1. Overview

For every multigraph G, we have $\chi'(G) \geq \left\lceil \frac{|E(G)|}{\lfloor |V(G)|/2 \rfloor} \right\rceil$, since each color class has size at most $\left\lfloor \frac{|V(G)|}{2} \right\rfloor$. Likewise, the same bound holds for any subgraph H. Thus, let $\mathcal{W}(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil$ (over all subgraphs H with at least two vertices). Now clearly $\chi'(G) \geq \mathcal{W}(G)$ for every multigraph G. Goldberg [] and Seymour [] each conjectured that this lower bound holds with equality, whenever $\chi'(G) > \Delta(G) + 1$.

Goldberg–Seymour Conjecture. *When $\mathcal{W}(G)$ is as above, every multigraph G satisfies*

$$\chi'(G) \leq \max\{\mathcal{W}(G), \Delta(G) + 1\}.$$

The Goldberg–Seymour conjecture is the major open problem in the area of edge-coloring multigraphs. The second author showed [**?**] that $\chi(G) \leq \max\{\omega(G), \frac{7\Delta(G)+10}{8}\}$ for every line graph G. In the same paper, he conjectured that $\chi(G) \leq \max\{\omega(G), \frac{5\Delta(G)+8}{6}\}$. This conjecture is best possible, as shown by replacing each edge in a 5-cycle by k parallel edges, and taking the line graph. In this paper we prove the latter inequality. Along the way, we develop more general techniques and results that will likely be of independent interest, due to their use in approaching the Goldberg–Seymour conjecture.

The main result of this paper is the following theorem.

Theorem 16 ($\frac{5}{6}$-Theorem)**.** *If Q the line graph of a multigraph G, then we have $\chi(Q) \leq \max\{\omega(Q), \frac{5\Delta(Q)+8}{6}\}$.*

Most of our work goes toward proving the following intermediate result, in Section 6.

Theorem 13 (Weak $\frac{5}{6}$-Theorem)**.** *If Q the line graph of a multigraph G, then $\chi(Q) \leq \max\{\mathcal{W}(G), \Delta(G) + 1, \frac{5\Delta(Q)+8}{6}\}$.*

Finally, in Section 8 we show that the Weak $\frac{5}{6}$-Theorem does indeed imply the $\frac{5}{6}$-Theorem.

2. Tashkinov Trees

Throughout this paper, graphs can have multiple edges unless stated otherwise. A graph G is *elementary* if $\chi'(G) = \mathcal{W}(G)$. Let $[k]$ denote $\{1, \ldots, k\}$. For a path or cycle Q, let $\ell(Q)$ denote the length of Q. A graph G is *critical* if $\chi'(G - e) < \chi'(G)$ for all $e \in E(G)$. For a graph G and a partial k-edge-coloring φ, for each vertex $v \in V(G)$, let $\varphi(v)$ denote the set of colors used in φ on edges incident to v. Let $\overline{\varphi}(v) = [k] \setminus \varphi(v)$. A color c is *seen* by a vertex v if $c \in \varphi(v)$ and c is *missed* by v if $c \in \overline{\varphi}(v)$. Given a partial k-edge-coloring φ, a set $W \subseteq V(G)$ is *elementary* with respect to φ (henceforth, *w.r.t.* φ) if each color in $[k]$ is missed by at most one vertex of W. More formally, $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$ for all distinct $u, v \in W$. A *defective color* for a set $X \subseteq V(G)$ (w.r.t. φ) is a color used on more than one edge from X to $V(G) \setminus X$. A set X is *strongly closed* w.r.t. φ if X has no defective color. Elementary and strongly closed sets are of particular interest because of the following theorem, proved implicitly by Andersen [] and Goldberg []; see also [**?**, Theorem 1.4].

Theorem 1. *Let G be a graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)$. If G is critical, then G is elementary if and only if there exists $uv \in E(G)$, a k-edge-coloring φ of $G - uv$, and a set X with $u, v \in X$ such that X is both elementary and strongly closed w.r.t. φ.*

A *Tashkinov tree* w.r.t. φ is a sequence $v_0, e_1, v_1, e_2, \ldots, v_{t-1}, e_t, v_t$ such that all v_i are distinct, $e_i = v_j v_i$ and $\varphi(e_i) \in \overline{\varphi}(v_\ell)$ for some j and ℓ with $0 \leq j < i$ and $0 \leq \ell < i$. A *Vizing fan* (or simply *fan*) is a Tashkinov tree that induces a star. Tashkinov trees are of interest because of the following lemma.

Tashkinov's Lemma. *Let G be a graph with $\chi'(G) = k+1$, for some integer $k \geq \Delta(G)+1$ and choose $e \in E(G)$ such that $\chi'(G-e) < \chi'(G)$. Let φ be a k-edge-coloring of $G - e$. If T is a Tashkinov tree w.r.t. φ and e, then $V(T)$ is elementary w.r.t. φ.*

In view of Theorem 1 and Tashkinov's Lemma, to prove that a graph G is elementary, it suffices to find an edge e, a k-edge-coloring φ of $G - e$, and a Tashkinov tree T containing e such that $V(T)$ is strongly closed. This motivates our next two lemmas. But first, we need a few more definitions.

Let $t(G)$ be the maximum number of vertices in a Tashkinov tree over all $e \in E(G)$ and all k-edge-colorings φ of $G - e$. Let $\mathcal{T}(G)$ be the set of all triples (T, e, φ) such that $e \in E(G)$, φ is a k-edge-coloring of $G - e$ and T is a Tashkinov tree with respect to e and φ with $|T| = t(G)$. Notice that, by definition, we have $\mathcal{T}(G) \neq \emptyset$. For a k-edge-coloring φ of $G - e$, a maximal Tashkinov tree starting with e may not be unique. However, if T_1 and T_2 are both such trees, then it is easy to show that $V(T_1) \subseteq V(T_2)$; by symmetry, also $V(T_2) \subseteq V(T_1)$, so $V(T_1) = V(T_2)$. Let G be a critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. Let φ be a k-edge-coloring of $G - e_0$ for some $e_0 \in E(G)$. For $v \in V(G)$ and colors α, β, let $P_v(\alpha, \beta)$ be the maximal connected subgraph of G that contains v and is induced by edges with color α or β. So $P_v(\alpha, \beta)$ is a path or a cycle. For a k-edge-coloring φ of $G - v_0v_1$, we often let $P = P_{v_1}(\alpha, \beta)$ for some $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Clearly P must end at v_0 (or we can swap colors α and β on P and color v_0v_1 with α), so let $v_1, \ldots, v_r, v_0$ denote the vertices of P in order. To *rotate the α, β coloring on $P \cup \{v_0v_1\}$ by one*, we uncolor v_1v_2 and use its color on v_0v_1. To *rotate the α, β coloring on $P \cup \{v_0v_1\}$ by j*, we rotate the α, β coloring by one j times in succession. (When we do not specify j, we allow j to take any value from 1 to r.)

Lemma 2. *Let G be a non-elementary critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. For every $v_0v_1 \in E(G)$, k-edge-coloring φ of $G - v_0v_1$, $\alpha \in \overline{\varphi}(v_0)$, and $\beta \in \overline{\varphi}(v_1)$, we have $|P_{v_1}(\alpha, \beta)| < t(G)$.*

Proof. Suppose the lemma is false and choose $v_0v_1 \in E(G)$, a k-edge-coloring φ of $G - v_0v_1$, $\alpha \in \overline{\varphi}(v_0)$, and $\beta \in \overline{\varphi}(v_1)$, such that $|P_{v_1}(\alpha, \beta)| \geq t(G)$. Let $P = P_{v_1}(\alpha, \beta)$. Let (T, v_0v_1, φ) be a Tashkinov tree that begins with edges $v_0v_1, v_1v_2, \ldots, v_{r-1}v_r$. Now $V(T) = V(P)$ since $t(G) \geq |T| \geq |P| \geq t(G)$. By hypothesis G is non-elementary, so Theorem 1 implies that $V(T)$ is not strongly closed; thus, T has a defective color δ with respect to φ. Choose $\tau \in \overline{\varphi}(v_2)$. Let $Q = P_{v_2}(\tau, \delta)$. Since T is maximal, δ is not missing at any vertex of T, and since $V(T)$ is elementary, τ is not missing at any vertex of T other than v_2. As a result, Q ends outside $V(T)$. Now Q could leave $V(T)$ and re-enter it repeatedly, but Q ends outside

$V(T)$, so there is a last vertex $w \in V(Q) \cap V(T)$; say Q ends at $z \in V(G) \setminus V(T)$. Let $\pi \notin \{\alpha, \beta\}$ be a color missing at w. Since $\tau \in \overline{\varphi}(v_2)$ and $\pi \in \overline{\varphi}(w)$ and $|T| = t(G)$, no edge colored τ or π leaves $V(T)$. So we can swap τ and π on every edge in $G - V(T)$ without changing the fact that T is a Tashkinov tree with $|T| = t(G)$. After swapping τ and π, we swap δ and π on the subpath of Q from w to z. Since π is missing at w, the $\delta - \pi$ path starting at z must end at w. Now δ is missing at w, but δ was defective in φ, so some other edge e colored δ still leaves $V(T)$. Adding e gets a larger Tashkinov tree, which is a contradiction. $\square$

3. Short vertices

Recall that a vertex $v \in V(G)$ is *short* if every Vizing fan rooted at v (taken over all k-colorings of $G - e$, over all edges e incident to v) has at most 3 vertices, including v. Otherwise, v is *long*. Let $\nu(T)$ be the number of long vertices in a Tashkinov tree T.

