## 1. Overview

For every multigraph $G$, we have $\chi'(G) \geq \left\lceil \frac{|E(G)|}{\lfloor |V(G)|/2 \rfloor} \right\rceil$, since each color class has size at most $\left\lfloor \frac{|V(G)|}{2} \right\rfloor$. Likewise, the same bound holds for any subgraph $H$. Thus, let $\mathcal{W}(G) = \max_{H \subseteq G} \left\lceil \frac{|E(H)|}{\lfloor |V(H)|/2 \rfloor} \right\rceil$ (over all subgraphs $H$ with at least two vertices). Now clearly $\chi'(G) \geq \mathcal{W}(G)$ for every multigraph $G$. Goldberg [] and Seymour [] each conjectured that this lower bound holds with equality, whenever $\chi'(G) > \Delta(G) + 1$.

**Goldberg–Seymour Conjecture.** *When $\mathcal{W}(G)$ is as above, every multigraph $G$ satisfies*

$$\chi'(G) \leq \max\{\mathcal{W}(G), \Delta(G) + 1\}.$$

The Goldberg–Seymour conjecture is the major open problem in the area of edge-coloring multigraphs. The second author showed [**?**] that $\chi(G) \leq \max\{\omega(G), \frac{7\Delta(G)+10}{8}\}$ for every line graph $G$. In the same paper, he conjectured that $\chi(G) \leq \max\{\omega(G), \frac{5\Delta(G)+8}{6}\}$. This conjecture is best possible, as shown by replacing each edge in a 5-cycle by $k$ parallel edges, and taking the line graph. In this paper we prove the latter inequality. Along the way, we develop more general techniques and results that will likely be of independent interest, due to their use in approaching the Goldberg–Seymour conjecture.

The main result of this paper is the following theorem.

**Theorem 16** ($\frac{5}{6}$-Theorem)**.** *If $Q$ the line graph of a multigraph $G$, then we have $\chi(Q) \leq \max\{\omega(Q), \frac{5\Delta(Q)+8}{6}\}$.*

Most of our work goes toward proving the following intermediate result, in Section 6.

**Theorem 13** (Weak $\frac{5}{6}$-Theorem)**.** *If $Q$ the line graph of a multigraph $G$, then $\chi(Q) \leq \max\{\mathcal{W}(G), \Delta(G) + 1, \frac{5\Delta(Q)+8}{6}\}$.*

Finally, in Section **??** we show that the Weak $\frac{5}{6}$-Theorem does indeed imply the $\frac{5}{6}$-Theorem.

## 2. Tashkinov Trees

Throughout this paper, graphs can have multiple edges unless stated otherwise. A graph $G$ is *elementary* if $\chi'(G) = \mathcal{W}(G)$. Let $[k]$ denote $\{1, \ldots, k\}$. For a path or cycle $Q$, let $\ell(Q)$ denote the length of $Q$. A graph $G$ is *critical* if $\chi'(G - e) < \chi'(G)$ for all $e \in E(G)$. For a graph $G$ and a partial $k$-edge-coloring $\varphi$, for each vertex $v \in V(G)$, let $\varphi(v)$ denote the set of colors used in $\varphi$ on edges incident to $v$. Let $\overline{\varphi}(v) = [k] \setminus \varphi(v)$. A color $c$ is *seen* by a vertex $v$ if $c \in \varphi(v)$ and $c$ is *missed* by $v$ if $c \in \overline{\varphi}(v)$. Given a partial $k$-edge-coloring $\varphi$, a set $W \subseteq V(G)$ is *elementary* with respect to $\varphi$ (henceforth, *w.r.t.* $\varphi$) if each color in $[k]$ is missed by at most one vertex of $W$. More formally, $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$ for all distinct $u, v \in W$. A *defective color* for a set $X \subseteq V(G)$ (w.r.t. $\varphi$) is a color used on more than one edge from $X$ to $V(G) \setminus X$. A set $X$ is *strongly closed* w.r.t. $\varphi$ if $X$ has no defective color. Elementary and strongly closed sets are of particular interest because of the following theorem, proved implicitly by Andersen [] and Goldberg []; see also [**?**, Theorem 1.4].

