notes on the Borodin-Kostochka conjecture

May 10, 2017

1 Introduction

Conjecture 1 (Borodin and Kostochka [3])**.** *Every graph G with $\Delta(G) \geq 9$ satisfies $\chi(G) \leq \max\{\omega(G), \Delta(G) - 1\}$.*

2 Techniques

Catlin/Mozhan shuffling, independent transversals and strong coloing, d_1-choosables, mules, fractional coloring, probabilistic method, kernels, Kostochka's method

claw-free, doubly-critical edges, squares, vertex-transitive

3 Excluded induced subgraphs by d_1-choosability

A graph G is *d_r-choosable* if G can be L-colored from every list assingment L with $|L(v)| \geq d_G(v) - r$ for all $v \in V(G)$. Every graph is d_{-1}-choosable. The d_0-choosable graphs were classified by Borodin [2] and independently by Erdős, Rubin, and Taylor [10] as those graphs whose every block is either complete or an odd cycle (a connected such graph is a *Gallai tree*). Classifying the d_r-choosable graphs for any $r \geq 1$ appears to be a hard problem. However, we can get useful sufficient conditions for a graph to be d_1-choosable. For example, all of the graphs here are d_1-choosable (the vertex color indicates components of the complement): `https://landon.github.io/graphdata/borodinkostochka/offline/index.html`

Cranston and Rabern [7] classified all d_1-choosable graphs of the form $A \vee B$.

4 Decompositions

4.1 Reed's decomposition

In [16], Reed proved the Borodin-Kostochka conjecture for graphs G with $\Delta(G) \geq 10^{14}$. A piece of that proof was a decomposition of G into dense chunks and one sparse chunk that also works for smaller $\Delta(G)$. The following tight form of this decomposition is given in [15]. Let $\mathcal{C}_t(G)$ be the maximal cliques in G having at least t vertices.

Reed's Decomposition. *Suppose G is a graph with $\Delta(G) \geq 8$ that contains no $K_{\Delta(G)}$ and has no d_1-choosable induced sugraph. If $\frac{\Delta(G)+5}{2} \leq t \leq \Delta(G) - 1$, then $\bigcup \mathcal{C}_t(G)$ can be partitioned into sets $D_1, \ldots, D_r$ such that for each $i \in [r]$ at least one of the following holds:*

1. *$D_i = C_i \in \mathcal{C}_t(G)$,*

2. *$D_i = C_i \cup \{x_i\}$ where $C_i \in \mathcal{C}_t(G)$ and $|N(x_i) \cap C_i| \geq t - 1$.*

4.2 Fajtlowicz's decomposition

In [11], Fajtlowicz proved that every graph has $\alpha(G) \geq \frac{2|G|}{\omega(G)+\Delta(G)+1}$. The proof of this result gives a decomposition which we state in the special case needed for the Borodin-Kostochka conjecture.

Fajtlowicz's Decomposition. *Suppose G is a vertex-critical graph with $\chi(G) = \Delta(G)$. Then $V(G)$ can be partitioned into sets M, T, and K such that*

1. *M contains a maximum independent set I of G; and*

2. *each $v \in T$ has $d_G(v) = \Delta(G)$, two neighbors in I and zero neighbors in $M \setminus I$; and*

3. *K can be covered by $\alpha(G)$ (or fewer) cliques; and*

4. *each $v \in K$ has exactly one neighbor in I and at most one neighbor in $M \setminus I$ (none if $d_G(v) < \Delta(G)$); and*

5. *the vertices in $M \setminus I$ can be ordered $v_1, \ldots, v_r$ such that for $i \in [r]$, either v_i has at least three neighbors in $I \cup \{v_1, \ldots, v_{i-1}\}$ or $d_G(v_i) < \Delta(G)$ and v_i has at least two neighbors in $I \cup \{v_1, \ldots, v_{i-1}\}$.*

Proof. Let I be a maximum independent set in G. Construct a maximal length sequence $I = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r$ such that for $j > 0$,

- every $v \in M_j$ with $d_G(v) = \Delta(G)$ either has at least three neighbors in M_{j-1} or at least two neighbors in $M_{j-1} \setminus I$; and

- every $v \in M_j$ with $d_G(v) = \Delta(G) - 1$ either has at least two neighbors in M_{j-1} or at least one neighbor in $M_{j-1} \setminus I$.

Now let $M = M_r$, let T be the vertices in $V(G) \setminus M$ with exactly two neighbors in I and let K be the vertices in $V(G) \setminus M$ with exactly one neighbor in I. The decomposition has the properties 1,2,4 and 5 since the sequence $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r$ was chosen to be maximal length. Property 3 follows since for each $v \in I$, the set of $x \in K$ adjacent to v must be a clique for otherwise we could get an independent set larger than I. $\square$

5 Results from strong coloring

In [15], using ideas from strong coloring [12, 1], Rabern showed that any counterexample to the Borodin-Kostochka conjecture must have some sparse neighborhood and large independence number.

