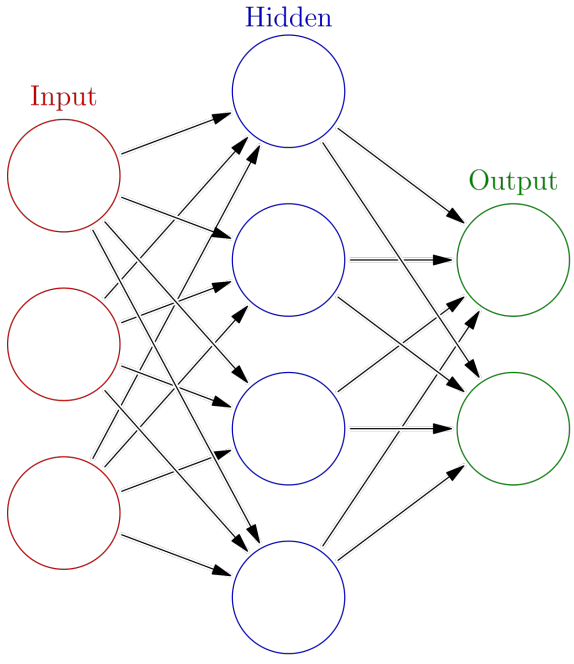
Deep and Convolutional Sparse Models

EN.580.709 - Fall 2019

What are artificial neural networks?



- Input features

$$\mathbf{x} \in \mathbb{R}^n$$

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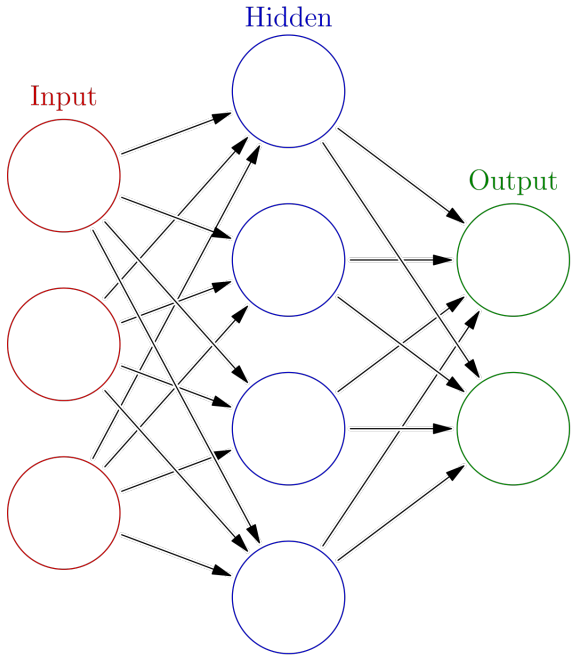

- Input features

$$\mathbf{x} \in \mathbb{R}^n$$

- Hidden neuron

$$h_j = \sigma\left(\sum_{i=1}^{n} w_{i,j} x_i + b_j\right)$$

$$\mathbf{h} = \sigma(\mathbf{W}\mathbf{x} + \mathbf{b}) \in \mathbb{R}^m$$

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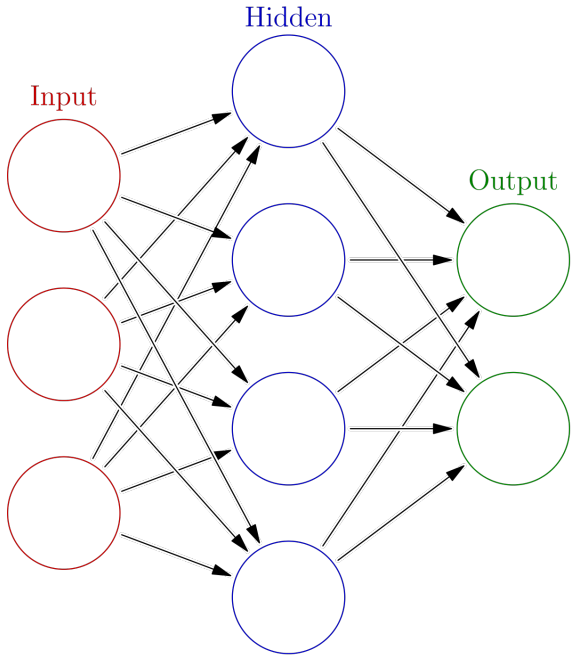

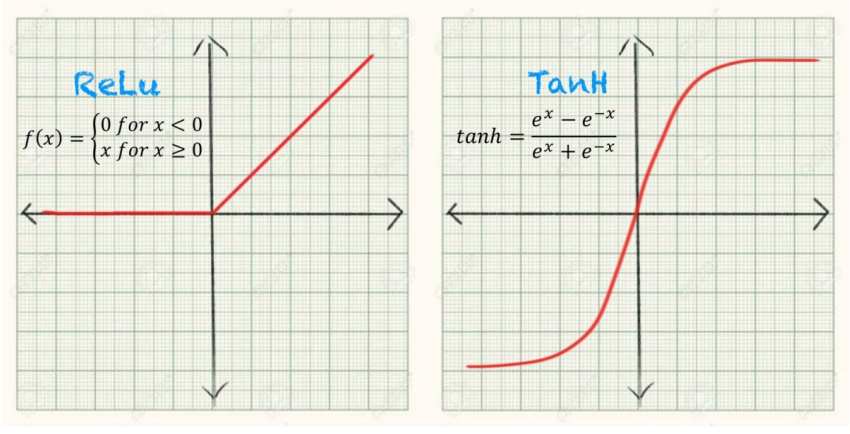

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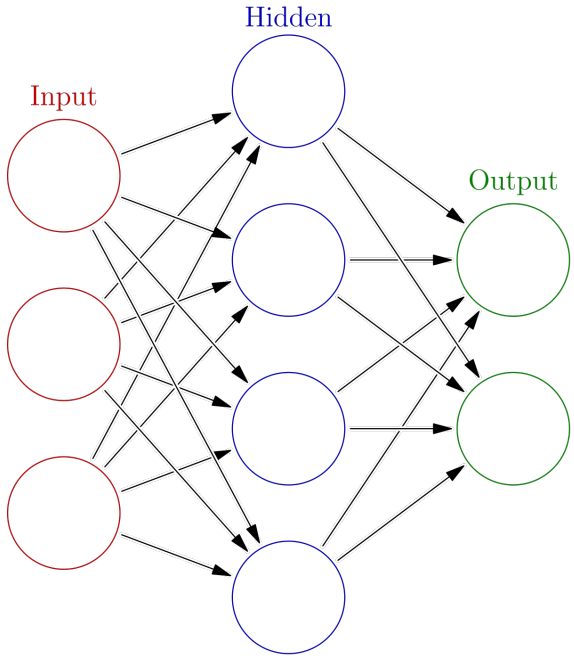

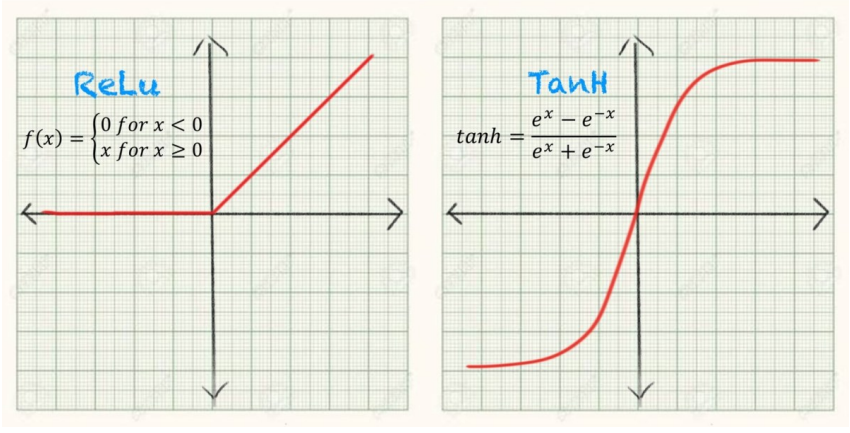

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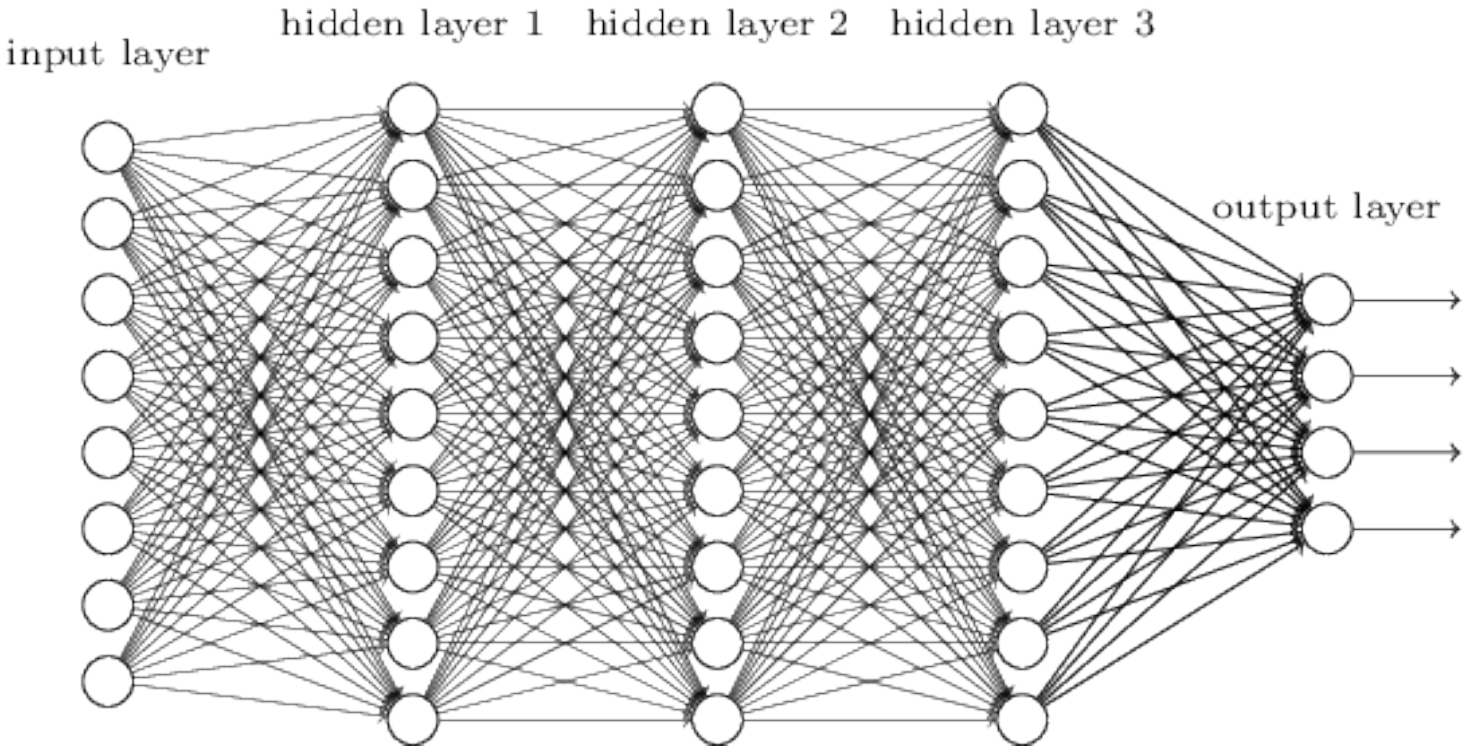
- Output

$$\mathbf{y} = \mathbf{W}_2\mathbf{h} \in \mathbb{R}^{m_2}$$

What are deep neural networks?



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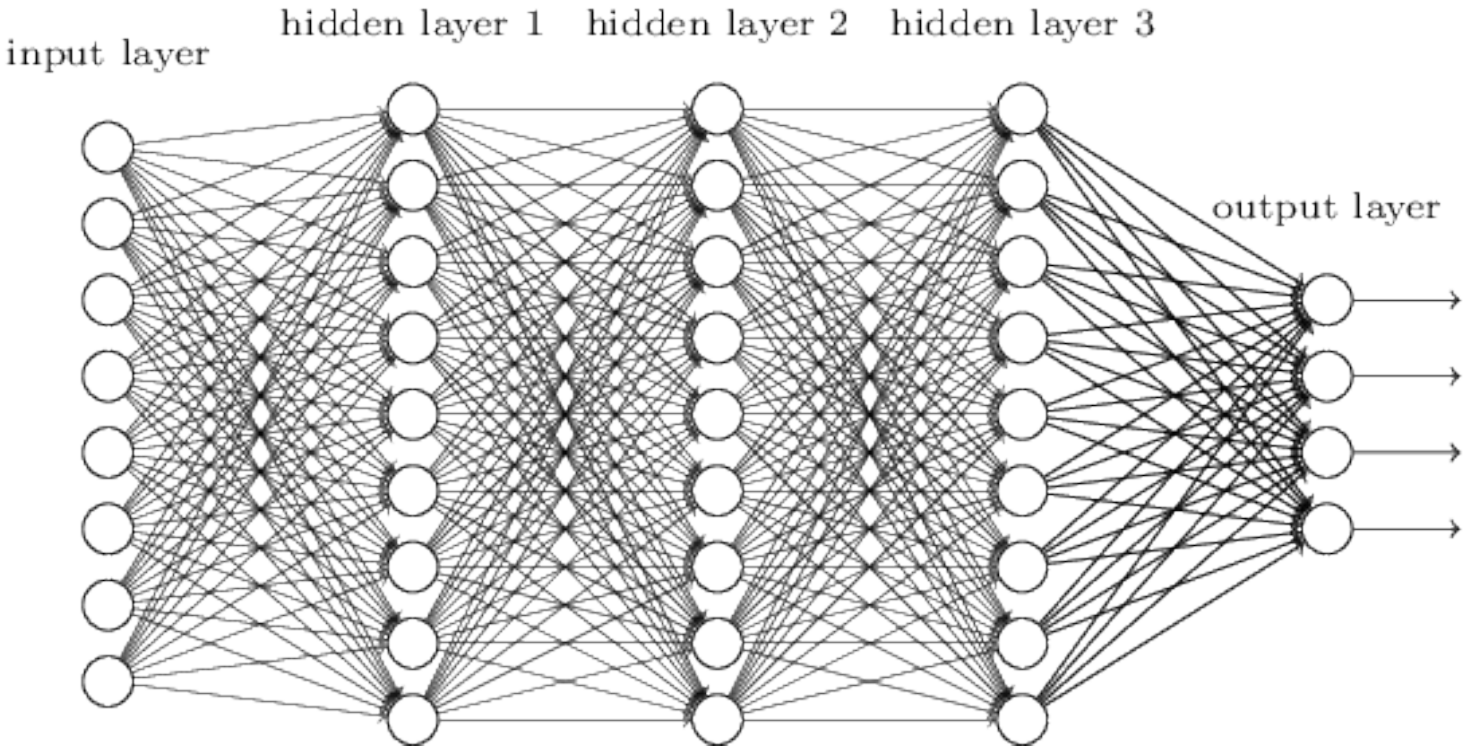
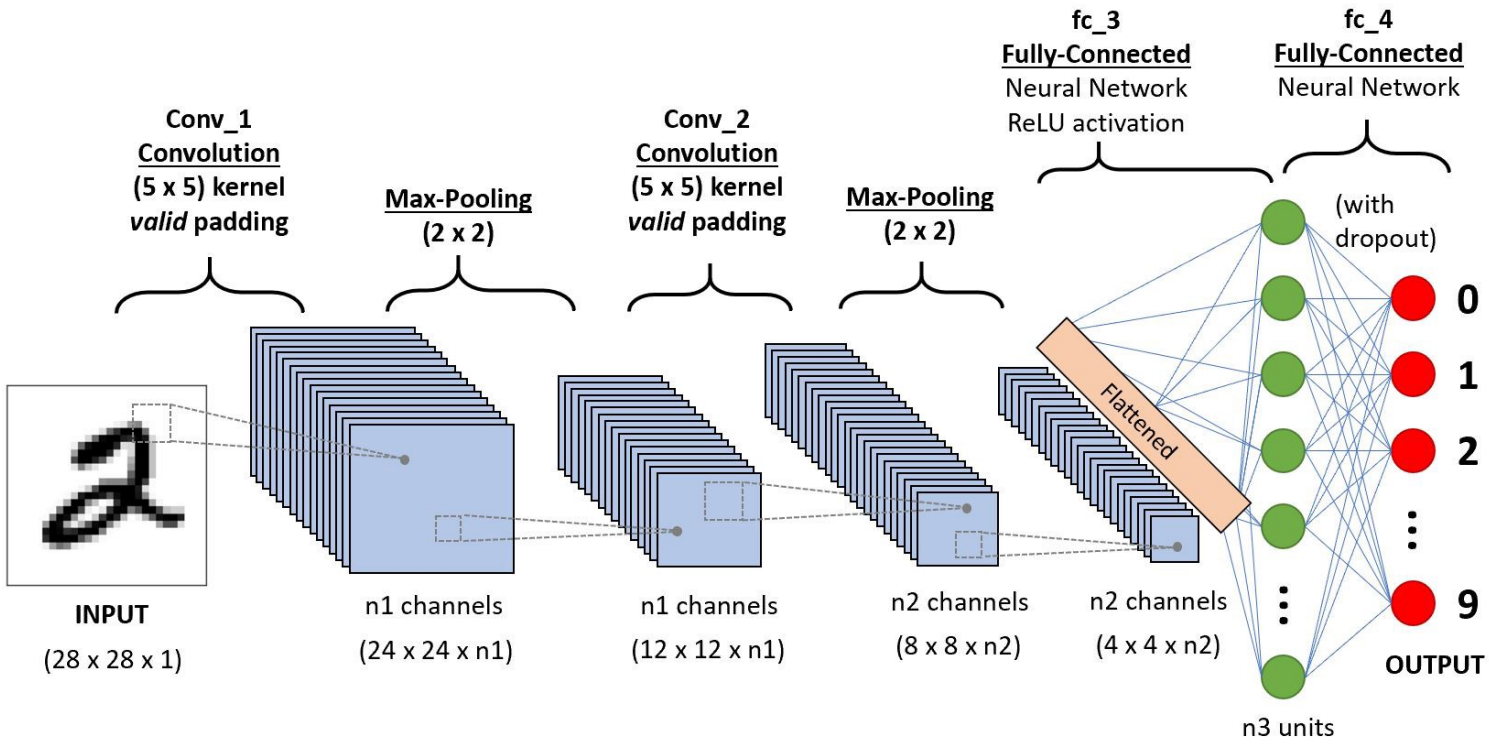


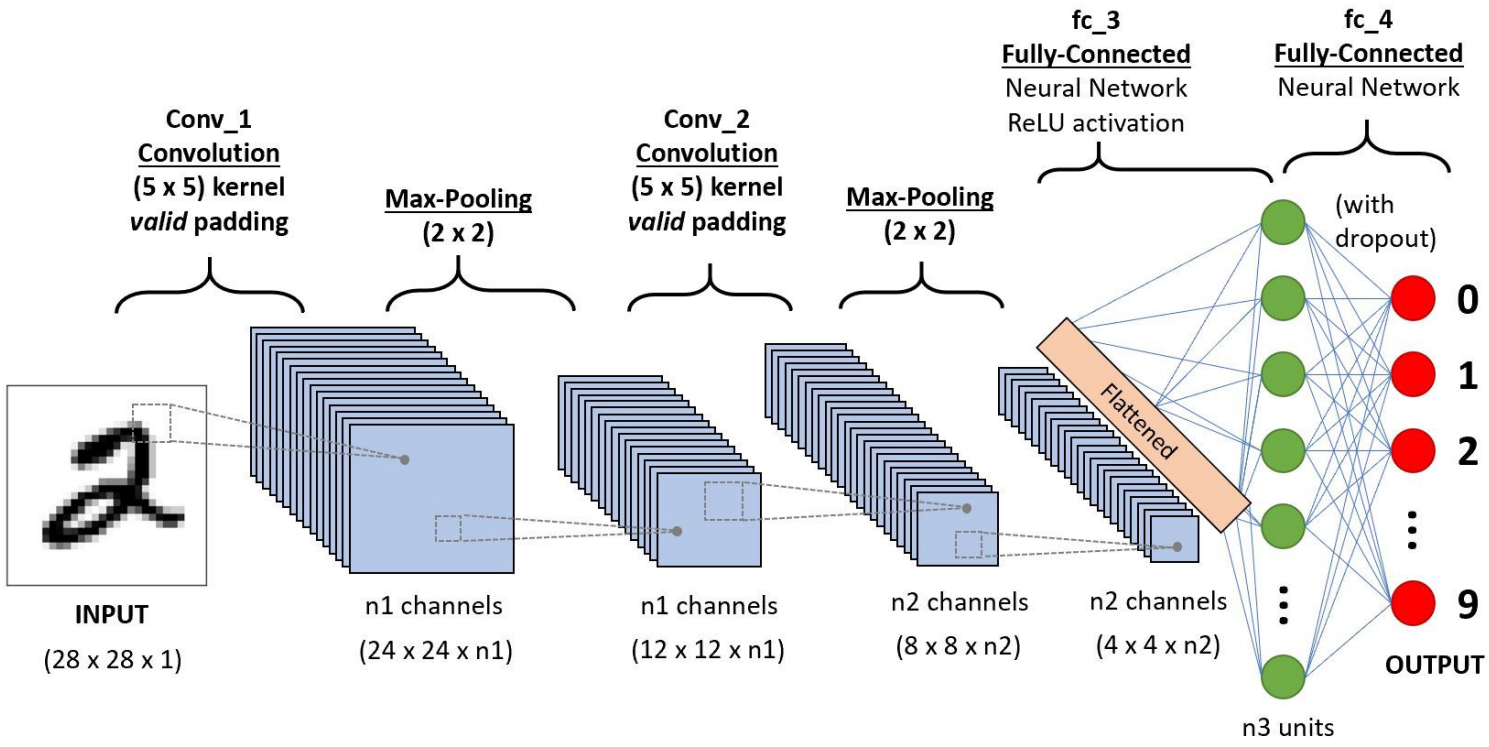
$$\mathbf{y} = \Phi(\mathbf{x}) = \mathbf{W}_L \sigma(\mathbf{W}_{L-1} \sigma(\dots \sigma(\mathbf{W}_1 \mathbf{x} + \mathbf{b}_1) \dots) + \mathbf{b}_{L-1})$$

What are deep convolutional neural networks?



