PARTITIONING AND COLORING GRAPHS WITH DEGREE CONSTRAINTS

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Abstract. We prove that if G is a vertex-critical graph with $\chi(G) \geq \Delta(G) + 1 - p \geq 4$ for some $p \in \mathbb{N}$ and $\omega(\mathcal{H}(G)) \leq \frac{\chi(G)+1}{p+1} - 2$, then $G = K_{\chi(G)}$ or $G = O_5$. Here $\mathcal{H}(G)$ is the subgraph of G induced on the vertices of degree at least $\chi(G)$. This simplifies the proofs and improves the results in the paper of Kostochka, Rabern and Stiebitz [8].

1. Introduction

Our notation follows Diestel [6] unless otherwise specified. The natural numbers include zero; that is, $\mathbb{N} := \{0, 1, 2, 3, \ldots\}$. We also use the shorthand $[k] := \{1, 2, \ldots, k\}$. The complete graph on t vertices is indicated by K_t and the edgeless graph on t vertices by E_t. A vertex $v \in V(G)$ is called *universal* in G if it is adjacent to every other vertex of G. We write $\mathcal{H}(G)$ for the subgraph of G induced on the vertices of degree at least $\chi(G)$.

The classical theorem of Brooks [4] gives the necessary and sufficient conditions for a graph G to be $\Delta(G)$-colorable.

Theorem 1.1 (Brooks [4] 1941)**.** *If G is a graph with $\chi(G) \geq \Delta(G)+1 \geq 4$ then G contains $K_{\chi(G)}$.*

In [7] Kierstead and Kostochka investigated the same question with the Ore-degree $\theta(G)$ in place of $\Delta(G)$.

Definition 1. The *Ore-degree* of an edge xy in a graph G is $\theta(xy) := d(x) + d(y)$. The *Ore-degree* of a graph G is $\theta(G) := \max_{xy \in E(G)} \theta(xy)$.

Theorem 1.2 (Kierstead and Kostochka [7] 2010)**.** *If G is a graph with $\chi(G) \geq \left\lfloor \frac{\theta(G)}{2} \right\rfloor + 1 \geq 7$ then G contains $K_{\chi(G)}$.*

This statement about Ore-degree is equivalent to the following statement about vertex-critical graphs.

Theorem 1.3 (Kierstead and Kostochka [7] 2010)**.** *The only vertex-critical graph G with $\chi(G) \geq \Delta(G) \geq 7$ such that $\mathcal{H}(G)$ is edgeless is $K_{\chi(G)}$.*

In [12], we improved the 7 to 6 by proving the following generalization.

Theorem 1.4 (Rabern [12] 2012)**.** *The only vertex-critical graph G with $\chi(G) \geq \Delta(G) \geq 6$ and $\omega(\mathcal{H}(G)) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2$ is $K_{\chi(G)}$.*

This result and those in [11] were improved by Kostochka, Rabern and Stiebitz in [8]. In particular, the following was proved.

Theorem 1.5 (Kostochka, Rabern and Stiebitz [8] 2012)**.** *The only vertex-critical graphs G with $\chi(G) \geq \Delta(G) \geq 5$ such that $\mathcal{H}(G)$ is edgeless are $K_{\chi(G)}$ and O_5.*

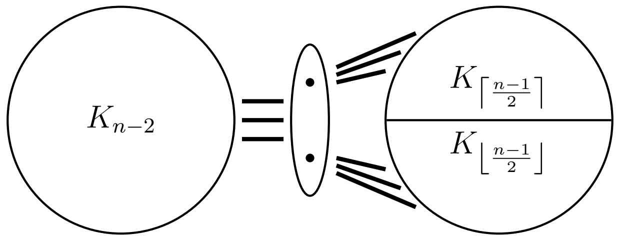

FIGURE 1. The graph O_n.

Here O_n is the graph formed from the disjoint union of $K_n - xy$ and K_{n-1} by joining $\lfloor \frac{n-1}{2} \rfloor$ vertices of the K_{n-1} to x and the other $\lceil \frac{n-1}{2} \rceil$ vertices of the K_{n-1} to y (see Figure 1). In this paper we prove a result which implies all of the results in [8]. The proof replaces an algorithm of Mozhan [10] with the original, more general, algorithm of Catlin [5] on which it is based. This allows for a considerable simplification. Moreover, we prove two preliminary partitioning results that are of independent interest. All coloring results follow from the first of these, the second is a generalization of a lemma due to Borodin [2] (and independently Bollobás and Manvel [1]) about partitioning a graph into degenerate subgraphs. The following is the main coloring result in this paper.

Corollary 3.3. *Let G be a vertex-critical graph with $\chi(G) \geq \Delta(G) + 1 - p \geq 4$ for some $p \in \mathbb{N}$. If $\omega(\mathcal{H}(G)) \leq \frac{\chi(G)+1}{p+1} - 2$, then $G = K_{\chi(G)}$ or $G = O_5$.*

