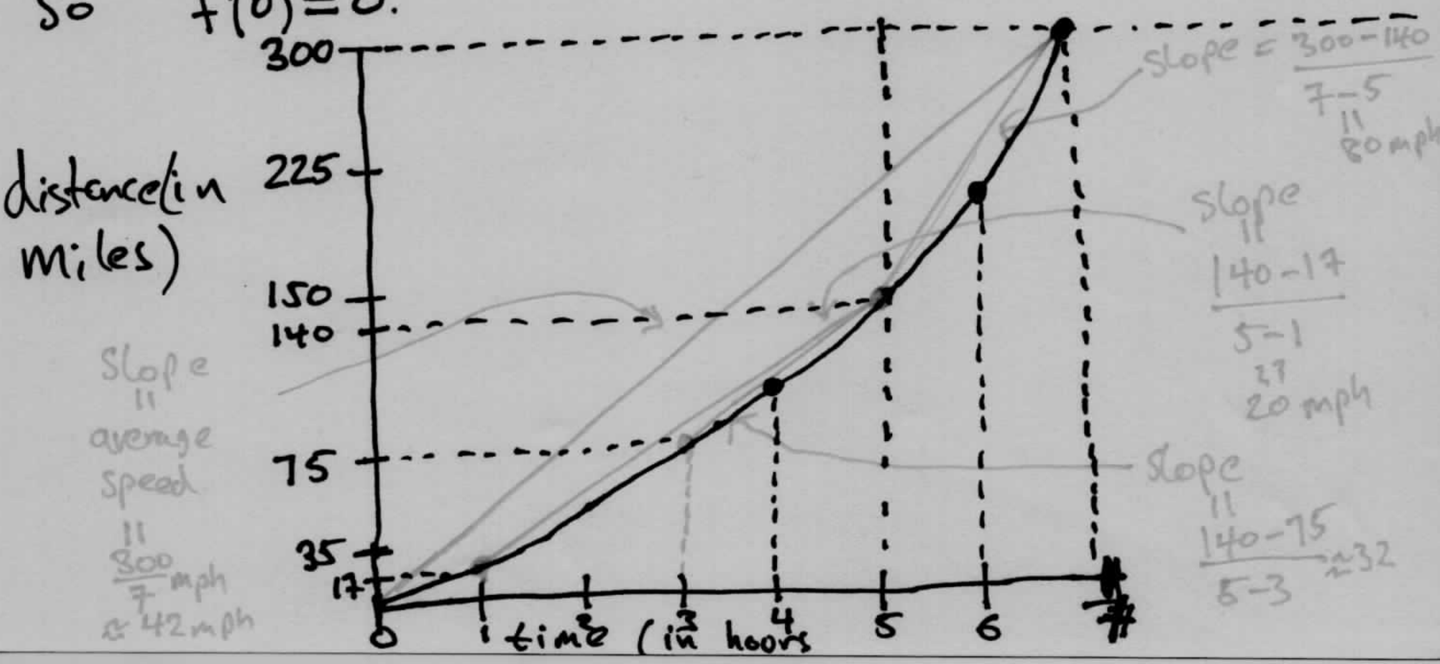
days 1,2,3: see math 105 notes.

section 2.1: the tangent and velocity problems

You enter a highway with speed limit 60 mph. Due to budget cuts, there are no highway patrol cars to check speed along ~~the~~ a certain 300 mi long stretch of the highway. Instead, the police have set up one booth at the entrance and one at the exit. At the entrance you get a ticket with the current time printed on it. At the exit, the booth operator checks the current time, subtracts the time on your ticket and divides by 300mi, if this is over 60 mph, you get a fine.

Let's say you enter at time $t=0$ hours and your position is given by the function $f(t)$ (in miles).

