notes for planar 5-AT

December 1, 2015

1 orientation tools

Let G be a graph and $\leq$ a total order on $V(G)$. An orientation of G is *even* if the number of directed edges vw with $v \leq w$ is even; otherwise, the orientation is *odd*. Let $\alpha\colon V(G) \to \mathbb{N}$. An orientation X of G is an *α-orientation* if $d_X^+(v) = \alpha(v)$ for all $v \in V$. Let $D_\alpha(G)$ be the set of α-orientations of G. We partition $D_\alpha(G)$ into even α-orientations $DE_\alpha(G)$ and odd α-orientations $DO_\alpha(G)$. For $X, Y \in D_\alpha(G)$, let $X \oplus Y$ be the spanning subgraph of X with edge set

$$\{x_1x_2 \in E(X) \mid x_2x_1 \in E(Y)\}.$$

Then $X \oplus Y$ is a spanning Eulerian subgraph of X. We say that a spanning Eulerian subgraph of X is *even* if it has an even number of edges and *odd* otherwise. Let $EL(X)$ be the set of spanning Eulerian subgraphs of X. We partition $EL(X)$ into even spanning Eulerian subgraphs $EE(X)$ and odd spanning Eulerian subgraphs $EO(X)$.

Lemma 1.1. *Let $X \in D_\alpha(G)$. For each $S \in EL(X)$ there is a unique $X_S \in D_\alpha(G)$ such that $S = X \oplus X_S$. Moreover, S is odd when X and X_S have opposite parity and even otherwise. Therefore, if X is even, then $|EE(X)| = |DE_\alpha(G)|$ and $|EO(X)| = |DO_\alpha(G)|$. If X is odd, then $|EE(X)| = |DO_\alpha(G)|$ and $|EO(X)| = |DE_\alpha(G)|$. So, up to sign, we always have*

$$|EE(X)| - |EO(X)| = |DE_\alpha(G)| - |DO_\alpha(G)|.$$

Since Lemma 1.1 was for any $X \in D_\alpha(G)$, we have the following.

Corollary 1.2. *If $X, Y \in D_\alpha(G)$ then, up to sign, we have*

$$|EE(X)| - |EO(X)| = |EE(Y)| - |EO(Y)|.$$

It will be useful to investigate α-orientations further. First, a basic fact about Eulerian graphs.

Lemma 1.3. *If D is an Eulerian directed graph, then D is an edge-disjoint union of directed cycles.*

Proof. If D is not edgeless, it must have a directed cycle, remove it and apply induction. $\square$

One important thing to note about Lemma 1.3 is there may be multiple different decompositions of D into directed cycles. Following Felsner [1], we say that $vw \in E(G)$ is *α-rigid* if vw is oriented the same way in every α-orientation of G.

Lemma 1.4. *If $X, Y \in D_\alpha(G)$ with $x_1x_2 \in E(X)$ and $x_2x_1 \in E(Y)$, then there is a directed cycle C in X containing x_1x_2 such that Y contains the directed cycle made from C by reversing all edges.*

Proof. Since $X \oplus Y$ is Eulerian, it is an edge-disjoint union of directed cycles. Let C be the directed cycle containing x_1x_2. □

From Lemma 1.4 we have the following.

Corollary 1.5. *An edge e of G is α-rigid if and only if no α-orientation of G has a directed cycle containing e.*

A graph G is *α-AT* if there is an α-orientation X of G with $EE(X) \neq EO(X)$. Note that by Lemma 1.1, if G is α-AT then $EE(X) \neq EO(X)$ for every $X \in D_\alpha(G)$. It is useful to see how α-AT behaves when we remove edges.

Lemma 1.6. *For any α-orientation of G and $vw \in E(G)$ with $v \leq w$, we have*

$$\begin{aligned}|D_\alpha(G)| &= |D_{\alpha-1_v}(G)| + |D_{\alpha-1_w}(G)|, \text{ and}\\ |DE_\alpha(G)| &= |DO_{\alpha-1_v}(G)| + |DE_{\alpha-1_w}(G)|, \text{ and}\\ |DO_\alpha(G)| &= |DE_{\alpha-1_v}(G)| + |DO_{\alpha-1_w}(G)|.\end{aligned}$$

Lemma 1.7. *Suppose G is α-AT and $vw \in E(G)$ with $v \leq w$. If vw is α-rigid (say always directed from v to w), then $G - vw$ is $(\alpha - 1_v)$-AT. Otherwise, $G - vw$ is either $(\alpha - 1_v)$-AT or $(\alpha - 1_w)$-AT.*

Proof. First, suppose vw is α-rigid. Let X be an α-orientation of G. Then vw is not contained in any $S \in EL(X)$ and hence removing it does not change parities. So, $G - vw$ is $(\alpha - 1_v)$-AT.

