

1. OVERVIEW

The biggest open problem in edge-coloring is the Goldberg–Seymour conjecture. Over the past two decades, the main tool for attacking this problem has become Tashkinov trees, a vast generalization of Vizing fans and Kierstead paths. The second author proved that if G is a line graph, then $\chi(G) \leq \max\{\omega(G), \frac{7\Delta(G)+10}{8}\}$. In the same paper, he conjectured that $\chi(G) \leq \max\{\omega(G), \frac{5\Delta(G)+8}{6}\}$, which is best possible. We call the latter inequality the $\frac{5}{6}$ -Conjecture, and in this paper we prove it. Along the way, we develop more general techniques and results that will likely be of independent interest, due to their use in approaching the Goldberg–Seymour conjecture.

A graph G is *elementary* if $\chi'(G) = \mathcal{W}(G)$; such graphs satisfy the Goldberg–Seymour Conjecture. (We begin by proving that every minimal counterexample to the $\frac{5}{6}$ -Conjecture is elementary. In Section 3, we conclude by also proving that every elementary graph satisfies the $\frac{5}{6}$ -conjecture.) A *defective color* for a Tashkinov tree T is a color used on more than one edge from $V(T)$ to $V(G) - V(T)$; a Tashkinov tree is *strongly closed* if it has no defective color. Andersen [1] and Goldberg [2] showed that if G is critical, then G is elementary if there exists $e \in E(G)$ and $X \subseteq V(G)$ and a k -edge-coloring φ of $G - e$ such that X contains the endpoints of e and X is elementary and strongly closed w.r.t. φ . Thus, to show that G is elementary, it suffices to show that if G is $(k+1)$ -critical, then there exists an edge $e \in E(G)$ and a k -coloring φ of $G - e$ such that some maximal Tashkinov tree containing e is strongly closed. The following definition is useful. A vertex $v \in V(G)$ is *special* if every Vizing fan rooted at v (taken over all k -colorings of $G - e$, over all edges e incident to v) has at most 3 vertices, including v . As a warmup, in Section 2 we prove that if $\chi'(G) \geq \Delta(G) + 2$ and every vertex of G is special, then G is elementary, i.e., $\chi'(G) = \mathcal{W}(G)$. Next, we push our methods further, allowing our maximal Tashkinov tree to have at most 3 non-special vertices.

In Section 3, we show that if G is a minimal counterexample to the $\frac{5}{6}$ -Conjecture, then every non-special vertex v has $d_G(v) < \frac{3}{4}\Delta(G)$. Since every maximal Tashkinov tree T is elementary, and every non-special vertex misses more than $k/4$ colors, we conclude that T has at most 3 non-special vertices. Thus our results from Section 2 apply. As a consequence, every minimal counterexample to the $\frac{5}{6}$ -Conjecture is elementary. To complete the proof of the $\frac{5}{6}$ -conjecture, we prove that it follows from the Goldberg–Seymour Conjecture. More precisely, we show for each graph G that if $\chi'(G) = \mathcal{W}(G)$, then $\chi'(G) \leq \max\{\omega(G), \frac{5\Delta(G)+8}{6}\}$.

2. USEFUL LEMMAS

Throughout this section, let G be a $(k+1)$ -edge-critical multigraph for some $k \geq \Delta(G) + 1$. We use the following notation. Let φ be a partial k -edge-coloring of G . For each vertex $v \in V(G)$, let $\varphi(v)$ be the set of colors used in φ on edges incident to v and let $\overline{\varphi}(v) = [k] \setminus \varphi(v)$. For an uncolored edge e_0 , a *Tashkinov tree* is a sequence $v_0, e_1, v_1, e_2, \dots, v_{t-1}, e_t, v_t$ such that all v_i are distinct and $e_i = v_j v_i$ for some j and ℓ with $0 \leq j < i$ and $0 \leq \ell < i$ such that $\varphi(e_i) \in \overline{\varphi}(v_\ell)$.

Lemma 0. *Let φ be a k -edge-coloring of $G - v_0 v_1$. Suppose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Let $P = v_1 v_2 \dots v_r$ be an $\alpha - \beta$ path with edges $e_i = v_i v_{i+1}$ for $1 \leq i \leq r - 1$. If v_i is special*

for all odd i , then for any $\tau \in \overline{\varphi}(v_0)$ there are edges $f_i = v_i v_{i+1}$ for $1 \leq i \leq r-1$ such that $f_i = e_i$ for i even and $\varphi(f_i) = \tau$ for i odd.