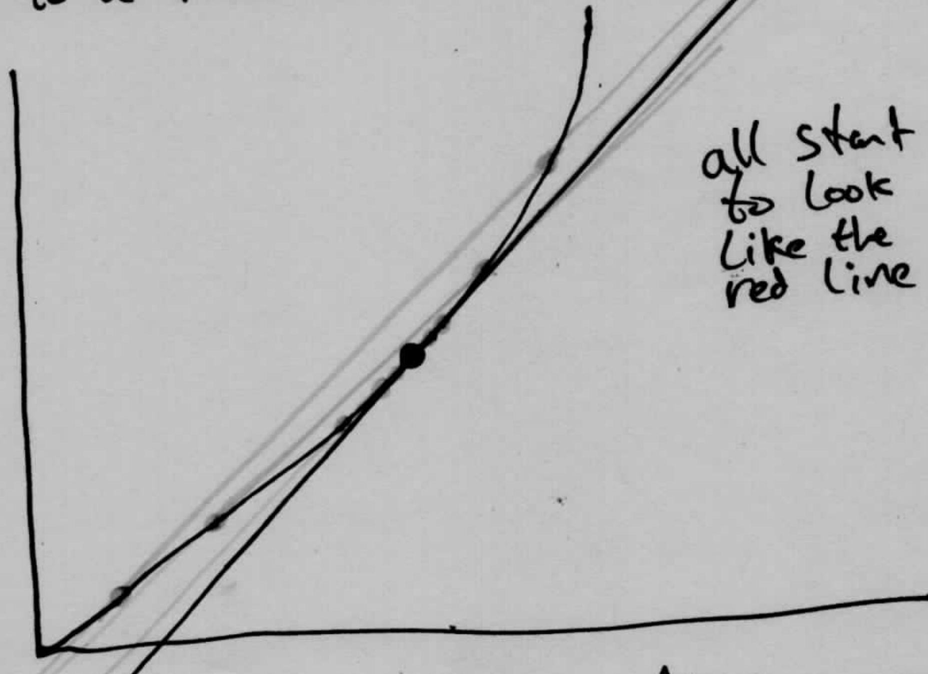
so $f(0)=0$.



- how well does this work at limiting speed?
- what could the police do to make it work better?

to answer those, need to know

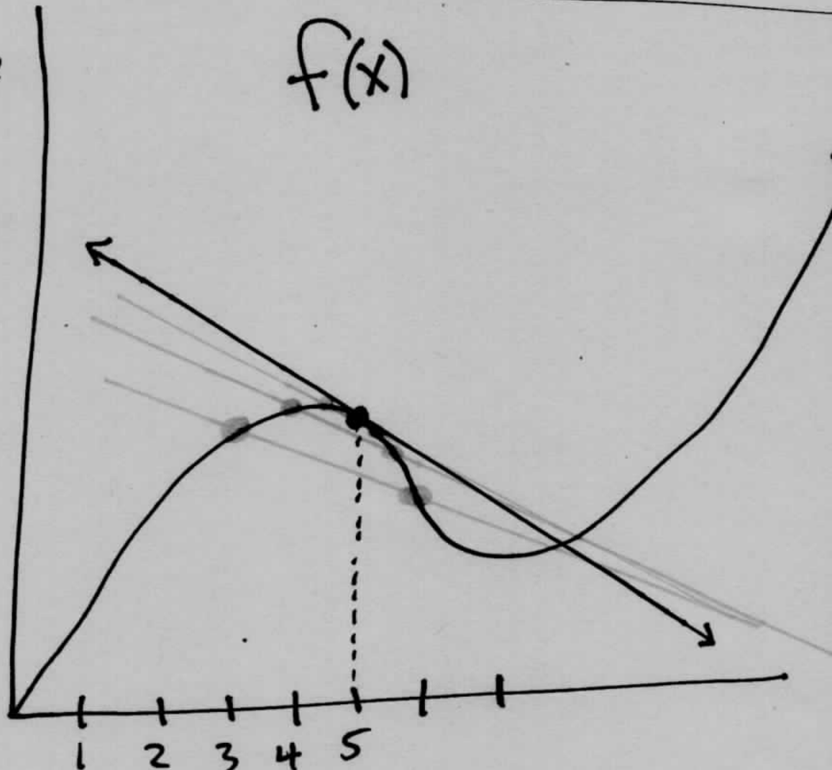
- given f, how do we determine what the car's speedometer reads at say $t=4$?
  - find average speed over intervals containing $t=4$.
  - shorter interval = better approximation to speedometer reading
  - average speed over interval = slope of line from start to end of interval
  - speedometer reading ≈ slope of line over tiny interval
  - keep making the interval smaller, these lines start to be indistinguishable from one another



- slope of red line = speedometer reading
- so, to find speedometer reading, need to find red line.