**Theorem 1.** *Let $G$ be a graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)$. If $G$ is critical, then $G$ is elementary if and only if there exists $uv \in E(G)$, a $k$-edge-coloring $\varphi$ of $G - uv$, and a set $X$ with $u, v \in X$ such that $X$ is both elementary and strongly closed w.r.t. $\varphi$.*

A *Tashkinov tree* w.r.t. $\varphi$ is a sequence $v_0, e_1, v_1, e_2, \ldots, v_{t-1}, e_t, v_t$ such that all $v_i$ are distinct, $e_i = v_j v_i$ and $\varphi(e_i) \in \overline{\varphi}(v_\ell)$ for some $j$ and $\ell$ with $0 \leq j < i$ and $0 \leq \ell < i$. A *Vizing fan* (or simply *fan*) is a Tashkinov tree that induces a star. Tashkinov trees are of interest because of the following lemma.

**Tashkinov's Lemma.** *Let $G$ be a graph with $\chi'(G) = k+1$, for some integer $k \geq \Delta(G)+1$ and choose $e \in E(G)$ such that $\chi'(G-e) < \chi'(G)$. Let $\varphi$ be a $k$-edge-coloring of $G - e$. If $T$ is a Tashkinov tree w.r.t. $\varphi$ and $e$, then $V(T)$ is elementary w.r.t. $\varphi$.*

In view of Theorem 1 and Tashkinov's Lemma, to prove that a graph $G$ is elementary, it suffices to find an edge $e$, a $k$-edge-coloring $\varphi$ of $G - e$, and a Tashkinov tree $T$ containing $e$ such that $V(T)$ is strongly closed. This motivates our next two lemmas. But first, we need a few more definitions.

Let $t(G)$ be the maximum number of vertices in a Tashkinov tree over all $e \in E(G)$ and all $k$-edge-colorings $\varphi$ of $G - e$. Let $\mathcal{T}(G)$ be the set of all triples $(T, e, \varphi)$ such that $e \in E(G)$, $\varphi$ is a $k$-edge-coloring of $G - e$ and $T$ is a Tashkinov tree with respect to $e$ and $\varphi$ with $|T| = t(G)$. Notice that, by definition, we have $\mathcal{T}(G) \neq \emptyset$. For a $k$-edge-coloring $\varphi$ of $G - e$, a maximal Tashkinov tree starting with $e$ may not be unique. However, if $T_1$ and $T_2$ are both such trees, then it is easy to show that $V(T_1) \subseteq V(T_2)$; by symmetry, also $V(T_2) \subseteq V(T_1)$, so $V(T_1) = V(T_2)$. Let $G$ be a critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. Let $\varphi$ be a $k$-edge-coloring of $G - e_0$ for some $e_0 \in E(G)$. For $v \in V(G)$ and colors $\alpha, \beta$, let $P_v(\alpha, \beta)$ be the maximal connected subgraph of $G$ that contains $v$ and is induced by edges with color $\alpha$ or $\beta$. So $P_v(\alpha, \beta)$ is a path or a cycle. For a $k$-edge-coloring $\varphi$ of $G - v_0v_1$, we often let $P = P_{v_1}(\alpha, \beta)$ for some $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Clearly $P$ must end at $v_0$ (or we can swap colors $\alpha$ and $\beta$ on $P$ and color $v_0v_1$ with $\alpha$), so let $v_1, \ldots, v_r, v_0$ denote the vertices of $P$ in order. To *rotate the $\alpha, \beta$ coloring on $P \cup \{v_0v_1\}$ by one*, we uncolor $v_1v_2$ and use its color on $v_0v_1$. To *rotate the $\alpha, \beta$ coloring on $P \cup \{v_0v_1\}$ by $j$*, we rotate the $\alpha, \beta$ coloring by one $j$ times in succession. (When we do not specify $j$, we allow $j$ to take any value from 1 to $r$.)

