Coloring from almost maximum degree sized palettes

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1 Introduction

My dissertation contains material on a few different topics all relating to graph coloring, but today I'll mostly talk about a conjecture of Borodin and Kostochka from 1977. But before i get to that, i want to put the problem in context. First, i need to define some terms.

Define, $\Delta(G)$, K_t, coloring. PICTURES.

A pair of complimentary results from the late 90s tell us that c-coloring transitions from being hard to being easy around around $c = \Delta - \sqrt{\Delta}$. The first is from Emden-Weinert, Hougardy and Kreuter and says:

Theorem 1.1. *Fix Δ. For any k such that $k^2 + k > \Delta$, the problem of determining whether a graph G of maximum degree Δ has a $(\Delta + 1 - k)$-coloring is NP-complete (also need, $\Delta + 1 - k \geq 3$).*

The second result is from Molloy and Reed.

Theorem 1.2. *There exists Δ_0 such that for fixed $\Delta \geq \Delta_0$ and k such that $k^2 + k \leq \Delta$ the problem of determining whether a graph G of maximum degree Δ has a $(\Delta + 1 - k)$-coloring is in P.*

The large Δ requirement comes from using the probabilistic method. The complexity situation for small Δ is open. For $k = 0$, this was solved in 1941 by Brooks:

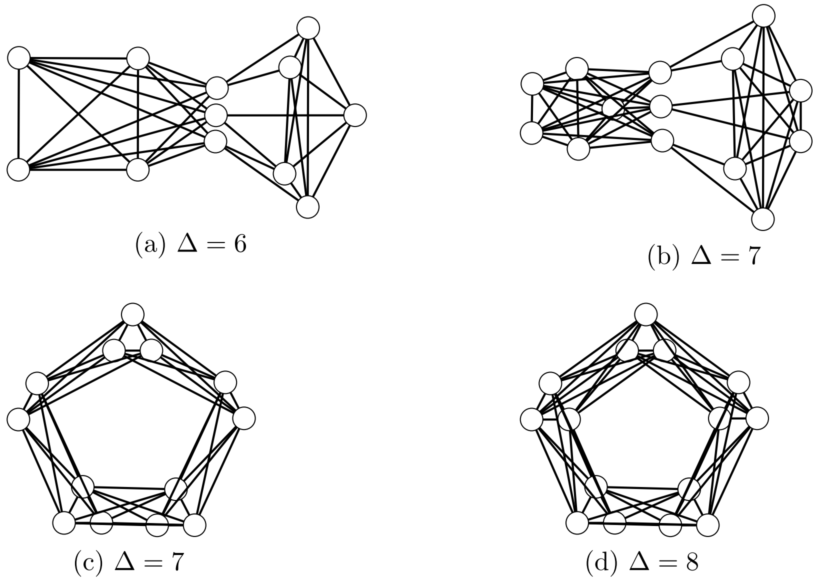
Theorem 1.3 (Brooks 1941)**.** *Every graph G with $\Delta(G) \geq 3$ that doesn't contain $K_{\Delta(G)+1}$ is $\Delta(G)$-colorable.*

So, an algorithm can just test for a $K_{\Delta+1}$ component. In 1977, Borodin and Kostochka conjectured a result that would solve the case when $k = 1$ and $\Delta \geq 9$.

Conjecture 1.4. *Every graph G with $\Delta(G) \geq 9$ that doesn't contain $K_{\Delta(G)}$ is $(\Delta(G) - 1)$-colorable.*

Talk, K_Δ is the obvious obstruction to $(\Delta - 1)$-coloring. The $\Delta \geq 9$ condition is necessary:

Known results:



(a) $\Delta = 6$

(b) $\Delta = 7$

(c) $\Delta = 7$

(d) $\Delta = 8$

Figure 1: Counterexamples to the Borodin-Kostochka Conjecture for small Δ.

- In 1980, Kostochka proved that if we exclude $K_{\Delta(G)-29}$ instead, then G is $(\Delta(G)-1)$-colorable.
- Later in the 1980s, Mozhan proved that if $\Delta(G) \geq 31$ and we exclude $K_{\Delta(G)-3}$ instead, then G is $(\Delta(G)-1)$-colorable.
- In 1999, Reed proved that the conjecture holds for $\Delta \geq 10^{14}$.

Highlights:

- We prove the full Borodin-Kostochka conjecture for claw-free graphs.
- We prove that the following conjecture is equivalent to the Borodin-Kostochka conjecture.

  Conjecture 1.5. *If G is a graph with $\Delta(G) \geq 9$ such that G doesn't contain $K_3 * \overline{K}_{\Delta(G)-3}$, then G is $(\Delta(G)-1)$-colorable.*

- We generalize Reed's result to list coloring:

  Theorem 1.6. *There exists Δ_0 such that every graph with $\Delta \geq \Delta_0$ that doesn't contain K_Δ is $(\Delta - 1)$-choosable.*

  Define k-choosable.

2 List coloring lemmas

We call a graph d_k-choosable if it colorable from any list assignment L with $|L(v)| \geq d(v) - k$ for each vertex v. Why do we care about these particular list assignments? Suppose we want to $(\Delta(G) - k)$-color a graph G and that we can $(\Delta(G) - k)$-color some proper induced subgraph H. PICTURE. Look at the lists of colors available on $G - H$. Each $v \in V(G - H)$ has at most $\Delta(G) - d_{G-H}(v)$ neighbors in H, so has at least $\Delta(G) - k - (\Delta(G) - d_{G-H}(v)) = d_{G-H}(v) - k$ colors available. So, if $G - H$ is d_k-choosable, we can finish.

