Bistatic SAR with Instantaneous Doppler: From Move–Stop–Move to a Constant Fast–Time Velocity Model

Incorporating instantaneous Doppler into the BSAR forward model and processing chain

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- Objective: replace the slow–platform move–stop–move approximation with a *constant fast–time velocity (CFV)* model that computes instantaneous Doppler to *every* scene position at a moment in slow-time.

Overview & Problem Statement

Motivation.

- Move–stop–move (MSM) is adequate for slow TX/RX but fails to capture high-speed BSAR performance (instantaneous Doppler, range–Doppler coupling, and spatial variance).
- We will *incorporate instantaneous Doppler* into the forward model via a *constant fast–time velocity (CFV)* formulation evaluated at each slow-time sample: positions/velocities vary in slow-time but are *constant within a fast-time processing window*.

Matched filters vs waveform tolerance.

- Original flow: one matched filter per pixel (costly for high bandwidth).
- Doppler-tolerant waveforms (per the *wideband auto-ambiguity function*) admit fewer filters.
- For LFM, a *single* matched filter at scene center plus a sampling pulse train can be sufficient.

Design questions.

- What is the *minimum* number of matched filters for new waveforms (e.g., PRO-FM) under the CFV instantaneous Doppler model?
- How do we size slow-time sampling to bound phase error across the scene?
- How does frequency agility interact with the CFV instantaneous Doppler to shape $TF(\vec{K})$ and preserve RGIQE?

Roadmap.

1. Review local LSI vs spatial variance.
2. MSM fast-time model (baseline).
3. Replace MSM with CFV instantaneous-Doppler model.
4. Filter-count reduction criteria for LFM and PRO-FM.

Spatial Invariance Assumption

Context. Main results use frequency-agile waveforms to shape the transfer-function passband in the spatial Fourier domain. This relies on a *linear spatially invariant (LSI)* model. **LSI model (near scene center).**

$$\tilde{\rho}(\vec{r}) \;=\; \rho(\vec{r})\,(*)_{\vec{r}}\;\mathsf{ipr}(\vec{r}) \;=\; \int_{\vec{r}'} \rho(\vec{r}')\;\mathsf{ipr}(\vec{r}-\vec{r}')\,\left|\mathrm{d}\vec{r}'\right|$$

Equivalently, with spectrum $P(\vec{K})$ and transfer function $TF(\vec{K})$:

$$\tilde{\rho}(\vec{r}) \;=\; \mathcal{F}^{-1}\big[P(\vec{K})\;TF(\vec{K})\big]$$

Frequency agility $\Rightarrow$ *passband de-skewing in* $TF(\vec{K})$ *enabling RGIQE gains.*

Reality: spatially variant (non-convolutional).

$$\tilde{\rho}(\vec{r}) \;=\; \int_{\vec{r}'} \rho(\vec{r}')\;\mathsf{psf}(\vec{r}',\vec{r})\,\left|\mathrm{d}\vec{r}'\right|$$

with point-spread function $\mathsf{psf}(\vec{r}',\vec{r})$ depending on both source $\vec{r}'$ and image $\vec{r}$ locations. **Local LSI approximation.** For a small neighborhood around the scene center $\vec{O}$,

$$\mathsf{psf}\left(\vec{O}-\tfrac{\vec{r}}{2},\;\vec{O}+\tfrac{\vec{r}}{2}\right) \;\approx\; \mathsf{ipr}(\vec{r})$$

This justifies modeling via convolution and a factorizable $TF(\vec{K})$ for passband design.
Implications (Appendix).

- Validity conditions and error terms for the LSI approximation.
- Impact on RGIQE and limits away from $\vec{O}$.