**Lemma 2.** *Let $G$ be a non-elementary critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. For every $v_0v_1 \in E(G)$, $k$-edge-coloring $\varphi$ of $G - v_0v_1$, $\alpha \in \overline{\varphi}(v_0)$, and $\beta \in \overline{\varphi}(v_1)$, we have $|P_{v_1}(\alpha, \beta)| < t(G)$.*

*Proof.* Suppose the lemma is false and choose $v_0v_1 \in E(G)$, a $k$-edge-coloring $\varphi$ of $G - v_0v_1$, $\alpha \in \overline{\varphi}(v_0)$, and $\beta \in \overline{\varphi}(v_1)$, such that $|P_{v_1}(\alpha, \beta)| \geq t(G)$. Let $P = P_{v_1}(\alpha, \beta)$. Let $(T, v_0v_1, \varphi)$ be a Tashkinov tree that begins with edges $v_0v_1, v_1v_2, \ldots, v_{r-1}v_r$. Now $V(T) = V(P)$ since $t(G) \geq |T| \geq |P| \geq t(G)$. By hypothesis $G$ is non-elementary, so Theorem 1 implies that $V(T)$ is not strongly closed; thus, $T$ has a defective color $\delta$ with respect to $\varphi$. Choose $\tau \in \overline{\varphi}(v_2)$. Let $Q = P_{v_2}(\tau, \delta)$. Since $T$ is maximal, $\delta$ is not missing at any vertex of $T$, and since $V(T)$ is elementary, $\tau$ is not missing at any vertex of $T$ other than $v_2$. As a result, $Q$ ends outside $V(T)$. Now $Q$ could leave $V(T)$ and re-enter it repeatedly, but $Q$ ends outside

$V(T)$, so there is a last vertex $w \in V(Q) \cap V(T)$; say $Q$ ends at $z \in V(G) \setminus V(T)$. Let $\pi \notin \{\alpha, \beta\}$ be a color missing at $w$. Since $\tau \in \overline{\varphi}(v_2)$ and $\pi \in \overline{\varphi}(w)$ and $|T| = t(G)$, no edge colored $\tau$ or $\pi$ leaves $V(T)$. So we can swap $\tau$ and $\pi$ on every edge in $G - V(T)$ without changing the fact that $T$ is a Tashkinov tree with $|T| = t(G)$. After swapping $\tau$ and $\pi$, we swap $\delta$ and $\pi$ on the subpath of $Q$ from $w$ to $z$. Since $\pi$ is missing at $w$, the $\delta - \pi$ path starting at $z$ must end at $w$. Now $\delta$ is missing at $w$, but $\delta$ was defective in $\varphi$, so some other edge $e$ colored $\delta$ still leaves $V(T)$. Adding $e$ gets a larger Tashkinov tree, which is a contradiction. $\square$

## 3. Short vertices

Recall that a vertex $v \in V(G)$ is *short* if every Vizing fan rooted at $v$ (taken over all $k$-colorings of $G - e$, over all edges $e$ incident to $v$) has at most 3 vertices, including $v$. Otherwise, $v$ is *long*. Let $\nu(T)$ be the number of long vertices in a Tashkinov tree $T$.

Now we can outline our proof of the $\frac{5}{6}$-Conjecture. We will show in Section **??** that the $\frac{5}{6}$-Conjecture is implied by the Goldberg–Seymour Conjecture. More precisely, if $G$ is a multigraph such that $\chi'(G) \leq \max\{\lceil \mathcal{W}(G) \rceil, \Delta(G)+1\}$, then also $\chi'(G) \leq \frac{5\Delta(G)+8}{6}$. So here it suffices to show that $\chi'(G) \leq \max\{\lceil \mathcal{W}(G) \rceil, \Delta(G)+1, \frac{5\Delta(G)+8}{6}\}$. We consider cases based on $\nu(T)$, for some Tashkinov tree $T \in \mathcal{T}(G)$.

