PROPOSAL FOR 2012 BLOCK GRANT: EDGE COLORING VIA TWO-PLAYER GAMES

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1. Introduction

We propose to study the following two-player game which has applications to edge coloring graphs. The game is played on a graph G by *Fixer* and *Breaker*. The game is set up by assigning a list of colors $L(v)$ to each $v \in V(G)$ and choosing a *pot* P with $\bigcup_{v \in V(G)} L(v) \subseteq P$. Fixer always gets the first move and he wins iff before moving on his turn he can construct a proper edge coloring of G, say $\pi\colon E(G) \to P$ such that $\pi(xy) \in L(x) \cap L(y)$ for each $xy \in E(G)$.

Fixer's turn. Pick $\alpha \in P$ and $v \in V(G)$ with $\alpha \notin L(v)$ and set $L(v) := L(v) \cup \{\alpha\} \smallsetminus \{\beta\}$ for some $\beta \in L(v)$.

Breaker's turn. If Fixer modified $L(v)$ by inserting α and removing β, Breaker can pick at most one vertex in $V(G) \smallsetminus \{v\}$ and modify its list by swapping α for β or β for α.

The game setup corresponds to the situation that arises after picking some edges in a (multi)graph, removing them and coloring the rest of the graph. A Fixer move followed by a Breaker move corresponds to switching colors along a two-colored path (see the proof of Vizing's theorem below for an example). Studying this game on its own apart from edge coloring has many advantages just as studying list coloring as a separate concept has advantages over merely viewing it as pre-coloring extension—more induction parameters, indirect minimizations and hypothesis generalizations to name a few.

In [13], we have carried out the analysis of winning strategies for Fixer when G is a fan $K_{1,t}$. The result is a simultaneous generalization of Hall's theorem and Vizing's theorem. Other results such as Vizing's adjacency lemma and generalizations also follow from the analysis. In the next section, we will outline this analysis as an example.

Our proposal is to take these results much further by analyzing the game on more classes of graphs. One goal will be to fold the acceptable paths of Kierstead [4] and the acceptable trees of Tashkinov [6] into this framework. We also hope that this approach will help make progress on the big conjecture of Goldberg [2]. For a graph G, define the *overfull parameter* of G as

$$w(G) := \max_{H \subseteq G} \left\lceil \frac{\|H\|}{\left\lfloor \frac{1}{2} |H| \right\rfloor} \right\rceil.$$

Goldberg's Conjecture. *For any graph G we have $\chi'(G) \leq \max\{\Delta(G) + 1, w(G)\}$.*

2. The game on a fan

As an example, we have completed the analysis of the game on a fan $K_{1,t}$. From this, all of the classical edge coloring results follow. We need a couple of preconditions to make the game tenable. First we always assume $|P| \geq t$ and second we assume that the root r of the $K_{1,t}$ has $L(r) = P$. With these assumptions, the game amounts to doing Hall's theorem with a little extra power. For the remainder of this proposal, let $\mathcal{S}$ be a finite family of finite sets. A *transversal* of $\mathcal{S}$ is an injection $f \colon \mathcal{S} \hookrightarrow \bigcup \mathcal{S}$ such that $f(S) \in S$ for each $S \in \mathcal{S}$. Hall's theorem [3] gives the precise conditions under which $\mathcal{S}$ has a transversal.

Theorem 2.1 (Hall [3])**.** *$\mathcal{S}$ has a transversal iff $\left|\bigcup \mathcal{W}\right| \geq |\mathcal{W}|$ for each $\mathcal{W} \subseteq \mathcal{S}$.*

The precise conditions under which Fixer has a winning strategy will look very similar. To state the main theorem we need a couple pieces of notation. Define the *degree* in $\mathcal{W} \subseteq \mathcal{S}$ of $x \in P$ as

$$d_{\mathcal{W}}(x) := \left|\{S \in \mathcal{W} \mid x \in S\}\right|.$$

Now define the *value* of $\mathcal{W} \subseteq \mathcal{S}$ as

$$\nu(\mathcal{W}) := \sum_{x \in P} \left\lfloor \frac{\max\{0, d_{\mathcal{W}}(x) - 1\}}{2} \right\rfloor.$$

Theorem 2.2. *Fixer has a winning strategy against Breaker iff $\left|\bigcup \mathcal{W}\right| \geq |\mathcal{W}| - \nu(\mathcal{W})$ for each $\mathcal{W} \subseteq \mathcal{S}$.*

Vizing's theorem follows from a special case of Theorem 2.2. In fact, the strategy employed by Fixer is based, in part, on Ehrenfeucht, Faber and Kierstead's proof of Vizing's theorem [1].