Move-stop-Move — Fast-Time Processing

Fast- vs. slow-time. Fast-time τ measures RF propagation delay; slow-time t indexes platform motion with $\vec{Pos}_{TX}(t)$ and $\vec{Pos}_{RX}(t)$. **Wideband transmit waveform.** Model $s_{TX}(t,\tau)$ as a frequency average of a complex phasor with slow-time–varying limits. Let $BW(t) = f_{\max}(t) - f_{\min}(t)$:

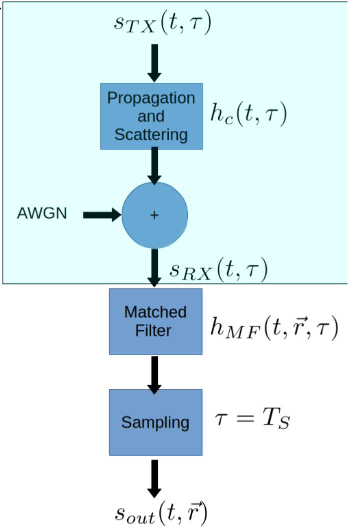
$$s_{TX}(t,\tau) = \mathbb{E}_f\left[e^{j2\pi f\tau}\right](t) = \frac{1}{BW(t)}\int_{f_{\min}(t)}^{f_{\max}(t)} e^{j2\pi f\tau}\,\mathrm{d}f.$$

Move–stop–move LTI channel (at fixed t). With reflectivity $\rho(\vec{r}')$ and bistatic range $R(t,\vec{r}')$,

$$h_c(t,\tau) = \int_{\vec{r}'} \rho(\vec{r}')\,\delta\!\left(\tau - \frac{R(t,\vec{r}')}{c}\right)\,|\mathrm{d}\vec{r}'|.$$

Received signal (fast-time convolution). Additive white Gaussian noise $n(t,\tau)$:

$$\begin{aligned} s_{RX}(t,\tau) &= \left[s_{TX}(t,\tau)(*)_\tau h_c(t,\tau)\right] + n(t,\tau) \\ &= \int_{\vec{r}'} \rho(\vec{r}')\, s_{TX}\!\left(t,\ \tau - \frac{R(t,\vec{r}')}{c}\right)\,|\mathrm{d}\vec{r}'| + n(t,\tau). \end{aligned}$$



Instantaneous Doppler — Fast-Time Processing (CFV Model)

CFV assumption. Within each fast-time window, platform positions and velocities are *constant*; they vary only across slow-time t.

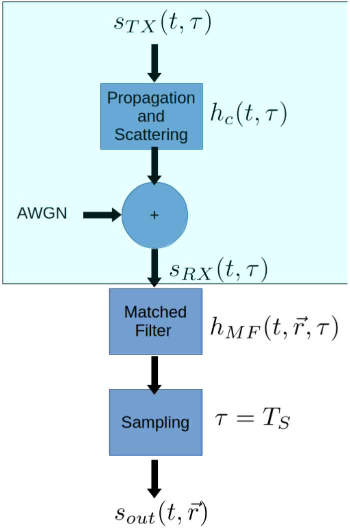
Instantaneous Doppler LTI channel (at fixed t). With reflectivity $\rho(\vec{r}')$ and Doppler time scale $\eta(t,\vec{r}')$,

$$h_c(t,\tau) = \int_{\vec{r}'} \rho(\vec{r}')\,\delta\left(\eta(t,\vec{r}')\cdot\left(\tau - \frac{R(t,\vec{r}')}{c}\right)\right)\,|\mathrm{d}\vec{r}'|.$$

Doppler time scale. $\eta(t,\vec{r}') = \frac{c+\dot{R}(t,\vec{r}')}{c-\dot{R}(t,\vec{r}')}$ (constant w.r.t. fast-time τ; depends on bistatic range rate $\dot{R}$).

Received signal (fast-time convolution). Additive white Gaussian noise $n(t,\tau)$:

$$\begin{aligned} s_{RX}(t,\tau) &= \left[s_{TX}(t,\tau)(*)_\tau h_c(t,\tau)\right] + n(t,\tau) \\ &= \int_{\vec{r}'} \rho(\vec{r}')\, s_{TX}\left(t,\ \eta(t,\vec{r}')\cdot\left(\tau - \frac{R(t,\vec{r}')}{c}\right)\right)\,|\mathrm{d}\vec{r}'| + n(t,\tau). \end{aligned}$$



Matched Filtering (CFV)

Objective. Use a fast-time matched filter to isolate scattering from image position $\vec{r}$. The matched filter varies with slow-time t and image position $\vec{r}$ and is designed to maximize SNR when sampled at $\tau = T_s$.

