Dirac planar map for paint and AT notes

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1 Basics

We consider graphs embedded on surfaces without boundary. These come in two flavors: the *orientable surfaces* surfaces Σ_g which is the sphere with g handles; and the *non-orientable surfaces* Π_h which is the sphere with h cross-caps. The *Euler genus* ε of Σ_g is $2g$ and the Euler genus of Π_h is h. The *Euler genus* of a graph G is the smallest Euler genus of a surface on which G can be embedded. By Euler's formula, if a graph G is embedded on a surface of Euler genus ε, then $n - m + f \geq 2 - \varepsilon$ where $n = |G|$, $m = \|G\|$ and f is the number of faces of G. It follows that if G is embedded on a surface of Euler genus ε, then $\|G\| \leq 3\,|G| - 6 + 3\varepsilon$. The *Heawood* number is given by

$$H(\varepsilon) = \left\lfloor \frac{7 + \sqrt{24\varepsilon + 1}}{2} \right\rfloor.$$

When $\varepsilon \geq 1$, the bound on $\|G\|$ above implies that if G is embedded on a surface of Euler genus ε, then G has a vertex of degree at most $H(\varepsilon) - 1$. In particular, the graphs embedded on a surface of Euler genus $\varepsilon \geq 1$ are $H(\varepsilon)$-AT and hence $H(\varepsilon)$-paintable. The goal is to show that the only obstruction to G being $(H(\varepsilon) - 1)$-AT is G containing $K_{H(\varepsilon)}$.

Conjecture 1.1. *Let G be a graph embedded on a surface of Euler genus $\varepsilon \geq 1$. If $K_{H(\varepsilon)} \not\subseteq G$, then G is $(H(\varepsilon) - 1)$-AT.*

2 With Kernel Magic

We can get almost all the way there for paint using the follow result from [1].

Definition 1. The *maximum independent cover number* of a graph G is the maximum $\text{mic}(G)$ of $\sum_{v \in I} d_G(v)$ over all independent sets I of G. A set I that witnesses this maximum is said to be optimal.

In [1] it was shown that $\text{mic}(G) \geq |G| - 1$ for all G and $\text{mic}(G) \geq |G|$ if G is a connected graph that is not a Gallai tree.

Definition 2. A graph G is *P-reducible* to H if H is a nonempty induced subgraph of G which is f_H-paintable where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not P-reducible to any nonempty induced subgraph, then it is *P-irreducible*.

Theorem 2.1. *Every P-irreducible graph G satisfies* $\operatorname{mic}(G) \leq 2\|G\| - (\delta(G) - 1)|G| - 1$.

A minimal counterexample G to Conjecture 1.1 for paint is clearly P-irreducible. So, if G has Euler genus ε, then we have

$$\operatorname{mic}(G) \leq 2\|G\| - (\delta(G) - 1)|G| - 1 \leq 2(3|G| - 6 + 3\varepsilon) - (\delta(G) - 1)|G| - 1.$$

Since $\delta(G) \geq H(\varepsilon) - 1$, this becomes

$$\operatorname{mic}(G) \leq (8 - H(\varepsilon))|G| + 6\varepsilon - 13.$$

Since G is 2-connected and not complete, it is not a Gallai tree, thus we have $\operatorname{mic}(G) \geq |G|$ which gives

$$0 \leq (7 - H(\varepsilon))|G| + 6\varepsilon - 13.$$

Since $H(2) = 7$, we conclude that $\varepsilon \neq 2$.

References

[1] Hal Kierstead and Landon Rabern, *Improved lower bounds on the number of edges in list critical and online list critical graphs*, arXiv preprint, `http://arxiv.org/abs/1406.7355` (2014).