Theorem 2. *Every graph G with $\chi(G) \geq \Delta(G) \geq 9$ such that every vertex is in a clique on $\frac{2}{3}\Delta(G) + 2$ vertices contains $K_{\Delta(G)}$.*

Theorem 3. *Every graph G with $\omega(G) < \Delta(G)$ such that $d(G[N(v)]) \geq \frac{2}{3}\Delta(G) + 4$ for each $v \in V(G)$ is $(\Delta(G) - 1)$-colorable.*

Theorem 4. *Every graph G satisfies $\chi(G) \leq \max\{\omega(G), \Delta(G) - 1, 4\alpha(G)\}$.*

In the next subsection, we improve Theorem 2 and Theorem 3 slightly.

5.1 An improvement

The proof of Theorem 2 in [15] uses techniques developed for strong coloring. Here we show how to use the best known bounds for strong coloring directly and hence get a small improvement.

For a positive integer r, a graph G with $|G| = rk$ is called *strongly r-colorable* if for every partition of $V(G)$ into parts of size r there is a proper coloring of G that uses all r colors on each part. If $|G|$ is not a multiple of r, then G is strongly r-colorable iff the graph formed by adding $r\left\lceil\frac{|G|}{r}\right\rceil - |G|$ isolated vertices to G is strongly r-colorable. The *strong chromatic number* $s\chi(G)$ is the smallest r for which G is strongly r-colorable.

Note that a strong r-coloring of G with respect to a partition $V_1, \dots, V_k$ of $V(G)$ with $|V_i| = r$ must partition $V(G)$ into r independent transversals of $V_1, \dots, V_k$. In [17], Szabó and Tardos constructed partitioned graphs with part sizes $2\Delta - 1$ that have no independent transversal. So we must have $s\chi(G) \geq 2\Delta(G)$. That the upper bound $s\chi(G) \leq 2\Delta(G)$ holds is the *strong coloring conjecture*.

Haxell [12] proved that $s\chi(G) \leq 3\Delta(G) - 1$.

Theorem 5. *Every graph G with $\chi(G) \geq \Delta(G) \geq 8$ such that every vertex is in a clique on $\frac{2}{3}\Delta(G) + 1$ vertices contains $K_{\Delta(G)}$.*

Proof. Suppose the theorem does not hold and let G be a counterexample minimizing $|G|$. Apply Reed's decomposition with $t := \left\lceil\frac{2}{3}\Delta(G)\right\rceil + 1$ to get sets $D_1, \dots, D_r$ with $\bigcup_{i\in[r]} D_i = V(G)$. Create G' from G by removing all the edges in $G[D_i]$ for each $i \in [r]$. By Haxell's bound, $s\chi(G') \leq 3\Delta(G') - 1 \leq 3(\Delta(G) - (t-1)) - 1 \leq \Delta(G) - 1$. If $|D_i| \leq \Delta(G) - 1$, this gives a $(\Delta(G) - 1)$-coloring of G, a contradiction. By symmetry, we may assume $|D_1| \geq \Delta(G)$. Then it must be that $D_1 = C_1 \cup \{x\}$ where C_1 is a $K_{\Delta(G)-1}$ and x has at least $\left\lceil\frac{2}{3}\Delta(G)\right\rceil$ neighbors in C_1. □

If the strong coloring conjecture holds, we get the following improvement.

Conjecture 6. *Every graph G with $\chi(G) \geq \Delta(G) \geq 8$ such that every vertex is in a clique on $\frac{\Delta(G)+5}{2}$ vertices contains $K_{\Delta(G)}$.*

6 Properties of minimum counterexamples

In [8] Cranston and Rabern used the d_1-choosable graphs in Section 3 to prove properties of a minimum counterexample to the Borodin-Kostochka conjecture. For example, the following improves a lemma Reed used in his proof [16].

Lemma 7. *Let G be a minimum counterexample to the Borodin-Kostochka conjecture. If X is a $K_{\Delta(G)-1}$ in G, then every $v \in V(G-X)$ has at most one neighbor in X.*

Lemma 8. *Let G be a minimum counterexample to the Borodin-Kostochka conjecture. Let A and B be disjoint subgraphs of G with $|A| + |B| = \Delta(G)$ such that $|A|, |B| \geq 3$. If G contains all edges between A and B, then $A = K_1 + K_{|A|-1}$ and $B = K_1 + K_{|B|-1}$.*

7 Results from kernel methods

In [13], Kierstead and Rabern proved a general lemma that allows the user to get list colorings for free from large independent sets. Specialized to the Borodin-Kostochka conjecture, this becomes.

