. Thus $\eta(A) \geq 1 + 1/\mu(A)$. ∎

Thus, with this result one has an optimality guarantee with a computable bound:

Theorem: If $\exists\, x: Ax = b$ and $\|b\|_0 < \frac{1}{2}\left(1 + 1/\mu(A)\right)$, then x is the sparsest possible solution – the optimal solution of (P_0).

proof: if $\|x\|_0 \leq \frac{1}{2}\left(1 + 1/\mu(A)\right) \leq \frac{1}{2}\eta(A)$ $\Rightarrow$ thus unique. ∎

The uniqueness guarantees with the spark are generally tight and far more powerfull, whereas the $\mu(A)$ only provides a lower bound on $\eta(A)$ is thus loose. Recall that $\mu(A) \geq 1/\sqrt{n}$. (for 2 ortho case) and thus $\|x\|_0 < \frac{\sqrt{n}}{2}$. However, spark$(A)$ can easily be as large as n, giving a bound of $\|x\|_0 < \frac{n}{2}$.

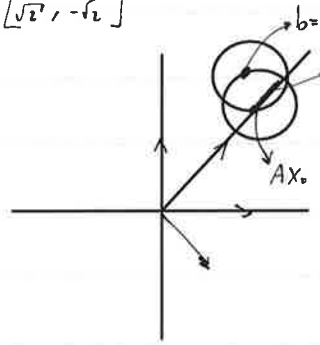
Stability of Sparse Solutions:

In many cases enforcing an exact constraint $Ax=b$ might be too stringent: what if the model for b is not really Ax? What if there were some contamination in the process of acquiring b? For these reasons, and more, we often relax the (P_0) problem to:

$$(P_0^\epsilon): \quad \min_x \|x\|_0 \ \text{ s.t. } \ \|b-Ax\|_2 \le \epsilon,$$

allowing for an ϵ-sized discrepancy between b and Ax. $(\epsilon > 0)$.
Thus, as a more general generative model, we will assume throughout that $b = Ax + e \ : \ \|e\|_2 \le \epsilon.$, and study the solution x^ϵ to (P_0^ϵ). Is it unique?

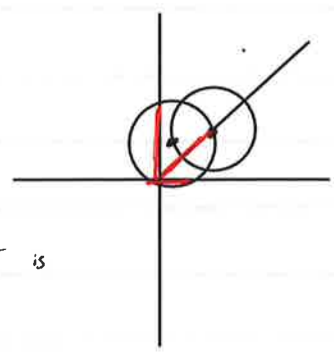
In such cases, the uniqueness of the solution is lost. Consider $A=[I, H]$,
$H = \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ \sqrt{2}, & -\sqrt{2} \end{bmatrix}$



area of feasible solutions: $z : \|Az-b\| \le \epsilon,$
$\|z\|_0 = 1.$

$\Rightarrow$ infinite z!

In this case, even the zero solution is included in the feasible set, and so it is the sparsest $\Rightarrow$ the solution to (P_0^ϵ).



In this way, the theoretical analysis of (P_0^ϵ) is concerned with the stability of the solution rather than with its uniqueness. To analyze this, we need a new characterization of the dictionary:

<u>Definition:</u> Restricted Isometry Property (RIP)

A matrix A (normalized) $(n\times m)$ satisfies the RIP of order $s\leq n$ with constant δ_s if

$$\forall\, x: \|x\|_0 \leq s, \quad (1-\delta_s)\|x\|_2^2 \leq \|Ax\|_2^2 \leq (1+\delta_s)\|x\|_2^2.$$

In this way, if A has the RIP of order s, any subset of 's' columns behaves "close" to an isometry/orthogonal transform.

Just as the spark(A), the RIP constant is not computable. However, it can be bounded with the mutual coherence:

Notice that if $s=1$, $\delta_s = 0$, as $\|a_i\|_2 = 1$. More generally, for $x: \|x\|_0 = s$, recall the subGram of A constructed with s-columns from A. From Gershgorin's circle Theorem,

$$|\lambda(\underbrace{A_s^T A_s}_{\tilde{G}}) - 1| \leq (s-1)\,\mu(A)$$

And so $\quad 1-(s-1)\mu(A) \leq \lambda(A_s^T A_s) \leq 1+(s-1)\mu(A).$

Thus, $\forall\, x: \|x\|_0 = s,$

$$1-(s-1)\mu\|x\| \leq \lambda_{min}(A_s^T A_s)\|x\|_2^2 \leq \|Ax\|_2^2 \leq \lambda_{max}(A_s^T A_s)\|x\|_2^2 \leq (1+(1-s)\mu)\|x\|_2^2$$

This implies that $\boxed{\delta_s \leq (s-1)\,\mu(A)}$

With such an assumption of restricted isometry, one can guarantee the stability of the solution to the (P_0^ϵ) problem:

<u>Theorem:</u> Consider a sparse vector $x: \|x\|_0 = k < \frac{1}{2}\left(1+\frac{1}{\mu(A)}\right)$, and the measurements $y = Ax+n$, where $\|n\|_2 \leq \epsilon$. If the matrix A satisfies the RIP with constant δ_{2k}, then a solution to (P_0^ϵ), $\hat{x}$, satisfies:

$$\|x-\hat{x}\|_2^2 \leq \frac{4\epsilon^2}{1-\delta_{2k}} \leq \frac{4\epsilon^2}{1-(2k-1)\mu(A)}.$$

proof: By definition, $\hat{x}$ satisfies $\|y - A\hat{x}\|_2 \leq \epsilon$. It also satisfies $\|\hat{x}\|_0 \leq \|x\|_0$, as it is the one with minimal l_0 norm. Let $\Delta = x - \hat{x}$, then:

$$\|A\Delta\|_2^2 = \|Ax - y + y - A\hat{x}\|_2^2 \leq 4\epsilon^2, \quad \text{by triang. inequality.}$$

Note that also

$$\|\Delta\|_0 \leq \|x\|_0 + \|\hat{x}\|_0 \leq 2k.$$

Thus, if A has δ_{2k}-RIP:

$$(1 - \delta_{2k})\|\Delta\|_2^2 \leq \|A\Delta\|_2^2 \leq 4\epsilon^2$$

and so:

$$\|x - \hat{x}\|_2^2 \leq \frac{4\epsilon^2}{1 - \delta_{2k}} \leq \frac{4\epsilon^2}{1 - (2k-1)\mu(A)}$$

from the previous bound.