Now, suppose vw is not α-rigid. By Lemma 1.6, we have

$$0 \neq |DE_\alpha(G)| - |DO_\alpha(G)| = |DO_{\alpha-1_v}(G)| - |DE_{\alpha-1_v}(G)| + |DE_{\alpha-1_w}(G)| - |DO_{\alpha-1_w}(G)|.$$

Hence either $|DO_{\alpha-1_v}(G)| - |DE_{\alpha-1_v}(G)| \neq 0$ or $|DE_{\alpha-1_w}(G)| - |DO_{\alpha-1_w}(G)| \neq 0$. By Lemma 1.1, $G - vw$ is either $(\alpha - 1_v)$-AT or $(\alpha - 1_w)$-AT. □

We will use the following to reverse an edge on a triangle cutset when the inductive hypothesis directs the triangle cyclically.

Lemma 1.8. *Suppose G is α-AT and X is an α-orientation of G. If Z is an induced subgraph of X such that $EE(Z) = EO(Z)$, then X has an induced cycle $C \not\subseteq Z$ containing an edge of Z.*

Proof. Otherwise, every spanning Eulerian subgraph of X is the edge-disjoint union of a spanning Eulerian subgraph of Z and a spanning Eulerian subgraph of $X - E(Z)$. But then $EE(Z) = EO(Z)$ implies $EE(X) = EO(X)$, a contradiction. □

2 planar graphs

We are going to try to prove Thomassen's stronger result about choosability of near-triangulations for AT. Precisely, our aim is the following.

Conjecture 2.1. *Let G be a plane near-triangulation with outer face C. Then for any $x_1x_2 \in E(C)$, there is an orientation X of $G - x_1x_2$ such that*

1. $d^+_X(x_1) = d^+_X(x_2) = 0$, *and*
2. $d^+_X(v) \le 2$ *for all* $v \in V(C)$, *and*
3. $d^+_X(v) \le 4$ *for all* $v \in V(G) \setminus V(C)$, *and*
4. $EE(X) \ne EO(X)$.

Suppose the conjecture is false and choose a counterexample G minimizing $|G|$ and subject to that, minimizing $|C|$.

Lemma 2.2. *G has no clique cutset.*

Proof. Let $S \subseteq V(G)$ be a minimal cutset. Then $|S| \le 4$. If $|S| \le 2$, we are done immediately by applying minimality to the lobes and patching the orientations together. The $|S| = 3$ case implies that G contains no K_4. So, all we need to do is show there is no triangle cutset. Say $S = \{a, b, c\}$. We apply minimality to each S-lobe of G. For the lobe containing the interior of abc we use abc as the outer face. For the other lobe we use C. Let X be the resulting orientation of the outer lobe. Suppose X does not orient abc cyclically. Then, by symmetry, we may assume that $ab, ac, bc \in E(X)$. For the inner lobe, apply minimality with abc as the other face and using edge bc, let Y be the resulting orientation of the inner lobe minus bc. Then $ab, ac \in E(Y)$. Adding bc back in does not change the Eulerian subgraph counts since c is a sink in Y. But now X and Y give the same orientation to abc, so we can patch them together to get an orientation Q of $G - x_1x_2$. Since all edges in Y incident to a, b, c point into a, b, c, our patching did not create any new directed cycles. So the spanning Eulerian subgraphs of Q are all pairings of spanning Eulerian subgraphs in X and in Y. Since $EE(X) \ne EO(X)$ and $EE(Y) \ne EO(Y)$, we conclude $EE(Q) \ne EO(Q)$. Also, our patching did not change the out-degree of any vertex, so the degree condition is still satisfied.

If X does orient abc cyclically, then we won't be able to match up the two orientations. But, we can change X to another α-orientation (where α is the out-degree sequence of X) that does not orient abc cyclically. Let Z be the induced subgraph of X on $\{a, b, c\}$. Then $EE(Z) = EO(Z)$, so by Lemma 1.8, X has an induced cycle $A \not\subseteq Z$ containing an edge of Z. Then A contains at most two edges of Z, so reversing all the edges on A produces a new α-orientation X' where abc is not oriented cyclically. By Lemma 1.2, we have $0 \ne EE(X) - EO(X) = \pm(EE(X') - EO(X'))$. So we can use X' in place of X in the argument in the previous paragraph to conclude that $G - x_1x_2$ has the desired orientation, a contradiction. □

Lemma 2.3. $|C| \ge 4$.

Proof. The Thomassen-style argument goes through when $|C| = 3$. □

References

[1] Stefan Felsner, *Lattice structures from planar graphs*, Electron. J. Combin **11** (2004), no. 1, R15.