most low Alon-Tarsi notes

August 20, 2015

1 Introduction

We consider graphs with vertices labeled by natural numbers; that is, pairs (G, h) where G is a graph and $h: V(G) \to \mathbb{N}$. We say that (G, h) *is AT* if G is $(d_G - h)$-AT. When H is an induced subgraph of G, we simplify notation by referring to the pair (H, h) where we really mean $\left(H, h\restriction_{V(H)}\right)$.

2 Subgraphs, subdivisions and cuts

Definition 1. A graph G is *h-minimal* if G is connected and (H, h) is not AT for every proper induced subgraph H of G. A graph G is *h-greedy-minimal* if G is connected and (H, h) is not AT for every proper induced subgraph H of G where $h(v) = 0$ for all $v \in V(G) \setminus V(H)$. Note that if G is h-minimal then it is also h-greedy-minimal.

Lemma 2.1. *If G is connected and (G, h) is not AT, then G is h-greedy-minimal.*

Proof. If there were a proper induced subgraph H such that $\left(H, h\restriction_{V(H)}\right)$ is AT, then by ordering the vertices of each component of $G - V(H)$ by increasing distance to H and directing all edges away from H in this order we conclude that (G, h) is AT. □

Lemma 2.2. *If (G', h') is formed from (G, h) by subdividing an edge e of G twice and having h' give zero on the two new vertices, then*

1. *if (G, h) is AT, then (G', h') is AT; and*
2. *if (G', h') is AT, then either (G, h) is AT or $(G - e, h)$ is AT.*

Proof. Suppose $e = xy$ and call the new vertices x' and y' so that G' contains the induced path $xx'y'y$. For (1), let D be an orientation of G showing that (G, h) is AT. By symmetry we may assume $xy \in E(D)$. Make an orientation D' of G' from D by replacing xy with the directed path $xx'y'y$. There is a natural parity preserving bijection between the spanning Eulerian subgraphs of D and D', so we conclude that (G', h') is AT.

For (2), let D' be an orientation of G' showing that (G', h') is AT. Suppose G' contains the directed path $xx'y'y$ or the directed path $yy'x'x$. By symmetry, we can assume it is $xx'y'y$. Then make an orientation D of G by replacing $xx'y'y$ with the directed edge xy. As above, we have a parity preserving bijection between the spanning Eulerian subgraphs of D and D', so we conclude that (G, h) is AT. Otherwise, no spanning Eulerian subgraph of D' contains a cycle passing through x' and y'. So, the spanning Eulerian subgraph counts of D' are the same as those of $D' - x' - y'$. But this gives an orientation of $G - e$ showing that $(G - e, h)$ is AT. □

Lemma 2.3. *Let $\{A_1, A_2\}$ be a separation of G such that $A_1 \cap A_2 = \{x\}$. If $G[A_i]$ is f_i-AT for $i \in [2]$, then G is f-AT where $f(v) = f_i(v)$ for $v \in V(A_i - x)$ and $f(x) = f_1(x) + f_2(x) - 1$. Going the other direction, if G is f-AT, then $G[A_i]$ is f_i-AT for $i \in [2]$ where $f_i(v) = f(v)$ for $v \in V(A_i - x)$ and $f_1(x) + f_2(x) \le f(x) + 1$.*

Proof. For $i \in [2]$, choose an orientation D_i of A_i showing that A_i is f_i-AT. Together these give an orientation D of G and since no cycle has vertices in both $A_1 - x$ and $A_2 - x$, we have

$$\begin{aligned} EE(D) - EO(D) &= EE(D_1)EE(D_2) + EO(D_1)EO(D_2) - (EE(D_1)EO(D_2) + EO(D_1)EE(D_2)) \\ &= (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2)) \\ &\neq 0. \end{aligned}$$

Hence G is f-AT.

