Math 105 notes

day 1:

Introductions

Logistics

Numbers

- $\overbrace{\text{three}}^{3}$ apples :

  $\underbrace{\text{two}}_{2}$ oranges:

  $+$ no further combining

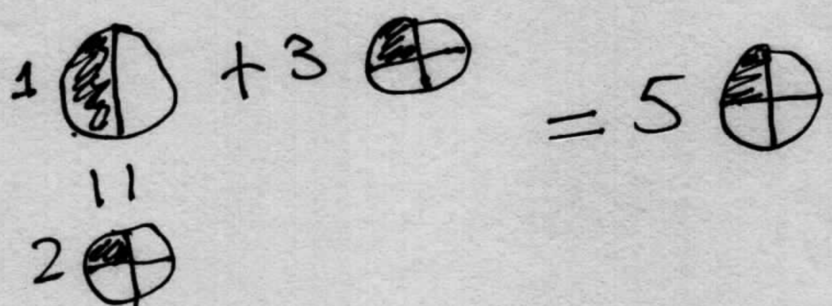
- $3 \text{ tens} + \overbrace{\text{two}}^{2} \text{ fives}$ $\quad$ 10, 10, + 5, 5. 10,

  1 ten = 2 fives : $10 = 5+5$

  $\begin{matrix}10\\10\\10\end{matrix} + \begin{matrix}5\\5\end{matrix} = \begin{matrix}5,5\\5,5\\5,5\end{matrix} + \begin{matrix}5,\\5\end{matrix} = \begin{matrix}5,5\\5,5\\5,5\\5,5\end{matrix} = 8 \text{ fives} = 40 \text{ ones}$

- $\frac{1}{2} + \frac{3}{4}$

  ~~or~~ 1 half + 3 fourths



5 fourths

$= \frac{5}{4}$

- $3x + 2x = 5x$

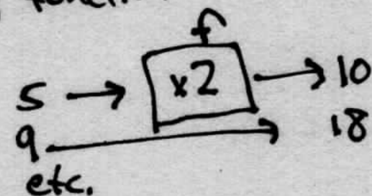
  $3x + 2x^2$ no further combining
  just like the apples and oranges

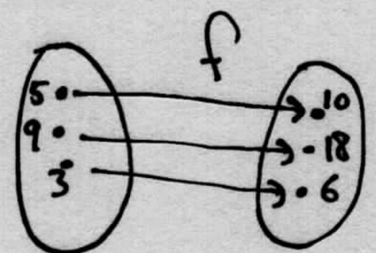
$\frac{x}{2} + \frac{x}{5} = \frac{1}{2}x + \frac{1}{5}x = \left(\frac{1}{2} + \frac{1}{5}\right)x$

