Subcubic edge-chromatic critical graphs have many edges

Daniel W. Cranston[*] Landon Rabern[†]

April 1, 2016

Abstract

We consider graphs G with $\Delta = 3$ such that $\chi'(G) = 4$ and $\chi'(G-e) = 3$ for every edge e, so-called *critical* graphs. Jakobsen noted that the Petersen graph with a vertex deleted, P^*, is such a graph and has average degree only $\frac{8}{3}$. He showed that every critical graph has average degree at least $\frac{8}{3}$, and asked if P^* is the only graph where equality holds. A result of Cariolaro and Cariolaro shows that this is true. We strengthen this average degree bound further. Our main result is that if G is a subcubic critical graph other than P^*, then G has average degree at least $\frac{46}{17} \approx 2.706$. This bound is best possible, as shown by the Hajós join of two copies of P^*.

1 Introduction

By a *coloring* of a graph G, we mean a *proper edge-coloring*, which assigns a color to each edge in $E(G)$ such that edges with a common endpoint receive distinct colors. The minimum number of colors needed for a proper edge-coloring is the *edge-chromatic number of* G, denoted $\chi'(G)$. Given a (partial) coloring of G, if a color w is used on an edge incident to a vertex v, then v *sees* w; otherwise v *misses* w. The maximum degree of G is denoted $\Delta(G)$, or simply Δ when G is clear from context. Its *average degree*, $2|E(G)|/|V(G)|$, is denoted $a(G)$. Note that always $\Delta(G) \leq \chi'(G)$. Vizing famously proved that always $\chi'(G) \leq \Delta(G) + 1$. A graph is *edge-chromatic critical* (also Δ-*critical*, or simply *critical*) if $\chi'(G) > \Delta(G)$ but $\chi'(G-e) = \Delta(G)$ for every edge e. A vertex of degree k is a k-*vertex*. If v is a k-vertex and v is adjacent to u, then v is a k-*neighbor* of u. The *order* of G is $|V(G)|$.

Vizing [9, 10] was the first to seek a lower bound on the number of edges in a critical graph, in terms of its order. This problem is now widely studied, for a large range of maximum degrees Δ. Woodall gives a nice history of this work, in the introduction to [11]. In this paper, we study the problem for subcubic graphs, i.e., when $\Delta = 3$.

It is easy to check that 2-critical graphs are precisely odd cycles, which are completely understood. So the first non-trivial case is 3-critical graphs. Let P^* denote the Petersen

[*]Department of Mathematics and Applied Mathematics, Viriginia Commonwealth University, Richmond, VA; `dcranston@vcu.edu`; The first author's research is partially supported by NSA Grant H98230-15-1-0013.
[†]LBD Data Solutions, Lancaster, PA; `landon.rabern@gmail.com`

graph with a vertex deleted. Jakobsen [5] observed that P^* is 3-critical and has average degree $\frac{8}{3}$. In the same paper he asked if every 3-critical graph G has $a(G) \geq \frac{8}{3}$. A year later [6], he answered this question affirmatively. However, in this second paper Jakobsen asked whether P^* is the only graph for which the bound holds with equality. The answer is yes. This follows easily from a result of Cariolaro and Cariolaro [1], as we now show.

Proposition 1. *If G is a 3-critical graph of order n, then $|E(G)| \geq \frac{4}{3}n$ and equality holds only if $G = P^*$.*

Proof. Let P^* be a 3-critical graph of order n. Let n_k denote the number of k-vertices in G, for each $k \in \{2, 3\}$; let $G^{(3)}$ denote the subgraph induced by all 3-vertices. An easy consequence of Vizing's Adjacency Lemma (see the start of Section 2) is that each 2-vertex is adjacent to two 3-vertices and each 3-vertex is adjacent to at most one 2-vertex. Thus $n_3 \geq 2n_2$ and equality implies $\Delta(G^{(3)}) \leq 2$. Letting $m = |E(G)|$, we get

$$\frac{2m}{n} = \frac{1}{n}(2n_2 + 3n_3) = 2 + \frac{n_3}{n} \geq \frac{8}{3}.$$

The last equality is equivalent to $\frac{n_3}{n} \geq \frac{2}{3}$. This is equivalent to $3n_3 \geq 2n = 2(n_2 + n_3)$, and hence to $n_3 \geq 2n_2$. So if $\frac{2m}{n} = \frac{8}{3}$, then $n_3 = 2n_2$, and thus $\Delta(G^{(3)}) \leq 2$. Now a result of Cariolaro and Cariolaro [1] (see also [8, Theorem 4.11]) implies that $G = P^*$. □

A natural extension of this question is to find the maximum α such that every 3-critical graph other than P^* has average degree at least $2 + \alpha$. A complementary question is to find the minimum β such that there exists an infinite sequence of 3-critical graphs with average degree at most $2 + \beta$. The first progress toward answering this question is due to Fiorini and Wilson [3, p. 43], who constructed an infinite family of 3-critical graphs with average degree approaching $2 + \frac{3}{4}$ from below. Woodall [11, p. 815] gave another family with the same number of edges and vertices; see Figure 1. Before presenting this construction, we need a definition.

Let G_1 and G_2 be two graphs with $v_1v_2 \in E(G_1)$ and $v_3v_4 \in E(G_2)$. A *Hajós join* of G_1 and G_2 is formed from the disjoint union of $G_1 - v_1v_2$ and $G_2 - v_3v_4$ by identifying vertices v_1 and v_3 and adding the edge v_2v_4.

Lemma 2. *If G_1 and G_2 are k-critical graphs, and G is a Hajós join of G_1 and G_2 that has maximum degree k, then G is also k-critical.*

This is an old result of Jakobsen [5]. It is a straightforward exercise, so we omit the details, which are available in Fiorini & Wilson [4, p. 82–83] and Stiebitz et al. [8, p. 94].

Corollary 3. *Let G_1 and G_2 be subcubic graphs, and let G_1 be 3-critical. If G is a subcubic graph that is a Hajós join of G_1 and G_2, then G is 3-critical if and only if G_2 is 3-critical.*

Proof. The "if" direction follows immediately from the previous lemma.

To prove the "only if" direction, assume that G_2 is not 3-critical. Either $\chi'(G_2) = 4$ or $\chi'(G_2) \leq 3$; first assume that $\chi'(G_2) = 4$. Since G_2 is not 3-critical, there exists $e \in E(G_2)$ such that $\chi'(G_2 - e) = 4$. Suppose that $\chi'(G_2 - v_3v_4) = 4$. Now $\chi'(G) \geq 4$, since $G_2 - v_3v_4 \subseteq G$. Further, G is not 3-critical, since $G_2 - v_3v_4 \subseteq G - e$ for every $e \in E(G_1) - v_1v_2$. So

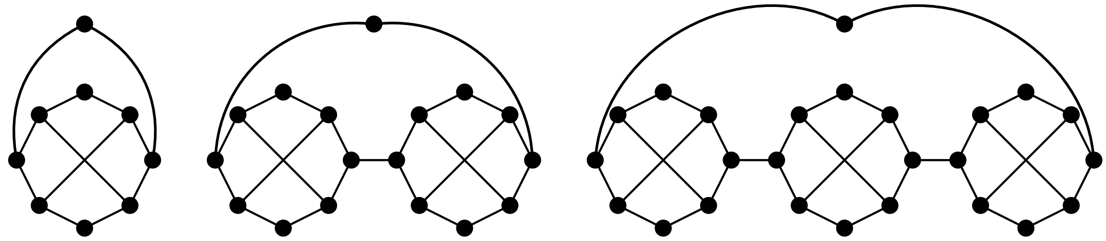

Figure 1: The first three examples in an infinite family of 3-critical graphs with $2|E(G)| < (2 + \frac{3}{4})|V(G)|$.

suppose there exists a 3-critical subgraph $H \subsetneq G_2$ such that $v_3v_4 \in E(H)$. By Lemma 2, the Hajós join of G_1 and H is 3-critical; but this is a proper subgraph of G, so G is not 3-critical.