In the present section, we show that if $G$ has a maximum Tashkinov tree $T$ that contains no short vertices, i.e., $\nu(T) = 0$, then $G$ is elementary. In fact, Lemma 7 implies that the same is true when $\nu(T) = 1$. In the proof of Theorem 14, we show that if $G$ is a minimal counterexample to the $\frac{5}{6}$-Conjecture, then every long vertex $v$ has $d(v) < \frac{3}{4}\Delta(G)$. This implies that $\nu(T) < 4$, since otherwise the number of colors missing at vertices of $T$ is more than $4(k - \frac{3}{4}\Delta(G)) > k$, which contradicts that $V(T)$ is elementary. So it remains to consider the case $\nu(T) \in \{2, 3\}$.

In Section 6, we introduce the notion of *$k$-thin graphs*, which are essentially those for which $\mu(G)$ is not too large. Using a lemma from [**?**], we show that every minimal counterexample to the $\frac{5}{6}$-Conjecture must be $k$-thin. We then extend the ideas of the present section to show handle the case when $\nu(T) \in \{2, 3\}$. Much like when $\nu(T) \geq 4$, we show that $T$ has too many colors missing at its vertices to be elementary. More precisely, $\sum_{v \in V(T)} |\overline{\varphi}(v)| > k$, which is a contradiction.

Short vertices were introduced in [**?**], where they were motivated by a version of the following lemma in the context of proving a strengthening of Reed's Conjecture for line graphs.

**Lemma 3.** *Let $G$ be a critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. Let $\varphi$ be a $k$-edge-coloring of $G - v_0v_1$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Let $P = v_1v_2 \cdots v_r$ be an $\alpha, \beta$ path with edges $e_i = v_iv_{i+1}$ for all $i \in [r-1]$. If $v_i$ is short for all odd $i$, then for each $\tau \in \overline{\varphi}(v_0)$ there are edges $f_i = v_iv_{i+1}$ for all $i \in [r-1]$ such that $f_i = e_i$ for $i$ even and $\varphi(f_i) = \tau$ for $i$ odd.*

*Proof.* Suppose not and choose a counterexample minimizing $r$. By minimality of $r$, we have $\varphi(v_{r-1}v_r) = \alpha$ and we have $f_i = v_iv_{i+1}$ for all $i \in [r-2]$ such that $f_i = e_i$ for $i$ even and $\varphi(f_i) = \tau$ for $i$ odd. Swap $\alpha$ and $\beta$ on $e_i$ for all $i \in [r-3]$ and then color $v_0v_1$ (call this edge $e_0$) with $\alpha$ and uncolor $e_{r-2}$. Let $\varphi'$ be the resulting coloring. Since $k \geq \Delta(G)+1$, some color other than $\alpha$ is missing at $v_{r-2}$; let $\gamma$ be such a color. Now $v_{r-1}$ is short since $r-1$ is odd (since $P$ starts and ends with $\alpha$), so there is an edge $e = v_{r-1}v_r$ with $\varphi'(e) = \gamma$. Swap $\tau$ and $\alpha$ on $e_i$ for all $i$ with $0 \leq i \leq r-3$ to get a new coloring $\varphi^*$. Now $\gamma$ and $\tau$ are both missing at $v_{r-2}$ in $\varphi^*$. Since $v_{r-1}$ is short, the fan with $v_{r-2}, v_{r-1}, v_r$ and $e$ implies that there is an edge $f_{r-1} = v_{r-1}v_r$ with $\varphi^*(f_{r-1}) = \tau$. But we have never recolored $f_{r-1}$, so $\varphi(f_{r-1}) = \tau$, which is a contradiction. □

**Lemma 4.** *Let $G$ be a non-elementary critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. Choose $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$ for some $v_0v_1 \in E(G)$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $P = P_{v_1}(\alpha, \beta)$. Now $P$ contains a long vertex. In particular, $\nu(T) \geq 1$.*

