SHUFFLE

Call multigraph *standard* if it is loopless with maximum multiplicity at most 2.

Definition 1. For $k \geq 1$, a standard multigraph is k-bad if

- $k = 1$ and G is a simple cycle; or
- $k = 2$ and G is an odd cycle with all multiplicities equal to 2; or
- G is a K_{k+1} with all multiplicities 2; or
- G is $K_{\Delta(G)+1}$.

Lemma 1. *Let G be a connected standard multigraph. If $\Delta(G) \leq 2k$ and G is not k-bad, then $V(G)$ can be partitioned into k (possibly empty) sets $V_1, \dots, V_k$ such that $G[V_i]$ induces a disjoint union of simple paths.*

Proof. Choose a partition $P = \{P_1, \dots, P_k\}$ of $V(G)$

(1) minimizing $\sum_i \|G[P_i]\|$,
(2) subject to (1), minimizing the total number of cycles within parts.

For all $i \in [k]$, we have $\Delta(G[P_i]) \leq 2$ since if $v \in P_i$ has degree > 2 in P_i then it has degree < 2 in some other P_j, moving v to P_j violates (1).

We are done unless some $G[P_i]$ contains a cycle, so by symmetry we may assume that $G[P_1]$ contains a cycle C_0. Put $P^0 = P$ and choose v_0 on C_0. Then, by the minimization in (1), v_0 has degree 2 is every other part. So, we can move v_0 to any other part and preserve (1). Let P^1 be the partition formed by moving v_0 to P_1^0. Since moving v_0 destroys C_0, it must be that v_0 is in a cycle component in $G[P_1^1]$, call this C_1. Now pick $v_1 \in P_1^1$ adjacent to v_0 and move it to P_0^1 to form P^2. Let C_2 be the created cycle. Since G is finite, continuing this way, there is a least t such that for some $s < t$ we have $C_s - v_s = C_t - v_{t-1}$.

By symmetry, we may assume that $C_s \subseteq G[P_0^s]$. Let $v_s x_s$ and $v_s y_s$ be edges in $G[P_0^s]$ (where we could have $x_s = y_s$ for a digon). Then, from the cycle in $G[P_0^t]$, there must also be edges $v_{t-1}x_s$ and $v_{t-1}y_s$. Let $v_{t-2}v_{t-1}$ and $v_{t-2}x_{t-1}$ be edges in $G[P_1^{t-1}]$. In P^s, consider moving x_s to P_1^s. Then, there must be edges $x_s v_{t-1}$ and $x_s x_{t-1}$.

Suppose C_{t-1} is a digon. Then $v_{t-1} = x_{t-1}$. But then the edge $v_{t-1}x_s$ is a double edge and hence $x_s = y_s$ showing that C_s is a digon as well. Now consider the fact that the sequence $P^s, \dots, P^t$ is cyclical, so we can start it from any point and even run it in reverse. This shows that either all or none of $C_s, \dots, C_t$ are digons. When they are all digons, we have an odd cycle with all edges doubled. In particular, $s = 0$.

Now, suppose C_{t-1} is not a digon. Further, suppose C_{t-1} is not a triangle and let y_{t-1} be a neighbor of x_{t-1} different from v_{t-2} and v_{t-1}. In P^t, move x_s to P_1^t. Since we have the edge $x_s x_{t-1}$, we see that x_{t-1} has different neighbors v_{t-2}, y_{t-1} and x_s, so there is some part we can move x_{t-1} to violate (1). Hence C_{t-1} is a triangle. As before, we can start the sequence $P^s, \dots, P^t$ anywhere to conclude that each of $C_s, \dots, C_t$ is a triangle. Since $x_s x_{t-1}$ is an edge, moving x_s to P_1^t shows that $x_s v_{t-2}$ is an edge. Similarly, $y_s v_{t-2}$ is an edge. But the only way for that to happen is for v_s to be v_{t-2}. So $t = s + 2$. Suppose $s > 0$, then by symmetry, we may assume that $v_{s-1} = x_s$. But then x_s has neighbors in two different components in P_1^s, so moving x_s to P_1^s does not create a cycle, violating (2). Hence $s = 0$, C_0 is a triangle joined to $C_1 - v_0$ which is a K_2.

So, the possible sequences come in two types, digon odd cycles or length one joins. Suppose C_0 is not a digon. Then applying the above argument to each part $P_1, \dots, P_k$, get that C_0 is a triangle joined to a K_2 in each of these parts. But we can move a vertex from C_0 to P_1 to form a triangle there and run the same argument to conclude that the K_2 in P_1 is joined to the rest of the K_2's, playing this game for all parts, we conclude that all the K_2's are joined to each other giving a $K_{\Delta(G)+1}$.

Instead, suppose C_0 is a digon and let v be on C_0. Then, by the above argument, moving v to any other part, we must create a digon. But, we can repeat this argument after moving v to any other part with v's neighbors and all their neighbors. We conclude that v's component has all doubled edges. Since G is connected, G has all doubled edges. Since $\Delta(G) \leq 2k$, Brooks' theorem gives a k-coloring of G unless G is a K_{k+1} with all multiplicities 2. $\square$