In more general terms, the red line is the tangent line to f at t=4.

- want tangent line to f(x) at $x=5$
- pick interval containing $x=5$, draw secant line
- pick smaller intervals draw secant lines
- repeat, those lines "converge" to the tangent line.



- Slope of the tangent line at $x=5$ tells us how fast $f(x)$ is changing near $x=5$.
  - that is, it tells us the speedometer reading.

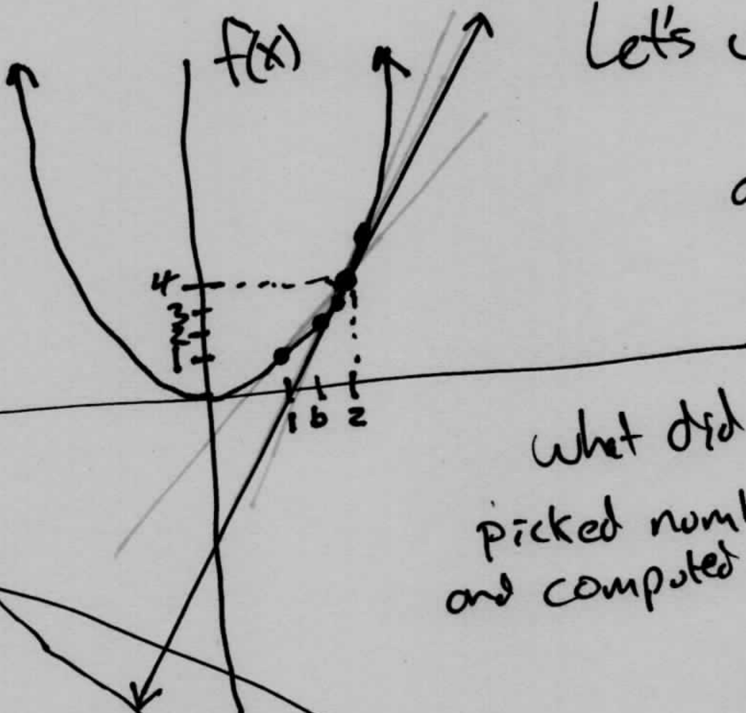
- finding the slope of the tangent line at a given point is useful (say, for the police to design their fine system).
- we can find this slope ~~by~~ for $f(x)$ at $x=a$ by
- drawing the picture of f.
- taking intervals containing $x=a$, drawing secant lines and finding their slope.

smaller and smaller

It would be convenient if we could find the slope of the tangent line to $f(x)$ without having to draw the picture.

<u>ex</u>. Find the slope of the line tangent to the graph of $f(x) = x^2$ at $x=2$.

one more picture so we can see how to do without.



Let's use $(2, f(2))$ $= (2,4)$ as one end of all our secant lines

what did we do?

picked numbers $b \neq a$ and computed slope $= \dfrac{f(2)-f(b)}{2-b}$

compute Slope of tangent line to graph of $f(x)$ at $x=a$:

pick numbers $b \neq a$ closer and closer to a and compute $\dfrac{f(a)-f(b)}{a-b}$ for each.

Let's try for $f(x)=x^2$ at $x=2$.

$a=2.$

$f(a)=4$

b	$f(b)$	$\frac{f(a)-f(b)}{a-b}$
3	9	$\frac{4-9}{2-3}=\frac{-5}{-1}=5$
1	1	$\frac{4-1}{2-1}=3$
2.1	4.41	$\frac{4-4.41}{2-2.1}=\frac{-0.41}{-0.1}=4.1$
1.9	3.61	$\frac{4-3.61}{2-1.9}=\frac{0.39}{0.1}=3.9$
2.01	4.0401	$\frac{4-4.0401}{2-2.01}=\frac{-0.0401}{-0.01}=4.01$

$\vdots$

Looks like slope of tangent line is going to be 4.

