fixable proofs

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1 Proofs

Lemma 1.1. *The graph in Figure 1 is reducible.*

Proof. Let $X = \{0, 1\}$, $Y = \{0, 2\}$ and $Z = \{1, 2\}$. Then with the vertex ordering in Figure 1, a string such as ZXYXXX, represents a possible list assignment on $V(H)$ arising from a 3-edge-coloring of $G - E(H)$. By an X-Kempe change, we mean flipping colors 0 and 1 on a two-colored path in $G - E(H)$. We call such a path an X-path. Any endpoint of an X-path in H must end at a Y or Z vertex. The meanings of Y-Kempe change, Z-Kempe change, Y-path and Z-path are analogous. Note that if there are an odd number of Y's and Z's, then at least one X-Kempe change has only one endpoint in H.

We need to handle all boards that are nearly colorable for edge e up to permutations of $\{X, Y, Z\}$, so it will suffice to handle all boards of the form $\bigstar Z\bigstar\bigstar YZ$, $\bigstar\bigstar\bigstar YYZ$, $Y\bigstar\bigstar\bigstar YZ$, $X\bigstar\bigstar YZZ$, $Z\bigstar\bigstar YZZ$, $X\bigstar\bigstar ZYZ$, $Z\bigstar\bigstar XYZ$, $X\bigstar YZZZ$, $Y\bigstar YZZZ$, $YY\bigstar YZZ$, $YX\bigstar YZZ$, $XYZZZZ$, $YYZZZZ$, $YZZZZZ$, $ZZZZZZ$ or $ZZYZZZ$.

Case 1. *B is one of $\bigstar Z\bigstar\bigstar YZ$, $\bigstar Y\bigstar YZZ$, $\bigstar X\bigstar YZZ$, $\bigstar\bigstar ZYYZ$, $\bigstar\bigstar XYYZ$, $Y\bigstar Y\bigstar YZ$, $\bigstar ZYZZZ$, $X\bigstar ZZYZ$, $X\bigstar XZYZ$, $Y\bigstar YZZZ$, $Y\bigstar ZZYZ$, $Y\bigstar XXYZ$, $Z\bigstar ZXYZ$, $Z\bigstar XXYZ$, $XYZZZZ$, $XZYYZZ$, $XZXYZZ$, $YYZZZZ$, $ZZZZZZ$, $ZZZYZZ$ or $ZZYYZZ$.* In all these cases, H is immediately colorable from the lists.

Case 2. *B is one of $XYY\bigstar YZ$, $XXYXYZ$, $XXYZYZ$, $XZZYZZ$, $XYYZZZ$, $YZZYZZ$, $YYXZYZ$, $YZZZZZ$, $ZYYZZZ$, $ZYZZYZ$, $ZYYYYZ$, $ZYXZYZ$, $ZXYZYZ$, $ZXZZYZ$, $ZXYZZZ$, $ZXYYYZ$ or $ZYZZZZ$.*

For YZZZZZ, if the Y-path starting at the third vertex doesn't end in H or ends at the fourth, fifth or sixth vertex of H, then doing a Y-Kempe change there yields XZYZZZ, XZYYZZ, XZYZYZ and XYZYYZ respectively, which are handled by Case 1. If the Y-path starting at the fifth vertex doesn't end in H or ends at the fourth or sixth vertex of H, then doing a Y-Kempe change there yields XZZZYZ, XZZYYZ and XYYYZZ respectively, which are handled by Case 1. If the Y-path starting at the second vertex doesn't end in

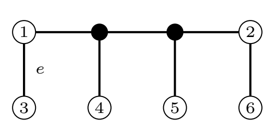

Figure 1: Solid vertices have lists of size 3 and the labeled vertices have lists of size 2.

H then doing a Y-Kempe change there yields XYZZZZ, which is handled by Case 1. Since we already handled the permutation of all resulting boards by (2 1 4 3 6 5), we have also handled ZYZZZZ.

