Destroying non-complete regular components in graph partitions

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July 11, 2011

Abstract

We prove that if G is a graph and $r_1, \ldots, r_k \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k$ then $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ and $G[V_i]$ contains no non-complete r_i-regular components for each $1 \leq i \leq k$. In particular, the vertex set of any graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ sets, each of which induces a disjoint union of triangles and paths.

1 Introduction

In [5] Kostochka modified an algorithm of Catlin to show that every triangle-free graph G can be colored with at most $\frac{2}{3}(\Delta(G) + 3)$ colors. In fact, his modification proves that the vertex set of any triangle-free graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ sets, each of which induces a disjoint union of paths. We generalize this as follows.

Main Lemma. *Let G be a graph and $r_1, \ldots, r_k \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k$. Then $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ and $G[V_i]$ contains no non-complete r_i-regular components for each $1 \leq i \leq k$.*

Setting $k = \left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ and $r_i = 2$ for each i gives a slightly more general form of Kostochka's theorem.

Corollary 1. *The vertex set of any graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ sets, each of which induces a disjoint union of triangles and paths.*

For coloring, this actually gives the bound $\chi(G) \leq 2 \left\lceil \frac{\Delta(G)+2}{3} \right\rceil$ for triangle free graphs. To get $\frac{2}{3}(\Delta(G)+3)$, just use $r_k = 0$ when $\Delta \equiv 2 (\mathrm{mod}\ 3)$. Similarly, for any $r \geq 2$, setting $k = \left\lceil \frac{\Delta(G)+2}{r+1} \right\rceil$ and $r_i = r$ for each i gives the following.

Corollary 2. *Fix $r \geq 2$. The vertex set of any K_{r+1}-free graph G can be partitioned into $\left\lceil \frac{\Delta(G)+2}{r+1} \right\rceil$ sets each inducing an $(r-1)$-degenerate subgraph with maximum degree at most r.*

For the purposes of coloring it is more economical to split off $\Delta + 2 - (r+1) \left\lfloor \frac{\Delta+2}{r+1} \right\rfloor$ parts with $r_j = 0$.

Corollary 3. *Fix $r \geq 2$. The vertex set of any K_{r+1}-free graph G can be partitioned into $\left\lfloor \frac{\Delta(G)+2}{r+1} \right\rfloor$ sets each inducing an $(r-1)$-degenerate subgraph with maximum degree at most r and $\Delta(G)+2-(r+1)\left\lfloor \frac{\Delta(G)+2}{r+1} \right\rfloor$ independent sets. In particular, $\chi(G) \leq \Delta(G) + 2 - \left\lfloor \frac{\Delta(G)+2}{r+1} \right\rfloor$.*

For $r \geq 3$, the bound on the chromatic number is only interesting in that its proof does not rely on Brooks' Theorem. In [7] Lovász proved a decomposition lemma of the same form as the Main Lemma. The Main Lemma gives a more restrictive partition at the cost of replacing $\Delta(G) + 1$ with $\Delta(G) + 2$.

Lovász's Decomposition Lemma. *Let G be a graph and $r_1, \ldots, r_k \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G) + 1 - k$. Then $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ for each $1 \leq i \leq k$.*

For $r \geq 3$, combining this with Brooks' Theorem gives the following better bound for a K_{r+1}-free graph G (first proved in [1], [3] and [6]):

$$\chi(G) \leq \Delta(G) + 1 - \left\lfloor \frac{\Delta(G)+1}{r+1} \right\rfloor .$$

2 The proofs

Instead of proving directly that we can destroy all non-complete r-regular components in the partition, we prove the theorem for the more general class of what we call r-permissible graphs and show that non-complete r-regular graphs are r-permissible.

Definition 1. For a graph G and $r \geq 0$, let G^r be the subgraph of G induced on the vertices of degree r in G.

Definition 2. Fix $r \geq 2$. A collection T of graphs is *r-permissible* if it satisfies all of the following conditions.

