better bound for edges in 4-list-critical graphs

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Abstract

1 Introduction

For a graph G and disjoint $A, B \subseteq V(G)$, let $\|A, B\|$ be the number of edges between A and B.

Definition 1. The *maximum independent cover number* of a graph G is the maximum $\operatorname{mic}(G)$ of $\|I, V(G) \setminus I\|$ over all independent sets I of G.

Theorem 1.1. *Every k-list-critical graph G satisfies*

$$2\|G\| \geq (k-2)|G| + \operatorname{mic}(G) + 1.$$

2 The Bound

Theorem 2.1. *For $k \geq 4$, every incomplete k-list-critical graph has average degree at least $k - 1 + \frac{k-3}{(k-1)^2}$.*

Proof. Let $G \neq K_k$ be a k-list-critical graph. Let $\mathcal{L} \subseteq V(G)$ be the vertices with degree $k-1$ and let $\mathcal{H} = V(G) \setminus \mathcal{L}$. Put $\|\mathcal{L}\| := \|G[\mathcal{L}]\|$ and $\|\mathcal{H}\| := \|G[\mathcal{H}]\|$. Then

$$\|\mathcal{H}, \mathcal{L}\| = (k-1)|\mathcal{L}| - 2\|\mathcal{L}\|. \tag{1}$$

By Lemma 2.2,

$$2\|\mathcal{L}\| \leq (k-2)|\mathcal{L}| + 2\beta(\mathcal{L}) \tag{2}$$

Combining 1 and 2 gives

$$\|\mathcal{H}, \mathcal{L}\| \geq |\mathcal{L}| - 2\beta(\mathcal{L}). \tag{3}$$

Also,

$$\begin{aligned}\|\mathcal{H},\mathcal{L}\| &= -2\,\|\mathcal{H}\| + \sum_{v\in\mathcal{H}} d_G(v)\\ &= (k-1)\,|\mathcal{H}| - 2\,\|\mathcal{H}\| + \sum_{v\in\mathcal{H}} (d_G(v)-(k-1))\\ &= (k-1)\,|\mathcal{H}| - 2\,\|\mathcal{H}\| + \sum_{v\in V(G)} (d_G(v)-(k-1))\\ &= (k-1)\,|\mathcal{H}| - 2\,\|\mathcal{H}\| + 2\,\|G\| - (k-1)\,|G|\,,\end{aligned}$$

that is

$$\|\mathcal{H},\mathcal{L}\| = (k-1)\,|\mathcal{H}| - 2\,\|\mathcal{H}\| + 2\,\|G\| - (k-1)\,|G|\,. \tag{4}$$

Combining 3 with 4 gives

$$2\,\|G\| \geq (k-1)\,|G| + |\mathcal{L}| + 2\,\|\mathcal{H}\| - (k-1)\,|\mathcal{H}| - 2\beta(\mathcal{L}).$$

Since $|G| = |\mathcal{L}| + |\mathcal{H}|$, this is

$$2\,\|G\| \geq (k-1)\,|G| + 2\,\|\mathcal{H}\| - k\,|\mathcal{H}| - 2\beta(\mathcal{L}). \tag{5}$$

Let M be the maximum of $\|I, V(G)\setminus I\|$ over all independent sets I of G with $I\subseteq\mathcal{H}$. Then

$$\text{mic}(G) \geq M + (k-1)\beta(\mathcal{L}).$$

Applying Lemma 1.1 gives

$$2\,\|G\| \geq (k-2)\,|G| + M + (k-1)\beta(\mathcal{L}) + 1. \tag{6}$$

Adding twice 5 to $k-1$ times 6 gives

$$(k+1)(2\,\|G\|) \geq ((k-1)^2 + 2(k-2))\,|G| + 2M + 2 + 2(k-1)\,\|\mathcal{H}\| - k(k-1)\,|\mathcal{H}|\,.$$

Hence

$$2\,\|G\| \geq \frac{k^2-3}{k+1}\,|G| + \frac{2(M+(k-1)\,\|\mathcal{H}\|+1) - k(k-1)\,|\mathcal{H}|}{k+1}. \tag{7}$$

Let $\mathcal{C}$ be the components of $G[\mathcal{H}]$. Then $\alpha(C) \geq \frac{|C|}{\chi(C)}$ for all $C\in\mathcal{C}$. Whence

$$M + (k-1)\,\|\mathcal{H}\| \geq \sum_{C\in\mathcal{C}} k\frac{|C|}{\chi(C)} + (k-1)\,\|C\| \tag{8}$$

