

## 1. OVERVIEW

For a multigraph  $G$ , we clearly have  $\chi'(G) \geq \lceil |E(G)| / \lfloor |V(G)|/2 \rfloor \rceil$ . Likewise, the same bound holds for any subgraph  $H$ . Thus, The biggest open problem in edge-coloring is the Goldberg–Seymour conjecture. Over the past two decades, the main tool for attacking this problem has become Tashkinov trees, a vast generalization of Vizing fans and Kierstead paths. The second author proved that if  $G$  is a line graph, then  $\chi(G) \leq \max\{\omega(G), \frac{7\Delta(G)+10}{8}\}$ . In the same paper, he conjectured that  $\chi(G) \leq \max\{\omega(G), \frac{5\Delta(G)+8}{6}\}$ . This conjecture is best possible, as shown by replacing each edge in a 5-cycle by  $k$  parallel edges, and taking the line graph. We call the latter inequality the  $\frac{5}{6}$ -Conjecture, and in this paper we prove it. Along the way, we develop more general techniques and results that will likely be of independent interest, due to their use in approaching the Goldberg–Seymour conjecture.

A graph  $G$  is *elementary* if  $\chi'(G) = \mathcal{W}(G)$ ; such graphs satisfy the Goldberg–Seymour Conjecture. A *defective color* for a Tashkinov tree  $T$  is a color used on more than one edge from  $V(T)$  to  $V(G) - V(T)$ ; a Tashkinov tree is *strongly closed* if it has no defective color. Andersen [1] and Goldberg [2] showed that if  $G$  is critical, then  $G$  is elementary if there exists  $e \in E(G)$  and  $X \subseteq V(G)$  and a  $k$ -edge-coloring  $\varphi$  of  $G - e$  such that  $X$  contains the endpoints of  $e$  and  $X$  is elementary and strongly closed w.r.t.  $\varphi$ . Thus, to show that  $G$  is elementary, it suffices to show that if  $G$  is  $(k+1)$ -critical, then there exists an edge  $e \in E(G)$  and a  $k$ -coloring  $\varphi$  of  $G - e$  such that some maximal Tashkinov tree containing  $e$  is strongly closed. The following definition is useful. A vertex  $v \in V(G)$  is *special* if every Vizing fan rooted at  $v$  (taken over all  $k$ -colorings of  $G - e$ , over all edges  $e$  incident to  $v$ ) has at most 3 vertices, including  $v$ . As a warmup, in Section 2 we prove that if  $\chi'(G) \geq \Delta(G) + 2$  and every vertex of  $G$  is special, then  $G$  is elementary, i.e.,  $\chi'(G) = \mathcal{W}(G)$ . Next, we push our methods further, allowing our maximal Tashkinov tree to have at most 3 non-special vertices.

In Section 3, we consider the  $\frac{5}{6}$ -Conjecture. As a consequence of results in Section 2, if  $G$  is a minimal counterexample, then every non-special vertex  $v$  has  $d_G(v) < \frac{3}{4}\Delta(G)$ . Since every maximal Tashkinov tree  $T$  is elementary, and every non-special vertex misses more than  $\frac{k}{4}$  colors, we conclude that  $T$  has at most 3 non-special vertices. Thus our results from Section 2 apply. As a consequence, every minimal counterexample to the  $\frac{5}{6}$ -Conjecture is elementary. To complete the proof of the  $\frac{5}{6}$ -conjecture, we show for each graph  $G$  that if  $\chi'(G) = \mathcal{W}(G)$ , then  $\chi'(G) \leq \max\{\omega(G), \frac{5\Delta(G)+8}{6}\}$ .

Graphs can have multiple edges.

## 2. TASHKINOV TREES

Recall that a graph  $G$  is *elementary* if  $\chi'(G) = \mathcal{W}(G)$ , as defined above. We also use the following notation. Let  $[k]$  denote  $\{1, \dots, k\}$ . A graph  $G$  is *critical* if  $\chi'(G - e) < \chi'(G)$  for all  $e \in E(G)$ . For a graph  $G$  and a partial  $k$ -edge-coloring  $\varphi$ , for each vertex  $v \in V(G)$ , let  $\varphi(v)$  denote the set of colors used in  $\varphi$  on edges incident to  $v$ . Let  $\overline{\varphi}(v) = [k] \setminus \varphi(v)$ . A color  $c$  is *seen* by a vertex  $v$  if  $c \in \varphi(v)$  and  $c$  is *missed* by  $v$  if  $c \in \overline{\varphi}(v)$ . Given a partial  $k$ -edge-coloring  $\varphi$ , a set  $W \subseteq V(G)$  is *elementary* (w.r.t.  $\varphi$ ) if each color in  $[k]$  is missed by at most one vertex of  $W$ . More formally,  $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$  for all distinct  $u, v \in W$ . A *defective color* for a set  $X \subseteq V(G)$  (w.r.t.  $\varphi$ ) is a color used on more than one edge from

$X$  to  $V(G) \setminus X$ . A set  $X$  is *strongly closed* w.r.t.  $\varphi$  if  $X$  has no defective color. Elementary and strongly closed sets are of particular interest because of the following theorem, proved implicitly by Andersen [1] and Goldberg [2].

**Theorem 1.** *Let  $G$  be a graph with  $\chi'(G) = k + 1$  for some integer  $k \geq \Delta(G)$ . If  $G$  is critical, then  $G$  is elementary if and only if there exists  $uv \in E(G)$ , a  $k$ -edge-coloring  $\varphi$  of  $G - uv$ , and a set  $X$  with  $u, v \in X$  such that  $X$  is both elementary and strongly closed w.r.t.  $\varphi$ .*

A *Tashkinov tree* w.r.t.  $\varphi$  is a sequence  $v_0, e_1, v_1, e_2, \dots, v_{t-1}, e_t, v_t$  such that all  $v_i$  are distinct,  $e_i = v_j v_i$  and  $\varphi(e_i) \in \overline{\varphi}(v_\ell)$  for some  $j$  and  $\ell$  with  $0 \leq j < i$  and  $0 \leq \ell < i$ . Tashkinov trees are of interest because of the following lemma.