Now, suppose G is f-AT and choose an orientation D of G showing this. Put $D_i = D[A_i]$ for $i \in [2]$. Then, as above, we have $0 \neq EE(D) - EO(D) = (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2))$ and hence $EE(D_1) - EO(D_1) \neq 0$ and $EE(D_2) - EO(D_2) \neq 0$. Since the in-degree of x in D is the sum of the in-degree of x in D_1 and the in-degree of x in D_2, the lemma follows. □

Corollary 2.4. *Let G be an h-greedy-minimal graph. If (G, h) is AT and G has an induced path $x_1x_2x_3x_4$ such that $d_G(x_2) = d_G(x_3) = 2$ and $h(x_2) = h(x_3) = 0$, then*

$$\big((G - x_2 - x_3) + x_1x_4, h\restriction_{V(G)\setminus\{x_2,x_3\}}\big) \text{ is AT.}$$

Proof. Suppose (G, h) is AT and G has such an induced path $x_1x_2x_3x_4$. Applying Lemma 2.2 part (2) shows that either $\big(G - x_2 - x_3, h\restriction_{V(G)\setminus\{x_2,x_3\}}\big)$ is AT or $\big((G - x_2 - x_3) + x_1x_4, h\restriction_{V(G)\setminus\{x_2,x_3\}}\big)$ is AT. But $G - x_2 - x_3$ is a proper induced subgraph of G, so the former cannot happen since G is h-greedy-minimal and $h(x_2) = h(x_3) = 0$. Hence $\big((G - x_2 - x_3) + x_1x_4, h\restriction_{V(G)\setminus\{x_2,x_3\}}\big)$ is AT. □

3 Extension lemma

This is a key lemma from [1], it generalizes a lemma from [2] from list coloring to Alon-Tarsi orientations. This is what i talked about in Baltimore. The basic idea is that in some cases we can pair off odd/even spanning Eulerian subgraphs via a parity reversing bijection.

Lemma 3.1. *Let G be a multigraph without loops and $f\colon V(G) \to \mathbb{N}$. If there are $F \subseteq G$ and $Y \subseteq V(G)$ such that:*

1. *any multiple edges in G are contained in $G[Y]$; and*

2. $f(v) \geq d_G(v)$ *for all* $v \in V(G) \setminus Y$*; and*

3. $f(v) \geq d_{G[Y]}(v) + d_F(v) + 1$ *for all* $v \in Y$*; and*

4. *For each component* T *of* $G - Y$ *there are different* $x_1, x_2 \in V(T)$ *where* $N_T[x_1] = N_T[x_2]$ *and* $T - \{x_1, x_2\}$ *is connected such that either:*

   (a) *there are* $x_1y_1, x_2y_2 \in E(F)$ *where* $y_1 \neq y_2$ *and* $N(x_i) \cap Y = \{y_i\}$ *for* $i \in [2]$*; or*

   (b) $|N(x_2) \cap Y| = 0$ *and there is* $x_1y_1 \in E(F)$ *where* $N(x_1) \cap Y = \{y_1\}$*,*

then G *is* f*-AT.*

Proof. Suppose not and pick a counterexample (G, f, F, Y) minimizing $|G - Y|$. If $|G - Y| = 0$, then $Y = V(G)$ and thus $f(v) \geq d_G(v) + 1$ for all $v \in V(G)$ by (3). Pick an acyclic orientation D of G. Then $EE(D) = 1$, $EO(D) = 0$ and $d^+_D(v) \leq d_G(v) \leq f(v) - 1$ for all $v \in V(D)$. Hence G is f-AT. So, we must have $|G - Y| > 0$.