- how can we know when to stop?
- maybe there is magic cancellation in $\frac{f(a)-f(b)}{a-b}$

$$\frac{a^2-b^2}{a-b}=\frac{(a-b)(a+b)}{a-b}=a+b$$

So pick numbers $b\neq a$ and plug into

$\frac{f(a)-f(b)}{a-b}=a+b$, when b gets close to a
$a+b$ gets close to $\underline{\underline{2a}}$

$2\cdot 2=4$, yay!

- easier to see if we make $b=a+h$ and make h get close to 0. $\frac{f(a)-f(a+h)}{a-(a+h)}=\frac{f(a)-f(a+h)}{-h}=\boxed{\frac{f(a+h)-f(a)}{h}}$

Slope of tangent line to graph of $f(x)$ at $x=a$:

Pick numbers $h \neq 0$ closer and closer to zero and compute

$$\frac{f(a+h)-f(a)}{h}.$$

- magic cancellation can happen and we can save ourselves the work of plugging in values forever.

ex. again $f(x)=x^2$ at $x=a$

$$\frac{f(a+h)-f(a)}{h} = \frac{(a+h)^2-a^2}{h} = \frac{(a+h-a)(a+h+a)}{h} = \frac{h(2a+h)}{h}$$

does it?

$$= 2a+h$$

as h gets close to 0 (as $h \to 0$)

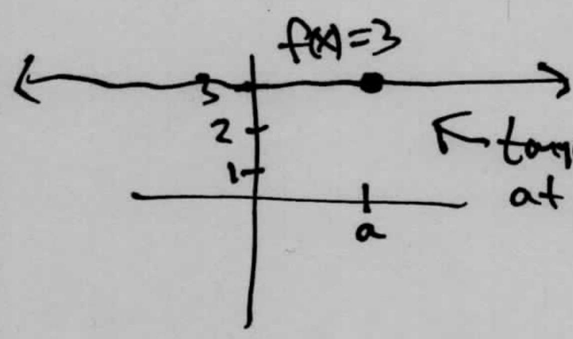
$$= 2a.$$

another ex

$f(x)=3$ at $x=a$

$$\frac{f(a+h)-f(a)}{h} = \frac{3-3}{h} = \frac{0}{h} = 0.$$

is this ok?



tangent line at any a has slope 0!

you try: What is the slope of the line tangent to the graph of $f(x)=x^3$ at $x=a$?

$$\frac{f(a+h)-f(a)}{h} = \frac{(a+h)^3-a^3}{h}$$

$$= \frac{(a+h-a)((a+h)^2-(a+h)a+a^2)}{h}$$

$$= \frac{h((a+h)^2+(a+h)a+a^2)}{h}$$

is this OK?

$$\stackrel{?}{=} (a+h)^2+(a+h)a+a^2$$

$$= a^2+2ah+h^2+a^2+ah+a^2$$

$$= 3a^2+h(3a+h)$$

as $h\to 0$

$$\to 3a^2.$$

how about $f(x)=x^n$ at $x=a$?

$$\frac{f(a+h)-f(a)}{h} = \frac{(a+h)^n-a^n}{h}$$

plug in $h=0$? $\frac{a^n-a^n}{0}=\frac{0}{0}$???

need to multiply it out to get magic cancellation.

how?

$(a+b)^2 = a^2+2ab+b^2$

$(a+b)^3 =$ not fun already

$(a+b)(a+b)$

$a^2+ab+ba+b^2$

when we multiply this out we get one term for each choice of either a or b from each $(a+b)$

$(a+b)(a+b)(a+b)$

all a's a^3

same deal:

$(a+b)$	$(a+b)$
a	a
a	b
b	a
b	b

$(a+b)$	$(a+b)$	$(a+b)$
a	a	a
a	a	b
a	b	a
a	b	b
b	a	a
b	a	b
b	b	a
b	b	b

$(a+b)(a+b)(a+b)(a+b)$

a	a	a	a
a	a	a	b
a	a	b	a
a	a	b	b
a	b	a	a
a	b	a	b
a	b	b	a
a	b	b	b
b	a	a	a
b	a	a	b
b	a	b	a
b	a	b	b
b	b	a	a
b	b	a	b
b	b	b	a
b	b	b	b

pick ~~which ones~~ how many will be a ~~or~~ the rest are b.