**Tashkinov's Lemma.** *Let  $G$  be a graph with  $\chi'(G) = k + 1$ , for some integer  $k \geq \Delta(G) + 1$  and choose  $e \in E(G)$  such that  $\chi'(G - e) < \chi'(G)$ . Let  $\varphi$  be a  $k$ -edge-coloring of  $G - e$ . If  $T$  is a Tashkinov tree w.r.t.  $\varphi$  and  $e$ , then  $V(T)$  is elementary w.r.t.  $\varphi$ .*

In view of Theorem 1 and Tashkinov's Lemma, to prove that a graph  $G$  is elementary, it suffices to find an edge  $e$ , a  $k$ -edge-coloring  $\varphi$  of  $G - e$ , and a Tashkinov tree  $T$  containing  $e$  such that  $V(T)$  is strongly closed. This motivates our next two lemmas. But first, we need a few more definitions.

Let  $t(G)$  be the maximum number of vertices in a Tashkinov tree over all  $e \in E(G)$  and all  $k$ -edge-colorings  $\varphi$  of  $G - e$ . Let  $\mathcal{T}(G)$  be the set of all triples  $(T, e, \varphi)$  such that  $e \in E(G)$ ,  $\varphi$  is a  $k$ -edge-coloring of  $G - e$  and  $T$  is a Tashkinov tree with respect to  $e$  and  $\varphi$  with  $|T| = t(G)$ . Notice that, by definition, we have  $\mathcal{T}(G) \neq \emptyset$ . For a  $k$ -edge-coloring  $\varphi$  of  $G - e$ , a maximal Tashkinov tree starting with  $e$  may not be unique. However, if  $T_1$  and  $T_2$  are both such trees, then it is easy to show that  $V(T_1) \subseteq V(T_2)$ ; by symmetry, also  $V(T_2) \subseteq V(T_1)$ , so  $V(T_1) = V(T_2)$ . Let  $G$  be a critical graph with  $\chi'(G) = k + 1$  for an integer  $k \geq \Delta(G) + 1$ . Let  $\varphi$  be a  $k$ -edge-coloring of  $G - e_0$  for some  $e_0 \in E(G)$ . For  $v \in V(G)$  and colors  $\alpha, \beta$ , let  $P_v(\alpha, \beta)$  be the maximal connected subgraph of  $G$  that contains  $v$  and is induced by edges with color  $\alpha$  or  $\beta$ . So  $P_v(\alpha, \beta)$  is a path or a cycle.

**Lemma 2.** *Let  $G$  be a non-elementary critical graph with  $\chi'(G) = k + 1$  for an integer  $k \geq \Delta(G) + 1$ . For every  $v_0 v_1 \in E(G)$ ,  $k$ -edge-coloring  $\varphi$  of  $G - v_0 v_1$  and for all  $\alpha \in \overline{\varphi}(v_0)$  and  $\beta \in \overline{\varphi}(v_1)$  we have  $|P_{v_1}(\alpha, \beta)| < t(G)$ .*

*Proof.* Suppose the lemma is false and choose  $v_0 v_1 \in E(G)$ , a  $k$ -edge-coloring  $\varphi$  of  $G - v_0 v_1$ ,  $\alpha \in \overline{\varphi}(v_0)$  and  $\beta \in \overline{\varphi}(v_1)$  such that  $|P_{v_1}(\alpha, \beta)| \geq t(G)$ . Put  $P = P_{v_1}(\alpha, \beta)$ . Clearly  $P$  must end at  $v_0$  (or we can swap colors  $\alpha$  and  $\beta$  on  $P$  and color  $v_0 v_1$ ), so let  $v_1, \dots, v_r, v_0$  denote the vertices of  $P$  in order. Let  $(T, v_0 v_1, \varphi)$  be a Tashkinov tree that begins with edges  $v_0 v_1, v_1 v_2, \dots, v_{r-1} v_r$ . Then  $V(T) = V(P)$  since  $t(G) \geq |T| \geq |P| \geq t(G)$ . Since  $G$  is non-elementary, Theorem 1 implies that  $V(T)$  is not strongly closed, so  $T$  has a defective color  $\delta$  with respect to  $\varphi$ . Choose  $\tau \in \overline{\varphi}(v_2)$ . Let  $Q = P_{v_2}(\tau, \delta)$ . Since  $T$  is maximal,  $\delta$  is not missing at any vertex of  $T$ ; since  $V(T)$  is elementary,  $\tau$  is not missing at any vertex of  $T$  other than  $v_2$ . As a result,  $Q$  ends outside  $V(T)$ . Now  $Q$  could leave  $V(T)$  and re-enter it repeatedly, but  $Q$  ends outside  $V(T)$ , so there is a last vertex  $w \in V(Q) \cap V(T)$ ; say  $Q$  ends at  $z \in V(G) \setminus V(T)$ . Let  $\pi \notin \{\alpha, \beta\}$  be a color missing at  $w$ . Since  $|T| = t(G)$ , no edge

colored  $\tau$  or  $\pi$  leaves  $V(T)$ . So, we can swap  $\tau$  and  $\pi$  on every edge in  $G - V(T)$  without changing the fact that  $T$  is a Tashkinov tree with  $|T| = t(G)$ . Now swap  $\delta$  and  $\pi$  on the subpath of  $Q$  from  $w$  to  $z$ ; since  $\pi$  is missing at  $w$ , the  $\delta - \pi$  path does end at  $w$ . Now  $\delta$  is missing at  $w$ , but  $\delta$  was defective in  $\varphi$ , so some other edge  $e$  colored  $\delta$  still leaves  $V(T)$ , adding  $e$  gets a larger Tashkinov tree, a contradiction.  $\square$

### 3. SPECIAL VERTICES

Recall that a vertex  $v \in V(G)$  is *special* if every Vizing fan rooted at  $v$  (taken over all  $k$ -colorings of  $G - e$ , over all edges  $e$  incident to  $v$ ) has at most 3 vertices, including  $v$ . Let  $\nu(T)$  be the number of non-special vertices in  $T$ .

