

# A better lower bound on average degree of 4-list-critical graphs

Landon Rabern

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## Abstract

This short note proves that every incomplete  $k$ -list-critical graph has average degree at least  $k - 1 + \frac{k-3}{k^2-2k+2}$ . This improves the best known bound for  $k = 4, 5, 6$ . The same bound holds for online  $k$ -list-critical graphs.

## 1 Introduction

A graph  $G$  is  *$k$ -list-critical* if  $G$  is not  $(k - 1)$ -choosable, but every proper subgraph of  $G$  is  $(k - 1)$ -choosable. For further definitions and notation, see [5, 2]. Table 1 shows some history of lower bounds on the average degree of  $k$ -list-critical graphs.

**Main Theorem.** *Every incomplete  $k$ -list-critical graph has average degree at least*

$$k - 1 + \frac{k - 3}{k^2 - 2k + 2}.$$

Main Theorem gives a lower bound of  $3 + \frac{1}{10}$  for 4-list-critical graphs. This is the first improvement over Gallai's bound of  $3 + \frac{1}{13}$ . The same proof shows that Main Theorem holds for online  $k$ -list-critical graphs as well. Our primary tool is a lemma proved with Kierstead [6] that generalizes a kernel technique of Kostochka and Yancey [8].

**Definition.** The *maximum independent cover number* of a graph  $G$  is the maximum  $\text{mic}(G)$  of  $\|I, V(G) \setminus I\|$  over all independent sets  $I$  of  $G$ .

**Kernel Magic** (Kierstead and R. [6]). *Every  $k$ -list-critical graph  $G$  satisfies*

$$2 \|G\| \geq (k - 2) |G| + \text{mic}(G) + 1.$$

The previous best bounds in Table 1 for  $k$ -list-critical graphs hold for  $k$ -Alon-Tarsi-critical graphs as well. Since Kernel Magic relies on the Kernel Lemma, our proof does not work for  $k$ -Alon-Tarsi-critical graphs. Any improvement over Gallai's bound of  $3 + \frac{1}{13}$  for 4-Alon-Tarsi-critical graphs would be interesting.

	$k$ -Critical $G$				$k$ -List Critical $G$			
$k$	Gallai [4] $d(G) \geq$	Kriv [9] $d(G) \geq$	KS [7] $d(G) \geq$	KY [8] $d(G) \geq$	KS [7] $d(G) \geq$	KR [5] $d(G) \geq$	CR [2] $d(G) \geq$	Here $d(G) \geq$
4	3.0769	3.1429	—	3.3333	—	—	—	<b>3.1000</b>
5	4.0909	4.1429	—	4.5000	—	4.0984	4.1000	<b>4.1176</b>
6	5.0909	5.1304	5.0976	5.6000	—	5.1053	5.1076	<b>5.1153</b>
7	6.0870	6.1176	6.0990	6.6667	—	6.1149	<b>6.1192</b>	6.1081
8	7.0820	7.1064	7.0980	7.7143	—	7.1128	<b>7.1167</b>	7.1000
9	8.0769	8.0968	8.0959	8.7500	8.0838	8.1094	<b>8.1130</b>	8.0923
10	9.0722	9.0886	9.0932	9.7778	9.0793	9.1055	<b>9.1088</b>	9.0853
15	14.0541	14.0618	14.0785	14.8571	14.0610	14.0864	<b>14.0884</b>	14.0609
20	19.0428	19.0474	19.0666	19.8947	19.0490	19.0719	<b>19.0733</b>	19.0469

Table 1: History of lower bounds on the average degree  $d(G)$  of  $k$ -critical and  $k$ -list-critical graphs  $G$ .

## 2 The Proof

The connected graphs in which each block is a complete graph or an odd cycle are called *Gallai trees*. Gallai [4] proved that in a  $k$ -critical graph, the vertices of degree  $k - 1$  induce a disjoint union of Gallai trees. The same is true for  $k$ -list-critical graphs [1, 3]. For a graph  $T$  and  $k \in \mathbb{N}$ , let  $\beta_k(T)$  be the independence number of the subgraph of  $T$  induced on the vertices of degree  $k - 1$ . When  $k$  is defined in the context, put  $\beta(T) := \beta_k(T)$ .