2. PARTITIONING

An *ordered partition* of a graph G is a sequence $(V_1, V_2, \ldots, V_k)$ where the V_i are pairwise disjoint and cover $V(G)$. Note that we allow the V_i to be empty. When there is no possibility of ambiguity, we call such a sequence a *partition.* For a vector $\mathbf{r} \in \mathbb{N}^k$ we take the coordinate labeling $\mathbf{r} = (r_1, r_2, \ldots, r_k)$ as convention. Define the *weight* of a vector $\mathbf{r} \in \mathbb{N}^k$ as $w(\mathbf{r}) := \sum_{i \in [k]} r_i$. Let G be a graph. An $\mathbf{r}$*-partition* of G is an ordered partition $P := (V_1, \ldots, V_k)$ of $V(G)$ minimizing

$$f(P) := \sum_{i \in [k]} \left(\|G[V_i]\| - r_i \left|V_i\right| \right).$$

It is a fundamental result of Lovász [9] that if $P := (V_1, \ldots, V_k)$ is an $\mathbf{r}$-partition of G with $w(\mathbf{r}) \geq \Delta(G) + 1 - k$, then $\Delta(G[V_i]) \leq r_i$ for each $i \in [k]$. As Catlin [5] showed, with the stronger condition $w(\mathbf{r}) \geq \Delta(G) + 2 - k$, a vertex of degree r_i in $G[V_i]$ can always be moved to some other part while maintaining $f(P)$. Since G is finite, a well-chosen sequence of such moves must always "wrap back on itself" in a sense that will become clear in the proofs. Many authors, including Catlin [5], Bollobás and Manvel [1] and Mozhan [10] have used such techniques to prove coloring results. We generalize these techniques by taking into account the degree in G of the vertex to be moved—a vertex of degree less than the maximum needs a weaker condition on $w(\mathbf{r})$ to be moved.

For $x \in V(G)$ and $D \subseteq V(G)$ we use the notation $N_D(x) := N(x) \cap D$ and $d_D(x) := |N_D(x)|$. Let $\mathcal{C}(G)$ be the components of G and $c(G) := |\mathcal{C}(G)|$. For an induced subgraph H of G, define $\delta_G(H) := \min_{v \in V(H)} d_G(v)$.

Definition 2. Let G be a graph and H an induced subgraph of G. For $d \in \mathbb{N}$, we let $H^{G,d}$ be the subgraph of G induced on $\{v \in V(H) \mid d_G(v) = d \text{ and } H - v \text{ is connected}\}$. When the containing graph G is clear from context, we just write H^d.

Note that when H is 2-connected, $V(H^d)$ is just $\{v \in V(H) \mid d_G(v) = d\}$. In the proof of Theorem 2.1, the H's for which we use H^d will be complete graphs or odd cycles and hence 2-connected. In the proof of Theorem 2.2 we need the more general definition. We prove two partition theorems of similar form. All of our coloring results will follow from the first theorem, the second theorem is a degeneracy result from which Borodin's result in [2] follows. For unification purposes, define a *t-obstruction* as an odd cycle when $t = 2$ and a K_{t+1} when $t \geq 3$.

Theorem 2.1. *Let G be a graph, $k, d \in \mathbb{N}$ with $k \geq 2$ and $\mathbf{r} \in \mathbb{N}^k_{\geq 2}$. If $w(\mathbf{r}) \geq \max\{\Delta(G) + 1 - k, d\}$, then at least one of the following holds:*

(1) *$w(\mathbf{r}) = d$ and G contains an induced subgraph Q with $|Q| = d + 1$ which can be partitioned into k cliques $F_1, \dots, F_k$ where*
  (a) *$|F_1| = r_1 + 1$, $|F_i| = r_i$ for $i \geq 2$,*
  (b) *$|F_1^d| \geq 2$, $|F_i^d| \geq 1$ for $i \geq 2$,*
  (c) *for $i \in [k]$, each $v \in V(F_i^d)$ is universal in Q;*

(2) *there exists an $\mathbf{r}$-partition $P := (V_1, \dots, V_k)$ of G such that if C is an r_i-obstruction in $G[V_i]$, then $\delta_G(C) \geq d$ and C^d is edgeless.*

Proof. For $i \in [k]$, call a connected graph C *i-bad* if C is an r_i-obstruction such that C^d has an edge. For a graph H and $i \in [k]$, let $b_i(H)$ be the number of i-bad components of H. For an $\mathbf{r}$-partition $P := (V_1, \dots, V_k)$ of G let

$$b(P) := \sum_{i \in [k]} b_i(G[V_i]).$$

Let $P := (V_1, \dots, V_k)$ be an $\mathbf{r}$-partition of $V(G)$ minimizing $b(P)$.

Let $i \in [k]$ and $x \in V_i$ with $d_{V_i}(x) \geq r_i$. Suppose $d_G(x) = d$. Then, since $w(\mathbf{r}) \geq d$, for every $j \neq i$ we have $d_{V_j}(x) \leq r_j$. Moving x from V_i to V_j gives a new partition P^* with $f(P^*) \leq f(P)$. Note that if $d_G(x) < d$ we would have $f(P^*) < f(P)$ contradicting the minimality of P.

Supppose (2) fails to hold. Then $b(P) > 0$. By symmetry, we may assume that there is a 1-bad component A_1 of $G[V_1]$. Put $P_1 := P$ and $V_{1,i} := V_i$ for $i \in [k]$. Since A_1 is 1-bad we have $x_1 \in V(A_1^d)$ which has a neighbor in $V(A_1^d)$. By the above we can move x_1 from $V_{1,1}$ to $V_{1,2}$ to get a new partition $P_2 := (V_{2,1}, V_{2,2}, \dots, V_{2,k})$ where $f(P_2) = f(P_1)$. Since removing x_1 from A_1 decreased $b_1(G[V_1])$, minimality of $b(P_1)$ implies that x_1 is in a 2-bad component A_2 in $V_{2,2}$. Now, we may choose $x_2 \in V(A_2^d) - \{x_1\}$ having a neighbor in A_2^d and move x_2 from $V_{2,2}$ to $V_{2,1}$ to get a new partition $P_3 := (V_{3,1}, V_{3,2}, \dots, V_{3,k})$ where $f(P_3) = f(P_1)$. We continue on this way to construct sequences $A_1, A_2, \dots$, $P_1, P_2, P_3, \dots$ and $x_1, x_2, \dots$.