Proof. Suppose not and choose a counterexample minimizing r . By minimality of r , we have $\varphi(v_{r-1}v_r) = \alpha$ and we have $f_i = v_i v_{i+1}$ for $1 \leq i \leq r-2$ such that $f_i = e_i$ for i even and $\varphi(f_i) = \tau$ for i odd. Swap α and β on e_i for $1 \leq i \leq r-3$ and then color $v_0 v_1$ (call this edge e_0) with α and uncolor e_{r-2} . Let φ' be the resulting coloring. Since $k \geq \Delta(G) + 1$, some color other than α is missing at v_{r-2} ; let γ be such a color. Now v_{r-1} is special since $r-1$ is odd (since P starts and ends with α), so there is an edge $e = v_{r-1}v_r$ with $\varphi'(e) = \gamma$. Swap τ and α on e_i for $0 \leq i \leq r-3$ to get a new coloring φ^* . Now γ and τ are both missing at v_{r-2} in φ^* . Since v_{r-1} is special, the fan with v_{r-2}, v_{r-1}, v_r and e implies that there is an edge $f_{r-1} = v_{r-1}v_r$ with $\varphi^*(f_{r-1}) = \tau$. But we have never recolored f_{r-1} , so $\varphi(f_{r-1}) = \tau$, a contradiction. \square

Recall that for a k -edge-coloring φ of $G - e$, a maximal Tashkinov tree starting with e may not be unique. However, if T_1 and T_2 are both such trees, then it is easy to show that $V(T_1) \subseteq V(T_2)$; by symmetry, also $V(T_2) \subseteq V(T_1)$, so $V(T_1) = V(T_2)$.

Lemma 1. *Let T be a maximal Tashkinov tree with respect to a k -edge-coloring φ of $G - xy$. If at most one $v \in V(T)$ is non-special, then, for all $\alpha \in \overline{\varphi}(x)$ and $\beta \in \overline{\varphi}(y)$, the $\alpha - \beta$ path P from y to x has $V(P) = V(T)$.*

Proof. Let φ be a k -edge-coloring of $G - xy$ and let T be a maximal Tashkinov tree containing xy . Choose $\alpha \in \overline{\varphi}(x)$ and $\beta \in \overline{\varphi}(y)$. Note that the $\alpha - \beta$ path P starting from y must end at x (or else we can perform a Kempe swap on P and color edge xy with color α). We show that P is a maximal Tashkinov tree; hence $V(P) = V(T)$. Say $P = v_0 \cdots v_r$, where $v_0 = y$ and $v_r = x$. Suppose P is not maximal; so some color τ is missing on P and some edge colored τ leaves P . Note that $E(P) \cup \{xy\}$ is an odd cycle. We can relabel $V(P)$ so that τ is missing at v_0 , the edge colored τ leaving $V(P)$ is incident to v_i and all vertices v_j with $0 < j < i$ are special. Further, if v_i is non-special, then we can choose i to be even (by possibly going around the cycle the other way). Also, we can recolor $E(P) \cup \{xy\}$ with α and β so that only $v_0 v_1$ is uncolored and α and β are missing at v_0 and v_1 , respectively. Now by Lemma 0, each of these edges colored α has a parallel edge colored τ . In particular, the edge colored τ incident to v_i ends in P , a contradiction. \square

A *defective color* for a Tashkinov tree T in a critical graph G is a color used on more than one edge from $V(T)$ to $V(G) - V(T)$. Let $t(G)$ be the maximum size of a Tashkinov tree over all $e \in E(G)$ and all k -edge-colorings φ of $G - e$. If a Tashkinov tree T has $|T| = t(G)$, then T is *maximum*.

