Maybe?

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1 Introduction

2 The decomposition

Let $\mathcal{D}_1$ be the collection of graphs without induced d_1-choosable subgraphs. Plainly, $\mathcal{D}_1$ is hereditary. For a graph G and $r \in \mathbb{N}$, let $\mathcal{C}_r$ be the maximal cliques in G having at least r vertices.

Lemma 2.1. *If B is a graph such that $K_4 * B$ is not d_1-choosable, then at least one of the following holds:*

- *$|B| \leq 4$ and B is $E_3 * K_{|B|-3}$,*
- *$\omega(B) \geq |B| - 1$.*

Lemma 2.2. *If $G \in \mathcal{D}_1$ and $\frac{\Delta(G)+5}{2} \leq r \leq \Delta(G) - 1$, then $\bigcup \mathcal{C}_r$ can be partitioned into sets $F_1, \ldots, F_t$ such that for each $i \in [t]$ at least one of the following holds:*

- *$F_i \in \mathcal{C}_r$,*
- *$F_i = C_i \cup \{x_i\}$ where $C_i \in \mathcal{C}_r$ and $|N(x_i) \cap C_i| \geq r - 1$.*

When $F_i \in \mathcal{C}_r$, we put $K_i := C_i := F_i$ and when $F_i = C_i \cup \{x_i\}$, we put $K_i := N(x_i) \cap C_i$.

3 The recoloring technique

Lemma 3.1. *Let H be a graph and $V_1 \cup \cdots \cup V_r$ a partition of $V(H)$. Suppose that $|V_i| \geq 2\Delta(H)$ for each $i \in [r]$. If a graph G is formed by attaching a new vertex x to fewer than $2\Delta(H)$ vertices of H, then G has an independent set $\{x, v_1, \ldots, v_r\}$ where $v_i \in V_i$ for each $i \in [r]$.*

Proof. Suppose not. Remove $\{x\} \cup N(x)$ from G to form H' with induced partition $V_1', V_2', \ldots, V_r'$. Then $V_1', V_2', \ldots, V_r'$ has no independent transversal since we could combine one with x to get our desired independent set in G. Note that $|V_i'| \geq 1$. Create a graph Q by removing edges from H' until it is edge minimal without an independent transversal. Pick

$yz \in E(Q)$ and apply Lemma **??** on yz with the induced partition to get the guaranteed $J \subseteq [r]$ and the totally dominating induced matching M with $|M| = |J| - 1$. Now $\left|\bigcup_{i \in J} V_i'\right| > 2\Delta(H)\,|J| - 2\Delta(H) = 2(|J| - 1)\Delta(H)$ and hence M cannot dominate, a contradiction. □

Call $v \in V(G)$ *big* if it is contained in a $\frac{2}{3}\Delta(G) + 5$ clique in G. If a vertex is not big, it is *small*.

Lemma 3.2. *If $G \in \mathcal{D}_1$ with $\Delta(G) \geq 50?$ and $\omega(G) < \Delta(G)$, then exactly one of the following holds:*

1. *$\chi(G) \leq \Delta(G) - 2$,*
2. *$\chi(G) = \Delta(G) - 1$ and G has a $(\Delta(G) - 1)$-coloring that uses only $\Delta(G) - 2$ colors on the small vertices.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Put $\Delta := \Delta(G)$, $\chi := \chi(G)$ $\omega := \omega(G)$. Plainly, $\chi \in \{\Delta - 1, \Delta\}$. If G has no big vertex, then Lemma 4.1 gives a contradiction since $\chi \geq \Delta - 1$.

Hence G contains a big vertices. First, suppose $\Delta(G - x) < \Delta$ for every big vertex x. Let H be the vertices with degree Δ in G and let B be the big vertices in G. Then B is joined to H in G. But then we must have $H \subseteq B$ since any vertex joined to all big vertices must be big. Thus it must be that the partition given by Lemma 2.2 for $r := \frac{2}{3}\Delta + 5$ has only one part F_1 and $H \subseteq K_1$. By Lemma 4.2, we have $(\Delta - 1)$-coloring of G that uses only $(\Delta - 2)$ colors on the small vertices, contradicting (2).

Otherwise, we have a big vertex x such that, with $Q := G - x$, we have $\Delta(Q) = \Delta$. Applying minimality of $|G|$ shows that either (1) or (2) holds for Q. It can't be (1), for then adding $\{x\}$ as a color class to a $(\Delta - 2)$-coloring of Q gives a $(\Delta - 1)$-coloring of G satisfying (2).

Hence, by symmetry, we have a $(\Delta - 1)$-coloring π of Q and such that every vertex in $M := \pi^{-1}(1)$ is big. We may as well have taken such a coloring π so as to minimize $|M|$. By minimality of M, every vertex in M has a neighbor in $\pi(i)$ for each $i \in J := [\Delta - 1] - \{1\}$. For each $z \in M$, put $O_z := \{v \in N_Q(z) \mid \pi(z) \notin \pi(N_Q(z) - \{v\})\}$. Then $|O_z| \geq \Delta - 4$ for each $z \in M$. Since z is big, it is in some F_j. If $z = x_j$, then put $P_z := K_j \cap O_z$, otherwise put $P_z := C_j \cap O_z$. In either case, $|P_z| \geq \frac{2}{3}\Delta + 5 - 1 - 2 \geq \frac{2}{3}\Delta$.

Suppose there are different $y, z \in M$ such that $P_y \cap P_z \neq \emptyset$. By the definition of our partition of the big vertices, it must be that $P_y, P_z \subseteq F_j$ for some j. But y and z are not adjacent so one of them must be x_j and the other in $C_j - K_j$. In particular, $|P_y \cap P_z| \geq |K_j| - 4 \geq \frac{2}{3}\Delta$. Plainly, no P_z intersects two others P_y and P_w.

For the $w \in M$ such that P_w does not intersect some other P_z, put $S_w := P_w$. The other vertices in M come in pairs w, w', put $S_w := P_w \cap P_{w'}$. Consider the subgraph T of G induced on $\{x\} \cup \bigcup_{w \in M} S_w$ with each S_w made into an independent set. For $w \in M$ and $z \in S_w$, we have $d_T(z) \leq \Delta - 1 - (\frac{2}{3}\Delta - 1) \leq \frac{1}{3}\Delta$.

□

Theorem 3.3. *Every graph satisfying $\chi \geq \Delta \geq 50?$ contains a K_Δ.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Then G is vertex critical and hence $G \in \mathcal{D}_1$. By Lemma 3.2, G has a $(\Delta(G) - 1)$-coloring, a contradiction. □

4 Some more details

Lemma 4.1. *Any graph satisfying* $\chi \geq \Delta - 1 \geq 20$? *contains a* $K_{\frac{2}{3}\Delta}$.

Lemma 4.2. *Let* $G \in \mathcal{D}_1$ *and put* $r := \frac{2}{3}\Delta(G) + 5$. *If the partition in Lemma 2.2 of* $\mathcal{C}_r$ *has only one part* F_1 *and* $\{v \in V(G) \mid d(v) = \Delta(G)\} \subseteq K_i$, *then* G *has a* $(\Delta(G) - 1)$*-coloring using only* $\Delta(G) - 2$ *colors on the small vertices.*