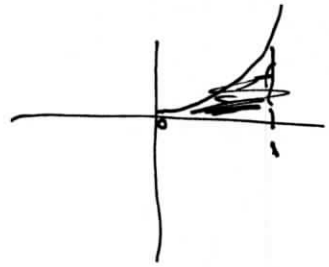
Math 109 notes

trying to find



$f(x) = x^2$

$$1^2 + 2^2 + 3^2 + \dots + n^2 = ?$$

Notation:

$$\sum_{k=1}^{n} k^2$$

Last time $\displaystyle\sum_{k=1}^{n} k = 1 + 2 + \dots + n = \frac{n(n+1)}{2}$

now consider

$$\sum_{k=1}^{n} \left((k+1)^3 - k^3\right) = (2^3 - 1^3) + (3^3 - 2^3) + (4^3 - 3^3) + \dots (n+1)^3 - n^3$$

$$= (n+1)^3 - 1^3$$

$$\parallel$$

$$\sum_{k=1}^{n} \left(k^3 + 3k^2 + 3k + 1 - k^3\right)$$

$$\parallel$$

$$\sum_{k=1}^{n} \left(3k^2 + 3k + 1\right)$$

$$\parallel$$

$$3\sum_{k=1}^{n} k^2 + 3\sum_{k=1}^{n} k + \sum_{k=1}^{n} 1$$

$$3\sum_{k=1}^{n} k^2 + 3\,\frac{n(n+1)}{2} + n = (n+1)^3 - 1^3$$

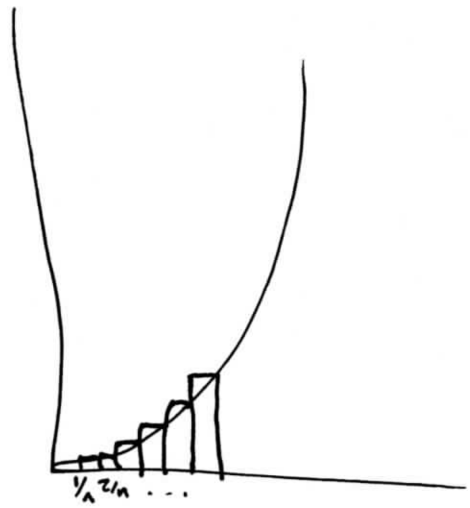
$$\sum_{k=1}^{n} k^2 = \frac{(n+1)^3 - 1}{3} - \frac{n(n+1)}{2} - \frac{n}{3}$$

$$= \frac{n^3 + 3n^2 + 3n - n}{3} - \frac{n(n+1)}{2}$$

$$= \frac{2n^3 + 6n^2 + 4n - 3n^2 - 3n}{6}$$

$$= \frac{2n^3 + 3n^2 + n}{6} = \frac{n(2n^2 + 3n + 1)}{6} = \boxed{\frac{n(n+1)(2n+1)}{6}}$$

Math 109 notes



$$\frac{1}{n}\cdot f\left(\frac{1}{n}\right)+\frac{1}{n}\cdot f\left(\frac{2}{n}\right)+\dots+\frac{1}{n}\cdot f\left(\frac{n}{n}\right)$$

$$=\frac{1}{n}\cdot\left(\frac{1}{n}\right)^2+\left(\frac{1}{n}\right)\left(\frac{2}{n}\right)^2+\dots+\frac{1}{n}\cdot\left(\frac{n}{n}\right)^2$$

$$=\frac{1}{n^3}\left(1^2+2^2+3^2+\dots+n^2\right)$$

$$=\frac{1}{n^3}\left(\frac{\cancel{n}(2n+1)(n+1)}{6}\right)$$

$$=\frac{(n+1)(2n+1)}{6n^2}$$

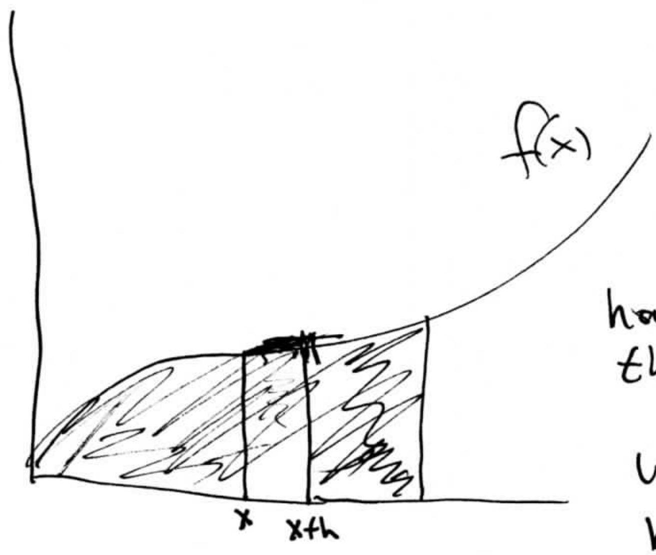
$$=\frac{2n^2+3n+1}{6n^2}$$

$$=\frac{1}{3}+\frac{3n+1}{n^2}$$

$$\lim_{n\to\infty}=\frac{1}{3}$$

ok, that was a good amount of work even for a parabola, more complicated functions are going to be harder or impossible this way.