**Lemma 3.** *Let  $G$  be a critical graph with  $\chi'(G) = k + 1$  for an integer  $k \geq \Delta(G) + 1$ . Let  $\varphi$  be a  $k$ -edge-coloring of  $G - v_0v_1$ . Suppose  $\alpha \in \overline{\varphi}(v_0)$  and  $\beta \in \overline{\varphi}(v_1)$ . Let  $P = v_1v_2 \cdots v_r$  be an  $\alpha - \beta$  path with edges  $e_i = v_iv_{i+1}$  for  $1 \leq i \leq r - 1$ . If  $v_i$  is special for all odd  $i$ , then for any  $\tau \in \overline{\varphi}(v_0)$  there are edges  $f_i = v_iv_{i+1}$  for  $1 \leq i \leq r - 1$  such that  $f_i = e_i$  for  $i$  even and  $\varphi(f_i) = \tau$  for  $i$  odd.*

*Proof.* Suppose not and choose a counterexample minimizing  $r$ . By minimality of  $r$ , we have  $\varphi(v_{r-1}v_r) = \alpha$  and we have  $f_i = v_iv_{i+1}$  for  $1 \leq i \leq r - 2$  such that  $f_i = e_i$  for  $i$  even and  $\varphi(f_i) = \tau$  for  $i$  odd. Swap  $\alpha$  and  $\beta$  on  $e_i$  for  $1 \leq i \leq r - 3$  and then color  $v_0v_1$  (call this edge  $e_0$ ) with  $\alpha$  and uncolor  $e_{r-2}$ . Let  $\varphi'$  be the resulting coloring. Since  $k \geq \Delta(G) + 1$ , some color other than  $\alpha$  is missing at  $v_{r-2}$ ; let  $\gamma$  be such a color. Now  $v_{r-1}$  is special since  $r - 1$  is odd (since  $P$  starts and ends with  $\alpha$ ), so there is an edge  $e = v_{r-1}v_r$  with  $\varphi'(e) = \gamma$ . Swap  $\tau$  and  $\alpha$  on  $e_i$  for  $0 \leq i \leq r - 3$  to get a new coloring  $\varphi^*$ . Now  $\gamma$  and  $\tau$  are both missing at  $v_{r-2}$  in  $\varphi^*$ . Since  $v_{r-1}$  is special, the fan with  $v_{r-2}, v_{r-1}, v_r$  and  $e$  implies that there is an edge  $f_{r-1} = v_{r-1}v_r$  with  $\varphi^*(f_{r-1}) = \tau$ . But we have never recolored  $f_{r-1}$ , so  $\varphi(f_{r-1}) = \tau$ , a contradiction.  $\square$

**Lemma 4.** *Let  $G$  be a non-elementary critical graph with  $\chi'(G) = k + 1$  for an integer  $k \geq \Delta(G) + 1$ . If  $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$  for some  $v_0v_1 \in E(G)$ , then  $\nu(T) \geq 1$ .*

*Proof.* Suppose  $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$  for some  $v_0v_1 \in E(G)$  and  $\nu(T) = 0$ . Let  $\alpha \in \overline{\varphi}(v_0)$  and  $\beta \in \overline{\varphi}(v_1)$  and put  $P = P_{v_1}(\alpha, \beta)$ . Applying Lemma 3 to  $P$  shows that every  $\tau \in \overline{\varphi}(v_0)$ , there is a  $\tau$ -edge in  $T$  incident to every  $v \in V(P - v_0)$ . By symmetry, the same is true of every  $v \in V(P)$ . Hence  $V(P) = V(T)$  contradicting Lemma 2.  $\square$

**Theorem 5.** *If  $G$  is a critical graph in which every vertex is special, then*

$$\chi'(G) \leq \max \{ \lceil \chi'_f(G) \rceil, \Delta(G) + 1 \}.$$

*Proof.* Suppose  $G$  is a critical graph in which every vertex is special and put  $k = \chi'(G) - 1$ . Then  $k \geq \Delta(G) + 1$ . Since  $\mathcal{T}(G) \neq \emptyset$ , applying Lemma 4, we conclude that  $G$  is elementary. Hence  $\chi'(G) = \lceil \chi'_f(G) \rceil$ , a contradiction.  $\square$

## 4. THE EASY BOUND

Let  $G$  be a graph. The *claw-degree* of  $x \in V(G)$  is

$$d_{\text{claw}}(x) := \max_{\substack{S \subseteq N(x) \\ |S|=3}} \frac{1}{4} \left( d(x) + \sum_{v \in S} d(v) \right).$$

The *claw-degree* of  $G$  is

$$d_{\text{claw}}(G) := \max_{x \in V(G)} d_{\text{claw}}(x).$$

**Theorem 6.** *If  $G$  is a graph, then*

$$\chi'(G) \leq \max \left\{ \lceil \chi'_f(G) \rceil, \Delta(G) + 1, \left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil \right\}.$$

*Proof.* Suppose not and choose a counterexample  $G$  minimizing  $\|G\|$ ; note that  $G$  critical. Let  $k = \chi'(G) - 1$ , so  $k \geq \lceil \frac{4}{3} d_{\text{claw}}(G) \rceil$ . By Theorem 5,  $G$  has a non-special vertex  $x$ . Choose  $xy_1 \in E(G)$  and a  $k$ -edge-coloring  $\varphi$  of  $G - xy_1$  such that  $\varphi$  has a fan  $F$  of length 3 rooted at  $x$  with leaves  $y_1, y_2, y_3$ . Since  $V(F)$  is elementary,

$$2 + k - d(x) + \sum_{i \in [3]} k - d(y_i) \leq k,$$

and hence

$$d_{\text{claw}}(x) \geq \frac{1}{4} \left( d(x) + \sum_{i \in [3]} d(y_i) \right) \geq \frac{3k + 2}{4}.$$

This gives the contradiction

$$\left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil \leq k \leq \frac{4}{3} d_{\text{claw}}(G) - \frac{2}{3}.$$

□

TODO: ADD REED, LOCAL REED AND SUPERLOCAL REED CONSEQUENCES.

