

most low Alon-Tarsi notes

August 22, 2015

1 Introduction

We consider graphs with vertices labeled by natural numbers; that is, pairs (G, h) where G is a graph and $h: V(G) \rightarrow \mathbb{N}$. We say that (G, h) is AT if G is $(d_G - h)$ -AT. When H is an induced subgraph of G , we simplify notation by referring to the pair (H, h) where we really mean $(H, h|_{V(H)})$.

2 Subgraphs, subdivisions and cuts

Definition 1. A graph G is *h -minimal* if G is connected and (H, h) is not AT for every proper induced subgraph H of G . A graph G is *h -greedy-minimal* if G is connected and (H, h) is not AT for every proper induced subgraph H of G where $h(v) = 0$ for all $v \in V(G) \setminus V(H)$. Note that if G is h -minimal then it is also h -greedy-minimal.

Lemma 2.1. *If G is connected and (G, h) is not AT, then G is h -greedy-minimal.*

Proof. If there were a proper induced subgraph H such that $(H, h|_{V(H)})$ is AT, then by ordering the vertices of each component of $G - V(H)$ by increasing distance to H and directing all edges away from H in this order we conclude that (G, h) is AT. \square

Lemma 2.2. *If (G', h') is formed from (G, h) by subdividing an edge e of G twice and having h' give zero on the two new vertices, then*

1. *if (G, h) is AT, then (G', h') is AT; and*
2. *if (G', h') is AT, then either (G, h) is AT or $(G - e, h)$ is AT.*

Proof. Suppose $e = xy$ and call the new vertices x' and y' so that G' contains the induced path $xx'y'y$. For (1), let D be an orientation of G showing that (G, h) is AT. By symmetry we may assume $xy \in E(D)$. Make an orientation D' of G' from D by replacing xy with the directed path $xx'y'y$. There is a natural parity preserving bijection between the spanning Eulerian subgraphs of D and D' , so we conclude that (G', h') is AT.

For (2), let D' be an orientation of G' showing that (G', h') is AT. Suppose G' contains the directed path $xx'y'y$ or the directed path $yy'x'x$. By symmetry, we can assume it is $xx'y'y$. Then make an orientation D of G by replacing $xx'y'y$ with the directed edge xy . As

above, we have a parity preserving bijection between the spanning Eulerian subgraphs of D and D' , so we conclude that (G, h) is AT. Otherwise, no spanning Eulerian subgraph of D' contains a cycle passing through x' and y' . So, the spanning Eulerian subgraph counts of D' are the same as those of $D' - x' - y'$. But this gives an orientation of $G - e$ showing that $(G - e, h)$ is AT. \square

Lemma 2.3. *Let $\{A_1, A_2\}$ be a separation of G such that $A_1 \cap A_2 = \{x\}$. If $G[A_i]$ is f_i -AT for $i \in [2]$, then G is f -AT where $f(v) = f_i(v)$ for $v \in V(A_i - x)$ and $f(x) = f_1(x) + f_2(x) - 1$. Going the other direction, if G is f -AT, then $G[A_i]$ is f_i -AT for $i \in [2]$ where $f_i(v) = f(v)$ for $v \in V(A_i - x)$ and $f_1(x) + f_2(x) \leq f(x) + 1$.*

Proof. For $i \in [2]$, choose an orientation D_i of A_i showing that A_i is f_i -AT. Together these give an orientation D of G and since no cycle has vertices in both $A_1 - x$ and $A_2 - x$, we have

$$\begin{aligned} EE(D) - EO(D) &= EE(D_1)EE(D_2) + EO(D_1)EO(D_2) - (EE(D_1)EO(D_2) + EO(D_1)EE(D_2)) \\ &= (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2)) \\ &\neq 0. \end{aligned}$$

Hence G is f -AT.