Convolutional Feature Maps

What are deep convolutional neural networks?



Convolutional Feature Maps

$$\mathbf{h}_i = \sigma(\mathbf{w}_1 * \mathbf{x} + b_i)$$

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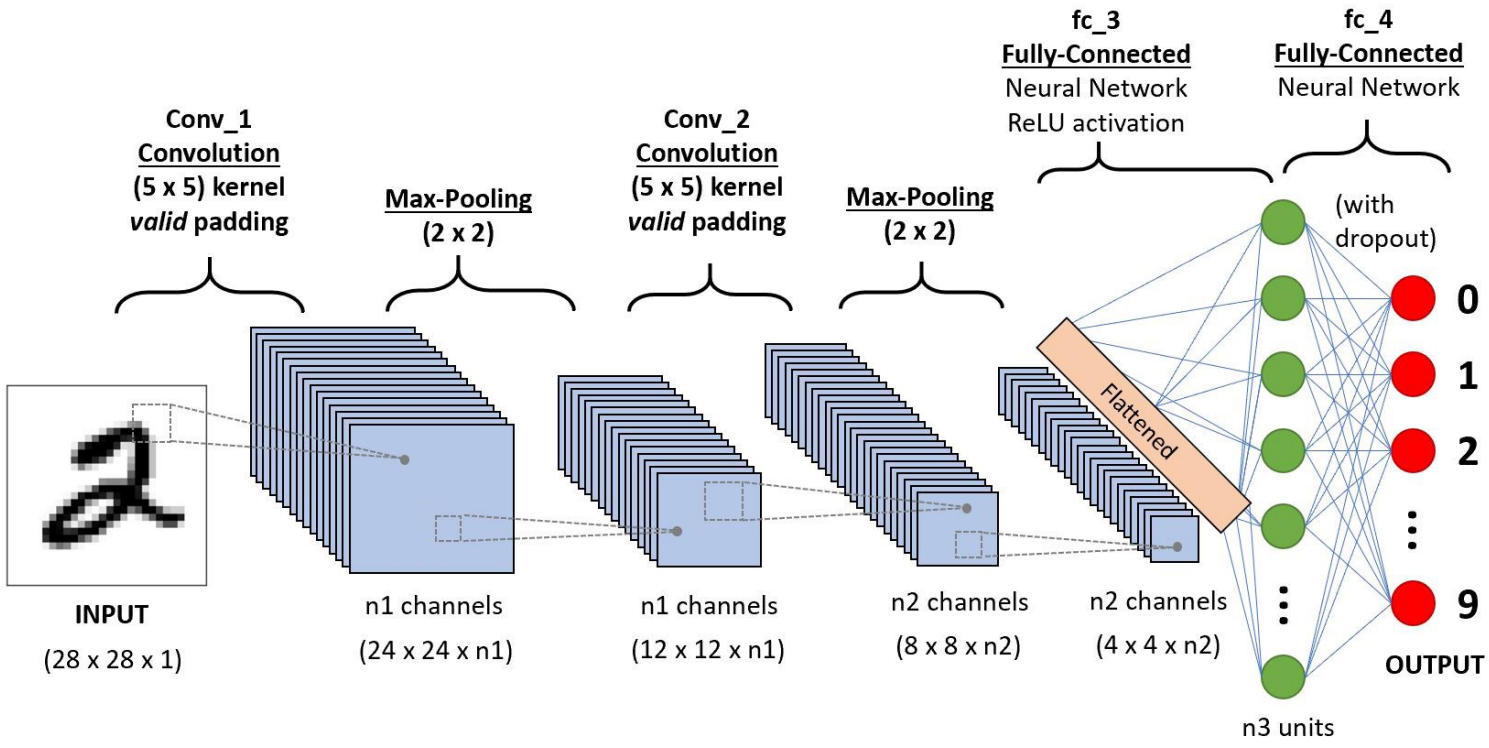
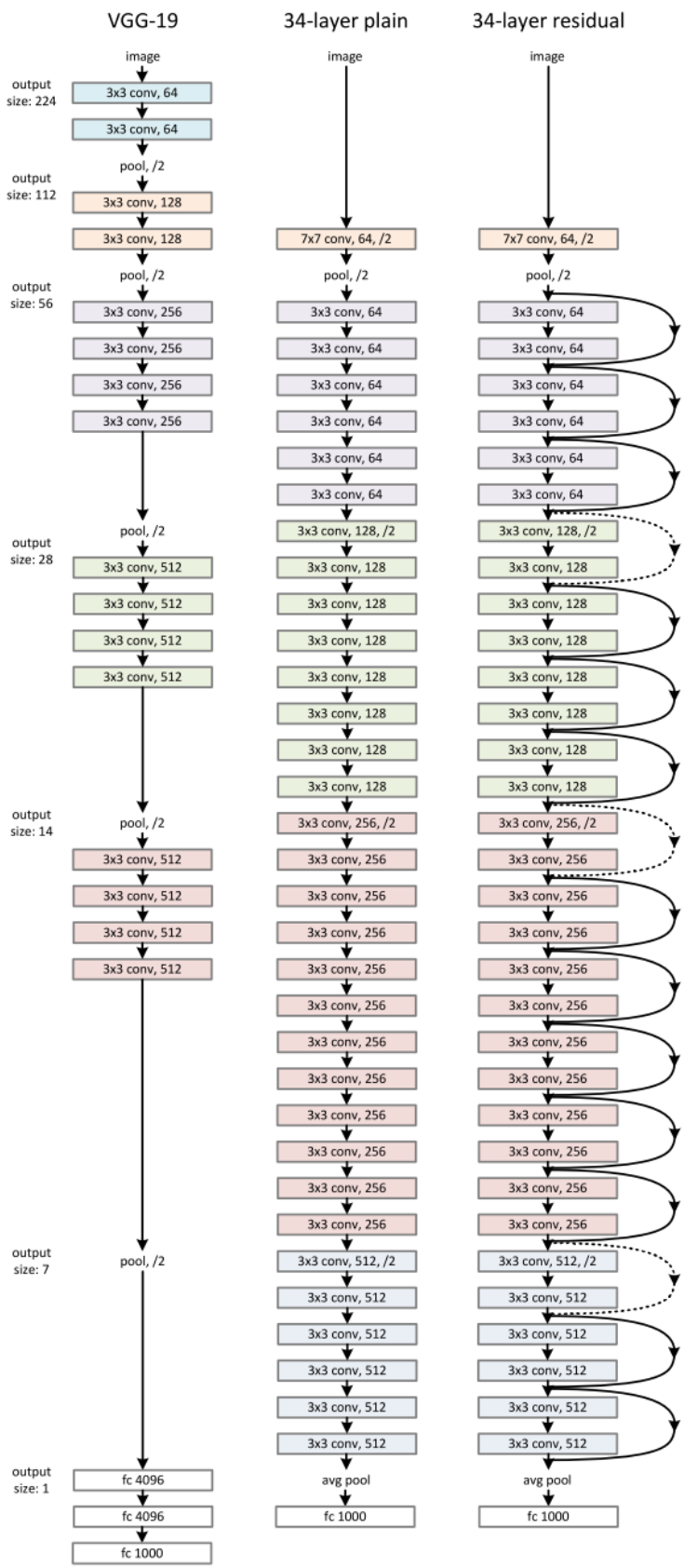


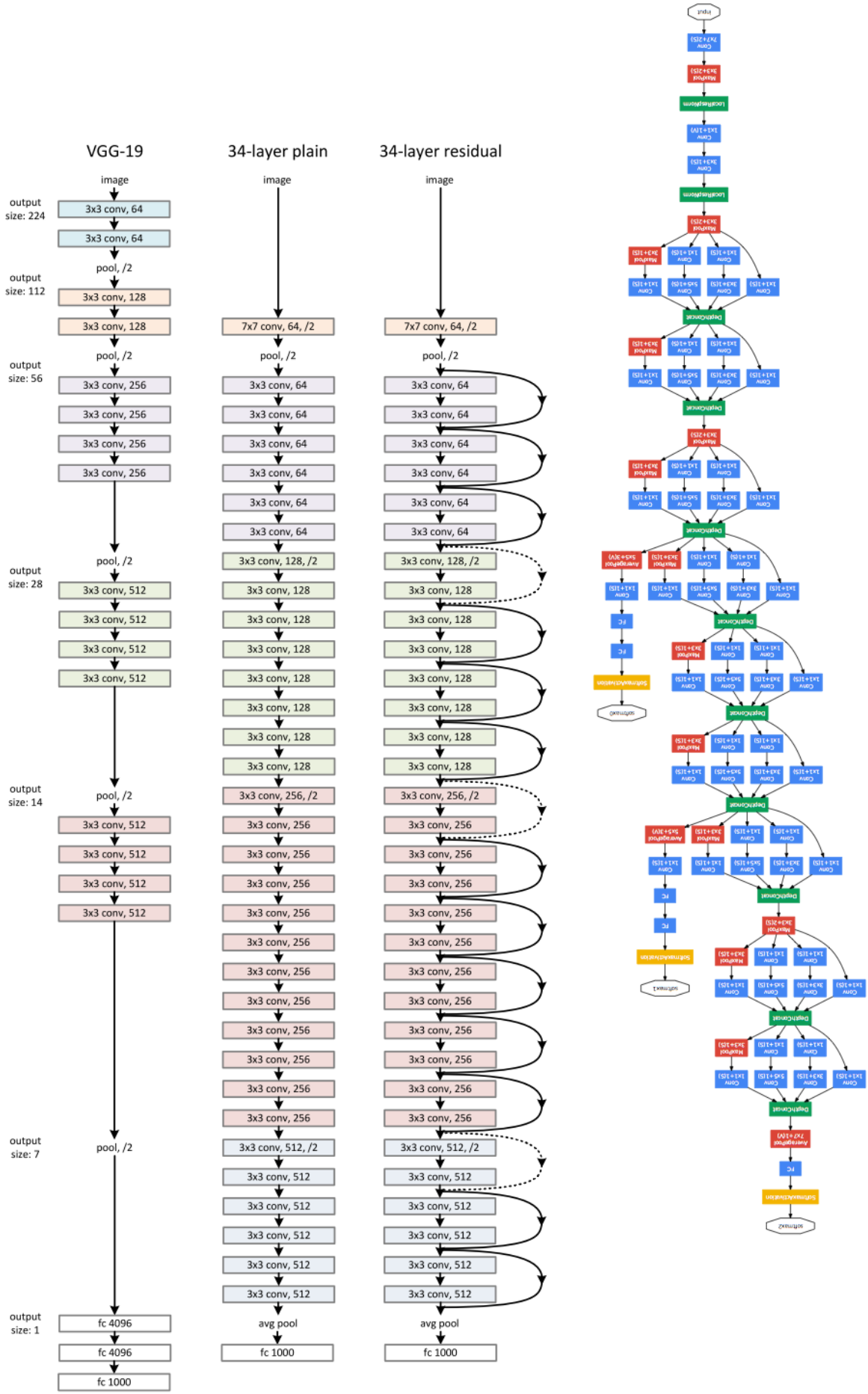
Convolutional Feature Maps

$$\mathbf{h}_i = \sigma(\mathbf{w}_1 * \mathbf{x} + b_i)$$

Max Pooling

$$\mathbf{h}(i) = \max_{j \in \mathcal{N}(s \cdot i)} \mathbf{x}(j)$$


VGG-19
34-layer plain
34-layer residual
image
output size: 224
3x3 conv, 64
3x3 conv, 64
pool, /2
output size: 112
3x3 conv, 128
3x3 conv, 128
pool, /2
output size: 56
3x3 conv, 256
3x3 conv, 256
3x3 conv, 256
3x3 conv, 256
pool, /2
output size: 28
3x3 conv, 512
3x3 conv, 512
3x3 conv, 512
3x3 conv, 512
pool, /2
output size: 14
3x3 conv, 512
3x3 conv, 512
3x3 conv, 512
3x3 conv, 512
pool, /2
output size: 7
fc 4096
fc 4096
fc 1000
output size: 1
7x7 conv, 64, /2
pool, /2
3x3 conv, 64
3x3 conv, 128, /2
3x3 conv, 128
3x3 conv, 256, /2
3x3 conv, 256
3x3 conv, 512, /2
3x3 conv, 512
avg pool
fc 1000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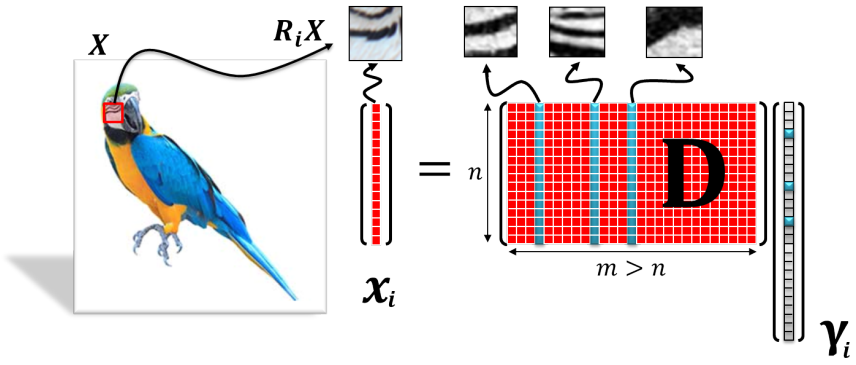
Guiding Questions

- What are the signal models behind these algorithms?
- Why these algorithms and not others?
- Understanding, theoretical guarantees, guidelines

Synthesis Sparse Modeling



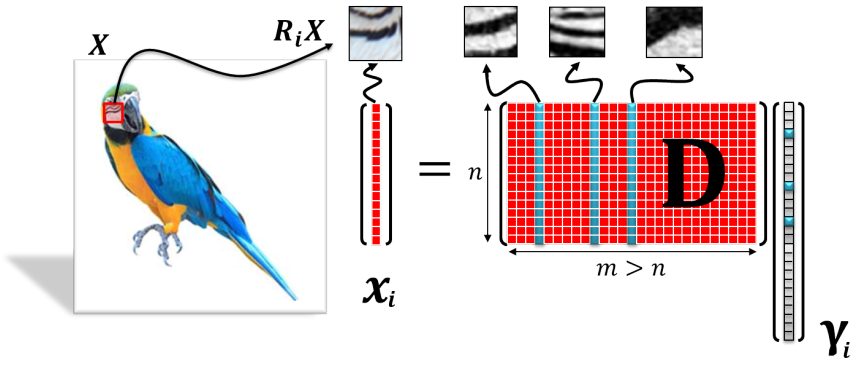
Model:

$$\mathbf{y} = \mathbf{D}\boldsymbol{\gamma} + \mathbf{w}, \quad \|\boldsymbol{\gamma}\|_0 \leq k$$

Pursuit:

$$\min_{\mathbf{x}} \|\mathbf{y} - \mathbf{D}\boldsymbol{\gamma}\|_2^2 \quad \text{s.t.} \quad \|\boldsymbol{\gamma}\|_0 \leq k$$

Synthesis Sparse Modeling



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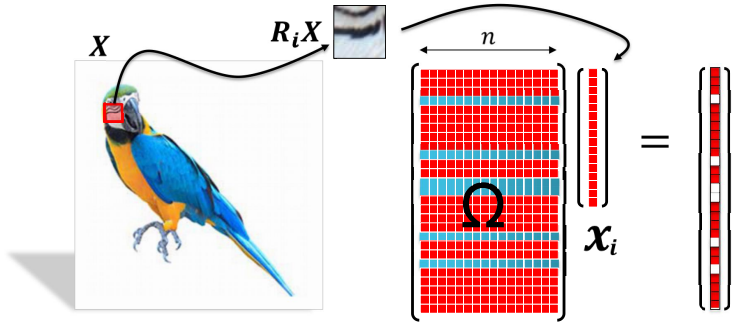
Analysis Sparse Modeling



Model:

$$\mathbf{y} = \mathbf{x} + \mathbf{w}, \quad \|\boldsymbol{\Omega}\mathbf{x}\|_0 \leq m - \ell$$

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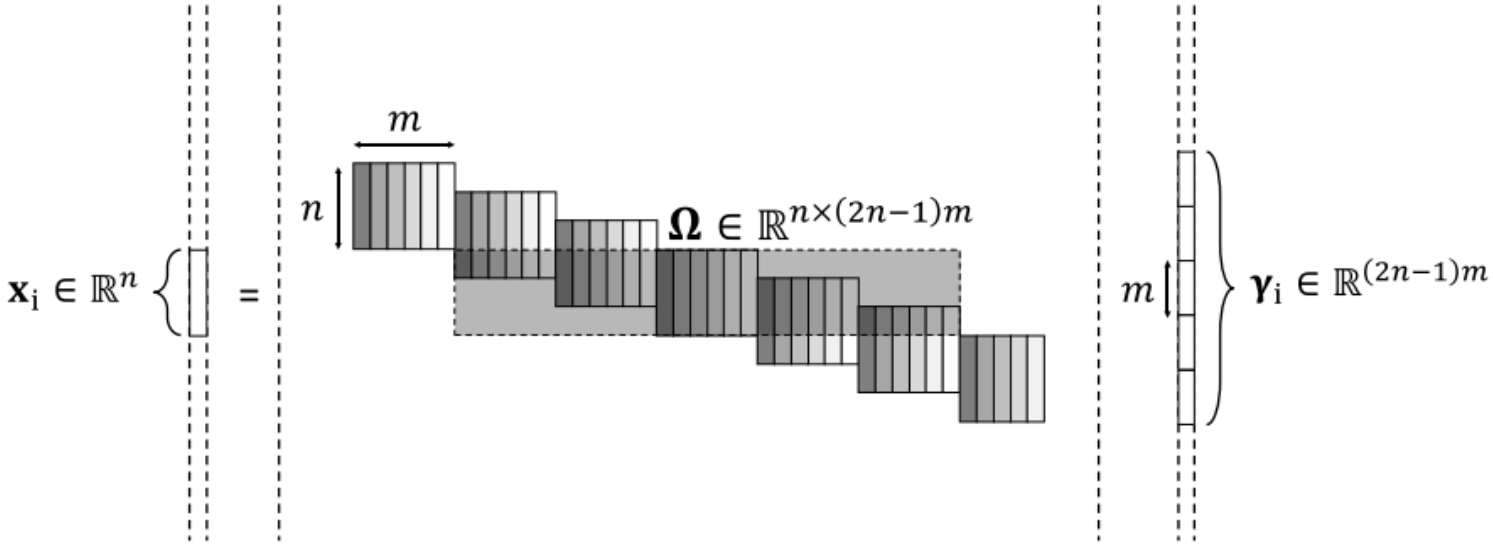
Recap on Convolutional Sparse Models



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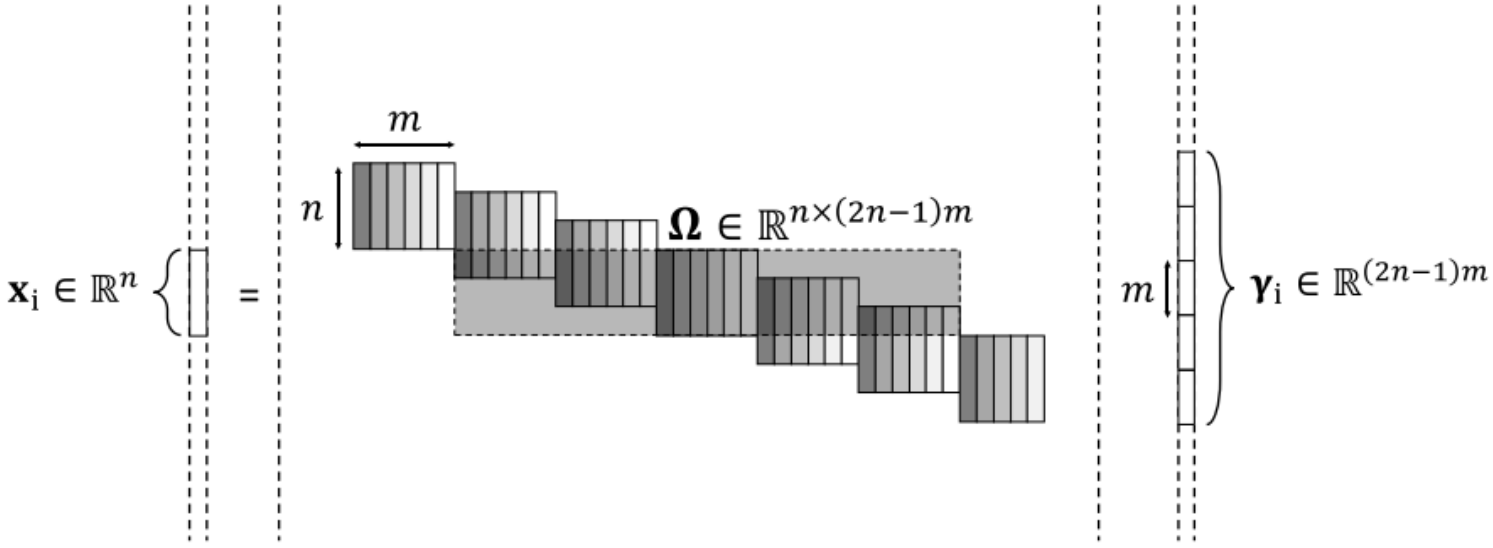

Shift-Invariant Model:

$$\mathbf{x}_i = \mathbf{R}_i\mathbf{X} = \boldsymbol{\Omega}\mathbf{S}_i\boldsymbol{\Gamma} = \boldsymbol{\Omega}\boldsymbol{\gamma}_i$$