Pick a component T of $G - Y$ and pick $x_1, x_2 \in V(T)$ as guaranteed by (4). First, suppose (4a) holds. Put $G' := (G - T) + y_1y_2$, $F' := F - T$, $Y' := Y$ and let f' be f restricted to $V(G')$. Then G' has an orientation D' where $f'(v) \geq d^+_{D'}(v) + 1$ for all $v \in V(D')$ and $EE(D') \neq EO(D')$, for otherwise (G', f', F', Y') would contradict minimality. By symmetry we may assume that the new edge y_1y_2 is directed toward y_2. Now we use the orientation of D' to construct the desired orientation of D. First, we use the orientation on $D' - y_1y_2$ on $G - T$. Now, order the vertices of T as $x_1, x_2, z_1, z_2, \ldots$ so that every vertex has at least one neighbor to the right. Orient the edges of T left-to-right in this ordering. Finally, we use y_1x_1 and x_2y_2 and orient all other edges between T and $G - T$ away from T. Plainly, $f(v) \geq d^+_D(v) + 1$ for all $v \in V(D)$. Since y_1x_1 is the only edge of D going into T, any Eulerian subgraph of D that contains a vertex of T must contain y_1x_1. So, any Eulerian subgraph of D either contains (i) neither y_1x_1 nor x_2y_2, (ii) both y_1x_1 and x_2y_2, or (iii) y_1x_1 but not x_2y_2. We first handle (i) and (ii) together. Consider the function h that maps an Eulerian subgraph Q of D' to an Eulerian subgraph $h(Q)$ of D as follows. If Q does not contain y_1y_2, let $h(Q) = \iota(Q)$ where $\iota(Q)$ is the natural embedding of $D' - y_1y_2$ in D. Otherwise, let $h(Q) = \iota(Q - y_1y_2) + \{y_1x_1, x_1x_2, x_2y_2\}$. Then h is a parity-preserving injection with image precisely the union of those Eulerian subgraphs of D in (i) and (ii). Hence if we can show that exactly half of the Eulerian subgraphs of D in (iii) are even, we will conclude $EE(D) \neq EO(D)$, a contradiction. To do so, consider an Eulerian subgraph A of D containing y_1x_1 and not x_2y_2. Since x_1 must have in-degree 1 in A, it must also have out-degree 1 in A. We show that A has a mate A' of opposite parity. Suppose $x_2 \notin A$ and $x_1z_1 \in A$; then we make A' by removing x_1z_1 from A and adding $x_1x_2z_1$. If $x_2 \in A$ and $x_1x_2z_1 \in A$, we make A' by removing $x_1x_2z_1$ and adding x_1z_1. Hence exactly half of the Eulerian subgraphs of D in (iii) are even and we conclude $EE(D) \neq EO(D)$, a contradiction.

Now suppose (4b) holds. Put $G' := G - T$, $F' := F - T$, $Y' := Y$ and define f' by $f'(v) = f(v)$ for all $v \in V(G' - y_1)$ and $f'(y_1) = f(y_1) - 1$. Then G' has an orientation D' where $f'(v) \geq d^+_{D'}(v) + 1$ for all $v \in V(D')$ and $EE(D') \neq EO(D')$, for otherwise (G', f', F', Y') would contradict minimality. We orient $G - T$ according to D, orient T as in the previous case, again use y_1x_1 and orient all other edges between T and $G - T$ away from T. Since we decreased $f'(y_1)$ by 1, the extra out edge of y_1 is accounted for and we have

$f(v) \geq d_D^+(v) + 1$ for all $v \in V(D)$. Again any additional Eulerian subgraph must contain y_1x_1 and since x_2 has no neighbor in $G - T$ we can use x_2 as before to build a mate of opposite parity for any additional Eulerian subgraph. Hence $EE(D) \neq EO(D)$ giving our final contradiction. □

4 Degree-AT graphs

A graph G is called *degree-AT* if (G, h) is *AT* where h is the constant zero function.

Lemma 4.1. *A connected graph G is degree-AT if it is not a Gallai tree.*

Proof. Suppose there exists a connected graph that is not a Gallai tree, but is also not degree-AT. Let G be such a graph with as few vertices as possible. Since G is not degree-AT, no induced subgraph H of G is degree-AT by Lemma 2.1. Hence, for any $v \in V(G)$ that is not a cutvertex, $G - v$ must be a Gallai tree by minimality of $|G|$.