This process can be defined recursively as follows. For $t \in \mathbb{N}$, put $j_t := 1$ for odd t and $j_t := 2$ for even t. Put $P_1 := P$ and $V_{1,i} := V_i$ for $i \in [k]$. Pick $x_1 \in V(A_1^d)$ which has a neighbor in $V(A_1^d)$. Move x_1 from $V_{1,1}$ to $V_{1,2}$ to get a new partition $P_2 := (V_{2,1}, V_{2,2}, \ldots, V_{2,k})$ where $f(P_2) = f(P_1)$ and let A_2 be the 2-bad component in $V_{2,2}$ containing x_1. Then for $t \geq 2$, pick $x_t \in V(A_t^d - x_{t-1})$ which has a neighbor in $V(A_t^d)$. Move x_t from V_{t,j_t} to $V_{t,3-j_t}$ to get a new partition $P_{t+1} := (V_{t+1,1}, V_{t+1,2}, \ldots, V_{t+1,k})$ where $f(P_{t+1}) = f(P_t)$ and let A_{t+1} be the $(3 - j_t)$-bad component in $V_{t+1,3-j_t}$ containing x_t.

Since G is finite, at some point we will need to reuse a leftover component; that is, there is a smallest t such that $A_{t+1} - x_t = A_s - x_s$ for some $s < t$. Let $j \in [2]$ be such that in $V(A_s) \subseteq V_{s,j}$. Then $V(A_t) \subseteq V_{t,3-j}$.

Claim 1. *$N(x_t) \cap V(A_s - x_s) = N(x_s) \cap V(A_s - x_s)$.*

This is immediate since A_s is r_j-regular.

Claim 2. *$s = 1$, $t = 2$, both A_s and A_t are complete, A_s^d is joined to $A_t - x_{t-1}$ and A_t^d is joined to $A_s - x_s$.*

Subclaim 2a. *$N(x_s) \cap V(A_s^d) \neq \emptyset$.*

In the construction of the sequence, x_s was chosen such that it had a neighbor in A_s^d.

Subclaim 2b. *For any $z \in N(x_s) \cap V(A_s^d)$ we have $N(z) \cap V(A_t - x_{t-1}) = N(x_{t-1}) \cap V(A_t - x_{t-1})$. Moreover, if x_s is adjacent to x_t, then $N(x_s) \cap V(A_t - x_{t-1}) = N(x_{t-1}) \cap V(A_t - x_{t-1})$ and $x_s = x_{t-1}$.*

In P_s, move z to $V_{s,3-j}$ to get a new partition $P^\gamma := (V_{\gamma,1}, V_{\gamma,2}, \ldots, V_{\gamma,k})$. Then z must create an r_{3-j}-obstruction with $A_t - x_{t-1}$ in $V_{\gamma,3-j}$ since z is adjacent to x_t by Claim 1. In particular, $N(z) \cap V(A_t - x_{t-1}) = N(x_{t-1}) \cap V(A_t - x_{t-1})$. If x_s is adjacent to x_t, the same argument (with x_s in place of z) gives $N(x_s) \cap V(A_t - x_{t-1}) = N(x_{t-1}) \cap V(A_t - x_{t-1})$ and $x_s = x_{t-1}$.

Subclaim 2c. *A_s is complete and x_s is adjacent to x_t.*

By Subclaim 2a, $N(x_s) \cap V(A_s^d) \neq \emptyset$. Pick $z \in N(x_s) \cap V(A_s^d)$ and let P^γ be as in Subclaim 2b. In P^γ, move x_t to $V_{\gamma,j}$ to get a new partition $P^{\gamma *} := (V_{\gamma *,1}, V_{\gamma *,2}, \ldots, V_{\gamma *,k})$. Since x_s has at least two neighbors in A_s, by Claim 1, x_t has a neighbor in $A_s - z$. Hence x_t must create an r_j-obstruction with $A_s - z$ in $V_{\gamma *,j}$. In particular, $N(z) \cap V(A_s - z) = N(x_t) \cap V(A_s - z)$. Thus x_s is adjacent to x_t and we have $N[z] \cap V(A_s) = N[x_s] \cap V(A_s)$. Thus, if A_s is an odd cycle, it must be a triangle. Hence A_s is complete.

Subclaim 2d. *A_s^d is joined to $N(x_{t-1}) \cap V(A_t - x_{t-1})$ and $x_s = x_{t-1}$.*

Since A_s is complete by Subclaim 2c, we have $N(x_s) \cap V(A_s^d) = V(A_s^d - x_s)$. Since x_s is adjacent to x_t by Subclaim 2c, applying Subclaim 2b shows that A_s^d is joined to $N(x_{t-1}) \cap V(A_t - x_{t-1})$ and $x_s = x_{t-1}$.

Subclaim 2e. *$s = 1$ and $t = 2$.*

Suppose $s > 1$. Then, since $x_{s-1} \in V(A_s^d)$, Subclaim 2d shows that x_{s-1} is joined to $N(x_{t-1}) \cap V(A_t - x_{t-1})$ and hence $A_t - x_{t-1} = A_{s-1} - x_{s-1}$ violating minimality of t. Whence, $s = 1$ and $t = 2$.