## 5. PROPERTIES OF NON-SPECIAL VERTICES

**Lemma 7.** *Let  $G$  be a non-elementary critical graph with  $\chi'(G) = k + 1$  for an integer  $k \geq \Delta(G) + 1$ . Let  $(T, v_0 v_1, \varphi) \in \mathcal{T}(G)$ . Suppose  $\alpha \in \overline{\varphi}(v_0)$  and  $\beta \in \overline{\varphi}(v_1)$  and let  $P = P_{v_1}(\alpha, \beta)$ . Then there are  $a, b \in [r]$  such that both  $v_a, v_b$  are non-special and  $b - a$  is odd.*

*Proof.* Let  $v_{i_1}, \dots, v_{i_p}$  be the non-special vertices in  $P$ . Suppose  $i_a - i_b$  is even for all  $a, b \in [p]$ . By Lemma 2,  $V(P) \subsetneq V(T)$ . Hence there are  $j, \ell \in [r]$  and  $\tau \in \overline{\varphi}(v_j)$  such that  $\varphi(v_\ell z) = \tau$  for some  $z \in V(T) \setminus V(P)$ . By rotating the  $\alpha - \beta$  coloring on  $P$ , we may assume that  $j = 0$ .

Let  $m$  be the least odd integer such that  $v_m$  is non-special (if it exists, otherwise, let  $m = r$ ). Let  $m'$  be the largest even integer such that  $v_{m'}$  is non-special (if it exists, otherwise, let  $m' = 0$ ). By Lemma 3, there is a  $\tau$ -colored edge between  $v_t$  and  $v_{t+1}$  for all odd  $t \in [m - 1]$ . By uncoloring  $v_r v_0$ , coloring  $v_0 v_1$  with  $\beta$  and then running the same argument going the

opposite direction on  $P$ , we conclude that there is a  $\tau$ -colored edge between  $v_t$  and  $v_{t+1}$  for all odd  $t$  with  $m' + 1 \leq t \leq r - 1$ .

But  $i_a - i_b$  is even for all  $a, b \in [p]$ , so either  $m = r$  or  $m' = 0$ . Hence there is a  $\tau$ -colored edge between  $v_{\ell-1}v_\ell$  or  $v_\ell v_{\ell+1}$  contradicting  $\varphi(v_\ell z) = \tau$ .  $\square$

As an immediate consequence of Lemma 7, we get the following.

**Corollary 8.** *Let  $G$  be a non-elementary critical graph with  $\chi'(G) = k + 1$  for an integer  $k \geq \Delta(G) + 1$ . If  $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$  for some  $v_0v_1 \in E(G)$ , then  $\nu(T) \geq 2$ .*

## 6. THIN GRAPHS

Let  $G$  be a critical graph with  $\chi'(G) = k + 1$  for an integer  $k \geq \Delta(G) + 1$ . For vertices  $x, y \in V(G)$ , we say that  $x$  is  $y$ -special if every Vizing fan rooted at  $x$ , with respect to any  $k$ -edge-coloring of  $G - xy$ , has at most 3 vertices. We say that  $G$  is  $k$ -thin if  $\mu(G) < 2k - d(x) - d(y)$  for all non-special  $x, y \in V(G)$ .

**Lemma 9.** *Let  $G$  be a non-elementary critical graph with  $\chi'(G) = k + 1$  for an integer  $k \geq \Delta(G) + 1$ . Suppose  $G$  is  $k$ -thin. Let  $\varphi$  be a  $k$ -edge-coloring of  $G - v_0v_1$ . Suppose  $\alpha \in \overline{\varphi}(v_0)$  and  $\beta \in \overline{\varphi}(v_1)$ . Let  $P = v_1v_2 \cdots v_r$  be an  $\alpha - \beta$  path. Let  $i, j \in [r]$  with  $i + 3 \leq j$  such that  $j - i$  is odd. Then there is an integer  $t$  with  $i < t < j$  such that  $v_t$  is non-special.*

*Proof.* TODO: ADD PROOF  $\square$

**Lemma 10.** *Let  $G$  be a non-elementary critical graph with  $\chi'(G) = k + 1$  for an integer  $k \geq \Delta(G) + 1$ . Suppose  $G$  is  $k$ -thin. Let  $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$ . Suppose  $\alpha \in \overline{\varphi}(v_0)$  and  $\beta \in \overline{\varphi}(v_1)$ . Then there are consecutive non-special vertices on  $P_{v_1}(\alpha, \beta)$ .*

*Proof.* Put  $P = P_{v_1}(\alpha, \beta)$  and let  $v_{i_1}, \dots, v_{i_p}$  be the non-special vertices in  $P$ . Suppose  $i_{j+1} - i_j > 1$  for all  $j \in [p - 1]$ . Since  $G$  is thin, Lemma 9 implies that  $i_{j+1} - i_j$  is even for all  $j \in [p - 1]$ . Hence  $i_a - i_b$  is even for all  $a, b \in [p]$  violating Lemma 7.  $\square$

**Lemma 11.** *Let  $G$  be a non-elementary critical graph with  $\chi'(G) = k + 1$  for an integer  $k \geq \Delta(G) + 1$ . Suppose  $G$  is  $k$ -thin. If  $(T, e, \varphi) \in \mathcal{T}(G)$  for  $e \in E(G)$ , then  $\nu(T) \geq 3$ .*

*Proof.* Lemma 8 gives  $\nu(T) \geq 2$ . Suppose  $\nu(T) = 2$ . Let  $\alpha \in \overline{\varphi}(v_0)$  and  $\beta \in \overline{\varphi}(v_1)$ . Let  $P = P_{v_1}(\alpha, \beta) = v_1v_2 \cdots v_rv_0$ . By Lemma 10, there is  $i \in [r - 1]$  such that both  $v_i$  and  $v_{i+1}$  are non-special. By rotating the  $\alpha - \beta$  coloring on  $P$ , we may assume that  $i = 1$ .

