notes on coloring cayley graphs

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1 Basics

Definition 1. For a group G and $A \subseteq G$, the *cayley graph* of G with respect to A is the directed graph with vertex set G and an edge from x to xa for each $x \in G$ and $a \in A$. Write $\mathcal{C}(G, A)$ for this digraph.

We are concerned with coloring undirected graphs without loops, so we want A to not contain the identity element of G and $\frac{1}{A} = A$, where

$$\frac{1}{A} = \left\{a^{-1} \mid a \in A\right\}.$$

Given this, $\mathcal{C}(G, A)$ has all edges directed both ways. Let G_A be the undirected graph with the structure of $\mathcal{C}(G, A)$. We call such G_A a *standard cayley graph*.

Remark. a and b are adjacent in a standard cayley graph G_A just in case $ab^{-1} \in A$.

Conjecture 1.1. *Let G be an abelian group and G_A a standard cayley graph. If $\Delta(G) \geq 9$ and $\omega(G) < \Delta(G)$, then $\chi(G) < \Delta(G)$.*

i am trying to make the $\Delta = 8$ example as a cayley graph of $C_5 \times C_3$, with the standard generators, its missing some edges though so need to throw more into A.

Lemma 1.2. *If a and b are adjacent in a standard cayley graph G_A, then for any independent set X in G_A*

$$\frac{1}{X}a \bigcap \frac{1}{X}b = \emptyset.$$

Proof. Suppose there is $c \in \frac{1}{X}a \bigcap \frac{1}{X}b$. Then $c = x^{-1}a$ and $c = y^{-1}b$ for some $x, y \in X$. So $yx^{-1} = ba^{-1} \in A$, so x and y are adjacent, but they can't be since both are in the independent set X. □

Since we are just working with abelian groups now, we can use a nicer form of Lemma 1.2.

Lemma 1.3. *Let G be an abelian group and G_A a standard cayley graph. Then for any clique K and independent set X in G_A,*

1. $|XK| = |X|\,|K|$, *and*

2. $\left|\frac{1}{X}K\right| = |X|\,|K|$

Proof. Part (2) is immediate from Lemma 1.2 since the sets $\left\{\frac{1}{X}a \mid a \in K\right\}$ are pairwise disjoint and $\left|\frac{1}{X}\right| = |X|$.

For (1), suppose $a, b \in K$ are different vertices such that $Xa \cap Xb \neq \emptyset$. Then for $c \in Xa \cap Xb$, we have $c = xa = yb$ for some $x, y \in X$. But then $ab^{-1} = x^{-1}y = yx^{-1}$. But a and b are adjacent, so $ab^{-1} \in A$, so $yx^{-1} \in A$, so x and y are adjacent, a contradiction. Now (1) follows in the same way as (2). □

Using this, we can get our first bound on the chromatic number.

Theorem 1.4. *Let G be an abelian group and G_A a standard cayley graph. Then*

$$\chi(G_A) \leq \omega(G_A) + |G| - \omega(G_A)\alpha(G_A).$$

Proof. Take a maximum independent set X in G_A and maximum clique K in G_A. By Lemma 1.3 $\{Xa \mid a \in K\}$ is a collection of pairwise disjoint maximum independent sets in G_A. Using one color for each of those and then one color for each vertex in $G_A - XK$ gives the bound. □

Generally, that is a terrible bound, but we have a lot of room for improvement in coloring the leftover bit $G_A - XK$. The case where $G_A - XK$ is empty matches up nicely with $\chi(G_A) = \omega(G_A)$ in the $\frac{5}{6}$-bound. We want to show that when there is some of the leftover bit $G_A - XK$, we can color it with something like $\frac{5}{6}\Delta(G_A) - \omega(G_A)$ colors. There is a lot to play with here. For example, we can swap a vertex in $G_A - XK$ that has only one neighbor in X for its neighbor to get X'. Now we get a new coloring by looking at $X'K$ which has a lot in common with our previous coloring. Right now i am trying to see what sorts of information we can get out of this.

2 Strong coloring cayley graphs

We know that to prove the $\frac{5}{6}$-bound for a class $\mathcal{C}$ of vertex transitive graphs, it suffices to prove the 2.5Δ upper bound for strong coloring and Reed's ω, Δ, χ-conjecture for $\mathcal{C}$. When $\mathcal{C}$ is the abelian standard cayley graphs, the needed strong coloring bound comes for free from $Haxell's$ 2Δ bound for independent transversals together with Lemma 1.3.

Lemma 2.1. *Let G be an abelian group and G_A a standard cayley graph. If $\omega(G_A) > \frac{2}{3}(\Delta(G_A) + 1)$, then $\chi(G_A) = \omega(G_A)$.*

Proof. Since G_A is vertex transitive, $\omega(G_A) > \frac{2}{3}(\Delta(G_A)+1)$ implies that G is a disjoint union of $\omega(G)$-cliques. Since $\omega(G_A) > \frac{2}{3}(\Delta(G_A) + 1)$, we have $\omega(G_A) > 2(\Delta(G_A) + 1 - \omega(G_A))$ and hence we may apply Haxell's independent transversal lemma to get an independent set X in G_A with $|X|\omega(G) = |G_A|$. So. $|G_A| = \omega(G_A)\alpha(G_A)$ and lemma 1.4 gives $\chi(G_A) = \omega(G_A)$. □

Theorem 2.2. *If Reed's conjecture holds for abelian standard cayley graphs, then such graphs satisfy the $\frac{5}{6}$-bound.*

Proof. When $\omega(G_A) > \frac{2}{3}(\Delta(G_A) + 1)$ we are done by Lemma 2.1. So we can apply Reed's conjecture using $\omega(G_A) \leq \frac{2}{3}(\Delta(G) + 1)$ to get the desired $\frac{5}{6}$-bound. □