If G has more than one block, then for endblocks B_1 and B_2, choose noncutvertices $w \in B_1$ and $x \in B_2$. By the minimality of $|G|$, both $G - w$ and $G - x$ are Gallai trees. Since every block of G appears either as a block of $G - w$ or as a block of $G - x$, every block of G is either complete or an odd cycle. Hence, G is a Gallai tree, a contradiction. So instead G has only one block, that is, G is 2-connected. Further, $G - v$ is a Gallai tree for all $v \in V(G)$.

Let v be a vertex of minimum degree in G. Since G is 2-connected, $d_G(v) \geq 2$ and v is adjacent to a noncutvertex in every endblock of $G - v$. If $G - v$ has a complete block B with noncutvertices x_1, x_2 where $v \leftrightarrow x_1$ and $v \not\leftrightarrow x_2$, then we can apply Lemma 3.1 with $Y = \{v\}$ and $F = vx_1$ to conclude that G is degree-AT, a contradiction. So, v must be adjacent to every noncutvertex in every complete endblock of $G - v$.

Suppose $d_G(v) \geq 3$. Then no endblock of $G - v$ can be an odd cycle of length at least 5 (there would be vertices of degree 3 but we'd have $d_G(v) \geq 4$). Let B be a smallest complete endblock of $G - v$. Then for a noncutvertex $x \in V(B)$, we have $d_G(x) = |B|$ and hence $d_G(v) \leq |B|$. If $G - v$ has at least two endblocks, then $2(|B| - 1) \leq |B|$ and hence $d_G(v) \leq |B| = 2$, a contradiction. Hence $G - v = B$ and v is joined to B, so G is complete, a contradiction.

Hence, we must have $d_G(v) = 2$. Suppose $G - v$ has at least 2 endblocks. Then, it has exactly 2 and v is adjacent to one noncutvertex in each. Neither of the endblocks can be odd cycles of length at least 5 since then we could get a smaller counterexample by Lemma 2.2. Since v is adjacent to every noncutvertex in every complete endblock of $G - v$, both endblocks must be K_2. But then either $G = C_4$ (which is trivially degree-AT) or we can get a smaller counterexample by Lemma 2.2. So, $G - v$ must be 2-connected. Since $G - v$ is a Gallai tree, it is either complete or an odd cycle. If $G - v$ is not complete, we can get a smaller counterexample by Lemma 2.2. So, $G - v$ is complete and v is adjacent to every noncutvertex of $G - v$; that is, G is complete, a contradiction. □

5 When h is 1 for at most one vertex

For a graph G and $x \in V(G)$ let $h_x \colon V(G) \to \mathbb{N}$ be defined by $h_x(x) = 1$ and $h_x(v) = 0$ for all $v \in V(G - x)$. We classify the connected h_x-minimal graphs G such that (G, h_x) is AT for some $x \in V(G)$.

To start we will reduce to the case when G is 2-connected.

Lemma 5.1. *Let G be h_x-minimal for $x \in V(G)$ and let $\mathcal{B}$ be the set of blocks of G containing x. Then (G, h_x) is AT if and only if*

1. *$\mathcal{B}$ contains at least two degree-AT graphs; or*

2. *G is 2-connected and (G, h_x) is AT.*

Proof. Since G is h_x-minimal, no block outside of $\mathcal{B}$ is degree-AT. The lemma follows since if G is not 2-connected, then (G, h_x) is AT if and only if (1) holds by Lemma 2.3. $\square$

Lemma 5.2. *If G is a connected graph and $x \in V(G)$ with $d_G(x) = 2$, then (G, h_x) is AT if and only if $G - x$ is degree-AT.*

Proof. Let D be an orientation of G showing that (G, h_x) is AT. Then $d_D^-(x) = 2$ and hence no spanning Eulerian subgraph contains a cycle passing through x. Therefore, the Eulerian subgraph counts in $G - x$ are different and $G - x$ is degree-AT. The other direction is immediate from Lemma 2.1. $\square$