Instead we assume that $\chi'(G_2) \leq 3$. Let C_1 be a 3-coloring of $G_1 - v_1v_2$ (by criticality); by symmetry, assume that v_2 misses color x. Note that v_1 must see x, since otherwise we get a 3-coloring of G_1. If $d_{G_1-v_1v_2}(v_1) = 2$, then assume also that v_1 sees color y. Let C_2 be a 3-coloring of G_2. By symmetry, assume that v_3v_4 uses color x; if $d_{G_2}(v_3) = 2$, then assume also by symmetry that v_3 sees color z. To get a 3-coloring of G, use C_1 on $G - v_1v_2$, use C_2 on $G_2 - v_3v_4$, and use color x on v_2v_4. $\square$

Now we present a construction of 3-critical graphs with few edges; see [3, p. 43] and [11, p. 815].

Example 1. *Form J_k by starting with P^* and taking the Hajós join with P^* a total of k times (successively), so that each intermediate graph has $\Delta = 3$. The resulting graph J_k is 3-critical, has $11k + 12$ edges and $8k + 9$ vertices. Thus, $a(J_k) \to \frac{11}{4}$ from below as $k \to \infty$.*

The vertex and edge counts follow immediately by induction. That J_k is 3-critical uses induction and also Lemma 2. $\square$

Our main result is the following.

Theorem 1. *Every 3-critical graph G other than P^* has average degree $a(G) \geq \frac{46}{17} \approx 2.706$.*

By using a computer, we are able to improve our edge bound further.

Theorem 2. *Let G be a 3-critical graph. If G is neither P^* nor the Hajós join of two copies of P^*, then G has average degree $a(G) \geq \frac{84}{31} \approx 2.710$.*

However, a human-readable proof is too long to include here (roughly 100 pages). We discuss this work a bit more in Section 4, as well as give a web link where that proof is available. The following conjecture is probably too good to be true, does it fail already for $k = 2$?

Conjecture 4. *For all $k \geq 1$, if G is a 3-critical graph that is not the Hajós join of k or fewer copies of P^*, then*

$$a(G) \geq \frac{22k + 24}{8k + 9}.$$

2 Poor subgraphs, discharging, proofs of the theorems

In this section, we introduce some basic tools for proving edge-coloring results. We then prove our main result, subject to some reducibility lemmas which we prove afterward.

Given a (partial) edge-coloring of a graph G, an (x, y)-*Kempe chain* (or simply (x, y)-*chain*) is a component of the subgraph induced by the edges colored x and y. Note that each Kempe chain is either a path or an even cycle. If vertices v_1 and v_2 lie in the same (x, y)-Kempe chain, then v_1 and v_2 are x, y-*linked*. Given a proper coloring of (some subgraph of) a graph G, if we interchange the colors on some (x, y)-chain, then the resulting coloring is again proper. This interchange is an (x, y)-*Kempe swap* (or simply (x, y)-*swap*) and plays a central role in most proofs of forbidden subgraphs in critical graphs.

Suppose that $d(v_1) = 2$, $d(v_2) = 3$, and $v_1v_2 \in E(G)$. Suppose also that we 3-color $G - v_1v_2$ with colors x, y, and z. If we cannot extend this coloring to G, then (by symmetry) we may assume that v_1 sees x and that v_2 sees y and z. Furthermore, v_1 and v_2 must be x, y-linked; otherwise we perform an (x, y)-swap at v_1 and afterwards color v_1v_2 with x. Similarly, v_1 and v_2 must be x, z-linked.

The quintessential tool for forbidding a subgraph in a critical graph is Vizing's Adjacency Lemma, which he proved using Kempe chains and a similar structure for recoloring, known as Vizing fans. The proof is available in Fiorini & Wilson [4, p. 72–74] and Stiebitz et al. [8].

Vizing's Adjacency Lemma. *Let G be a Δ-critical graph. If $u, v \in V(G)$ and $uv \in E(G)$, then the number of Δ-neighbors of u different from v is at least $\Delta - d(v) + 1$.*

We recall the following basic facts, which are well known.

Lemma 5. *For every 3-critical graph G, the following hold:*

(a) G is 2-connected; and

(b) G has no adjacent 2-vertices, and G has no 3-vertex with two or more 2-neighbors.

Proof. Note that (b) follows immediately from Vizing's Adjacency Lemma. Hence, it suffices to prove (a). If G is disconnected, then we can 3-color each component by criticality, a contradiction. Similarly, suppose that G has a cut-edge v_1v_2 and let G_1 and G_2 be the components of $G - v_1v_2$. By criticality, we can 3-color both G_1 and G_2. Further, we can permute the colors on G_2 so that v_1 and v_2 miss a common color, say x. Now we color v_1v_2 with x to get a 3-coloring of G, a contradiction. □

Let G be a 3-critical graph. Lemma 5(a) implies that every vertex of G has degree 2 or 3. We call a 3-vertex *poor* if it has a 2-neighbor and *rich* otherwise. The *poor subgraph* of G is the subgraph H induced by the poor vertices; its components are the *poor fragments* of G. Every poor vertex has exactly one 2-neighbor, so at most two poor neighbors. Thus, every poor fragment has maximum degree at most 2, so is a path or a cycle.

For a rich vertex w, let $p(w)$ denote the sum of the orders of all the poor fragments in which w has neighbors, accounting for multiplicity (so if w is adjacent to both end-vertices of a path-fragment H_i, then $|V(H_i)|$ counts twice towards $p(w)$). Let $p(G) = \max\{p(w) : w$ is a rich vertex of $G\}$. The next result is the foundation of our proof.

Lemma 6. *Let G be a 3-critical graph such that every poor fragment is a path (not a cycle) and let $p = p(G)$. Now G has average degree $a(G) \geq \frac{8p+12}{3p+4}$.*

Proof. We use the discharging method with initial $\mathrm{ch}(v) = d(v)$ for each vertex v and the following three discharging rules.

(R1) Each 2-vertex takes charge $\frac{p+2}{3p+4}$ from each neighbor.

(R2) Each rich vertex gives charge $\frac{t}{3p+4}$ to each neighbor in a poor fragment of order t.

(R3) All vertices within a poor fragment share charge equally.