Subclaim 2f. *A_t is complete and A_s^d is joined to $A_t - x_{t-1}$.*

Pick $z \in N(x_s) \cap V(A_s^d)$. Then z is joined to $A_t - x_t$ by Subclaim 2d. In P_{t+1}, move z to $V_{t+1,3-j}$ to get a new partition $P^\beta := (V_{\beta,1}, V_{\beta,2}, \ldots, V_{\beta,k})$. Then z must create an r_{3-j}-obstruction with $A_t - x_t$ in $V_{\beta,3-j}$. In particular, $V(A_t - x_t) = N(z) \cap V(A_t - x_t) =$

$N(x_t) \cap V(A_t - x_t)$. Thus, if A_t is an odd cycle, it must be a triangle. Hence A_t is complete. Now Subclaim 2d gives that A_s^d is joined to $A_t - x_{t-1}$.

Subclaim 2g. *A_t^d is joined to $A_s - x_s$.*

Since $x_s = x_{t-1}$, the statement is clear for x_{t-1}. Pick $y \in V(A_t^d - x_{t-1})$ and $z \in V(A_s^d)$. In P_t, move y to $V_{t,j}$. Since y is adjacent to z by Subclaim 2f, y must create an r_j-obstruction with $A_s - x_s$ and since A_s is complete, y must be joind to $A_s - x_s$. Hence A_t^d is joined to $A_s - x_s$.

Claim 3. *(1) holds.*

We can play the same game with V_1 and V_i for any $3 \leq i \leq k$ as we did with V_1 and V_2 above. Let $B_1 := A_1$, $B_2 := A_2$ and for $i \geq 3$, let B_i be the r_i-obstruction made by moving x_1 into V_i. Then B_i is complete for each $i \in [k]$. Applying Claim 2 to all pairs B_i, B_j shows that for any distinct $i, j \in [k]$, B_i^d is joined to $B_j - x_1$. Put $F_1 = B_1$ and $F_i = B_i - x_1$ for $i \geq 2$. Let Q be the union of the F_i. Then (a), (b) and (c) of (1) are satisfied. Note that $|Q| = w(\mathbf{r}) + 1$ and since any $v \in B_1^d$ is universal in Q, $|Q| \leq d + 1$. By assumption $w(\mathbf{r}) \geq d$, whence $w(\mathbf{r}) = d$. Hence, (1) holds. □

The following result generalizes a lemma due to Borodin [2]. This lemma of Borodin was generalized in another direction in [3]. The proof that follows is basically the same as that of Theorem 2.1. For a reader that is only interested in the coloring results, this theorem can be safely skipped.

Theorem 2.2. *Let G be a graph, $k, d \in \mathbb{N}$ with $k \geq 2$ and $\mathbf{r} \in \mathbb{N}_{\geq 1}^k$ where at most one of the r_i is one. If $w(\mathbf{r}) \geq \max\{\Delta(G) + 1 - k, d\}$, then at least one of the following holds:*

1. *$w(\mathbf{r}) = d$ and G contains a $K_t * E_{d+1-t}$ where $t \geq d + 1 - k$, for each $v \in V(K_t)$ we have $d_G(v) = d$ and for each $v \in V(E_{d+1-t})$ we have $d_G(v) > d$; or,*
2. *there exists an $\mathbf{r}$-partition $P := (V_1, \ldots, V_k)$ of G such that if C is an r_i-regular component of $G[V_i]$, then $\delta_G(C) \geq d$ and there is at most one $x \in V(C^d)$ with $d_{C^d}(x) \geq r_i - 1$. Moreover, P can be chosen so that either:*
   - (a) *for all $i \in [k]$ and r_i-regular component C of $G[V_i]$, we have $\left|C^d\right| \leq 1$; or,*
   - (b) *for some $i \in [k]$ and some r_i-regular component C of $G[V_i]$, there is $x \in V(C^d)$ such that $\{y \in N_C(x) \mid d_G(y) = d\}$ is a clique.*

Proof. For $i \in [k]$, call a connected graph C *i-bad* if C is r_i-regular and there are at least two $x \in V(C^d)$ with $d_{C^d}(x) \geq r_i - 1$. We say that such an x *witnesses* the i-badness of C. For a graph H and $i \in [k]$, let $b_i(H)$ be the number of i-bad components of H. For an $\mathbf{r}$-partition $P := (V_1, \ldots, V_k)$ of G let

$$c(P) := \sum_{i \in [k]} c(G[V_i]),$$

$$b(P) := \sum_{i \in [k]} b_i(G[V_i]).$$

Let $P := (V_1, \ldots, V_k)$ be an $\mathbf{r}$-partition of $V(G)$ minimizing $c(P)$ and subject to that $b(P)$.

Let $i \in [k]$ and $x \in V_i$ with $d_{V_i}(x) \geq r_i$. Suppose $d_G(x) = d$. Then, since $w(\mathbf{r}) \geq d$, for every $j \neq i$ we have $d_{V_j}(x) \leq r_j$. Moving x from V_i to V_j gives a new partition P^* with

$f(P^*) \leq f(P)$. Note that if $d_G(x) < d$ we would have $f(P^*) < f(P)$ contradicting the minimality of P.

Suppose $b(P) > 0$. By symmetry, we may assume that there is a 1-bad component A_1 of $G[V_1]$. Put $P_1 := P$ and $V_{1,i} := V_i$ for $i \in [k]$. Since A_1 is 1-bad we have $x_1 \in V(A_1^d)$ with $d_{A_1^d}(x) \geq r_1 - 1$. By the above we can move x_1 from $V_{1,1}$ to $V_{1,2}$ to get a new partition $P_2 := (V_{2,1}, V_{2,2}, \ldots, V_{2,k})$ where $f(P_2) = f(P_1)$. By the minimality of $c(P_1)$, x_1 is adjacent to only one component C_2 in $G[V_{1,2}]$. Let $A_2 := G[V(C_2) \cup \{x_1\}]$. Since removing x_1 from A_1 decreased $b_1(G[V_1])$, minimality of $b(P_1)$ implies that A_2 is 2-bad. Now, we may choose $x_2 \in V(A_2^d) - \{x_1\}$ with $d_{A_2^d}(x) \geq r_2 - 1$ and move x_2 from $V_{2,2}$ to $V_{2,1}$ to get a new partition $P_3 := (V_{3,1}, V_{3,2}, \ldots, V_{3,k})$ where $f(P_3) = f(P_1)$.