Now we can outline our proof of the $\frac{5}{6}$-Conjecture. We will show in Section 8 that the $\frac{5}{6}$-Conjecture is implied by the Goldberg–Seymour Conjecture. More precisely, if G is a multigraph such that $\chi'(G) \le \max\{\lceil \mathcal{W}(G) \rceil, \Delta(G)+1\}$, then also $\chi'(G) \le \frac{5\Delta(G)+8}{6}$. So here it suffices to show that $\chi'(G) \le \max\{\lceil \mathcal{W}(G) \rceil, \Delta(G)+1, \frac{5\Delta(G)+8}{6}\}$. We consider cases based on $\nu(T)$, for some Tashkinov tree $T \in \mathcal{T}(G)$.

In the present section, we show that if G has a maximum Tashkinov tree T that contains no short vertices, i.e., $\nu(T) = 0$, then G is elementary. In fact, Lemma 7 implies that the same is true when $\nu(T) = 1$. In the proof of Theorem 14, we show that if G is a minimal counterexample to the $\frac{5}{6}$-Conjecture, then every long vertex v has $d(v) < \frac{3}{4}\Delta(G)$. This implies that $\nu(T) < 4$, since otherwise the number of colors missing at vertices of T is more than $4(k - \frac{3}{4}\Delta(G)) > k$, which contradicts that $V(T)$ is elementary. So it remains to consider the case $\nu(T) \in \{2, 3\}$.

In Section 6, we introduce the notion of *k-thin graphs*, which are essentially those for which $\mu(G)$ is not too large. Using a lemma from [**?**], we show that every minimal counterexample to the $\frac{5}{6}$-Conjecture must be k-thin. We then extend the ideas of the present section to show handle the case when $\nu(T) \in \{2, 3\}$. Much like when $\nu(T) \ge 4$, we show that T has too many colors missing at its vertices to be elementary. More precisely, $\sum_{v \in V(T)} |\overline{\varphi}(v)| > k$, which is a contradiction.

Short vertices were introduced in [**?**], where they were motivated by a version of the following lemma in the context of proving a strengthening of Reed's Conjecture for line graphs.

Lemma 3. *Let G be a critical graph with $\chi'(G) = k + 1$ for some integer $k \ge \Delta(G) + 1$. Let φ be a k-edge-coloring of $G - v_0v_1$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Let $P = v_1v_2 \cdots v_r$ be an α, β path with edges $e_i = v_iv_{i+1}$ for all $i \in [r-1]$. If v_i is short for all odd i, then for each $\tau \in \overline{\varphi}(v_0)$ there are edges $f_i = v_iv_{i+1}$ for all $i \in [r-1]$ such that $f_i = e_i$ for i even and $\varphi(f_i) = \tau$ for i odd.*

Proof. Suppose not and choose a counterexample minimizing r. By minimality of r, we have $\varphi(v_{r-1}v_r) = \alpha$ and we have $f_i = v_iv_{i+1}$ for all $i \in [r-2]$ such that $f_i = e_i$ for i even and $\varphi(f_i) = \tau$ for i odd. Swap α and β on e_i for all $i \in [r-3]$ and then color v_0v_1 (call this edge e_0) with α and uncolor e_{r-2}. Let φ' be the resulting coloring. Since $k \geq \Delta(G)+1$, some color other than α is missing at v_{r-2}; let γ be such a color. Now v_{r-1} is short since $r-1$ is odd (since P starts and ends with α), so there is an edge $e = v_{r-1}v_r$ with $\varphi'(e) = \gamma$. Swap τ and α on e_i for all i with $0 \leq i \leq r-3$ to get a new coloring φ^*. Now γ and τ are both missing at v_{r-2} in φ^*. Since v_{r-1} is short, the fan with v_{r-2}, v_{r-1}, v_r and e implies that there is an edge $f_{r-1} = v_{r-1}v_r$ with $\varphi^*(f_{r-1}) = \tau$. But we have never recolored f_{r-1}, so $\varphi(f_{r-1}) = \tau$, which is a contradiction. □

Lemma 4. *Let G be a non-elementary critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. Choose $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$ for some $v_0v_1 \in E(G)$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $P = P_{v_1}(\alpha, \beta)$. Now P contains a long vertex. In particular, $\nu(T) \geq 1$.*

Proof. Suppose every vertex of P is short. Applying Lemma 3 to P shows that for every $\tau \in \overline{\varphi}(v_0)$, there is an edge in T colored τ incident to every $v \in V(P - v_0)$. The same is also true of every $v \in V(P)$; to see this, we rotate the α, β coloring of $P \cup \{v_0v_1\}$ and repeat the same argument. Hence $V(P) = V(T)$, which contradicts Lemma 2. □

Theorem 5. *If G is a critical graph in which every vertex is short, then*

$$\chi'(G) \leq \max\{\mathcal{W}(G), \Delta(G)+1\}.$$

Proof. Suppose not and let G be a counterexample. Let $k = \chi'(G) - 1$, and note that $k \geq \Delta(G)+1$. Since $\mathcal{T}(G) \neq \emptyset$, by applying Lemma 4 we conclude that G is elementary. Hence $\chi'(G) = \mathcal{W}(G)$, which is a contradiction. □

4. An easy bound

In this section, we apply the results of Section 3 to prove an easy bound on $\chi'(G)$. We also show how those results imply Reed's Conjecture, as well as Local and Superlocal strengthenings of Reed's Conjecture, for the class of line graphs.

Let G be a graph. The *claw-degree* of $x \in V(G)$ is

$$d_{\text{claw}}(x) := \max_{\substack{S \subseteq N(x) \\ |S|=3}} \frac{1}{4}\left(d(x) + \sum_{v \in S} d(v)\right).$$

The *claw-degree* of G is

$$d_{\text{claw}}(G) := \max_{x \in V(G)} d_{\text{claw}}(x).$$

Theorem 6. *If G is a graph, then*

$$\chi'(G) \leq \max\left\{\mathcal{W}(G), \Delta(G)+1, \left\lceil \frac{4}{3} d_{claw}(G) \right\rceil\right\}.$$

Proof. Suppose not and choose a counterexample G with the fewest edges; note that G is critical. Let $k = \chi'(G) - 1$, so $k \geq \left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil$. By Theorem 5, G has a long vertex x. Choose $xy_1 \in E(G)$ and a k-edge-coloring φ of $G - xy_1$ such that φ has a fan F of length 3 rooted at x with leaves y_1, y_2, y_3. Since $V(F)$ is elementary,

$$2 + k - d(x) + \sum_{i \in [3]} k - d(y_i) \leq k,$$

and hence

$$d_{\text{claw}}(x) \geq \frac{1}{4}\left(d(x) + \sum_{i\in[3]} d(y_i)\right) \geq \frac{3k+2}{4}.$$

This gives the contradiction

$$\left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil \leq k \leq \frac{4}{3} d_{\text{claw}}(G) - \frac{2}{3}.$$

□

Reed [?] conjectured that $\chi(G) \leq \left\lceil \frac{\omega(G)+\Delta+1}{2} \right\rceil$ for every graph G. This is the average of a trivial lower bound $\omega(G)$ and a trivial upper bound $\Delta(G)+1$. King [?] conjecture the stronger bound $\chi(G) \leq \max_{v\in V(G)} \left\lceil \frac{\omega(v)+d(v)+1}{2} \right\rceil$, where $\omega(v)$ is the size of the largest clique containing v, which is now known to hold for many classes of graphs, including line graphs [?]. Here we show that for line graphs this bound is an easy consequence of our more general lemmas from Section 3. The following is essentially Lemma 10 from [?].

Corollary 7. *Let G be a graph. For $uv \in E(G)$, let $f(uv) = \max\{d(u) + \frac{1}{2}(d(v) - \mu(uv)), d(v) + \frac{1}{2}(d(u) - \mu(uv)\}$. Let $f(G) = \max_{uv,vw\in E(G)} \left\lceil \frac{1}{2}(f(uv) + f(uw)) \right\rceil$. Now*

$$\chi'(G) \leq \max\left\{\mathcal{W}(G), \Delta(G)+1, f(G)\right\}.$$

In particular, the Superlocal version of Reed's Conjecture holds for every line graph.