Now we show that each vertex v finishes with final charge $\mathrm{ch}^*(v) \geq \frac{8p+12}{3p+4}$, which proves the lemma. If v is a 2-vertex, then $\mathrm{ch}^*(v) = 2 + 2(\frac{p+2}{3p+4}) = \frac{8p+12}{3p+4}$. If v is a rich vertex, then $\mathrm{ch}^*(v) \geq 3 - \frac{p}{3p+4} = \frac{8p+12}{3p+4}$. Now suppose that v is a poor vertex in a poor fragment H, and let $t = |V(H)|$. Since H is a path, each of its two end-vertices must be adjacent to a rich vertex. Thus, these end-vertices receive in total $2(\frac{t}{3p+4})$. So by (R3), each vertex of H receives $\frac{2}{3p+4}$. Thus $\mathrm{ch}^*(v) = 3 - \frac{p+2}{3p+4} + \frac{2}{3p+4} = \frac{8p+12}{3p+4}$, as desired. □

If we let $f(p) = \frac{8p+12}{3p+4}$, then $f(10) = \frac{46}{17} \approx 2.706$ and $f(9) = \frac{84}{31} \approx 2.710$. So to prove Theorems 1 and 2 it suffices to prove that (for appropriate graphs G) always $p(G) \leq 10$ and $p(G) \leq 9$, respectively. Subject to some lemmas that we prove below and in the following sections, we can now prove the two main theorems. For convenience, we restate them.

Theorem 1. *Every 3-critical graph G other than P^* has average degree $a(G) \geq \frac{46}{17} \approx 2.706$.*

Proof. We will use Lemma 6. So it suffices to show that (i) every poor fragment is a path and (ii) $p(G) \leq 10$. Since $\frac{46}{17} < \frac{30}{11}$, Lemma 7 shows that every poor fragment is a path with at most 5 vertices. Thus, it suffice to show that $p(G) \leq 10$.

Consider an arbitrary rich vertex w. If every poor fragment adjacent to w has order at most 3, then $p(w) \leq 3(3) = 9$. So suppose w has some adjacent poor fragment H_1 with order at least 4; recall from above that H_1 has order at most 5. By Claim 6, each poor neighbor of w is in a distinct poor fragment. By Lemma 8, each adjacent poor fragment other than H_1 has order at most 3. Thus $p(w) \leq 5 + 3 + 3 = 11$. Further, equality holds only if w is adjacent to poor fragments of orders 5, 3, and 3. By Lemma 11, this is impossible. Thus, $p(w) \leq 10$, as desired. □

Theorem 2. *Let G be a 3-critical graph. If G is neither P^* nor the Hajós join of two copies of P^*, then G has average degree $a(G) \geq \frac{84}{31} \approx 2.710$.*

Proof. The proof is nearly the same as that of Theorem 1. Now, in addition, we use a computer to show that if G is not the Hajós join of two copies of P^*, then no rich vertex is adjacent to two poor fragments, each of order at least 3. Thus, for each rich vertex w, the largest order of an adjacent poor fragment is at most 5; if this order is at least 4, then each of the remaining adjacent poor fragments has order at most 2. Hence, $p(w) \leq 5 + 2 + 2 = 9$ or $p(w) \leq 3 + 3 + 3 = 9$; so, $p(G) \leq 9$. □

We summarize the results of this section in the following lemma. For the proofs of Theorems 1 and 2, we need to know that this result holds when $\alpha = \frac{46}{17} \approx 2.706$ and $\alpha = \frac{84}{31} \approx 2.710$ respectively, which it does, since $\frac{8}{3} \approx 2.667 < \frac{46}{17} < \frac{84}{31} < 2.75 = \frac{11}{4}$. If H_i is a poor fragment of G, then let H_i^+ denote the subgraph of G induced by the vertices of H_i and the neighboring 2-vertices.

Lemma 7. *Let $\frac{8}{3} \le \alpha \le \frac{11}{4}$, let G be a 3-critical graph such that $a(G) < \alpha$ and G is not the Hajós join of copies of P^*, and assume that G has the fewest veritces among all 3-critical graphs with these properties. Assume that either G is triangle-free or $\alpha \le \frac{30}{11} = 2.\overline{72}$. Now every poor fragment H_i of G is a path on at most 5 vertices, and H_i^+ has one of the forms shown in Figure 3. In particular, no rich vertex is adjacent to both end-vertices of H_i.*

We prove Lemma 7 in a sequence of six claims.

Claim 1. *Suppose $0 < n < |V(G)|$ and J is a connected subcubic graph with $|V(G)| - n$ vertices and $|E(G)| - m$ edges. Suppose that one of the following holds: (i) $\frac{m}{n} \ge \frac{\alpha}{2}$ and J is not a Hajós join of copies of P^* or (ii) $\frac{12+m}{9+n} \ge \frac{\alpha}{2}$. Now J is not 3-critical.*

Proof. Suppose first that (i) holds. Since $\frac{2m}{n} \ge \alpha > a(G) = \frac{2|E(G)|}{|V(G)|}$, it follows that $a(J) = \frac{2|E(G)|-2m}{|V(G)|-n} < \frac{2|E(G)|}{|V(G)|} = a(G) < \alpha$. Since J is not a Hajós join of copies of P^* by (i), it follows from the minimality of G that J is not 3-critical.

We now prove that (ii) implies (i). Since $\frac{12}{9} = \frac{4}{3} \le \frac{\alpha}{2}$, the hypothesis implies that $\frac{m}{n} \ge \frac{\alpha}{2}$. Suppose now that J is a Hajós join of k copies of P^*, so that $|V(J)| = 8k + 1$ and $|E(J)| = 11k + 1$. Recall that $\frac{2(12+m)}{9+n} \ge \alpha$ by (ii), and $\frac{11}{4} \ge \alpha$. Thus, $a(G) = \frac{2(|E(J)|+m)}{|V(J)|+n} = \frac{2(11k+1+m)}{8k+1+n} = \frac{2(11(k-1)+(12+m))}{8(k-1)+(9+n)} \ge \alpha$. This contradiction shows that J is not a Hajós join of copies of P^*, and so (i) holds. □

We now prove some structural properties of G.

Claim 2. *G does not have a subgraph F containing exactly two vertices, v_1 and v_2, with neighbors outside F such that $F + v_1v_2 \cong P^*$.*

Proof. Suppose it does. Since every edge of P^* has at least one end-vertex of degree 3, we may assume that v_2 has degree 3 in P^* and degree 2 in F. Since G is subcubic and 2-connected by Lemma 5, there is exactly one edge v_2v_4 in G with $v_4 \notin F$. It is now easy to see that G is the Hajós join of P^* and a connected subcubic graph J smaller than G. Note that J is not a Hajós join of copies of P^*, since otherwise G would be as well, which it is not.

By Corollary 3, J is 3-critical. Now since $|V(J)| = |V(G)| - 8$ and $|E(J)| = |E(G)| - 11$ and $\frac{11}{8} \ge \frac{\alpha}{2}$, it follows from Claim 1(i) that J is not 3-critical. This is the required contradiction. □

Claim 3. *If $\alpha \le \frac{30}{11} = 2.\overline{72}$, then G is triangle-free.*

Proof. Suppose that G has a 3-cycle $v_1v_2v_3$. Suppose first that no edge of $v_1v_2v_3$ lies in another triangle. Form G' from G by contracting the three edges of triangle $v_1v_2v_3$. Since $|V(G')| = |V(G)| - 2$ and $|E(G')| = |E(G)| - 3$ and $\frac{12+3}{9+2} \ge \frac{\alpha}{2}$, Claim 1(ii) implies that G' is not 3-critical. But it is easy to see that if any two subcubic graphs G and G' are related in this way (with a vertex of G' corresponding to a triangle in G), then G' is 3-critical if and only if G is, so we have a contradiction.

