SPARSE GRAPHS ADMIT HOMOMORPHISMS INTO ODD CYCLES

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Abstract.

1. Introduction

All graphs under consideration are nonempty finite simple graphs. For graphs G and H, we indicate the existence of a homomorphism from G to H or lack thereof by $G \to H$ and $G \not\to H$, respectively. We write $H \trianglelefteq G$ to indicate that H is an induced subgraph of G, when we want the containment to be proper, we write $H \triangleleft G$.

2. Potential functions

Kostochka and Yancey [3] used "potential functions" to great effect in proving lower bounds on the number of edges in critical graphs. Here we generalize this idea and prove some basic facts.

Definition 1. For positive integers a and b, the (a, b)-*potential function* is the function from graphs to $\mathbb{Z}$ given by $\rho_{a,b}(G) := a\,|G| - b\,\|G\|$. Additionally, put

$$\hat{\rho}_{a,b}(G) := \min_{H \trianglelefteq G} \rho_{a,b}(H).$$

The invariant $\hat{\rho}_{a,b}(G)$ is a measure of the sparseness of G, the larger $\hat{\rho}_{a,b}(G)$ is, the sparser G is. For example, if $\hat{\rho}_{a,b}(G) \geq 0$, then $\operatorname{mad}(G) \leq \frac{2a}{b}$ where $\operatorname{mad}(G)$ is the maximum average degree of G.

For any fixed graph T, we are interested in proving results of the form: any sufficiently sparse graph admits a homomorphism into T. To do so, it will be useful to get the benefits of having a minimum counterexample without being bound to a fixed inductive context. To achieve this, we use *mules* as introduced in [2, 4].

2.1. Mules.

Definition 2. If G and H are graphs, an *epimorphism* is a graph homomorphism $f\colon G \twoheadrightarrow H$ such that $f(V(G)) = V(H)$. We indicate this with the arrow $\twoheadrightarrow$.

Definition 3. Let G be a graph. A graph A is called a *child* of G if $A \neq G$ and there exists $H \trianglelefteq G$ and an epimorphism $f\colon H \twoheadrightarrow A$.

Note that the child-of relation is a strict partial order on the set of (finite simple) graphs $\mathcal{G}$. We call this the *child order* on $\mathcal{G}$ and denote it by '$\prec$'. By definition, if $H \triangleleft G$ then $H \prec G$.

$$\begin{tikzcd} H \arrow[r, hook, "\iota"] \arrow[d, two heads, "h"'] & G \arrow[d, two heads, "h'"] \\ Q \arrow[r, hook, "\iota"'] & G_h \end{tikzcd}$$

FIGURE 1. The commutative diagram for G_h.

Lemma 1. *The ordering $\prec$ is well-founded on $\mathcal{G}$; that is, every nonempty subset of $\mathcal{G}$ has a minimal element under $\prec$.*

Proof. Let $\mathcal{T}$ be a nonempty subset of $\mathcal{G}$. Pick $G \in \mathcal{T}$ minimizing $|V(G)|$ and then maximizing $|E(G)|$. Since any child of G must have fewer vertices or more edges (or both), we see that G is minimal in $\mathcal{T}$ with respect to $\prec$. $\square$

Definition 4. Let $\mathcal{T}$ be a collection of graphs. A minimal graph in $\mathcal{T}$ under the child order is called a $\mathcal{T}$*-mule*.

2.2. **Basic facts.**

For a graph T together with positive integers a, b and c, let $\mathcal{C}_{T,a,b,c}$ be the set of graphs G such that $G \not\to T$ and $\hat{\rho}_{a,b}(G) \geq c$.

Lemma 2. *Let G be a $\mathcal{C}_{T,a,b,c}$-mule. If $H \lhd G$, then $H \to T$.*

Proof. Since $\hat{\rho}_{a,b}(H) \geq \hat{\rho}_{a,b}(G) \geq c$ and $H \prec G$, we must have $H \to T$ since G is a $\mathcal{C}_{T,a,b,c}$-mule. $\square$

Definition 5. Let H be an induced subgraph of a graph G and $h \colon H \twoheadrightarrow Q$ an epimorphism onto some graph Q. Let G_h be the image of the natural extension of h to an epimorphism h' defined on G; that is, G_h and h' are such that the diagram in Figure 1 commutes (where ι indicates the inclusion map).