$\boxed{\text{functions}}$ machines that take input, give output

$$\text{in} \longrightarrow \boxed{f} \longrightarrow \text{out}$$

ex. doubling function





name output

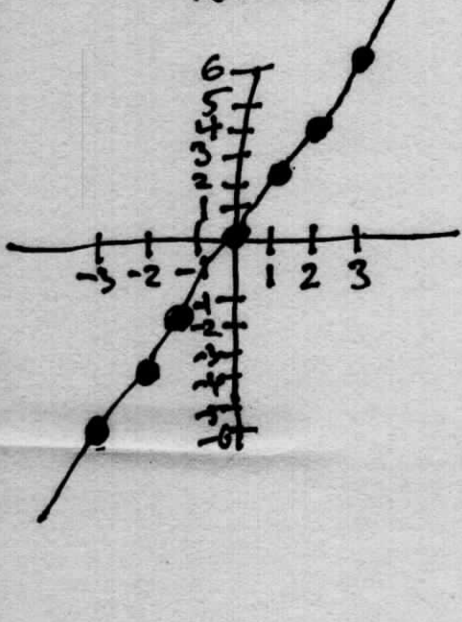
$$f(x) = 2x$$

input

move to day 2 → $\boxed{\text{domain}}$: set of all inputs that make sense

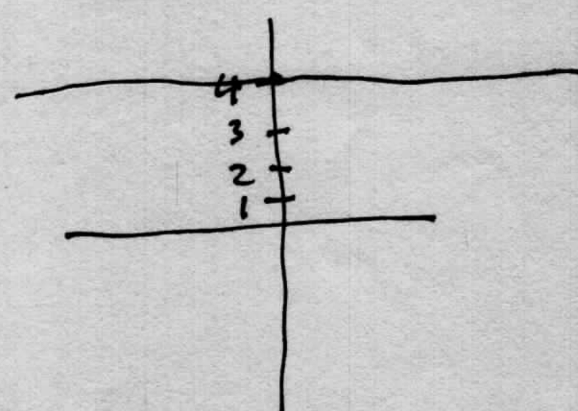
pictures of functions (graphs)

$f(x) = 2x$

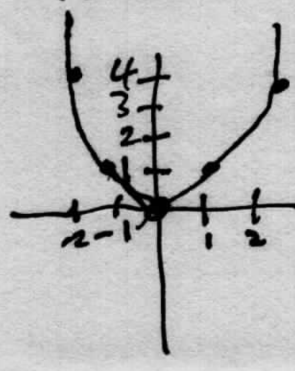


x	f(x)	2x
0	f(0)	0
1	f(1)	2
2	f(2)	4
⋮	⋮	⋮

$f(x) = 4$

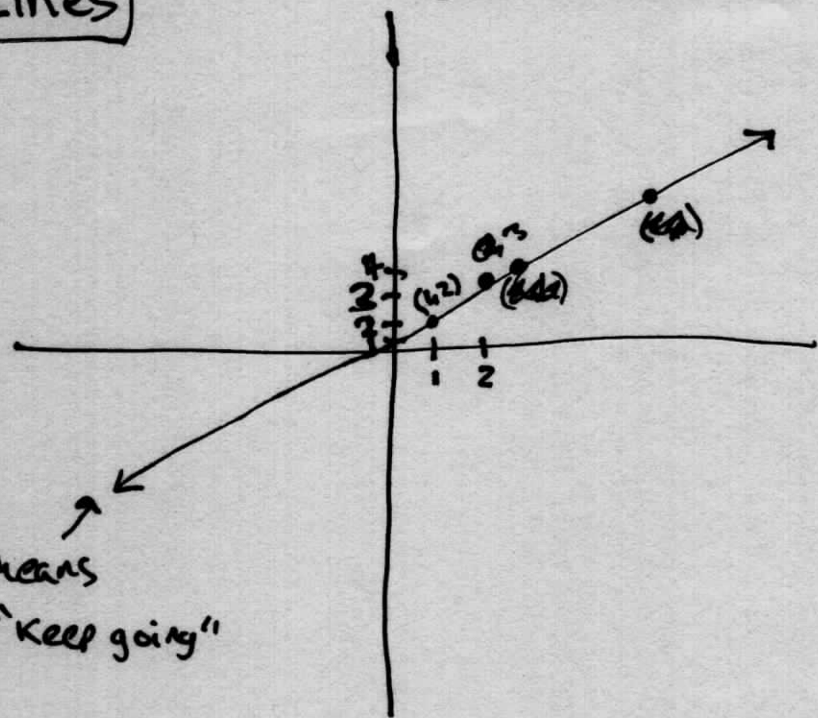


$f(x) = x^2$



Lines

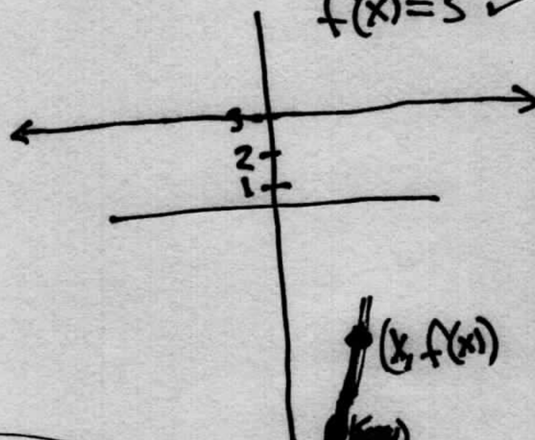
"Coordinate plane"