Now, suppose G is f -AT and choose an orientation D of G showing this. Put $D_i = D[A_i]$ for $i \in [2]$. Then, as above, we have $0 \neq EE(D) - EO(D) = (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2))$ and hence $EE(D_1) - EO(D_1) \neq 0$ and $EE(D_2) - EO(D_2) \neq 0$. Since the in-degree of x in D is the sum of the in-degree of x in D_1 and the in-degree of x in D_2 , the lemma follows. \square

Corollary 2.4. *Let G be an h -greedy-minimal graph. If (G, h) is AT and G has an induced path $x_1x_2x_3x_4$ such that $d_G(x_2) = d_G(x_3) = 2$ and $h(x_2) = h(x_3) = 0$, then*

$$((G - x_2 - x_3) + x_1x_4, h|_{V(G) \setminus \{x_2, x_3\}}) \text{ is AT.}$$

Proof. Suppose (G, h) is AT and G has such an induced path $x_1x_2x_3x_4$. Applying Lemma 2.2 part (2) shows that either $(G - x_2 - x_3, h|_{V(G) \setminus \{x_2, x_3\}})$ is AT or $((G - x_2 - x_3) + x_1x_4, h|_{V(G) \setminus \{x_2, x_3\}})$ is AT. But $G - x_2 - x_3$ is a proper induced subgraph of G , so the former cannot happen since G is h -greedy-minimal and $h(x_2) = h(x_3) = 0$. Hence $((G - x_2 - x_3) + x_1x_4, h|_{V(G) \setminus \{x_2, x_3\}})$ is AT. \square

3 Extension lemma

This is a key lemma from [1], it generalizes a lemma from [2] from list coloring to Alon-Tarsi orientations. This is what I talked about in Baltimore. The basic idea is that in some cases we can pair off odd/even spanning Eulerian subgraphs via a parity reversing bijection.

Lemma 3.1. *Let G be a multigraph without loops and $f: V(G) \rightarrow \mathbb{N}$. If there are $F \subseteq G$ and $Y \subseteq V(G)$ such that:*

1. *any multiple edges in G are contained in $G[Y]$; and*

2. $f(v) \geq d_G(v)$ for all $v \in V(G) \setminus Y$; and
3. $f(v) \geq d_{G[Y]}(v) + d_F(v) + 1$ for all $v \in Y$; and
4. For each component T of $G - Y$ there are different $x_1, x_2 \in V(T)$ where $N_T[x_1] = N_T[x_2]$ and $T - \{x_1, x_2\}$ is connected such that either:
 - (a) there are $x_1y_1, x_2y_2 \in E(F)$ where $y_1 \neq y_2$ and $N(x_i) \cap Y = \{y_i\}$ for $i \in [2]$; or
 - (b) $|N(x_2) \cap Y| = 0$ and there is $x_1y_1 \in E(F)$ where $N(x_1) \cap Y = \{y_1\}$,

then G is f -AT.

Proof. Suppose not and pick a counterexample (G, f, F, Y) minimizing $|G - Y|$. If $|G - Y| = 0$, then $Y = V(G)$ and thus $f(v) \geq d_G(v) + 1$ for all $v \in V(G)$ by (3). Pick an acyclic orientation D of G . Then $EE(D) = 1$, $EO(D) = 0$ and $d_D^+(v) \leq d_G(v) \leq f(v) - 1$ for all $v \in V(D)$. Hence G is f -AT. So, we must have $|G - Y| > 0$.