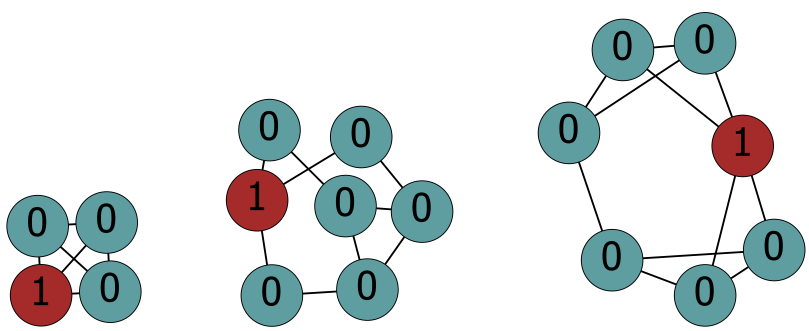

Figure 1: The seed blocks.

Lemma 2.2 part (2) suggests a way to construct G such that (G, h) is not AT from smaller graphs. Specifically, we have the following.

Corollary 5.3. *If e is an edge in G such that (G, h) is not AT and $(G - e, h)$ is not AT, then (G', h') is not AT where (G', h') is formed from (G, h) by subdividing e twice and having h' give zero on the two new vertices.*

Let $\mathcal{D}$ be the smallest collection of pairs (G, h) containing the pairs in Figure 1 that is closed under the operation in Corollary 5.3.

Lemma 5.4. *Let G be h_x-minimal for $x \in V(G)$ with $d_G(x) \geq 3$. If (G, h_x) is not AT, then every induced subdivision of $K_{1,3}$ in G contains at most two vertices in $N(x)$. In particular, every induced path in G contains at most two vertices in $N(x)$.*

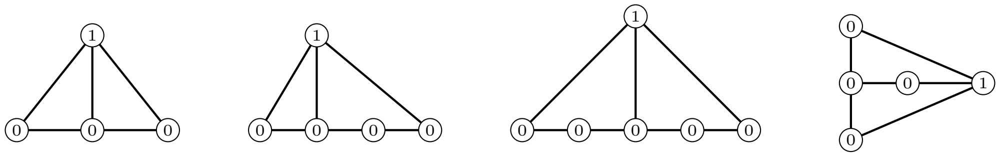

Figure 2: These are AT.

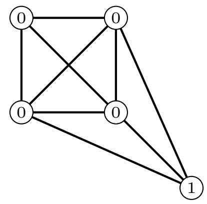

Figure 3: This is AT.

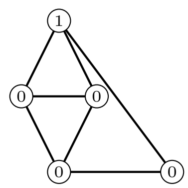

Figure 4: This is AT.

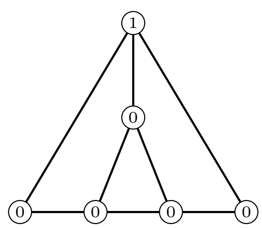

Figure 5: This is AT.

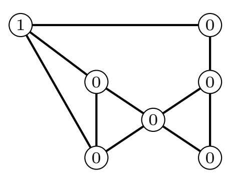

Figure 6: This is AT.

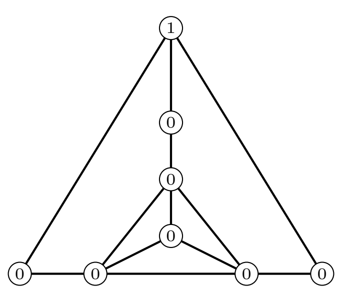

Figure 7: This is AT.

