Math 109
Exam 2
November 3, 2015

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1. (20 points) Differentiate each of the following functions with respect to x.

   (a) (4 points) $f(x) = x^{2048} + 2x^{1024} + 4x^{512} + 8x^{256} + 16x^{128} + 32x^{64} + 64x^{32} + 128x^{16} + 256x^8 + 512x^4 + 1024x^2 + 2048x + \pi$.

   $$f'(x) = 2048(x^{2047} + x^{1023} + x^{511} + x^{255} + x^{127} + x^{63} + x^{31} + x^{15} + x^7 + x^3 + x + 1)$$

   (b) (4 points) $g(x) = e^{(x^5+47)}$.

   $$g'(x) = e^{x^5+47} \cdot (5x^4)$$

   (c) (4 points) $r(x) = (\ln(x))^{\ln(x)}$.

   $$\ln(r(x)) = \ln(\ln(x)^{\ln(x)}) = \ln(x)\ln(\ln(x))$$

   $$\frac{r'(x)}{r(x)} = \frac{1}{x} \cdot \frac{1}{\ln(x)} \cdot \ln(x) + \frac{1}{x}\ln(\ln(x)) = \frac{1}{x}(1 + \ln(\ln(x)))$$

   $$r'(x) = \frac{\ln(x)^{\ln(x)} \cdot (1 + \ln(\ln(x)))}{x}$$

   (d) (4 points) $f(x) = (\arctan(x)\sin(x))^{17}$.

   $$f'(x) = 17(\arctan(x) \cdot \sin(x))^{16} \cdot \left(\arctan(x) \cdot \cos(x) + \frac{\sin(x)}{1+x^2}\right)$$

   (e) (4 points) $f(x) = \frac{\sin^2(x)+\sin(x)+\cos^2(x)-1}{\cos(x)} = \frac{\sin(x)}{\cos(x)} = \tan(x)$

   $$f'(x) = \frac{1}{\cos^2(x)}$$

2. (10 points) If f is a differentiable function such that $\sin(f(x)) = \tan(x)$ and $f(2) = \pi$, what is $f'(2)$?

$$\cos(f(x)) \cdot f'(x) = \frac{1}{\cos^2(x)}$$

$$f'(x) = \left(\cos(f(x)) \cdot \cos^2(x)\right)^{-1}$$

$$f'(2) = \left(\cos(\pi) \cdot \cos^2(2)\right)^{-1}$$

$$f'(2) = \frac{-1}{\cos^2(2)}.$$

3. (10 points) Find the equation of the line tangent to the curve $x^3y^4 - 5 = x^3 - x^2 + y$ at the point $(2, -1)$. have point, need slope.

$$x^3 \cdot \left(4y^3 \frac{dy}{dx}\right) + y^4(3x^2) = 3x^2 - 2x + \frac{dy}{dx}$$

$$4x^3y^3 \frac{dy}{dx} - \frac{dy}{dx} = 3x^2 - 2x - 3x^2y^4$$

$$(4x^3y^3 - 1)\frac{dy}{dx} = 3x^2 - 2x - 3x^2y^4$$

$$\frac{dy}{dx} = \frac{3x^2 - 2x - 3x^2y^4}{4x^3y^3 - 1}$$ at $(2,-1)$ is

$$\frac{3(2)^2 - 2(2) - 3(2)^2(-1)^4}{4(2)^3(-1)^3 - 1} = \frac{4}{33}$$

$$\boxed{y - (-1) = \frac{4}{33}(x-2)}$$

4. (10 points) Circle True or False for each question. Each correctly answered question gets you 1 point.

(a) True False If $f(x) = x^3$, then $\frac{d}{dx}((f \circ f \circ f)(x)) = 9x^8$.

(b) True False $\sin\left(\arctan\left(\frac{1}{2}\right)\right) = \frac{1}{\sqrt{5}}$.

(c) True False $\sqrt{x+y} = \sqrt{x} + \sqrt{y}$ for all non-negative integers x and y with $x + y = 0$.

(d) True False $\frac{d}{dt}(t^t) = tt^{t-1}$.

(e) True False $\frac{d}{dt}(t^t) = t^t \ln(t)$.

(f) True False $t = \pi^{\log_\pi(t)}$ for all $t > 0$.

(g) True False $e = 1 + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots$

(h) True False If n is a positive integer such that $\frac{n-1}{4}$ is an integer, then the n-th derivative of $\sin(x)$ with respect to x is $\cos(x)$.

(i) True False $\pi \leq 3.141592$

(j) True False I answered every question on this page correctly.

5. (2π bonus points) What is $f'(1)$ if $f(x) = (g \circ g \circ g \circ g \circ g \circ g)(x)$ where $g(x) = x^x$?

$g(1) = 1^1 = 1$

$(g \circ g)(1) = g(1) = 1$

$((g \circ g) \circ (g \circ g))(1) = (g \circ g)(1) = 1$

$((g \circ g) \circ (g \circ g) \circ (g \circ g))(1) = ((g \circ g) \circ (g \circ g))(1) = 1$

$g'(x) = (1 + \ln(x))\, x^x$, so $g'(1) = (1+0)\, 1^1 = 1.$

$(g \circ g)'(1) = g'(g(1)) \cdot g'(1) = g'(1) \cdot g'(1) = 1 \cdot 1 = 1$

$((g \circ g) \circ (g \circ g))'(1) = (g \circ g)'((g \circ g)(1)) \cdot (g \circ g)'(1) = (g \circ g)'(1) \cdot (g \circ g)'(1) = 1.$

$f'(1) = ((g \circ g) \circ (g \circ g) \circ (g \circ g))'(1) = ((g \circ g) \circ (g \circ g))'(1) \cdot (g \circ g)'(1) = 1 \cdot 1 = 1.$

$f'(1) = 1.$

Another way. Let $g^n(x) = (\overbrace{g \circ g \circ \cdots \circ g}^{n})(x)$. Clearly, $g^n(1) = 1$ for all $n \geq 1$.

Claim. $(g^n)'(1) = 1$ for all $n \geq 1$.

Proof. If not, then there is a least n for which $(g^n)'(1) \neq 1$. If $n = 1$, then $g'(1) \neq 1$, but $g'(1) = (1+0)\, 1^1 = 1$, so we must have $n \geq 2$.

$$(g^n)'(1) = (g \circ g^{n-1})'(1) = g'(g^{n-1}(1)) \cdot (g^{n-1})'(1) = g'(1) \cdot (g^{n-1})'(1) = (g^{n-1})'(1).$$

But we chose n to be smallest such that $(g^n)'(1) \neq 1$, so $(g^{n-1})'(1) = 1$ and we get $(g^n)'(1) = (g^{n-1})'(1) = 1$, a contradiction ∎