Pick a component T of $G - Y$ and pick $x_1, x_2 \in V(T)$ as guaranteed by (4). First, suppose (4a) holds. Put $G' := (G - T) + y_1y_2$, $F' := F - T$, $Y' := Y$ and let f' be f restricted to $V(G')$. Then G' has an orientation D' where $f'(v) \geq d_{D'}^+(v) + 1$ for all $v \in V(D')$ and $EE(D') \neq EO(D')$, for otherwise (G', f', F', Y') would contradict minimality. By symmetry we may assume that the new edge y_1y_2 is directed toward y_2 . Now we use the orientation of D' to construct the desired orientation of D . First, we use the orientation on $D' - y_1y_2$ on $G - T$. Now, order the vertices of T as $x_1, x_2, z_1, z_2, \dots$ so that every vertex has at least one neighbor to the right. Orient the edges of T left-to-right in this ordering. Finally, we use y_1x_1 and x_2y_2 and orient all other edges between T and $G - T$ away from T . Plainly, $f(v) \geq d_D^+(v) + 1$ for all $v \in V(D)$. Since y_1x_1 is the only edge of D going into T , any Eulerian subgraph of D that contains a vertex of T must contain y_1x_1 . So, any Eulerian subgraph of D either contains (i) neither y_1x_1 nor x_2y_2 , (ii) both y_1x_1 and x_2y_2 , or (iii) y_1x_1 but not x_2y_2 . We first handle (i) and (ii) together. Consider the function h that maps an Eulerian subgraph Q of D' to an Eulerian subgraph $h(Q)$ of D as follows. If Q does not contain y_1y_2 , let $h(Q) = \iota(Q)$ where $\iota(Q)$ is the natural embedding of $D' - y_1y_2$ in D . Otherwise, let $h(Q) = \iota(Q - y_1y_2) + \{y_1x_1, x_1x_2, x_2y_2\}$. Then h is a parity-preserving injection with image precisely the union of those Eulerian subgraphs of D in (i) and (ii). Hence if we can show that exactly half of the Eulerian subgraphs of D in (iii) are even, we will conclude $EE(D) \neq EO(D)$, a contradiction. To do so, consider an Eulerian subgraph A of D containing y_1x_1 and not x_2y_2 . Since x_1 must have in-degree 1 in A , it must also have out-degree 1 in A . We show that A has a mate A' of opposite parity. Suppose $x_2 \notin A$ and $x_1z_1 \in A$; then we make A' by removing x_1z_1 from A and adding $x_1x_2z_1$. If $x_2 \in A$ and $x_1x_2z_1 \in A$, we make A' by removing $x_1x_2z_1$ and adding x_1z_1 . Hence exactly half of the Eulerian subgraphs of D in (iii) are even and we conclude $EE(D) \neq EO(D)$, a contradiction.

Now suppose (4b) holds. Put $G' := G - T$, $F' := F - T$, $Y' := Y$ and define f' by $f'(v) = f(v)$ for all $v \in V(G' - y_1)$ and $f'(y_1) = f(y_1) - 1$. Then G' has an orientation D' where $f'(v) \geq d_{D'}^+(v) + 1$ for all $v \in V(D')$ and $EE(D') \neq EO(D')$, for otherwise (G', f', F', Y') would contradict minimality. We orient $G - T$ according to D , orient T as in the previous case, again use y_1x_1 and orient all other edges between T and $G - T$ away from T . Since we decreased $f'(y_1)$ by 1, the extra out edge of y_1 is accounted for and we have

$f(v) \geq d_D^+(v) + 1$ for all $v \in V(D)$. Again any additional Eulerian subgraph must contain y_1x_1 and since x_2 has no neighbor in $G - T$ we can use x_2 as before to build a mate of opposite parity for any additional Eulerian subgraph. Hence $EE(D) \neq EO(D)$ giving our final contradiction. \square

4 Degree-AT graphs

A graph G is called *degree-AT* if (G, h) is AT where h is the constant zero function.

Lemma 4.1. *A connected graph G is degree-AT if it is not a Gallai tree.*

Proof. Suppose there exists a connected graph that is not a Gallai tree, but is also not degree-AT. Let G be such a graph with as few vertices as possible. Since G is not degree-AT, no induced subgraph H of G is degree-AT by Lemma 2.1. Hence, for any $v \in V(G)$ that is not a cutvertex, $G - v$ must be a Gallai tree by minimality of $|G|$.