Thus we may assume that some edge of $v_1v_2v_3$ lies in another triangle, say $v_1v_2v_4$. Note that v_3 and v_4 are not adjacent and have no common neighbor of degree 2, since this would imply $(|V(G)|, |E(G)|)$ is $(4, 6)$ or $(5, 7)$ and $a(G) \geq \frac{14}{5} = 2.8 > \alpha$. Since G is 2-connected, v_3 and v_4 have distinct neighbors v_5 and v_6, respectively.

A *ladder* is any graph that is formed from the disjoint union of paths $w_1 \ldots w_\ell$ and $z_1 \cdots z_\ell$ by adding all edges z_iw_i for $2 \leq i \leq \ell - 1$ as well as possibly adding w_1z_1 or $w_\ell z_\ell$ (or both). Let H be a maximal induced subgraph of G containing $v_1, \ldots, v_6$ such that $H - \{v_1, v_2\}$ is a ladder. Let x_1, x_2 be the vertices on the opposite end of the ladder from v_3, v_4.

First, suppose $x_1 \not\leftrightarrow x_2$. Form H' from H by identifying x_1 and x_2 to create a new vertex x'. Note that G is a Hajós join of H' and another graph J. Suppose the ladder $H - \{v_1, v_2\}$ has t rungs. Then $|V(J)| = |V(G)| - 2(t + 2)$ and $|E(J)| = |E(G)| - 3(t + 2)$. Since $\frac{12+3(t+2)}{9+2(t+2)} \geq \frac{\alpha}{2}$ for all $t \geq 0$, Claim 1(ii) shows that J is not 3-critical. Since since G is 3-critical, Corollary 3 shows that H' is not 3-critical. It is easy to see that H' is 4-chromatic, since it is 3-regular, except for a single 2-vertex. So, H' has a 3-critical proper subgraph Q. Then $x' \in V(Q)$ since $H' - x'$ is a proper subgraph of G. By Lemma 5(b), Q contains every edge in $G[v_1, v_2, v_3, v_4]$. By Lemma 5(a), Q contains all edges of the ladder that are not rungs. If Q is missing a rung of the ladder, then Q contains an edge-cut $\{e_1, e_2\}$ with each e_i incident to a 2-vertex, which is impossible since Q is 3-critical. Hence $Q = H'$, a contradiction.

So, we must have $x_1 \leftrightarrow x_2$. Now, x_1 and x_2 cannot have distinct neighbors outside H, by maximality of H. But x_1 and x_2 have no common neighbor either since then either G would have a cut-edge or a triangle with no edge in two triangles. To avoid a cut-edge, the only remaining possibility is that x_1 and x_2 are both 2-vertices. But then G is easily 3-colored, a contradiction. $\square$

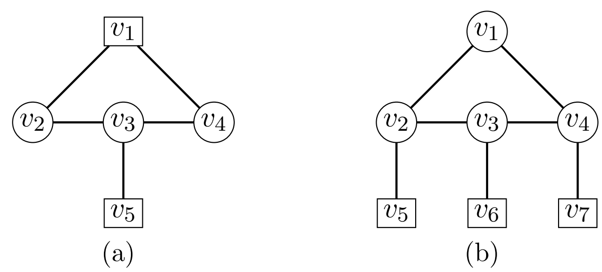

Figure 2: Two subgraphs forbidden from a 3-critical graph G. Vertices drawn as rectangles have degree 2 in G and those drawn as circles have degree 3 in G.

Claim 4. *Neither of the configurations in Figures 2(a) and 2(b) is a subgraph of G.*

Proof. We begin with Figure 2(a). Since G is 3-critical, $\chi'(G) = 4$, but there is a 3-coloring φ of $G - v_3v_5$ using colors x, y, and z. Since v_3v_5 cannot be colored, we may assume that v_5

sees color x, $\varphi(v_2v_3) = y$, and $\varphi(v_3v_4) = z$. First suppose that v_1 sees x. By symmetry (in the graph and also between colors y and z), assume $\varphi(v_1v_2) = x$ and $\varphi(v_1v_4) = y$. Recolor v_2v_3 with x, recolor v_1v_2 and v_3v_4 with y, and recolor v_1v_4 with z. Now we can color v_3v_5 with z. So assume instead that v_1 misses x; thus $\varphi(v_1v_2) = z$ and $\varphi(v_1v_4) = y$. Now do an (x, y)-swap at v_5. Edge v_3v_5 will be colorable with x unless the (x, y)-path starting at v_5 ends at v_3; so assume that it does. Now v_5 sees y and v_1 sees y, so we can extend the coloring to G as above (with y in the role of x).

Now consider Figure 2(b). As above, we use colors x, y, z to 3-color $G - v_3v_6$; call the coloring φ. As before, we assume that v_6 sees color x, $\varphi(v_2v_3) = y$, and $\varphi(v_3v_4) = z$. We assume by symmetry that $\varphi(v_1v_2) \neq x$, so that $\varphi(v_1v_2) = z$ and $\varphi(v_2v_5) = x$. We may assume that v_3 and v_6 are x, y-linked. Thus, v_5 sees y. If $\varphi(v_1v_4) = y$, then do a (y, z)-swap at v_3 (the entire component is just the 4-cycle $v_1v_2v_3v_4$). Now v_3 and v_6 are no longer x, z-linked, so do an (x, z)-swap at v_3, and color v_3v_6 with z. Thus, we assume that $\varphi(v_1v_4) = x$. Now again, do an (x, z)-swap at v_3, then color v_3v_6 with z. □

Claim 5. *Every poor fragment H_i of G is a path on at most 5 vertices, and H_i^+ has one of the forms in Figure 3.*

Proof. Suppose not. By construction, $\Delta(H) \leq 2$; since G has no 3-cycle (by Claim 3), assume that some poor fragment H_1 induces a path or cycle $v_1 \cdots v_t$ on four or more vertices. Since G has no 3-cycles, no successive 3-vertices on H_1 have a common 2-neighbor. Similarly, since Figure 2(a) is forbidden, no vertices at distance two on H_1 have a common 2-neighbor. By Lemma 10, no four consecutive vertices on H_1 have distinct 2-neighbors. Thus, each vertex $v_i \in V(H_1)$ (for $i \in \{1, \dots, t-3\}$) must share a common 2-neighbor with v_{i+3}. This immediately gives that $t \leq 6$, since otherwise v_4 must share a common 2-neighbor with both v_1 and v_7, a contradiction.