For XYYZZZ, if the X-path starting at the third vertex doesn't end in H or ends at the fourth or sixth vertex of H, then doing an X-Kempe change there yields XYZZZZ, XYZYZZ and XZYYYZ respectively, which are handled by Case 1. If the X-path starting at the fourth vertex doesn't end in H or ends at the sixth vertex of H, then doing an X-Kempe change there yields XYYYZZ and XZZZYZ respectively, which are handled by Case 1. If the X-path starting at the second vertex doesn't end in H then doing an X-Kempe change there yields XZYZZZ, which is handled by Case 1. If the X-path starting at the sixth vertex doesn't end in H then doing an X-Kempe change there yields XZZYYZ, which is handled by Case 1. Since we already handled the permutation of all resulting boards by (2 1 6 3 4 5), we have also handled ZXYYYZ.

For XZZYZZ, if the X-path starting at the third vertex doesn't end in H or ends at the second, fourth, fifth or sixth vertex of H, then doing an X-Kempe change there yields XZYYZZ, XYYYZZ, XZYZZZ, XZYYYZ and XYZZYZ respectively, which are handled by Case 1. If the X-path starting at the fifth vertex doesn't end in H or ends at the fourth vertex of H, then doing an X-Kempe change there yields XZZYYZ and XZZZYZ respectively, which are handled by Case 1. If the X-path starting at the second vertex doesn't end in H then doing an X-Kempe change there yields XYZYZZ, which is handled by Case 1. Since we already handled the permutation of all resulting boards by (1 2 3 6 5 4), (2 1 6 3 4 5) and (2 1 6 5 4 3), we have also handled XYYYYZ, ZXYZZZ and ZXZZYZ.

For XYYZYZ, if the X-path starting at the third vertex doesn't end in H or ends at the second, fourth, fifth or sixth vertex of H, then doing an X-Kempe change there yields XYZZYZ, XZZZYZ, XYZYYZ, XYZZZZ and XZYYZZ respectively, which are handled by Case 1. If the X-path starting at the second vertex doesn't end in H then doing an X-Kempe change there yields XZYZYZ, which is handled by Case 1. If the X-path starting at the fifth vertex ends at the fourth or sixth vertex of H, then doing an X-Kempe change there yields XYYYZZ and XZZYYZ respectively, which are handled by Case 1. Since we already handled the permutation of all resulting boards by (2 1 6 3 4 5), we have also handled ZXYZYZ.

For XXYZYZ, if the X-path starting at the fourth vertex ends at the fifth or sixth vertex of H, then doing an X-Kempe change there yields XXYYZZ and YYZZZZ respectively, which are handled by Case 1. If the X-path starting at the fifth vertex ends at the sixth vertex of H, then doing an X-Kempe change there yields XXZYYZ, which is handled by Case 1.

For YYXZYZ, if the X-path starting at the fourth vertex doesn't end in H or ends at the second, fifth or sixth vertex of H, then doing an X-Kempe change there yields YYXYYZ, YZXYYZ, YYXYZZ and ZZYZZZ respectively, which are handled by Case 1. If the X-path starting at the second vertex doesn't end in H or ends at the fifth or sixth vertex of H, then doing an X-Kempe change there yields YZXZYZ, XZYZZZ and ZYXYZZ respectively, which are handled by Case 1. If the X-path starting at the fifth vertex ends at the sixth vertex of H, then doing an X-Kempe change there yields ZZXYYZ, which is handled by Case 1. Since we already handled the permutation of all resulting boards by (1 4 3 2 6 5), (2 1 6 5 4 3) and (2 5 6 1 3 4), we have also handled ZYXZYZ, XXYXYZ and XYYXYZ.