1. Every $G \in T$ is connected.
2. $\Delta(G) = r$ for each $G \in T$.
3. $\delta(G^r) > 0$ for each $G \in T$.
4. If $G \in T$ and $x \in V(G^r)$, then $G - x \notin T$.
5. If $G \in T$ and $x \in V(G^r)$, then there exists $y \in V(G^r) - (\{x\} \cup N_G(x))$ such that $G - y$ is connected.
6. Let $G \in T$ and $x \in V(G^r)$. Put $H := G - x$. Let $A \subseteq V(H)$ with $|A| = r$. Let y be some new vertex and form H_A by joining y to A in H; that is, $V(H_A) := V(H) \cup \{y\}$ and $E(H_A) := E(H) \cup \{xy \mid x \in A\}$. If $H_A \in T$, then $A \cap N_G(x) \cap V(G^r) \neq \emptyset$.

For $r = 0, 1$ the empty set is the only r-permissible collection.

Lemma 4. *Fix $r \geq 2$ and let T be the collection of all non-complete connected r-regular graphs. Then T is r-permissible.*

Proof. Let $G \in T$. We have $G^r = G$ and (1), (2), (3) and (4) are clearly satisfied. That (6) holds is immediate from regularity. It remains to check (5). Let $x \in V(G)$. First, suppose G is 2-connected. If (5) did not hold, then x would need to be adjacent to every other vertex in G. But then $|G| \leq \Delta(G) + 1 = r + 1$ and hence $G = K_r$ violating our assumption. Otherwise G has at least two end blocks and so we can pick some y in an end block not containing x such that $G - y$ is connected. Hence (5) holds. Therefore T is r-permissible. $\square$

Now to prove the Main Lemma we just need to prove the following result. For a graph G, $x \in V(G)$ and $D \subseteq V(G)$ we use the notation $N_D(x) := N(x) \cap D$ and $d_D(x) := |N_D(x)|$.

Lemma 5. *Let G be a graph and $r_1, \ldots, r_k \in \mathbb{Z}_{\geq 0}$ such that $\sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k$. If T_i is an r_i-permissible collection for each $1 \leq i \leq k$, then $V(G)$ can be partitioned into sets $V_1, \ldots, V_k$ such that $\Delta(G[V_i]) \leq r_i$ and $G[V_i]$ contains no element of T_i as a component for each $1 \leq i \leq k$.*

Proof. For a graph H, let $c(H)$ be the number of components in H and let $p_i(H)$ be the number of components of H that are members of T_i. For a partition $P := (V_1, \ldots, V_k)$ of $V(G)$ let

$$f(P) := \sum_{i=1}^{k} \left(|E(G[V_i])| - r_i|V_i|\right),$$

$$c(P) := \sum_{i=1}^{k} c(G[V_i]),$$

$$p(P) := \sum_{i=1}^{k} p_i(G[V_i]).$$

Let $P := (V_1, \ldots, V_k)$ be a partition of $V(G)$ minimizing $f(P)$, and subject to that $c(P)$, and subject to that $p(P)$.

Let $1 \leq i \leq k$ and $x \in V_i$ with $d_{V_i}(x) \geq r_i$. Since $\sum_{i=1}^{k} r_i \geq \Delta(G) + 2 - k$ there is some $j \neq i$ such that $d_{V_j}(x) \leq r_j$. Moving x from V_i to V_j gives a new partition P^* with $f(P^*) \leq f(P)$. Note that if $d_{V_i}(x) > r_i$ we would have $f(P^*) < f(P)$ contradicting the minimality of P. This proves that $\Delta(G[V_i]) \leq r_i$ for each $1 \leq i \leq k$.