Note that the last inequality is true only if $k \leq \frac{1}{2}\left(1 + \frac{1}{\mu(A)}\right)$. //

Possible Introduction / Motivation to this Chapter

Consider a general signal $b \in \mathbb{R}^n$. It could be audio, images, etc.
Recall their expansion in an orthonormal basis: $\{\phi_i\}_{i=1}^n$

i.e: $\Phi = [\phi_1, \phi_2, \dots] \in \mathbb{R}^{n\times n}$, and $\Phi^T\Phi = I$.

Then we can write:

$$b = \sum_{i=1}^n x_i \cdot \phi_i \quad ; \quad x_i \in \mathbb{R}. \qquad (b = \Phi x).$$

and the coefficients are obtained simply by $x_i = \langle b, \phi_i \rangle = \sum_{j=1}^n b_i \cdot \phi_{ij}$

$$\Rightarrow \quad x = \Phi^T b.$$

Some signals that are dense, might become sparse under a change of basis. Consider, e.g., a sine function: $b = \left[\sin\left(\frac{2\pi w}{n} \cdot j\right)\right]_{j=1}^n = \sin(\omega t)$

While b is dense, if one takes $\Phi = \mathcal{F}$: (discrete) Fourier transform, then

Recall that: $\Phi = \mathcal{F} = \{e^{-\frac{i2\pi}{n}jk}\}_{k=0}^{k=n}$

$\Phi = [\phi_{k=0}, \phi_{k=2}, \dots]$

$$x = \Phi^T b = \delta(w - \theta) = [1, 0, \dots, 1, \dots 0].$$

$(b = e^{-\frac{i2\pi}{n}wj} + b_0)$ some frequency.

$b = e^{-\frac{i2\pi}{n}jk^*} + b_0$, $k^* > 0$.

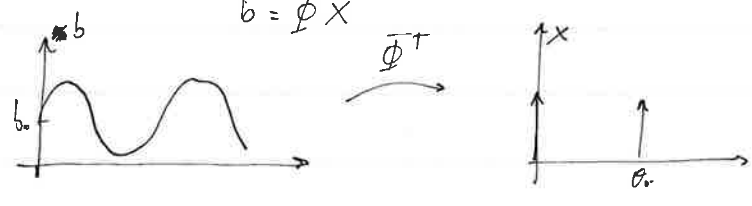
So if we wanted to compress the signal b (i.e: retain as much information with as few coefficients as possible) we would have to solve a problem like:

$$\hat{x} = \arg\min_{x\in\mathbb{R}^n} \|b - \Phi x\|_2^2 \quad \text{s.t.} \quad \|x\|_0 \le k.$$

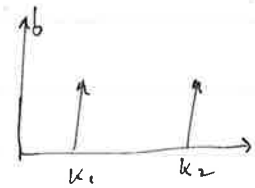
This is easy: since Φ: unitary, $\|b - \Phi x\|_2^2 = \|\Phi^T b - x\|_2^2 = \|\varepsilon - x\|_2^2$

and $\min_x \|\varepsilon - x\|_2^2$ s.t. $\|x\|_0 \le k$ is solved by taking the k-largest entries in ε, say $\hat{x} = \mathcal{T}_k(\varepsilon)$. Finally, one can reconstruct

$$\hat{b} = \Phi \hat{x}$$



Now, what if $b = \delta(i - k_1) + \delta(i - k_2)$?

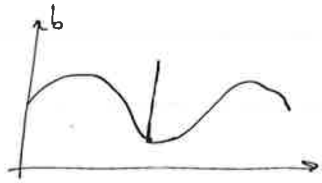


$\Rightarrow$ b is already sparse in its natural (canonical basis), and $\Phi^T b = x$ will not be sparse if Φ is, say, $\mathcal{F}$!

$\Rightarrow$ $\boxed{\text{The basis is central to obtain sparsity}}$

Consider now the case where $b = b_0 \mathbb{1} + e^{j\frac{2\pi}{n}\theta i} + \delta(i-a)$?

Now b will not be sparse in either base, Φ or I. So what can we do?



A natural solution would be to have

$$A = [\Phi, I].$$

Then $b = Ax = [\Phi, I]\begin{bmatrix} x_\Phi \\ x_I \end{bmatrix}$, $\quad x_\Phi = [1, 0 0, \ldots 1, 0 0]$.
$\quad x_I = [0, 0 \ldots 1, 0 0 \ldots]$.

The problem is, that to solve the "compression problem" from before, we'd need to solve:

$$\min_x \|b - Ax\|_2^2 \quad \text{s.t.} \quad \|x\|_0 \leq k$$

and now this is combinatorial, as we saw before. Is this a lost cause? In principle, there could $\exists$ many x: $b = Ax$.
Can we then have a unique solution?

Q: What happens if b is white noise?