Proof. The first statement follows directly from Theorem 6, by showing that $\left\lceil \frac{4}{3} d_{claw}(G) \right\rceil \leq f(G)$. Choose $x \in V(G)$ and $S \in N(x)$ such that x and S achieve maximality in the definition of $d_{claw}(G)$. Now

$$\begin{aligned}\left\lceil \frac{4}{3}\frac{1}{4}(d(x) + d(v_1) + d(v_2) + d(v_3)) \right\rceil &\leq \Bigg\lceil \frac{1}{3} \Bigg(d(v_1) + \frac{1}{2}(d(x) - \mu(xv_1)) \\ &\quad + d(v_2) + \frac{1}{2}(d(x) - \mu(xv_2)) \\ &\quad + d(v_3) + \frac{1}{2}(d(x) - \mu(xv_3)) \Bigg) \Bigg\rceil \\ &\leq \left\lceil \frac{1}{3}(f(xv_1) + f(xv_2) + f(xv_3)) \right\rceil \\ &\leq f(G).\end{aligned}$$

This proves the first statement. For the second statement, we show that $\mathcal{W}(G) \leq f(G)$, as follows. For each vertex v, let $v_1, v_2, \ldots$ denote the neighbors of v (with subscripts modulo

$|N(v)|$). Also, let $\overline{d} = \frac{2|E(H)|}{|V(H)|}$.

$$
\begin{aligned}
f(G)2|E(H)| &\geq \sum_{v \in V} \sum_{i=1}^{|N(v)|} \frac{1}{2}(d(v) + \frac{1}{2}(d(v_i) - \mu(vv_i)) + d(v) + \frac{1}{2}(d(v_{i+1}) - \mu(vv_{i+1}))) \\
&= \sum_{v \in V} \sum_{i=1}^{|N(v)|} d(v) + \frac{1}{2}(d(v_i) - \mu(vv_i)) \\
&= \sum_{uv \in E(H)} \frac{3}{2}d(u) + \frac{3}{2}d(v) - \mu(uv) \\
&= \sum_{v \in V(H)} \frac{3}{2}d(v)^2 - |E(H)| \\
&\geq \frac{3}{2}\overline{d}^2 |V(H)| - |E(H)| \\
&= 6\frac{|E(H)|^2}{|V(H)|} - |E(H)|
\end{aligned}
$$

Thus $f(G) \geq \frac{3|E(H)|}{|V(H)|} - \frac{1}{2}$. Since $\mathcal{W}(G) = \left\lceil \frac{2|E(H)|}{|V(H)|-1} \right\rceil \leq \frac{2|E(H)|+|V(H)|-3}{|V(H)|-1}$, it suffices to have $\frac{3|E(H)|}{|V(H)|} - \frac{1}{2} \geq \frac{2|E(H)|+|V(H)|-3}{|V(H)|-1}$. Now solving for $|E(H)|$ gives $|E(H)| \geq \frac{3}{2}|V(H)|\frac{|V(H)|-\frac{7}{3}}{|V(H)|-3}$. Taking $|V(H)| \geq 5$, it suffices to have $2|V(H)| \leq |E(H)|$. Suppose, to the contrary, that we have $|E(H)| < 2|V(H)|$. It will suffice to show that $\frac{2|E(H)|}{|V(H)|-1} \leq \Delta(H)$. Now solving $\frac{4|V(H)|}{|V(H)|-1} \leq \Delta(H)$ (using $|V(H)| \geq 5$), shows that it suffices to have $\Delta(H) \geq 5$.
TODO: HANDLE $\Delta(H) \in \{3, 4\}$. □

5. Properties of long vertices

For a path Q, recall that $\ell(Q)$ denotes the length of Q. For $x, y \in V(Q)$, let xQy denote the subpath of Q with endvertices x and y, and let $d_Q(x, y) = \ell(xQy)$, i.e., the distance from x to y along Q.

Lemma 8. *Let G be a critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. Let φ be a k-edge-coloring of $G - v_0v_1$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. If $\tau \in \overline{\varphi}(x)$ for some $x \in V(C)$ and there is a τ-colored edge from $y \in V(C)$ to $w \in V(G) \setminus V(C)$, then C has a subpath Q with long endpoints z_1, z_2 such that $x \in V(Q)$, $y \notin V(Q - z_1 - z_2)$ and the distance from x to z_i along Q is odd for each $i \in [2]$. Moreover, for each $i \in [2]$, there are no τ-colored edges between z_i and its neighbors along C.*

Proof. Let G, α, β, τ, x, and y be as in the statement of the lemma. Choose z_1 (resp. z_2) to be the first vertex at an odd distance from x along C in the clockwise (resp. counterclockwise) direction with no incident τ-colored edge parallel to some edge of C. Let Q be the subpath of C with endpoints z_1 and z_2 that contains x. By the choice of z_1 each vertex w between x and z_1 with $d_Q(xw)$ odd has a τ-colored edge parallel to some edge of C. The presence of these edges implies the same for each w for which $d_Q(xw)$ is even. By the proof of the

Parallel Edge Lemma, z_1 must be long, since otherwise it would have an incident τ-colored edge parallel to some edge of C. The same argument applies to z_2. □

6. Thin graphs

Let G be a critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. For vertices $x \in V(G)$ and $S \subseteq V(G) \setminus \{x\}$, we say that x is *S-short* if every Vizing fan F rooted at x with $S \subseteq V(F)$, has $|F| \leq 3$ (with respect to any k-edge-coloring of $G - xy$). Otherwise, x is *S-long*. For brevity, when $S = \{y\}$, we may write y-short instead of $\{y\}$-short. It is worth noting that in Lemma 3 we can weaken the hypothesis that v_i is short for all odd i to require only that v_i is v_{i-1}-short for all odd i, since this is what we use in the proof.

A graph G is *k-thin* if $\mu(G) < 2k - d(x) - d(y)$ for all long $x, y \in V(G)$. In the proof of Theorem 14, we will show that every counterexample to the $\frac{5}{6}$-Conjecture must be k-thin.

Lemma 9. *Let G be a k-thin, critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. Let φ be a k-edge-coloring of $G - v_0v_1$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. Let Q be a subpath of C with long end vertices. If all internal vertices of Q are short and $2 \leq \ell(Q) \leq \ell(C) - 2$, then $\ell(Q)$ is even.*

Proof. Suppose to the contrary that we have a subpath Q of C with end vertices long, all internal vertices short, $2 \leq \ell(Q) \leq \ell(C) - 2$, and $\ell(Q)$ odd. Let x and y be the end vertices of Q. Say $C = v_1v_2 \cdots v_rv_0v_1$. By rotating the α, β coloring of C, we may assume that $x = v_0$ and $y = v_a$, where $a \geq 3$ is odd.

We now apply Lemma 3 twice, to show that $\mu(v_1v_2) \geq 2k - d(v_0) - d(v_a)$, which contradicts that G is k-thin. More specifically, assume that the edges $v_0v_1, v_1v_2, \ldots$ go clockwise around C. We apply Lemma 3 once going clockwise starting from v_0 and once going counterclockwise starting from v_a. The first application implies that every color in $\overline{\varphi}(v_0)$ appears on some edge parallel to v_1v_2; the second implies the same for every color in $\overline{\varphi}(v_a)$. Since $|\overline{\varphi}(v_i)| = k - d(v_i)$ for each $i \in \{0, a\}$ and $\overline{\varphi}(v_0) \cap \overline{\varphi}(v_a) = \emptyset$, the conclusion follows. □

Lemma 10. *Let G be a k-thin, critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. Let φ be a k-edge-coloring of $G - v_0v_1$. Suppose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. If C contains exactly 3 long vertices, then $C = xyAzBx$ where A and B are paths of even length and x, y, z are all long. Moreover, x is y-long and y is x-long.*

Proof. Let G be a graph satisfying the hypotheses, and let x, y, z be the three long vertices. The three subpaths of C with endpoints x, y, and z either (i) all have odd length or (ii) include two paths of even length and one of odd length. First assume that $\ell(C) \geq 5$. If we are in (i), then the longest of these three subpaths violates Lemma 9; so we are in (ii), and also the path of odd length is simply an edge. This proves the first statement. For the second statement, assume to the contrary that x is y-short. By rotating the α, β coloring, we can assume that $y = v_0$ and $x = v_1$. As in the previous lemma, we use Lemma 3 (and the comment in the first paragraph of Section 6) to conclude that $\mu(v_1v_2) \geq 2k - d(v_0) - d(z)$. As above, this contradicts that G is k-thin; this contradiction proves the second statement. □

Lemma 11. *Let G be a non-elementary, k-thin, critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. Choose $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$. If $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$, then $P_{v_1}(\alpha, \beta) + v_0v_1$ contains consecutive long vertices.*

Proof. Let $C = P_{v_1}(\alpha,\beta) + v_0v_1$. By Lemma 2, there is $x \in V(C)$ and $\tau \in \overline{\varphi}(x)$ such that there is a τ-colored edge from $y \in V(C)$ to $w \in V(T) \setminus V(C)$. Lemma 8 implies that C has a subpath Q with $x \in V(Q)$ and long endpoints z_1, z_2 such that the distance from x to z_i along Q is odd for each $i \in [2]$. Let Q' be the subpath of C with endpoints z_1 and z_2 that does not contain x. Since C is an odd cycle, $\ell(Q')$ is odd. Let Q^* be a minimum length subpath of Q' with long ends. Now $\ell(Q^*) = 1$ by Lemma 9, as desired. □

Lemma 12. *Let G be a non-elementary, k-thin, critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. If $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$ and $\nu(T) \leq 3$, then T contains long vertices z_1, z_2, z_3 such that either*

1. *z_1 is $\{z_2, z_3\}$-long and z_2 is z_1-long; or*
2. *z_i is z_j-long and z_j is z_i-long for each $(i,j) \in \{(1,2),(2,3)\}$.*

Proof. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ so that $P_{v_1}(\alpha,\beta)$ contains as many long vertices as possible; let $C = P_{v_1}(\alpha,\beta) + v_0v_1$. By Lemma 2, there is $x \in V(C)$ and $\tau \in \overline{\varphi}(x)$ such that there is a τ-colored edge from $y \in V(C)$ to $w \in V(T) \setminus V(C)$. By Lemma 11, C has at least two long vertices.