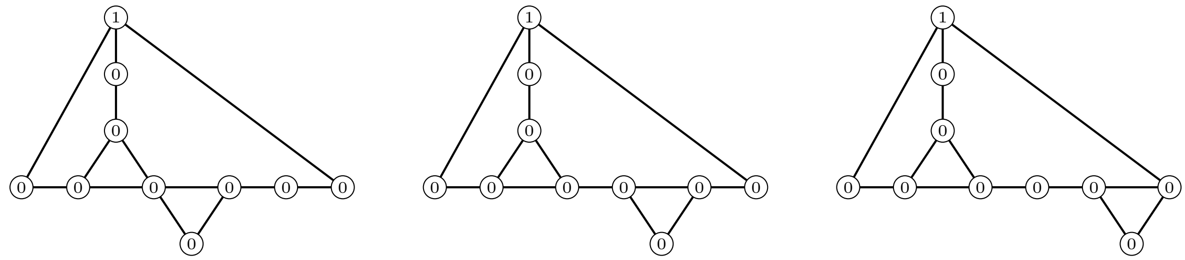

Figure 8: These are AT.

Proof. This is immediate from Lemma 2.2 and the graphs in Figure 2. □

For a connected graph G and endblock B of G, let z_B be the cutvertex of G contained in B.

Lemma 5.5. *Let G be a connected graph and $v \in V(G)$ a cutvertex of G. For any endblock B of G there is another endblock D and an induced path from x_B to x_D containing v.*

Proof. Let B be an endblock of G. Then we can pick to be any endblock that v separates from B. Now any shortest path from x_B to x_D is the desired induced path. □

Lemma 5.6. *Let G be h_x-minimal for $x \in V(G)$. If G is 2-connected, then (G, h_x) is AT if and only if*

1. *$d_G(x) \geq 3$; and*

2. *G is not complete and not an odd cycle; and*

3. *$(G, h_x) \notin \mathcal{D}$.*

Proof. Suppose the lemma is false and choose a counterexample G minimizing $|G|$. If $d_G(x) \leq 2$, then (G, h_x) is not AT by Lemma 5.2 since G is h_x-minimal. So, we must have $d_G(x) \geq 3$. Since (G, h_x) is not AT if $(G, h_x) \in \mathcal{D}$ by construction, it must be that $(G, h_x) \notin \mathcal{D}$ and (G, h_x) is not AT. Since G is h_x-minimal and 2-connected, $G - x$ is a Gallai tree.

Since G is 2-connected, x is adjacent to a noncutvertex in every endblock of $G - x$. If $G - x$ has a complete block B with noncutvertices v_1, v_2 where $x \leftrightarrow v_1$ and $x \not\leftrightarrow v_2$, then we can apply Lemma 3.1 with $Y = \{x\}$ and $F = xv_1$ to conclude that (G, h_x) is AT, a contradiction. So, x must be adjacent to every noncutvertex in every complete endblock of $G - x$.

If $G - x$ contains an induced path $v_1v_2v_3v_4$ such that $d_G(v_2) = d_G(v_3) = 2$, then we get a smaller counterexample by applying Lemma 2.2 part (1). So, $G - x$ has no such induced path.

Suppose G has an endblock B that is an odd cycle of length $\ell \geq 5$. Then x is adjacent to at least $\left\lfloor \frac{\ell}{2} \right\rfloor \geq 2$ noncutvertices in B. If $G - x$ had another endblock B', then we can pick two neighbors of x in B and one neighbor of x in B' all on an induced path in $G - x$, violating Lemma 5.4. But if $G - x = B$, we can again violate Lemma 5.4. Hence every endblock of $G - x$ is complete.

Since $G - x$ is not complete (otherwise G would be complete), this also implies that $G - x$ has at least two endblocks. Suppose x is adjacent to a cutvertex v of $G - x$. Using Lemma 5.5, we get an induced path from x_B to x_D containing v, where B and D are different endblocks. But then adding one noncutvertex from each of B and D to this path, we have an induced path violating Lemma 5.4. Therefore, x is not adjacent to any cutvertex of $G - x$.

