

# most low Alon-Tarsi notes

August 12, 2015

## 1 Introduction

We consider graphs with vertices labeled by natural numbers; that is, pairs  $(G, h)$  where  $G$  is a graph and  $h: V(G) \rightarrow \mathbb{N}$ . We say that  $(G, h)$  is AT if  $G$  is  $(d_G - h)$ -AT.

## 2 Subgraphs, subdivisions and cuts

**Lemma 2.1.** *If  $G$  is connected and  $(G, h)$  is not AT, then  $(H, h|_{V(H)})$  is not AT for every induced subgraph  $H$  of  $G$  where  $h(v) = 0$  for all  $v \in V(G) \setminus V(H)$ .*

*Proof.* If there were such an  $H$  such that  $(H, h|_{V(H)})$  is AT, then by ordering the vertices of each component of  $G - V(H)$  by increasing distance to  $H$  and directing all edges away from  $H$  in this order we conclude that  $(G, h)$  is AT.  $\square$

**Lemma 2.2.** *For any  $(G', h')$  formed from  $(G, h)$  by subdividing an edge of  $G$  twice and letting  $h'$  give zero on the new vertices,  $(G, h)$  is AT if and only if  $(G', h')$  is AT.*

*Proof.* This is immediate since there is a parity preserving bijection between the spanning Eulerian subgraphs of  $G$  and  $G'$ .  $\square$

**Lemma 2.3.** *Let  $\{A_1, A_2\}$  be a separation of  $G$  such that  $A_1 \cap A_2 = \{x\}$ . If  $G[A_i]$  is  $f_i$ -AT for  $i \in [2]$ , then  $G$  is  $f$ -AT where  $f(v) = f_i(v)$  for  $v \in V(A_i - x)$  and  $f(x) = f_1(x) + f_2(x) - 1$ . Going the other direction, if  $G$  is  $f$ -AT, then  $G[A_i]$  is  $f_i$ -AT for  $i \in [2]$  where  $f_i(v) = f(v)$  for  $v \in V(A_i - x)$  and  $f_1(x) + f_2(x) \leq f(x) + 1$ .*

*Proof.* For  $i \in [2]$ , choose an orientation  $D_i$  of  $A_i$  showing that  $A_i$  is  $f_i$ -AT. Together these give an orientation  $D$  of  $G$  and since no cycle has vertices in both  $A_1 - x$  and  $A_2 - x$ , we have

$$\begin{aligned} EE(D) - EO(D) &= EE(D_1)EE(D_2) + EO(D_1)EO(D_2) - (EE(D_1)EO(D_2) + EO(D_1)EE(D_2)) \\ &= (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2)) \\ &\neq 0. \end{aligned}$$

Hence  $G$  is  $f$ -AT.

Now, suppose  $G$  is  $f$ -AT and choose an orientation  $D$  of  $G$  showing this. Put  $D_i = D[A_i]$  for  $i \in [2]$ . Then, as above, we have  $0 \neq EE(D) - EO(D) = (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2))$  and hence  $EE(D_1) - EO(D_1) \neq 0$  and  $EE(D_2) - EO(D_2) \neq 0$ . Since the in-degree of  $x$  in  $D$  is the sum of the in-degree of  $x$  in  $D_1$  and the in-degree of  $x$  in  $D_2$ , the lemma follows.  $\square$

### 3 Extension lemma

This is a key lemma from [1], it generalizes a lemma from [2] from list coloring to Alon-Tarsi orientations. This is what I talked about in Baltimore. The basic idea is that in some cases we can pair off odd/even spanning Eulerian subgraphs via a parity reversing bijection.

**Lemma 3.1.** *Let  $G$  be a multigraph without loops and  $f: V(G) \rightarrow \mathbb{N}$ . If there are  $F \subseteq G$  and  $Y \subseteq V(G)$  such that:*

1. *any multiple edges in  $G$  are contained in  $G[Y]$ ; and*
2.  *$f(v) \geq d_G(v)$  for all  $v \in V(G) \setminus Y$ ; and*
3.  *$f(v) \geq d_{G[Y]}(v) + d_F(v) + 1$  for all  $v \in Y$ ; and*
4. *For each component  $T$  of  $G - Y$  there are different  $x_1, x_2 \in V(T)$  where  $N_T[x_1] = N_T[x_2]$  and  $T - \{x_1, x_2\}$  is connected such that either:*
  - (a) *there are  $x_1y_1, x_2y_2 \in E(F)$  where  $y_1 \neq y_2$  and  $N(x_i) \cap Y = \{y_i\}$  for  $i \in [2]$ ; or*
  - (b)  *$|N(x_2) \cap Y| = 0$  and there is  $x_1y_1 \in E(F)$  where  $N(x_1) \cap Y = \{y_1\}$ ,*