If G has more than one block, then for endblocks B_1 and B_2 , choose noncutvertices $w \in B_1$ and $x \in B_2$. By the minimality of $|G|$, both $G - w$ and $G - x$ are Gallai trees. Since every block of G appears either as a block of $G - w$ or as a block of $G - x$, every block of G is either complete or an odd cycle. Hence, G is a Gallai tree, a contradiction. So instead G has only one block, that is, G is 2-connected. Further, $G - v$ is a Gallai tree for all $v \in V(G)$.

Let v be a vertex of minimum degree in G . Since G is 2-connected, $d_G(v) \geq 2$ and v is adjacent to a noncutvertex in every endblock of $G - v$. If $G - v$ has a complete block B with noncutvertices x_1, x_2 where $v \leftrightarrow x_1$ and $v \not\leftrightarrow x_2$, then we can apply Lemma 3.1 with $Y = \{v\}$ and $F = vx_1$ to conclude that G is degree-AT, a contradiction. So, v must be adjacent to every noncutvertex in every complete endblock of $G - v$.

Suppose $d_G(v) \geq 3$. Then no endblock of $G - v$ can be an odd cycle of length at least 5 (there would be vertices of degree 3 but we'd have $d_G(v) \geq 4$). Let B be a smallest complete endblock of $G - v$. Then for a noncutvertex $x \in V(B)$, we have $d_G(x) = |B|$ and hence $d_G(v) \leq |B|$. If $G - v$ has at least two endblocks, then $2(|B| - 1) \leq |B|$ and hence $d_G(v) \leq |B| = 2$, a contradiction. Hence $G - v = B$ and v is joined to B , so G is complete, a contradiction.

Hence, we must have $d_G(v) = 2$. Suppose $G - v$ has at least 2 endblocks. Then, it has exactly 2 and v is adjacent to one noncutvertex in each. Neither of the endblocks can be odd cycles of length at least 5 since then we could get a smaller counterexample by Lemma 2.2. Since v is adjacent to every noncutvertex in every complete endblock of $G - v$, both endblocks must be K_2 . But then either $G = C_4$ (which is trivially degree-AT) or we can get a smaller counterexample by Lemma 2.2. So, $G - v$ must be 2-connected. Since $G - v$ is a Gallai tree, it is either complete or an odd cycle. If $G - v$ is not complete, we can get a smaller counterexample by Lemma 2.2. So, $G - v$ is complete and v is adjacent to every noncutvertex of $G - v$; that is, G is complete, a contradiction. \square

5 When h is 1 for at most one vertex

For a graph G and $x \in V(G)$ let $h_x: V(G) \rightarrow \mathbb{N}$ be defined by $h_x(x) = 1$ and $h_x(v) = 0$ for all $v \in V(G - x)$. We classify the connected h_x -minimal graphs G such that (G, h_x) is AT

for some $x \in V(G)$.

To start we will reduce to the case when G is 2-connected.

Lemma 5.1. *Let G be h_x -minimal for $x \in V(G)$ and let \mathcal{B} be the set of blocks of G containing x . Then (G, h_x) is AT if and only if*

1. \mathcal{B} contains at least two degree-AT graphs; or
2. G is 2-connected and (G, h_x) is AT.

Proof. Since G is h_x -minimal, no block outside of \mathcal{B} is degree-AT. The lemma follows since if G is not 2-connected, then (G, h_x) is AT if and only if (1) holds by Lemma 2.3. \square

Lemma 5.2. *If G is a connected graph and $x \in V(G)$ with $d_G(x) = 2$, then (G, h_x) is AT if and only if $G - x$ is degree-AT.*

Proof. Let D be an orientation of G showing that (G, h_x) is AT. Then $d_D^-(x) = 2$ and hence no spanning Eulerian subgraph contains a cycle passing through x . Therefore, the Eulerian subgraph counts in $G - x$ are different and $G - x$ is degree-AT. The other direction is immediate from Lemma 2.1. \square