*Proof.* Suppose every vertex of $P$ is short. Applying Lemma 3 to $P$ shows that for every $\tau \in \overline{\varphi}(v_0)$, there is an edge in $T$ colored $\tau$ incident to every $v \in V(P - v_0)$. The same is also true of every $v \in V(P)$; to see this, we rotate the $\alpha, \beta$ coloring of $P \cup \{v_0v_1\}$ and repeat the same argument. Hence $V(P) = V(T)$, which contradicts Lemma 2. □

**Theorem 5.** *If $G$ is a critical graph in which every vertex is short, then*

$$\chi'(G) \leq \max\{\mathcal{W}(G), \Delta(G)+1\}.$$

*Proof.* Suppose not and let $G$ be a counterexample. Let $k = \chi'(G) - 1$, and note that $k \geq \Delta(G)+1$. Since $\mathcal{T}(G) \neq \emptyset$, by applying Lemma 4 we conclude that $G$ is elementary. Hence $\chi'(G) = \mathcal{W}(G)$, which is a contradiction. □

## 4. An easy bound

In this section, we apply the results of Section 3 to prove an easy bound on $\chi'(G)$. We also show how those results imply Reed's Conjecture, as well as Local and Superlocal strengthenings of Reed's Conjecture, for the class of line graphs.

Let $G$ be a graph. The *claw-degree* of $x \in V(G)$ is

$$d_{\text{claw}}(x) := \max_{\substack{S \subseteq N(x) \\ |S|=3}} \frac{1}{4}\left(d(x) + \sum_{v \in S} d(v)\right).$$

The *claw-degree* of $G$ is

$$d_{\text{claw}}(G) := \max_{x \in V(G)} d_{\text{claw}}(x).$$

**Theorem 6.** *If $G$ is a graph, then*

$$\chi'(G) \leq \max\left\{\mathcal{W}(G), \Delta(G)+1, \left\lceil \frac{4}{3} d_{claw}(G) \right\rceil\right\}.$$

*Proof.* Suppose not and choose a counterexample $G$ with the fewest edges; note that $G$ is critical. Let $k = \chi'(G) - 1$, so $k \geq \left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil$. By Theorem 5, $G$ has a long vertex $x$. Choose $xy_1 \in E(G)$ and a $k$-edge-coloring $\varphi$ of $G - xy_1$ such that $\varphi$ has a fan $F$ of length 3 rooted at $x$ with leaves $y_1, y_2, y_3$. Since $V(F)$ is elementary,

$$2 + k - d(x) + \sum_{i \in [3]} k - d(y_i) \leq k,$$

and hence

$$d_{\text{claw}}(x) \geq \frac{1}{4}\left(d(x) + \sum_{i\in[3]} d(y_i)\right) \geq \frac{3k+2}{4}.$$

This gives the contradiction

$$\left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil \leq k \leq \frac{4}{3} d_{\text{claw}}(G) - \frac{2}{3}.$$

□

Reed [?] conjectured that $\chi(G) \leq \left\lceil \frac{\omega(G)+\Delta+1}{2} \right\rceil$ for every graph $G$. This is the average of a trivial lower bound $\omega(G)$ and a trivial upper bound $\Delta(G)+1$. King [?] conjecture the stronger bound $\chi(G) \leq \max_{v\in V(G)} \left\lceil \frac{\omega(v)+d(v)+1}{2} \right\rceil$, where $\omega(v)$ is the size of the largest clique containing $v$, which is now known to hold for many classes of graphs, including line graphs [?]. Here we show that for line graphs this bound is an easy consequence of our more general lemmas from Section 3. The following is essentially Lemma 10 from [?].