First suppose that C contains only 2 long vertices, z_1 and z_2. By Lemma 11, z_1 and z_2 are consecutive on C. Lemma 8 implies that C has a subpath Q with endpoints z_1, z_2 such that $x \in V(Q)$ and $y \notin V(Q - z_1 - z_2)$ and for each $i \in [2]$ there are no τ-colored edges between z_i and its neighbors on C. By rotating the α,β coloring of C, we can assume that $x = v_0$ and $\alpha, \tau \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Note that $P_{v_1}(\tau,\beta)$ must end at v_0 (since otherwise we can recolor the Kempe chain and color v_0v_1 with τ). Let $C' = P_{v_1}(\tau,\beta) + v_0v_1$. Note that C' must include v_1Qz_1 and also v_0Qz_2 (the β-colored edges are present by definition and the τ-colored edges are present by the Parallel Edge Lemma). Thus, $z_1, z_2 \in V(C')$. Since z_1 and z_2 are not consecutive on C' and C' contains no other long vertices by the maximality condition on C, Lemma 11 gives a contradiction.

So instead C contains exactly 3 long vertices, z_1, z_2, and z_3. By Lemma 10, $C = z_1z_2Az_3Bz_1$ where A and B are paths of even length. Also, z_1 is z_2-long and z_2 is z_1-long.

By Lemma 8, C has a subpath Q with endpoints z_1, z_3 and with $x \in V(Q)$ and $y \notin V(Q - z_1 - z_3)$ such that there are no τ-colored edges between z_i and its neighbors along C for each $i \in \{1,3\}$ (it could happen that z_3 has a τ-colored edge parallel to an edge of C, so the endpoints of Q are z_1, z_2, but now we get a contradiction as in the previous case, by letting $C' = P_{v_1}(\tau,\beta) + v_0v_1$). By rotating the α,β coloring of C, we may assume that $x = v_0$. Again, let $C' = P_{v_1}(\tau,\beta) + v_0v_1$. We know that C' contains z_1 and z_3 and that z_1 and z_2 are not consecutive on C'. Note also that all long vertices in $V(C')$ must be among z_1, z_2, z_3, since otherwise $\nu(T) \geq 4$, contradicting our hypothesis. So by Lemma 11, either z_1 and z_3 are consecutive on C' or z_2 and z_3 are consecutive on C'.

Suppose that z_2 and z_3 are consecutive on C', and thus connected by a τ-colored edge. Now applying Lemma 10 shows that z_2 is z_3-long and z_3 is z_2-long, so we satisfy (2) in the conclusion of the lemma (by swapping the names of z_1 and z_2).

So instead z_1 and z_3 must be consecutive on C', and thus connected by a τ-colored edge. If $z_1 = v_1$, then we have a fan with an α-colored edge from z_1 to z_2 and a τ-colored edge from z_1 to z_3, so z_1 is $\{z_2, z_3\}$-long.

Now assume that $z_1 \neq v_1$. Let z_1' be the predecessor of z_1 on the path from v_0 (through v_1) to z_1. We can shift the coloring so that $z_1'z_1$ is uncolored and z_1z_2 is colored α (as in the proof of the Parallel Edge Lemma). In fact, we can shift either the α, β edges or the τ, β edges. This gives the options that either $\alpha \in \overline{\varphi}(z_1')$ or $\tau \in \overline{\varphi}(z_1')$, whichever we prefer. Suppose we shift the τ, β edges. Now choose $\gamma \in \overline{\varphi}(z_1') - \alpha - \tau$. Consider the γ-colored edge e incident to z_1. If e goes to z_2, then we z_1 is $\{z_2, z_3\}$-long, by colors γ and τ; so we satisfy (1) in the conclusion of the lemma. If instead e goes to z_3, then instead of shifting the τ, β edges we shift the α, β edges; note that this recoloring preserves the fact that γ is missing at z_1'. Now again z_1 is $\{z_2, z_3\}$-long, this time by colors α and γ; so we again satisfy (1) in the conclusion of the lemma.

Finally, assume that the γ-colored edge incident to z_1 goes to some vertex other than z_2 and z_3. Now let $C'' = P_{z_1}(\gamma, \beta) + z_1z_1'$. Since $V(C'') \subseteq V(T)$, Lemmas 11 and 10 imply that z_2 and z_3 are adjacent on C'' and furthermore z_2 is z_3-long and z_3 is z_2-long; thus, we satisfy (2) in the conclusion of the lemma. □

We need the following result from [**?**], which we use to handle the case when G is not k-thin.

Theorem 13 ([?])**.** *If Q is the line graph of a graph G and Q is vertex critical, then*

$$\chi(Q) \leq \max\left\{\omega(Q), \Delta(Q) + 1 - \frac{\mu(G) - 1}{2}\right\}.$$

Now we prove the main result of this section.

Theorem 14. *If Q is the line graph of G, then*

$$\chi(Q) \leq \max\left\{\lceil \chi_f(Q) \rceil, \Delta(G) + 1, \left\lceil \frac{5\Delta(Q) + 3}{6} \right\rceil\right\}.$$

Proof. Suppose the theorem is false and choose a counterexample minimizing $|Q|$. Let $k = \max\left\{\lceil \chi_f(Q) \rceil, \Delta(G) + 1, \left\lceil \frac{5\Delta(Q)+3}{6} \right\rceil\right\}$. Say $Q = L(G)$ for a graph G. The minimality of Q implies that G is critical and $\chi(Q) = k + 1$, for some $k \geq \Delta(G) + 1$.

The heart of the proof is Claim 1, which roughly says that if x is long, then $d(x) < \frac{3}{4}\Delta(G)$. Moreover, we can improve this bound further if x is the root of a long fan F such that either (i) F has length more than 3 or (ii) some of the other vertices in F have degree less than $\Delta(G)$. The claims thereafter are all essentially applications of Claim 1.

Claim 1. *Let F be a fan rooted at x with respect to a k-edge-coloring of $G - xy$. If $S \subseteq V(F) - x$ and $|S| \geq 3$, then*

$$d(x) \leq \frac{1}{5|S| - 11}\left(2|S| - 12 + \sum_{v \in S} d(v)\right).$$

In particular, if $|S| = 3$, then $d(x) \leq \frac{1}{4}\left(-6 + \sum_{v \in S} d(v)\right)$.

Proof: Since F is elementary, we have

$$2+k-d(x)+\sum_{v\in S} k-d(v)\le k,$$

so

$$2+|S|k\le d(x)+\sum_{v\in S} d(v).$$

Using $k\ge \frac{5}{6}(\Delta(Q)+1)-\frac{1}{3}\ge \frac{5}{6}(d(x)+d(v)-\mu(xv))-\frac{1}{3}$ for each $v\in S$, we get

$$2+\sum_{v\in S}\left(\frac{5}{6}(d(x)+d(v)-\mu(xv))-\frac{1}{3}\right)\le d(x)+\sum_{v\in S} d(v),$$

so multiplying by 6 and rearranging terms gives

$$12+(5|S|-6)\,d(x)-2|S|\le \sum_{v\in S} 5\mu(xv)+\sum_{v\in S} d(v).$$

Now $\sum_{v\in S}\mu(xv)\le d(x)$, so this implies

$$12+(5|S|-11)\,d(x)-2|S|\le \sum_{v\in S} d(v).$$

Solving for $d(x)$ gives

$$d(x)\le \frac{1}{5|S|-11}\left(2|S|-12+\sum_{v\in S} d(v)\right),$$

and when $|S|=3$, we get $d(x)\le \frac{1}{4}\left(-6+\sum_{v\in S} d(v)\right).$ □

Claim 2. *If $x\in V(G)$ is long, then $d(x)\le \frac{3}{4}\Delta(G)-1$.*
Proof: This is immediate from Claim 1, since $d(v)\le\Delta(G)$ for all $v\in S$. □