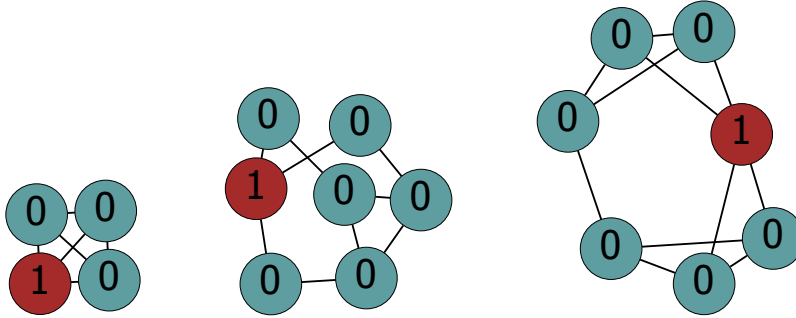


Figure 1: The seed blocks.

Lemma 2.2 part (2) suggests a way to construct G such that (G, h) is not AT from smaller graphs. Specifically, we have the following.

Corollary 5.3. *If e is an edge in G such that (G, h) is not AT and $(G - e, h)$ is not AT, then (G', h') is not AT where (G', h') is formed from (G, h) by subdividing e twice and having h' give zero on the two new vertices.*

Let \mathcal{D} be the smallest collection of pairs (G, h) containing the pairs in Figure 1 that is closed under the operation in Corollary 5.3.

For a connected graph G and endblock B of G , let x_B be the cutvertex of G contained in B .

Lemma 5.4. *Let G be a connected graph and $v \in V(G)$ a cutvertex of G . If $G - v$ has t components, then there are endblocks B_1, \dots, B_t and an induced subdivision of $K_{1,t}$ where the root is v and the leaves are x_{B_1}, \dots, x_{B_t} .*

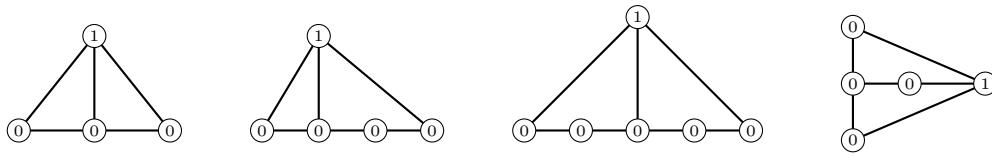


Figure 2: These are AT.

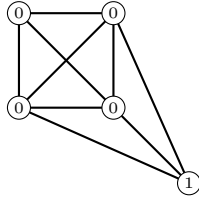


Figure 3: This is AT.

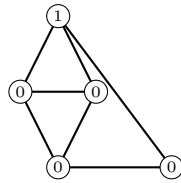


Figure 4: This is AT.

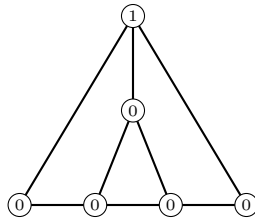


Figure 5: This is AT.

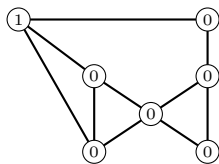


Figure 6: This is AT.

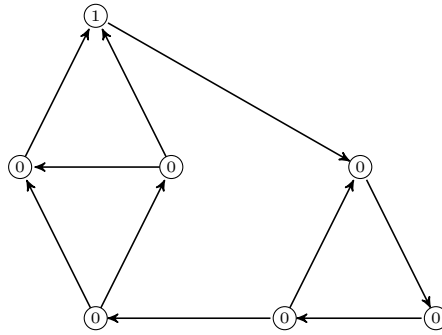


Figure 7: This is AT.

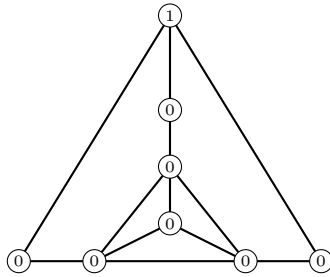


Figure 8: This is AT.