If H_1 is a 6-cycle, then (since G is subcubic and connected), $G = H_1^+$, and also $H_1^+ \cong P^*$; this contradicts the definition of G in Lemma 7. (Recall that P^* can alternatively be drawn as a 6-cycle C, where each pair of vertices at distance 3 on C have a commong 2-neighbor.) If H_1 is a path of order 6, then v_1 and v_6 are the only vertices of H_1^+ that can have neighbors outside H_1^+, so $H_1^+ + v_1v_6 \cong P^*$; this contradicts Claim 2. If H_1 is a 5-cycle, then v_1 and v_4 have a common 2-neighbor; similarly, v_2 and v_5 have a common 2-neighbor. Thus, the edge from v_3 to its 2-neighbor is a cut-edge, contradicting Lemma 5(a). If H_1 is a 4-cycle, then v_1 and v_4 have a common 2-neighbor, but they are also adjacent. This 3-cycle contradicts Claim 3. Thus, H_1 is a path on at most five vertices. Further, since Figure 8 is forbidden by Lemma 10, H_i^+ has one of the forms in Figure 3. □

Claim 6. *No 3-vertex has two neighbors in the same poor fragment.*

Proof. Suppose that G contains such a 3-vertex v, and let H_1 be the poor fragment containing two neighbors. Claim 5 implies that H_1 is a path on at most 5 vertices; further, v must be adjacent to the endvertices of H_1. If H_1 has order 2, then G contains a 3-cycle, contradicting Claim 3. If H_1 has order 3, then G contains Figure 2(b), contradicting Claim 4. If H_1 has order 4, then we can color $G - H_1$ by criticality, and extend the coloring to G via one of the two extensions shown in Figure 4, depending on which colors are available at the 2-vertices. (By symmetry, assume that v sees color x. If x is also seen by both 2-vertices in H_1 with

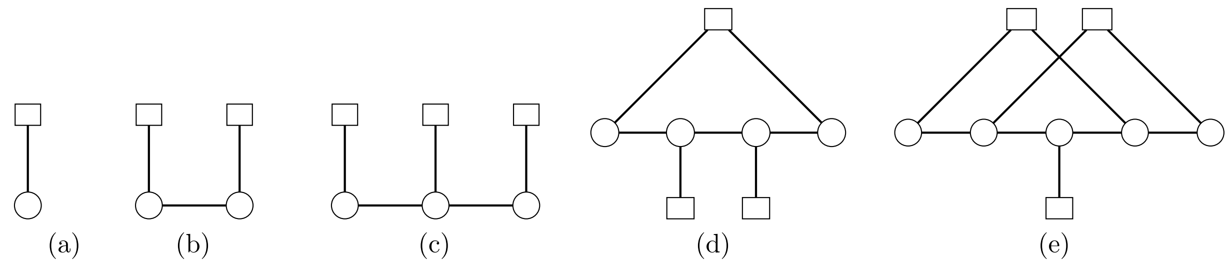

Figure 3: The five possibilities for H_i^+.

edges to vertices outside H_1, then we use the extension on the left; otherwise, the extension on the right.) Finally, suppose that H_1 has order 5. Now G is the Hajós join of P^* and a smaller graph J; the copy of $P^* - e$ in G consists of H_1^+, v, and v's neighbor outside of H_1. Corollary 3 implies that J is 3-critical. Now $a(J) < a(G)$, which contradicts our choice of G as a minimal counterexample. □

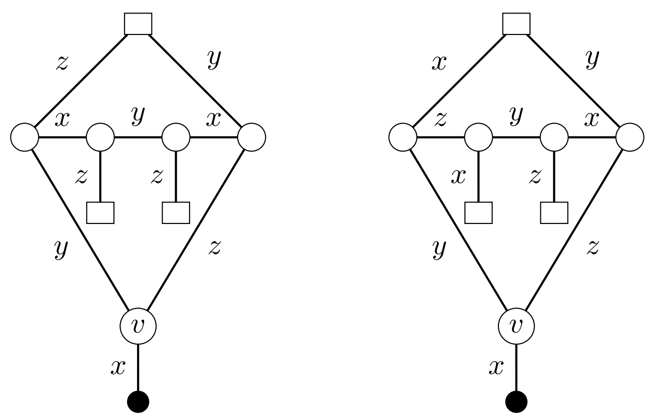

Figure 4: How to extend a coloring of $G \setminus H_1$ to G when H_1 has order 4 and a 3-vertex v has two neighbors in H_1.

Claim 6 completes the proof of Lemma 7, which ends this section.

3 Reducibility

Lemma 8. *The subgraph shown in Figure 5 cannot appear in a 3-critical graph. Nor can it appear if we identify one or two vertex pairs in* $\{v_6, v_7, v_{14}, v_{15}\}$. *Thus, no rich vertex has neighbors in two distinct poor fragments each of order at least 4.*

Proof. We first consider the case where no pairs of 2-vertices are identified. Let L and R denote the subgraphs of Figure 5 induced by vertices $v_1, \ldots, v_7$ and $v_9, \ldots, v_{15}$, respectively. Note that the R and L are symmetric. By criticality, construct a partial 3-coloring $G - E(L)$ The four vertices where colored and uncolored edges meet are v_2, v_6, v_7, v_5, each of which sees exactly one color. We have numerous possibilities for the ordered 4-tuple of colors seen

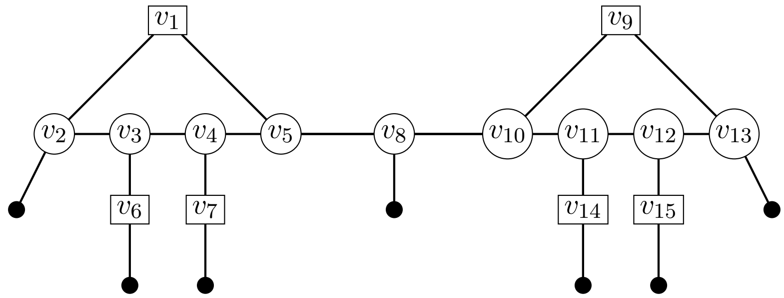

Figure 5: A subgraph forbidden from appearing in 3-critical graph G.

by these vertices (up to permuting color classes, we have 14 such possibilities). To show that we can extend these partial colorings to G, we can assume by permuting color classes that v_2 sees x. We begin by showing that we can extend the coloring to all of G unless the ordered 4-tuple of colors seen by v_2, v_6, v_7, v_5 is (x, x, y, y) or (x, y, y, x).

If v_2 and v_5 see distinct colors, then Figure 6(a) shows (possibly by permuting color classes) how to extend the coloring unless the ordered 4-tuple of colors seen by v_2, v_6, v_7, v_5 is (x, x, y, y): simply color greedily along the path of uncolored edges, starting at v_6 and ending at v_1. If v_7 and v_5 see distinct colors, this is clear. If not, then we can assume that v_2 and v_6 see distinct colors, so we can swap the roles of v_2 and v_5. More formally, we reflect the coloring in Figure 6(a) across a vertical line running through v_1.

Now suppose instead that v_2 and v_5 see the same color, x. If v_6 or v_7 sees x, then we can extend the coloring as in Figure 6(b): now color greedily along the path of uncolored edges, ending at v_7 (again, if v_6 see x, but v_7 does not, reflect the coloring across a vertical line through v_1). Further, if v_6 and v_7 see distinct colors (other than x), then we can color as in Figure 6(c). Thus, we conclude that we can extend the partial coloring to G unless the ordered 4-tuple of colors seen is (x, x, y, y) or (x, y, y, x). Note that these two bad possibilities differ in the colors used on *two* pendant edges, even up to all permutations of color classes. Thus, if at least one pendant edge is not yet colored, we can always find an extension of the partial coloring.