**Corollary 7.** *Let $G$ be a graph. For $uv \in E(G)$, let $f(uv) = \max\{d(u) + \frac{1}{2}(d(v) - \mu(uv)), d(v) + \frac{1}{2}(d(u) - \mu(uv)\}$. Let $f(G) = \max_{uv,vw\in E(G)} \left\lceil \frac{1}{2}(f(uv) + f(uw)) \right\rceil$. Now*

$$\chi'(G) \leq \max\{\mathcal{W}(G), \Delta(G)+1, f(G)\}.$$

*In particular, the Superlocal version of Reed's Conjecture holds for every line graph.*

*Proof.* The first statement follows directly from Theorem 6, by showing that $\left\lceil \frac{4}{3} d_{claw}(G) \right\rceil \leq f(G)$. Choose $x \in V(G)$ and $S \in N(x)$ such that $x$ and $S$ achieve maximality in the definition of $d_{claw}(G)$. Now

$$\begin{aligned}
\left\lceil \frac{4}{3}\frac{1}{4}(d(x) + d(v_1) + d(v_2) + d(v_3)) \right\rceil \leq & \left\lceil \frac{1}{3}\left( d(v_1) + \frac{1}{2}(d(x) - \mu(xv_1)) \right.\right. \\
& + d(v_2) + \frac{1}{2}(d(x) - \mu(xv_2)) \\
& \left.\left. + d(v_3) + \frac{1}{2}(d(x) - \mu(xv_3)) \right) \right\rceil \\
\leq & \left\lceil \frac{1}{3}(f(xv_1) + f(xv_2) + f(xv_3)) \right\rceil \\
\leq & f(G).
\end{aligned}$$

This proves the first statement. For the second statement, we show that $\mathcal{W}(G) \leq f(G)$, as follows. For each vertex $v$, let $v_1, v_2, \ldots$ denote the neighbors of $v$ (with subscripts modulo

$|N(v)|$). Also, let $\overline{d} = \frac{2|E(H)|}{|V(H)|}$.

$$\begin{aligned}
f(G)2|E(H)| &\geq \sum_{v\in V}\sum_{i=1}^{|N(v)|}\frac{1}{2}(d(v)+\frac{1}{2}(d(v_i)-\mu(vv_i))+d(v)+\frac{1}{2}(d(v_{i+1})-\mu(vv_{i+1})))\\
&=\sum_{v\in V}\sum_{i=1}^{|N(v)|} d(v)+\frac{1}{2}(d(v_i)-\mu(vv_i))\\
&=\sum_{uv\in E(H)}\frac{3}{2}d(u)+\frac{3}{2}d(v)-\mu(uv)\\
&=\sum_{v\in V(H)}\frac{3}{2}d(v)^2-|E(H)|\\
&\geq \frac{3}{2}\overline{d}^2|V(H)|-|E(H)|\\
&=6\frac{|E(H)|^2}{|V(H)|}-|E(H)|
\end{aligned}$$

Thus $f(G) \geq \frac{3|E(H)|}{|V(H)|} - \frac{1}{2}$. Since $\mathcal{W}(G) = \left\lceil \frac{2|E(H)|}{|V(H)|-1} \right\rceil \leq \frac{2|E(H)|+|V(H)|-3}{|V(H)|-1}$, it suffices to have $\frac{3|E(H)|}{|V(H)|} - \frac{1}{2} \geq \frac{2|E(H)|+|V(H)|-3}{|V(H)|-1}$. Now solving for $|E(H)|$ gives $|E(H)| \geq \frac{3}{2}|V(H)|\frac{|V(H)|-\frac{7}{3}}{|V(H)|-3}$. Taking $|V(H)| \geq 5$, it suffices to have $2|V(H)| \leq |E(H)|$. Suppose, to the contrary, that we have $|E(H)| < 2|V(H)|$. It will suffice to show that $\frac{2|E(H)|}{|V(H)|-1} \leq \Delta(H)$. Now solving $\frac{4|V(H)|}{|V(H)|-1} \leq \Delta(H)$ (using $|V(H)| \geq 5$), shows that it suffices to have $\Delta(H) \geq 5$.
TODO: HANDLE $\Delta(H) \in \{3,4\}$. □

## 5. Properties of long vertices

For a path $Q$, recall that $\ell(Q)$ denotes the length of $Q$. For $x, y \in V(Q)$, let $xQy$ denote the subpath of $Q$ with endvertices $x$ and $y$, and let $d_Q(x,y) = \ell(xQy)$, i.e., the distance from $x$ to $y$ along $Q$.