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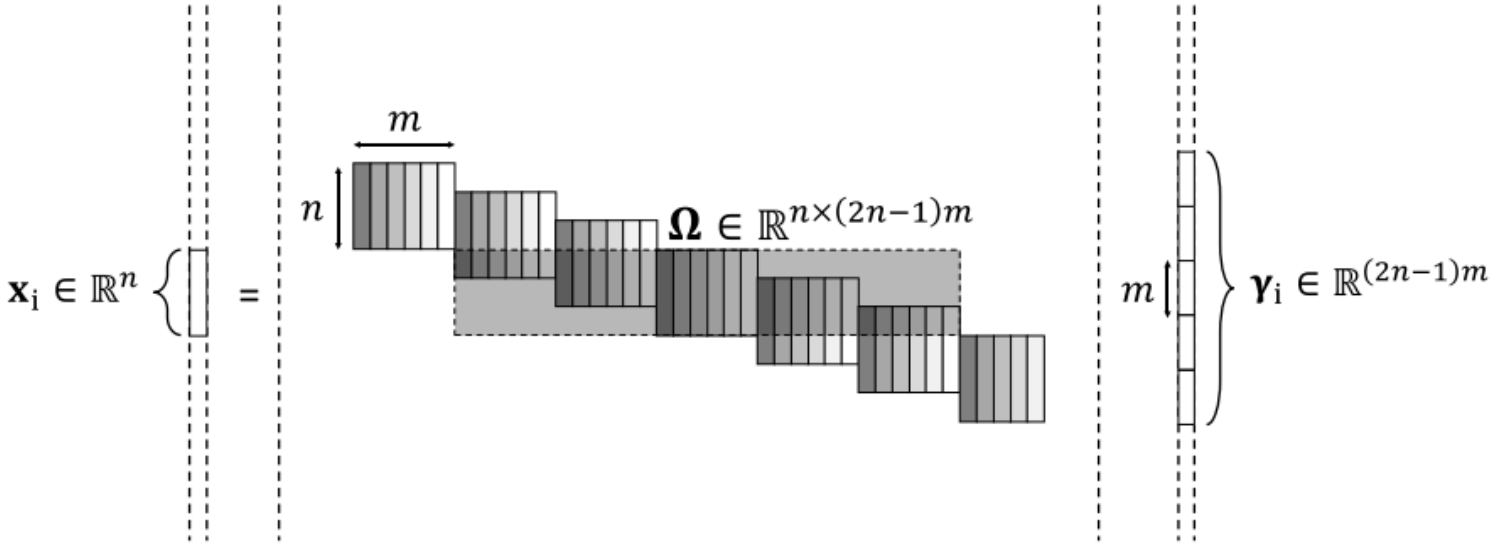

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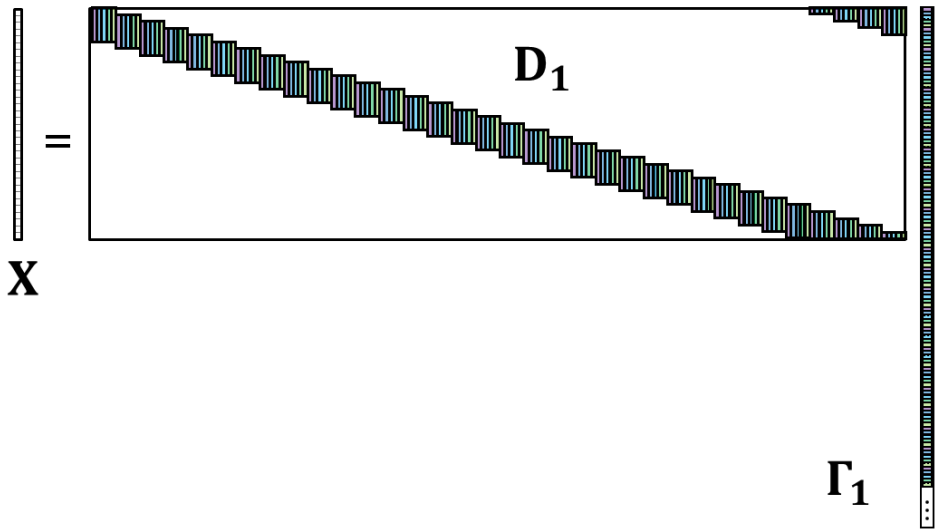
New sparsity measure:

$$\|\boldsymbol{\Gamma}\|_{0,\infty} := \max_i \|\boldsymbol{\gamma}_i\|_0 = \max_i \|\mathbf{S}_i\boldsymbol{\Gamma}\|_0$$

Multilayer (& convolutional) sparse models

Given $\{\mathbf{D}_i\}_{i=1}^{L}$, a signal $\mathbf{X} \in \mathbb{R}^N$,

$$\mathbf{X} = \mathbf{D}_1\boldsymbol{\Gamma}_1, \quad \|\boldsymbol{\Gamma}_1\|_{0,\infty}^{s} \leq \lambda_1,$$



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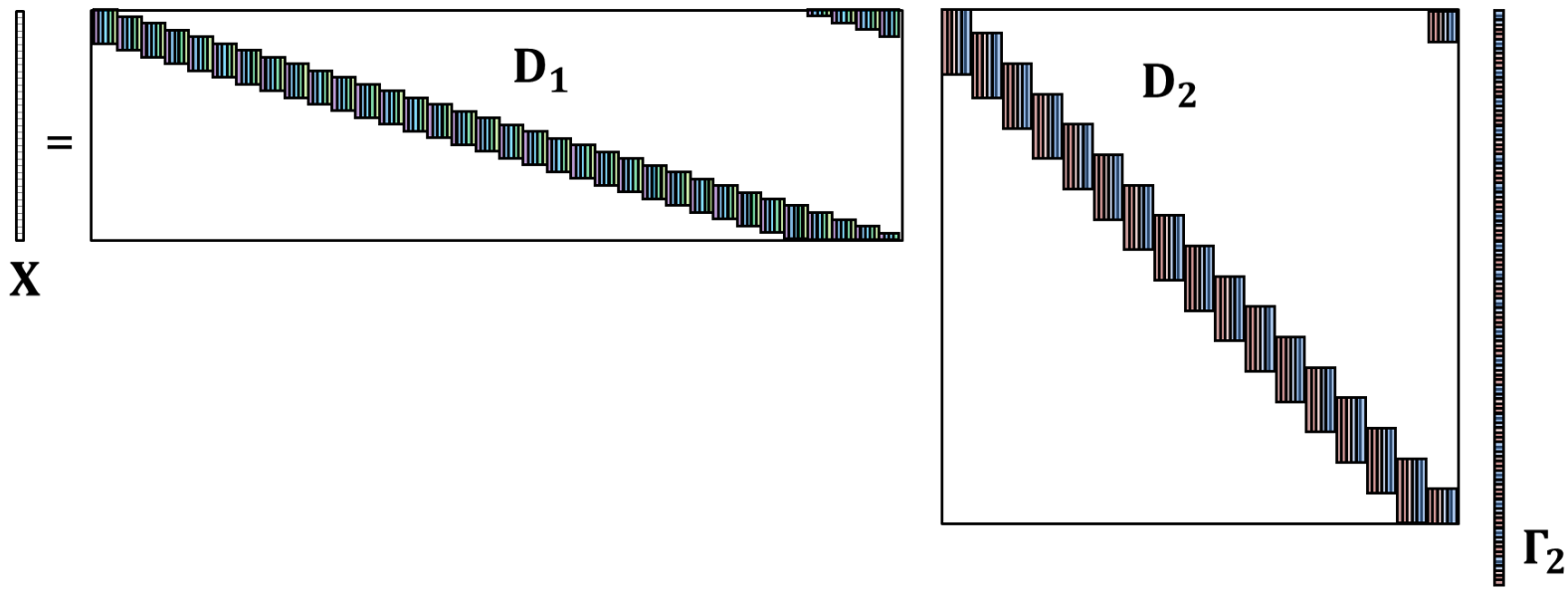

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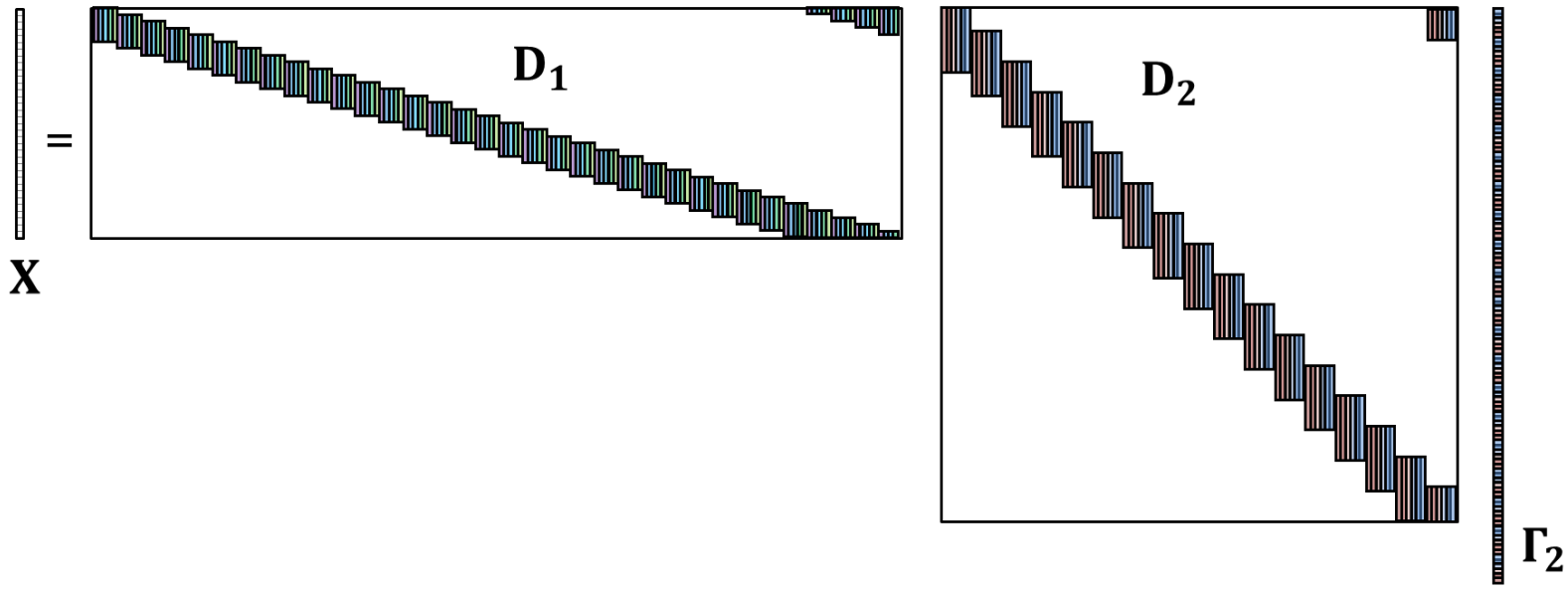

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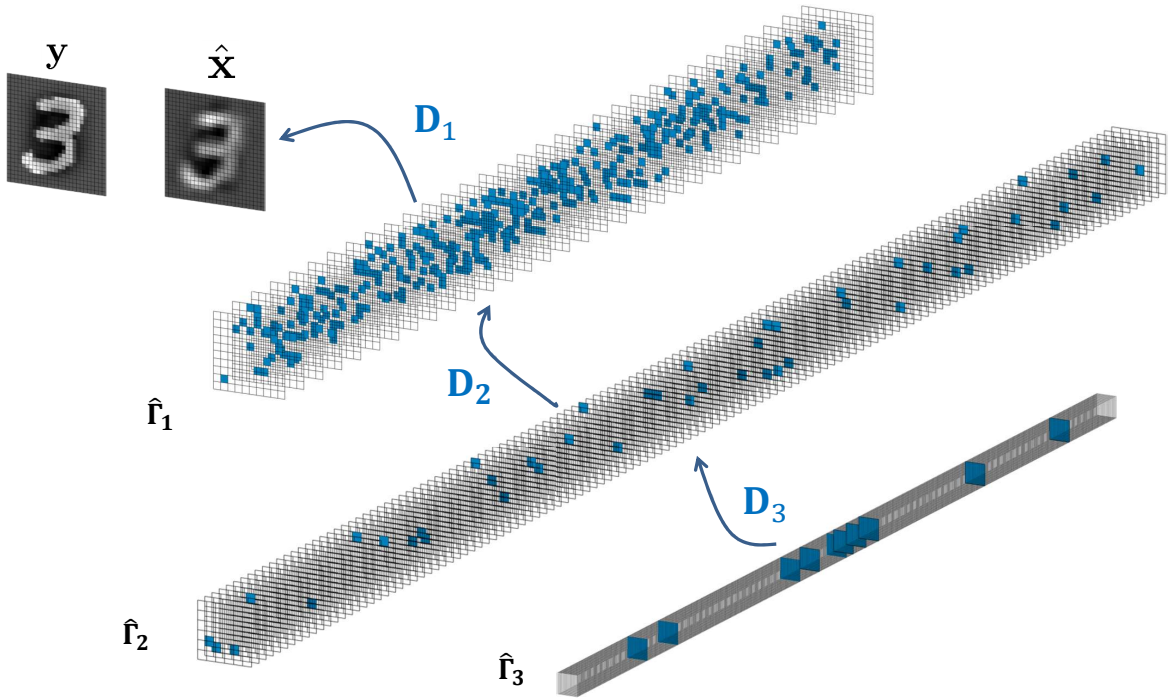

A Multi-Layer Pursuit

Say we get $\mathbf{Y} = \mathbf{X}(\mathbf{\Gamma}_i) + \mathbf{E}$, how to (deep) sparse code?

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Consider the first layer: $\mathbf{Y} = \mathbf{D}_1\mathbf{\Gamma}_1 + \mathbf{E}$, how to find $\mathbf{\Gamma}_1$?

Simplest Pursuit: $\hat{\mathbf{\Gamma}}_1 \leftarrow \mathcal{P}_{\beta_1}(\mathbf{D}_1^T\mathbf{Y})$

How good is this solution?

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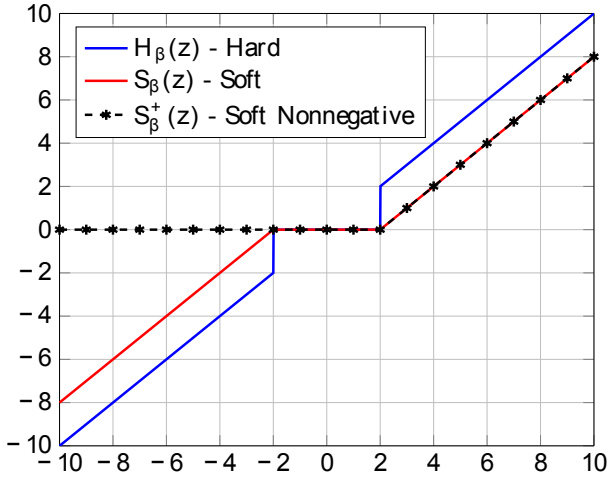

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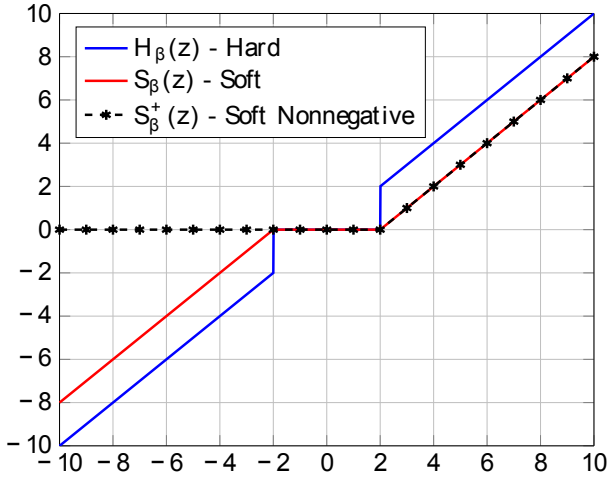
How good is this solution?

Consider then $\hat{\mathbf{\Gamma}}_1 = \mathbf{D}_2\mathbf{\Gamma}_2 + \mathbf{E}_1$, how to find $\mathbf{\Gamma}_2$?

Second Layer pursuit: $\hat{\mathbf{\Gamma}}_2 \leftarrow \mathcal{P}_{\beta_2}(\mathbf{D}_2^T\hat{\mathbf{\Gamma}}_1)$



Layered Thresholding algorithm

$$\hat{\mathbf{\Gamma}}_1 = \mathcal{P}_{\beta_1}(\mathbf{D}_1^T \mathbf{Y})$$

Layered Thresholding algorithm

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Written differently,

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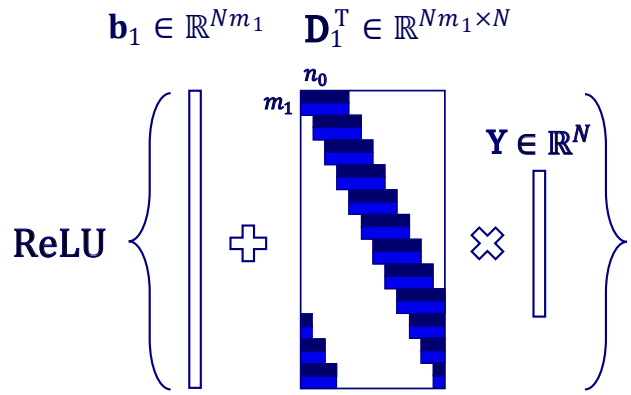

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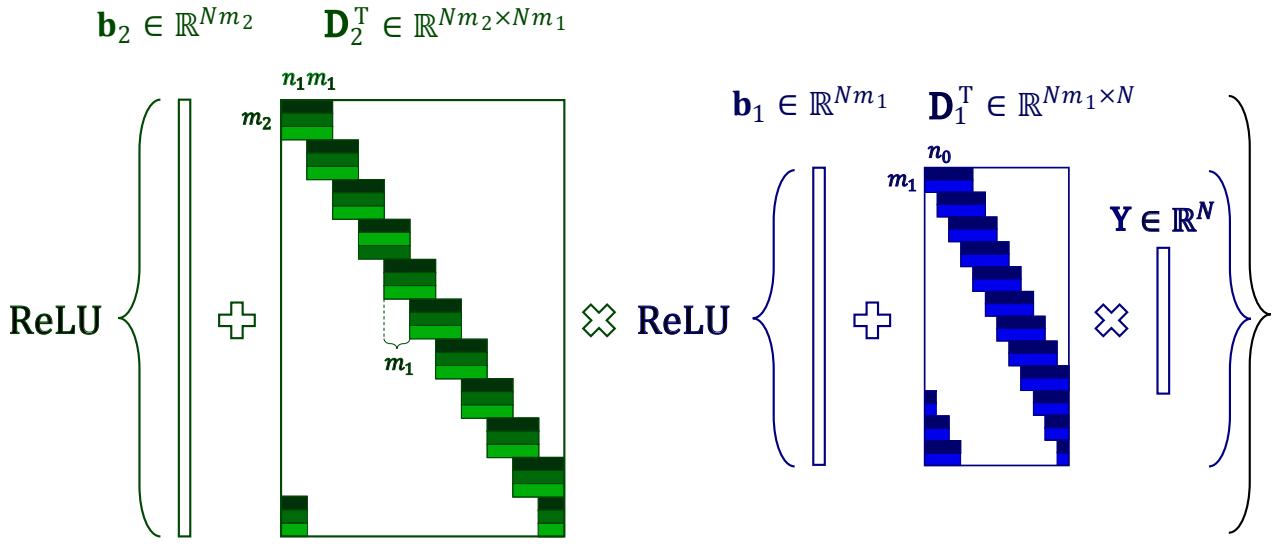

Layered Thresholding algorithm

$$\hat{\mathbf{\Gamma}}_2 = \mathcal{P}_{\beta_2}(\mathbf{D}_2^T \mathcal{P}_{\beta_1}(\mathbf{D}_1^T \mathbf{Y}))$$

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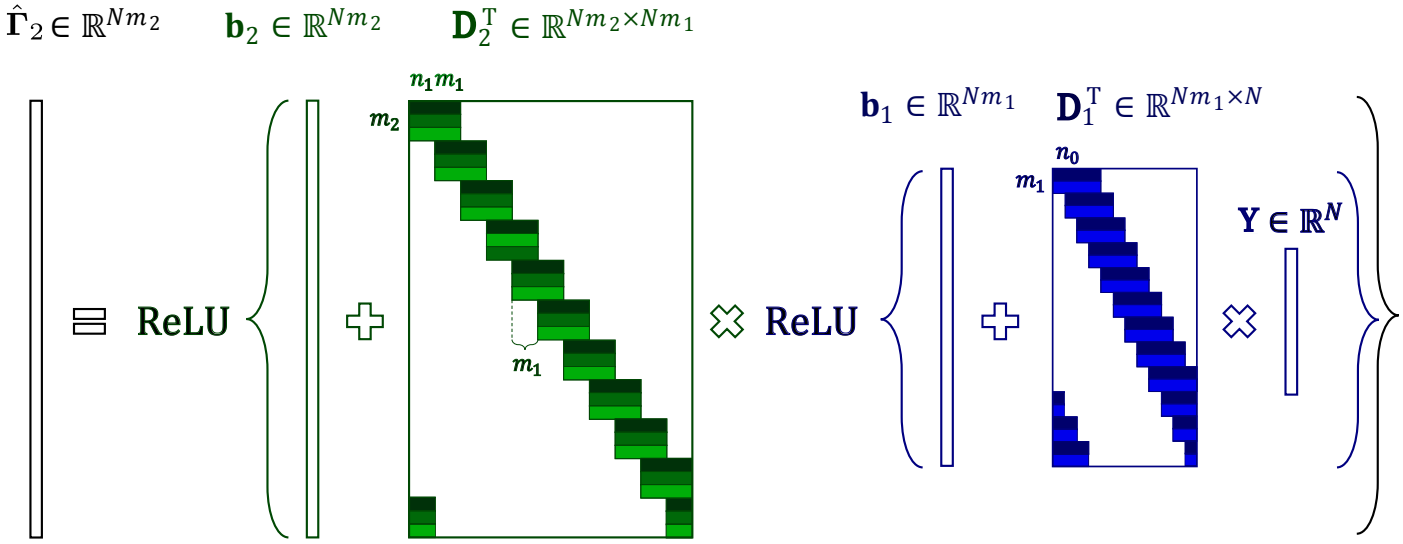