Matched-filter impulse response. With fast-time coherency window T_W,

$$h_{MF}(t, \vec{r}, \tau) = \frac{1}{T_W} s_{TX}^*\left(t,\ \eta(t, \vec{r}) \cdot \left(T_s - \tau + \frac{R(t, \vec{r})}{c}\right)\right), \qquad BW(t) \gg \frac{1}{T_W}.$$

Sampled MF output.

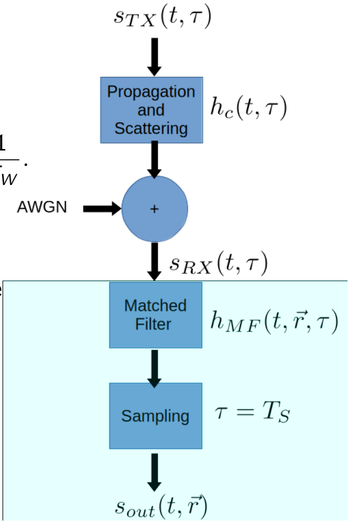
$$s_{out}(t, \vec{r}) = \left(s_{RX}(t, \tau)\,(*)_\tau\, h_{MF}(t, \vec{r}, \tau)\right)\Big|_{\tau = T_s}.$$

Correlation view (windowed fast-time average). Define a windowed average around T_s:

$$\mathbb{E}_\tau[g(\tau)] \triangleq \frac{1}{T_W}\int_{T_s - \frac{T_W}{2}}^{T_s + \frac{T_W}{2}} g(\tau)\,\mathrm{d}\tau.$$

Then sampling the matched filter at $\tau = T_s$ is equivalent to a fast-time cross-correlation with the reference:

$$\mathbb{E}_\tau\left[s_{TX}\left(t, \eta(t, \vec{r}') \cdot \left(\tau - \frac{R(t,\vec{r}')}{c}\right)\right)\ s_{TX}^*\left(t, \eta(t, \vec{r}) \cdot \left(\tau - \frac{R(t,\vec{r})}{c}\right)\right)\right].$$



Slow-Time Processing (CFV)

Slow-time averaging. The SAR image is the slow-time average of the sampled matched-filter output:

$$\tilde{\rho}(\vec{r}) \;=\; \mathbb{E}_t[s_{out}(t,\vec{r})]\,, \qquad \mathbb{E}_t[a(t)] \;\triangleq\; \frac{1}{T_C}\int_{t_{\min}}^{t_{\max}} a(t)\,\mathrm{d}t, \quad T_C = t_{\max} - t_{\min}.$$

Decomposition into signal and noise.

$$\tilde{\rho}(\vec{r}) \;=\; \tilde{\rho}_s(\vec{r}) \;+\; \tilde{\rho}_n(\vec{r}).$$

Noise term (slow-/fast-time cross-correlation). Using the windowed fast-time average $\mathbb{E}_\tau[\cdot]$, define the joint average $\mathbb{E}_{t,\tau}[\cdot] \triangleq \mathbb{E}_t\big[\mathbb{E}_\tau[\cdot]\big]$. Then

$$\tilde{\rho}_n(\vec{r}) \;=\; \mathbb{E}_{t,\tau}\left[n(t,\tau)\; s^*_{TX}\Big(t,\;\eta(t,\vec{r})\cdot\Big(\tau - \tfrac{R(t,\vec{r})}{c}\Big)\Big)\right].$$

Signal term (non-convolution spatial integral). After change of integration order,

$$\tilde{\rho}_s(\vec{r}) \;=\; \int_{\vec{r}'} \rho(\vec{r}')\;\; \mathsf{psf}(\vec{r}',\vec{r})\;\left|\mathrm{d}\vec{r}'\right|,$$

with point-spread function

$$\mathsf{psf}(\vec{r}',\vec{r}) \;\triangleq\; \mathbb{E}_{t,\tau}\left[s_{TX}\Big(t,\;\eta(t,\vec{r}')\cdot\Big(\tau - \tfrac{R(t,\vec{r}')}{c}\Big)\Big)\;\; s^*_{TX}\Big(t,\;\eta(t,\vec{r})\cdot\Big(\tau - \tfrac{R(t,\vec{r})}{c}\Big)\Big)\right],$$

which depends on both source $\vec{r}'$ and image $\vec{r}$ locations.