**Lemma 8.** *Let $G$ be a critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. Let $\varphi$ be a $k$-edge-coloring of $G - v_0v_1$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $C = P_{v_1}(\alpha,\beta) + v_0v_1$. If $\tau \in \overline{\varphi}(x)$ for some $x \in V(C)$ and there is a $\tau$-colored edge from $y \in V(C)$ to $w \in V(G)\setminus V(C)$, then $C$ has a subpath $Q$ with long endpoints $z_1, z_2$ such that $x \in V(Q)$, $y \notin V(Q - z_1 - z_2)$ and the distance from $x$ to $z_i$ along $Q$ is odd for each $i \in [2]$. Moreover, for each $i \in [2]$, there are no $\tau$-colored edges between $z_i$ and its neighbors along $C$.*

*Proof.* Let $G$, $\alpha$, $\beta$, $\tau$, $x$, and $y$ be as in the statement of the lemma. Choose $z_1$ (resp. $z_2$) to be the first vertex at an odd distance from $x$ along $C$ in the clockwise (resp. counterclockwise) direction with no incident $\tau$-colored edge parallel to some edge of $C$. Let $Q$ be the subpath of $C$ with endpoints $z_1$ and $z_2$ that contains $x$. By the choice of $z_1$ each vertex $w$ between $x$ and $z_1$ with $d_Q(xw)$ odd has a $\tau$-colored edge parallel to some edge of $C$. The presence of these edges implies the same for each $w$ for which $d_Q(xw)$ is even. By the proof of the

Parallel Edge Lemma, $z_1$ must be long, since otherwise it would have an incident $\tau$-colored edge parallel to some edge of $C$. The same argument applies to $z_2$. $\square$

## 6. Thin graphs

Let $G$ be a critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. For vertices $x \in V(G)$ and $S \subseteq V(G) \setminus \{x\}$, we say that $x$ is *$S$-short* if every Vizing fan $F$ rooted at $x$ with $S \subseteq V(F)$, has $|F| \leq 3$ (with respect to any $k$-edge-coloring of $G - xy$). Otherwise, $x$ is *$S$-long*. For brevity, when $S = \{y\}$, we may write $y$-short instead of $\{y\}$-short. It is worth noting that in Lemma 3 we can weaken the hypothesis that $v_i$ is short for all odd $i$ to require only that $v_i$ is $v_{i-1}$-short for all odd $i$, since this is what we use in the proof.

A graph $G$ is *$k$-thin* if $\mu(G) < 2k - d(x) - d(y)$ for all long $x, y \in V(G)$. In the proof of Theorem 14, we will show that every counterexample to the $\frac{5}{6}$-Conjecture must be $k$-thin.

**Lemma 9.** *Let $G$ be a $k$-thin, critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. Let $\varphi$ be a $k$-edge-coloring of $G - v_0v_1$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $C = P_{v_1}(\alpha,\beta) + v_0v_1$. Let $Q$ be a subpath of $C$ with long end vertices. If all internal vertices of $Q$ are short and $2 \leq \ell(Q) \leq \ell(C) - 2$, then $\ell(Q)$ is even.*

*Proof.* Suppose to the contrary that we have a subpath $Q$ of $C$ with end vertices long, all internal vertices short, $2 \leq \ell(Q) \leq \ell(C) - 2$, and $\ell(Q)$ odd. Let $x$ and $y$ be the end vertices of $Q$. Say $C = v_1v_2 \cdots v_rv_0v_1$. By rotating the $\alpha, \beta$ coloring of $C$, we may assume that $x = v_0$ and $y = v_a$, where $a \geq 3$ is odd.