Now suppose that G contains a copy of Figure 5. By criticality, we get a 3-coloring of all of G except the edges with both endpoints in $v_1, \ldots, v_{15}$. Our goal is to color the two remaining edges incident to v_8 so that both L and R can be colored as above. As we already noted, we must thus color v_5v_8 and v_8v_{10} so that neither L or R has as its ordered 4-tuple of colors seen either (x, x, y, y) or (x, y, y, x).

Given the colors seen by v_2, v_6, v_7, at most one choice of color for v_5v_8 gives a bad 4-tuple for L. Similarly, at most one choice of color for v_8v_{10} gives a bad 4-tuple for R. We can color the edges as desired unless the color that is bad on v_5v_8 for L is the same as the color that is bad on v_8v_{10} for R, and that color, say x, is different from the color y seen by v_8. So suppose this is true. Now perform an (x, y)-swap at v_8. If this Kempe chain ends in neither L nor R, then we color v_5v_8 and v_8v_{10} arbitrarily. Now we can extend the coloring to both L and R. So suppose instead that the Kempe chain ends in L (by symmetry). Now we can color L, since v_5v_8 is uncolored. Afterward, the color for v_8v_{10} is determined, and we can color R. This completes the case where no pairs of 2-vertices are identified.

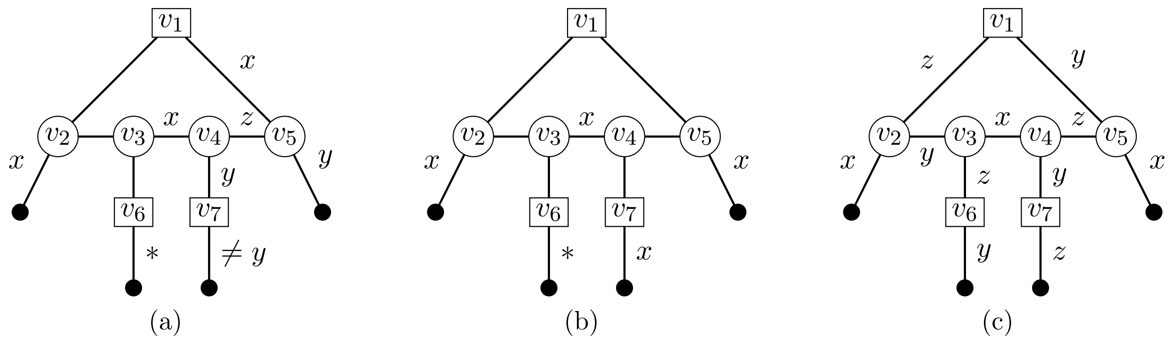

Figure 6: Extensions for part of Figure 5, based on the colors seen by v_2, v_5, v_6, and v_7.

Now we consider the case where two vertex pairs in $\{v_6, v_7, v_{14}, v_{15}\}$ are identified. Since G has no 3-cycles, each of v_6, v_7 must be identified with one of v_{14}, v_{15}. By criticality, color all edges except those with both endpoints in $v_1, \ldots, v_{15}$. Note that we have only 3 incident colored edges. As above, we now extend the coloring to G, using Figure 6.

Suppose that all colored incident edges use the same color, say x. Now color v_5v_8 and v_4v_7 with y and color v_8v_{10} and $v_{11}v_{14}$ with z. (Perhaps $v_7 = v_{14}$, but this is okay.) Now we can extend the coloring to each side, as in Figure 6(a). A similar strategy works in every case except when v_2 and v_{13} see a common color, say x, and v_8 sees some other color, say y. We always color v_5v_8 and v_8v_{10} so that their colors differ from those seen by v_2 and v_{13}, respectively. Next, we color v_4v_7 and $v_{11}v_{14}$ to match v_5v_8 and v_8v_{10}, respectively. Finally, we can color each side as in Figure 6(a), reflected. So suppose that v_2 and v_{13} see x and v_8 sees y. Color v_5v_8 with x and v_8v_{10} with z. Now color v_3v_4 with x and $v_{14}v_i$ with y, where $i \in \{3, 4\}$. To extend L, we assume by symmetry that $v_{14} = v_7$, so v_7v_4 is colored y and v_3v_6 is uncolored. Greedily color the path $v_4v_5v_1v_2v_3v_6$, starting from v_4. Now since v_{10} and v_{14} see different colors, we extend R as in Figure 6(a). This completes the case of two pairs of identified vertices.

Now suppose that one vertex pair in $\{v_6, v_7, v_{14}, v_{15}\}$ is identified; we consider three cases. The identified pair is either (v_6, v_{15}), (v_7, v_{14}), or (v_6, v_{14}); we call these cases "outside", "inside", and "mixed". In each case, five vertices in $v_1, \ldots, v_{15}$ see colors, but we initially consider only the colors seen by v_2, v_8, and v_{13}. Thus, for example, we write the 3-tuple (x, y, z) to signify that v_2 sees x, v_8 sees y, and v_{13} sees z.

First consider outside. Suppose we have the 3-tuple (x, x, y). Color v_3v_6 with x, color v_5v_8 and $v_{12}v_{15}$ with y, and color v_8v_{10} with z. We can extend the coloring on each side as in Figure 6(a). A similar strategy works for 3-tuples (x, y, y) and (x, y, z). Consider instead (x, y, x). Now color v_5v_8 and $v_{12}v_{15}$ with x and color v_8v_{10} with z. We can color R as in Figure 6(a) and L as in Figure 6(b). Finally, consider the 3-tuple (x, x, x). If v_7 sees x, then color $v_{12}v_{15}$ with x, color v_5v_8 with y, and color v_8v_{10} with z. Now color both L and R as in Figure 6(a). Otherwise, by symmetry v_7 sees y. Now color $v_{12}v_{15}$ with x, v_8v_{10} with y, and v_5v_8 with z. We can again color both L and R as in Figure 6(a).

Now consider inside. This case is similar to above. Suppose that v_2, v_8, v_{13} see some ordered triple *other* than (x, y, x). Color v_5v_8 and v_8v_{10} to differ from the colors seen by v_2 and v_{13}, respectively. Now color v_4v_7 and $v_{11}v_{14}$ to match v_5v_8 and v_8v_{10}, respectively. Finally, color both L and R as in Figure 6(a). So suppose v_2, v_8, v_{13} sees (x, y, x). If v_6 sees x, then color v_5v_8 with x and v_8v_{10} with z. Now we can color R first, then color L, as in Figure 6(b), reflected. So assume v_6 does not see x. Now color v_5v_8 with z and v_8v_{10} with x. Color R first, then color L as in Figure 6(a) reflected, since v_2 and v_6 see distinct colors.