Lemma 3. *Let G be a $\mathcal{C}_{T,a,b,c}$-mule and Q an arbitrary graph. If $H \unlhd G$ with $H \neq Q$ such that $H \twoheadrightarrow Q$, then $\rho_{a,b}(H) > \hat{\rho}_{a,b}(Q)$.*

Proof. Suppose to the contrary that there is $H \unlhd G$ with $H \neq Q$ such that $H \twoheadrightarrow Q$ and $\rho_{a,b}(H) \leq \hat{\rho}_{a,b}(Q)$. Let h be an epimorphism from H onto Q. Since G is a $\mathcal{C}_{T,a,b,c}$-mule, G_h cannot be a child of G. But we have an epimorphism h' from G onto G_h and $G_h \neq G$ since $H \neq Q$, so it must be that $G_h \notin \mathcal{C}_{T,a,b,c}$. Since $G \to G_h$ and $G \not\to T$, we must have $G_h \not\to T$. Therefore $\hat{\rho}_{a,b}(G_h) < c$. Pick $W \unlhd G_h$ with $\rho_{a,b}(W) < c$. Since $W \ntrianglelefteq G$, we must have $V(W) \cap V(Q) \neq \emptyset$. Hence $\rho_{a,b}\left(G\left[(V(W) - V(Q)) \cup V(H)\right]\right) \leq \rho_{a,b}(W) - \hat{\rho}_{a,b}(Q) + \rho_{a,b}(H) \leq \rho_{a,b}(W) - \hat{\rho}_{a,b}(Q) + \hat{\rho}_{a,b}(Q) = \rho_{a,b}(W) < c$, a contradiction since $\hat{\rho}_{a,b}(G) \geq c$. $\square$

Lemma 4. *Let G be a $\mathcal{C}_{T,a,b,c}$-mule and Q an arbitrary graph. If $H \unlhd G$ is not isomorphic to an induced subgraph of Q and $H \to Q$, then $\rho_{a,b}(H) > \hat{\rho}_{a,b}(Q)$.*

We have the following basic bound on the potential of non-complete subgraphs of G.

Corollary 5. *Let G be a $\mathcal{C}_{T,a,b,c}$-mule. If $H \trianglelefteq G$ is not complete and $\chi(H) \leq \frac{2a}{b}$, then $\rho_{a,b}(H) > a$.*

Proof. Suppose $\chi(H) = k \leq \frac{2a}{b}$. Then there is an epimorphism from H onto K_k given by contracting all color classes in a k-coloring of H. Since $H \neq K_k$, Lemma 3 gives $\rho_{a,b}(H) > \hat{\rho}_{a,b}(K_k)$. But $\hat{\rho}_{a,b}(K_k) = \min_{t \in [k]} at - b\binom{t}{2} = a$ since $k \leq \frac{2a}{b}$, so we have the desired bound. □

We need that mules cannot have uniquely T-colorable cutsets.

Lemma 6. *Let G be a $\mathcal{C}_{T,a,b,c}$-mule. If $X \subset V(G)$ is a cutset, then there is no $\pi \in \mathrm{Hom}(G[X],T)$ such that every element of $\mathrm{Hom}(G[X],T)$ is of the form $\tau \circ \pi$ for some $\tau \in \mathrm{Aut}(T)$.*

Proof. Suppose $X \subset V(G)$ is a cutset and there is $\pi \in \mathrm{Hom}(G[X],T)$ such that every element of $\mathrm{Hom}(G[X],T)$ is of the form $\tau \circ \pi$ for some $\tau \in \mathrm{Aut}(T)$.

Let $\{A, B\}$ be a separation of G with $A \cap B = X$. By Lemma 2 we have $\zeta_A \in \mathrm{Hom}(G[A],T)$ and $\zeta_B \in \mathrm{Hom}(G[B],T)$. Now ζ_A restricted to $G[X]$ is $\tau_A \circ \pi$ for some $\tau_A \in \mathrm{Aut}(T)$ and ζ_B restricted to $G[X]$ is $\tau_B \circ \pi$ for some $\tau_B \in \mathrm{Aut}(T)$. But then $\zeta_A \cup (\tau_A \circ \tau_B^{-1} \circ \zeta_B)$ is a homomorphism from G to T, a contradiction. □

Lemma 6 immediately implies the following.

Corollary 7. *If T is vertex-transitive, then all $\mathcal{C}_{T,a,b,c}$-mules are 2-connected.*

Definition 6. Put $\tilde{\rho}_{a,b}(G) := \min\{\rho_{a,b}(H) \mid H \trianglelefteq G \text{ with } |H| \geq 2\}$.