By Lemma 2,  $V(P) \subsetneq V(T)$ . Hence we have  $a, b \in [r]$  and  $\tau \in \overline{\varphi}(v_a)$  such that  $\varphi(v_bz) = \tau$  for some  $z \in V(T) \setminus V(P)$ . Let  $Q = P_{v_b}(\tau, \beta)$ . By Lemma 10,  $v_1$  and  $v_2$  (the only non-special vertices in  $T$ ) must be consecutive on  $Q$ . But there cannot be a  $\tau$  or  $\beta$  edge between  $v_1$  and  $v_2$ , a contradiction.  $\square$

**Lemma 12.** *Let  $G$  be a non-elementary critical graph with  $\chi'(G) = k + 1$  for an integer  $k \geq \Delta(G) + 1$ . Suppose  $G$  is  $k$ -thin. If  $(T, e, \varphi) \in \mathcal{T}(G)$  for  $e \in E(G)$  and  $\nu(T) = 3$ , then there are non-special  $x_1, x_2, x_3 \in V(T)$  such that  $x_ix_j \in E(G)$  and  $x_i$  is not  $x_j$ -special for all  $i, j \in [3]$  with  $i \neq j$ .*

*Proof.* Choose  $\alpha \in \overline{\varphi}(v_0)$  and  $\beta \in \overline{\varphi}(v_1)$  so that  $P = P_{v_1}(\alpha, \beta) = v_1v_2 \cdots v_rv_0$  contains the maximum number of non-special vertices and subject to that, maximizes  $|P|$ . Then

$P$  contains 3 non-special vertices since otherwise we can use the argument in Lemma 11. By Lemma 10 and symmetry we may assume that  $v_a, v_{a+1}$  and  $v_b$  are non-special where  $a + 1 < b$ . By Lemma 2,  $V(P) \subsetneq V(T)$ . Hence there are  $j, \ell \in [r]$  and  $\tau \in \overline{\varphi}(v_j)$  such that  $\varphi(v_\ell z) = \tau$  for some  $z \in V(T) \setminus V(P)$ . By rotating the  $\alpha - \beta$  coloring on  $P$ , we may assume that  $j = 0$ . Let  $Q = P_{v_1}(\tau, \beta)$ . Let  $b' = b$  if  $b$  is even and  $b' = a + 1$  if  $b$  is odd.

**Claim 1.** *There is a  $\tau$  edge between  $v_t$  and  $v_{t+1}$  for all odd  $t$  with  $t < a$  or  $t \geq b'$ . In particular,  $v_a \in V(Q)$ .*

Proof: Immediate from Lemma 3. □

**Claim 2.**  $a \leq \ell \leq b$

Proof: By Claim 1, each  $v_t$  with  $0 < t < a$  or  $t > b$  is incident to a  $\tau$  edge contained in  $P$ , so  $a \leq \ell \leq b$ . □

**Claim 3.**  $b > a + 2$ , in particular  $|P| \geq 5$ .

Proof: Suppose  $b = a + 2$ . By symmetry, we may assume that  $a$  is odd. By Claim 1 every vertex except  $v_0$  and possibly  $v_a, v_{a+1}$  is incident to a  $\tau$ -edge in  $G[V(T)]$ . So  $\ell \in \{a, a + 1\}$ . The only possibility is that  $Q$  contains a  $\tau - \beta$  path of length at least three leaving  $P$  at  $v_a$  and returning at  $v_{a+1}$ . Since there is a  $\beta$ -edge between  $v_{a+1}$  and  $v_{a+2}$ , this means that  $Q$  contains three non-special vertices and  $|Q| > |P|$ , contradicting our maximality condition on  $P$ . □

**Claim 4.**  $a$  is odd,  $b$  is even

Proof: By Lemma 9, either  $b = a + 2$  or  $b - a$  is odd, so  $b - a$  is odd by Claim 3. Hence it will suffice to show that  $b$  is even. Suppose  $b$  is odd. Then Claim 1 says that there is a  $\tau$  edge between  $v_t$  and  $v_{t+1}$  for all odd  $t$  with  $t < a$  or  $t \geq a + 1$ . That rules out all possibilities for  $\ell$ , a contradiction. □

**Claim 5.** *There is a  $\tau$  edge between  $v_{a+1}$  and  $v_b$ .*

Proof: By Lemma 10,  $Q$  must have consecutive non-special vertices. It cannot be that  $v_a$  and  $v_{a+1}$  are connected by a  $\tau$ -edge because then  $v_a$  does not block after all (TODO: PUT THIS CHANGE INTO Lemma 3).

Suppose  $v_a$  and  $v_b$  are connected by a  $\tau$ -edge.

□

□

**Lemma 13.** *Suppose  $G$  is  $k$ -thin. If  $(T, e, \varphi) \in \mathcal{T}(G)$  for  $e \in E(G)$  and  $\nu(T) \leq 4$ , then there are non-special  $x_1, x_2 \in V(T)$  such that  $x_1 x_2 \in E(G)$  and  $x_1$  is not  $x_2$ -special.*

*Proof.* By Lemma 11 and Lemma 12 we may assume  $\nu(T) = 4$ . Choose  $\alpha \in \overline{\varphi}(v_0)$  and  $\beta \in \overline{\varphi}(v_1)$  so that the  $\alpha - \beta$  path  $P = v_1 v_2 \cdots v_r v_0$  from  $v_1$  to  $v_0$  so as to maximize the number of non-special vertices in  $P$ . Then  $P$  contains 4 non-special vertices since otherwise we can use the arguments in Lemma 11 and in Lemma 12.

