notes on the Borodin-Kostochka conjecture

May 9, 2017

1 Introduction

Conjecture 1 (Borodin and Kostochka [2])**.** *Every graph G with $\Delta(G) \geq 9$ satisfies $\chi(G) \leq \max\{\omega(G), \Delta(G) - 1\}$.*

2 Excluded induced subgraphs by d_1-choosability

A graph G is *d_r-choosable* if G can be L-colored from every list assingment L with $|L(v)| \geq d_G(v) - r$ for all $v \in V(G)$. Every graph is d_{-1}-choosable. The d_0-choosable graphs were classified by Borodin [1] and independently by Erdős, Rubin, and Taylor [9] as those graphs whose every block is either complete or an odd cycle (a connected such graph is a *Gallai tree*). Classifying the d_r-choosable graphs for any $r \geq 1$ appears to be a hard problem. However, we can get useful sufficient conditions for a graph to be d_1-choosable. For example, all of the graphs here are d_1-choosable (the vertex color indicates components of the complement): `https://landon.github.io/graphdata/borodinkostochka/offline/index.html`

Cranston and Rabern [6] classified all d_1-choosable graphs of the form $A \vee B$.

3 Decompositions

3.1 Reed's decomposition

In [14], Reed proved the Borodin-Kostochka conjecture for graphs G with $\Delta(G) \geq 10^{14}$. A piece of that proof was a decomposition of G into dense chunks and one sparse chunk that also works for smaller $\Delta(G)$. The following tight form of this decomposition is proved in [13]. Let $\mathcal{C}_t(G)$ be the maximal cliques in G having at least t vertices.

Reed's Decomposition. *Suppose G is a graph with $\Delta(G) \geq 8$ that contains no $K_{\Delta(G)}$ and has no d_1-choosable induced sugraph. If $\frac{\Delta(G)+5}{2} \leq t \leq \Delta(G) - 1$, then $\bigcup \mathcal{C}_t(G)$ can be partitioned into sets $D_1, \dots, D_r$ such that for each $i \in [r]$ at least one of the following holds:*

1. *$D_i = C_i \in \mathcal{C}_t(G)$,*
2. *$D_i = C_i \cup \{x_i\}$ where $C_i \in \mathcal{C}_t(G)$ and $|N(x_i) \cap C_i| \geq t - 1$.*

3.2 Fajtlowicz's decomposition

In [10], Fajtlowicz proved that every graph has $\alpha(G) \geq \frac{2|G|}{\omega(G)+\Delta(G)+1}$. The proof of this result gives a decomposition which we state in the special case needed for the Borodin-Kostochka conjecture.

Fajtlowicz's Decomposition. *Suppose G is a vertex-critical graph with $\chi(G) = \Delta(G)$. Then $V(G)$ can be partitioned into sets M, T, and K such that*

1. *M contains a maximum independent set I of G; and*
2. *each $v \in T$ has $d_G(v) = \Delta(G)$, two neighbors in I and zero neighbors in $M \setminus I$; and*
3. *K can be covered by $\alpha(G)$ (or fewer) cliques; and*
4. *each $v \in K$ has exactly one neighbor in I and at most one neighbor in $M \setminus I$ (none if $d_G(v) < \Delta(G)$); and*
5. *the vertices in $M \setminus I$ can be ordered $v_1, \ldots, v_r$ such that for $i \in [r]$, either v_i has at least three neighbors in $I \cup \{v_1, \ldots, v_{i-1}\}$ or $d_G(v_i) < \Delta(G)$ and v_i has at least two neighbors in $I \cup \{v_1, \ldots, v_{i-1}\}$.*

Proof. Let I be a maximum independent set in G. Construct a maximal length sequence $I = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r$ such that for $j > 0$,

- every $v \in M_j$ with $d_G(v) = \Delta(G)$ either has at least three neighbors in M_{j-1} or at least two neighbors in $M_{j-1} \setminus I$; and
- every $v \in M_j$ with $d_G(v) = \Delta(G) - 1$ either has at least two neighbors in M_{j-1} or at least one neighbor in $M_{j-1} \setminus I$.