Claim 3. *If $x_1x_2\in E(G)$ such that x_1 is x_2-long and x_2 is x_1-long, then*

$$d(x_i)\le \frac{2}{3}\Delta(G)-2 \text{ for all } i\in[2].$$

Proof: By Claim 1, for each $i\in[2]$,

$$d(x_i)\le \frac{1}{4}\left(-6+\sum_{v\in S} d(v)\right)\le \frac{1}{4}\left(-6+d(x_{3-i})+2\Delta(G)\right),$$

Substituting the bound on $d(x_{3-i})$ into that on $d(x_i)$ and simplifying gives for each $i\in[2]$,

$$d(x_i)\le -2+\frac{2}{3}\Delta(G).$$

□

Claim 4. *If $x_1x_2, x_1x_3\in E(G)$ such that x_1 is $\{x_2,x_3\}$-long, x_2 is x_1-long and x_3 is long, then*

$$d(x_1)\le -\frac{8}{5}+\frac{3}{5}\Delta(G),$$

$$d(x_2) \leq -\frac{7}{5} + \frac{13}{20}\Delta(G).$$

Proof: By Claim 1, we have

$$d(x_1) \leq \frac{1}{4}\left(-6 + \sum_{v \in S} d(v)\right) \leq \frac{1}{4}\left(-6 + d(x_2) + d(x_3) + \Delta(G)\right),$$

$$d(x_2) \leq \frac{1}{4}\left(-6 + \sum_{v \in S} d(v)\right) \leq \frac{1}{4}\left(-6 + d(x_1) + 2\Delta(G)\right).$$

By the same calculation as in Claim 3, these together imply

$$d(x_1) \leq -2 + \frac{2}{5}\Delta(G) + \frac{4}{15}d(x_3).$$

Since x_3 is long, using Claim 2, we get

$$d(x_1) \leq -\frac{34}{15} + \frac{3}{5}\Delta(G),$$

and hence

$$d(x_2) \leq -\frac{61}{15} + \frac{13}{20}\Delta(G).$$

□

Claim 5. *The theorem is true.*

Proof: Let $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$. By Lemma 12, one of the following holds:

(1) G is elementary; or
(2) G is not k-thin; or
(3) $\nu(T) = 3$ and $V(T)$ contains vertices x_1, x_2, x_3 such that x_1 is x_2-long, x_2 is x_1-long, x_2 is x_3-long, and x_3 is x_2-long; or
(4) $\nu(T) = 3$ and $V(T)$ contains vertices x_1, x_2, x_3 such that x_1 is $\{x_2, x_3\}$-long, x_2 is x_1-long, and x_3 is long; or
(5) $V(T)$ contains four long vertices x_1, x_2, x_3, x_4.

If (1) holds, then $\chi(Q) = \lceil \chi_f(Q) \rceil$, which contradicts our choice of Q as a counterexample.

If (2) holds, then Claim 2 implies that $\mu(G) \geq 2k - \frac{3}{2}\Delta(G) + 2$. Now Theorem 13 gives

$$\begin{aligned} k + 1 &\leq \Delta(Q) + 1 - \frac{2k - \frac{3}{2}\Delta(G) + 2}{2} \\ &= \Delta(Q) + 1 - k + \frac{3}{4}\Delta(G) - 1, \end{aligned}$$

so

$$2(k+1) \leq \Delta(Q) + 1 + \frac{3}{4}\Delta(G).$$

Substituting $\Delta(G) \leq k$ and solving for k gives

$$k \leq \frac{4}{5}\Delta(Q) - \frac{4}{5} < \frac{5}{6}\Delta(Q) + \frac{1}{2} \leq k,$$

which is a contradiction.

Suppose (3) holds. Now

$$2 + \sum_{i \in [3]} k - d(x_i) \leq k,$$

so Claim 3 implies

$$3\left(\frac{2}{3}\Delta(G) - 2\right) \geq 2k + 2,$$

which is a contradiction, since $\Delta(G) \leq k$.

Suppose (4) holds. Now

$$2 + \sum_{i \in [3]} k - d(x_i) \leq k,$$

so Claims 2 and 4 give

$$\left(\frac{3}{5} + \frac{13}{20} + \frac{3}{4}\right)\Delta(G) - \left(\frac{34}{15} + \frac{16}{15} + 1\right) \geq 2k + 2,$$

which is

$$2\Delta(G) - \frac{13}{3} \geq 2k + 2,$$

again a contradiction, since $\Delta(G) \leq k$.

So (5) must hold. But now

$$2 + \sum_{i \in [4]} k - d(x_i) \leq k,$$

so using Claim 2 gives

$$4\left(\frac{3}{4}\Delta(G) - 1\right) \geq 3k + 2,$$

a contradiction since $\Delta(G) \leq k$. $\square$

This finishes the final case of Claim 5, which proves the theorem. $\square$

In the previous theorem, we showed that $\chi(Q) \leq \max\left\{\lceil \chi_f(Q) \rceil, \Delta(G) + 1, \left\lceil \frac{5\Delta(Q)+3}{6} \right\rceil\right\}$. Now we show that if the maximum is attained by the second argument, then G satisfies the $\frac{5}{6}$-Conjecture. We use the following lemma, which is implicit in [?].

Lemma 15. *If Q is the line graph of a graph G and Q is vertex critical, then*

$$\chi(Q) \leq \max\left\{\Delta(G), \Delta(Q) + 1 + 2\mu(G) - \Delta(G)\right\}.$$

Proof. The fan equation implies this (see the proof in strengthening Brooks paper). $\square$

Corollary 16. *If Q is the line graph of G, then*

$$\chi(Q) \leq \max\left\{\lceil \chi_f(Q) \rceil, \frac{5}{6}\Delta(Q) + \frac{4}{3}\right\}.$$

Proof. Since $\lceil \frac{5}{6}\Delta + \frac{3}{6} \rceil \leq \frac{5}{6}\Delta + \frac{8}{6}$, the bound follows directly from Theorem 14 unless we have $k+1 = \chi'(G) = \Delta(G)+1$. So assume this is true. Now Lemma 15 gives

$$k+1 = \chi(Q) \leq \Delta(Q) + 1 + 2\mu(G) - k,$$

so solving for $\mu(G)$ gives

$$\mu(G) \geq k - \frac{\Delta(Q)}{2}.$$

Applying Theorem 13 gives

$$k+1 = \chi(Q) \leq \Delta(Q) + 1 - \frac{k - \frac{\Delta(Q)}{2} - 1}{2},$$

and solving for $k+1$ yields

$$k+1 \leq \frac{5}{6}\Delta(Q) + \frac{4}{3}.$$

$\square$

7. With slack variables

Lemma 17. *Let G be a critical, elementary graph with $\chi'(G) = k+1$ where $k \geq \Delta(G)+1$. Put $Q := L(G)$. If $k = \epsilon\,(\Delta(Q)+1)+\beta$, then for all $x \in V(G)$,*

$$|N(x)| = \frac{\epsilon\,(|G| - \Delta(G) - d_G(x) - 1 + S_1 + S_2 + S_3\,(|G|-1))}{(1-\epsilon)\Delta(G) - \epsilon d_G(x) + 1 - \beta + S_3},$$

where

$$S_1 := \sum_{v \in N(x)} \Delta(Q) - d_Q(xv),$$

$$S_2 := 2 + \sum_{v \in V(G)\setminus N(x)} \Delta(G) - d_G(v),$$

$$S_3 = k - (\Delta(G)+1).$$

Proof. Since G is critical and elementary, $|G|$ is odd and

$$k = \frac{2(\|G\|-1)}{|G|-1}. \tag{1}$$

Let $x \in V(G)$, put $M := |N(x)|$ and

$$P := \sum_{v \in N(x)} d_G(v).$$

Then

$$2(\|G\|-1) = \Delta(G)(|G| - M) - S_2 + P. \tag{2}$$

Since

$$\frac{2(\|G\|-1)}{|G|-1} = k = \Delta(G) + 1 + S_3,$$

using (2), we get

$$P = (|G|-1)(\Delta(G)+1+S_3) - \Delta(G)(|G|-M) + S_2,$$

which is

$$P = \Delta(G)(M-1) + |G| - 1 + S_2 + S_3(|G|-1). \tag{3}$$

Also, using $k = \epsilon\,(\Delta(Q)+1)+\beta$, we get

$$kM = \beta M + \epsilon S_1 + \epsilon \sum_{v \in N(x)} d_G(x) + d_G(v) - \mu(xv),$$

Since $\sum_{v \in N(x)} \mu(xv) = d_G(x)$,we have

$$kM = \beta M + \epsilon S_1 + \epsilon d_G(x)(M-1) + \epsilon P. \tag{4}$$

Plugging (3) into (4) and solving for M gives

$$M = \frac{\epsilon\,(|G| - \Delta(G) - d_G(x) - 1 + S_1 + S_2 + S_3\,(|G|-1))}{(1-\epsilon)\Delta(G) - \epsilon d_G(x) + 1 - \beta + S_3},$$

as desired. $\square$

Using $\epsilon = \frac{5}{6}$, we get the following.