By Lemma 10 there are contiguous non-special vertices in  $P$ . By symmetry, we may assume that  $v_1$  and  $v_2$  are non-special. Let  $v_a$  and  $v_b$  with  $2 < a < b$  be the other two non-special vertices in  $P$ . Suppose  $v_i$  is  $v_{3-i}$ -special for  $i \in [2]$ .

SKETCH: Look at the fans both ways, we violate thinness on either  $v_0 v_1$  or  $v_2 v_3$  depending on the parity of  $b$ . □

**Theorem 14** (from strengthening Brooks paper). *If  $Q$  is the line graph of a graph  $G$  and  $Q$  is vertex critical, then*

$$\chi(Q) \leq \max \left\{ \omega(Q), \Delta(Q) + 1 - \frac{\mu(G) - 1}{2} \right\}.$$

**Theorem 15.** *If  $Q$  is a line graph, then*

$$\chi(Q) \leq \max \{ \lceil \chi_f(Q) \rceil, \lceil \epsilon(\Delta(Q) + 1) \rceil \}.$$

*Proof.* Suppose the theorem is false and choose a counterexample minimizing  $|Q|$ . Let  $k = \max \{ \lceil \chi_f(Q) \rceil, \lceil \epsilon(\Delta(Q) + 1) \rceil \}$ . Say  $Q = L(G)$  for a graph  $G$ . The minimality of  $Q$  implies that  $G$  is  $k$ -edge-critical.

**Claim 0.** *Let  $F$  be a fan rooted at  $x$  with respect to a  $k$ -edge-coloring of  $G - xy$ . If  $|F| = 4$ , then*

$$d(x) < \frac{1 - \epsilon}{2\epsilon - 1} \sum_{v \in V(F-x)} d(v).$$

Proof: Since  $F$  is elementary, we have

$$2 + k - d(x) + \sum_{v \in V(F-x)} k - d(v) \leq k,$$

so

$$2 + (|F| - 1)k \leq d(x) + \sum_{v \in V(F-x)} d(v).$$

Using  $k \geq \epsilon(\Delta(Q) + 1) \geq \epsilon(d(x) + d(v) - \mu(xv))$  for each  $v \in V(F - x)$ , we get

$$2 + \sum_{v \in V(F-x)} \epsilon(d(x) + d(v) - \mu(xv)) \leq d(x) + \sum_{v \in V(F-x)} d(v),$$

so

$$2 + (\epsilon|F| - 1 - \epsilon) d(x) \leq \sum_{v \in V(F-x)} \epsilon \mu(xv) + \sum_{v \in V(F-x)} (1 - \epsilon) d(v).$$

Now  $\sum_{v \in V(F-x)} \mu(xv) \leq d(x)$ , so this implies

$$2 + (\epsilon|F| - 1 - 2\epsilon) d(x) \leq \sum_{v \in V(F-x)} (1 - \epsilon) d(v).$$

Using  $|F| = 4$  gives

$$d(x) < \frac{1 - \epsilon}{2\epsilon - 1} \sum_{v \in V(F-x)} d(v).$$

□

**Claim 1.** *If  $x \in V(G)$  with  $d(x) \geq \frac{3(1-\epsilon)}{2\epsilon-1} \Delta(G)$ , then  $x$  is special.*

Proof: This is immediate from Claim 0, since  $d(v) \leq \Delta(G)$  for all  $v \in V(F - x)$ . □

Note that when  $\epsilon = \frac{5}{6}$ , every non-special vertex  $x$  has  $d(x) < \frac{3}{4} \Delta(G)$ . Since  $k \geq \Delta(G) + 1$ , every non-special vertex  $x$  misses at least  $k - d(x) > k - \frac{3}{4} \Delta(G) > \frac{k}{4}$  colors. Since  $T$  is

elementary,  $T$  has at most 3 non-special vertices. This observation is not needed for the present proof, but suggests how we will prove the  $\frac{5}{6}$ -conjecture in the next section.

**Claim 2.** *If  $x_1x_2 \in E(G)$  with*

$$d(x_i) \geq \frac{2(1-\epsilon)}{3\epsilon-2}\Delta(G),$$

*for at least one  $i \in [2]$ , then  $x_1$  is  $x_2$ -special or  $x_2$  is  $x_1$ -special.*

Proof: Suppose  $x_1$  is not  $x_2$ -special and  $x_2$  is not  $x_1$ -special. By Claim 0, for each  $i \in [2]$ ,

$$d(x_i) < \frac{1-\epsilon}{2\epsilon-1} \sum_{v \in V(F-x)} d(v) \leq \frac{1-\epsilon}{2\epsilon-1} (d(x_{3-i}) + 2\Delta(G)),$$

Substituting the bound on  $d(x_{3-i})$  into that on  $d(x_i)$  and simplifying gives for each  $i \in [2]$ ,

$$d(x_i) < \frac{2(1-\epsilon)}{3\epsilon-2}\Delta(G).$$

□

**Claim 3.** *If  $xy \in E(G)$  with*

$$d(x) \geq \frac{1-\epsilon^2}{(2\epsilon-1)^2}\Delta(G),$$

*then  $y$  is special or  $x$  is  $y$ -special.*

Proof: Suppose  $y$  is not special and  $x$  is not  $y$ -special. By Claim 0, we have

$$d(x) < \frac{1-\epsilon}{2\epsilon-1} \sum_{v \in V(F-x)} d(v) \leq \frac{1-\epsilon}{2\epsilon-1} (d(y) + 2\Delta(G)),$$

Since  $y$  is not special, Claim 1 gives  $d(y) < \frac{3(1-\epsilon)}{2\epsilon-1}\Delta(G)$  and thus

$$d(x) < \frac{1-\epsilon^2}{(2\epsilon-1)^2}\Delta(G).$$

□

**Claim 4.** *The theorem is true.*

Proof: Let  $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$ . By Lemmas 11, 12, and 13 one of the following holds:

- (1)  $G$  is elementary; or
- (2)  $G$  is not thin; or
- (3)  $\nu(T) = 3$  and  $E(T)$  contains non-special  $x_1, x_2, x_3 \in V(T)$  such that  $x_ix_j \in E(G)$  and  $x_i$  is not  $x_j$ -special for all  $i, j \in [3]$  with  $i \neq j$ ; or
- (4)  $\nu(T) = 4$  and  $E(T)$  contains an edge  $x_1x_2$  where  $x_2$  is not special and  $x_1$  is not  $x_2$ -special; or
- (5)  $V(T)$  contains five non-special vertices  $x_1, x_2, x_3, x_4, x_5$ .