Kernel Magic. *Suppose G is a vertex-critical graph with $\chi(G) = \Delta(G)$. For every induced subgraph H of G and independent set I in H, we have*

$$\sum_{v \in I} d_H(v) < \sum_{v \in V(H)} \Delta(G) + 2 - d_G(v).$$

Applied with $H = G$, this gives:

Corollary 9. *If G is a vertex-critical graph with $\chi(G) = \Delta(G)$, then $\alpha(G) < \frac{2|G|}{\Delta(G)-1}$.*

8 Mozhan partitions

Extending ideas of Mozhan [14], Cranston and Rabern [9] proved the following.

Theorem 10. *If G is a vertex-critical graph with $\chi(G) = \Delta(G) \geq 13$, then $\omega(G) \geq \Delta(G) - 3$.*

9 Vertex-transitive graphs

In [5] Cranston and Rabern used Reed's decomposition and the ideas in Sections 5 and 8 to prove the Borodin-Kostochka conjecture for vertex-transitive graphs with $\Delta(G) \geq 13$. It would be interesting to improve this to $\Delta(G) \geq 9$.

Theorem 11. *Every vertex-transitive graph G with $\Delta(G) \geq 13$ satisfies $\chi(G) \leq \max\{\omega(G), \Delta(G) - 1\}$.*

10 Claw-free graphs

In [6], Cranston and Rabern proved the Borodin-Kostochka conjecture for claw-free graphs using some of the d_1-choosable graphs in Section 3 combined with the structure theorem for quasi-line graphs of Chudnovsky and Seymour [4].

References

[1] R. Aharoni, E. Berger, and R. Ziv. Independent systems of representatives in weighted graphs. *Combinatorica*, 27(3):253–267, 2007. 3

[2] O.V. Borodin. Criterion of chromaticity of a degree prescription. In *Abstracts of IV All-Union Conf. on Th. Cybernetics*, pages 127–128, 1977. 1

[3] O.V. Borodin and A.V. Kostochka. On an upper bound of a graph's chromatic number, depending on the graph's degree and density. *Journal of Combinatorial Theory, Series B*, 23(2-3):247–250, 1977. 1

[4] M. Chudnovsky and P. Seymour. The structure of claw-free graphs. *Surveys in combinatorics*, 327:153–171, 2005. 4

[5] Daniel W Cranston and Landon Rabern. A note on coloring vertex-transitive graphs. *arXiv preprint arXiv:1404.6550*, 2014. 4

[6] D.W. Cranston and L. Rabern. Coloring claw-free graphs with $\Delta - 1$ colors. *Arxiv preprint arXiv:1206.1269*, 2012. 4

[7] D.W. Cranston and L. Rabern. Conjectures equivalent to the Borodin-Kostochka Conjecture that appear weaker. *Arxiv preprint arXiv:1203.5380*, 2012. 1

[8] D.W. Cranston and L. Rabern. Conjectures equivalent to the Borodin-Kostochka conjecture that appear weaker. *Arxiv preprint arXiv:1203.5380*, 2012. 4

[9] D.W. Cranston and L. Rabern. Graphs with $\chi = \Delta$ have big cliques. *arXiv preprint* *http://arxiv.org/abs/1305.3526*, 2013. 4

[10] P. Erdős, A.L. Rubin, and H. Taylor. Choosability in graphs. In *Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium*, volume 26, pages 125–157, 1979. 1

[11] S. Fajtlowicz. Independence, clique size and maximum degree. *Combinatorica*, 4(1):35–38, 1984. 2

[12] P. Haxell. On the strong chromatic number. *Combinatorics, Probability and Computing*, 13(06):857–865, 2004. 3

[13] H.A. Kierstead and L. Rabern. Extracting list colorings from large independent sets. *arXiv:1512.08130*, 2015. 4

[14] N.N. Mozhan. Chromatic number of graphs with a density that does not exceed two-thirds of the maximal degree. *Metody Diskretn. Anal.*, 39:52–65 (in Russian), 1983. 4

[15] L. Rabern. Coloring graphs with dense neighborhoods. *Journal of Graph Theory*, 2013. 1, 3

[16] B. Reed. A strengthening of Brooks' theorem. *Journal of Combinatorial Theory, Series B*, 76(2):136–149, 1999. 1, 4

[17] T. Szabó and G. Tardos. Extremal problems for transversals in graphs with bounded degree. *Combinatorica*, 26(3):333–351, 2006. 3