Point Spread Function (PSF)

Definition (CFV). The PSF is the fast-time/slow-time cross-correlation between two copies of the transmitted waveform, modulated for the Doppler and propagation delays to $\vec{r}$ and $\vec{r}'$:

$$\mathrm{psf}(\vec{r}', \vec{r}) = \mathbb{E}_{t,\tau}\left[s_{TX}\left(t, \eta(t,\vec{r}') \cdot \left(\tau - \frac{R(t,\vec{r}')}{c}\right)\right)\, s_{TX}^{*}\left(t, \eta(t,\vec{r}') \cdot \left(\tau - \frac{R(t,\vec{r})}{c}\right)\right)\right].$$

Mathematical formulation.

$$= \mathbb{E}_{t,\tau}\left[\mathbb{E}_{v}\left[\mathbb{E}_{f}\left[e^{j2\pi v\,\eta(t,\vec{r}')\cdot\left(\tau - \frac{R(t,\vec{r}')}{c}\right)}\, e^{-j2\pi f\,\eta(t,\vec{r})\cdot\left(\tau - \frac{R(t,\vec{r})}{c}\right)}\right]\right]\right]$$

$$= \mathbb{E}_{t,\tau,f,v}\left[\exp\left(j2\pi\left[v\,\eta(t,\vec{r}')\cdot\left(\tau - \frac{R(t,\vec{r}')}{c}\right) - f\,\eta(t,\vec{r})\cdot\left(\tau - \frac{R(t,\vec{r})}{c}\right)\right]\right)\right]$$

$$= \mathbb{E}_{t,f,v}\left[\mathbb{E}_{\tau}\left[e^{j2\pi\left(v\,\eta(t,\vec{r}') - f\,\eta(t,\vec{r})\right)\tau}\right] e^{j\frac{2\pi}{c}\left(f\,\eta(t,\vec{r})\,R(t,\vec{r}) - v\,\eta(t,\vec{r}')\,R(t,\vec{r}')\right)}\right]$$

Objective We need an approximation such that $\mathrm{psf}(\vec{r}', \vec{r}) \approx \mathrm{ipr}(\vec{r} - \vec{r}')$.
Tools: Rank-reduce the frequency averages from f and v to only an average over f.
$\mathbb{E}_{\tau}\left[e^{j2\pi\left(v\,\eta(t,\vec{r}') - f\,\eta(t,\vec{r})\right)\tau}\right] = \delta\left(v\,\eta(t,\vec{r}') - f\,\eta(t,\vec{r})\right)$, set $v = f\,\frac{\eta(t,\vec{r})}{\eta(t,\vec{r}')}$ and only average along f; or approximate $\delta(v - f)$, set $v = f$ and only average along f.
Use a truncated Taylor series of the Doppler time scale $\eta(t,\vec{r})$ w.r.t. $\dot{R}(t,\vec{r})$. $f(x) = \frac{c+x}{c-x} \approx f(0) + f'(0)\,x + f''(0)\,\frac{x^2}{2}$.
With this approximation, $\eta(t,\vec{r}) \approx 1 + \frac{2\,\dot{R}(t,\vec{r})}{c} + \frac{2\,\dot{R}^2(t,\vec{r})}{c^2}$.

Point Spread Function (PSF) — Approximation 1 (CFV)

Rank Reduction $\mathbb{E}_\tau \left[e^{j2\pi\left(v\,\eta(t,\vec{r}')-f\,\eta(t,\vec{r})\right)\tau} \right] = \delta\left(v\,\eta(t,\vec{r}') - f\,\eta(t,\vec{r})\right)$ or set $v = f\,\frac{\eta(t,\vec{r})}{\eta(t,\vec{r}')}$ and only average over f.