Lemma 2. *Let T be a Tashkinov tree with respect to a k -edge-coloring φ of $G - v_0 v_1$. If T is maximum and at most one $v \in V(T)$ is non-special, then $V(T)$ has no defective colors.*

Proof. Use Lemma 1 to get an $\alpha - \beta$ path $P = v_1 \cdots v_r v_0$ with $V(P) = V(T)$; recall that P is a maximum size Tashkinov tree. Suppose that P has a defective color δ with respect to φ . Let τ be missing at v_2 . Consider a maximal $\tau - \delta$ path Q , starting at v_2 . Since P is maximal, δ is not missing at any vertex of P ; since $V(P)$ is elementary, τ is not missing at any vertex of P other than v_2 . As a result, Q ends outside $V(P)$. Now Q could leave $V(T)$ and re-enter

it repeatedly, but Q ends outside $V(P)$, so there is a last vertex $w \in V(Q) \cap V(P)$; say Q ends at $z \in V(G) - V(P)$. Let $\pi \notin \{\alpha, \beta\}$ be a color missing at w . Since P is maximal, no edge colored τ or π leaves $V(P)$. So, we can swap τ and π on every edge in $G - V(P)$ without changing the fact that P is a maximum size Tashkinov tree. Now swap δ and π on the subpath of Q from w to z ; since π is missing at w , the $\delta - \pi$ path does end at w . Now δ is missing at w , but δ was defective in φ , so there are still edges colored δ leaving $V(P)$, adding such an edge gets a larger Tashkinov tree, a contradiction. \square

Theorem 3. *Let G be a multigraph. If at most one $v \in V(G)$ is non-special, then $\chi'(G) \leq \max \{ \lceil \chi'_f(G) \rceil, \Delta(G) + 1 \}$.*

Proof. We may assume that G is critical. If $\chi'(G) \leq \Delta(G) + 1$, then we are done. So instead, assume that $\chi'(G) \geq \Delta(G) + 2$ and let $k = \chi'(G) - 1$. Choose $e \in E(G)$. Now by Lemma 2, $G - e$ has a k -edge-coloring φ and a maximal Tashkinov tree T such that $V(T)$ has no defective color, i.e., $V(T)$ is strongly closed. By a Theorem of Andersen and Goldberg (see also [Stiebitz, p. 8–9]), this implies that $\chi'(G) = \lceil \chi'_f(G) \rceil$. \square

3. THE EASY BOUND

Let G be a multigraph. The *claw-degree* of $x \in V(G)$ is

$$d_{\text{claw}}(x) := \max_{\substack{S \subseteq N(x) \\ |S|=3}} \frac{1}{4} \left(d(x) + \sum_{v \in S} d(v) \right).$$

The *claw-degree* of G is

$$d_{\text{claw}}(G) := \max_{x \in V(G)} d_{\text{claw}}(x).$$

Theorem 4. *If G is a multigraph, then*

$$\chi'(G) \leq \max \left\{ \lceil \chi'_f(G) \rceil, \Delta(G) + 1, \left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil \right\}.$$

Proof. Suppose not and choose a counterexample G minimizing $\|G\|$; note that G is edge-critical. Let $k = \chi'(G) - 1$, so $k \geq \lceil \frac{4}{3} d_{\text{claw}}(G) \rceil$. By Theorem 3, G has a non-special vertex x . Choose $xy_1 \in E(G)$ and a k -edge-coloring φ of $G - xy_1$ such that φ has a fan F of length 3 rooted at x with leaves y_1, y_2, y_3 . Since $V(F)$ is elementary,

$$2 + k - d(x) + \sum_{i \in [3]} k - d(y_i) \leq k,$$

and hence

$$d_{\text{claw}}(x) \geq \frac{1}{4} \left(d(x) + \sum_{i \in [3]} d(y_i) \right) \geq \frac{3k + 2}{4}.$$

This gives the contradiction

$$\left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil \leq k \leq \frac{4}{3} d_{\text{claw}}(G) - \frac{2}{3}.$$

\square

4. A STRONGER BOUND

For vertices $x, y \in V(G)$, we say that x is y -special if every Vizing fan rooted at x , with respect to any k -edge-coloring of $G - xy$, has at most 3 vertices. An edge $xy \in E(G)$ is *special* if x is y -special or y is x -special (or both).