Suppose $G-x$ has a complete endblock $B = K_t$ for $t \geq 4$. Let $v \in V(B)$ be a noncutvertex in $G - x$. Then $G - v$ is 2-connected and h_x-minimal, so by minimality of $|G|$, we conclude that $d_{G-v}(x) \leq 2$, $G - v$ is complete or an odd cycle, or $(G - v, h_x) \in \mathcal{D}$. First, suppose $d_{G-v}(x) \leq 2$. Then $d_G(v) = 3$ and hence $G-x$ has only one endblock, impossible. If $G-v$ is complete, then so is G. Also, $G - v$ cannot be a noncomplete odd cycle since it contains K_3. Hence, we must have $(G-v, h_x) \in \mathcal{D}$. Since all of v's neighbors in $G-x$ have degree at least 3 in $G-v$, removing v cannot create an induced path $v_1v_2v_3v_4$ such that $d_G(v_2) = d_G(v_3) = 2$. Hence $G - v$ must be one of the three graphs in Figure 1. The leftmost graph is complete and every endblock of the middle graph is K_2, so $G - v$ must be the rightmost graph in Figure 1. But then G has the graph in Figure 3 as an induced subgraph, impossible. Hence, every complete endblock of $G - x$ has at most three vertices.

Suppose $G - x$ has a noncutvertex v with $v \not\leftrightarrow x$. Then $G - v$ is 2-connected and h_x-minimal, so by minimality of $|G|$, we conclude that $d_{G-v}(x) \leq 2$, $G - v$ is complete or an odd cycle, or $(G - v, h_x) \in \mathcal{D}$. The first three clearly cannot occur, so we have $(G - v, h_x) \in \mathcal{D}$. First, suppose that removing v did not create an induced path $v_1v_2v_3v_4$ such that $d_G(v_2) = d_G(v_3) = 2$. Then $G - v$ is one of the graphs in Figure 1. But $G - v$ cannot be the leftmost, middle, or rightmost graph in Figure 1 because then G would contain the graph in Figure 3, Figure 7, and Figure 5 as an induced subgraph, respectively.

So, removing v created an induced path $v_1v_2v_3v_4$ such that $d_G(v_2) = d_G(v_3) = 2$. Let B be the block containing v. First, suppose B is complete. Since v is only adjacent to vertices in B, the only way for this to happen is if $B = K_3$. So, $G - v$ is either one of the graphs in Figure 1 or the result of applying the operation in Corollary 5.3 to a graph in Figure 1 one time. The former case was handled above.

For the rightmost graph, removing any edge leaves an AT graph, so Corollary 5.3 cannot be applied. [ADD (easy) DETAILS]

For the middle graph, removing any edge in the triangle leaves an AT graph, so Corollary 5.3 cannot be applied to those edges. But then G is one of the graphs in Figure 8, impossible.

For leftmost graph, removing any of the edges not incident to the vertex labeled 1 leaves an AT graph, so Corollary 5.3 cannot be applied to those edges. But then G contains an induced Figure 4 or Figure 6, impossible.

So, B cannot be complete. Suppose instead that B is an odd cycle of length at least 5. Let $N(v) = \{w, z\}$. By Lemma 2.2, we see that w and z are either cutvertices of $G-x$ or adjacent to x. Since removing v created an induced path $v_1v_2v_3v_4$ such that $d_G(v_2) = d_G(v_3) = 2$, we must have $\{w, z\} \cap \{v_2, v_3\} \neq \emptyset$. By symmetry, we may assume that $w = v_2$.

Therefore, every noncutvertex of $G - x$ is adjacent to x. So, at this point, we know that

- every endblock of $G - x$ is K_2 or K_3; and
- x is adjacent to all noncutvertices of $G - x$
- x is nonadjacent to all cutvertices in $G - x$.

$\square$

References

[1] Hal Kierstead and Landon Rabern, *Improved lower bounds on the number of edges in list critical and online list critical graphs*, arXiv preprint arXiv:1406.7355 (2014).

[2] A.V. Kostochka and M. Stiebitz, *A new lower bound on the number of edges in colour-critical graphs and hypergraphs*, Journal of Combinatorial Theory, Series B **87** (2003), no. 2, 374–402.