Continue on this way to construct sequences $A_1, A_2, \ldots$, $P_1, P_2, P_3, \ldots$ and $x_1, x_2, \ldots$. Since G is finite, at some point we will need to reuse a leftover component; that is, there is a smallest t such that $A_{t+1} - x_t = A_s - x_s$ for some $s < t$. Let $j \in [2]$ be such that in $V(A_s) \subseteq V_{s,j}$. Then $V(A_t) \subseteq V_{t,3-j}$. Note that, since A_s is r_j-regular, $N(x_t) \cap V(A_s - x_s) = N(x_s) \cap V(A_s - x_s)$.

We claim that $s = 1$, $t = 2$, both A_s and A_t are complete, A_s^d is joined to $A_t - x_{t-1}$ and A_t^d is joined to $A_s - x_s$.

Put $X := N(x_s) \cap V(A_s^d)$. Since x_s witnesses the j-badness of A_s, $|X| \geq \max\{1, r_j - 1\}$. Pick $z \in X$. In P_s, move z to $V_{s,3-j}$ to get a new partition $P^\gamma := (V_{\gamma,1}, V_{\gamma,2}, \ldots, V_{\gamma,k})$. Then z must create an r_{3-j}-regular component with $A_t - x_{t-1}$ in $V_{\gamma,3-j}$ since z is adjacent to x_t. In particular, $N(z) \cap V(A_t - x_{t-1}) = N(x_{t-1}) \cap V(A_t - x_{t-1})$. Since z is adjacent to x_t, so is x_{t-1}.

Suppose $r_j \geq 2$. In P^γ, move x_t to $V_{\gamma,j}$ to get a new partition $P^{\gamma*} := (V_{\gamma*,1}, V_{\gamma*,2}, \ldots, V_{\gamma*,k})$. Then x_t must create an r_j-regular component with $A_s - z$ in $V_{\gamma*,j}$. In particular, $N(z) \cap V(A_s - z) = N(x_t) \cap V(A_s - z)$. Thus x_s is adjacent to x_t and we have $N[z] \cap V(A_s) = N[x_s] \cap V(A_s)$. Put $K := X \cup \{x_s\}$. Then $|K| \geq r_j$ and K induces a clique. If $|K| > r_j$, then $A_s = K$ is complete. Otherwise, the vertices of K have a common neighbor $y \in V(A_s) - K$ and again A_s is complete. Also, since x_s is adjacent to x_t, using x_s in place of z in the previous paragraph, we conclude that K is joined to $N(x_{t-1}) \cap V(A_t - x_{t-1})$ and $x_s = x_{t-1}$.

Suppose $s > 1$. Then x_{s-1} is joined to $N(x_{t-1}) \cap V(A_t - x_{t-1})$ and hence $A_t - x_{t-1} = A_{s-1} - x_{s-1}$ violating minimality of t. Whence, if $r_j \geq 2$ then $s = 1$.

Note that $K = V(A_s^d)$ and hence if $r_j \geq 2$ then A_s is complete and A_s^d is joined to $N(x_{t-1}) \cap V(A_t - x_{t-1})$. If $r_{3-j} = 1$, then A_t is a K_2 and $N(x_{t-1}) \cap V(A_t - x_{t-1}) = V(A_t - x_{t-1}) = \{x_t\}$. We already know that x_t is joined to $A_s - x_s$. Thus the cases when $r_j \geq 2$ and $r_{3-j} = 1$ are taken care of. By assumption, at least one of r_j or r_{3-j} is at least two. Hence it remains to handle the cases with $r_{3-j} \geq 2$.

Suppose $r_{3-j} \geq 2$. In P_{t+1}, move z to $V_{t+1,3-j}$ to get a new partition $P^\beta := (V_{\beta,1}, V_{\beta,2}, \ldots, V_{\beta,k})$. Then z must create an r_{3-j}-regular component with $A_t - x_t$ in $V_{\beta,3-j}$. In particular, $N(z) \cap V(A_t - x_t) = N(x_t) \cap V(A_t - x_t)$. Since $N(z) \cap V(A_t - x_{t-1}) = N(x_{t-1}) \cap V(A_t - x_{t-1})$, we have $N[x_{t-1}] \cap V(A_t) = N(z) \cap V(A_t) = N[x_t] \cap V(A_t)$. Put $W := N[x_t] \cap V(A_t^d)$. Each $w \in W$ is adjacent to z and running through the argument above with w in place of x_t shows that W is a clique joined to z. Moreover, since x_t witnesses the $(3-j)$-badness of A_t, $|W| \geq r_{3-j}$. As with A_s above, we conclude that A_t is complete. Since $x_s \in V_{t+1,3-j}$ and x_s is adjacent to z, it must be that $x_s \in V(A_t - x_t)$. Thence x_s is joined to W and $x_s = x_{t-1}$.

Suppose that $r_j \geq 2$ as well. We know that $s = 1$, A_s is complete and A_s^d is joined to $N(x_{t-1}) \cap V(A_t - x_{t-1}) = A_t - x_{t-1}$. Also, we just showed that A_t is complete and A_t^d is joined to $A_s - x_s$.