Lemma 8. *Let G be a $\mathcal{C}_{T,a,b,c}$-mule where T is vertex-transitive and $\tilde{\rho}_{a,b}(T) \geq a+1 \geq b+c$. If $H \triangleleft G$ and H is not isomorphic to an induced subgraph of T, then $\rho_{a,b}(H) > a+1$.*

Proof. Suppose to the contrary that we have $H \triangleleft G$ where H is not isomorphic to an induced subgraph of T and $\rho_{a,b}(H) \leq a+1$. Note that the hypotheses imply $\hat{\rho}_{a,b}(T) = a$. By Lemma 2, $H \to T$, so $\rho_{a,b}(H) = a+1$ by Lemma 4. Let F be all $x \in V(H)$ with neighbors in $G - V(H)$. Since G is 2-connected by Lemma 7, we have $|F| \geq 2$. Pick different $x, y \in F$ and let $H' = H + xy$ if $xy \notin E(H)$ and $H' = H$ otherwise. Then $\hat{\rho}_{a,b}(H') \geq \min\{a, a+1-b\} \geq c$. Since $H' \prec G$ and G is a $\mathcal{C}_{T,a,b,c}$-mule, we must have $H' \to T$.

So, we have a homomorphism $h \colon H \to T$ such that $h(x) \neq h(y)$. Put $Q = \mathrm{im}(h)$. Then $H \twoheadrightarrow Q$. Since G is a $\mathcal{C}_{T,a,b,c}$-mule, G_h cannot be a child of G. But we have an epimorphism h' from G onto G_h and $G_h \neq G$ since H is not isomorphic to Q, so it must be that $G_h \notin \mathcal{C}_{T,a,b,c}$. Since $G \to G_h$ and $G \not\to T$, we must have $G_h \not\to T$. Therefore $\hat{\rho}_{a,b}(G_h) < c$. Pick $W \trianglelefteq G_h$ with $\rho_{a,b}(W) < c$. Since $W \not\trianglelefteq G$, we must have $V(W) \cap V(Q) \neq \emptyset$. Hence $\rho_{a,b}(G[(V(W) - V(Q)) \cup V(H)]) \leq \rho_{a,b}(W) - \hat{\rho}_{a,b}(Q) + \rho_{a,b}(H) < \rho_{a,b}(H)$ since $\hat{\rho}_{a,b}(Q) \geq \hat{\rho}_{a,b}(T) \geq c$. Since H is not isomorphic to an induced subgraph of T, neither is $G[(V(W) - V(Q)) \cup V(H)]$. But then, by Lemma 4, we must have $G[(V(W) - V(Q)) \cup V(H)] \not\to T$ and hence $(V(W) - V(Q)) \cup V(H) = V(G)$.

Suppose $|V(W) \cap V(Q)| = 1$ and let $S = h^{-1}(V(W) \cap V(Q))$. Then S is an independent set in H and hence $x \notin S$ or $y \notin S$. By symmetry, we may assume $y \notin S$. Since $y \in F$, there is an edge yz with $z \in V(G) \setminus V(H)$. Using this extra edge in our estimate from before gives $\rho_{a,b}(G) = \rho_{a,b}\left(G\left[(V(W) - V(Q)) \cup V(H)\right]\right) \leq \rho_{a,b}(W) - \hat{\rho}_{a,b}(Q) + \rho_{a,b}(H) - b \leq \rho_{a,b}(W) + 1 - b \leq \rho_{a,b}(W) < c$ since $b \geq 1$, a contradiction.

So, we must have $|V(W) \cap V(Q)| \geq 2$. Then our estimate is $\rho_{a,b}\left(G\left[(V(W) - V(Q)) \cup V(H)\right]\right) \leq \rho_{a,b}(W) - \tilde{\rho}_{a,b}(Q) + \rho_{a,b}(H) \leq \rho_{a,b}(W) + \rho_{a,b}(H) - (a+1)$. Since $\rho_{a,b}(W) < c$, we must have $\rho_{a,b}(H) - (a+1) \geq 1$. That is, $a + 1 = \rho_{a,b}(H) \geq a + 2$, a contradiction. □

This is a useful form for excluding small subgraphs.

Corollary 9. *Let G be a $\mathcal{C}_{T,a,b,c}$-mule where T is vertex-transitive, $\tilde{\rho}_{a,b}(T) \geq a + 1 \geq b + c$, and $a > b$. If $H \lhd G$ and H is not isomorphic to an induced subgraph of T, then*

$$|H| \geq \frac{a + 2 + (\|H\| - |H|)\, b}{a - b}.$$

References

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