Finally, consider mixed. Recall that v_6 and v_{14} are identified. Suppose the triple seen by v_2, v_8, v_{13} is something other than other than (x, x, x) and (x, y, y). If v_{13} and v_{15} see distinct colors, then color v_8v_{10} with a color not seen by v_{13}. Now color L, then extend to R as in Figure 6(a). Otherwise v_{13} and v_{15} see the same color, so use that color on v_8v_{10}. Now color L, then extend to R, as in Figure 6(b). Instead, consider (x, x, x). Color v_3v_6 with x, color v_5v_8 with y, and both v_8v_{10} and $v_{11}v_{14}$ with z. Now extend both sides as in Figure 6(a). Finally, consider (x, y, y). If v_{15} sees a color other than y, then color v_8v_{10} to avoid the color seen by v_{15}. Now color L, followed by R, as in Figure 6(a). Similarly, if v_7 sees x, then color v_5v_8 with x and color R, followed by L, asi in Figure 6(a) Likewise, if v_7 sees y, then color v_5v_8 with z and color R, followed by L. Thus, we conclude that v_7 sees z and v_{15} sees y. Now perform an (x, y)-swap at v_8. The resulting coloring will be one of the cases above. □

Lemma 9. *Let G' be a 3-colorable graph having the configuration in Figure 7 as a subgraph. Now G' has a 3-coloring such that at least one of the Kempe chains through edge $w'w$ is a path, not a cycle.*

Proof. By possibly permuting colors, assume that the three edges incident with w are given the colors indicated; also assume that, for every 3-coloring of G' with these colors, both the (x, y)-chain and the (x, z)-chain through edges $w'w$ are cycles. This implies that swapping colors in any (x, y)-chain or (x, z)-chain that is not a cycle will not change the color of any edge at w. Let the unshown neighbors of u_1, u_2, u_3, v_3 be u_1', u_2', u_3', v_3', respectively.

Let $C_{x,y}$ denote the (x, y)-Kempe chain (which must be a cycle) containing the path $w'wv_1$. If $C_{x,y}$ includes $v_1u_1u_1'$, then an (x, z)-swap at u_1 will reroute it along v_1v_2; so we assume that $C_{x,y}$ includes v_1v_2. So v_1u_1 has color z.

We now prove the following: *If $C_{x,y}$ contains the path $w'wv_1v_2$, then (a) $C_{x,y}$ contains edge v_2v_3, and (b) edges u_1u_1' and u_2u_2' have the same color and are the ends-edges of an (x, y)-chain.*

To prove (a), suppose to the contrary that $C_{x,y}$ includes $v_2u_2u_2'$. By an (x, y)-swap at u_1 if necessary, we can assume that u_1u_1' has color x. If the (x, z)-chain $P_{x,z}(u_2)$ starting along u_2u_2' does not include the path $u_1'u_1v_1v_2v_3$, then swapping colors on $P_{x,z}(u_2)$ will terminate $C_{x,y}$ at u_2. And if $P_{x,z}(u_2)$ does include this path, then swapping colors on $P_{x,z}(u_2)$ will reroute $C_{x,y}$ along $w'wv_1u_1$, where it terminates. Both of these possibilities contradict our assumptions, and this proves (a). Thus edges v_1u_1 and v_2u_2 are both colored z.

Suppose now that (b) is false. Now by (x, y)-interchanges at u_1 and/or u_2, if necessary, we can make u_1u_1' and u_2u_2' have colors x and y, respectively. This contradiction proves (b). By swapping colors in the (x, y)-chain linking u_1u_1' and u_2u_2', if necessary, assume that these edges both have color y. Note that an (x, z)-interchange at u_3 now cannot change the color of edge v_1v_2.

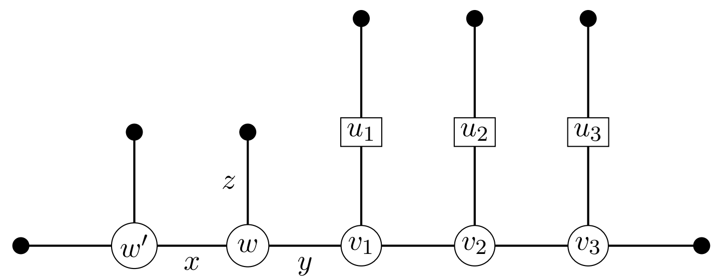

Figure 7: If a graph contains this configuration and has a 3-coloring, then it has a 3-coloring where one Kempe chain through edge $w'w$ is a path, not a cycle.

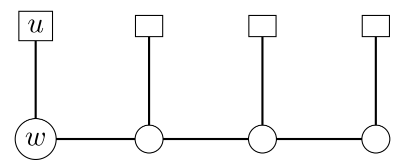

Figure 8: This configuration cannot appear in a 3-critical graph.

If $C_{x,y}$ includes $v_3u_3u'_3$, then an (x,z)-swap at u_3 will reroute it along $v_3v'_3$; so we will assume that $C_{x,y}$ contains the path $w'wv_1v_2v_3v'_3$ and that all three edges v_iu_i are colored z. By (b) applied the the current coloring, there is still an (x,y)-chain linking $u_1u'_1$ and $u_2u'_2$. Thus the (x,y)-chains containing $u_2u'_2$ and $u_3u'_3$ are disjoint. By swapping colors on these chains if necessary, we can assume that both of these edges have color x. Now interchanging colors y and z on the Kempe chain $u_2v_2v_3u_3$ will reroute $C_{x,y}$ along v_2u_2, which contradicts (a). This contradiction completes the proof. □

Lemma 10. *The configuration shown in Figure 8 is not a subgraph of any 3-critical graph.*

Proof. Suppose G is a 3-cirtical graph containing this configuration as a subgraph. Let w' be the unshown neighbor of u in Figure 8. Since G is 3-critical, $G - uw$ has a 3-coloring that cannot be extended to a 3-coloring of G, and which therefore gives a 3-coloring of the graph G' obtained from G by contracting edge uw (so replacing path $w'uw$ by a single edge $w'w$).

By Lemma 9, G' has a 3-coloring such that one of the Kempe chains through $w'w$ is a path P. Transferring this coloring to G, it is proper at every vertex except u, where the edges $w'u$ and uw have the same color. However, swapping colors in a maximal segment of P ending at u creates a proper coloring of G, and this contradiction completes the proof. □

Lemma 11. *Let G be as in Section 2. No rich vertex of G has neighbors in three poor fragments with orders 5, 3, and 3.*

Proof. Suppose that w is a vertex of G with neighbors in three poor fragments H_1, H_2, H_3 of orders 5, 3, 3, respectively. Exactly one 2-vertex has a single neighbor in H_1, and its other neighbor cannot be in both H_2 and H_3; assume it is not in H_2, so that H_1^+ and H_2^+ are disjoint. (We will make no further use of H_3.)

Consider a 3-coloring of $G - E(H_1^+)$. There are three colored edges that are incident with vertices of H_1^+; call them *boundary* edges. It is easy to see that if any two of them have the

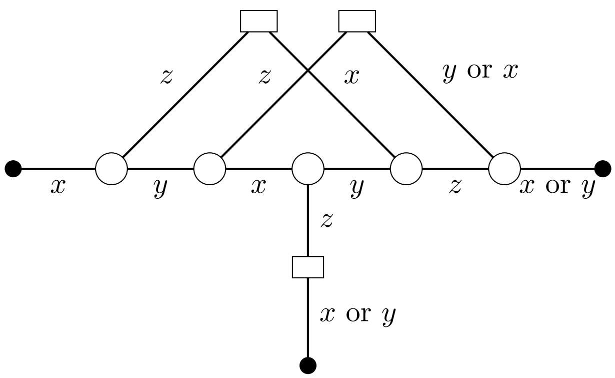

Figure 9: How to extend a 3-coloring of $G - E(H_1^+)$ to G when H_1 is a poor fragment of order 5 and at least two of its boundary edges have the same color.

same color, then the coloring can be extended to all of G; see Figure 9. This contradiction shows that the boundary edges must have distinct colors. Thus, there is a 3-coloring of the graph G' formed from G by contracting H_1^+ to a single vertex w'. Note that w', w, and H_2^+ form in G' the configuration shown in Figure 7. By Lemma 9, there is a coloring of G' such that one of the Kempe chains containing $w'w$ is a path P.