Thus, we must have $r_j = 1$ and $r_{3-j} \geq 2$. Then, since A_s is a K_2, by the above, A_s is joined to W. Since $W = A_t^d$, it only remains to show that $s = 1$. Suppose $s > 1$. Then x_{s-1} is joined to W and hence $A_t - x_{t-1} = A_{s-1} - x_{s-1}$ violating minimality of t.

Therefore $s = 1$, $t = 2$, both A_s and A_t are complete, A_s^d is joined to $A_t - x_{t-1}$ and A_t^d is joined to $A_s - x_s$. But we can play the same game with V_1 and V_i for any $3 \leq i \leq k$ as well. Let $B_1 := A_1$, $B_2 := A_2$ and for $i \geq 3$, let B_i be the r_i-regular component made by moving x_1 into V_i. Then B_i is complete for each $i \in [k]$. Applying what we just proved to all pairs B_i, B_j shows that for any distinct $i, j \in [k]$, B_i^d is joined to $B_j - x_1$. Since $\left|B_i^d\right| \geq r_i$ and $x_1 \in V(B_i^d)$ for each i, this gives a $K_t * E_{w(\mathbf{r})+1-t}$ in G where $t \geq w(\mathbf{r}) + 1 - k$. Take such a subgraph Q maximizing t. Since all the B_i are complete, any vertex of degree d will be in B_i^d; therefore, for each $v \in V(K_t)$ we have $d_G(v) = d$ and for each $v \in V(E_{w(\mathbf{r})+1-t})$ we have $d_G(v) > d$. Note that $|Q| = w(\mathbf{r}) + 1$ and since $d_G(v) = d$ for any $v \in V(K_t)$, $|Q| \leq d + 1$. By assumption $w(\mathbf{r}) \geq d$, whence $w(\mathbf{r}) = d$. Thus if (1) fails, then the first part of (2) holds.

It remains to prove that we can choose P to satisfy one of (a) or (b). Suppose that (1) fails and P cannot be chosen to satisfy either (a) or (b). For $i \in [k]$, call a connected graph C *i-ugly* if C is r_i-regular and $\left|C^d\right| \geq 2$ let $u_i(H)$ be the number of i-ugly components of H. Note that if C is i-bad, then it is i-ugly. For an $\mathbf{r}$-partition $P := (V_1, \ldots, V_k)$ of G let

$$u(P) := \sum_{i \in [k]} u_i(G[V_i]).$$

Choose an $\mathbf{r}$-partition $Q := (V_1, \ldots, V_k)$ of G first minimizing $c(Q)$, then subject to that requiring $b(Q) \leq 1$ and then subject to that minimizing $u(Q)$. Since Q does not satisfy (a), at least one of $b(Q) = 1$ or $u(Q) \geq 1$ holds. By symmetry, we may assume that $G[V_1]$ contains a component D_1 which is either 1-bad or 1-ugly (or both). If D_1 is 1-bad, pick $w_1 \in V(D_1^d)$ witnessing the 1-badness of D_1; otherwise pick $w_1 \in V(D_1^d)$ arbitrarily. Move w_1 to V_2, to form a new $\mathbf{r}$-partition. This new partition still satisfies all of our conditions on Q. As above we construct a sequence of vertex moves that will wrap around on itself. This can be defined recursively as follows. For $t \geq 2$, if D_t is bad pick $w_t \in V(D_t^d - w_{t-1})$ witnessing the badness of D_t; otherwise, if D_t is ugly pick $w_t \in V(D_t^d - w_{t-1})$ arbitrarily. Now move w_t to the part from which w_{t-1} came to form D_{t+1}. Let $Q_1 := Q, Q_2, Q_3, \ldots$ be the partitions created by a run of this process. Note that the process can never create a component which is not ugly lest we violate the minimality of $u(Q)$.

Since G is finite, at some point we will need to reuse a leftover component; that is, there is a smallest t such that $D_{t+1} - x_t = D_s - x_s$ for some $s < t$. First, suppose D_s is not bad, but merely ugly. Then D_{t+1} is not bad and hence $b(Q_{t+1}) = 0$ and $u(Q_{t+1}) < u(Q)$, a contradiction. Hence D_s is bad.

Suppose D_t is not bad. As in the proof of the first part of (2), we can conclude that $x_s = x_{t-1}$. Pick $z \in N(x_s) \cap V(D_s^d)$. Since z is adjacent to x_t, by moving z to the part containing x_t in P_s we conclude $N(z) \cap V(D_t - x_s) = N(x_s) \cap V(D_t - x_s)$. Put $T := \{y \in N_{D_t}(x_s) \mid d_G(y) = d\}$. Suppose T is not a clique and let $w_1, w_2 \in T$ be nonadjacent.

Now, in P_t, since z is adjacent to both w_1 and w_2, swapping w_1 and w_2 with z contradicts minimality of $f(Q)$. Hence T is a clique and (b) holds, a contradiction.

Thus we may assume that D_t is bad as well. Now we may apply the same argument as in the proof of the first part of (2) to show that (1) holds. This final contradiction completes the proof. $\square$

Corollary 2.3 (Borodin [2])**.** *Let G be a graph not containing a $K_{\Delta(G)+1}$. If $r_1, r_2 \in \mathbb{N}_{\geq 1}$ with $r_1 + r_2 \geq \Delta(G) \geq 3$, then $V(G)$ can be partitioned into sets V_1, V_2 such that $\Delta(G[V_i]) \leq r_i$ and $\operatorname{col}(G[V_i]) \leq r_i$ for $i \in [2]$.*

Proof. Apply Theorem 2.2 with $\mathbf{r} := (r_1, r_2)$ and $d = \Delta(G)$. Since G doesn't contain a $K_{\Delta(G)+1}$ and no vertex in G has degree larger than d, (1) cannot hold. Thus (2) must hold. Let $P := (V_1, V_2)$ be the guaranteed partition and suppose that for some $j \in [2]$, $G[V_j]$ contains an r_j-regular component H. Then every vertex of H has degree d in G and hence H^d contains all noncutvertices of H. But H has maximum degree r_j and thus contains at least r_j noncutvertices. If $r_j = 1$, then H is K_2 and hence has 2 noncutvertices. In any case, we have $\left|H^d\right| \geq 2$. Hence (a) cannot hold for P. Thus, by (b), we have $i \in [2]$, an r_i-regular component C of $G[V_i]$ and $x \in V(C)$ such that $N_C(x)$ is a clique. But then C is K_{r_i+1} violating (2), a contradiction.