aa
ab
ba
bb

2 a's: a^2
one a: $2ab$
zero a: b^2

$\boxed{a^2+2ab+b^2}$

1 2 1

aaa
aab
aba
abb
baa
bab
bba
bbb

three a's: a^3
two a's: $3a^2b$
one a : $3ab^2$
zero a: b^3

$\boxed{a^3+3a^2b+3ab^2+b^3}$

1 3 3 1

aaaa
aaab
aaba
aabb
abaa
abab
abba
abbb
baaa
baab
baba
babb
bbaa
bbab
bbba
bbbb

four a's: a^4
three a's: $4a^3b$
two a's: $6a^2b^2$
one a : $4ab^3$
zero a: b^4

$\boxed{a^4+4a^3b+6a^2b^2+4ab^3+b^4}$

1 4 6 4 1

Pascal's triangle

1
1 1
1 2 1
1 3 3 1
1 4 6 4 1
1 5 10 10 5 1

number of ways to choose 1 person out of 2

1 out of 3

2 out of 3

1 out of 4

2 out of 4

$(a+b)^1 = a+b$

1 1

tim mary sally john

number of ways to pick 3 people out of 5

$3=2+1$
$4=3+1$
$6=3+3$ why?
$10=4+6$

number of ways to pick 2 from 4 + number of ways to pick 3 from 4 = mike + 0000 (tim mary sally john) two of these mike 3 of these 0000 (tim mary sally john)

||

number of ways to pick 2 of the 4 not-mike people + number of ways to pick 3 not-mike people back to $\dfrac{(a+h)^n - a^n}{h}$:

is this ok?

$n=1$: $\dfrac{(a+h)^1 - a^1}{h} = \dfrac{a+h-a}{h} = \dfrac{h}{h} = 1$

ok?

$n=2$: $\dfrac{(a+h)^2 - a^2}{h} = \dfrac{a^2+2ah+h^2-a^2}{h} = \dfrac{2ah+h^2}{h} = \dfrac{h(2a+h)}{h} = 2a+h$
$= 2a,$
as $h \to 0$

$n=3$: $\dfrac{(a+h)^3 - a^3}{h} = \dfrac{a^3+3a^2h+3ah^2+h^3-a^3}{h} = \dfrac{3a^2h+3ah^2+h^3}{h}$
$= \dfrac{h(3a^2+3ah+h^2)}{h}$

ok?

$= 3a^2+3ah+h^2$
$= 3a^2 + h(3a+h)$
$= 3a^2,$
as $h \to 0$

$n=4$: $\dfrac{(a+h)^4 - a^4}{h} = \dfrac{a^4+4a^3h+6a^2h^2+4ah^3+h^4-a^4}{h}$
$= \dfrac{4a^3h+6a^2h^2+4ah^3+h^4}{h}$
$= \dfrac{h(4a^3+6a^2h+4ah^2+h^3)}{h}$

ok?

$= 4a^3+6a^2h+4ah^2+h^3$
$= 4a^3+h(6a^2+4ah+h^2)$
$= 4a^3$
as $h \to 0$.

$\dfrac{(a+h)^n - a^n}{h} = \dfrac{a^n + \square a^{n-1}h + \square a^{n-2}h^2 + \dots + h^n - a^n}{h}$
$= \dfrac{\square a^{n-1}h + \square a^{n-2}h^2 + \dots + h^n}{h}$

ok?

$= \dfrac{h(\square a^{n-1} + \square a^{n-2}h + \dots + h^{n-1})}{h}$
$= \square a^{n-1} + \square a^{n-2}h + \dots + h^{n-1}$
$= \square a^{n-1} + h(a\square a^{n-2} + \dots + h^{n-2})$

$$= \boxed{}\, a^{n-1}.$$

as $h \to 0$

what is this something?

number of ways to pick 1 person out of n people $= n$.

$$= n a^{n-1}.$$