The d_0-choosable graphs were classified in the 70s by Borodin and independently Erdos-Rubin-Taylor. For the Borodin-Kostochka conjecture, we want to know about d_1-choosable graphs.

We classified the d_1-choosable graph joins $A * B$ where $|A|, |B| \geq 2$, this gives a lot of structure about a counterexample. The classification takes 45 pages to prove, we won't go into it, but will use the results to prove Borodin-Kostochka for claw-free graphs. One example, $K_6 * B$ is d_1-choosable unless $\omega(B) \geq |B| - 1$, so intersections of cliques are severely restricted. PICTURE.

3 Borodin-Kostochka for claw-free graphs

We outline the proof of the following.

Theorem 3.1. *Every claw-free graph with $\Delta \geq 9$ that doesn't contain K_Δ can be $(\Delta - 1)$-colored.*

The proof uses the structure theorem for claw-free graphs proved by Chudnovsky and Seymour. We actually only need a simpler part of it: the structure theorem for quasi-line graphs; graphs where the neighborhood of every vertex can be covered by two cliques. PICTURE.

We use the following structure theorem for quasi-line graphs.

Lemma 3.2. *Every connected skeletal quasi-line graph is a circular interval graph or a composition of linear interval strips.*

We need to define the terms in this lemma.

A *homogeneous pair of cliques* (A_1, A_2) in a graph G is a pair of disjoint nonempty cliques such that for each $i \in [2]$, every vertex in $G - (A_1 \cup A_2)$ is either joined to A_i or misses all of A_i and $|A_1| + |A_2| \geq 3$. PICTURES.

A homogeneous pair of cliques (A_1, A_2) is *skeletal* if for any $e \in E(A, B)$ we have $\omega(G[A \cup B] - e) < \omega(G[A \cup B])$. A graph is *skeletal* if it contains no nonskeletal homogeneous pair of cliques.

Given a set V of points on the unit circle together with a set of closed intervals C on the unit circle we define a graph with vertex set V and an edge between two different vertices if and only if they are both contained in some element of C. Any graph isomorphic to such a graph is a *circular interval graph*. Similarly, by replacing the unit circle with the unit interval, we get the class of *linear interval graphs*.

It remains to define *compositions of linear interval strips*. These are a generalization of line graphs. A *linear interval strip* (S, A_1, A_2) is a linear interval graph S together with end cliques A_1 and A_2. PICTURE.

Let H be a directed multigraph (possibly with loops) and suppose for each edge e of H we have a strip (S_e, X_e, Y_e). For each $v \in V(H)$ define

$$C_v := \left(\bigcup\{X_e \mid e \text{ is directed out of } v\}\right) \cup \left(\bigcup\{Y_e \mid e \text{ is directed into } v\}\right)$$

The graph formed by taking the disjoint union of $\{S_e \mid e \in E(H)\}$ and making C_v a clique for each $v \in V(H)$ is the composition of the strips (S_e, X_e, Y_e). Any graph formed in such a manner is called a *composition of linear interval strips*. PICTURE.

Taking all strips to have a single vertex gives the line graph construction.

Now we can outline the proof.

1. Prove for circular interval graphs.
2. Reduce from quasi-line graphs to line graphs as follows:
   (a) It is always possible to make skeletal counterexample from a given counterexample just by removing edges in nonskeletal homogeneous pairs of cliques. Do so.
   (b) We must have a composition of linear interval strips by the structure theorem.
   (c) Take a composition representation using the maximum number of strips.
   (d) Show that for each strip (S, A_1, A_2) we must have $V(S) = A_1 = A_2$ and thus we have a line graph.
3. Prove for line graphs of multigraphs.
4. Reduce from claw-free graphs to quasi-line graphs.

Steps (1), (2) and (4) all rely heavily on our classification of d_1-choosable joins. Step (4) uses some d_1-choosability results outside this classification, for example, the following graph D_8 is d_1-choosable:

The reduction from claw-free graphs to quasi-line graphs works for list coloring as well. Also, the circular interval graphs proof works for list coloring. So, the following generalization seems within reach.

Conjecture 3.3. *Every claw-free graph with $\Delta \geq 9$ that doesn't contain K_Δ is $(\Delta - 1)$-choosable.*

Borodin and Kostochka conjecture that this holds with the claw-free restriction removed. As evidence of this, we generalized Reed's proof of Borodin-Kostochka for large Δ to list coloring, proving:

Theorem 3.4. *There exists Δ_0 such that every graph with $\Delta \geq \Delta_0$ that doesn't contain K_Δ is $(\Delta - 1)$-choosable.*

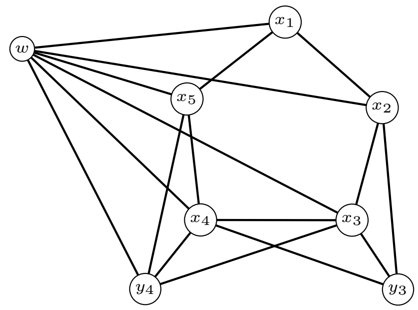


Figure 2: The graph D_8.

4 Future directions

1. BK for list coloring for claw-free graphs.

2. improved bounds on the number of edges in (online) list-critical. These can be used to prove Ore degree bounds for (online) list coloring; we have so far that the Ore degree version of Brooks' theorem for (online) list coloring holds for $\Delta \geq 11$.

3. Improve Mozhan's methods to get down to $\Delta - 2$, we can now prove his $\Delta - 3$ result for $\Delta \geq 13$ instead of $\Delta \geq 31$, and the proof is relatively simple.

4. BK for large Δ for online list coloring (and Alon-Tarsi number)