Lemma 5. *Let φ be a k -edge-coloring of $G - v_0v_1$. Suppose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Let $P = v_1v_2 \cdots v_r$ be an $\alpha - \beta$ path. Let $i, j \in [r]$ with $i + 3 \leq j$ such that $j - i$ is odd. If v_t is special for all $i < t < j$, then $\mu(G) \geq 2k - d(v_i) - d(v_j)$.*

Corollary 6. *Let φ be a k -edge-coloring of $G - v_0v_1$. Suppose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Let $P = v_1v_2 \cdots v_r$ be an $\alpha - \beta$ path. Let $i, a, j \in [r]$ with $i + 2 \leq a \leq j - 2$ such that $j - i$ is odd. If v_t is special for all $i < t < a$ and $a < t < j$, then $\mu(G) \geq 2k - d(v_i) - d(v_a)$ or $\mu(G) \geq 2k - d(v_a) - d(v_j)$.*

The notion of a *normal Tashkinov tree* of Scheide (in Stiebitz book, page 160) generalizes what we have been doing with the maximal $\alpha - \beta$ cycle. A Tashkinov tree T with respect to a k -edge-coloring φ of $G - v_0v_1$ is *normal* if it starts with a maximal $\alpha - \beta$ path where $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. It is not hard to show that for any $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$, if we start with the maximal $\alpha - \beta$ path, we can extend this to a Tashkinov tree with vertex set $V(T)$ (see Stiebitz book, Theorem 5.3d on page 122). The *height* $h(T)$ of T is the length of the $\alpha - \beta$ cycle. The *height* $h(G)$ of G is the maximum height over all Tashkinov trees T with $|T| = t(G)$. We say that T has *maximum height* if $|T| = t(G)$ and $h(T) = h(G)$. Clearly, $h(G)$ is odd. Scheide showed (Stiebitz book page 175) that $h(G) \geq 5$ for non-elementary G . So, we can skip the worry i had about triangles with all non-special vertices.

A multigraph G is k -thin if $\mu(G) < 2k - d(x) - d(y)$ for all non-special $x, y \in V(G)$. For a Tashkinov tree T , let $\nu(T)$ be the number of non-special vertices in T .

Lemma 7. *Let G be a critical multigraph with $\chi'(G) = k + 1 \geq \Delta(G) + 2$. Suppose G is k -thin and non-elementary. Let T be a Tashkinov tree with respect to a k -edge-coloring φ of $G - v_0v_1$. If T has maximum height, then*

- (1) $\nu(T) = 3$ and there is a non-special edge in $E(T)$; or
- (2) $\nu(T) = 4$ and there is $xy \in E(T)$ where y is not special and x is not y -special; or
- (3) $\nu(T) \geq 5$.

Proof. Let $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ so that for $\alpha - \beta$ path $P = v_1v_2 \cdots v_r$ we have $E(P) \subseteq E(T)$. By lemmas of Scheide (Lemmas 6.14, 6.15), we know that $t(G) - 4 \geq h(G) \geq 5$ since G is non-elementary. So $r \geq 4$ and $|V(T) \setminus V(P)| \geq 4$.

Suppose (1), (2) and (3) do not hold. By Lemma 2 there must be non-special vertices v_i, v_j , where $i < j$. Without loss of generality, there is $\tau \in \overline{\varphi}(v_0)$ a τ -colored edge leaving P from v_b . Then, by Lemma 0, i is odd, j is even and $i \leq b \leq j$. By shifting the coloring, we may assume $i = 1$.

Suppose v_t is special for $1 < t < j$. Then, since G is thin, Lemma 5 implies $j = 2$. By symmetry, we may assume $b = 1$. Consider the $\tau - \beta$ path Q starting at v_1 . Plainly, Q must end at v_0 . Since P is an $\alpha - \beta$ path, Q must re-enter P along a τ -edge. But we just showed that τ edges can only leave at v_1 or v_2 . So, the $\tau - \beta$ path re-enters P at v_2 . But then we replaced a single edge v_1v_2 with a path of length at least three, so Q is longer than

the $\alpha - \beta$ path. Let T' be a maximal Tashkinov tree starting with Q . Then $V(T') = V(T)$, so $t(T') = t(T)$ and $h(T') > h(T)$, a contradiction since T has maximal height.

Hence there is $1 < a < j$ such that v_a is non-special. Suppose $\nu(T) = 3$. Since G is thin, Lemma 6 implies that $a = 2$ or $a = j - 1$. By symmetry, we may assume $a = 2$. Uncolor v_1v_2 and color v_0v_1 with α . Let F_i be a maximal fan rooted at v_i for $i \in [2]$. If $|F_i| \leq 3$, then all the colors in $\bar{\varphi}(v_{3-i})$ must appear on edges between v_i and $v_{3(i-1)}$. But this is impossible since G is thin and all the colors in $\bar{\varphi}(v_j)$ also appear on edges between v_i and $v_{3(i-1)}$. Hence $|F_i| \geq 4$ for $i \in [2]$. So, v_1v_2 is a non-special edge and $\nu(T) = 3$, so (1) holds, a contradiction.