Updated $\mathrm{psf}(\vec{r}', \vec{r}) = \mathbb{E}_{t,f}\left[e^{j\frac{2\pi f}{c}\eta(t,\vec{r})\left(R(t,\vec{r})-R(t,\vec{r}')\right)} \right]$

Key points.

- Two waveform copies are expressed as frequency averages of phasors: one with f as the frequency variable, the other with v.

Point Spread Function (PSF) — Approximation 2 (CFV)

Approximation of Rank Reduction $\mathbb{E}_\tau\left[e^{j2\pi\left(v\,\eta(t,\vec{r}')-f\,\eta(t,\vec{r})\right)\tau}\right] \approx \delta\left(v-f\right)$ or set $v=f$ and only average over f. Taylor series: $\eta(t,\vec{r}) \approx 1+\frac{2}{c}\dot{R}(t,\vec{r})$.

Updated

$$\begin{aligned}
\mathrm{psf}(\vec{r}',\vec{r}) &= \mathbb{E}_{t,f}\left[e^{j\frac{2\pi f}{c}\left(\eta(t,\vec{r})\,R(t,\vec{r})-\eta(t,\vec{r}')\,R(t,\vec{r}')\right)}\right] \\
&\approx \mathbb{E}_{t,f}\left[e^{j\frac{2\pi f}{c}\left(\left(1+\frac{2}{c}\dot{R}(t,\vec{r})\right)R(t,\vec{r})-\left(1+\frac{2}{c}\dot{R}(t,\vec{r}')\right)R(t,\vec{r}')\right)}\right] \\
&\approx \mathbb{E}_{t,f}\left[e^{j\frac{2\pi f}{c}\left(R(t,\vec{r})-R(t,\vec{r}')\right)}\,e^{j\frac{4\pi f}{c^2}\left(R(t,\vec{r})\,\dot{R}(t,\vec{r})-R(t,\vec{r}')\,\dot{R}(t,\vec{r}')\right)}\right] \\
&\approx \mathbb{E}_{t,f}\left[e^{j\frac{2\pi f}{c}\left(R(t,\vec{r})-R(t,\vec{r}')\right)}\,e^{j\frac{4\pi f}{c^2}R(t,\vec{O})\left(\dot{R}(t,\vec{r})-\dot{R}(t,\vec{r}')\right)}\right]
\end{aligned}$$

Key points.

- Move-stop-move PSF captured geometric diversity as a difference in range signatures $R(t,\vec{r})-R(t,\vec{r}')$, CFV model captured geometric diversity as a difference in weighted range signatures $\eta(t,\vec{r})\cdot R(t,\vec{r})-\eta(t,\vec{r}')\cdot R(t,\vec{r}')$.
- Using Taylor Series, we have one phasor with $R(t,\vec{r})-R(t,\vec{r}')$ and another with $R(t,\vec{r})\dot{R}(t,\vec{r})-R(t,\vec{r}')\dot{R}(t,\vec{r}')$.
- We want $\dot{R}(t,\vec{r})-\dot{R}(t,\vec{r}')$ so we approximate $R(t,\vec{r})\approx R(t,\vec{O})$ and $R(t,\vec{r}')\approx R(t,\vec{O})$ and factor it out on the second phasor.

Range/Range Rate Gradients

$$R(t,\vec{r}) \approx R(t,\vec{O}) + \langle \vec{r}-\vec{O}, \vec{\nabla} R(t,\vec{O})\rangle$$

$$\dot{R}(t,\vec{r}) \approx \dot{R}(t,\vec{O}) + \langle \vec{r}-\vec{O}, \vec{\nabla} \dot{R}(t,\vec{O})\rangle$$

$$R(t,\vec{r}) - R(t,\vec{r}') \approx \langle \vec{r}-\vec{r}', \vec{\nabla} R(t,\vec{O})\rangle$$

$$\dot{R}(t,\vec{r}) - \dot{R}(t,\vec{r}') \approx \langle \vec{r}-\vec{r}', \vec{\nabla} \dot{R}(t,\vec{O})\rangle$$