Corollary 2.3. *If $\mathcal{S} = \{S_1, \dots, S_k\}$ with $|S_k| \geq 1$ and $|S_i| \geq 2$ for all $i \in [k-1]$, then Fixer has a winning strategy against Breaker.*

Proof. Let $\mathcal{W} \subseteq \mathcal{S}$. Then $\nu(\mathcal{W}) \geq \sum_{x \in \bigcup \mathcal{W}} \frac{d_{\mathcal{W}}(x) - 2}{2} = \frac{1}{2} \sum_{S \in \mathcal{W}} |S| - \left|\bigcup \mathcal{W}\right| \geq \frac{1}{2}(2|\mathcal{W}| - 1) - \left|\bigcup \mathcal{W}\right|$. Hence $\nu(\mathcal{W}) \geq |\mathcal{W}| - \left|\bigcup \mathcal{W}\right|$ as desired. □

Corollary 2.4 (Vizing [7])**.** *Every simple graph satisfies $\chi' \leq \Delta + 1$.*

Proof. Suppose not and let G be a counterexample minimizing $|G|$. Put $\Delta := \Delta(G)$. Pick $v \in V(G)$ with degree Δ, say $v_1, \dots, v_\Delta$ are the neighbors of v in G. By minimality of $|G|$, we have a $(\Delta+1)$-edge-coloring of $G - v$. Let S_i be the colors not incident with v_i in this coloring. Each v_i has degree at most $\Delta - 1$ in $G - v$ and hence $|S_i| \geq 2$. Also, if $a \in S_i$ and $b \notin S_i$ we may exchange colors on a maximum length path starting at v_i and alternating between colors b and a. This gives a Fixer move followed by a Breaker move. Apply Corollary 2.3 to get a transversal of the S_i. Now we may complete the $(\Delta+1)$-edge-coloring to all of G by using the corresponding element of the transversal on vv_i for each $i \in [\Delta]$. □

To deal with edge coloring multigraphs, we need to generalize our game slightly. Instead of looking for a transversal, we will look for a system of disjoint representatives. For $\eta \colon \mathcal{S} \to \mathbb{N}^+$ an *η-transversal* of $\mathcal{S}$ is a function $f \colon \mathcal{S} \to \mathcal{P}\left(\bigcup \mathcal{S}\right)$ such that $f(S) \subseteq S$, $|f(S)| = \eta(S)$ for

$S \in \mathcal{S}$ and $f(A) \cap f(B) = \emptyset$ for different $A, B \in \mathcal{S}$. By making $\eta(S)$ copies of each $S \in \mathcal{S}$ and applying Hall's theorem we get the following.

Theorem 2.5. *$\mathcal{S}$ has an η-transversal iff $|\bigcup \mathcal{W}| \geq \sum_{W \in \mathcal{W}} \eta(W)$ for each $\mathcal{W} \subseteq \mathcal{S}$.*

Call the game where Fixer wins iff he creates an η-transversal *the η-game*. Using the above simple generalization of Hall's theorem in the proof of Theorem 2.2 gives the following.

Theorem 2.6. *Fixer has a winning strategy against Breaker in the η-game iff $|\bigcup \mathcal{W}| \geq \sum_{W \in \mathcal{W}} \eta(W) - \nu(\mathcal{W})$ for each $\mathcal{W} \subseteq \mathcal{S}$.*

With this theorem in hand, many edge coloring results such as Vizing's adjacency lemma and generalizations follow.

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