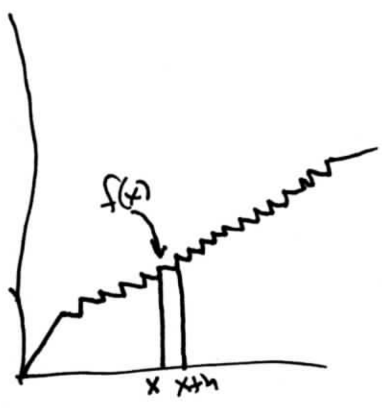
need better way.



Area function A

how much gets added to the area between x and x+h?

what if we could make h small enough so that the curve was flat between x and x+h



$A(x)$ vs. $A(x+h)$

$$A(x+h) - A(x) \doteq h \cdot f(x)$$

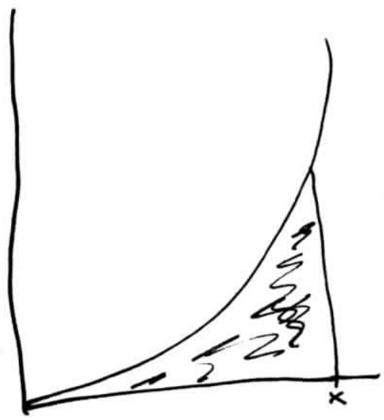
$$f(x) \approx \frac{A(x+h) - A(x)}{h}$$

So value of f gives change in area

$$A'(x) = f(x)$$

Math 109 notes

our example $f(x) = x^2$



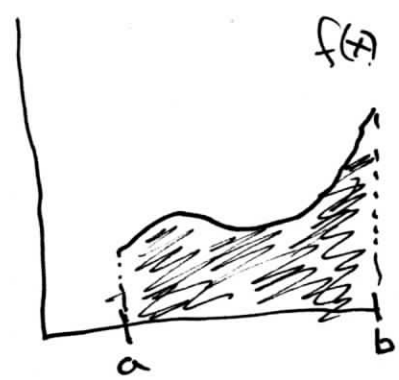
let $A(x)$ give area under f from 0 to x

$$A'(x) = f(x) = x^2$$

$$A(x) = \frac{1}{3}x^2$$

$$A(1) = \frac{1}{3}$$

General tool: Suppose $F(x)$ is an antiderivative of $f(x)$ on $[a,b]$ (so $F'(x) = f(x)$)



then the area under $f(x)$ between a and b is

$$F(b) - F(a).$$

example once more with tool.

area under $f(x) = x^2$ from 0 to 1.

$$F(x) = \frac{1}{3}x^3$$

$$F(1) - F(0) = \frac{1}{3} - 0 = \frac{1}{3}.$$

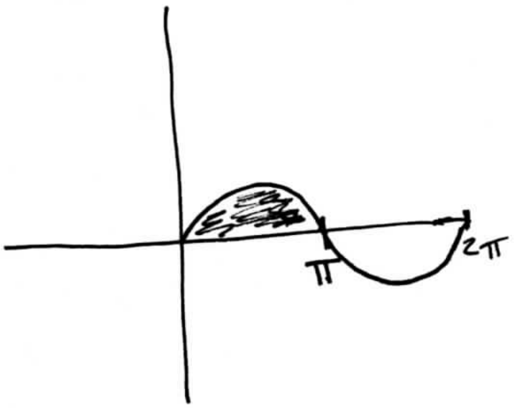
Notation: Area under $f(x)$ from a to b $\longleftarrow$ $\int_a^b f(x)\,dx$

later

Math 109 notes

__example:__

find the area under sin(x) from 0 to π $\int_0^{\pi} \sin(x)\,dx$



$$F(x) = -\cos(x)$$

$$F(\pi) - F(0)$$

$$\parallel$$

$$-\cos(\pi) - (-\cos(0))$$

$$\parallel$$

$$1 + 1 = 2.$$

__new notation:__ $\int \ldots$

previously we reduced finding arc length to finding area, in our new notation this becomes

arc length of f(x) from a to b is $\int_a^b \sqrt{1 + (f'(x))^2}\,dx$