Therefore, for $i \in [2]$, each component of $G[V_i]$ contains a vertex of degree at most $r_i - 1$. Whence $\operatorname{col}(G[V_i]) \leq r_i$ for $i \in [2]$. $\square$

3. Coloring

Using Theorem 2.1, we can prove coloring results for graphs with only small cliques among the vertices of high degree. To make this precise, for $d \in \mathbb{N}$ define $\omega_d(G)$ to be the cardinality of the largest clique in G containing only vertices of degree larger than d; that is, $\omega_d(G) := \omega\left(G\left[\{v \in V(G) \mid d_G(v) > d\}\right]\right)$.

Corollary 3.1. *Let G be a graph, $k, d \in \mathbb{N}$ with $k \geq 2$ and $\mathbf{r} \in \mathbb{N}^k$. If $w(\mathbf{r}) \geq \max\{\Delta(G) + 1 - k, d\}$ and $r_i \geq \omega_d(G) + 1$ for all $i \in [k]$, then at least one of the following holds:*

1. *$w(\mathbf{r}) = d$ and G contains an induced subgraph Q with $|Q| = d + 1$ which can be partitioned into k cliques $F_1, \dots, F_k$ where*
   - (a) *$|F_1| = r_1 + 1$, $|F_i| = r_i$ for $i \geq 2$,*
   - (b) *$\left|F_i^d\right| \geq |F_i| - \omega_d(G)$ for $i \in [k]$,*
   - (c) *for $i \in [k]$, each $v \in V(F_i^d)$ is universal in Q;*
2. *$\chi(G) \leq w(\mathbf{r})$.*

Proof. Apply Theorem 2.1 to conclude that either (1) holds or there exists an $\mathbf{r}$-partition $P := (V_1, \dots, V_k)$ of G such that if C is an r_i-obstruction in $G[V_i]$, then $\delta_G(C) \geq d$ and C^d is edgeless. Since $\Delta(G[V_i]) \leq r_i$ for all $i \in [k]$, it will be enough to show that no $G[V_i]$ contains an r_i-obstruction. Suppose otherwise that we have an r_i-obstruction C in some $G[V_i]$. First, if $r_i \geq 3$, then C is K_{r_i+1} and hence C contains a $K_{\omega_d(G)+2}$. But C^d is edgeless, so $\omega_d(G) \geq \omega_d(C) \geq \omega(C) - 1 \geq \omega_d(G) + 1$, a contradiction. Thus $r_i = 2$ and C is an odd cycle. Since C^d is edgeless and $\omega_d(C) \leq \omega_d(G) \leq 1$, we have a 2-coloring $\left\{V(C^d), V(C - C^d)\right\}$ of the odd cycle C, a contradiction. $\square$

For a vertex-critical graph G, call $v \in V(G)$ *low* if $d(v) = \chi(G) - 1$ and *high* otherwise. Let $\mathcal{H}(G)$ be the subgraph of G induced on the high vertices of G.

Corollary 3.2. *Let G be a vertex-critical graph with $\chi(G) = \Delta(G) + 2 - k$ for some $k \geq 2$. If $k \leq \frac{\chi(G)-1}{\omega(\mathcal{H}(G))+1}$, then G contains an induced subgraph Q with $|Q| = \chi(G)$ which can be partitioned into k cliques $F_1, \ldots, F_k$ where*

(1) $|F_1| = \chi(G) - (k-1)(\omega(\mathcal{H}(G)) + 1)$, $|F_i| = \omega(\mathcal{H}(G)) + 1$ *for $i \geq 2$;*
(2) *for each $i \in [k]$, F_i contains at least $|F_i| - \omega(\mathcal{H}(G))$ low vertices which are all universal in Q.*

Proof. Suppose $k \leq \frac{\chi(G)-1}{\omega(\mathcal{H}(G))+1}$. Put $r_i := \omega(\mathcal{H}(G)) + 1$ for $i \in [k] - \{1\}$ and $r_1 := \chi(G) - 1 - (k-1)(\omega(\mathcal{H}(G)) + 1)$. Set $\mathbf{r} := (r_1, r_2, \ldots, r_k)$. Then $w(\mathbf{r}) = \chi(G) - 1 = \Delta(G) + 1 - k$. Now applying Corollary 3.1 with $d := \chi(G) - 1$ proves the corollary. □

Corollary 3.3. *Let G be a vertex-critical graph with $\chi(G) \geq \Delta(G) + 1 - p \geq 4$ for some $p \in \mathbb{N}$. If $\omega(\mathcal{H}(G)) \leq \frac{\chi(G)+1}{p+1} - 2$, then $G = K_{\chi(G)}$ or $G = O_5$.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Put $\chi := \chi(G)$, $\Delta := \Delta(G)$ and $h := \omega(\mathcal{H}(G))$. Then $p \geq 1$ and $h \geq 1$ by Brooks' theorem. Hence $\chi \geq 5$. By assumption, we have $h \leq \frac{\chi+1}{p+1} - 2 = \frac{\chi-2p-1}{p+1} \leq \frac{\chi-p-2}{p+1}$ since $p \geq 1$. Thus $p + 1 \leq \frac{\chi-1}{h+1}$ and we may apply Corollary 3.2 with $k := p + 1$ to get an induced subgraph Q of G with $|Q| = \chi$ which can be partitioned into $p + 1$ cliques $F_1, \ldots, F_{p+1}$ where

(1) $|F_1| = \chi - p(h + 1)$, $|F_i| = h + 1$ for $i \geq 2$;
(2) for each $i \in [p + 1]$, F_i contains at least $|F_i| - h$ low vertices which are all universal in Q.