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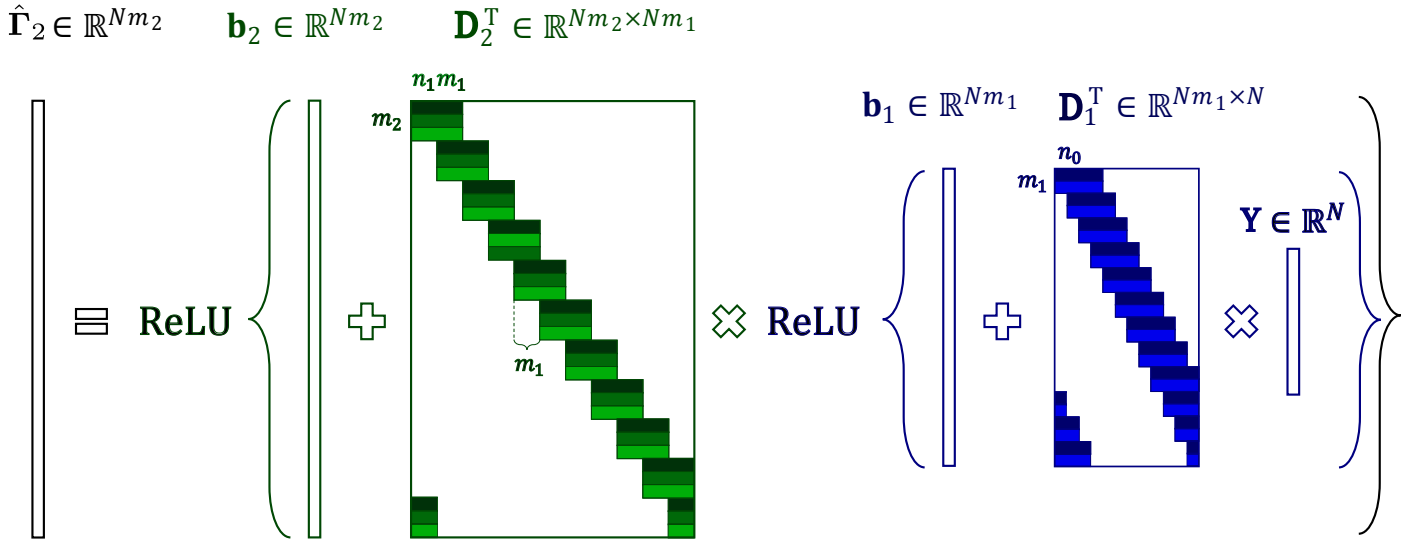

The forward pass can be seen as a pursuit seeking for the sparse representations under the ML-CSC model

Looking into the Networks

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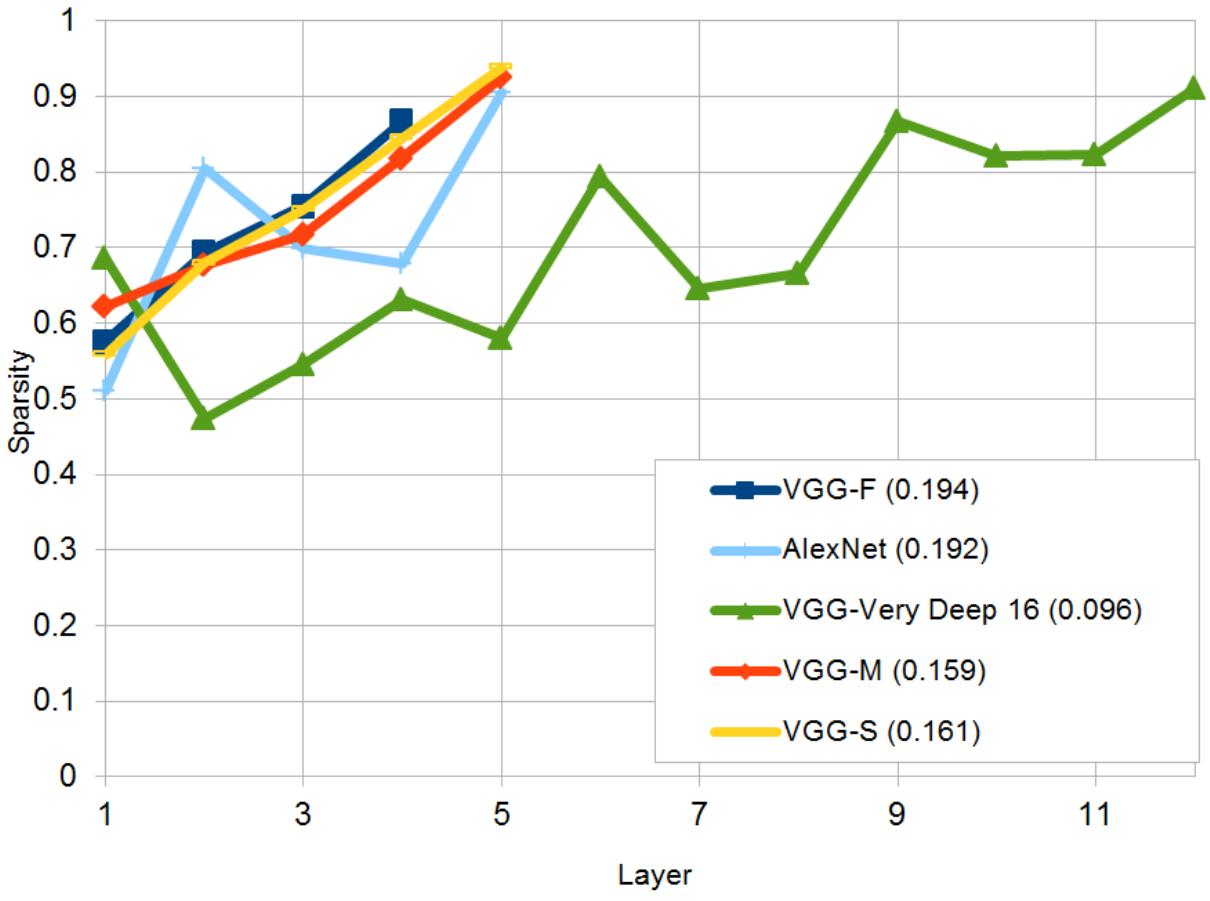

Stability of the Multi-Layer Thresholding (a.k.a Forward Pass)

If a set of solutions $\{\mathbf{\Gamma}_i\}_{i=1}^K$ satisfy

$$\|\mathbf{\Gamma}_i\|_{0,\infty} \leq \frac{1}{2}\left(1+\frac{1}{\mu(\mathbf{D}_i)}\frac{|\mathbf{\Gamma}_i^{min}|}{|\mathbf{\Gamma}_i^{max}|}\right) - \frac{1}{\mu(\mathbf{D}_i)}\frac{\epsilon_L^{i-1}}{|\mathbf{\Gamma}_i^{max}|}$$

then, for proper thresholds/biases β_i^*

- $Supp(\hat{\mathbf{\Gamma}}_i) = Supp(\mathbf{\Gamma}_i)$
- $\|\hat{\mathbf{\Gamma}}_i - \mathbf{\Gamma}_i\|_{2,\infty}^p \leq \sqrt{\|\mathbf{\Gamma}_i\|_{0,\infty}^p}\left(\epsilon_L^{i-1} + \mu(\mathbf{D}_i)\left(\|\mathbf{\Gamma}_i\|_{0,\infty}^s - 1\right)|\mathbf{\Gamma}_i^{\max}| + \beta_i\right)$

✓ Recovery of the support

✓ Stable recovery of representations

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✓ Recovery of the support
✓ Stable recovery of representations

× Signal contrast
× No perfect recovery in noiseless case
× Bounds increase with depth

Layered Basis Pursuit

$$\hat{\mathbf{\Gamma}}_1 = \arg\min_{\mathbf{\Gamma}} \quad \frac{1}{2}\|\mathbf{Y} - \mathbf{D}_1\mathbf{\Gamma}\|_2^2 + \lambda_1\|\mathbf{\Gamma}\|_1$$

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Stability

If $\{\mathbf{\Gamma}_i\}_{i=1}^K$ satisfy $\|\mathbf{\Gamma}_i\|_{0,\infty} \leq \frac{1}{3}\left(1 + \frac{1}{\mu(\mathbf{D}_i)}\right)$, then

- $Supp\{\hat{\mathbf{\Gamma}}_i\} \subseteq Supp\{\mathbf{\Gamma}_i\}$
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Layered Basis Pursuit

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× Bound increases with depth

ML-CSC Projection ($\mathcal{P}_{\mathcal{M}_{\boldsymbol{\lambda}}}$)

Given $\mathbf{Y} = \mathbf{X}(\boldsymbol{\Gamma}_i) + \mathbf{E}$, $\mathbf{X} \in \mathcal{M}_{\boldsymbol{\lambda}}$ and convolutional dictionaries $\{\mathbf{D}_i\}_{i=1}^K$,

$$\mathcal{P}_{\mathcal{M}_{\boldsymbol{\lambda}}}: \quad \min_{\{\boldsymbol{\Gamma}_i\}} \quad \|\mathbf{Y} - \mathbf{X}(\boldsymbol{\Gamma}_i)\|_2 \quad \text{s.t.} \quad \mathbf{X}(\boldsymbol{\Gamma}_i) \in \mathcal{M}_{\boldsymbol{\lambda}}.$$

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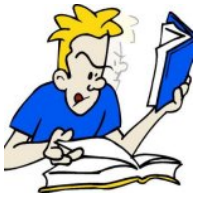
Stability of the $P_{\mathcal{M}}$ problem

If $\|\mathbf{E}\|_2 \leq \mathcal{E}_0$, and $\|\mathbf{\Gamma}_i\|_{0,\infty} < \frac{1}{2}\left(1 + \frac{1}{\mu(\mathbf{D}^{(i)})}\right)$, for $1 \leq i \leq K$, then

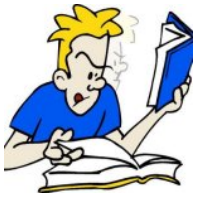
$$\|\mathbf{\Gamma}_i - \hat{\mathbf{\Gamma}}_i\|_2^2 \leq \frac{4\mathcal{E}_0^2}{1 - (2\|\mathbf{\Gamma}_i\|_{0,\infty} - 1)\mu(\mathbf{D}^{(i)})}$$

- ✓ Bound is not cumulative across layers
- ✓ Dependence on $\mu(\mathbf{D}^{(L)})$ - Global effective dictionary



How to Learn?

The existence of $\mathbf{X}(\mathbf{\Gamma}_i) \in \mathcal{M}_{\boldsymbol{\lambda}}$ depends on proper dictionaries $\mathbf{D}_i$.



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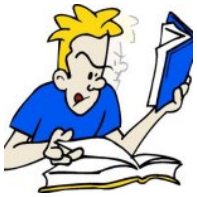

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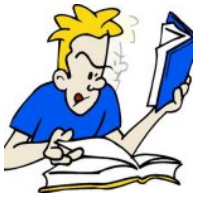

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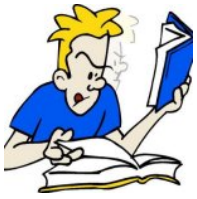
Sparsity Proxies

$$\mathbf{\Gamma}_{L-1} = \mathbf{D}_L \mathbf{\Gamma}_L. \qquad \Rightarrow \|\mathbf{\Gamma}_{K-1}\|_{0,\infty}^s \leq c_L \|\mathbf{D}_L\|_0 \|\mathbf{\Gamma}_L\|_{0,\infty}^s$$



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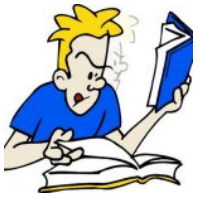

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Problem formulation

$$\min_{\{\mathbf{\Gamma}_L^t\},\{\mathbf{D}_i\}} \sum_{t=1}^{T} \|\mathbf{Y}^t - \mathbf{D}_1 \mathbf{D}_2 \dots \mathbf{D}_L \mathbf{\Gamma}_L^t\|_2^2 + \sum_{i=2}^{K} \zeta_i \|\mathbf{D}_i\|_0 \quad \text{s.t.} \quad \|\mathbf{\Gamma}_L^t\|_{0,\infty}^s \leq \lambda_L$$

Multi-Layer Convolutional Dictionary Learning

$$\min_{\{\mathbf{\Gamma}_L^t\},\{\mathbf{D}_i\}} \sum_{t=1}^{T} \|\mathbf{Y}^t - \mathbf{D}_1\mathbf{D}_2\ldots\mathbf{D}_L\mathbf{\Gamma}_L^t\|_2^2 + \sum_{i=2}^{K} \zeta_i \|\mathbf{D}_i\|_0 \quad \text{s.t.} \quad \|\mathbf{\Gamma}_L^t\|_{0,\infty}^s \leq \lambda_L$$

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Algorithm

Data: Training samples $\{\mathbf{Y}_i\}$, initial convolutional dictionaries $\mathbf{D}_i^0$

for $t = 1, \ldots, T$ **do**

Draw $\mathbf{Y}^t$ at random

Sparse Coding: $\hat{\mathbf{\Gamma}}_L \leftarrow \arg\min_{\mathbf{\Gamma}} \|\mathbf{Y}^t - \mathbf{D}^{(K)}\mathbf{\Gamma}\|_2$ s.t. $\|\mathbf{\Gamma}\|_{0,\infty}^s \leq \lambda_L$

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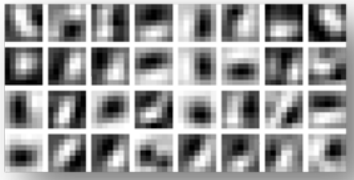
- Update Dictionaries:
- **for** $l = 1, \ldots, L$ **do**
  - $\mathbf{D}_l \leftarrow \arg\min_{\mathbf{D}_l} \|\mathbf{Y}^t - \mathbf{D}_1 \ldots \mathbf{D}_l \ldots \mathbf{D}_L\mathbf{\Gamma}_L\|_2 + \zeta_L\|\mathbf{D}_l\|_0$
- **end**

end

Learning Multi-Layer CSC Models

Learning Multi-Layer CSC Models

$\mathbf{D}_1$



Learning Multi-Layer CSC Models

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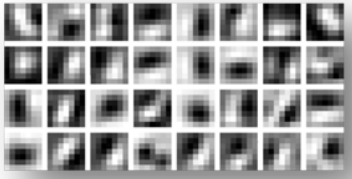
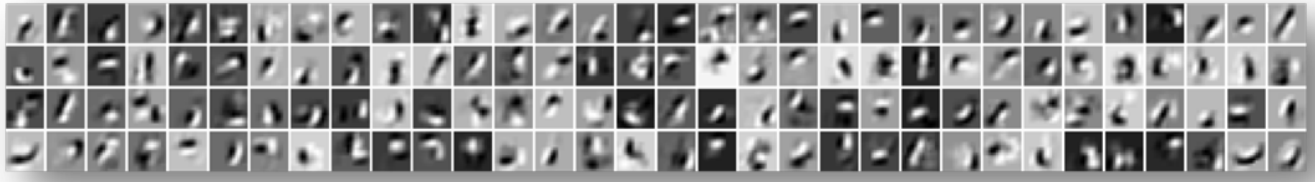


$\mathbf{D}_1\mathbf{D}_2$



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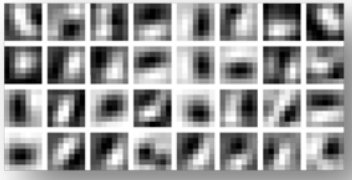
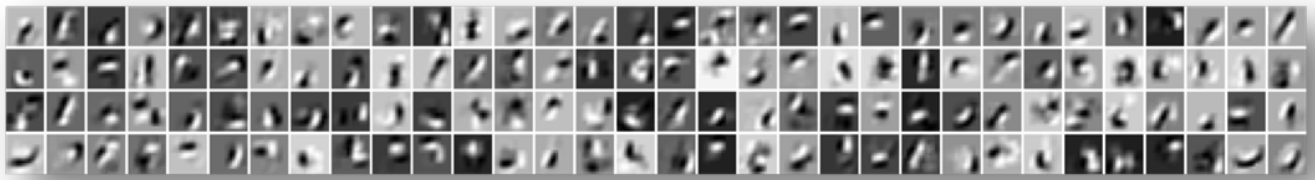
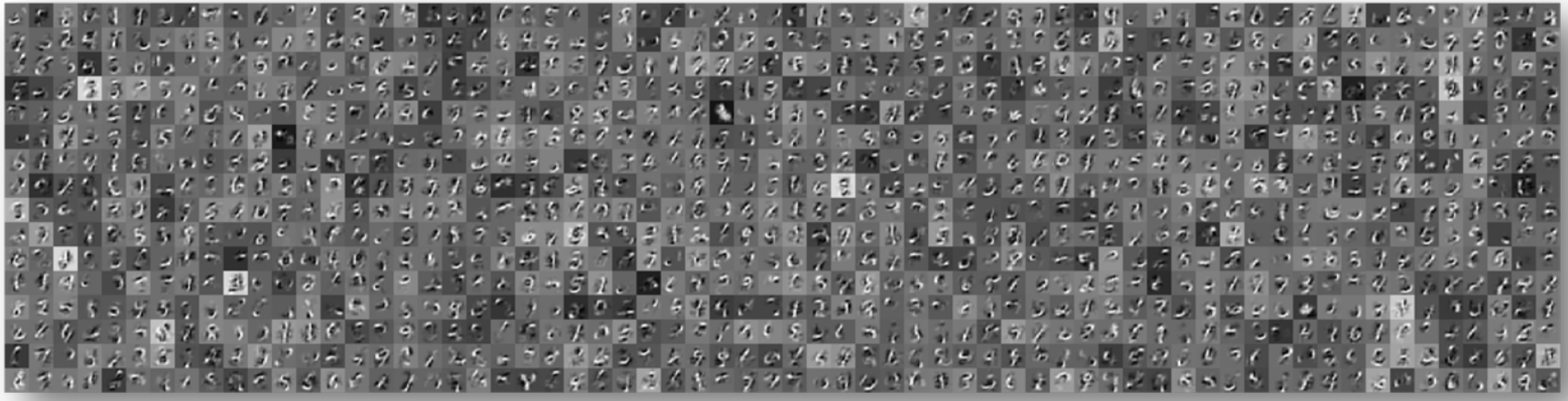


$\mathbf{D}_1\mathbf{D}_2$



$\mathbf{D}_1\mathbf{D}_2\mathbf{D}_3$



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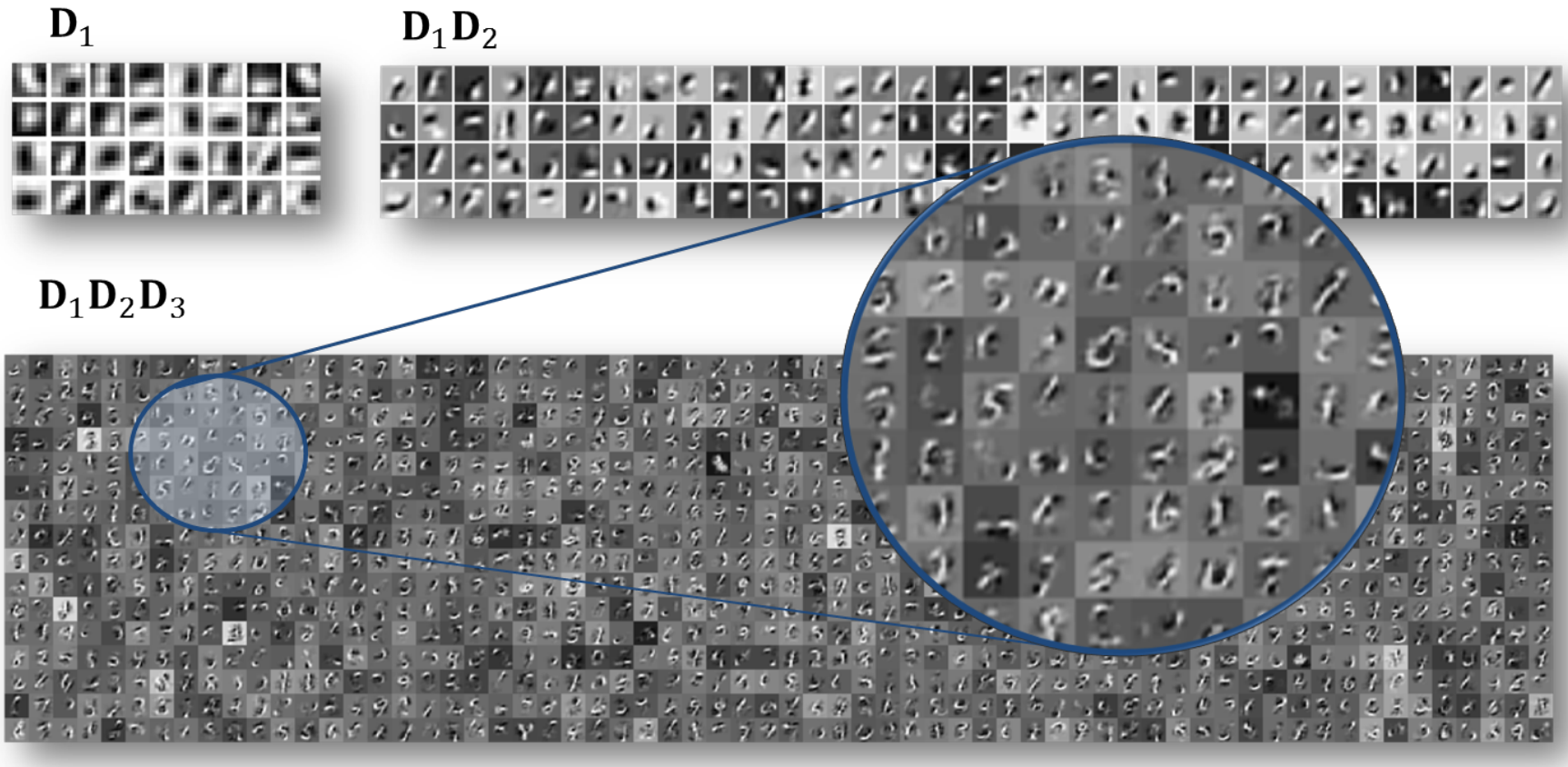