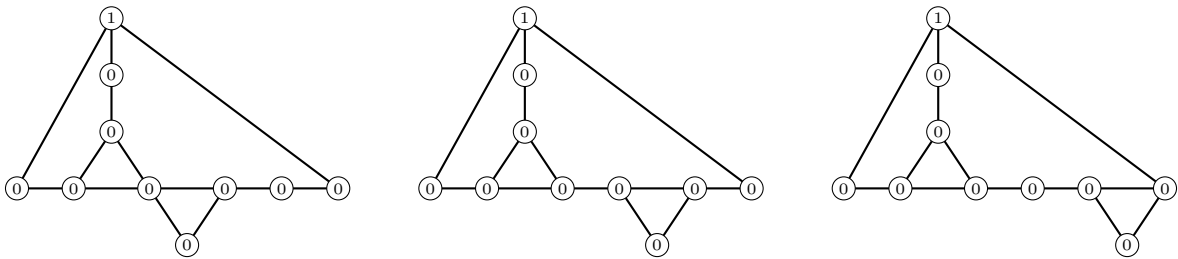


Figure 9: These are AT.

Proof. Pick endblocks B_1, \dots, B_t , one in each component of $G - v$. Now the desired induced subdivision of $K_{1,t}$ is the union of shortest paths from x_{B_1} to x_{B_i} for $2 \leq i \leq t$. \square

Lemma 5.4 will be really useful in applying the following lemma. Note that we can always extend the induced subdivision of $K_{1,3}$ or induced path we get one vertex into each endblock.

Lemma 5.5. *Let G be h_x -minimal for $x \in V(G)$ with $d_G(x) \geq 3$. If (G, h_x) is not AT, then every induced subdivision of $K_{1,3}$ in G contains at most two vertices in $N(x)$. In particular, every induced path in G contains at most two vertices in $N(x)$.*

Proof. This is immediate from Lemma 2.2 and the graphs in Figure 2. \square

Lemma 5.6. *Let G be h_x -minimal for $x \in V(G)$. If G is 2-connected, then (G, h_x) is AT if and only if*

1. $d_G(x) \geq 3$; and
2. G is not complete and not an odd cycle; and
3. $(G, h_x) \notin \mathcal{D}$.

Proof. Suppose the lemma is false and choose a counterexample G minimizing $|G|$. If $d_G(x) \leq 2$, then (G, h_x) is not AT by Lemma 5.2 since G is h_x -minimal. So, we must have $d_G(x) \geq 3$. Since (G, h_x) is not AT if $(G, h_x) \in \mathcal{D}$ by construction, it must be that $(G, h_x) \notin \mathcal{D}$ and (G, h_x) is not AT.

Claim 0. $G - x$ is a Gallai tree and x is adjacent to a noncutvertex in every endblock of $G - x$. This follows since G is h_x -minimal and 2-connected.

Claim 1. $G - x - v$ has at most two components for any $v \in V(G - x)$. Suppose $G - x$ has a cutvertex v such that $G - x - v$ has at least three components. Then, by Lemma 5.4, $G - x$ contains an induced $K_{1,3}$ violating Lemma 5.5.

Claim 2. x is not adjacent to any cutvertex v of $G - x$. Using Lemma 5.4, we get an induced path from x_B to x_D containing v , where B and D are different endblocks violating Lemma 5.5.

Claim 3. $G - x$ does not contain an induced path $v_1 v_2 v_3 v_4$ such that $d_G(v_2) = d_G(v_3) = 2$. If it did, then we could get a smaller counterexample by applying Lemma 2.2 part (1).

Claim 4. Every block of $G - x$ is complete. Suppose $G - x$ has a block B that is an odd cycle $v_1 v_2 \dots v_t v_1$ with $t \geq 5$.

Subclaim 4a. B contains at most two cutvertices of $G - x$. Otherwise there are $a, b, c \in [t]$ such that $v_a, \dots, v_b, \dots, v_c$ contains exactly three cutvertices v_a, v_b and v_c . Apply Lemma 5.4 to the component of $G - \{x, v_1, \dots, v_{a-1}, v_{b+1}, \dots, v_t\}$ containing v_a, v_b, v_c with $v = v_b$ to get an induced $K_{1,3}$ violating Lemma 5.5.

