# MULTIGRAPH VARIABLE DEGENERACY

In [2] Borodin, Kostochka and Toft proved a common generalization of Gallai's structure theorem for the low vertex subgraph of a critical graph (and also the classification of degree-choosable graphs proved independently by Borodin [3] and Erdős, Rubin and Taylor [4]) and Borodin's [1] result on decomposing a graph into degenerate parts. Here we generalize this result to multigraphs.

We reformulate the result from [2] so it looks more like coloring. Let $G$ be a loopless multigraph. To each $v \in V(G)$, assign a list $L(v)$ of pairs $(c, d) \in \mathbb{N} \times \mathbb{N}$ which we call *graded colors*. For $(c, d) \in L(v)$, we will think of $c$ as the color and $d$ as a measure of how hard it is to color $v$ with $c$ (where $d = 0$ means it is impossible). Let $\text{Pot}(L) = \{c \mid (c, d) \in L(v)\}$. We say that $G$ is $L$-colorable if there is a (possibly improper) coloring $\pi$ of $V(G)$ where $\pi(v) \in L(v)$ for each $v \in V(G)$ such that for each color $c \in \text{Pot}(L)$, the vertices colored with $c$, that is $V_c = \{v \mid \pi(v) \in \{c\} \times \mathbb{N}\}$ there exists an ordering of $V_c$ such that each $v \in V_c$ has fewer than $d$ edges going to the left (where $\pi(v) = (c, d)$).

That definition can surely be said better. Note that no vertex can be colored with $(c, 0)$ for any $c$, so we can remove any $(c, 0)$'s from lists without changing anything. By convention we will always remove $(c, 0)$'s from the lists. Also, in the case that $L(v) \subseteq \mathbb{N} \times \{1\}$ for all $v \in V(G)$, we recover normal list coloring. Now we can state the result from [2].

**Theorem 1** (Borodin, Kostochka and Toft [2])**.** *Let $G$ be a connected graph and $L(v)$ an assignment on $G$ such that $\sum_{(c,d) \in L(v)} d \geq d_G(v)$ for each $v \in V(G)$. Then $G$ is $L$-colorable if and only if $(G, L)$ is not "hard-constructible".*

Here "hard-constructible" is just giving the few exceptions. In the case that $L(v) \subseteq \mathbb{N} \times \{1\}$ for all $v \in V(G)$, this says that if each vertex has a list of at least $d_G(v)$ colors, then $G$ can be colored from the lists (unless exceptions, which coincide with Gallai trees); this is the classification of degree-choosable graphs. If instead, we strengthen the condition to $\sum_{(c,d) \in L(v)} d \geq \Delta(G)$, we get Borodin's [1] result. The goal is to extend the result to loopless multigraphs. This will imply the directed versions of the desired results by just considering multigraphs with maximum multiplicity 2. We'll determine what "harder-constructible" should mean in the process of proving.

**Theorem 2.** *Let $G$ be a connected loopless multigraph and $L(v)$ an assignment on $G$ such that $\sum_{(c,d) \in L(v)} d \geq d_G(v)$ for each $v \in V(G)$. Then $G$ is $L$-colorable if and only if $(G, L)$ is not "harder-constructible".*

*Proof.* We just consider the 'if' direction now. Suppose the theorem is false and let $G$ be a counterexample minimizing $|G|$.

Let $v$ be a noncutvertex of $G$. For any $(c, d) \in L(v)$ with $d > 0$, consider the list assignment $L'$ on $G - v$ created from $L$ by changing $(c, d') \in L(w)$ to $(c, \max\{0, d' - \mu(vw)\})$ (and so, removing the pair if $d' \leq \mu(vw)$) for each neighbor $w$ of $v$. Suppose $(G - v, L')$ is not "harder-constructible". Then, by minimality of $|G|$, we have an $L'$-coloring of $G - v$. We can extend this to an $L$-coloring of $G$ by coloring $v$ with $c$ by putting $v$ first in the ordering of $V_c$. Hence $(G - v, L')$ is "harder-constructible" for any $(c, d) \in L(v)$ with $d > 0$.

Suppose $G$ is not 2-connected. Consider two end-blocks, removing a noncutvertex from one shows that all but its block is hard, and the other one shows all. So $G$ is 2-connected.

Suppose $\mathcal{C}_L(v) \neq \mathcal{C}_L(w)$ for some $v, w \in V(G)$, then since $G$ is connected, there are adjacent vertices $u, z \in V(G)$ with $\mathcal{C}_L(u) - \mathcal{C}_L(z) \neq \emptyset$. Pick $c \in \mathcal{C}_L(u) - \mathcal{C}_L(z)$. Consider $(G - z, L')$ where $L'$ is created from $L$ by changing $(c, d') \in L(x)$ to $(c, \max\{0, d' - \mu(zx)\})$ for each neighbor $x$ of $z$. Then $u$ is a $+1$ vertex while the rest are still 0 vertices, so we win. $\square$

## References


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