Now let $M = M_r$, let T be the vertices in $V(G) \setminus M$ with exactly two neighbors in I and let K be the vertices in $V(G) \setminus M$ with exactly one neighbor in I. The decomposition has the properties 1,2,4 and 5 since the sequence $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r$ was chosen to be maximal length. Property 3 follows since for each $v \in I$, the set of $x \in K$ adjacent to v must be a clique for otherwise we could get an independent set larger than I. $\square$

4 Properties of minimum counterexamples

In [7] Cranston and R. used the d_1-choosable graphs in Section 2 to prove properties of a minimum counterexample to the Borodin-Kostochka conjecture. Almost all of the proofs there (specifically, the proofs only involving edge addition and not vertex set contraction) work for minimum counterexamples within a given collection of graphs that is closed under taking induced subgraphs and adding edges. Call such a collection of graphs *permissible*. For example, the following improves a lemma Reed used in his proof [14].

Lemma 2. *Let $\mathcal{A}$ be a permissible collection of graphs for which the Borodin-Kostochka conjecture does not hold. Let $G \in \mathcal{A}$ be a counterexample with the minimum number of vertices (of graphs in $\mathcal{A}$).*

1. *If X is a $K_{\Delta(G)-1}$ in G, then every $v \in V(G-X)$ has at most one neighbor in X; and*

2. *Let A and B be disjoint subgraphs of G with $|A|+|B| = \Delta(G)$ such that $|A|, |B| \geq 4$. If G contains all edges between A and B, then $A = K_1 + K_{|A|-1}$ and $B = K_1 + K_{|B|-1}$.*

5 Counterexamples have some sparse neighborhoods

In [13], R. showed that any counterexample to the Borodin-Kostochka conjecture must have some sparse neighborhoods and large independence number (increasing with $\Delta(G)$). For example,

Lemma 3. *If G is a counterexample to the Borodin-Kostochka conjecture, then*

1. *there exists $v \in V(G)$ such that v is not contained in any clique with at least $\frac{2}{3}\Delta(G)+2$ vertices; and*

2. *there exists $v \in V(G)$ such that $G[N(v)]$ has average degree at most $\frac{2}{3}\Delta(G)+3$; and*

3. *$\alpha(G) \geq \frac{\Delta(G)}{4}$; and*

4. *$|G| \geq 16\Delta(G)^2 - 528\Delta(G) + 3527$.*

6 Results from kernel methods

In [11], Kierstead and R. proved a general lemma that allows the user to get list colorings for free from large independent sets. Specialized to the Borodin-Kostochka conjecture, this becomes.

Kernel Magic. *Suppose G is a vertex-critical graph with $\chi(G) = \Delta(G)$. For every induced subgraph H of G and independent set I in H, we have*

$$\sum_{v \in V(I)} d_H(v) < \sum_{v \in V(H)} \Delta(G) + 2 - d_G(v).$$

Applied with $H = G$, this gives:

Corollary 4. *If G is a vertex-critical graph with $\chi(G) = \Delta(G)$, then $\alpha(G) < \frac{2|G|}{\Delta(G)}$.*

7 Mozhan partitions

Extending ideas of Mozhan [12], Cranston and R. [8] proved the following.

Theorem 5. *If G is a vertex-critical graph with $\chi(G) = \Delta(G) \geq 13$, then $\omega(G) \geq \Delta(G) - 3$.*

8 Vertex-transitive graphs

In [4] Cranston and R. used Reed's decomposition and the ideas in Sections 5 and 7 to prove the Borodin-Kostochka conjecture for vertex-transitive graphs with $\Delta(G) \geq 13$. It would be interesting to improve this to $\Delta(G) \geq 9$.

Theorem 6. *Every vertex-transitive graph G with $\Delta(G) \geq 13$ satisfies $\chi(G) \leq \max\{\omega(G), \Delta(G) - 1\}$.*

9 Claw-free graphs

In [5], Cranston and R. proved the Borodin-Kostochka conjecture for claw-free graphs using some of the d_1-choosable graphs in Section 2 combined with the structure theorem for quasi-line graphs of Chudnovsky and Seymour [3].

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