Lemma 18. *Let G be a critical, elementary graph with $\chi'(G) = k+1$ where $k \geq \Delta(G)+1$. Put $Q := L(G)$. If $k = \frac{5}{6}(\Delta(Q)+1)+\beta$, then for all $x \in V(G)$,*

$$|N(x)| = \frac{5\left(|G| - \Delta(G) - d_G(x) - 1 + S_1 + S_2 + S_3\left(|G|-1\right)\right)}{\Delta(G) - 5d_G(x) + 6(1-\beta+S_3)},$$

where

$$S_1 := \sum_{v \in N(x)} \Delta(Q) - d_Q(xv),$$

$$S_2 := 2 + \sum_{v \in V(G)\setminus N(x)} \Delta(G) - d_G(v),$$

$$S_3 = k - (\Delta(G)+1).$$

Lemma 19. *Let G be a critical, elementary graph with $\chi'(G) = k+1$ where $k \geq \Delta(G)+1$. Put $Q := L(G)$. If $k = \frac{5}{6}(\Delta(Q)+1)+\beta$ where $\beta \geq -\frac{1}{3}$, then for all $x \in V(G)$ with $|N(x)| \geq 3$,*

$$d_G(x) \leq \frac{|N(x)|}{5\left(|N(x)|-2\right)}\Delta(G) - \frac{1}{|N(x)|-2}\sum_{v \in V(G)\setminus N[x]} \Delta(G) - d_G(v).$$

Proof. Say $|N(x)| = 2 + S_4$ for some $S_4 \geq 1$. Applying Lemma 18 and simplifying using $S_1 \geq 0$ and $\beta \geq -\frac{1}{3}$ gives

$$(5+5S_4)d_G(x) \leq (7+S_4)\Delta(G) - 5|G| + 21 + S_3(-5|G| + 17 + 6S_4) + 8S_4 - 5S_2. \tag{5}$$

Put

$$t := \sum_{v \in V(G)\setminus N[x]} \Delta(G) - d_G(v).$$

Then $S_2 = t + 2 + \Delta(G) - d_G(x)$. Using this in (5), we get

$$5S_4 d_G(x) \leq (2+S_4)\Delta(G) - 5|G| + 11 + S_3(-5|G| + 17 + 6S_4) + 8S_4 - 5t. \tag{6}$$

The desired bound follows when $S_4 \leq \frac{5}{8}|G| - 2$, since then

$$-5|G| + 11 + S_3(-5|G| + 17 + 6S_4) + 8S_4 \leq 0.$$

So, suppose $S_4 > \frac{5}{8}|G| - 2$. Rearranging, we get

$$5S_4 d_G(x) \leq 3S_4\Delta(G) - 5|G| + 11 + S_3(-5|G| + 17 + 6S_4) + 8S_4 - (2S_4 - 2)\Delta(G) \tag{7}$$

We know that

$$-5|G| + 10 + 5S_4 + S_3(-5|G| + 10 + 5S_4) \leq 0,$$

so

$$5S_4 d_G(x) \leq 3S_4\Delta(G) + 1 + 7S_3 + (S_3+3)S_4 - (2S_4-2)\Delta(G) \tag{8}$$

Since $d_G(x) \geq |N(x)| \geq \frac{5}{8}|G| \geq \frac{5}{8}\Delta(G)$, we have a contradiction unless

$$1 + 7S_3 + (S_3+3)S_4 - (2S_4-2)\Delta(G) > 0$$

By Shannon's bound $S_3 \leq \frac{\Delta(G)}{2}$, so

$$1 + \left(\frac{7}{2} + 2\right)\Delta(G) + \frac{\Delta(G)+6}{2}S_4 - 2S_4\Delta(G) > 0,$$

which is

$$1 + \left(\frac{7}{2} + 2\right)\Delta(G) > \frac{3\Delta(G)-6}{2}S_4,$$

so

$$\frac{5}{8}|G| - 2 < S_4 < \frac{11\Delta(G)+2}{3\Delta(G)-6}.$$

□

Corollary 20. *Let G be a critical, elementary graph with $\chi'(G) = k+1$ where $k \geq \Delta(G)+1$. Put $Q := L(G)$. If $k = \frac{5}{6}(\Delta(Q)+1)+\beta$ where $\beta \geq -\frac{1}{3}$, then there are at most two $x \in V(G)$ with $|N(x)| \geq 3$ and if there are two x_1, x_2, then $|N(x_1)| = |N(x_2)| = 3$ and $x_1 \leftrightarrow x_2$.*

Proof. Since G is critical and elementary, $|G|$ is odd and

$$\frac{2(\|G\|-1)}{|G|-1} = k \geq \Delta(G) + 1,$$

so

$$2\,\|G\| \geq \Delta(G)\,|G| + |G| - \Delta(G) + 1.$$

In particular,

$$\sum_{v \in V(G)} \Delta(G) - d_G(v) \leq \Delta(G) - 1 - |G|\,.$$

By Lemma 19, every $x \in V(G)$ with $3 \leq |N(x)| \leq \frac{5}{8}|G|$ has $d_G(x) \leq \frac{3}{5}\Delta(G)$, so there are at most two such x since $\frac{2}{5} + \frac{2}{5} + \frac{2}{5} > 1$.

Suppose there are two such x_1, x_2 with $x_1 \not\leftrightarrow x_2$. If $|N(x_1)| \geq 4$, then Lemma 19 gives $d_G(x_1) \leq \frac{2}{5}\Delta(G)$ which is impossible because $d_G(x_2) \leq \frac{3}{5}\Delta(G)$. So, we must have $|N(x_1)| = |N(x_2)| = 3$. Since $x_1 \not\leftrightarrow x_2$, Lemma 19 gives for $i \in [2]$,

$$d_G(x_i) \leq \frac{3}{5}\Delta(G) - (\Delta(G) - d_G(x_{3-i})),$$

so

$$d_G(x_i) - d_G(x_{3-i}) \leq -\frac{2}{5}\Delta(G),$$

so $d_G(x_1) < d_G(x_2) < d_G(x_1)$, a contradiction. □

8. The $\frac{5}{6}$-Conjecture

Lemma 21. *Let H be an elementary multigraph and let J be the underlying simple graph. If H is a minimal counterexample to the $\frac{5}{6}$-Conjecture, then J is the a subdivision of (i) a cycle with a chord, (ii) a copy of K_4, (iii) a cycle with two chords with disjoint sets of endpoints, or (iv) a cycle with two chords with a common endpoint.*

We assume that H is elementary, and that H is a minimal counterexample to the $\frac{5}{6}$-Conjecture. We will show that if a vertex $v \in V(H)$ has more than two neighbors, then $d(v)$ is small. In particular, if $|N(v)| \geq 3$, then $d(v) <$ and if $N(v)| \geq 4$, then $d(v) <$. An immediate consequence of the first inequality is that H has at most four vertices v with $|N(v)| \geq 3$.

Let H be a minimal counterexample to $\frac{5}{6}$-conjecture. Then $\chi'(H) \geq \Delta(H) + 2$, so by Goldberg we have $\chi'(H) = \lceil \mathcal{W}(H) \rceil > \left\lceil \frac{5\Delta(L(H))+3}{6} \right\rceil$. Suppose $Q \subseteq V(H)$ achieves max in $\lceil \mathcal{W}(H) \rceil$. If there is $e \in E(H) \setminus E(Q)$, then removing it we get $\chi'(H-e) < \chi'(H)$, but $\lceil \mathcal{W}(H-e) \rceil = \lceil \mathcal{W}(H) \rceil$, so $\chi'(H) > \chi'(G-e) \geq \lceil \mathcal{W}(H-e) \rceil = \lceil \mathcal{W}(H) \rceil = \chi'(H)$, a contradiction. Hence $Q = H$.

Choose $x \in H$, and let $M = |N(x)|$ and $p = \sum_{v \in N(x)} d_H(v)$.

$$\begin{aligned} \mathcal{W}(H) &= \left\lceil \frac{2||H||}{|H|-1} \right\rceil \\ &\leq \frac{2||H|| + |H| - 3}{|H|-1} \\ &\leq \frac{\Delta(H)(|H|-M) + p + |H| - 3}{|H|-1} \end{aligned} \tag{9}$$

Also, summing $\mathcal{W}(H) > \frac{5}{6}(\Delta(G)+1)$ over all $v \in N(x)$, and using $\sum_{y \in N(x)} \mu(xy) \leq d(x)$, gives

$$\begin{aligned} \mathcal{W}(H)M &> \frac{5}{6} \sum_{y \in N(x)} d(x) + d(y) - \mu(xy) \\ &\geq \frac{5}{6}((M-1)d(x) + p). \end{aligned} \tag{10}$$

So, with (1),

$$\frac{5}{6}((M-1)d(x) + p) < M \frac{\Delta(H)(|H|-M) + p + |H| - 3}{|H|-1}.$$

Thus

$$\frac{(\Delta(H)(|H|-M) + |H| - 3)M}{|H|-1} - \frac{5}{6}(M-1)d(x) > \left(\frac{5}{6} - \frac{M}{|H|-1} \right) p \tag{11}$$