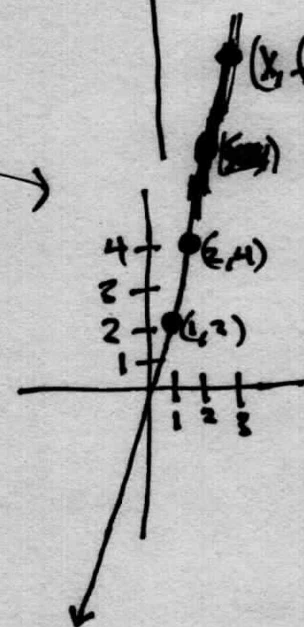


- is the line a picture of some function?
- which one?

<u>easy one:</u>

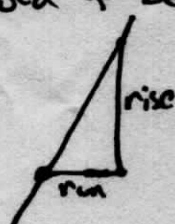
$f(x)=3$ ✓





if the line where a picture of a function f, what would f be?

Line has fixed $\boxed{\text{slope}} = \frac{\text{rise}}{\text{run}}$



in our case

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{4-2}{2-1} = \frac{2}{1} = 2$$

$\parallel$

but also $\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(x)-2}{x-1}$

so $\frac{f(x)-2}{x-1} = 2$, so solving $f(x)-2 = 2(x-1)$

$$f(x) = 2(x-1)+2 = 2x$$

so if the line where a picture of a function f, that function would have to be $f(x)=2x$.

<u>but</u> the line <u>is</u> the picture of $f(x)=2x$.

In the same way we can go from a picture of (almost!) any line to its function

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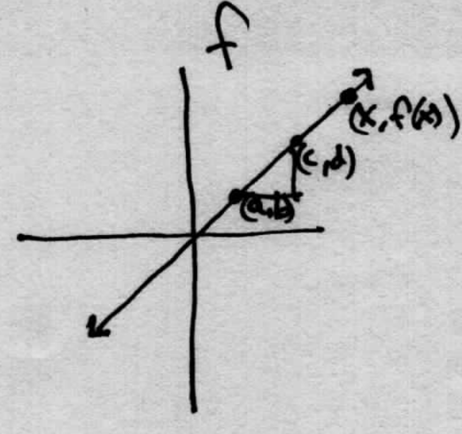
lines cont.



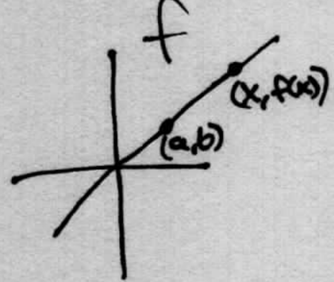
$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{d-b}{c-a}$$

$$\text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(x)-b}{x-a}$$

$$\frac{f(x)-b}{x-a} = \left(\frac{d-b}{c-a}\right)$$

$$f(x)-b = \left(\frac{d-b}{c-a}\right)(x-a)$$

$$\boxed{f(x) = \underbrace{\left(\frac{d-b}{c-a}\right)}_{\text{slope}}(x-a)+b}$$



told line has slope m

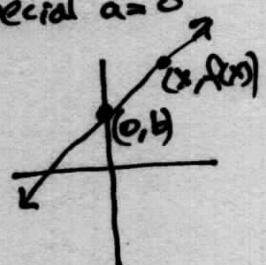
$$m = \text{slope} = \frac{\text{rise}}{\text{run}} = \frac{f(x)-b}{x-a}$$