Updated

$$\mathrm{psf}(\vec{r}',\vec{r}) \approx \mathrm{ipr}(\vec{r}-\vec{r}')$$

$$\mathrm{ipr}(\vec{r}) = \mathbb{E}_{t,f}\left[e^{j\langle \vec{r},\vec{F}(f,t)\rangle}\right]$$

$$\vec{F}(f,t) = \frac{2\pi f}{c}\left(\vec{\nabla} R(t,\vec{O}) + \frac{2\,R(t,\vec{O})}{c}\vec{\nabla}\dot{R}(t,\vec{O})\right)$$

Key points.

- Both Move-stop-move and CFV IPRs expressed in the form $\mathrm{ipr}(\vec{r}) = \mathbb{E}_{t,f}\left[e^{j\langle \vec{r},\vec{F}(f,t)\rangle}\right]$.
- Phasor with projection of $\vec{r}$ along surface $\vec{F}(f,t)$.
- Move-stop-move surface was function of frequency and range gradient : $\vec{F}(f,t) = \frac{2\pi f}{c}\vec{\nabla} R(t,\vec{O})$.
- CFV surface functin of frequency, Range and Range Rate gradients : $\vec{F}(f,t) = \frac{2\pi f}{c}\left(\vec{\nabla} R(t,\vec{O}) + \frac{2\,R(t,\vec{O})}{c}\vec{\nabla}\dot{R}(t,\vec{O})\right)$.

Spatial Convolution and Inverse Fourier Transform

Convolution in Spatial Domain

$$\tilde{\rho}(\vec{r}) \approx \rho(\vec{r})\,(*)_{\vec{r}}\,\mathrm{ipr}(\vec{r}) = \int_{\vec{r}'} \rho(\vec{r}')\,\mathrm{ipr}(\vec{r}-\vec{r}')\,\left|\vec{r}'\right|$$

Multiplication in Fourier Domain

$$\tilde{\rho}(\vec{r}) = \mathcal{F}_{\vec{r}}^{-1}\left[P(\vec{K})\,TF(\vec{K})\right]$$

$$P(\vec{K}) = \mathcal{F}_{\vec{r}}\left[\rho(\vec{r})\right]$$

Transfer Function - Fourier Transfrom of IPR

$$TF(\vec{K}) = \mathcal{F}_{\vec{r}}\left[\mathrm{ipr}(\vec{r})\right] = \mathcal{F}_{\vec{r}}\left[\mathbb{E}_{t,f}\left[e^{j\langle\vec{r},\vec{F}(f,t)\rangle}\right]\right]$$

$$= \mathbb{E}_{t,f}\left[\mathcal{F}_{\vec{r}}\left[e^{j\langle\vec{r},\vec{F}(f,t)\rangle}\right]\right] = \frac{1}{(2\pi)^3}\mathbb{E}_{t,f}\left[\delta\left(\vec{K}-\vec{F}(f,t)\right)\right]$$

Key points.

- TF represents a time–frequency modulation of the instantaneous position in the Fourier domain (k-space), expressed as $\vec{K} = \vec{F}(f,t)$.
- The passband is the set of all k-space positions traversed as slow time varies over $t \in [t_{\min}, t_{\max}]$ and frequency sweeps over $f \in [f_{\min}(t),\, f_{\max}(t)]$.

Projection–Slice Theory for Ground-Plane Imaging

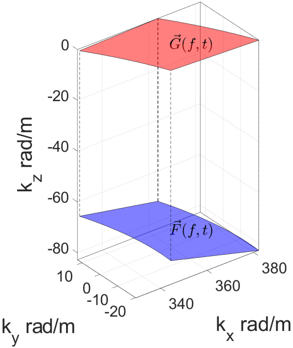
- If the SAR image $\tilde{\rho}(\vec{r})$ is confined to the ground plane (Slice) $\langle \vec{r} - \vec{O},\, \vec{z} \rangle = 0$, then imaging performance can be analyzed in the Fourier domain via *projection–slice* theory.
- Project the k-space surface $\vec{F}(f,t)$ into the ground plane using

$$\vec{G}(f,t) \;=\; \vec{z}\times\vec{F}(f,t)\times\vec{z} \qquad \text{(equivalently } \vec{G}(f,t) = (\mathbf{I} - \vec{z}\vec{z}^{\top})\,\vec{F}(f,t)\text{)}.$$