Now suppose $\nu(T) = 4$. Let v_b be the other non-special vertex besides v_1, v_a, v_j . First, suppose $b = 0$ or $b > j$. Since G is thin, Lemma 6 implies that $a = 2$ or $a = j - 1$. By symmetry, we may assume $a = 2$. Uncolor v_1v_2 and color v_0v_1 with α . Let F_2 be a maximal fan rooted at v_2 . If $|F_2| \leq 3$, then all the colors in $\bar{\varphi}(v_1)$ must appear on edges between v_2 and v_3 . But this is impossible since G is thin and all the colors in $\bar{\varphi}(v_j)$ also appear on edges between v_2 and v_3 . Hence v_2 is not v_1 -special, so (2) holds, a contradiction.

So, we must have $\nu(T) = 4$ and $1 < b < j$. By symmetry, we may assume $a < b$. If $a \neq 2$ and $b \neq a + 1$, then since G is thin, Lemma 5 shows that $a - 1$ is even and $b - a$ is even and hence $b - 1$ is even. But $j - 1$ is odd, so $j - b$ is odd. Since G is thin, Lemma 5 implies that $j = b + 1$. Hence $a = 2$, or $b = a + 1$ or $j = b + 1$. An argument similar to the previous paragraph yields a contradiction to G being thin. \square

Theorem 8 (from strengthening Brooks paper). *If Q is the line graph of a multigraph G and Q is vertex critical, then*

$$\chi(Q) \leq \max \left\{ \omega(Q), \Delta(Q) + 1 - \frac{\mu(G) - 1}{2} \right\}.$$

Theorem 9. *If Q is the line graph of a multigraph, then*

$$\chi(Q) \leq \max \{ \lceil \chi_f(Q) \rceil, \lceil \epsilon(\Delta(Q) + 1) \rceil \}.$$

Proof. Suppose the theorem is false and choose a counterexample minimizing $|Q|$. Put $k = \max \{ \lceil \chi_f(Q) \rceil, \lceil \epsilon(\Delta(Q) + 1) \rceil \}$. Say $Q = L(G)$ for a multigraph G . Then G is k -edge-critical.

Claim 0. *Let F be a fan rooted at x with respect to a k -edge-coloring of $G - xy$. If $|F| = 4$, then*

$$d(x) < \frac{1 - \epsilon}{2\epsilon - 1} \sum_{v \in V(F-x)} d(v).$$

Since F is elementary, we have

$$2 + k - d(x) + \sum_{v \in V(F-x)} k - d(v) \leq k,$$

so

$$2 + (|F| - 1)k \leq d(x) + \sum_{v \in V(F-x)} d(v).$$

Using $k \geq \epsilon(\Delta(Q) + 1) \geq \epsilon(d(x) + d(v) - \mu(xv))$ for each $v \in V(F - x)$, we get

$$2 + \sum_{v \in V(F-x)} \epsilon(d(x) + d(v) - \mu(xv)) \leq d(x) + \sum_{v \in V(F-x)} d(v),$$

so

$$2 + (\epsilon|F| - 1 - \epsilon) d(x) \leq \sum_{v \in V(F-x)} \epsilon \mu(xv) + \sum_{v \in V(F-x)} (1 - \epsilon) d(v).$$

Now $\sum_{v \in V(F-x)} \mu(xv) \leq d(x)$, so this becomes

$$2 + (\epsilon|F| - 1 - 2\epsilon) d(x) \leq \sum_{v \in V(F-x)} (1 - \epsilon) d(v).$$

Using $|F| = 4$ gives

$$d(x) < \frac{1 - \epsilon}{2\epsilon - 1} \sum_{v \in V(F-x)} d(v).$$

Claim 1. *If $x \in V(G)$ with $d(x) \geq \frac{3(1-\epsilon)}{2\epsilon-1} \Delta(G)$, then x is special.*

Immediate from Claim 0.