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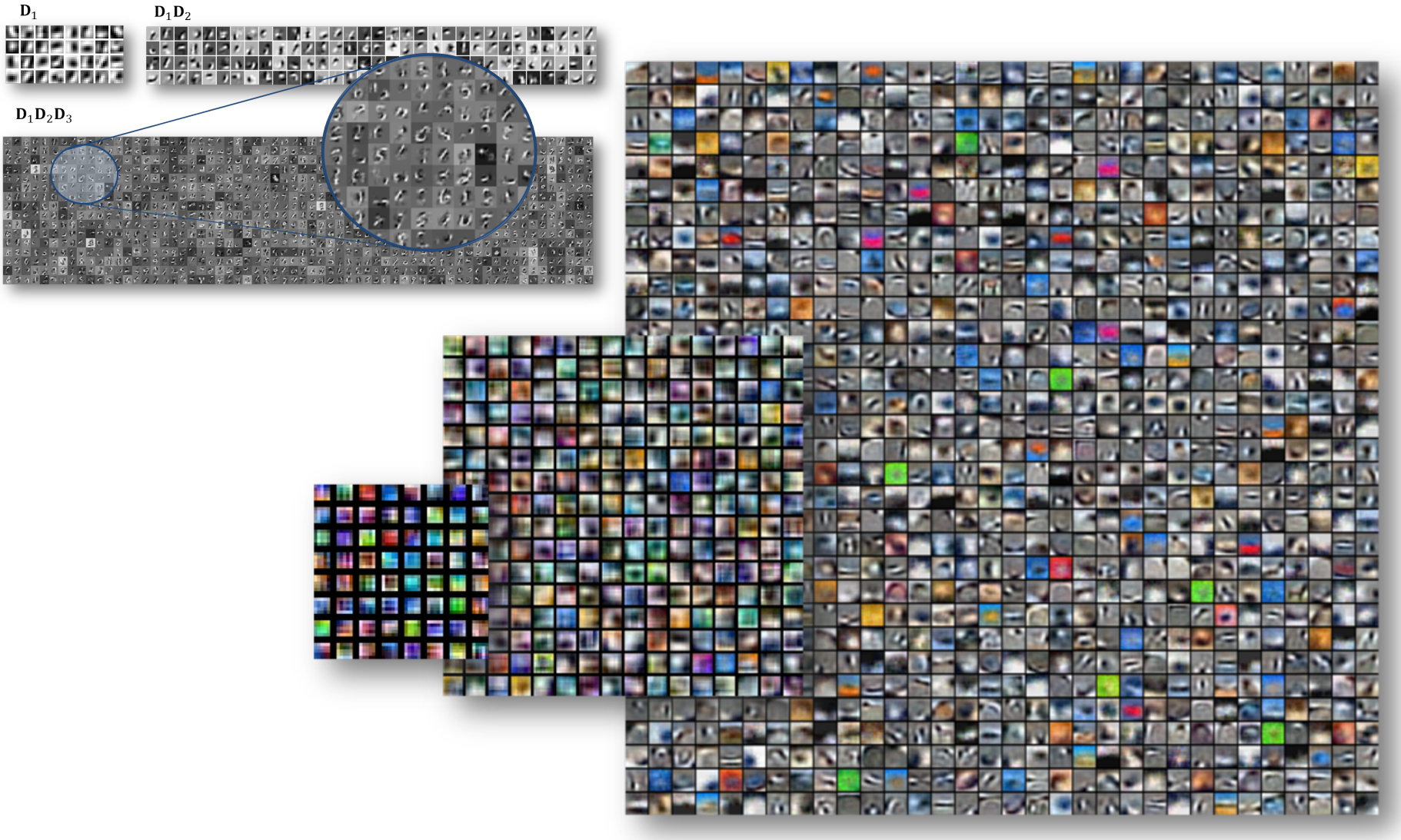
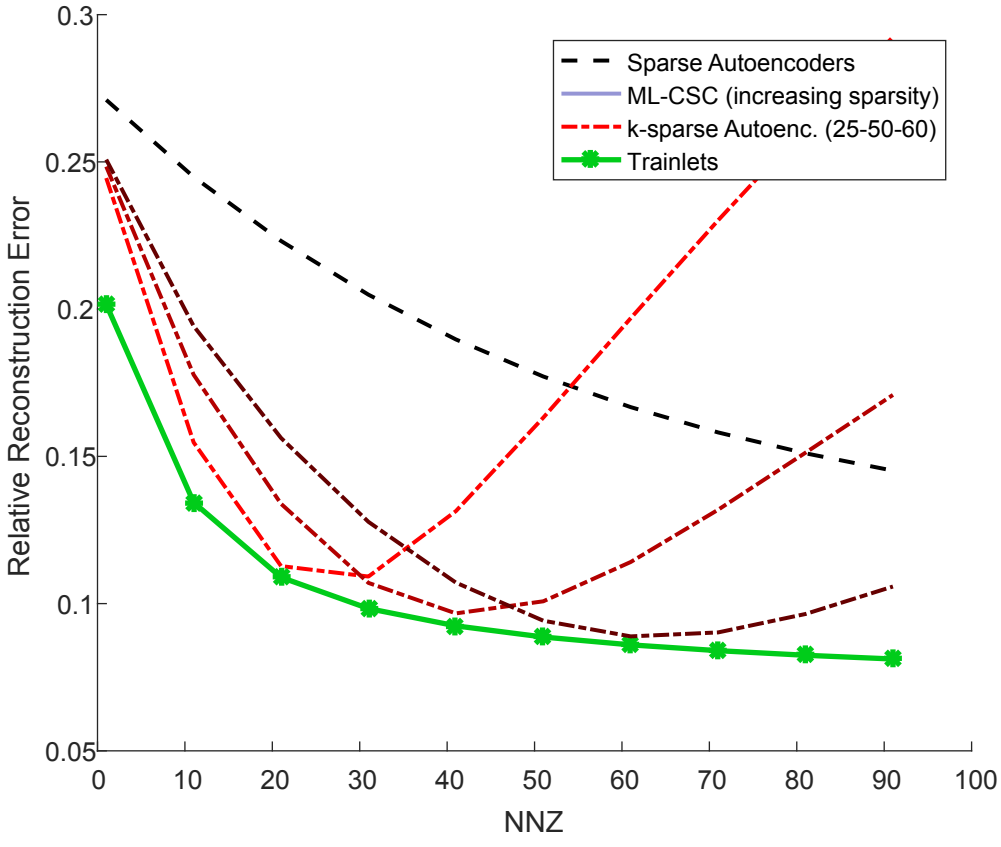


ML-CSC Decompositions

Signal Approximation:



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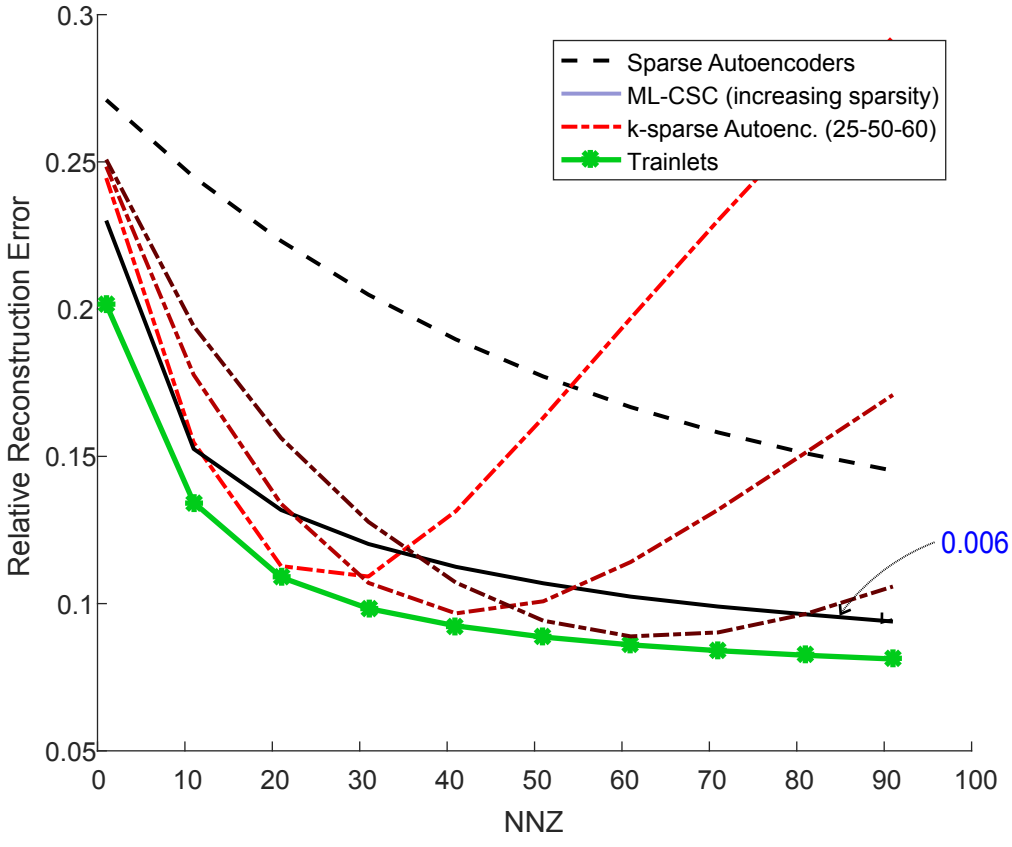

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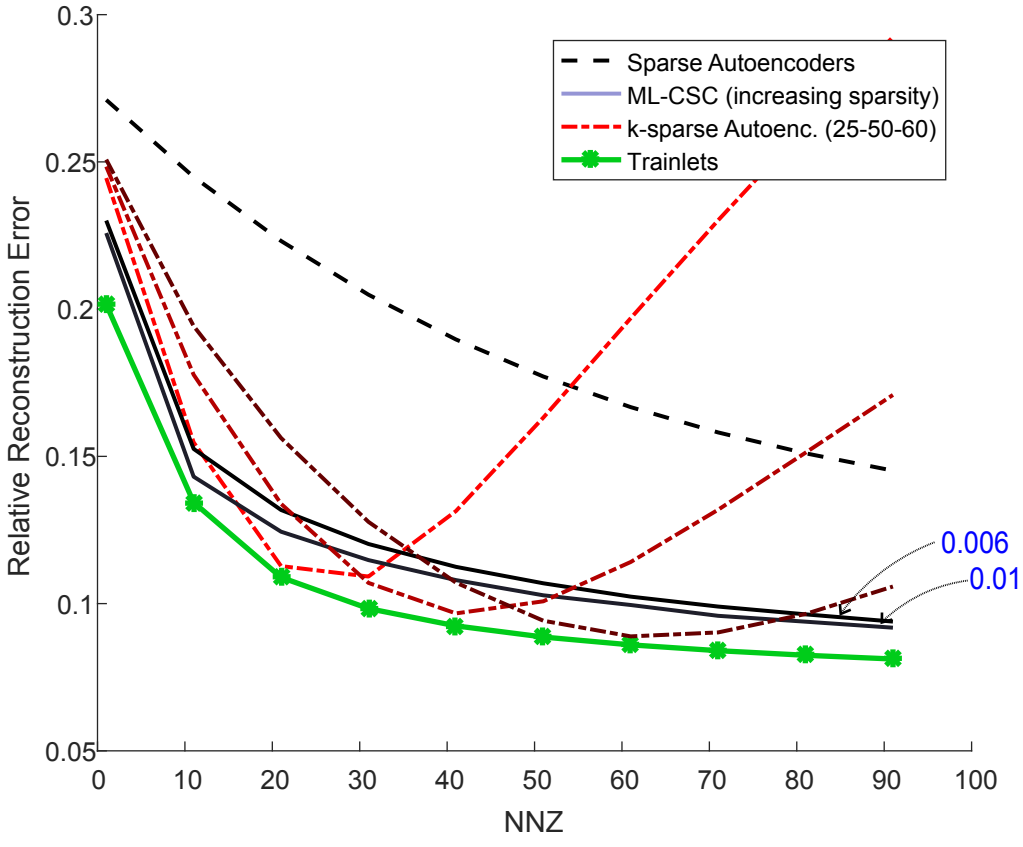

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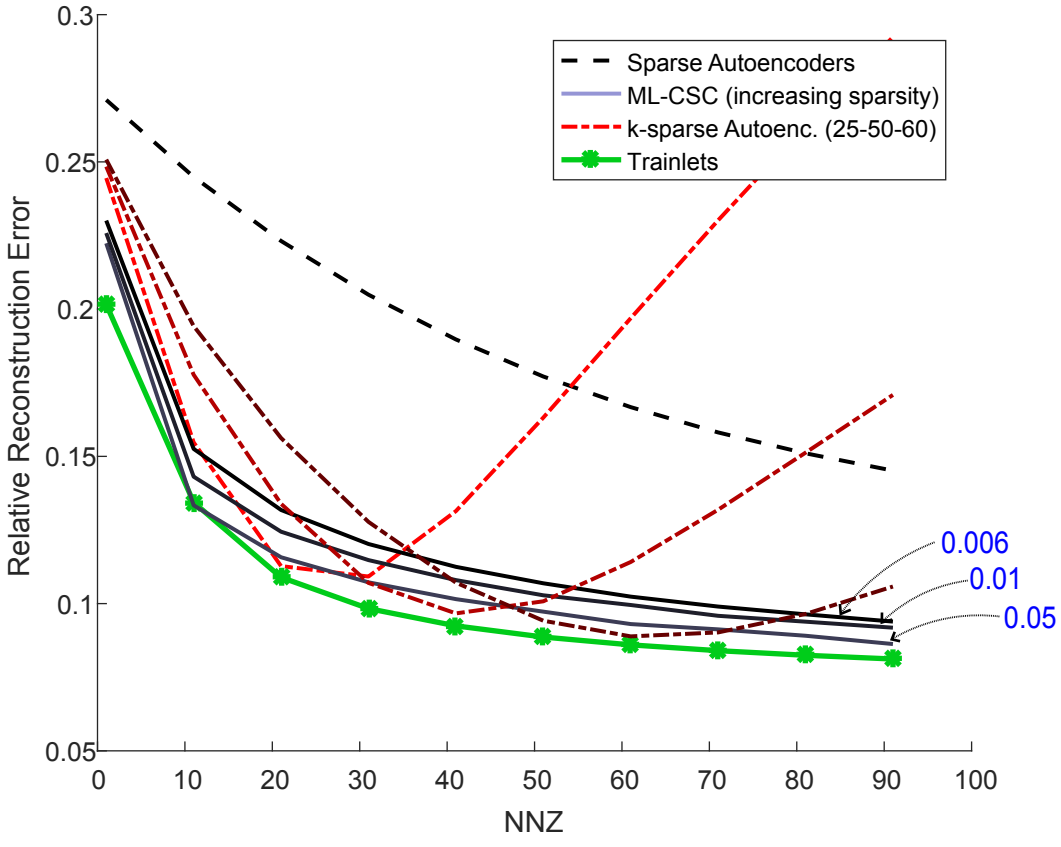

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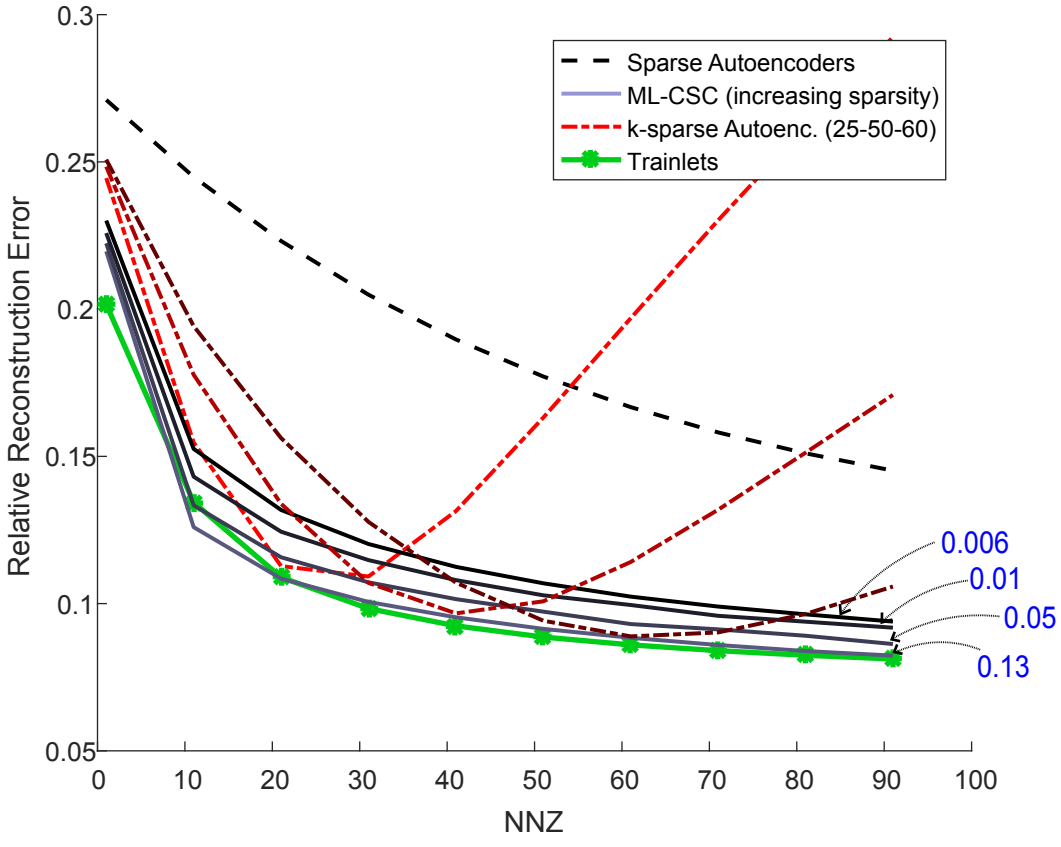

ML-CSC Decompositions

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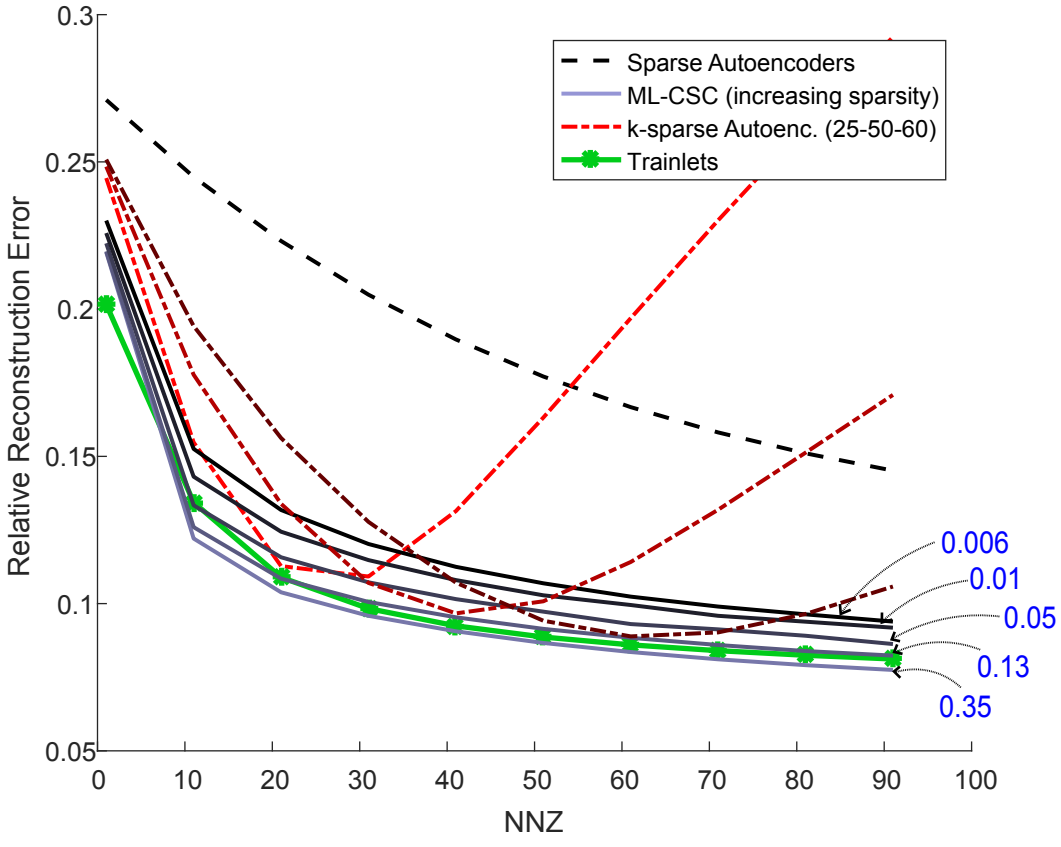

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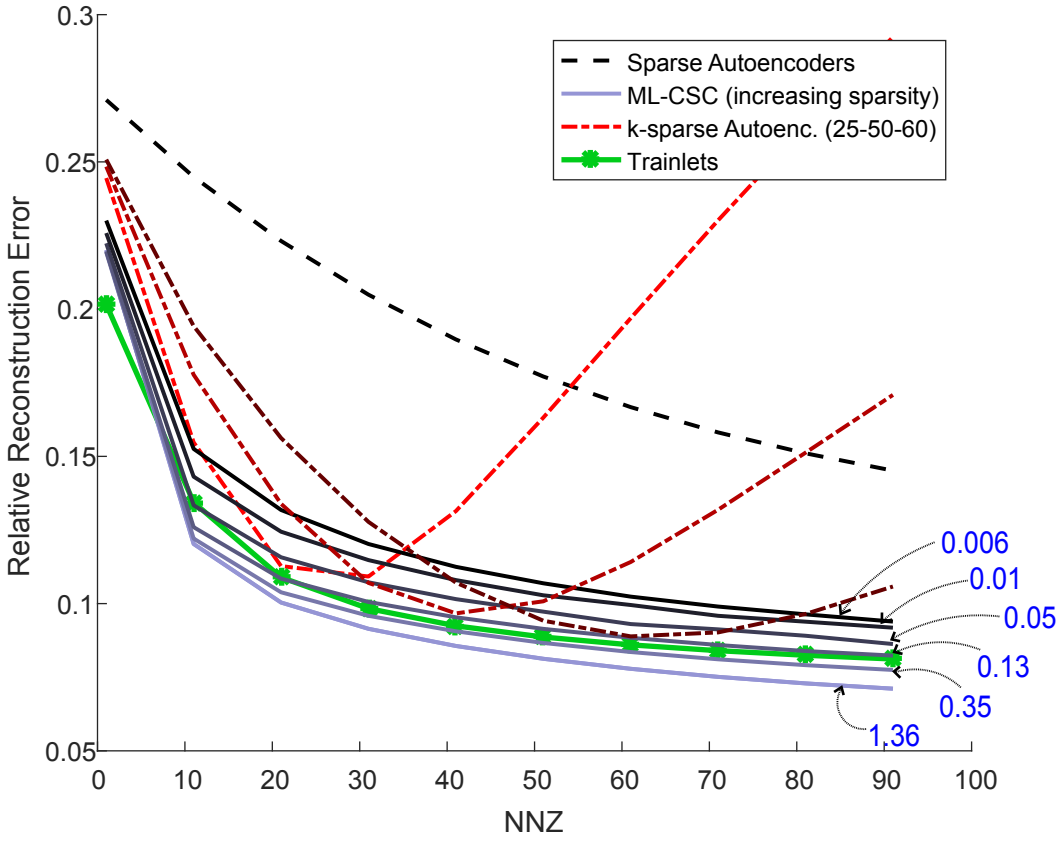