Now suppose that for some i_1 there is $A_1 \in T_{i_1}$ which is a component of $G[V_{i_1}]$. Plainly, we may assume that $r_{i_1} \geq 2$. Put $P_1 := P$ and $V_{1,i} := V_i$ for $1 \leq i \leq k$. Take $x_1 \in V(A_1^{r_{i_1}})$ such that $A_1 - x_1$ is connected (this exists by condition (5) of r-permissibility). By the above we have $i_2 \neq i_1$ such that moving x_1 from V_{1,i_1} to V_{1,i_2} gives a new partition $P_2 := (V_{2,1}, V_{2,2}, \ldots, V_{2,k})$ such that $f(P_2) = f(P_1)$. By the minimality of $c(P_1)$, x_1 is adjacent to only one component C_2 in $G[V_{2,i_2}]$. Let $A_2 := G[V(C_2) \cup \{x_1\}]$. Since (by condition (4)) we destroyed a T_{i_1} component when we moved x_1 out of V_{1,i_1}, by the minimality of $p(P_1)$, it must be that $A_2 \in T_{i_2}$. Now pick $x_2 \in A_2^{r_{i_2}}$ not adjacent to x_1 such that $A_2 - x_2$ is connected (again by condition (5)). Continue on this way to construct sequences $i_1, i_2, \ldots$, $A_1, A_2, \ldots$, $P_1, P_2, P_3, \ldots$ and $x_1, x_2, \ldots$. Since G is finite, this process cannot continue forever. At some point we will need to reuse a destroyed component; that is, there is a smallest t such that $A_{t+1} - x_t = A_s - x_s$ for some $s < t$. Put $B := V(A_s - x_s)$. Notice that A_{t+1} is constructed from $A_s - x_s$ by joining the vertex x_t to $N_B(x_t)$. By condition (6) of r_{i_s}-permissibility, we have $z \in N_B(x_t) \cap N_B(x_s) \cap A_s^{r_{i_s}}$.

We now modify P_s to contradict the minimality of $f(P)$. At step $t+1$, x_t was adjacent to exactly r_{i_s} vertices in V_{t+1,i_s}. This is what allowed us to move x_t into V_{t+1,i_s}. Our goal is to modify P_s so that we can move x_t into the i_s part without moving x_s out. Since z is adjacent to both x_s and x_t, moving z out of the i_s part will then give us our desired contradiction.

So, consider the set X of vertices that could have been moved out of V_{s,i_s} between step s and step $t+1$; that is, $X := \{x_{s+1}, x_{s+2}, \ldots, x_{t-1}\} \cap V_{s,i_s}$. For $x_j \in X$, since $x_j \in A_j^{r_{i_s}}$ and x_j is not adjacent to x_{j-1} we see that $d_{V_{s,i_s}}(x_j) \geq r_{i_s}$. Similarly, $d_{V_{s,i_t}}(x_t) \geq r_{i_t}$. Also, by the minimality of t, X is an independent set in G. Thus we may move all elements of X out of V_{s,i_s} to get a new partition $P^* := (V_{*,1}, \ldots, V_{*,k})$ with $f(P^*) = f(P)$. Since x_t is adjacent to exactly r_{i_s} vertices in V_{t+1,i_s} and the only possible neighbors of x_t that were moved out of V_{s,i_s} between steps s and $t+1$ are the elements of X, we see that $d_{V_{*,i_s}}(x_t) = r_{i_s}$. Since $d_{V_{*,i_t}}(x_t) \geq r_{i_t}$ we can move x_t from V_{*,i_t} to V_{*,i_s} to get a new partition $P^{**} := (V_{**,1}, \ldots, V_{**,k})$ with $f(P^{**}) = f(P^*)$. Now, recall that $z \in V_{**,i_s}$. Since z is adjacent to x_t we have $d_{V_{**,i_s}}(z) \geq r_{i_s} + 1$. Thus we may move z out of V_{**,i_s} to get a new partition P^{***} with $f(P^{***}) < f(P^{**}) = f(P)$. This contradicts the minimality of $f(P)$. □

Question. Are there any other interesting r-permissible collections?

Question. The definition of r-permissibility can be weakened in various ways and the proof will still go through. Does this yield anything interesting?

Acknowledgments

Thanks to Dieter Gernert for finding and sending me a copy of Kostochka's paper.

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