Since, $\lceil \mathcal{W}(H) \rceil = \chi'(H) \geq \Delta(H) + 2$, we have $\frac{2||H||}{|H|-1} + 1 \geq \Delta(H) + 2$ (here the '+1' on the left comes from the ceiling). So,

$$2||H|| \geq |H|\Delta(H) + |H| - 1 - \Delta(H),$$

which implies that

$$p \geq (M-1)\Delta(H) + |H| - 1. \tag{12}$$

Now substituting (4) into (3) gives

$$\frac{(\Delta(H)(|H|-M) + |H| - 3)M}{|H|-1} - \frac{5}{6}(M-1)d(x) > \left(\frac{5}{6} - \frac{M}{|H|-1}\right)((M-1)\Delta(H) + |H| - 1)$$

$$\frac{(\Delta(H)(|H|-M) + |H| - 3)M}{|H|-1} + M\frac{(M-1)\Delta(H)}{|H|-1} + M > \frac{5}{6}(M-1)(d(x) + \Delta(H)) + \frac{5}{6}(|H|-1)$$

$$\begin{aligned}
M(\Delta(H)+2) - \frac{2M}{|H|-1} &> \frac{5}{6}(M-1)(d(x)+\Delta(H)) + \frac{5}{6}(|H|-1) \\
&= \frac{5}{6}M(d(x)+\Delta(H)) - \frac{5}{6}(d(x)+\Delta(H)) + \frac{5}{6}(|H|-1) \\
\left(\frac{\Delta(H)}{6} - \frac{5}{6}d(x) + 2\right)M &> \frac{2M}{|H|-1} + \frac{5}{6}(|H|-1) - \frac{5}{6}(d(x)+\Delta(H)) \\
(\Delta(H) - 5d(x) + 12)M &> \frac{12M}{|H|-1} + 5(|H|-1) - 5(d(x)+\Delta(H)) \\
&> 5(|H|-1) - 5(d(x)+\Delta(H)) \\
M &< 5\frac{d(x)+\Delta(H)+1-|H|}{5d(x)-\Delta(H)-12}
\end{aligned}$$

The final inequality follows from the previous one because $\Delta(H) - 5d(x) + 12 < 0$, since: WHY???
We want to find bounds on $d(x)$ that ensure M is small. To this end, we write $M < 2 + Y$, for some expression Y and solve $Y < 1$ to get the desired bounds on $d(x)$.

$$\begin{aligned}
M &< \frac{10d(x) - 2\Delta(H) - 24}{5d(x) - \Delta(H) - 12} + \frac{7\Delta(H) - 5d(x) + 29 - 5|H|}{5d(x) - \Delta(H) - 12} \\
&= 2 + \frac{7\Delta(H) - 5d(x) + 29 - 5|H|}{5d(x) - \Delta(H) - 12}.
\end{aligned}$$

So $M \leq 2$ when

$$7\Delta(H) - 5d(x) + 29 - 5|H| < 5d(x) - \Delta(H) - 12$$

which simplifies to

$$d(x) > \frac{8\Delta(H) + 41 - 5|H|}{10}.$$

I think we should be able to prove that the conjecture follows from Goldberg–Seymour. That lemma you proved is pretty useful. We can assume that H is critical, which implies that $|N(x)| \geq 2$ for all x in H. Now let J be the simple graph underlying H. We know that $\delta(J) \geq 2$. Let $B = \{x \in H s.t. d_J(x) \geq 3\}$. That lemma implies that $|B| \leq 4$. Further, if $|B| = 4$, then each vertex of B has degree 3 in J. If $|B| = 3$, then two vertices of B have degree 3 in J and one has degree 4 in J. Otherwise $|B| \leq 2$. Now if J has a vertex x of degree at least 5, and $|B| = 2$, then the other vertex in B has degree 3 in J. Now x must be a cut-vertex (since J is formed by identifying one vertex in multiple disjoint cycles, exactly one of which has a chord). But a cut-vertex in J is also a cut-vertex in H, which is a contradiction. Thus, we only need consider the cases when $|B| = 3$ and $|B| = 4$, which have degree sequences $3, 3, 3, 3, 2, \ldots 2.$ and $4, 3, 3, 2, \ldots, 2.$ $|B| = 4$ is a subdivided K_4 or a subdivision of a 4-cycle where one matching has multiplicity 2. $|B| = 3$ is a subdivision of a triangulated 5-cycle. I haven't worked out those cases, but I don't think they should be too hard.

Lemma 22. *If H is a minimal counterexample to the $\frac{5}{6}$-Conjecture, then H has no path of three or more vertices of degree two.*

Proof Ideas. That lemma from strengthening Brooks is helpful because it shows that $\mu(G) \leq \Delta(G)/3$. Using that I showed that G can't have any path of 3 or more 2-vertices. For 4 or more, I think you can just delete those edges, then extend the coloring. For exactly 3, it looks like we can contract two of those edges on the path (then possibly reduce the multiplicity of the remaining edges on the path, so we don't increase $\Delta(G)$). From that coloring, it seems like we can extend fairly easily (after possibly uncoloring some of them). $\square$

Lemma 23. *If H is an elementary multigraph that is critical, then $|V(H)|$ is odd.*

Proof. Proof is from Scheinerman and Ullman, Section 4.2. The basic idea is to show that if $|V(H)|$ is odd, and $v \in V(H)$ with $d(v) = \delta(H)$, then $\chi'(H - v) \geq \chi'(H)$, so H is not critical. $\square$

Theorem 24. *The $\frac{5}{6}$-Conjecture is true. That is, if Q is the line graph of a multigraph, then $\chi(Q) \leq \max\{\omega(Q), \frac{5\Delta(Q)+8}{6}\}$.*

Proof. Suppose the theorem is false and let H be a minimal counterexample. By Corollary 16, we know that H is elementary. By Lemma 22, we also know that the underlying simple graph J of H has no path of three or more consecutive 2-vertices. By Lemma 21, we know that J is a subdivision of (i) a cycle with a chord, (ii) a copy of K_4, (iii) a cycle with two chords with disjoint sets of endpoints, or (iv) a cycle with two chords with a common endpoint. We consider these four possibilities in succession. In each case we find a small dominating set S in the line graph Q of H and sum $\frac{5}{6}(1 + d_Q(v))$ over all $v \in S$ to show that $\chi(Q) = \left\lceil \frac{2|E(H)|}{|V(H)|-1} \right\rceil \leq \left\lceil \frac{5\Delta(Q)+3}{6} \right\rceil$.

(i) Suppose that H is a subdivision of a cycle with a chord, i.e., J consists of two 3-vertices joined by three internally disjoint paths, each with length between 1 and 3 (by Lemma 22). Denote these three paths by A, B, and C. By symmetry, assume that $|A| \leq |B| \leq |C|$. A priori, we have $\binom{5}{3}$ possibilities for the lengths of A, B, and C. However, by

Lemma 23, we know that $|V(H)|$ is odd, so $|A|+|B|+|C|$ is even. Thus, $(|A|,|B|,|C|) \in \{(1,1,2),(1,2,3),(2,2,2),(2,3,3)\}$. The first case is trivial, since Q is a clique. So assume $|A| = 1$, $|B| = 2$, and $|C| = 3$. By considering an initial edge e_1 and a final edge e_2 of C, we get $2(\frac{5}{6}(\Delta(Q)+1)) \geq \frac{5}{6}((d_Q(e_1)+1)+(d_Q(e_2)+1)) \geq \frac{5}{6}(|E(H)|+2)$. Now $\frac{5}{6}(\Delta(Q)+1) \geq \frac{5}{12}(|E(H)|+2)$, which gives $\frac{5\Delta(Q)+3}{6} > \frac{|E(H)|}{2}$, so taking ceilings gives the desired bound on $\chi'(H)$. When $|A|=|B|=|C|=2$, essentially the same argument works, but now we consider an initial edge on A and a final edge on B. Finally, suppose $|A|=2$ and $|B|=|C|=3$. Now we take an initial edge on B and a final edge on C. This gives $2(\frac{5}{6}(\Delta(Q)+1)) \geq \frac{5}{6}|E(H)|$, so $\frac{5\Delta(Q)+3}{6} \geq \frac{5}{12}|E(H)| - \frac{1}{3} \geq \frac{1}{3}|E(H)|$, since $|E(H)| \geq 4$. Again, taking ceilings gives the desired bound on $\chi'(H)$.

Before considering further possibilities for J we prove the following claim, which generalizes the approach we took in (i) above. We use this claim repeatedly in the rest of the proof.

Claim 1. Let S be a subset of j vertices in Q such that each vertex w of Q appears in the closed neighborhood of at least k vertices in S and such that $\sum_{v\in S}|N[v]| \geq k|V(Q)|+\ell$, for some positive integers j, k, ℓ. Now H satisfies the $\frac{5}{6}$-Conjecture when $\frac{5k}{6j} \geq \frac{2}{|V(H)|-1}$ and $|E(H)|(\frac{5k}{6j} - \frac{2}{|V(H)|-1}) \geq \frac{5}{6}\left(\frac{2}{5} - \frac{l}{j}\right)$. In particular, this is true when $\frac{5k}{6j} \geq \frac{2}{|V(H)|-1}$ and $\frac{\ell}{j} \geq \frac{2}{5}$.