  Here $\vec{z}$ is the unit normal to the ground plane.
- The 2D ground-plane transfer function over spatial frequencies $\vec{K}$ is

$$TF(\vec{K}) \;=\; \frac{1}{(2\pi)^2}\mathbb{E}_{t,f}\left[\,\delta\big(\vec{K} - \vec{G}(f,t)\big)\,\right],$$

  describing the sampling density in k-space induced by (f,t) after projection onto the ground plane.



Tall/narrow illustration of $\vec{F}(f,t)$ projected to $\vec{G}(f,t)$.

K-Space Passband Shaped by Modulation Vectors

- The shape of the k-space passband depends on the **direction** and **length** of the time and frequency modulation vectors.
- A differential change in slow time dt or frequency df results in a change in k-space position:

$$d\vec{K} \;=\; \vec{G}_f(f,t)\,df \;+\; \vec{G}_t(f,t)\,dt.$$

- The modulation vectors $\vec{G}_f(f,t)$ and $\vec{G}_t(f,t)$ are the **partial derivatives** of the k-space surface $\vec{G}(f,t)$ with respect to frequency and slow time:

$$\vec{G}_f(f,t) = \frac{\partial}{\partial f}\vec{G}(f,t), \quad \vec{G}_t(f,t) = \frac{\partial}{\partial t}\vec{G}(f,t)$$

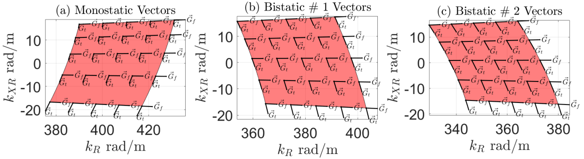


Illustration of $\vec{G}(f,t)$ showing modulation vectors $\vec{G}_f$ and $\vec{G}_t$ shaping the passband in k-space.

Modulation Vectors for Ground-Plane Projected K-Space

Ground-plane k-space surface:

$$\vec{G}(f,t) = \frac{2\pi f}{c}\left(\vec{\nabla} R_{GP}(t,\vec{O}) + \frac{2R(t,\vec{O})}{c}\,\vec{\nabla}\dot{R}_{GP}(t,\vec{O})\right)$$

Projected gradients:

$$\begin{aligned}
\vec{\nabla} R_{GP}(t,\vec{O}) &= \vec{z} \times \vec{\nabla} R(t,\vec{O}) \times \vec{z},\\
\vec{\nabla}\dot{R}_{GP}(t,\vec{O}) &= \vec{z} \times \tfrac{d}{dt}\vec{\nabla} R(t,\vec{O}) \times \vec{z},\\
\vec{\nabla}\ddot{R}_{GP}(t,\vec{O}) &= \vec{z} \times \tfrac{d^2}{dt^2}\vec{\nabla} R(t,\vec{O}) \times \vec{z}.
\end{aligned}$$

Modulation vectors:

$$\vec{G}_t(f,t) = \frac{2\pi f}{c}\left(\vec{\nabla}\dot{R}_{GP}(t,\vec{O}) + \frac{2\dot{R}(t,\vec{O})}{c}\,\vec{\nabla}\dot{R}_{GP}(t,\vec{O}) + \frac{2R(t,\vec{O})}{c}\,\vec{\nabla}\ddot{R}_{GP}(t,\vec{O})\right),$$

$$\vec{G}_f(f,t) = \frac{2\pi}{c}\left(\vec{\nabla} R_{GP}(t,\vec{O}) + \frac{2R(t,\vec{O})}{c}\,\vec{\nabla}\dot{R}_{GP}(t,\vec{O})\right).$$

Differential displacement:

$$d\vec{K} = \vec{G}_f(f,t)\,df + \vec{G}_t(f,t)\,dt$$