$$\frac{f(x)-b}{x-a} = m$$

$$f(x)-b = m(x-a)$$

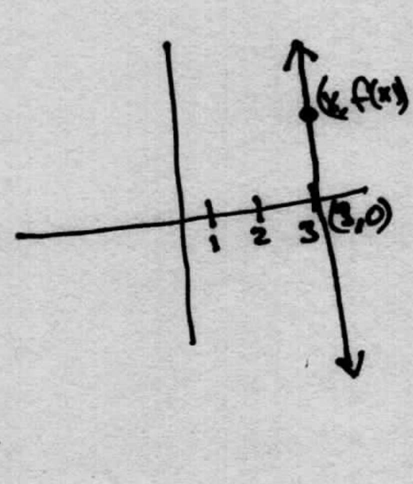
$$\boxed{f(x) = m(x-a)+b}$$ "point slope form"

pick special a = 0



$$\rightarrow \boxed{f(x) = mx+b}$$ "y-intercept form"

Why "almost"?



Let's try as before

$$\text{slope} = ?$$

$$\text{slope} = \frac{f(x)-0}{x-3}$$

oops, above only made sense for $x \neq a$

but there is no point on this line where $x \neq 3$

FAIL!

not all lines are the picture of some function. all but the vertical lines though.

turn paper 90°, good to go!

$\boxed{\text{"parallel"} = \text{same slope}}$

perpendicular lines later after we define distance

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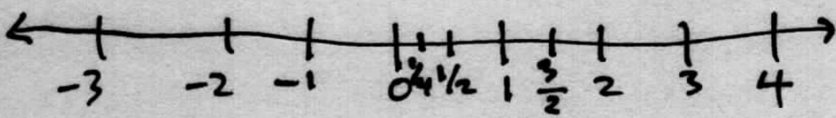
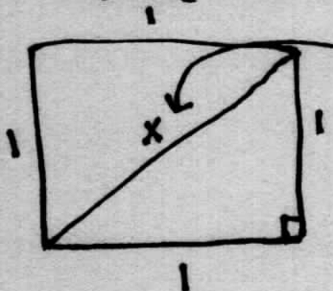
2.7.

integers, rationals, irrationals, reals

number line

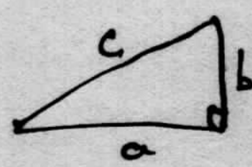


are there other numbers on this line?



what is length of this side?

→ next page, Pythagorean theorem



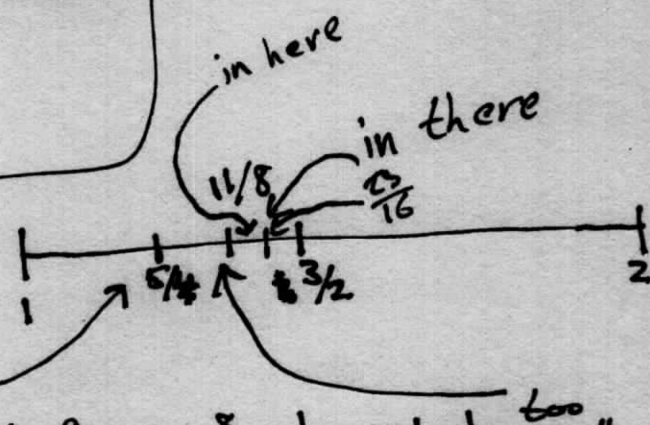
$a^2+b^2=c^2$

$1^2+1^2=x^2$

$2=x^2$ X is some number that when you multiply it by itself, you get 2.

are there integers a and b such that $\left(\frac{a}{b}\right)^2=2$?

try to find some: $\left(\frac{1}{1}\right)^2=1$ too small



$\left(\frac{2}{1}\right)^2=4$ too big $\left(\frac{3}{2}\right)^2=\frac{9}{4}=\frac{8}{4}+\frac{1}{4}=2+\frac{1}{4}$ too big