Let T be the low vertices in Q, put $H := Q - T$ and $t := |T|$. Then $Q = K_t * H$ and $t \geq \chi - p(h + 1) + p(h + 1) - (p + 1)h = \chi - (p + 1)h$.

Take any $(\chi - 1)$-coloring π of $G - Q$ and let L be the resulting list assignment on Q; that is, for $v \in V(Q)$ we put $L(v) := [\chi - 1] - \pi(N(v) \cap V(G - Q))$. Then $|L(v)| = d_Q(v)$ for each $v \in T$ and $|L(v)| \geq d_Q(v) - p$ for each $v \in V(H)$. Since $t \geq \chi - (p + 1)h \geq 2p + 1 \geq p + 1$, if there are nonadjacent $x, y \in V(H)$ and $c \in L(x) \cap L(y)$, then we may color x and y both with c and then greedily complete the coloring to the rest of H and then greedily to all of Q, a contradiction. Hence any nonadjacent pair in H have disjoint lists.

Let I be a maximal independent set in H. If there is an induced P_3 in H with ends in I, set $o_I := 1$, otherwise set $o_I := 0$. Since each pair of vertices in I have disjoint lists, we must have

$$\begin{aligned}
\chi - 1 &\geq \sum_{v \in I} |L(v)| \\
&\geq \sum_{v \in I} t + d_H(v) - p \\
&= (t - p)\,|I| + \sum_{v \in I} d_H(v) \\
&\geq (t - p)\,|I| + |H| - |I| + o_I
\end{aligned}$$

$$= (t - (p+1))\,|I| + \chi - t + o_I.$$

Hence $|I| \leq \frac{t-1-o_I}{t-(p+1)} = 1 + \frac{p-o_I}{t-(p+1)} \leq 1 + \frac{p-o_I}{2p+1-(p+1)} \leq 2$ as $t \geq 2p+1$. Since G is not K_χ, we must have $|I| = 2$ and thus $t = 2p+1$ and $o_I = 0$. Thence H is the disjoint union of two complete subgraphs. We then have $\frac{\chi-2p-1}{p+1} \geq h \geq \frac{|H|}{2} = \frac{\chi-2p-1}{2}$. Hence $p = 1$, $h = \frac{\chi-3}{2}$ and $Q = K_3 * 2K_h$.

Let $x, y \in V(H)$ be nonadjacent. Then $d_Q(x) + d_Q(y) = \chi + 1$. Let A be the subgraph of G induced on $V(G-Q) \cup \{x, y\}$. Then $d_A(x) + d_A(y) \leq 2\Delta - (\chi+1) = \chi - 1$. Let A' be the graph obtained by collapsing $\{x, y\}$ to a single vertex v_{xy}. If $\chi(A') \leq \chi - 1$, then we have a $(\chi-1)$-coloring of A in which x and y receive the same color. This is impossible as then we could complete the $(\chi-1)$-coloring to all of G greedily as above. Hence $\chi(A') = \chi$ and thus we have a vertex-critical subgraph Z of A' with $\chi(Z) = \chi$. We must have $v_{xy} \in V(Z)$ and since $d_A(x) + d_A(y) \leq \chi - 1$, v_{xy} is low. Hence, by minimality of $|G|$, $Z = K_\chi$ or $Z = O_5$.

First, suppose $\chi \geq 6$. Then $h \geq 2$ and thus we have $z \in V(H) - \{x, y\}$ nonadjacent to x. Apply the previous paragraph to both pairs $\{x, y\}$ and $\{x, z\}$. The case $Z = O_5$ cannot happen, for then we would have $\chi = \chi(Z) = 5$, a contradiction. Put $X_1 := N(x) \cap V(G-Q)$, $X_2 := N(y) \cap V(G-Q)$, $X_3 := N(z) \cap V(G-Q)$. Then $|X_i| = \frac{\chi-1}{2}$ for $i \in [3]$ and X_1 is joined to both X_2 and X_3. Since $|X_i| - h > 0$, each X_i contains a low vertex v_i. But then $N(v_1) = X_1 \cup X_2 \cup \{x\}$ and we must have $X_3 = X_2$. Whence $N(v_2) = X_1 \cup X_2 \cup \{y, z\}$ giving $d(v_2) \geq \chi$, a contradiction.

Therefore $\chi = 5$, $h = 1$ and $V(H) = \{x, y\}$. If $Z = K_5$, then $N[x] \cup N[y]$ induces an O_5 in G and hence $G = O_5$, a contradiction. Thus $Z = O_5$. But $h = 1$, so all of the neighbors of both x and y are low and hence all of the neighbors of v_{xy} in Z are low. But O_5 has no such low vertex v_{xy} with all low neighbors, so this is impossible. □

Question. The condition on k needed in Corollary 3.2 is weaker than that in Corollary 3.3. What do the intermediate cases look like? What are the extremal examples?

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