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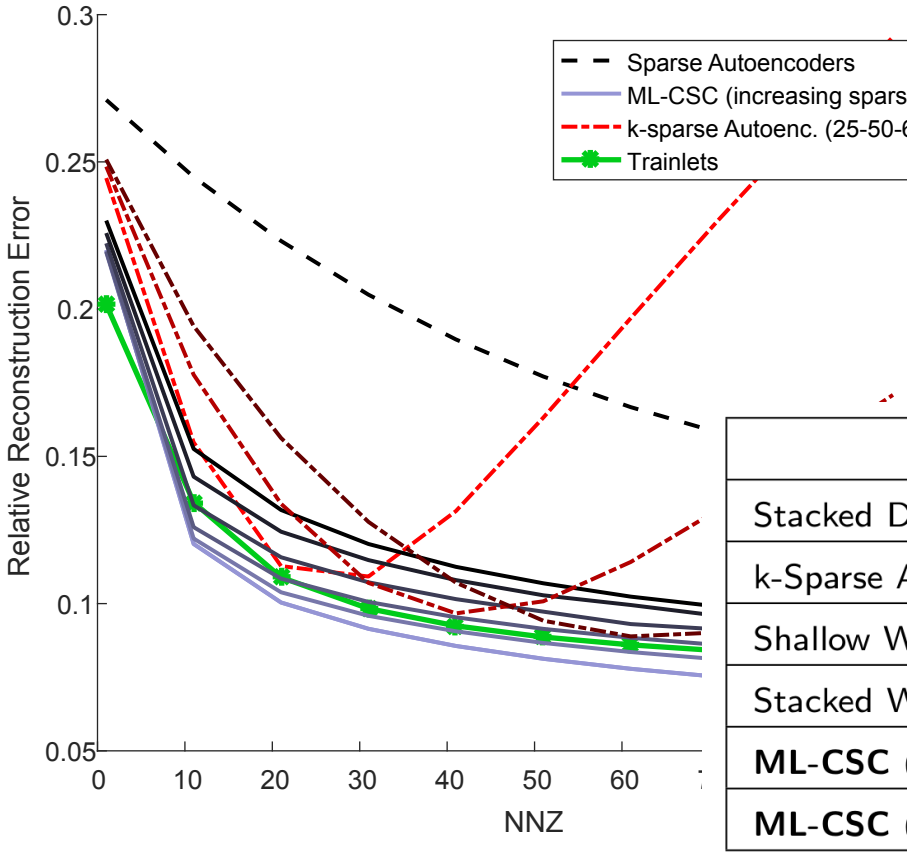

Method	Classification Error
Stacked Denoising Autoencoder (3 layers)	1.28%
k-Sparse Autoencoder (1K units)	1.35%
Shallow WTA Autoencoder (2K units)	1.20%
Stacked WTA Autoencoder (2K units)	1.11%
ML-CSC (1K units) - 2nd Layer Rep.	1.30%
ML-CSC (2K units) - 2nd&3rd Layer Rep.	1.15%

Multi-Layer Sparse Modeling of Signals

Model

$$\mathbf{y} = \mathbf{D}_1\boldsymbol{\gamma}_1 + \mathbf{w}, \qquad \{\boldsymbol{\gamma}_{i-1} = \mathbf{D}_i\boldsymbol{\gamma}_i, \quad \|\boldsymbol{\gamma}_i\|_0 \leq s_i\}_{i=1}^{L}$$

Multi-Layer Sparse Modeling of Signals

Model

$$\mathbf{y} = \mathbf{D}_{(1,L)}\boldsymbol{\gamma}_L + \mathbf{w}, \qquad \{\boldsymbol{\gamma}_{i-1} = \mathbf{D}_i\boldsymbol{\gamma}_i, \quad \|\boldsymbol{\gamma}_i\|_0 \leq s_i\}_{i=1}^{L}$$

$$\mathbf{D}_{(1,L)} = \mathbf{D}_1\mathbf{D}_2\ldots\mathbf{D}_L.$$

Multi-Layer Sparse Modeling of Signals

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$$\mathbf{y} = \mathbf{D}_{(1,L)}\boldsymbol{\gamma}_L + \mathbf{w}, \qquad \|\boldsymbol{\gamma}_L\|_0 \leq s_L,\ \{\|\mathbf{D}_{(i,L)}\boldsymbol{\gamma}_L\|_0 \leq s_{i-1}\}_{i=1}^{L}$$

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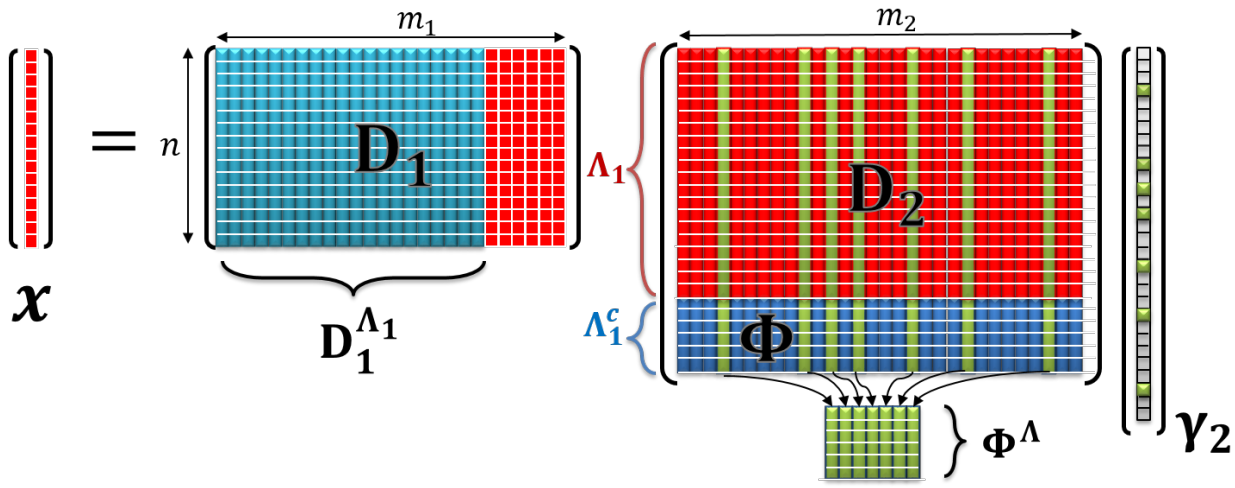
Pursuit

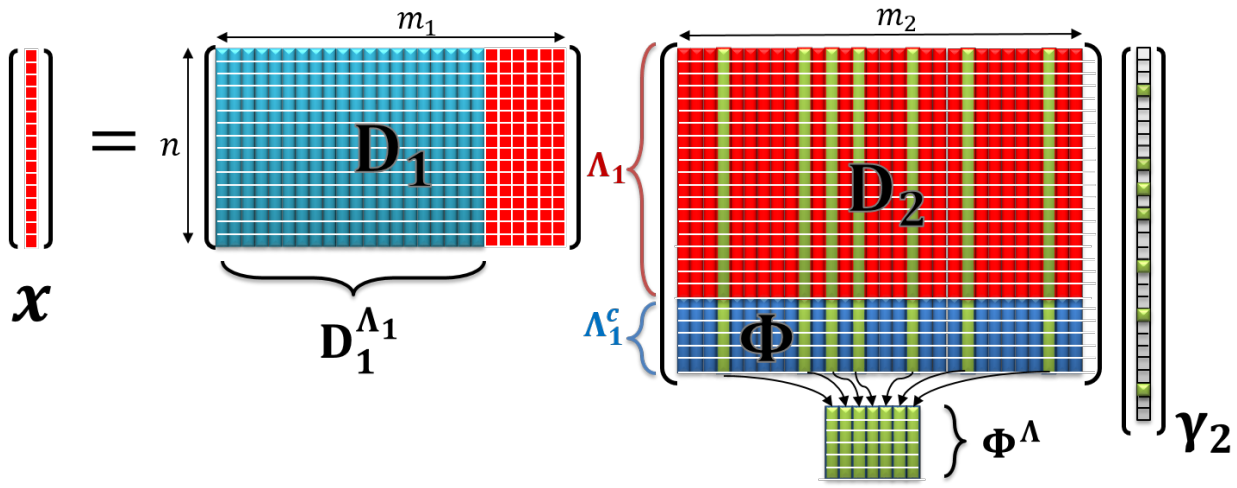
$$\min_{\boldsymbol{\gamma}_L} \quad \|\mathbf{y} - \mathbf{D}_{(1,L)}\boldsymbol{\gamma}_L\|_2^2 \quad \text{s.t.} \quad \|\boldsymbol{\gamma}_L\|_0 \le s_L,\ \left\{\|\mathbf{D}_{(i,L)}\boldsymbol{\gamma}_L\|_0 \le s_{i-1}\right\}_{i=1}^{L}$$

$\Rightarrow$ Coupled Synthesis and Analysis priors!

Synthesis-Analysis Interpretation



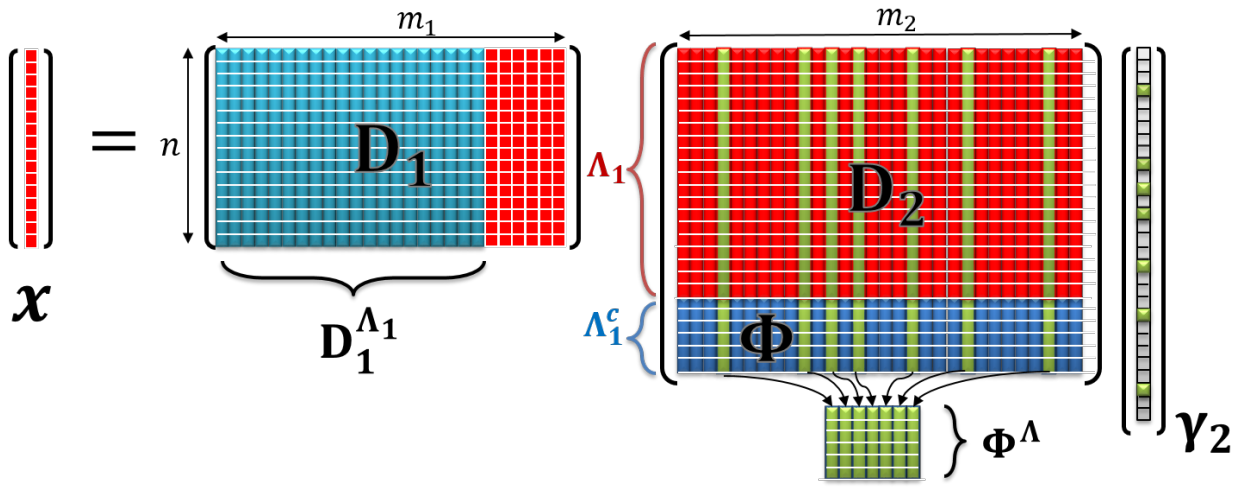
Synthesis-Analysis Interpretation



Uniqueness

- Degrees of freedom $= s_2 - \text{rank}(\Phi^{\Lambda})$

Synthesis-Analysis Interpretation



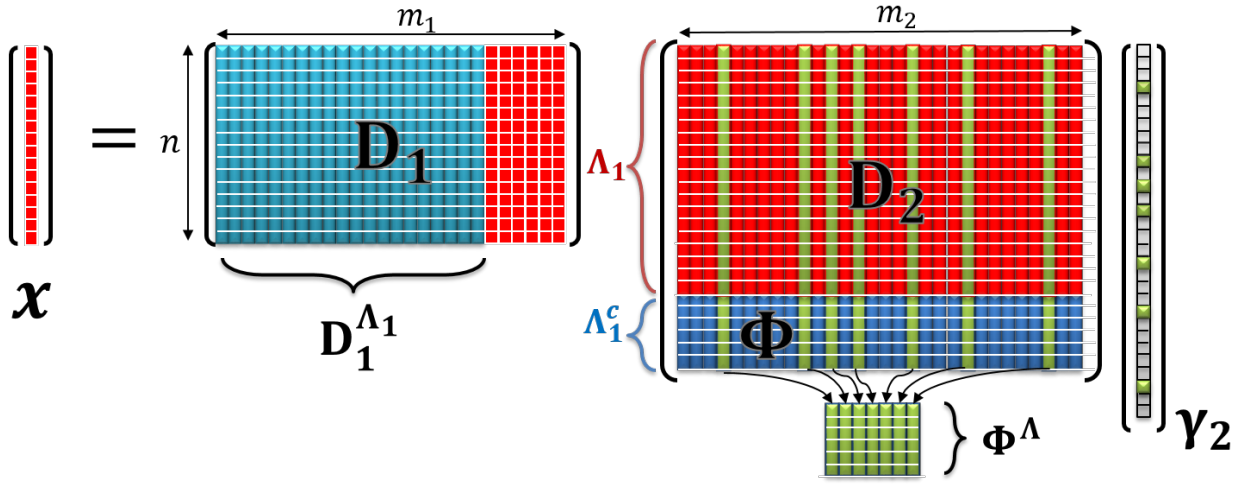
Uniqueness

- Degrees of freedom $= s_2 - \text{rank}(\Phi^{\Lambda})$
- Representations are **unique** if

$$\|\boldsymbol{\gamma}_2\|_0 \leq \frac{\eta(\mathbf{D}_{(1,2)}) - 1}{2}$$

$\eta(\mathbf{D}_i)$: *spark* of $\mathbf{D}_i$ – minimal number of *linearly dependent* columns.

Synthesis-Analysis Interpretation



Uniqueness

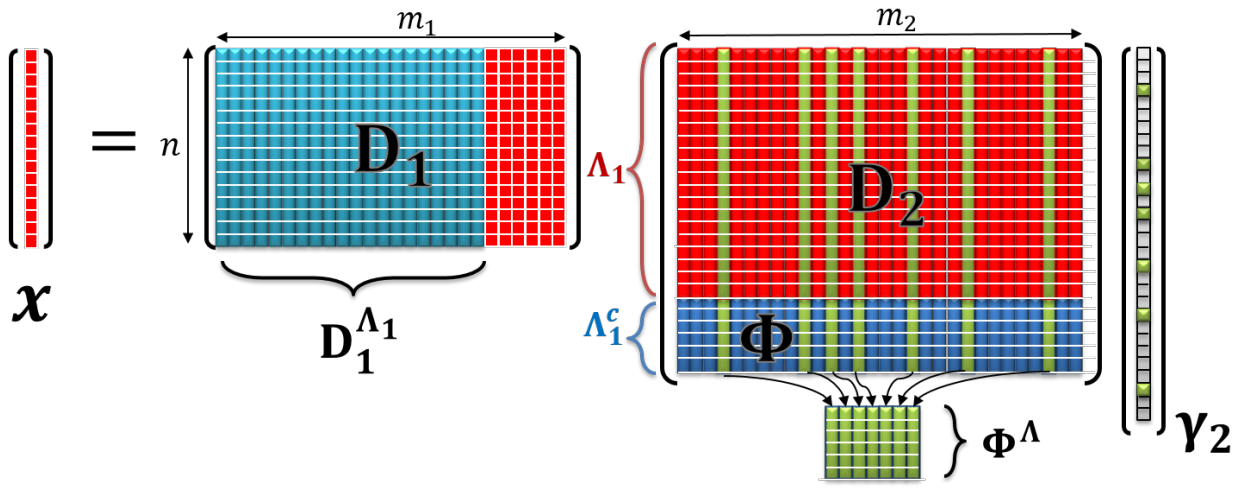
- Degrees of freedom $= s_2 - \text{rank}(\Phi^\Lambda)$
- Representations are **unique** if

$$\|\boldsymbol{\gamma}_2\|_0 \leq \frac{\eta(\mathbf{D}_{(1,2)}) - 1}{2} + \text{rank}(\Phi^\Lambda)$$

$\eta(\mathbf{D}_i)$: *spark* of $\mathbf{D}_i$ – minimal number of *linearly dependent* columns.

- ✓ Improved guarantees
- ✓ Intermediate representations *not as sparse*

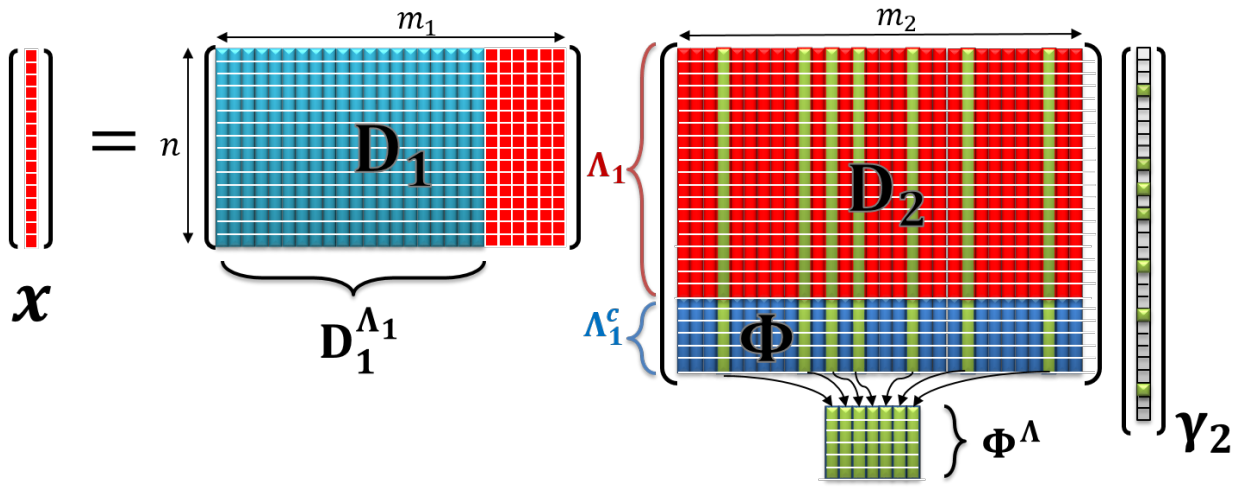
Synthesis-Analysis Interpretation



Stability

$$\mathbf{y} = \mathbf{D}_{(1,L)}\boldsymbol{\gamma}_L + \mathbf{w}, \quad \mathbf{w} \sim \mathcal{N}(0, \sigma^2\mathbf{I})$$

Synthesis-Analysis Interpretation



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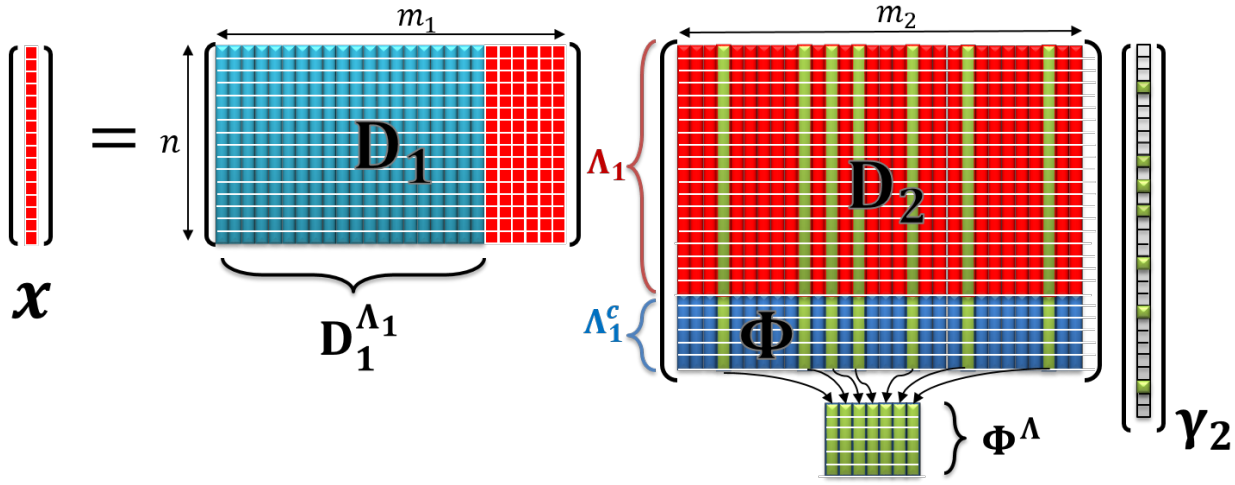
- Layered Synthesis

$$\mathbb{E}\ \|\boldsymbol{\gamma}_i - \hat{\boldsymbol{\gamma}}_i\|_2^2 \lesssim \frac{\sigma^2}{(1-\delta_{s_L})} s_L$$

Synthesis-Analysis Interpretation



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$$\mathbb{E}\ \|\boldsymbol{\gamma}_i - \hat{\boldsymbol{\gamma}}_i\|_2^2 \lesssim \frac{\sigma^2}{(1-\delta_{s_L})} s_L$$

- Synthesis-Analysis

$$\mathbb{E}\ \|\boldsymbol{\gamma}_i - \hat{\boldsymbol{\gamma}}_i\|_2^2 \lesssim \frac{\sigma^2}{(1-\delta_{s_L})} (s_L - rank(\Phi^{\Lambda_L}))$$

Multi-Layer Basis Pursuit

Pursuit

$$\min_{\boldsymbol{\gamma}_L} \quad \|\mathbf{y} - \mathbf{D}_{(1,L)}\boldsymbol{\gamma}_L\|_2^2 \quad \text{s.t.} \quad \|\boldsymbol{\gamma}_L\|_0 \leq s_L, \ \left\{\|\mathbf{D}_{(i,L)}\boldsymbol{\gamma}_L\|_0 \leq s_{i-1}\right\}_{i=1}^{L}$$

with $\mathbf{D}_{(i,L)} = \mathbf{D}_i \dots \mathbf{D}_L$

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with $\mathbf{D}_{(i,L)} = \mathbf{D}_i \dots \mathbf{D}_L$