Claim 2. *If $x_1x_2 \in E(G)$ with*

$$d(x_i) \geq \frac{2(1 - \epsilon)}{3\epsilon - 2} \Delta(G),$$

for at least one $i \in [2]$, then x_1x_2 is special. Suppose x_1x_2 is not special. Then by Claim 0, we have for $i \in [2]$,

$$d(x_i) < \frac{1 - \epsilon}{2\epsilon - 1} \sum_{v \in V(F-x)} d(v) \leq \frac{1 - \epsilon}{2\epsilon - 1} (d(3 - x_i) + 2\Delta(G)),$$

Solving the system gives for $i \in [2]$,

$$d(x_i) < \frac{2(1 - \epsilon)}{3\epsilon - 2} \Delta(G).$$

Claim 3. *If $xy \in E(G)$ with*

$$d(x) \geq \frac{1 - \epsilon^2}{(2\epsilon - 1)^2} \Delta(G),$$

then y is special or x is y -special. Suppose y is not special and x is not y -special. Then by Claim 0, we have

$$d(x) < \frac{1 - \epsilon}{2\epsilon - 1} \sum_{v \in V(F-x)} d(v) \leq \frac{1 - \epsilon}{2\epsilon - 1} (d(y) + 2\Delta(G)),$$

Since y is not special, Claim 1 gives $d(y) < \frac{3(1-\epsilon)}{2\epsilon-1} \Delta(G)$ and thus

$$d(x) < \frac{1 - \epsilon^2}{(2\epsilon - 1)^2} \Delta(G).$$

Claim 4. *The theorem is true.*

Let T be a Tashkinov tree with respect to a k -edge-coloring φ of $G - v_0v_1$ in G such that $|T| = t(G)$ and $h(T) = h(G)$. By Lemma 7 one of the following holds:

- (1) G is elementary; or
- (2) G is not thin; or
- (3) $\nu(T) = 3$ and $E(T)$ contains a non-special edge x_1x_2 ; or

(4) $\nu(T) = 4$ and $E(T)$ contains an edge x_1x_2 where x_2 is not special and x_1 is not x_2 -special; or

(5) $V(T)$ contains five non-special vertices x_1, x_2, x_3, x_4, x_5 .

If (1) holds, then $k + 1 = \lceil \chi_f(Q) \rceil \leq k$, a contradiction.

If (2) holds, then by Claim 1 we have $\mu(G) \geq 2k - 2\frac{3(1-\epsilon)}{2\epsilon-1}\Delta(G)$. Hence Theorem 8 gives

$$\leq k + 1 \leq \Delta(Q) + 1 - k - \frac{3(1-\epsilon)}{2\epsilon-1}\Delta(G) + \frac{1}{2},$$

so

$$2(k + 1) \leq \Delta(Q) + \frac{5}{2} - \frac{3(1-\epsilon)}{2\epsilon-1}\Delta(G).$$

Since $k \geq \Delta(G) + 1$, this gives

$$k + 1 < \frac{\Delta(Q) + \frac{5}{2}}{2 + \frac{3(1-\epsilon)}{2\epsilon-1}},$$

which is a contradiction when $\epsilon > \frac{1}{2}$.

Suppose (3) holds. Let x_3 be a non-special vertex in $V(T) \setminus \{x_1, x_2\}$. Then

$$2 + \sum_{i \in [3]} k - d(x_i) \leq k,$$

using Claim 1 and Claim 2, this gives

$$2 \left(\frac{2(1-\epsilon)}{3\epsilon-2} \right) \Delta(G) + \frac{3(1-\epsilon)}{2\epsilon-1} \Delta(G) \geq 2k + 2,$$

which is a contradiction when $\epsilon \geq \frac{41+\sqrt{57}}{58} \approx 0.837 > \frac{5}{6}$.

Suppose (4) holds. Let x_3, x_4 be non-special vertices in $V(T) \setminus \{x_1, x_2\}$.

$$2 + \sum_{i \in [4]} k - d(x_i) \leq k,$$

using Claim 1 and Claim 2 gives

$$\left(3 \frac{3(1-\epsilon)}{2\epsilon-1} + \frac{1-\epsilon^2}{(2\epsilon-1)^2} \right) \Delta(G) \geq 3k + 2,$$

which is a contradiction when $\epsilon \geq \frac{39+\sqrt{157}}{62} \approx 0.831$.

