Generalizing Fajtlowicz to mic

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Abstract

Generalizing Fajtlowicz's lower bound on the independence number, we show that every graph G has an independent set incident to at least $2|G| - \alpha(G)(\omega(G)+1)$ edges. Combined with the magic kernel lemma, this implies lower bounds on the average degree of online k-list-critical graphs. In particular, an online k-list-critical graph with n vertices, independence number α and clique number ω has average degree at least $k - \frac{\alpha(\omega+1)}{n}$ and also has average degree at least $k - 1 + \frac{k-\omega-2}{k+\omega}$.

1 Introduction

For a graph G and disjoint $A, B \subseteq V(G)$, let $\|A, B\|$ be the number of edges between A and B.

Definition 1. The *maximum independent cover number* of a graph G is the maximum $\text{mic}(G)$ of $\|I, V(G) \setminus I\|$ over all independent sets I of G.

2 The bound on mic

Theorem 2.1. *Every graph G satisfies* $\text{mic}(G) \geq 2|G| - \alpha(G)(\omega(G)+1)$.

Proof. Let t be the maximum $\text{mic}(G)$ of $\|M, V(G) \setminus M\|$ over all *maximum* independent sets M of G. Then $\text{mic}(G) \geq t$, so it suffices to show that $t \geq 2|G| - \alpha(G)(\omega(G)+1)$. Let A be a maximum independent set in G with $\|A, V(G) \setminus A\| = t$. For $i \in [\Delta(G)]$, let c_i be the number of vertices in $V(G) \setminus A$ with exactly i neighbors in A. Then

$$\sum_{i \in [\Delta(G)]} c_i = |G| - |A|, \tag{1}$$

and

$$\sum_{i \in [\Delta(G)]} ic_i = t. \tag{2}$$

Subtracting twice 1 from 2 gives

$$-c_1 \leq -c_1 + \sum_{i \in [\Delta(G)-2]} ic_{i+2} = t - 2|G| + 2|A|. \tag{3}$$

If $x, y \in V(G) \setminus A$ with $N(x) \cap A = N(y) \cap A = \{z\}$, then x and y are adjacent since otherwise $A \cup \{x, y\} \setminus \{z\}$ is an independent set in G which is larger than A. Therefore

$$c_1 \leq (\omega(G) - 1)|A|. \tag{4}$$

Plugging 4 into 3 gives

$$-(\omega(G) - 1)|A| \leq t - 2\,|G| + 2\,|A|\,,$$

and hence

$$t \geq 2\,|G| - \alpha(G)(\omega(G) + 1).$$

□

When $\delta(G) \geq \frac{\Delta(G)}{2}$, a similar argument works using $t = \text{mic}(G)$, which gets a better bound when some non-maximum independent set A is incident to the maximum number of edges. The bound can also be improved when $c_i > 0$ for some $i \geq 3$ since we just disregarded those terms.

Corollary 2.2. *Every graph G satisfies* $\text{mic}(G) \geq \frac{2\delta(G)}{\delta(G)+\omega(G)+1}\,|G|$.

Proof. For every graph G,

$$\text{mic}(G) \geq \delta(G)\alpha(G). \tag{5}$$

Adding $\frac{\omega(G)+1}{\delta(G)}$ times 5 to the bound in Theorem 2.1 gives

$$\left(1 + \frac{\omega(G) + 1}{\delta(G)}\right) \text{mic}(G) \geq 2\,|G|\,,$$

which is

$$\text{mic}(G) \geq \frac{2\delta(G)}{\delta(G) + \omega(G) + 1}\,|G|\,.$$

□

3 Online list coloring

Theorem 3.1. *Every OC-irreducible graph G satisfies*

$$\text{mic}(G) \leq 2\,\|G\| - (\delta(G) - 1)\,|G| - 1.$$

Combining Theorem 3.1 with Theorem 2.1 gives the following.

Corollary 3.2. *Every OC-irreducible graph has average degree at least*

$$2\,\|G\| \geq (\delta(G) + 1)\,|G| + 1 - \alpha(G)(\omega(G) + 1).$$

Combining Theorem 3.1 with Corollary 2.2 gives the following.

Corollary 3.3. *Every OC-irreducible graph G satisfies*

$$2\,\|G\| \geq \left(\delta(G) + \frac{\delta(G) - \omega(G) - 1}{\delta(G) + \omega(G) + 1}\right)|G| + 1.$$

References