*then  $G$  is  $f$ -AT.*

*Proof.* Suppose not and pick a counterexample  $(G, f, F, Y)$  minimizing  $|G - Y|$ . If  $|G - Y| = 0$ , then  $Y = V(G)$  and thus  $f(v) \geq d_G(v) + 1$  for all  $v \in V(G)$  by (3). Pick an acyclic orientation  $D$  of  $G$ . Then  $EE(D) = 1$ ,  $EO(D) = 0$  and  $d_D^+(v) \leq d_G(v) \leq f(v) - 1$  for all  $v \in V(D)$ . Hence  $G$  is  $f$ -AT. So, we must have  $|G - Y| > 0$ .

Pick a component  $T$  of  $G - Y$  and pick  $x_1, x_2 \in V(T)$  as guaranteed by (4). First, suppose (4a) holds. Put  $G' := (G - T) + y_1y_2$ ,  $F' := F - T$ ,  $Y' := Y$  and let  $f'$  be  $f$  restricted to  $V(G')$ . Then  $G'$  has an orientation  $D'$  where  $f'(v) \geq d_{D'}^+(v) + 1$  for all  $v \in V(D')$  and  $EE(D') \neq EO(D')$ , for otherwise  $(G', f', F', Y')$  would contradict minimality. By symmetry we may assume that the new edge  $y_1y_2$  is directed toward  $y_2$ . Now we use the orientation of  $D'$  to construct the desired orientation of  $D$ . First, we use the orientation on  $D' - y_1y_2$  on  $G - T$ . Now, order the vertices of  $T$  as  $x_1, x_2, z_1, z_2, \dots$  so that every vertex has at least one neighbor to the right. Orient the edges of  $T$  left-to-right in this ordering. Finally, we use  $y_1x_1$  and  $x_2y_2$  and orient all other edges between  $T$  and  $G - T$  away from  $T$ . Plainly,  $f(v) \geq d_D^+(v) + 1$  for all  $v \in V(D)$ . Since  $y_1x_1$  is the only edge of  $D$  going into  $T$ , any Eulerian subgraph of  $D$  that contains a vertex of  $T$  must contain  $y_1x_1$ . So, any Eulerian subgraph of  $D$  either contains (i) neither  $y_1x_1$  nor  $x_2y_2$ , (ii) both  $y_1x_1$  and  $x_2y_2$ , or (iii)  $y_1x_1$  but not  $x_2y_2$ . We first handle (i) and (ii) together. Consider the function  $h$  that maps

an Eulerian subgraph  $Q$  of  $D'$  to an Eulerian subgraph  $h(Q)$  of  $D$  as follows. If  $Q$  does not contain  $y_1y_2$ , let  $h(Q) = \iota(Q)$  where  $\iota(Q)$  is the natural embedding of  $D' - y_1y_2$  in  $D$ . Otherwise, let  $h(Q) = \iota(Q - y_1y_2) + \{y_1x_1, x_1x_2, x_2y_2\}$ . Then  $h$  is a parity-preserving injection with image precisely the union of those Eulerian subgraphs of  $D$  in (i) and (ii). Hence if we can show that exactly half of the Eulerian subgraphs of  $D$  in (iii) are even, we will conclude  $EE(D) \neq EO(D)$ , a contradiction. To do so, consider an Eulerian subgraph  $A$  of  $D$  containing  $y_1x_1$  and not  $x_2y_2$ . Since  $x_1$  must have in-degree 1 in  $A$ , it must also have out-degree 1 in  $A$ . We show that  $A$  has a mate  $A'$  of opposite parity. Suppose  $x_2 \notin A$  and  $x_1z_1 \in A$ ; then we make  $A'$  by removing  $x_1z_1$  from  $A$  and adding  $x_1x_2z_1$ . If  $x_2 \in A$  and  $x_1x_2z_1 \in A$ , we make  $A'$  by removing  $x_1x_2z_1$  and adding  $x_1z_1$ . Hence exactly half of the Eulerian subgraphs of  $D$  in (iii) are even and we conclude  $EE(D) \neq EO(D)$ , a contradiction.