For ZYYYYZ, if the X-path starting at the third vertex ends at the second, fifth or sixth vertex of H, then doing an X-Kempe change there yields ZZZYYZ, ZYZYZZ and YZYZZZ respectively, which are handled by Case 1. If the X-path starting at the fifth vertex ends at the second or sixth vertex of H, then doing an X-Kempe change there yields ZZYYZZ and YZZZYZ respectively, which are handled by Case 1. If the X-path starting at the second vertex ends at the sixth vertex of H, then doing an X-Kempe change there yields YYZZZZ, which is handled by Case 1. Since we already handled the permutation of all resulting boards by (1 2 3 6 5 4), (2 1 4 3 6 5) and (2 1 4 5 6 3), we have also handled YZZYZZ, ZYZZYZ and ZYYZZZ.

Case 3. *B is one of $XYXXYZ$, $XYZXYZ$, $XXXXYZ$, $XXZXYZ$, $XXYYYZ$, $XXYZZZ$, $YZXYZZ$, $YXXZYZ$, $YYZXYZ$, $ZXXZYZ$, $ZYYXYZ$, $ZXYXYZ$ or $ZZXYZZ$.*

Since XXYZZZ has an odd number of X's and Y's, there is a Z-path with exactly one end in H. If this is the first, second or third vertex of H, then doing a Z-Kempe change there yields YXYZZZ, XYYZZZ and YYYZZZ respectively, which are handled by Cases 1 and 2. Since we already handled the permutation of all resulting boards by (1 2 6 3 4 5), (3 6 4 1 2 5) and (3 6 5 1 2 4), we have also handled XXYYYZ, YYZXYZ and XXZXYZ.

For YXXZYZ, if the Z-path starting at the second vertex ends at the third or fifth vertex of H, then doing a Z-Kempe change there yields YYYZYZ and XXYZYZ respectively, which are handled by Cases 1 and 2. If the Z-path starting at the third vertex ends at the fifth vertex of H, then doing a Z-Kempe change there yields XYXZYZ, which is handled by Case 1. Since we already handled the permutation of all resulting boards by (1 2 4 3 6 5) and (1 3 6 2 4 5), we have also handled ZXYXYZ and XYZXYZ.

For ZYYXYZ, if the Z-path starting at the third vertex ends at the fourth or fifth vertex of H, then doing a Z-Kempe change there yields ZYXYYZ and ZXYYYZ respectively, which are handled by Cases 1 and 2. If the Z-path starting at the fourth vertex ends at the fifth vertex of H, then doing a Z-Kempe change there yields ZXXXYZ, which is handled by Case 1. Since we already handled the permutation of all resulting boards by (1 2 5 3 6 4), (2 1 3 6 4 5) and (2 1 4 5 6 3), we have also handled YZXYZZ, XYXXYZ and ZXXZYZ.

For ZZXYZZ, if the Y-path starting at the third vertex doesn't end in H or ends at the first, second or fifth vertex of H, then doing a Y-Kempe change there yields ZZZYZZ, XZZYZZ, ZXZYZZ and ZZZXYZ respectively, which are handled by Cases 1 and 2. If the Y-path starting at the first vertex doesn't end in H then doing a Y-Kempe change there yields XZXYZZ, which is handled by Case 1. If the Y-path starting at the second vertex doesn't end in H then doing a Y-Kempe change there yields ZXXYZZ, which is handled by Case 1. If the Y-path starting at the fifth vertex doesn't end in H then doing a Y-Kempe change there yields ZZYXYZ, which is handled by Case 1. Since we already handled the permutation of all resulting boards by (1 2 6 5 4 3), we have also handled XXXXYZ.

Case 4. *B is one of $YXZXYZ$.*

For YXZXYZ, if the X-path starting at the third vertex ends at the fifth or sixth vertex of H, then doing an X-Kempe change there yields XYXYZZ and ZYZYZZ respectively, which are handled by Case 1. If the X-path starting at the fifth vertex ends at the sixth vertex of H, then doing an X-Kempe change there yields ZXYXYZ, which is handled by Case 3.

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