So (5) must hold. But then

$$2 + \sum_{i \in [5]} k - d(x_i) \leq k,$$

using Claim 1 gives

$$\frac{15(1-\epsilon)}{2\epsilon-1} \Delta(G) \geq 4k + 2,$$

which is a contradiction when $\epsilon \geq \frac{19}{23} \approx 0.826$.

□

5. THE $\frac{5}{6}$ -CONJECTURE

Lemma 10. *If H is a connected multigraph and $G = L(H)$, then $\mathcal{W}(H) \leq \max\{\omega(G), 5(\Delta(G) + 1)/6 + 3/6\}$.*

Proof. Let $d = d_H(x)$, $\Delta = \Delta(H)$, and $h = |H|$. Also, let $p = \sum_{v \in N(x)} d_H(v)$ and let $t = \Delta h - 2||H||$. Note that $0 < t \leq \Delta$. Also $p \geq Md - t$. Now summing over $N_H(x)$ gives

$$|N(x)|(\Delta h - t)/(h - 1) > 5/6((|N(x)| - 1)d + |N(x)|\Delta - t) + |N(x)|/2$$

Solving for $|N(x)|$ gives

$$|N(x)| < (5d + 5t)/(3 + 5d + 5\Delta - 6(\Delta h - t)/(h - 1)).$$

Since the numerator and denominator are linear in t , the right side is maximized at one end of the interval $1 \leq t \leq D$. Letting $t = D$, gives $|N(x)| < (5d + 5\Delta)/(3 + 5d - \Delta)$, like you had originally. Letting $t = 1$, gives $|N(x)| < (5d + 5)/(3 + 5d + 5\Delta - 6(\Delta h - 1)/(h - 1))$, which requires a little more analysis, akin to what you wrote in your most recent email.

Does that look right to you?

I did the analysis a little differently, but I got to the same conclusion: Substituting $d \geq 4D/5$ gives that if $M \geq 3$, then we must have $h \leq 4$, which implies $h \leq 3$, which contradicts $M \geq 3$.

So, I think I believe it. I also agree there must be an easier way. One thing that seems a little magical is that when $5/6 - M/(h - 1) \geq 0$ all of the h 's go away.

$w(H)$ really has a ceiling in its definition, not sure how much that changes things. without, it is the fractional chromatic index.

i think we get some gain as well from the $\Delta(H) + 2$ in place of $\Delta(H)$ we get as i wrote in the previous emails. Maybe this helps with the ceiling.

We can use $|H|$ odd to get a bit better on the ceiling in what you wrote since the top is even (divide both by two before doing ceiling approximation).

Thinking about your comment that we can assume H is critical, we can, but not how i was setting it up. Probably you are already thinking something like this:

Assume Goldberg. Take minimum counterexample to $5/6$ conjecture, say $G = L(H)$. The H is critical. From the argument like in strengthening of Brooks, we get $\chi(G) \geq \Delta(H) + 2$. By Goldberg this implies

$$\chi(G) = \max_{Q \subseteq H \text{ s.t. } |Q| \geq 3 \text{ and odd}} \left\lceil \frac{2||Q||}{|Q| - 1} \right\rceil$$

If the max is achieved at a proper subgraph of H , then there is an edge we can remove without decreasing the max, but this decreases the chromatic number by criticality and the max is a lower bound, so impossible. Therefore, $|H|$ is odd and