If $\mathcal{L}=\emptyset$, then G has average degree at least $k \geq k-1+\frac{k-3}{(k-1)^2}$. So, assume $\mathcal{L}\neq\emptyset$. Then $G[\mathcal{H}]$ is $(k-1)$-colorable by k-list-criticality of G. In particular, $\chi(C)\leq k-1$ for every $C\in\mathcal{C}$. If $C\in\mathcal{C}$ is not a tree, then $\|C\|\geq|C|$ and hence $k\frac{|C|}{\chi(C)}+(k-1)\,\|C\|\geq k\,|C|$. If C is a tree, then $\chi(C)\leq 2$ and hence $k\frac{|C|}{\chi(C)}+(k-1)\,\|C\|\geq k\frac{|C|}{2}+(k-1)(|C|-1)\geq k\,|C|$ when $|C|\geq\frac{2(k-1)}{k-2}$. But $k\geq 4$, so $\frac{2(k-1)}{k-2}\leq 3$

□

Lemma 2.2. *If $k\geq 4$ and $T\neq K_k$ is a Gallai tree with maximum degree at most $k-1$, then*

$$2||T|| \leq (k-2)|T| + 2\beta(T).$$

3 notes

Let G be OC-irreducible. Let $\mathcal{L}$ be the subgraph of G induced on the vertices of degree $\delta := \delta(G)$. Let $\mathcal{H}$ be $G - V(\mathcal{L})$. Let β be the maximum size of an independent set $A \subseteq V(\mathcal{L})$ such that each $v \in A$ has no neighbors in $V(\mathcal{H})$. Let $\text{mic}_G(\mathcal{H})$ be the maximum of $\|I, V(G) \setminus I\|$ over all independent sets I fo G with $I \subseteq V(G) \setminus \mathcal{L}$. Then

Observation. $\text{mic}(G) \geq \text{mic}_G(\mathcal{H}) + \delta\beta$.

We need a couple bounds on $\|\mathcal{H}, \mathcal{L}\|$.

Observation. $\|\mathcal{H}, \mathcal{L}\| = \delta\left|\mathcal{L}\right| - 2\left\|L\right\|$.

Lemma 3.1. $\|\mathcal{H}, \mathcal{L}\| = \delta\left|\mathcal{H}\right| - 2\left\|\mathcal{H}\right\| + 2\left\|G\right\| - \delta\left|G\right|$.

Proof. $\|\mathcal{H}, \mathcal{L}\| = -2\left\|\mathcal{H}\right\| + \sum_{v \in V(\mathcal{H})} d_G(v) = \delta\left|\mathcal{H}\right| - 2\left\|\mathcal{H}\right\| + \sum_{v \in V(\mathcal{H})}\left(d_G(v) - \delta\right) = \delta\left|\mathcal{H}\right| - 2\left\|\mathcal{H}\right\| + \sum_{v \in V(G)}\left(d_G(v) - \delta\right)$. □

Lemma 3.2. *If T is a Gallai tree with max degree δ, not equal to K_δ, then*

$$2||T|| \leq (\delta - 1)|T| + 2\beta(T).$$

Lemma 3.3. $\|\mathcal{H}, \mathcal{L}\| \geq \left|\mathcal{L}\right| - 2\beta$.

Lemma 3.4.

$$2\left\|G\right\| \geq \delta\left|G\right| + \left|\mathcal{L}\right| + 2\left\|\mathcal{H}\right\| - \delta\left|\mathcal{H}\right| - 2\beta.$$

Lemma 3.5.

$$2\left\|G\right\| \geq (\delta - 1)\left|G\right| + \text{mic}_G(\mathcal{H}) + \delta\beta + 1.$$

Lemma 3.6.

$$(2 + \delta)(2\left\|G\right\|) \geq (\delta^2 + 3\delta - 2)\left|G\right| + 2\,\text{mic}_G(\mathcal{H}) + 2 + 2\delta\left\|\mathcal{H}\right\| - \delta(\delta + 1)\left|\mathcal{H}\right|.$$

Lemma 3.7. $\text{mic}_G(\mathcal{H}) \geq \frac{\delta+1}{\delta}\left|\mathcal{H}\right|$.

Lemma 3.8. $\text{mic}_G(\mathcal{H}) + \delta\left\|\mathcal{H}\right\| \geq (\delta + 1)\left|\mathcal{H}\right|$.

Lemma 3.9.

$$(2 + \delta)(2\left\|G\right\|) \geq (\delta^2 + 3\delta - 2)\left|G\right| + 2 - (\delta - 2)(\delta + 1)\left|\mathcal{H}\right|.$$

Lemma 3.10. $2\left\|G\right\| \geq \delta\left|G\right| + \left|\mathcal{H}\right|$.

Lemma 3.11.

$$(\delta + 2 + (\delta - 2)(\delta + 1))(2\left\|G\right\|) \geq (\delta^2 + 3\delta - 2 + \delta(\delta - 2)(\delta + 1))\left|G\right| + 2$$

Lemma 3.12.

$$d(G) > \delta + \frac{1}{\delta} - \frac{2}{\delta^2}$$