If (1) holds, then  $k+1 = \lceil \chi_f(Q) \rceil \leq k$ , a contradiction.

If (2) holds, then by Claim 1 we have  $\mu(G) \geq 2k - 2\frac{3(1-\epsilon)}{2\epsilon-1}\Delta(G)$ . Hence Theorem 14 gives

$$k+1 \leq \Delta(Q) + 1 - k + \frac{3(1-\epsilon)}{2\epsilon-1}\Delta(G) + \frac{1}{2},$$



so

$$2(k+1) \leq \Delta(Q) + \frac{5}{2} + \frac{3(1-\epsilon)}{2\epsilon-1} \Delta(G).$$

Since  $k \geq \Delta(G) + 1$ , this gives

$$k+1 < \frac{\Delta(Q) + \frac{5}{2}}{2 - \frac{3(1-\epsilon)}{2\epsilon-1}},$$

which is a contradiction when  $\epsilon > \frac{4}{5}$ .

Suppose (3) holds. So

$$2 + \sum_{i \in [3]} k - d(x_i) \leq k,$$

using Claim 2, this gives

$$3 \left( \frac{2(1-\epsilon)}{3\epsilon-2} \right) \Delta(G) \geq 2k+2,$$

which is a contradiction when  $\epsilon \geq \frac{5}{6}$ .

Suppose (4) holds. Let  $x_3, x_4$  be non-special vertices in  $V(T) \setminus \{x_1, x_2\}$ .

$$2 + \sum_{i \in [4]} k - d(x_i) \leq k,$$

using Claim 1 and Claim 3 gives

$$\left( 3 \frac{3(1-\epsilon)}{2\epsilon-1} + \frac{1-\epsilon^2}{(2\epsilon-1)^2} \right) \Delta(G) \geq 3k+2,$$

which is a contradiction when  $\epsilon \geq \frac{39+\sqrt{157}}{62} \approx 0.831$ .

So (5) must hold. But then

$$2 + \sum_{i \in [5]} k - d(x_i) \leq k,$$

using Claim 1 gives

$$\frac{15(1-\epsilon)}{2\epsilon-1} \Delta(G) \geq 4k+2,$$

which is a contradiction when  $\epsilon \geq \frac{19}{23} \approx 0.826$ .

□  
□

7. THE  $\frac{5}{6}$ -CONJECTURE

**Lemma 16.** *If  $H$  is a connected multigraph and  $G = L(H)$ , then  $\mathcal{W}(H) \leq \max\{\omega(G), \frac{5}{6}(\Delta(G) + 1) + \frac{3}{6}\}$ .*

*Proof.* Let  $d = d_H(x)$ ,  $\Delta = \Delta(H)$ , and  $h = |H|$ . Also, let  $p = \sum_{v \in N(x)} d_H(v)$  and let  $t = \Delta h - 2||H||$ . Note that  $0 < t \leq \Delta$ . Also  $p \geq Md - t$ . Now summing over  $N_H(x)$  gives

$$|N(x)|(\Delta h - t)/(h - 1) > 5/6((|N(x)| - 1)d + |N(x)|\Delta - t) + |N(x)|/2$$

Solving for  $|N(x)|$  gives

$$|N(x)| < (5d + 5t)/(3 + 5d + 5\Delta - 6(\Delta h - t)/(h - 1)).$$

Since the numerator and denominator are linear in  $t$ , the right side is maximized at one end of the interval  $1 \leq t \leq D$ . Letting  $t = D$ , gives  $|N(x)| < (5d + 5\Delta)/(3 + 5d - \Delta)$ , like you had originally. Letting  $t = 1$ , gives  $|N(x)| < (5d + 5)/(3 + 5d + 5\Delta - 6(\Delta h - 1)/(h - 1))$ , which requires a little more analysis, akin to what you wrote in your most recent email.

Does that look right to you?

I did the analysis a little differently, but I got to the same conclusion: Substituting  $d \geq 4D/5$  gives that if  $M \geq 3$ , then we must have  $h \leq 4$ , which implies  $h \leq 3$ , which contradicts  $M \geq 3$ .

So, I think I believe it. I also agree there must be an easier way. One thing that seems a little magical is that when  $5/6 - M/(h - 1) \geq 0$  all of the  $h$ 's go away.

$w(H)$  really has a ceiling in its definition, not sure how much that changes things. without, it is the fractional chromatic index.

i think we get some gain as well from the  $\Delta(H) + 2$  in place of  $\Delta(H)$  we get as i wrote in the previous emails. Maybe this helps with the ceiling.

We can use  $|H|$  odd to get a bit better on the ceiling in what you wrote since the top is even (divide both by two before doing ceiling approximation).