**Lemma 1.** *If  $k \geq 4$  and  $T \neq K_k$  is a Gallai tree with maximum degree at most  $k - 1$ , then*

$$2||T|| \leq (k - 2)|T| + 2\beta(T).$$

*Proof.* Suppose the lemma is false and choose a counterexample  $T$  minimizing  $|T|$ . Plainly,  $T$  has more than one block. Let  $A$  be an endblock of  $T$  and let  $x$  be the unique cutvertex of  $T$  with  $x \in V(A)$ . Consider  $T' := T - (V(A) \setminus \{x\})$ . By minimality of  $|T|$ ,

$$2||T|| - 2||A|| \leq (k - 2)(|T| + 1 - |A|) + 2\beta(T').$$

Since  $T$  is a counterexample,  $2||A|| > (k - 2)(|A| - 1)$ . So, if  $k > 4$ , then  $A = K_{k-1}$  and if  $k = 4$ , then  $A$  is an odd cycle. In both cases,  $d_T(x) = k - 1$ . Consider  $T^* := T - V(A)$ . By minimality of  $|T|$ ,

$$2||T|| - 2||A|| - 2 \leq (k - 2)(|T| - |A|) + 2\beta(T^*).$$

Since  $T$  is a counterexample,  $2||A|| + 2 > (k - 2)|A| + 2(\beta(T) - \beta(T^*))$ . In  $T^*$ , all of  $x$ 's neighbors have degree at most  $k - 2$ . But  $d_T(x) = k - 1$ , so some vertex in  $\{x\} \cup N(x)$  is in a maximum independent set of degree  $k - 1$  vertices in  $T$ . Hence  $\beta(T^*) \leq \beta(T) - 1$ , which gives

$$2||A|| > (k - 2)|A|,$$

a contradiction since  $k \geq 4$ . □

*Proof of Main Theorem.* Let  $G \neq K_k$  be a  $k$ -list-critical graph. The theorem is trivially true if  $k \leq 3$ , so suppose  $k \geq 4$ . Let  $\mathcal{L} \subseteq V(G)$  be the vertices with degree  $k - 1$  and let  $\mathcal{H} = V(G) \setminus \mathcal{L}$ . Put  $\|\mathcal{L}\| := \|G[\mathcal{L}]\|$  and  $\|\mathcal{H}\| := \|G[\mathcal{H}]\|$ . By Lemma 1,

$$2\|\mathcal{L}\| \leq (k-2)|\mathcal{L}| + 2\beta(\mathcal{L})$$

Hence,

$$\begin{aligned} 2\|G\| &= 2\|\mathcal{H}\| + 2\|\mathcal{H}, \mathcal{L}\| + 2\|\mathcal{L}\| \\ &= 2\|\mathcal{H}\| + 2((k-1)|\mathcal{L}| - 2\|\mathcal{L}\|) + 2\|\mathcal{L}\| \\ &= 2\|\mathcal{H}\| + 2(k-1)|\mathcal{L}| - 2\|\mathcal{L}\| \\ &\geq 2\|\mathcal{H}\| + k|\mathcal{L}| - 2\beta(\mathcal{L}), \end{aligned}$$

which is

$$\beta(\mathcal{L}) \geq \|\mathcal{H}\| + \frac{k}{2}|\mathcal{L}| - \|G\|. \quad (1)$$