Now suppose (4b) holds. Put  $G' := G - T$ ,  $F' := F - T$ ,  $Y' := Y$  and define  $f'$  by  $f'(v) = f(v)$  for all  $v \in V(G' - y_1)$  and  $f'(y_1) = f(y_1) - 1$ . Then  $G'$  has an orientation  $D'$  where  $f'(v) \geq d_{D'}^+(v) + 1$  for all  $v \in V(D')$  and  $EE(D') \neq EO(D')$ , for otherwise  $(G', f', F', Y')$  would contradict minimality. We orient  $G - T$  according to  $D$ , orient  $T$  as in the previous case, again use  $y_1x_1$  and orient all other edges between  $T$  and  $G - T$  away from  $T$ . Since we decreased  $f'(y_1)$  by 1, the extra out edge of  $y_1$  is accounted for and we have  $f(v) \geq d_D^+(v) + 1$  for all  $v \in V(D)$ . Again any additional Eulerian subgraph must contain  $y_1x_1$  and since  $x_2$  has no neighbor in  $G - T$  we can use  $x_2$  as before to build a mate of opposite parity for any additional Eulerian subgraph. Hence  $EE(D) \neq EO(D)$  giving our final contradiction.  $\square$

## 4 Degree-AT graphs

A graph  $G$  is called *degree-AT* if  $(G, h)$  is *AT* where  $h$  is the constant zero function.

**Lemma 4.1.** *A connected graph  $G$  is degree-AT if it is not a Gallai tree.*

*Proof.* Suppose there exists a connected graph that is not a Gallai tree, but is also not degree-AT. Let  $G$  be such a graph with as few vertices as possible. Since  $G$  is not degree-AT, no induced subgraph  $H$  of  $G$  is degree-AT by Lemma 2.1. Hence, for any  $v \in V(G)$  that is not a cutvertex,  $G - v$  must be a Gallai tree by minimality of  $|G|$ .

If  $G$  has more than one block, then for endblocks  $B_1$  and  $B_2$ , choose noncutvertices  $w \in B_1$  and  $x \in B_2$ . By the minimality of  $|G|$ , both  $G - w$  and  $G - x$  are Gallai trees. Since every block of  $G$  appears either as a block of  $G - w$  or as a block of  $G - x$ , every block of  $G$  is either complete or an odd cycle. Hence,  $G$  is a Gallai tree, a contradiction. So instead  $G$  has only one block, that is,  $G$  is 2-connected. Further,  $G - v$  is a Gallai tree for all  $v \in V(G)$ .

Let  $v$  be a vertex of minimum degree in  $G$ . Since  $G$  is 2-connected,  $d_G(v) \geq 2$ . By Lemma 2.3,  $v$  is adjacent to a noncutvertex in every endblock of  $G - v$ . If  $G - v$  has a complete block  $B$  with noncutvertices  $x_1, x_2$  where  $v \leftrightarrow x_1$  and  $v \not\leftrightarrow x_2$ , then we can apply Lemma 3.1 with  $Y = \{v\}$  and  $F = vx_1$  to conclude that  $G$  is degree-AT, a contradiction. So,  $v$  must be adjacent to every noncutvertex in every complete endblock of  $G - v$ .

Suppose  $d_G(v) \geq 3$ . Then no endblock of  $G - v$  can be an odd cycle of length at least 5 (there would be vertices of degree 3 but we'd have  $d_G(v) \geq 4$ ). Let  $B$  be a smallest complete endblock of  $G - v$ . Then for a noncutvertex  $x \in V(B)$ , we have  $d_G(x) = |B|$  and

hence  $d_G(v) \leq |B|$ . If  $G - v$  has at least two endblocks, then  $2(|B| - 1) \leq |B|$  and hence  $d_G(v) \leq |B| = 2$ , a contradiction. Hence  $G - v = B$  and  $v$  is joined to  $B$ , so  $G$  is complete, a contradiction.

Hence, we must have  $d_G(v) = 2$ . Suppose  $G - v$  has at least 2 endblocks. Then, it has exactly 2 and  $v$  is adjacent to one noncutvertex in each. Neither of the endblocks can be odd cycles of length at least 5 since then we could get a smaller counterexample by Lemma 2.2. Since  $v$  is adjacent to every noncutvertex in every complete endblock of  $G - v$ , both endblocks must be  $K_2$ . But then we can get a smaller counterexample by Lemma 2.2. So,  $G - v$  must be 2-connected. Since  $G - v$  is a Gallai tree, it is either complete or an odd cycle. If  $G - v$  is not complete, we can get a smaller counterexample by Lemma 2.2. So,  $G - v$  is complete and  $v$  is adjacent to every noncutvertex of  $G - v$ ; that is,  $G$  is complete, a contradiction.  $\square$

## References

- [1] Hal Kierstead and Landon Rabern, *Improved lower bounds on the number of edges in list critical and online list critical graphs*, arXiv preprint arXiv:1406.7355 (2014).
- [2] A.V. Kostochka and M. Stiebitz, *A new lower bound on the number of edges in colour-critical graphs and hypergraphs*, Journal of Combinatorial Theory, Series B **87** (2003), no. 2, 374–402.