Thinking about your comment that we can assume  $H$  is critical, we can, but not how i was setting it up. Probably you are already thinking something like this:

Assume Goldberg. Take minimum counterexample to  $5/6$  conjecture, say  $G = L(H)$ . The  $H$  is critical. From the argument like in strengthening of Brooks, we get  $\chi(G) \geq \Delta(H) + 2$ . By Goldberg this implies

$$\chi(G) = \max_{Q \subseteq H \text{ s.t. } |Q| \geq 3 \text{ and odd}} \left\lceil \frac{2||Q||}{|Q| - 1} \right\rceil$$

If the max is achieved at a proper subgraph of  $H$ , then there is an edge we can remove without decreasing the max, but this decreases the chromatic number by criticality and the max is a lower bound, so impossible. Therefore,  $|H|$  is odd and

$$\chi(G) = \left\lceil \frac{2||H||}{|H| - 1} \right\rceil$$

so,

$$\left\lceil \frac{2||H||}{|H| - 1} \right\rceil \geq \Delta(H) + 2$$

$$2||H||/(|H| - 1) \geq \Delta(H) + 1$$

using

$$\Delta(H)|H| \geq 2||H||,$$

using  $\Delta(H)|H| \geq 2||H||$ , I get

$$\Delta(H) \geq |H| - 1,$$

□

I think we should be able to prove that the conjecture follows from Goldberg–Seymour. That lemma you proved is pretty useful. We can assume that  $H$  is critical, which implies that  $|N(x)| \geq 2$  for all  $x$  in  $H$ . Now let  $J$  be the simple graph underlying  $H$ . We know that  $\delta(J) \geq 2$ . Let  $B = \{x \in H \text{ s.t. } d_J(x) \geq 3\}$ . That lemma implies that  $|B| \leq 4$ . Further, if  $|B| = 4$ , then each vertex of  $B$  has degree 3 in  $J$ . If  $|B| = 3$ , then two vertices of  $B$  have degree 3 in  $J$  and one has degree 4 in  $J$ . Otherwise  $|B| \leq 2$ . Now if  $J$  has a vertex  $x$  of degree at least 5, and  $|B| = 2$ , then the other vertex in  $B$  has degree 3 in  $J$ . Now  $x$  must be a cut-vertex (since  $J$  is formed by identifying one vertex in multiple disjoint cycles, exactly one of which has a chord). But a cut-vertex in  $J$  is also a cut-vertex in  $H$ , which is a contradiction. Thus, we only need consider the cases when  $|B| = 3$  and  $|B| = 4$ , which have degree sequences  $3, 3, 3, 3, 2, \dots, 2$ . and  $4, 3, 3, 2, \dots, 2$ .  $|B| = 4$  is a subdivided  $K_4$  or a subdivision of a 4-cycle where one matching has multiplicity 2.  $|B| = 3$  is a subdivision of a triangulated 5-cycle. I haven't worked out those cases, but I don't think they should be too hard.

**Lemma 17.** *Suppose  $G = L(H)$  and  $G$  is a minimal counterexample to the  $\frac{5}{6}$ -Conjecture. Let  $k = \frac{5}{6}(\Delta(G) + 1)$ . If  $T$  is a Tashkinov tree w.r.t. a  $k$ -edge-coloring  $\varphi$  of  $H - e$ , then*

$$\sum_{v \in V(T)} d_H(v)(5d_T(v) - 6) \leq -12 + 5 \sum_{e \in E(T)} \mu_H(e)$$

*Proof.* Since  $T$  is elementary, the sets of colors missing at vertices of  $T$  are disjoint, so  $2 + \sum_{v \in V(T)} (k - d_H(v)) \leq k$ . Rewriting this gives  $k(|V(T)| - 1) \leq -2 + \sum_{v \in V(T)} d_H(v)$ . For each edge  $xy \in E(T)$ , we have  $k = \frac{5}{6}(\Delta(G) + 1) \geq \frac{5}{6}(d_H(x) + d_H(y) - \mu_H(xy))$ . Summing over all  $|T| - 1$  edges gives

$$\begin{aligned} -2 + \sum_{v \in V(T)} d_H(v) &\geq k(|V(T)| - 1) \\ &\geq \frac{5}{6}(\Delta(G) + 1)(|T| - 1) \\ &\geq \frac{5}{6} \sum_{uv \in E(T)} (d_H(u) + d_H(v) - \mu_H(uv)) \\ &= \frac{5}{6} \sum_{v \in V(T)} d_H(v)d_T(v) - \frac{5}{6} \sum_{uv \in E(T)} \mu_H(uv) \end{aligned}$$

To prove the lemma, we take the first and last expressions in the inequality chain, multiply by 6, then rearrange terms.  $\square$

**Corollary 18.** *If  $G = L(H)$  and  $G$  is a minimal counterexample to the  $\frac{5}{6}$ -Conjecture, then each  $x \in V(H)$  is special if  $d_H(x) > \frac{3}{4}\Delta(H) - 3$ .*

*Proof.* Suppose that  $x$  is a non-special vertex. Choose  $e$  incident to  $x$  and a  $k$ -edge-coloring  $\varphi$  of  $G - e$  such that there exists a Vizing fan  $T$  rooted at  $x$  with  $|T| \geq 4$ . Since every edge in  $F$  is incident to  $x$ , we have  $\sum_{e \in E(T)} \mu_H(e) \leq d_H(x)$ . From Lemma 17, we have

$$\begin{aligned} -12 + 5d_H(x) &\geq -12 + 5 \sum_{e \in E(T)} \mu_H(e) \\ &\geq \sum_{v \in T} (5d_T(v) - 6)d_H(v) \\ &\geq (5d_T(x) - 6)d_H(x) + \sum_{v \in T-x} (5d_T(v) - 6)d_H(v) \\ &= (5(|T| - 1) - 6)d_H(x) - \sum_{v \in V(T-x)} d_H(v), \end{aligned}$$

where the final equality holds because each vertex  $v \in T - x$  is a leaf. Now rearranging terms gives

$$\begin{aligned}
-12 + \sum_{v \in V(T-x)} d_H(v) &\geq (5(|T| - 1) - 11)d_H(x) \\
-12 + (|T| - 1)\Delta(H) &\geq (5(|T| - 16)d_H(x) \\
d_H(x) &\leq \frac{-12 + (|T| - 1)\Delta(H)}{5|T| - 16} \\
d_H(x) &\leq \frac{-12 + 3\Delta(H)}{4} = \frac{3}{4}\Delta(H) - 3,
\end{aligned}$$

where the final inequality holds because  $|T| \geq 4$  and the right side decreases as a function of  $|T|$ .  $\square$