Conjectures that should be true*

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1 Edges in list-critical graphs

A graph G is *k-list-critical* if G is not $(k-1)$-choosable, but every proper subgraph of G is $(k-1)$-choosable. Replace '$(k-1)$-' with 'online $(k-1)$-' and 'k-' with '*online* k-' in the previous sentence and read it.

Conjecture 1. *Every incomplete k-list-critical graph has average degree at least*

$$k - 1 + \frac{k-3}{(k-1)^2}.$$

Background. The connected graphs in which each block is a complete graph or an odd cycle are called *Gallai trees.* Gallai [7] proved that in a k-critical graph, the vertices of degree $k-1$ induce a disjoint union of Gallai trees. The same is true for k-list-critical graphs [1, 6]. This quickly implies a lower bound on the average degree of k-list-critical graphs of

$$k - 1 + \frac{k-3}{k^2-3}.$$

In [14], R. improved this to

$$k - 1 + \frac{k-3}{k^2-2k+2}$$

using a lemma from Kierstead and R. [9] that generalizes a kernel technique of Kostochka and Yancey [10]. As noted at the end of [14], a small improvement to the argument would yield Conjecture 1. □

Conjecture 2. *Every incomplete online k-list-critical graph G has*

$$2\,\|G\| \geq (k-1)\,|G| + k - 3.$$

Background. Dirac [5] proved this for k-critical graphs. Kostochka and Stiebitz [11] proved it for k-list-critical graphs. Their proof does not seem to generalize. When $|G|$ is large compared with k, the conjecture holds by Gallai-type bounds on the average degree of online k-list-critical graphs [8, 2]. □

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2 Maximum degree, clique number and colorings

2.1 Around Borodin-Kostochka

Conjecture 3. *Every graph with $\chi \geq \Delta \geq 8$ contains a $K_3 \vee H$ where H is some graph on $\Delta - 3$ vertices.*

Background. By results in [4], for $\Delta \geq 9$ the existence of $K_3 \vee H$ implies the existence of K_Δ. So, this (seemingly weaker) conjecture for $\Delta \geq 9$ implies the Borodin-Kostochka conjecture. The one known connected counterexample to the Borodin-Kostochka conjecture for $\Delta = 8$ is a 5-cycle with each vertex blown up to a triangle. This graph is not a counterexample to Conjecture 3. $\square$

Conjecture 4. *Every graph with $\chi \geq \Delta$ contains $K_{\Delta-3}$.*

Background. Results in [3] show that this holds with $K_{\Delta-4}$ instead of $K_{\Delta-3}$. Moreover, [3] proves the conjecture for all but $\Delta \in \{6, 8, 9, 11, 12\}$. $\square$

Conjecture 5. *Every graph with $\chi \geq \Delta$ either contains K_Δ or contains a $K_{\Delta-4}$ with all Δ-vertices.*

Background. Results in [3] show that this holds with $K_{\Delta-5}$ instead of $K_{\Delta-4}$. For $\Delta \leq 7$, the conjecture holds by [13, 12]. Also by [3], it holds when $\Delta = 3r + 1$ for $r \geq 3$. $\square$

Conjecture 6. *Every graph with $\Delta \geq 8$ and $\omega < \Delta$ is 2-fold $(2\Delta - 1)$-colorable.*

Background. The one known connected counterexample to the Borodin-Kostochka conjecture for $\Delta = 8$ is a 5-cycle with each vertex blown up to a triangle. This graph is not a counterexample to Conjecture 6. $\square$

Conjecture 7. *Every graph with $\theta \geq 10$ and $\omega \leq \frac{\theta}{2}$ is $\left\lfloor \frac{\theta}{2} \right\rfloor$-choosable.*

Background. $\square$

Conjecture 8. *Every claw-free graph with $\Delta \geq 9$ and $\omega < \Delta$ is $(\Delta - 1)$-choosable.*

Background. $\square$

Conjecture 9. *There is a polynomial time graph algorithm that finds either a $(\Delta - 1)$-coloring or a $K_{\Delta-3}$.*

Background. $\square$

2.2 The $\frac{5}{6}$ bound

Conjecture 10. *If G is vertex-transitive, then $\chi(G) \leq \max\left\{\omega(G), \left\lceil \frac{5\Delta(G)+3}{6} \right\rceil\right\}$.*

Background. $\square$

Conjecture 11. *If G is the line graph of a multigraph, then $\chi(G) \leq \max\left\{\omega(G), \left\lceil \frac{5\Delta(G)+3}{6} \right\rceil\right\}$.*

Background. $\square$

3 Planar graphs

Conjecture 12. *Every planar graph with no K_5-subdivision is 2-fold 9-colorable.*

Background. □

References

[1] O.V. Borodin, *Criterion of chromaticity of a degree prescription*, Abstracts of IV All-Union Conf. on Th. Cybernetics, 1977, pp. 127–128.

[2] D. Cranston and L. Rabern, *Edge lower bounds for list critical graphs, via discharging*, arXiv:1602.02589 (2016).

[3] Daniel W. Cranston and Landon Rabern, *Graphs with $\chi=\Delta$ Have Big Cliques*, SIAM Journal on Discrete Mathematics **29** (2015), no. 4, 1792–1814.

[4] D.W. Cranston and L. Rabern, *Conjectures equivalent to the Borodin-Kostochka Conjecture that appear weaker*, Arxiv preprint arXiv:1203.5380 (2012).

[5] G.A. Dirac, *A theorem of R.L. Brooks and a conjecture of H. Hadwiger*, Proceedings of the London Mathematical Society **3** (1957), no. 1, 161–195.

[6] P. Erdős, A.L. Rubin, and H. Taylor, *Choosability in graphs*, Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium, vol. 26, 1979, pp. 125–157.

[7] T. Gallai, *Kritische Graphen I.*, Publ. Math. Inst. Hungar. Acad. Sci **8** (1963), 165–192 (in German).

[8] H.A. Kierstead and L. Rabern, *Improved lower bounds on the number of edges in list critical and online list critical graphs*, arXiv preprint arXiv:1406.7355 (2014).

[9] ———, *Extracting list colorings from large independent sets*, arXiv:1512.08130 (2015).

[10] Alexandr Kostochka and Matthew Yancey, *Ore's conjecture on color-critical graphs is almost true*, J. Combin. Theory Ser. B **109** (2014), 73–101. MR 3269903

[11] Alexandr V Kostochka and Michael Stiebitz, *A list version of dirac's theorem on the number of edges in colour-critical graphs*, Journal of Graph Theory **39** (2002), no. 3, 165–177.

[12] A.V. Kostochka, L. Rabern, and M. Stiebitz, *Graphs with chromatic number close to maximum degree*, Discrete Mathematics **312** (2012), no. 6, 1273–1281.

[13] L. Rabern, *Δ-critical graphs with small high vertex cliques*, Journal of Combinatorial Theory, Series B **102** (2012), no. 1, 126–130.

[14] Landon Rabern, *A better lower bound on average degree of 4-list-critical graphs*, arXiv:1602.08532 (2016).