Just relax!

$$(P): \quad \min_{\boldsymbol{\gamma}_L} \|\mathbf{y} - \mathbf{D}_{(1,L)}\boldsymbol{\gamma}_L\|_2^2 + \sum_{i=2}^{L} \lambda_{i-1}\|\mathbf{D}_{(i,L)}\boldsymbol{\gamma}_L\|_1 + \lambda_L\|\boldsymbol{\gamma}_L\|_1$$

Multi-Layer Basis Pursuit

Pursuit

$$\min_{\boldsymbol{\gamma}_L} \quad \|\mathbf{y} - \mathbf{D}_{(1,L)}\boldsymbol{\gamma}_L\|_2^2 \quad \text{s.t.} \quad \|\boldsymbol{\gamma}_L\|_0 \leq s_L, \ \left\{\|\mathbf{D}_{(i,L)}\boldsymbol{\gamma}_L\|_0 \leq s_{i-1}\right\}_{i=1}^{L}$$

with $\mathbf{D}_{(i,L)} = \mathbf{D}_i \ldots \mathbf{D}_L$

Just relax!

$$(P): \quad \min_{\boldsymbol{\gamma}_2} \frac{1}{2}\|\mathbf{y} - \mathbf{D}_1\mathbf{D}_2\boldsymbol{\gamma}_2\|_2^2 + \lambda_1\|\mathbf{D}_2\boldsymbol{\gamma}_2\|_1 + \lambda_2\|\boldsymbol{\gamma}_2\|_1$$

Multi-Layer ISTA

$$\min_{\boldsymbol{\gamma}_2} F(\boldsymbol{\gamma}_2) = f(\mathbf{D}_2\boldsymbol{\gamma}_2) + g_1(\mathbf{D}_2\boldsymbol{\gamma}_2) + g_2(\boldsymbol{\gamma}_2)$$

Multi-Layer ISTA

$$\min_{\boldsymbol{\gamma}_2} F(\boldsymbol{\gamma}_2) = \underbrace{f(\mathbf{D}_2\boldsymbol{\gamma}_2) + g_1(\mathbf{D}_2\boldsymbol{\gamma}_2)}_{h(\boldsymbol{\gamma}_2)} + g_2(\boldsymbol{\gamma}_2)$$

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Proximal Gradient

$$\boldsymbol{\gamma}_2^{k+1} = \text{prox}_{tg_2}\left(\boldsymbol{\gamma}_2^k - t\nabla h(\boldsymbol{\gamma}_2^k)\right) \quad ??$$

Multi-Layer ISTA

$$\min_{\boldsymbol{\gamma}_2} F(\boldsymbol{\gamma}_2) = f(\mathbf{D}_2\boldsymbol{\gamma}_2) + g_1(\mathbf{D}_2\boldsymbol{\gamma}_2) + g_2(\boldsymbol{\gamma}_2)$$

Proximal Gradient-Mapping

$$\boldsymbol{\gamma}_2^{k+1} = \text{prox}_{tg_2}\left(\boldsymbol{\gamma}_2^k - tG_{1/\mu}^{f,g_1(\mathbf{D}_2\cdot)}(\boldsymbol{\gamma}_2^k)\right)$$

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where

$$G_{1/\mu}^{f,g_1}(\boldsymbol{\gamma}_2^k) = \frac{1}{\mu}\left[\boldsymbol{\gamma}_2^k - \text{prox}_{\mu g_1(\mathbf{D}_2\cdot)}\left(\boldsymbol{\gamma}_2^k - \mu\nabla f(\mathbf{D}_2\boldsymbol{\gamma}_2)\right)\right]$$

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$$\min_{\boldsymbol{\gamma}_2} F(\boldsymbol{\gamma}_2) = f(\mathbf{D}_2\boldsymbol{\gamma}_2) + g_1(\mathbf{D}_2\boldsymbol{\gamma}_2) + g_2(\boldsymbol{\gamma}_2)$$

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$$\boldsymbol{\gamma}_2^{k+1} = \text{prox}_{tg_2}\left(\boldsymbol{\gamma}_2^k - t\mathbf{D}_2^T G_{1/\mu}^{f,g_1}(\boldsymbol{\gamma}_1^k)\right)$$

where $\boldsymbol{\gamma}_1^k = \mathbf{D}_2\boldsymbol{\gamma}_2^k$ and

$$G_{1/\mu}^{f,g_1}(\boldsymbol{\gamma}_1^k) = \frac{1}{\mu}\left[\boldsymbol{\gamma}_1^k - \text{prox}_{\mu g_1}\left(\boldsymbol{\gamma}_1^k - \mu\nabla f(\boldsymbol{\gamma}_1^k)\right)\right]$$

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$$G_{1/\mu}^{f,g_1}(\boldsymbol{\gamma}_1^k) = \frac{1}{\mu}\left[\boldsymbol{\gamma}_1^k - \text{prox}_{\mu g_1}\left(\boldsymbol{\gamma}_1^k - \mu\nabla f(\boldsymbol{\gamma}_1^k)\right)\right]$$

In particular:

$$\boldsymbol{\gamma}_2^{k+1} = \mathcal{T}_{t\lambda_2}\left(\boldsymbol{\gamma}_2^k - \frac{t}{\mu}\mathbf{D}_2^T\left(\boldsymbol{\gamma}_1^k - \mathcal{T}_{\mu\lambda_1}(\boldsymbol{\gamma}_1^k - \mu\mathbf{D}_1^T(\mathbf{D}_1\boldsymbol{\gamma}_1^k - \mathbf{y}))\right)\right)$$

ML-ISTA

Set $\boldsymbol{\gamma}_0^k = \mathbf{y}\ \forall\ k$ and $\boldsymbol{\gamma}_L^1 = 0$

for $k = 1 : K$ **do**

  % for each iteration

    $\hat{\boldsymbol{\gamma}}_i \leftarrow \mathbf{D}_{(i,L)}\boldsymbol{\gamma}_L^k \quad \forall i \in [0, k-1]$

  for $i = 1 : L$ **do**

    % for each layer

      $\boldsymbol{\gamma}_i^{k+1} \leftarrow \mathcal{T}_{\mu_i\lambda_i}\left(\hat{\boldsymbol{\gamma}}_i - \mu_i\mathbf{D}_i^T(\mathbf{D}_i\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_{i-1}^{k+1})\right)$

  end

end

ML-ISTA

Set $\boldsymbol{\gamma}_0^k = \mathbf{y}\ \forall\ k$ and $\boldsymbol{\gamma}_L^1 = 0$
for $k = 1 : K$ **do**
    % for each iteration
        $\hat{\boldsymbol{\gamma}}_i \leftarrow \mathbf{D}_{(i,L)}\boldsymbol{\gamma}_L^k \quad \forall i \in [0, k-1]$
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        % for each layer
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    end
end

✓ *Nested Proximal-Gradient updates*

ML-FISTA

Set $\boldsymbol{\gamma}_0^k = \mathbf{y}\ \forall\ k$ and $\boldsymbol{\gamma}_L^1 = 0$
for $k = 1 : K$ **do**
  % for each iteration
    $\hat{\boldsymbol{\gamma}}_i \leftarrow \mathbf{D}_{(i,L)}\mathbf{z} \quad \forall i \in [0, k-1]$
  for $i = 1 : L$ **do**
    % for each layer
      $\boldsymbol{\gamma}_i^{k+1} \leftarrow \mathcal{T}_{\mu_i \lambda_i}\left(\hat{\boldsymbol{\gamma}}_i - \mu_i \mathbf{D}_i^T(\mathbf{D}_i\hat{\boldsymbol{\gamma}}_i - \boldsymbol{\gamma}_{i-1}^{k+1})\right)$
  end
    $\mathbf{z} \leftarrow \boldsymbol{\gamma}_L^{k+1} + \rho^k(\boldsymbol{\gamma}_L^{k+1} - \boldsymbol{\gamma}_L^k)$
end

✓ *Nested Proximal-Gradient updates with momentum*

Convergence for ML-ISTA

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Theorem

Suppose $\{\boldsymbol{\gamma}_2^k\}$ *generated by ML-ISTA with* $\mu \in \left(0, \frac{1}{\|\mathbf{D}_1\|_2^2}\right)$ *and* $t \in \left(0, \frac{4\mu}{3\|\mathbf{D}_2\|_2}\right)$.

If $\|\boldsymbol{\gamma}_2^{k+1} - \boldsymbol{\gamma}_2^k\|_2 \leq t\varepsilon$, *then*

$$F(\boldsymbol{\gamma}_2^{k+1}) - F_{opt} \leq \eta\varepsilon + (\beta + \kappa t)\mu,$$

where η, β *and* κ *are constants depending on* $\mathbf{D}_1$, $\mathbf{D}_2$, g_1, g_2.

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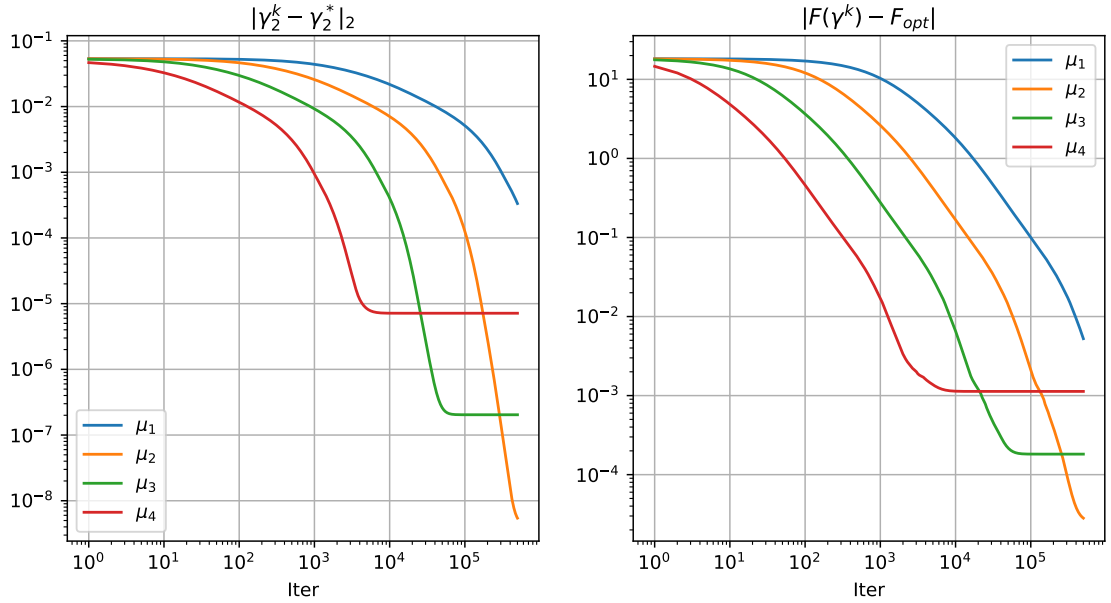

Convergence for ML-FISTA

Theorem

Suppose $\{\boldsymbol{\gamma}_2^k\}$ *generated by ML-ISTA with* $\mu \in \left(0, \frac{1}{\|\mathbf{D}_1\|_2^2}\right)$ *and* $t \in \left(0, \frac{4\mu}{3\|\mathbf{D}_2\|_2}\right)$.

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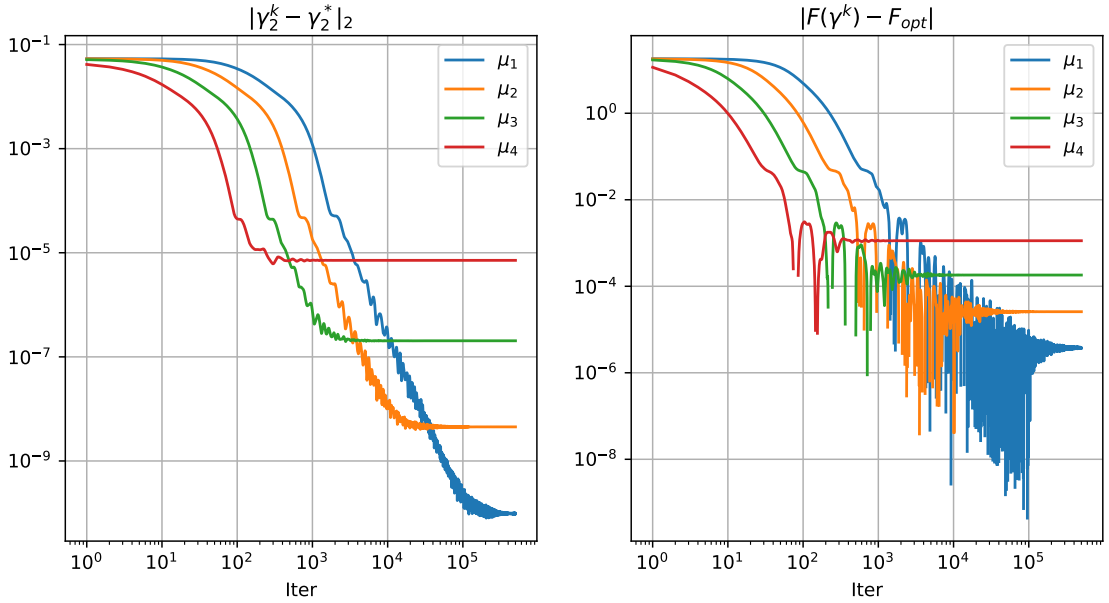

Generalizing Recurrent CNNs

Revisiting the ML-ISTA

$$\boldsymbol{\gamma}_2^{k+1} = \mathcal{T}_{t\lambda_2}\left(\boldsymbol{\gamma}_2^k - \mu_2 \mathbf{D}_2^T\left(\boldsymbol{\gamma}_1^k - \mathcal{T}_{\mu_1\lambda_1}\left(\boldsymbol{\gamma}_1^k - \mu \mathbf{D}_1^T\left(\mathbf{D}_1\boldsymbol{\gamma}_1^k - \mathbf{y}\right)\right)\right)\right)$$

Generalizing Recurrent CNNs

Revisiting the ML-ISTA

$$\boldsymbol{\gamma}_2^{(1)} = \mathcal{T}_{t\lambda_2}\Big(\quad - \mu_2 \mathbf{D}_2^T \big(\quad - \mathcal{T}_{\mu_1\lambda_1} \big(\quad - \mu \mathbf{D}_1^T \big(\qquad - \mathbf{y} \big) \big) \big) \Big)$$

Generalizing Recurrent CNNs

Revisiting the ML-ISTA

$$\boldsymbol{\gamma}_2^{(1)} = \mathcal{T}_{t\lambda_2}\left(\mu_2 \mathbf{D}_2^T \mathcal{T}_{\mu\lambda_1}(\mu_1 \mathbf{D}_1^T \mathbf{y})\right)$$

Generalizing Recurrent CNNs

Revisiting the ML-ISTA

$$\boldsymbol{\gamma}_2^{(1)} = \mathsf{ReLU}\Big(\mu_2 \mathbf{D}_2^T \mathsf{ReLU}(\mu_1 \mathbf{D}_1^T \mathbf{y} + \mathbf{b}_1) + \mathbf{b}_2\Big) \qquad \text{(F.P.)}$$

Generalizing Recurrent CNNs

Revisiting the ML-ISTA

$$\boldsymbol{\gamma}_2^{(1)} = \mathsf{ReLU}\Big(\mu_2 \mathbf{D}_2^T \mathsf{ReLU}(\mu_1 \mathbf{D}_1^T \mathbf{y} + \mathbf{b}_1) + \mathbf{b}_2\Big) \qquad \text{(F.P.)}$$

$$\boldsymbol{\gamma}_2^{(2)} = \mathsf{ReLU}\Big(\boldsymbol{\gamma}_2^{(1)} - \mu_2 \mathbf{D}_2^T \big(\boldsymbol{\gamma}_1^{(1)} - \mathsf{ReLU}(\mu_1 \mathbf{D}_1^T (\mathbf{D}_1 \boldsymbol{\gamma}_1^{(1)} - \mathbf{y}) + \mathbf{b}_1)\big) + \mathbf{b}_2\Big)$$

$$\vdots$$

Generalizing Recurrent CNNs

Revisiting the ML-ISTA

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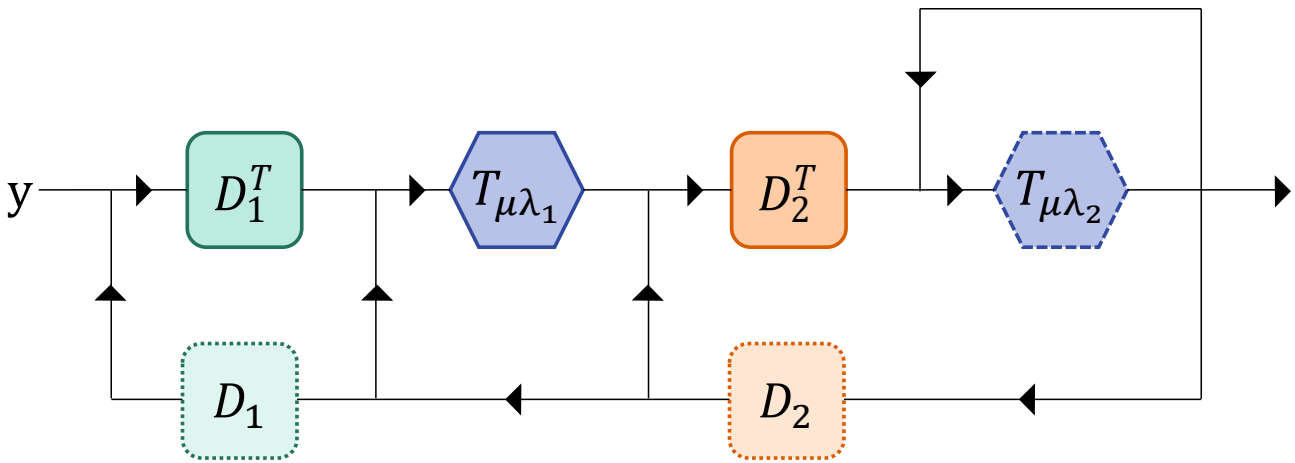

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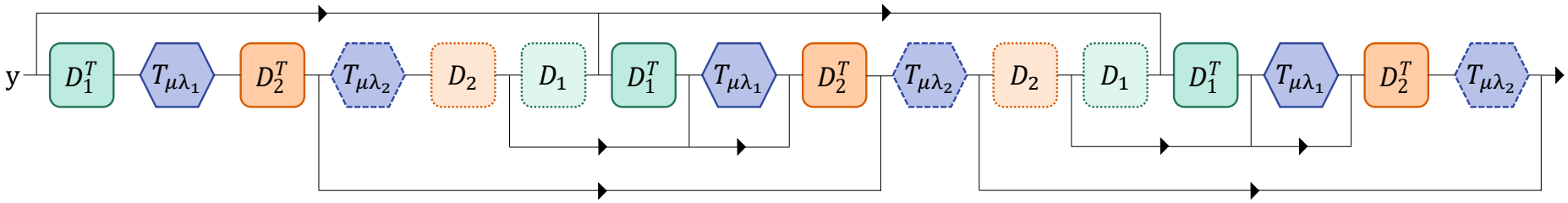
$$\vdots$$



Supervised Learning Formulation

Training examples: $\{(\mathbf{y}_i, h_i)\}_{i=1}^{N}$, $\mathbf{y}$: inputs, h_i : labels

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$$\min_{\theta, \{\mathbf{D}_i, \lambda_i\}} \frac{1}{N} \sum_{1=1}^{N} \mathcal{L}\left(h_i, \zeta_\theta(\boldsymbol{\gamma}^*)\right) \quad \text{s.t.}$$

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$$\boldsymbol{\gamma}^* = \arg\min_{\boldsymbol{\gamma}} \|\mathbf{y} - \mathbf{D}_{(1,L)}\boldsymbol{\gamma}\|_2^2 + \sum_{i=1}^{L-1} \lambda_i \|\mathbf{D}_{(i+1,L)}\boldsymbol{\gamma}\|_1 + \lambda_L \|\boldsymbol{\gamma}\|_1.$$

Supervised Learning Formulation

Training examples: $\{(\mathbf{y}_i, h_i)\}_{i=1}^N$, $\mathbf{y}$: inputs, h_i : labels

$$\min_{\theta, \{\mathbf{D}_i, \lambda_i\}} \frac{1}{N} \sum_{1=1}^{N} \mathcal{L}\left(h_i, \zeta_\theta(\boldsymbol{\gamma}^*)\right) \quad \text{s.t.}$$

$$\boldsymbol{\gamma}^* = \arg\min_{\boldsymbol{\gamma}} \|\mathbf{y} - \mathbf{D}_{(1,L)}\boldsymbol{\gamma}\|_2^2 + \sum_{i=1}^{L-1} \lambda_i \|\mathbf{D}_{(i+1,L)}\boldsymbol{\gamma}\|_1 + \lambda_L \|\boldsymbol{\gamma}\|_1.$$

Relaxed to

$$\min_{\theta, \{\mathbf{D}_i, \lambda_i\}} \frac{1}{N} \sum_{1=1}^{N} \mathcal{L}\left(h_i, \zeta_\theta(\hat{\boldsymbol{\gamma}}^k)\right)$$

$\boldsymbol{\gamma}^k$: k^{th} iteration of ML-ISTA.