In G, path P consists of two disjoint paths, each ending with one of the boundary edges. Swapping colors in one of these paths will create a 3-coloring of $G - E(H_1^+)$ such that two of the boundary edges have the same color. As we have seen, this coloring can be extended to the whole graph G, and this contradiction completes the proof. □

4 An Improved Bound using a Computer

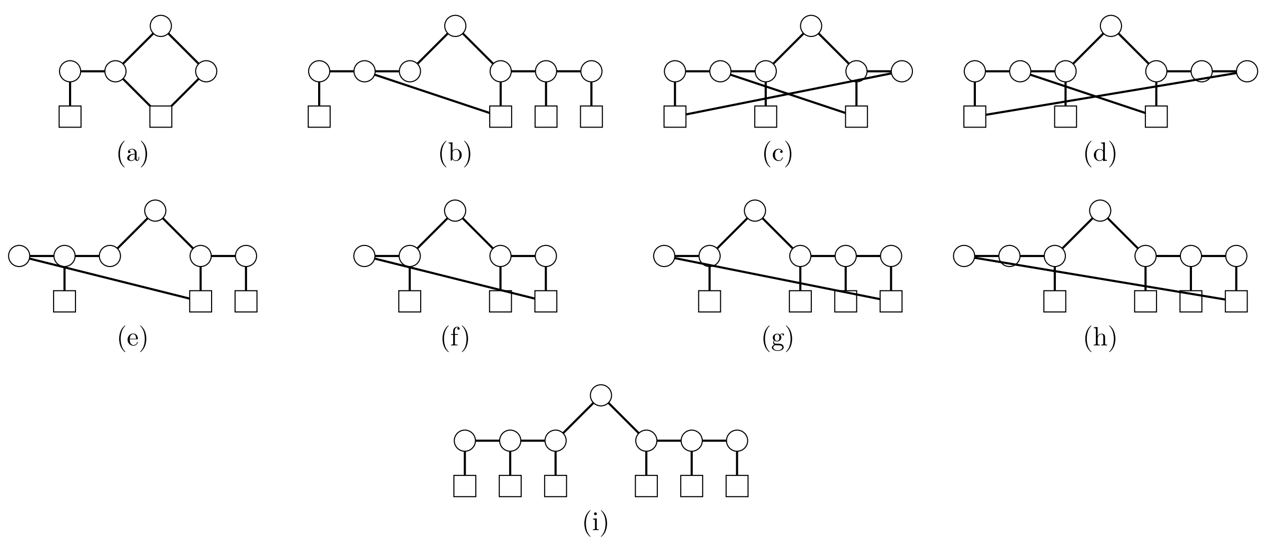

Figure 10: Extra reducible configurations.

The reducible configurations in this paper were originally found by computer. The computer uses an abstract definition capturing the notion of "colorable after performing some Kempe swaps", which frees it from considering an embedding in an ambient critical graph. This is called *fixability* and extends the idea in [7] from stars to arbitrary graphs. The computer is able to prove many reducibility results for which we have yet to find short proofs. Here we show how to use some of these reducible configurations to further improve the bound on the average degree of 3-critical graphs. We give a larger survey of what can be proved with these computer results in [2].

To conclude the paper we prove, modulo computer verification, that every 3-critical graph has $p(G) \leq 9$. This is the final piece needed for our improved bound on $a(G)$ in Theorem 2.

Lemma 12. *The configurations in Figure 10 are not subgraphs of any 3-critical graph. In particular,* $p(G) \leq 9$*.*

Proof. The first statement is proved by computer. We prove the second statement, assuming the first. Suppose that v is a rich vertex with neighbors in poor fragments H_1, H_2, H_3. If at most one H_i has order at least 3, then $p(w) \leq 2 + 2 + 5 = 9$. Suppose instead that H_1 and H_2 have order at least 3. If H_1 and H_2 have no common 2-neighbors, then G has a copy of Figure 10(i). Otherwise, the poor fragments share one or more common 2-neighbors, and G contains one of Figure 10(a)–(h). Choose a common 2-neighbor w such that the shortest cycle C through v and w is as short as possible. If C has length 4, then we have Figure 10(a); if length 5, then one of Figures 10(b–d); if length 6, then one of Figures 10(e–f); if length 7, then Figure 10(g); if length 8, then Figure 10(h). The computer is able to generate proofs in LaTeX, but at about 100 pages this one is not a fun read: `https://dl.dropboxusercontent.com/u/8609833/Papers/big%20tree.pdf` □

Acknowledgments

Thanks to two anonymous for their constructive feedback. In particular, one referee's extensive comments led to a significantly improved exposition, a much simpler proof of Lemma 10, and a slightly stronger bound in Theorem 1 (showing $p(G) \leq 10$, rather than $p(G) \leq 11$ as we had done).

References

[1] D. Cariolaro and G. Cariolaro, *Colouring the petals of a graph*, Electron. J. Combin. **10** (2003), Research Paper 6, 11 pp. (electronic). MR 1975756

[2] D.W. Cranston and L. Rabern, *Edge-coloring via Fixable Subgraphs*, In preparation (2015).

[3] S. Fiorini and R.J. Wilson, *On the chromatic index of a graph. II*, Combinatorics (Proc. British Combinatorial Conf., Univ. Coll. Wales, Aberystwyth, 1973), Cambridge Univ. Press, London, 1974, pp. 37–51. London Math. Soc. Lecture Note Ser., No. 13.

[4] ———, *Edge-colourings of graphs*, Pitman, London; distributed by Fearon-Pitman Publishers, Inc., Belmont, Calif., 1977, Research Notes in Mathematics, No. 16.

[5] I.T. Jakobsen, *Some remarks on the chromatic index of a graph*, Arch. Math. (Basel) **24** (1973), 440–448.

[6] ———, *On critical graphs with chromatic index* 4, Discrete Math. **9** (1974), 265–276.

[7] L. Rabern, *A game generalizing Hall's theorem*, Discrete Math. **320** (2014), 87–91.

[8] M. Stiebitz, D. Scheide, B. Toft, and L.M. Favrholdt, *Graph edge coloring: Vizing's theorem and Goldberg's conjecture*, Wiley Series in Discrete Mathematics and Optimization, John Wiley & Sons, Inc., Hoboken, NJ, 2012.

[9] V.G. Vizing, *The chromatic class of a multigraph*, Kibernetika (Kiev) **1965** (1965), no. 3, 29–39.

[10] ———, *Some unsolved problems in graph theory*, Uspehi Mat. Nauk **23** (1968), no. 6 (144), 117–134.

[11] D.R. Woodall, *The average degree of an edge-chromatic critical graph*, Discrete Math. **308** (2008), no. 5-6, 803–819.