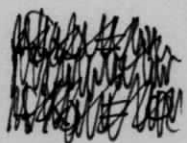


Is there a function that is its own derivative?

• yes $f(x) = 0$.

an interesting one?

We really only know how to differentiate polynomials
(and fractions of them)



$$f(x) = 1 + x + \frac{1}{2}x^2 \text{ here}$$

$$f'(x) = 1 + x$$

needs x here, so

but then needs $\frac{1}{2}x^2$

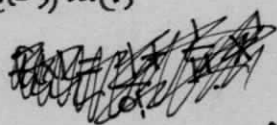
$$f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3} \cdot \frac{1}{2}x^3$$

$$f'(x) = 1 + x + \frac{1}{2}x^2$$

Can we keep going?

$$f(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \frac{1}{5 \cdot 4 \cdot 3 \cdot 2}x^5 + \dots$$

Notation: $n! = n(n-1)(n-2)(n-3)\dots(1)$
 $0! = 1$



$$f(x) = \frac{1}{0!}x^0 + \frac{1}{1!}x^1 + \frac{1}{2!}x^2 + \frac{1}{3!}x^3 + \frac{1}{4!}x^4 + \dots$$

what is this f ?

$$f(0) = 1$$

$$f(1) = 1 + 1 + \frac{1}{2} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3 \cdot 2} + \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} + \dots$$

$$= 2 + \frac{1}{2} + \frac{1}{6} + \frac{1}{24} + \frac{1}{120} + \frac{1}{720} + \frac{1}{5040} + \dots$$

$$\approx 2.7182$$

Let's call this number e .

$$e = 1 + \frac{1}{2} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3 \cdot 2} + \frac{1}{5 \cdot 4 \cdot 3 \cdot 2} + \frac{1}{6 \cdot 5 \cdot 4 \cdot 3 \cdot 2} + \dots$$

exponential functions

$$2^n = \overbrace{2 \cdot 2 \cdot 2 \cdots 2}^n$$

$$2^{1/n} = \sqrt[n]{2}$$

$$2^{a/b} = \sqrt[b]{2^a}$$

want to define 2^x for all x .

$$2^a \cdot 2^b = 2^{a+b}$$

$$(2^a)^b = 2^{ab} \quad \text{etc.} \quad \leftarrow \text{proofs}$$

2^x when x is not rational?

$2^x =$ approximate x by better and better rationals $\frac{a}{b}$ and compute $2^{a/b}$...

Can change the base to whatever, say a .

What is $f'(x)$?

$$f(x) = a^x \quad f'(x) = \lim_{h \rightarrow 0} \frac{a^{x+h} - a^x}{h} = \lim_{h \rightarrow 0} \frac{a^x \cdot a^h - a^x}{h} = \lim_{h \rightarrow 0} \frac{a^x(a^h - 1)}{h} = a^x \cdot \lim_{h \rightarrow 0} \frac{a^h - 1}{h}$$

So, if $\lim_{h \rightarrow 0} \frac{a^h - 1}{h} = 1$, then

$$f'(x) = f(x).$$

if you try to find the a for which this holds, you get $a \approx 2.7182$.

Can prove that $a = 1 + \frac{1}{2} + \frac{1}{3 \cdot 2} + \frac{1}{4 \cdot 3 \cdot 2} + \dots$ works!
that is, $a = e$.

interesting
two functions that are their own derivative

$$f(x) = \cancel{e^x} = e^x$$

$$\text{and } g(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \dots$$

$$f(1) = e$$

$$f'(x) = f(x)$$

$$f'(1) = f(1) = e$$

$$g(1) = e$$

$$g'(x) = g(x), \text{ so}$$

$$g'(1) = g(1) = e$$

$$f''(1) = e$$

$$f''(1) = e$$

$$g''(1) = e$$

$$g''(1) = e$$

at $x=1$ f and g have same value and same values for all derivatives
change, change in change, ... all same must be $f=g$.

$$e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3 \cdot 2}x^3 + \frac{1}{4 \cdot 3 \cdot 2}x^4 + \dots$$