<u>Proof:</u> The second statement follows from the first, since the left side is nonnegative and the right side is nonpositive. Now we prove the first statement. We have

$$\begin{aligned} j\frac{5}{6}(\Delta(Q)+1) &\geq \sum_{v\in S}\frac{5}{6}(d_Q(v)+1) \\ &= \frac{5}{6}(k|V(Q)|+\ell) \\ &= \frac{5k}{6}|E(H)| + \frac{5\ell}{6} \\ \frac{5\Delta(Q)+3}{6} &\geq \frac{5k}{6j}|E(H)| + \frac{5}{6}\left(\frac{\ell}{j} - \frac{2}{5}\right). \end{aligned}$$

To show that H satisfies the $\frac{5}{6}$-Conjecture, it suffices to show that the final expression is at least $\frac{2|E(H)|}{|V(H)|-1}$, and this follows immediately from the hypothesis. □

(ii) and (iii) Suppose that J is a subdivision of K_4 or a subdivision of a cycle with two chords with no common endpoint; for brevity, we write these two cases as K_4' and C_4'. Now we have $|V(H)| \in \{15,13,11,9,7,5\}$. Note that always $|E(J)| = |V(J)|+2$. Let T denote the four vertices of degree 3 in J. It is useful to observe that if at most four of the paths in J with endpoints in T have length 3, then Q has a dominating set of size four. For reference, we call this Fact 1. The idea of the proof is to choose an edge incident to each vertex in S such that we choose one edge from each path of length 3. It is straightforward to check that we can do this (up to symmetry, when J is K_4' we have only two possibilities for the paths of length 3 and when J is C_4' we have four possibilities).

Suppose $|V(H)| = 15$. Choose an arbitrary middle edge e of some path of length 3 and contract it. By Fact 1, $Q-e$ has a dominating set S of size 4, so $S\cup\{e\}$ is a dominating set of size 5 with $\sum_{v\in S\cup\{e\}} |N_Q(v)| \geq |V(Q)|+2$. Thus, we are done by Claim 1, with $j=5$, $k=1$, and $\ell=2$, since $\frac{2}{|V(H)|-1}=\frac{1}{7}$.

Suppose $|V(H)|=13$. By Fact 1, Q has a dominating set of size 4. We use Claim 1 with $j=4$, $k=1$, and $\ell\geq 0$. Cleary, $\frac{5}{24}\geq\frac{1}{6}$. Also $|E(H)|\geq\frac{1}{3}/(\frac{5}{24}-\frac{1}{6})=8$, so we are done.

Suppose $|V(H)|=11$. Again, by Fact 1, Q has a dominating set of size 4. Now we use Claim 1 with $j=4$, $k=1$, and $\ell\geq 3$. (That $\ell\geq 3$ comes from the fact that $|E(J)|=13$ but the sum of sizes of closed neighborhoods in Q is at least $4(4)=16$.)

Suppose $|V(H)|=9$. Now $|E(H)|=9+2=11$. So among paths in J joining vertices of T, the number with length 1 is one more than that with length 3. In particular, at least one such path has length 1 and at most two have length 3. We use these observations to find a dominating set S in Q of size 3. We choose one edge e on a path of length 1, say with endpoints v_1 and v_2, where $T=\{v_1,v_2,v_3,v_4\}$. Now we must also choose edges incident to v_3 and v_4 such that we choose one edge from each path of length 3. To succeed, we must choose e that (a) does not form a triangle with two paths of length 3, when $J=K_4'$, (b) does not share both endpoints with a path of length 3, when $J=C_4'$, and (c) does not share a common endpoint v_i with two paths of length 3, when $J=C_4'$. That we can choose such an e follows from the fact that J has more paths of length 1 than of length 3. Now we apply Claim 1 with $j=3$, $k=1$, and $\ell\geq 2$. (Note that $\ell\geq 5+4+4-|E(H)|=2$, since the size of each closed neighborhood in Q of an edge in S is at least 4 and that of e is 5.)

Suppose $|V(H)|=7$. Now the lengths of the paths, with multiplicity, are either (a) $2,2,2,1,1,1$ or (b) $3,2,1,1,1,1$. First assume that $J=C_4'$. Note that no two paths of length 1 can share both endpoints, since J would be non-simple, which is a contradiction. Suppose that we are in (a). Now there exist two paths of length 1 that do not share any endpoints, so the edges of these paths form a dominating set in Q; we apply Claim 1 with $j=2$, $k=1$, and $\ell=0$ (using that $|E(H)|\geq 4$). So now suppose that we are in (b). Now J contains a 7-cycle, which consists of the edges of all paths except for two of length 1. Now taking S to be the edges of the 7-cycle, we apply Claim 1 with $j=7$, $k=3$, and $\ell\geq 2$ (since each path of length 1 not in the 7-cycle has four incident edges on that cycle). This concludes the case $J=C_4'$.

Now assume that $J=K_4'$. First suppose that J contains a 7-cycle such that the two edges excluded from it each have four incident edges on the cycle. In this case we, we apply Claim 1 with $j=7$, $k=3$, and $\ell=2$. This covers all of (b), as well as the case of (a) when the three paths of length 2 together form a path of length 6. So assume instead that we are in (a) and either the three paths of length 2 have a common endpoint or they form a 6-cycle. In the former case, let S consist of the three edges of these paths not incident to their common endpoint. In the latter case, let S be the edges of the three paths of length 1. In each case we apply Claim 1 with $j=3$, $k=1$, and $\ell=3$.

Finally, suppose that $|V(H)|=5$. As above, no two paths of length 1 can share both endpoints, since J would be non-simple, which is a contradiction. Thus, we must have $J=K_4'$. Let e be the edge of J with no endpoint in common with the path of length 2, and let $S=E(J)-e$. Now we apply Claim 1 with $j=6$, $k=4$, and $\ell=0$ (and using $|E(H)|\geq 2$).

(iv) Suppose that H is a subdivision of a cycle with two chords with a common endpoint. In J, let u_1, u_2, u_3 denote the vertices of degree greater than two, and assume $d(u_1) = 4$ and $d(u_2) = d(u_3) = 3$. Let A and B denote paths from u_1 to u_2, let C and D denote paths from u_1 to u_3, and let E denote a path from u_2 to u_3. Recall that each path has length at most 3. Trivially, we can assume that $|V(H)| > 3$, since otherwise the line graph of H is a clique. Thus, by Lemma 23, we know that $|V(H)| \in \{13, 11, 9, 7, 5\}$. Let v_{a_1} denote a vertex of Q corresponding to a first edge on A in J; define $v_{b_i}, \ldots, v_{e_i}$ analogously.

Suppose $|V(H)| = 13$. Let $S = \{v_{a_1}, v_{b_2}, v_{c_2}, v_{d_3}, v_{e_1}\}$ and note that S is a dominating set in Q. Furthermore, both v_{b_3} and v_{c_3} have two neighbors in S, so $\sum_{v\in S} |N[v]| \geq |V(Q)|+2 = |E(H)| + 2$. Thus, $5(\frac{5}{6}(\Delta(Q) + 1) \geq \frac{5}{6}(|E(H)| + 2)$, so $\frac{5\Delta(Q)+3}{6} \geq \frac{|E(H)|}{6}$, which implies $\left\lceil \frac{5\Delta(Q)+3}{6} \right\rceil \geq \left\lceil \frac{|E(H)|}{6} \right\rceil \geq \chi(Q)$, as desired.

Suppose $|V(H)| \in \{11, 9\}$. First assume that J has four paths of length 3; since $|V(H)| \leq 11$, the fifth path has length 1. By considering an initial and a final edge on each path of length 3, we have $8(\frac{5}{6}(\Delta(Q) + 1)) \geq \frac{5}{6}(2|E(H)| + 2)$, so $\frac{5\Delta(Q)+3}{6} \geq \frac{5}{24}|E(H)| - \frac{3}{24} \geq \frac{|E(H)|}{5}$, since $|E(H)| \geq 15$. So assume instead that J has at most three paths of length 3.

Now we show that Q has a dominating set of size 3. Form a bipartite graph $\mathcal{B}$ with u_1, u_2, u_3 as one part and the paths of length three as the other part. It is easy to check that $\mathcal{B}$ has a matching saturating the paths of length three. This matching corresponds to a subset of Q. If the subset has size less than 3, then extend it to include one edge of H incident to each of u_1, u_2, u_3. Now, $3(\frac{5}{6}(\Delta(Q) + 1)) \geq \frac{5}{6}|E(H)|$, so $\frac{5\Delta(Q)+3}{6} \geq \frac{5}{18}|E(H)| - \frac{1}{3} \geq \frac{1}{4}|E(H)|$, so taking ceilings yields the desired bound (the final inequality holds since $|E(H)| \geq 12$, because $\frac{|E(H)|}{\lfloor \frac{|V(H)|}{2} \rfloor} > \Delta(H) \geq d(u_1) = 4$). $\square$