Proof of Brooks' theorem. Suppose the theorem is false and choose a counterexample G minimizing $|G|$. Plainly, G is connected. Let $\{A_1, A_2\}$ be a separation of G minimizing $k := |A_1 \cap A_2|$.

First suppose there is an induced P_3 xyz in G such that $G - x - z$ is connected. Then we may order $V(G)$ as $x, z, v_1, \ldots, v_n, y$ so that each v_i has a neighbor to the right. But this is a contradiction since greedily coloring in this order uses at most Δ colors as x and z get the same color, each v_i has at most $\Delta - 1$ neighbors to the left and y has two neighbors (x and z) colored the same.

Thus G contains no such P_3. In particular, $k \leq 2$. If there are nonadjacent $u, v \in A_1 \cap A_2$, put $H_i := G[A_i] + uv$ noting that $\Delta(H_i) \leq \Delta(G)$ since both u and v have neighbors on both sides of the separation by minimality of k. Otherwise put $H_i := G[A_i]$. By minimality of $|G|$, each H_i is either $\Delta(G)$-colorable or contains a $K_{\Delta(G)+1}$. If both H_i are $\Delta(G)$-colorable, then we have $\Delta(G)$-colorings of $G[A_1]$ and $G[A_2]$ where $A_1 \cap A_2$ receives k colors. By permuting color names if necessary we can combine these to get a $\Delta(G)$-coloring of G, a contradiction.

Otherwise, $k = 2$ and G contains an induced subgraph H which is a $K_{\Delta(G)+1}$ with one edge missing, call it xy. By minimality of $|G|$, we may $\Delta(G)$-color $G - H$, then color x and y the same (as they both have at least $\Delta(G) - 1$ legal colors left and $2(\Delta(G) - 1) > \Delta(G)$) and finally greedily finish the $\Delta(G)$-coloring on the rest of H. This gives a $\Delta(G)$-coloring of G, a contradiction. $\square$