So in here

$\frac{1+\frac{3}{2}}{2}=\frac{\frac{2}{2}+\frac{3}{2}}{2}=\frac{\frac{5}{2}}{2}$ $\left(\frac{5}{4}\right)^2=\frac{25}{16}=\frac{16}{16}+\frac{9}{16}=1+\frac{8}{16}+\frac{1}{16}=1+\frac{1}{2}+\frac{1}{16}$ too small

$\frac{5}{4}=1.25$

$\frac{\frac{5}{4}+\frac{3}{2}}{2}=\frac{\frac{5}{4}+\frac{6}{4}}{2}=\frac{\frac{11}{4}}{2}=\frac{11}{8}=\frac{8}{8}+\frac{3}{8}=1+\frac{2}{8}+\frac{1}{8}=1+\frac{1}{4}+\frac{1}{8}=1.375$

$\left(\frac{11}{8}\right)^2=\frac{121}{64}=\frac{64}{64}+\frac{57}{64}=1+\frac{32}{64}+\frac{25}{64}=1+\frac{1}{2}+\frac{16}{64}+\frac{9}{64}=1+\frac{1}{2}+\frac{1}{4}+\frac{8}{64}+\frac{1}{64}$

$=1+\frac{1}{2}+\frac{1}{4}+\frac{1}{8}+\frac{1}{64}$

$\left(\frac{23}{16}\right)^2=2.06640625$ too big $\frac{23}{16}=1.4375$

$=1.890625$

$\left(\frac{45}{32}\right)^2=1.9775390625$ too small $\frac{45}{32}=1.40625$

$\sqrt{2}=1.41421356237\ldots$ too small

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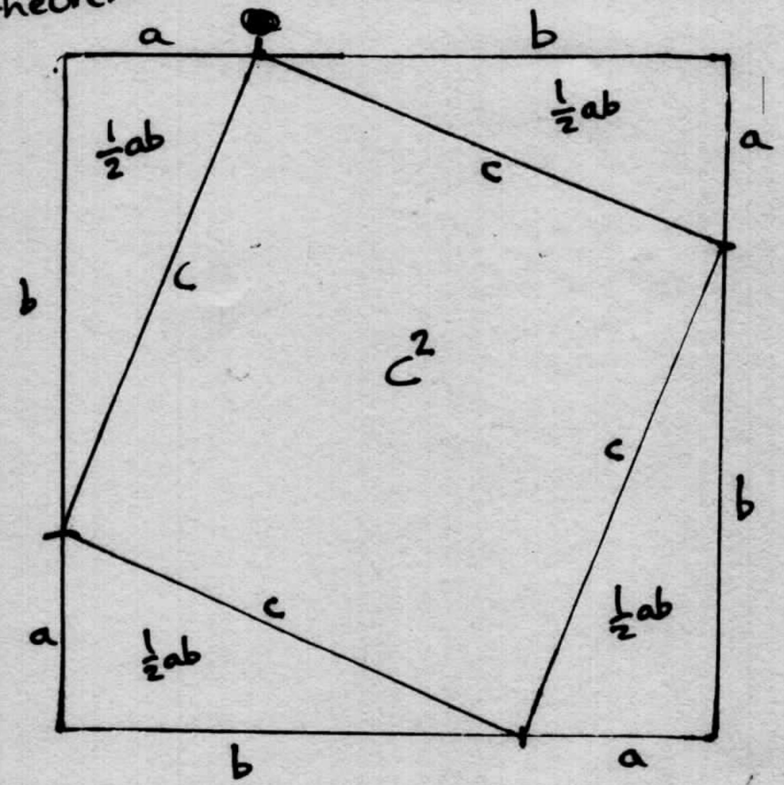
pythagorean theorem



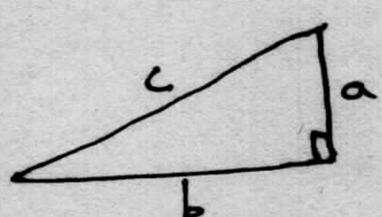
$$(a+b)^2 = c^2 + 4\left(\tfrac{1}{2}ab\right) = c^2 + 2ab$$

$\parallel$

$a^2+2ab+b^2$

$a^2+2ab+b^2 = c^2+2ab$

$\boxed{a^2+b^2=c^2.}$



$\sqrt{2}$ continued

seems like we can just keep getting closer and never get there.

maybe there are no integers a and b such that $\left(\frac{a}{b}\right)^2 = 2$?

how could we show that?