Let  $M$  be the maximum of  $\|I, V(G) \setminus I\|$  over all independent sets  $I$  of  $G$  with  $I \subseteq \mathcal{H}$ . Then

$$\text{mic}(G) \geq M + (k-1)\beta(\mathcal{L}).$$

Applying Kernel Magic and using (1) gives

$$\begin{aligned} 2\|G\| &\geq (k-2)|G| + M + (k-1)\beta(\mathcal{L}) + 1 \\ &\geq (k-2)|G| + M + (k-1)\left(\|\mathcal{H}\| + \frac{k}{2}|\mathcal{L}| - \|G\|\right) + 1 \\ &= (k-2)|G| + M + (k-1)\|\mathcal{H}\| + \frac{k(k-1)}{2}|\mathcal{L}| - (k-1)\|G\| + 1. \end{aligned}$$

Hence

$$(k+1)\|G\| \geq (k-2)|G| + M + (k-1)\|\mathcal{H}\| + \frac{k(k-1)}{2}|\mathcal{L}| + 1 \quad (2)$$

Let  $\mathcal{C}$  be the components of  $G[\mathcal{H}]$ . Then  $\alpha(C) \geq \frac{|C|}{\chi(C)}$  for all  $C \in \mathcal{C}$ . Whence

$$M + (k-1)\|\mathcal{H}\| \geq \sum_{C \in \mathcal{C}} k \frac{|C|}{\chi(C)} + (k-1)\|C\|. \quad (3)$$

If  $\mathcal{L} = \emptyset$ , then  $G$  has average degree at least  $k \geq k - 1 + \frac{k-3}{k^2-2k+2}$ . So, assume  $\mathcal{L} \neq \emptyset$ . Then  $G[\mathcal{H}]$  is  $(k-1)$ -colorable by  $k$ -list-criticality of  $G$ . In particular,  $\chi(C) \leq k-1$  for every  $C \in \mathcal{C}$ . For every  $C \in \mathcal{C}$ ,

$$k \frac{|C|}{\chi(C)} + (k-1)\|C\| \geq \left(k - \frac{1}{2}\right)|C|. \quad (4)$$

To see this, first suppose  $C \in \mathcal{C}$  is not a tree. Then  $\|C\| \geq |C|$  and hence  $k \frac{|C|}{\chi(C)} + (k-1)\|C\| \geq k \frac{|C|}{k-1} + (k-1)|C| \geq (k - \frac{1}{2})|C|$ . If  $C$  is a tree, then  $\chi(C) \leq 2$  and hence

$k \frac{|C|}{\chi(C)} + (k-1) \|C\| \geq k \frac{|C|}{2} + (k-1)(|C| - 1) \geq (k - \frac{1}{2}) |C|$  unless  $|C| = 1$ . This proves (4) since the bound is trivially satisfied when  $|C| = 1$ .

Now combining (2), (3) and (4) with the basic bound

$$|\mathcal{L}| \geq k |G| - 2 \|G\|,$$

gives

$$\begin{aligned} (k+1) \|G\| &\geq (k-2) |G| + \left(k - \frac{1}{2}\right) |\mathcal{H}| + \frac{k(k-1)}{2} |\mathcal{L}| + 1 \\ &= \left(2k - \frac{5}{2}\right) |G| + \frac{k^2 - 3k + 1}{2} |\mathcal{L}| + 1 \\ &\geq \left(2k - \frac{5}{2}\right) |G| + \frac{k^2 - 3k + 1}{2} (k |G| - 2 \|G\|) + 1. \end{aligned}$$

After some algebra, this becomes

$$2 \|G\| \geq \left(k - 1 + \frac{k-3}{k^2 - 2k + 2}\right) |G| + \frac{2}{k^2 - 2k + 2}.$$

That proves the theorem.  $\square$

The right side of equation (4) in the above proof can be improved to  $k |C|$  unless  $C$  is a  $K_2$  where both vertices have degree  $k$  in  $G$ . If these  $K_2$ 's could be handled, the average degree bound would improve to  $k - 1 + \frac{k-3}{(k-1)^2}$ .

**Conjecture.** *Every incomplete (online)  $k$ -list-critical graph has average degree at least*

$$k - 1 + \frac{k-3}{(k-1)^2}.$$

## References

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