$$\chi(G) = \left\lceil \frac{2||H||}{|H| - 1} \right\rceil$$

so,

$$\left\lceil \frac{2||H||}{|H| - 1} \right\rceil \geq \Delta(H) + 2$$

$$2||H||/(|H| - 1) \geq \Delta(H) + 1$$

using

$$\Delta(H)|H| \geq 2||H||,$$

using $\Delta(H)|H| \geq 2||H||$, I get

$$\Delta(H) \geq |H| - 1,$$

□

I think we should be able to prove that the conjecture follows from Goldberg–Seymour. That lemma you proved is pretty useful. We can assume that H is critical, which implies that $|N(x)| \geq 2$ for all x in H . Now let J be the simple graph underlying H . We know that $\delta(J) \geq 2$. Let $B = \{x \in H \text{ s.t. } d_J(x) \geq 3\}$. That lemma implies that $|B| \leq 4$. Further, if $|B| = 4$, then each vertex of B has degree 3 in J . If $|B| = 3$, then two vertices of B have degree 3 in J and one has degree 4 in J . Otherwise $|B| \leq 2$. Now if J has a vertex x of degree at least 5, and $|B| = 2$, then the other vertex in B has degree 3 in J . Now x must be a cut-vertex (since J is formed by identifying one vertex in multiple disjoint cycles, exactly one of which has a chord). But a cut-vertex in J is also a cut-vertex in H , which is a contradiction. Thus, we only need consider the cases when $|B| = 3$ and $|B| = 4$, which have degree sequences $3, 3, 3, 3, 2, \dots, 2$. and $4, 3, 3, 2, \dots, 2$. $|B| = 4$ is a subdivided K_4 or a subdivision of a 4-cycle where one matching has multiplicity 2. $|B| = 3$ is a subdivision of a triangulated 5-cycle. I haven't worked out those cases, but I don't think they should be too hard.

Lemma 11. *Suppose $G = L(H)$ and G is a minimal counterexample to the $\frac{5}{6}$ -Conjecture. Let $k = \frac{5}{6}(\Delta(G) + 1)$. If T is a Tashkinov tree w.r.t. a k -edge-coloring φ of $H - e$, then*

$$\sum_{v \in V(T)} d_H(v)(5d_T(v) - 6) \leq -12 + 5 \sum_{e \in E(T)} \mu_H(e)$$

Proof. Since T is elementary, the sets of colors missing at vertices of T are disjoint, so $2 + \sum_{v \in V(T)} (k - d_H(v)) \leq k$. Rewriting this gives $k(|V(T)| - 1) \leq -2 + \sum_{v \in V(T)} d_H(v)$. For each edge $xy \in E(T)$, we have $k = \frac{5}{6}(\Delta(G) + 1) \geq \frac{5}{6}(d_H(x) + d_H(y) - \mu_H(xy))$. Summing over all $|T| - 1$ edges gives

$$\begin{aligned} -2 + \sum_{v \in V(T)} d_H(v) &\geq k(|V(T)| - 1) \\ &\geq \frac{5}{6}(\Delta(G) + 1)(|T| - 1) \\ &\geq \frac{5}{6} \sum_{uv \in E(T)} (d_H(u) + d_H(v) - \mu_H(uv)) \\ &= \frac{5}{6} \sum_{v \in V(T)} d_H(v)d_T(v) - \frac{5}{6} \sum_{uv \in E(T)} \mu_H(uv) \end{aligned}$$

To prove the lemma, we take the first and last expressions in the inequality chain, multiply by 6, then rearrange terms. \square

Corollary 12. *If $G = L(H)$ and G is a minimal counterexample to the $\frac{5}{6}$ -Conjecture, then each $x \in V(H)$ is special if $d_H(x) > \frac{3}{4}\Delta(H) - 3$.*

Proof. Suppose that x is a non-special vertex. Choose e incident to x and a k -edge-coloring φ of $G - e$ such that there exists a Vizing fan T rooted at x with $|T| \geq 4$. Since every edge in F is incident to x , we have $\sum_{e \in E(T)} \mu_H(e) \leq d_H(x)$. From Lemma 11, we have

$$\begin{aligned} -12 + 5d_H(x) &\geq -12 + 5 \sum_{e \in E(T)} \mu_H(e) \\ &\geq \sum_{v \in T} (5d_T(v) - 6)d_H(v) \\ &\geq (5d_T(x) - 6)d_H(x) + \sum_{v \in T-x} (5d_T(v) - 6)d_H(v) \\ &= (5(|T| - 1) - 6)d_H(x) - \sum_{v \in V(T-x)} d_H(v), \end{aligned}$$

where the final equality holds because each vertex $v \in T - x$ is a leaf. Now rearranging terms gives

$$\begin{aligned}
-12 + \sum_{v \in V(T-x)} d_H(v) &\geq (5(|T| - 1) - 11)d_H(x) \\
-12 + (|T| - 1)\Delta(H) &\geq (5(|T| - 16)d_H(x) \\
d_H(x) &\leq \frac{-12 + (|T| - 1)\Delta(H)}{5|T| - 16} \\
d_H(x) &\leq \frac{-12 + 3\Delta(H)}{4} = \frac{3}{4}\Delta(H) - 3,
\end{aligned}$$

where the final inequality holds because $|T| \geq 4$ and the right side decreases as a function of $|T|$. \square