Let's just ~~assume~~ see what we could conclude if there were

$\left(\frac{2a}{2b}\right)^2 = 2$, then $\left(\frac{3a}{3b}\right)^2 = 2$, $\left(\frac{3a}{3b}\right)^2 = 2$, ... so if there is one way there are many

these really aren't very interesting,

Lets get rid of all the common factors on top and bottom

$\sqrt{2}$ continued

Suppose we have integers a and b such that

$$\left(\frac{a}{b}\right)^2 = 2.$$

then $2 = \left(\frac{a}{b}\right)^2 = \frac{a^2}{b^2}$

$2b^2 = a^2$, so a is even. that is, $a = 2c$ for some integer c.

$2b^2 = a^2 = (2c)^2 = 2^2c^2 = 4c^2$

$2b^2 = 4c^2$

$b^2 = 2c^2$

so b is even.

We just showed that if $\left(\frac{a}{b}\right)^2 = 2$ for any integers a and b,

then a and b are both even.

$a = 2c$
$b = 2d$, so $\left(\frac{c}{d}\right)^2 = \left(\frac{2c}{2d}\right)^2 = \left(\frac{a}{b}\right)^2 = 2$, so c and d are even!

$c = 2g$
$d = 2h$

$a = 2c = 2(2g) = 4g$

$b = 2d = 2(2h) = 4h$

a is a multiple of 4.

but we can just keep doing this

a is a multiple of 8

of 16

of 32

of 64

of 128

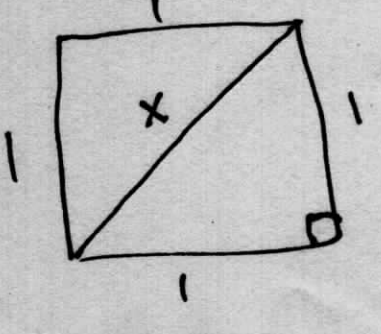
of ⋮

absurdity!

a would then be bigger than every number!

So,
there are not integers a & b
such that $\left(\frac{a}{b}\right)^2 = 2$

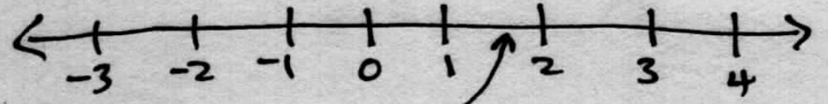
but



x is some length
some number

and $x^2 = 2$

So, there are numbers that cannot be written as $\frac{a}{b}$ for integers a & b.



between 1.40625 and 1.4375

there is a number x such that $x^2=2$.

Let us call x the square root of 2 and write it as $\sqrt{2}$

we can approximate $\sqrt{2}$ further like we were

$\sqrt{2} \approx 1.41421356237\ldots$ ← more digits forever

if $\sqrt{2}=1.41421356237$, then

$$\left(\frac{141421356237}{100000000000}\right)^2=2 \text{ no!!}$$

if $\sqrt{2}=1.4$

$\sqrt{2}=\frac{14}{10} \Rightarrow \left(\frac{14}{10}\right)^2=2$, no!

numbers that can be written as $\frac{a}{b}$ for integers a and b are called rational. numbers that cannot be so written are irrational.

$\sqrt{2}$ is irrational. So is $\sqrt{3}, \sqrt{5}, \sqrt{7}, \sqrt{11}, \ldots$

We will see other irrational numbers that are not square roots such as

$e \approx 2.718281828 45\ldots$

$\pi \approx 3.14159265358979323846\ldots$

there are "more" irrationals than rationals

both infinite, one infinity larger than the other.