Subclaim 4b. B contains at most one cutvertex of $G - x$. Otherwise, by Subclaim 4a, B has exactly two cutvertices v_a and v_b . By Claim 3, x is adjacent to a noncutvertex $v \in V(B)$. Consider the induced path given by applying Lemma 5.4 to v_a . If this path does not contain v , then have it go the other way around B . Now we have an induced path violating Lemma 5.5.

Subclaim 4c. Claim 4 is true. By Claim 3, x must be adjacent to at least every other noncutvertex of B . So, if $G - x = B$, we immediately violate Lemma 5.5. If instead, $G - x$

has another endblock B' then we can pick two neighbors of x in B and one neighbor of x in B' all on an induced path in $G - x$, violating Lemma 5.5.

Claim 5. *If x is adjacent to a noncutvertex in a block, then x is adjacent to all noncutvertices in that block. In particular, x is adjacent to every noncutvertex in every endblock of $G - x$.* Suppose $G - x$ has a block B with noncutvertices v_1, v_2 where $x \leftrightarrow v_1$ and $x \nleftrightarrow v_2$. By Claim 4, B is complete, so we can apply Lemma 3.1 with $Y = \{x\}$ and $F = xv_1$ to conclude that (G, h_x) is AT, a contradiction.

Claim 6. *$G - x$ has at least two endblocks.* If not, then $G - x$ is complete by Claim 0 and Claim 4. But then G is complete by Claim 5, a contradiction.

Claim 7. *The endblocks of $G - x$ are all K_2 , except possibly one K_3 .* By Claim 4, every endblock is complete. Suppose $G - x$ has an endblock $B = K_t$ for $t \geq 4$. Then by Claim 2, Claim 5 and Claim 6, G has an induced Figure 3, impossible. So every endblock of $G - x$ is K_2 or K_3 . Suppose $G - x$ has two K_3 endblocks B_1 and B_2 . Then $G[\{x\} \cup V(B_i)]$ is degree-AT for $i \in [2]$. If there is no edge between B_1 and B_2 , then, by Lemma 5.1, G contains an induced subgraph H such that (H, h_x) is AT, a contradiction. If there is an edge between B_1 and B_2 , then by Claim 1, G is the rightmost graph in Figure 1, a contradiction.

Claim 8. *Every noncutvertex of $G - x$ is adjacent to x .*

Suppose $G - x$ has a noncutvertex v with $v \nleftrightarrow x$. Then $G - v$ is 2-connected and h_x -minimal, so by minimality of $|G|$, we conclude that $d_{G-v}(x) \leq 2$, $G - v$ is complete or an odd cycle, or $(G - v, h_x) \in \mathcal{D}$. The first three clearly cannot occur, so we have $(G - v, h_x) \in \mathcal{D}$.

Subclaim 8a. *$G - v$ has an induced path $v_1v_2v_3v_4$ such that $d_G(v_2) = d_G(v_3) = 2$.*

Otherwise, $G - v$ is one of the graphs in Figure 1. But $G - v$ cannot be the leftmost, middle, or rightmost graph in Figure 1 because then G would contain the graph in Figure 3, Figure 8, and Figure 5 as an induced subgraph, respectively.

Subclaim 8b. *The block B containing v is K_3 .*

By Claim 4, B is complete. Some neighbor w of v must have gone from degree 3 to degree 2. Since v is only adjacent to vertices in B , the only way for this to happen is if $B = K_3$.

Subclaim 8c. *$G - v$ is the result of applying the operation in Corollary 5.3 to a graph F in Figure 1 one time.* None of the graphs in Figure 1 have an induced path $v_1v_2v_3v_4$ such that $d_G(v_2) = d_G(v_3) = 2$.

Subclaim 8d. *F is not the rightmost graph in Figure 1.* For this one, removing any edge leaves an AT graph, so Corollary 5.3 cannot be applied.