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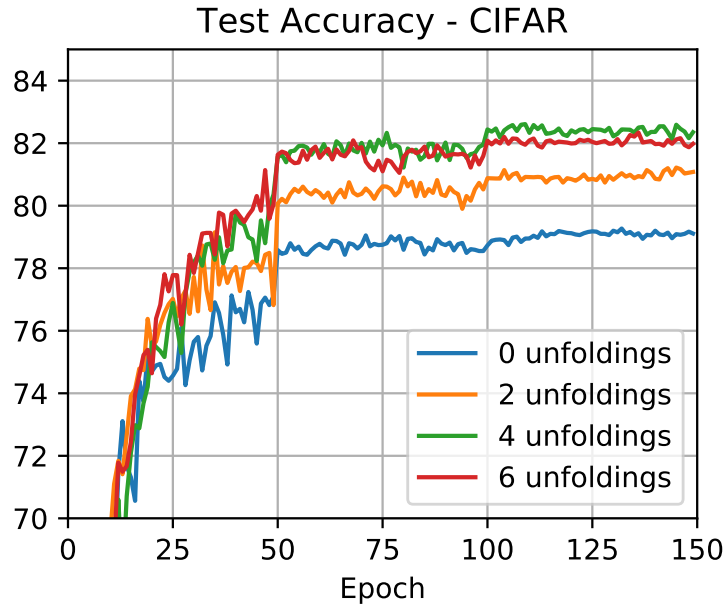
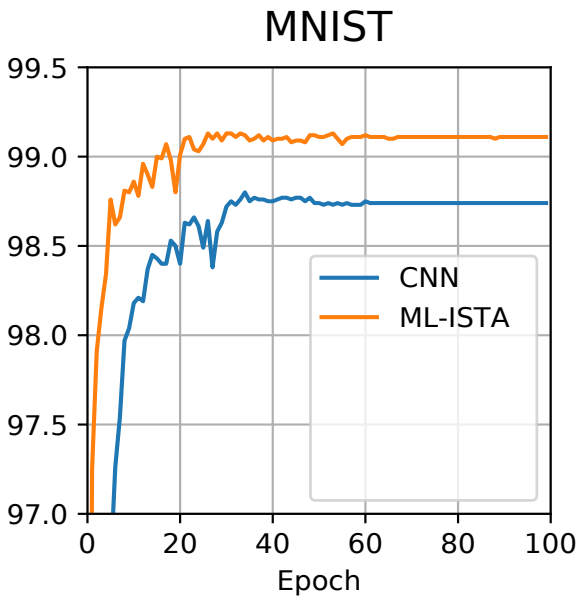
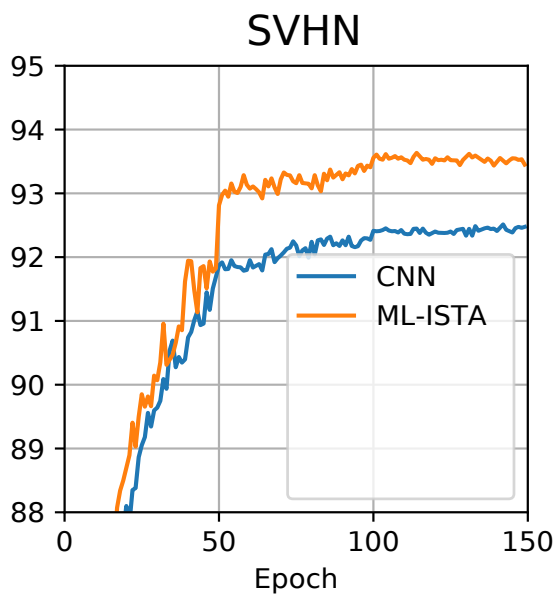
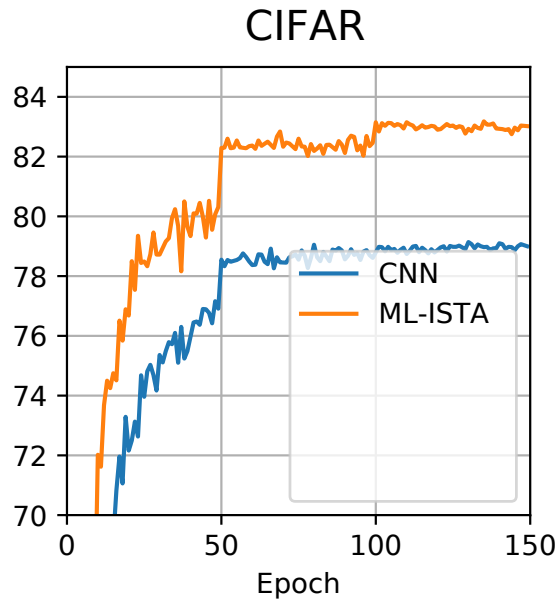


Image Classification







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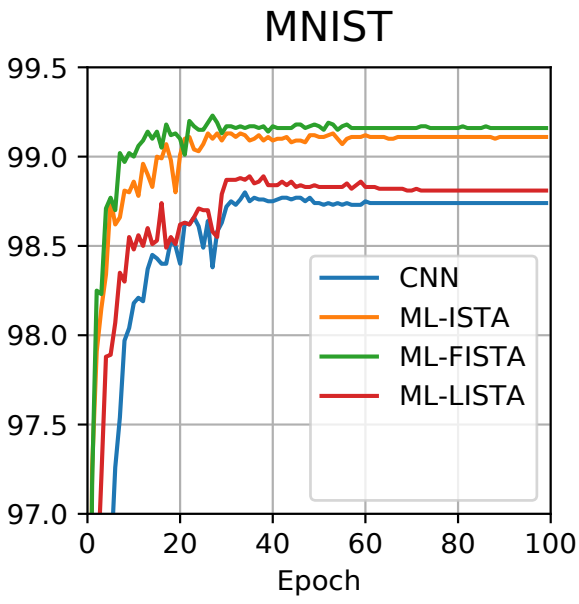

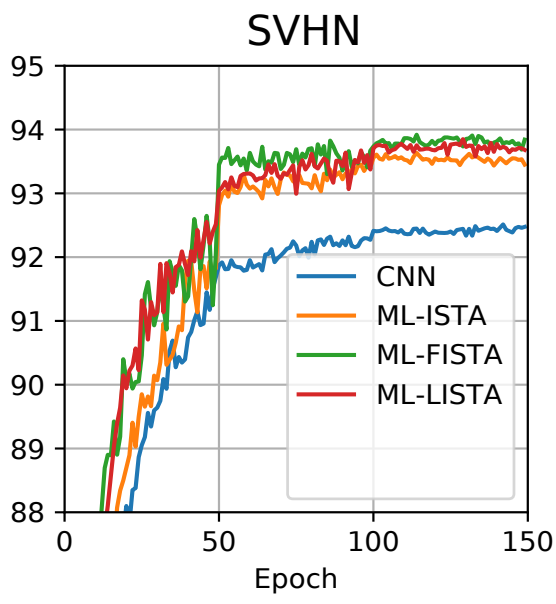

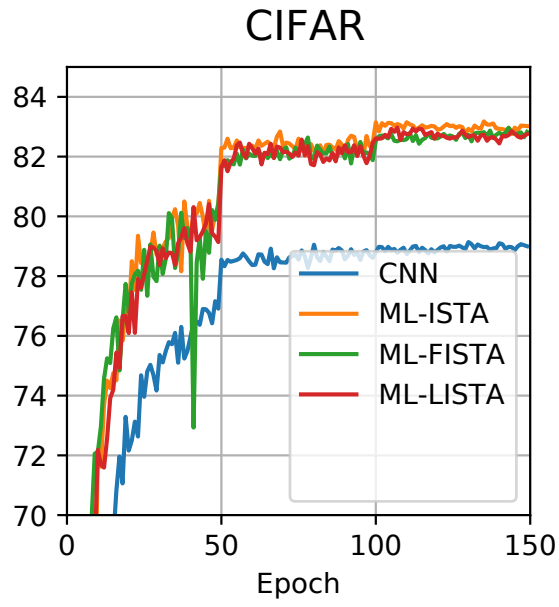

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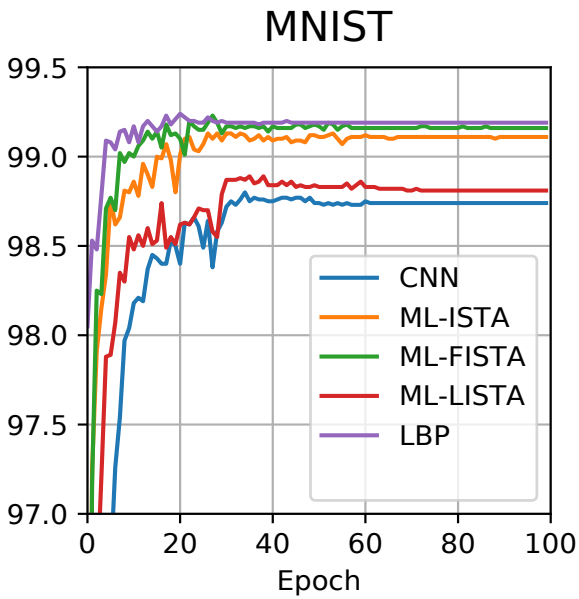

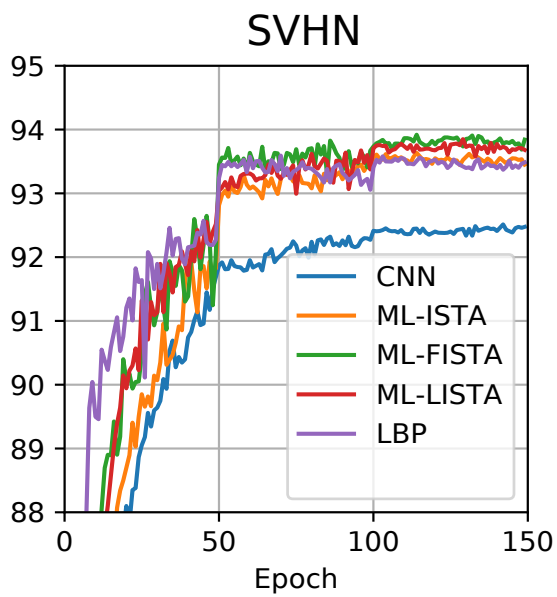

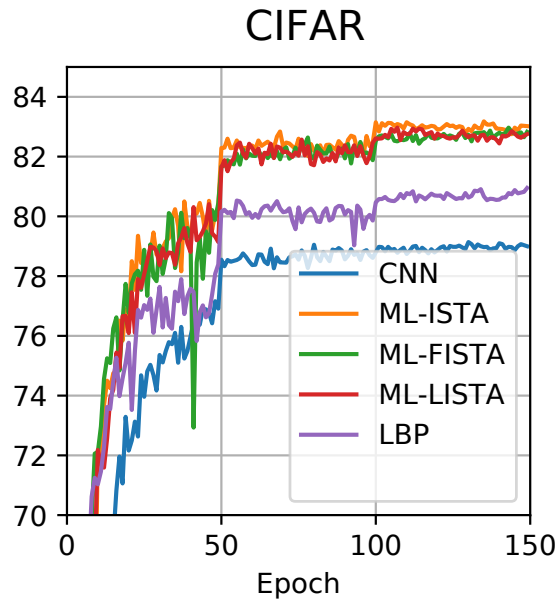

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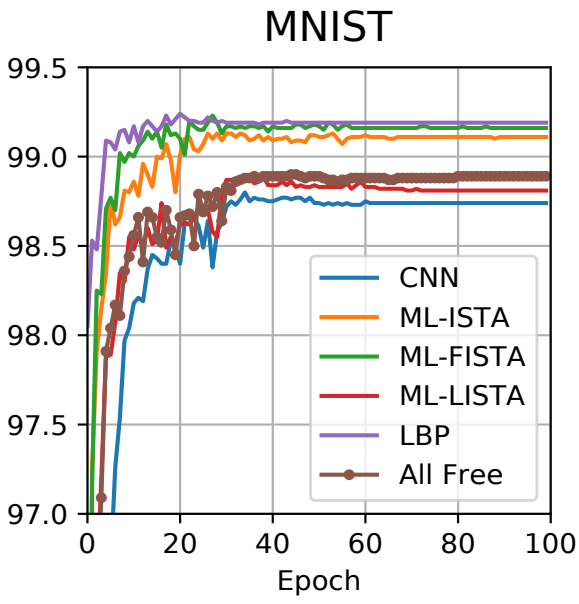

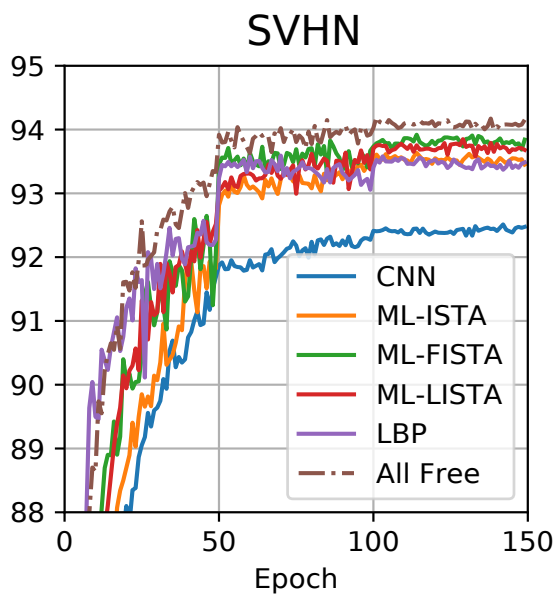

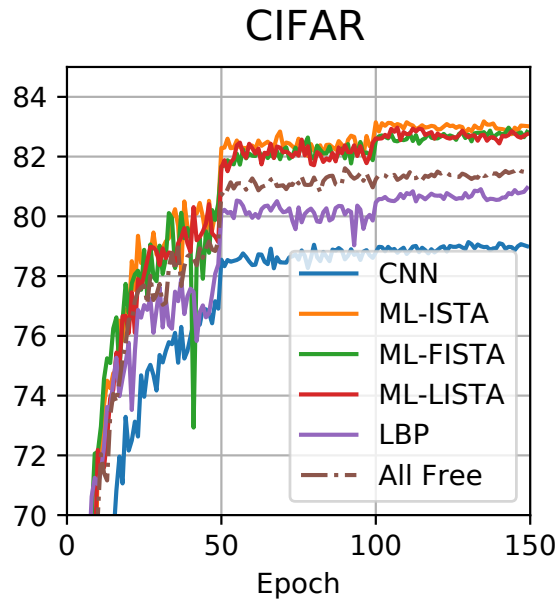

Adversarial Robustness Analysis

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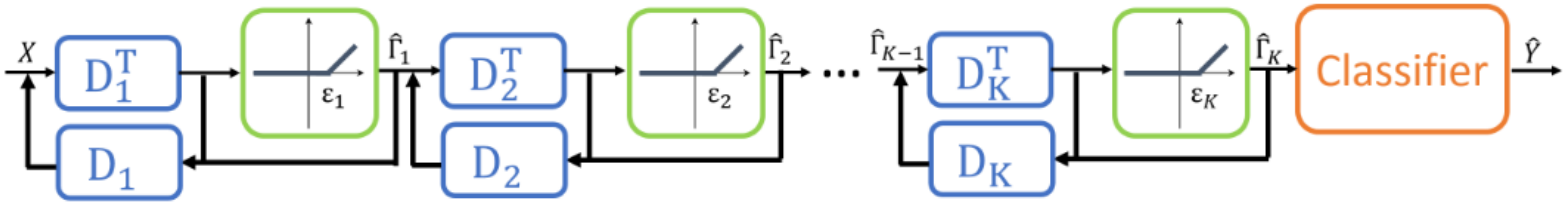

- Compute Representation: $\hat{\boldsymbol{\gamma}}_i = \arg\min_{\boldsymbol{\gamma}} \frac{1}{2}\|\hat{\boldsymbol{\gamma}}_{i-1} - \mathbf{D}_i\boldsymbol{\gamma}\|_2^2 + \lambda_i\|\boldsymbol{\gamma}\|_1$
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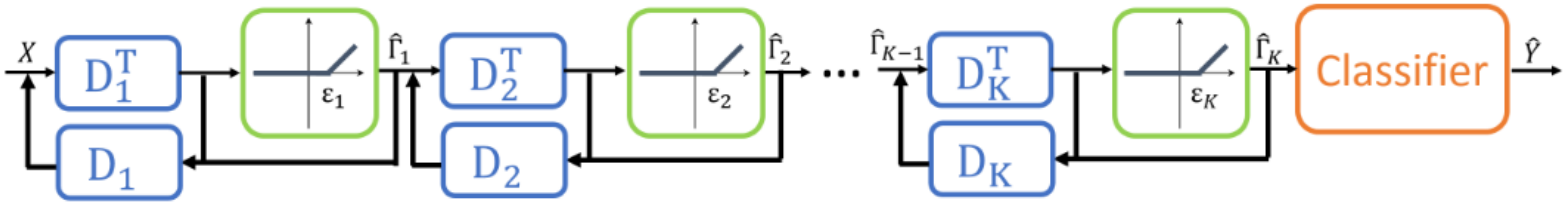

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Theorem (1 layer, FC): Let a signal with label y that is (adversarily) contaminated as $\mathbf{x} = \mathbf{D}\boldsymbol{\gamma} + \mathbf{v}$, such that $\|\mathbf{v}\|_2 \leq \epsilon$.

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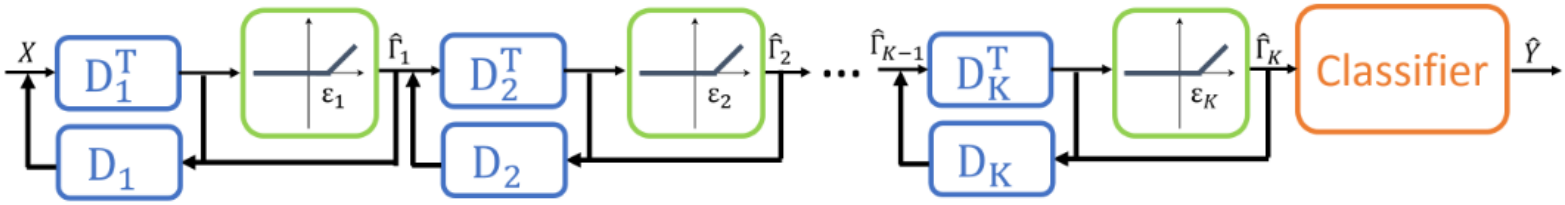

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$$\epsilon \leq \frac{O_{\mathcal{B}}}{7.5\|\boldsymbol{\gamma}\|_0\|\mathbf{w}\|_2}$$

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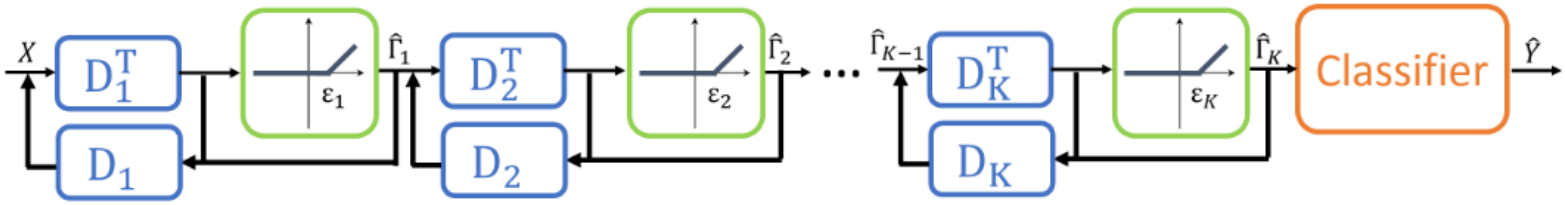

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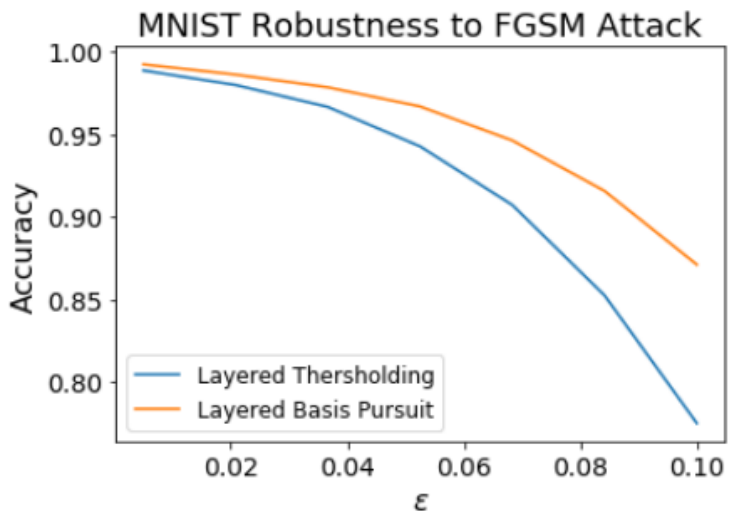

Take-Aways

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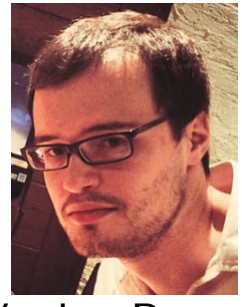
- Multi-Layer Sparse Model puts forward a generative sparse models for real signals
- Deep learning architectures can be understood analyzed as inference algorithms under this model

Joint work with

Aviad Aberdam
Technion

Amir Beck
Tel Aviv University

Yaniv Romano
Stanford



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Stanford

Miki Elad
Technion

References

- V. Papyan, Y. Romano, J. Sulam and M. Elad, **Theoretical Foundations of Deep Learning via Sparse Representations**, in IEEE Signal Processing Magazine, vol. 35, no. 4, pp. 72-89, July 2018.

- V. Papyan, J. Sulam, M. Elad. **Working Locally Thinking Globally: Theoretical Guarantees for Convolutional Sparse Coding**. IEEE Transactions on Signal Processing, 65(21), 5687-5701, 2017.
- V. Papyan, Y. Romano, and M. Elad. **Convolutional Neural Networks Analyzed via Convolutional Sparse Coding**. Journal of Machine Learning Research (JMLR), vol. 18, no. 83, pp. 1-52, 2017.
- J. Sulam, V. Papyan, Y. Romano, M. Elad. **Multi-Layer Convolutional Sparse Modeling: Pursuit and Dictionary Learning**. in IEEE TSP, vol. 66, no. 15, pp. 4090-4104, Aug.1, 1 2018.
- A. Aberdam, J. Sulam, M. Elad. **Multi Layer Sparse Coding: the Holistic Way**. SIAM Journal on Mathematics of Data Science, 2018.
- J. Sulam, A. Aberdam, A. Beck, M. Elad. **On Multi-Layer Basis Pursuit, Efficient Algorithms and Convolutional Neural Networks**. IEEE TPAMI (2019).
- Y. Romano, A. Aberdam, J. Sulam, M. Elad. **Adversarial Noise Attacks of Deep Learning Architectures-Stability Analysis via Sparse Modeled Signals**. arXiv preprint arXiv:1805.11596. (2019)