better bound for edges in 4-list-critical graphs

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Abstract

1 Introduction

For a graph G and disjoint $A, B \subseteq V(G)$, let $\|A, B\|$ be the number of edges between A and B.

Definition 1. The *maximum independent cover number* of a graph G is the maximum $\operatorname{mic}(G)$ of $\|I, V(G) \setminus I\|$ over all independent sets I of G.

Theorem 1.1. *Every OC-irreducible graph G satisfies*

$$\operatorname{mic}(G) \leq 2\,\|G\| - (\delta(G) - 1)\,|G| - 1.$$

2 initial improvement

Let G be OC-irreducible. Let $\mathcal{L}$ be the subgraph of G induced on the vertices of degree $\delta := \delta(G)$. Let $\mathcal{H}$ be $G - V(\mathcal{L})$. Let β be the maximum size of an independent set $A \subseteq V(\mathcal{L})$ such that each $v \in A$ has no neighbors in $V(\mathcal{H})$. Let $\operatorname{mic}_G(\mathcal{H})$ be the maximum of $\|I, V(G) \setminus I\|$ over all independent sets I fo G with $I \subseteq V(G) \setminus \mathcal{L}$. Then

Observation. $\operatorname{mic}(G) \geq \operatorname{mic}_G(\mathcal{H}) + \delta\beta$.

We need a couple bounds on $\|\mathcal{H}, \mathcal{L}\|$.

Observation. $\|\mathcal{H}, \mathcal{L}\| = \delta\,|\mathcal{L}| - 2\,\|L\|$.

Lemma 2.1. $\|\mathcal{H}, \mathcal{L}\| = \delta\,|\mathcal{H}| - 2\,\|\mathcal{H}\| + 2\,\|G\| - \delta\,|G|$.

Proof. $\|\mathcal{H}, \mathcal{L}\| = -2\,\|\mathcal{H}\| + \sum_{v \in V(\mathcal{H})} d_G(v) = \delta\,|\mathcal{H}| - 2\,\|\mathcal{H}\| + \sum_{v \in V(\mathcal{H})} (d_G(v) - \delta) = \delta\,|\mathcal{H}| - 2\,\|\mathcal{H}\| + \sum_{v \in V(G)} (d_G(v) - \delta)$. □

Lemma 2.2. *If T is a Gallai tree with max degree δ, not equal to K_δ, then*

$$2||T|| \leq (\delta - 1)|T| + 2\beta(T).$$

Lemma 2.3. $\|\mathcal{H}, \mathcal{L}\| \geq |\mathcal{L}| - 2\beta$.

Lemma 2.4.
$$2\|G\| \geq \delta |G| + |\mathcal{L}| + 2\|\mathcal{H}\| - \delta |\mathcal{H}| - 2\beta.$$

Lemma 2.5.
$$2\|G\| \geq (\delta - 1)|G| + \operatorname{mic}_G(\mathcal{H}) + \delta\beta + 1.$$

Lemma 2.6.
$$(2+\delta)(2\|G\|) \geq (\delta^2 + 3\delta - 2)|G| + 2\operatorname{mic}_G(\mathcal{H}) + 2 + 2\delta\|\mathcal{H}\| - \delta(\delta+1)|\mathcal{H}|.$$

Lemma 2.7. $\operatorname{mic}_G(\mathcal{H}) \geq \frac{\delta+1}{\delta}|\mathcal{H}|$.

Lemma 2.8. $\operatorname{mic}_G(\mathcal{H}) + \delta\|\mathcal{H}\| \geq (\delta+1)|\mathcal{H}|$.

Lemma 2.9.
$$(2+\delta)(2\|G\|) \geq (\delta^2 + 3\delta - 2)|G| + 2 - (\delta-2)(\delta+1)|\mathcal{H}|.$$

Lemma 2.10. $2\|G\| \geq \delta|G| + |\mathcal{H}|$.

Lemma 2.11.
$$(\delta + 2 + (\delta-2)(\delta+1))(2\|G\|) \geq (\delta^2 + 3\delta - 2 + \delta(\delta-2)(\delta+1))|G| + 2$$

Lemma 2.12.
$$d(G) > \delta + \frac{1}{\delta} - \frac{2}{\delta^2}$$