Subclaim 8e. *F is not the middle graph in Figure 1.* For this one, removing any edge in the triangle leaves an AT graph, so Corollary 5.3 cannot be applied to those edges. But then G is one of the graphs in Figure 9, impossible.

Subclaim 8f. *Claim 8 is true.* By the previous subclaims, F must be the leftmost graph in Figure 1. For this one, removing any of the edges not incident to the vertex labeled 1 leaves an AT graph, so Corollary 5.3 cannot be applied to those edges. But then G contains an induced Figure 4 or Figure 6, impossible.

Claim 9. *Every internal block of $G - x$ consists entirely of cutvertices.* Suppose otherwise that we have an internal block B of $G - x$ containing a noncutvertex v . By Claim 8, $x \leftrightarrow v$. Note that by Lemma 2.2 and Figure 4, we get that Figure 5 is AT with either or both of the bottom left and bottom right edge subdivided once. But G contains at least one of these with edges subdivided twice some number of times as an induced subgraph, a contradiction.

Claim 10. *x 's neighbors are precisely the noncutvertices in the endblocks of $G - x$.* By Claim 5, x is adjacent to all these vertices. By Claim 9 and Claim 2, x is not adjacent to any other vertex.

Claim 11. *$G - x$ has at least three endblocks.* If not, then by Claim 6, $G - x$ has exactly two endblocks B_1 and B_2 . Since $d_G(x) \geq 3$, Claim 7, Claim 9 and Claim 10 show that $G - x$ is a triangle $w_1w_2w_3$ with a path $w_1y_1y_2 \dots y_t$ emanating from w_1 . Since (G, h_x) is not AT, Lemma 2.2 and Figure 4 show that t must be even. But then $(G, h_x) \in \mathcal{D}$ since $G - x$ is formed from the leftmost graph in Figure 1 by applying Corollary 5.3 some number of times, a contradiction.

Claim 12. *Every endblock of $G - x$ is K_2 .* Suppose $G - x$ has a K_3 endblock B .

Subclaim 12a. *The component of $G - N(x_B)$ containing x is not degree-AT.* Since G is h_x -minimal, this follows by Lemma 5.1.

Subclaim 12b. *The component of $G - N(x_B)$ containing x is triangle-free.* If not, then by Claim 11, $G - N(x_B)$ contains an induced subgraph containing x that is a cycle $y_1 \dots y_t y_1$ plus the edge $y_1 y_3$ with $t \geq 4$. If t is even, then $G - N(x_B)$ contains an induced even cycle with at most one chord which is degree-AT, otherwise $G - N(x_B)$ contains the induced even cycle $y_1 y_3 \dots y_t y_1$ which is also degree-AT; this contradicts Subclaim 12a.

Subclaim 12c. *The other block D containing x_B is K_2 .* If not, then G must have either an induced even subdivision of Figure 6 or an induced even subdivision of Figure 4 (both path length parities are covered).

Subclaim 12d. *Claim 12 is true.* By Subclaim 12b and 12c, $G - x$ has exactly one K_3 internal component, call it Q , and the rest of the internal components are K_2 . Moreover, Q intersects D and in particular, a shortest path from Q to x passing through B has length three. By Claim 1 and Claim 11, $G - x$ has exactly three endblocks $B_1 = B, B_2$ and B_3 . For $i \in [3]$, let ℓ_i be the length of the shortest path from Q to x passing through B_i . We know $\ell_1 = 3$. If both ℓ_2 and ℓ_3 are even, then G would have an induced even subdivision of Figure 5, a contradiction. So, at least one of ℓ_2, ℓ_3 are odd and hence G contains an induced even subdivision of Figure 7, a contradiction.

Claim 13. *$G - x$ has at least four endblocks.* If not, then by Claim 11, $G - x$ has exactly three endblocks B_1, B_2 and B_3 . By Claim 1 and Claim 10, $G - x$ has exactly one K_3 internal block and the rest of the internal blocks must be K_2 .

□

References

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