We now apply Lemma 3 twice, to show that $\mu(v_1v_2) \geq 2k - d(v_0) - d(v_a)$, which contradicts that $G$ is $k$-thin. More specifically, assume that the edges $v_0v_1, v_1v_2, \ldots$ go clockwise around $C$. We apply Lemma 3 once going clockwise starting from $v_0$ and once going counterclockwise starting from $v_a$. The first application implies that every color in $\overline{\varphi}(v_0)$ appears on some edge parallel to $v_1v_2$; the second implies the same for every color in $\overline{\varphi}(v_a)$. Since $|\overline{\varphi}(v_i)| = k - d(v_i)$ for each $i \in \{0, a\}$ and $\overline{\varphi}(v_0) \cap \overline{\varphi}(v_a) = \emptyset$, the conclusion follows. $\square$

**Lemma 10.** *Let $G$ be a $k$-thin, critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. Let $\varphi$ be a $k$-edge-coloring of $G - v_0v_1$. Suppose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $C = P_{v_1}(\alpha,\beta) + v_0v_1$. If $C$ contains exactly 3 long vertices, then $C = xyAzBx$ where $A$ and $B$ are paths of even length and $x, y, z$ are all long. Moreover, $x$ is $y$-long and $y$ is $x$-long.*

*Proof.* Let $G$ be a graph satisfying the hypotheses, and let $x$, $y$, $z$ be the three long vertices. The three subpaths of $C$ with endpoints $x$, $y$, and $z$ either (i) all have odd length or (ii) include two paths of even length and one of odd length. First assume that $\ell(C) \geq 5$. If we are in (i), then the longest of these three subpaths violates Lemma 9; so we are in (ii), and also the path of odd length is simply an edge. This proves the first statement. For the second statement, assume to the contrary that $x$ is $y$-short. By rotating the $\alpha, \beta$ coloring, we can assume that $y = v_0$ and $x = v_1$. As in the previous lemma, we use Lemma 3 (and the comment in the first paragraph of Section 6) to conclude that $\mu(v_1v_2) \geq 2k - d(v_0) - d(z)$. As above, this contradicts that $G$ is $k$-thin; this contradiction proves the second statement. $\square$

**Lemma 11.** *Let $G$ be a non-elementary, $k$-thin, critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. Choose $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$. If $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$, then $P_{v_1}(\alpha,\beta) + v_0v_1$ contains consecutive long vertices.*

*Proof.* Let $C = P_{v_1}(\alpha,\beta) + v_0v_1$. By Lemma 2, there is $x \in V(C)$ and $\tau \in \overline{\varphi}(x)$ such that there is a $\tau$-colored edge from $y \in V(C)$ to $w \in V(T) \setminus V(C)$. Lemma 8 implies that $C$ has a subpath $Q$ with $x \in V(Q)$ and long endpoints $z_1, z_2$ such that the distance from $x$ to $z_i$ along $Q$ is odd for each $i \in [2]$. Let $Q'$ be the subpath of $C$ with endpoints $z_1$ and $z_2$ that does not contain $x$. Since $C$ is an odd cycle, $\ell(Q')$ is odd. Let $Q^*$ be a minimum length subpath of $Q'$ with long ends. Now $\ell(Q^*) = 1$ by Lemma 9, as desired. □

**Lemma 12.** *Let $G$ be a non-elementary, $k$-thin, critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. If $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$ and $\nu(T) \leq 3$, then $T$ contains long vertices $z_1, z_2, z_3$ such that either*

(1) *$z_1$ is $\{z_2, z_3\}$-long and $z_2$ is $z_1$-long; or*
(2) *$z_i$ is $z_j$-long and $z_j$ is $z_i$-long for each $(i,j) \in \{(1,2),(2,3)\}$.*

*Proof.* Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ so that $P_{v_1}(\alpha,\beta)$ contains as many long vertices as possible; let $C = P_{v_1}(\alpha,\beta) + v_0v_1$. By Lemma 2, there is $x \in V(C)$ and $\tau \in \overline{\varphi}(x)$ such that there is a $\tau$-colored edge from $y \in V(C)$ to $w \in V(T) \setminus V(C)$. By Lemma 11, $C$ has at least two long vertices.

First suppose that $C$ contains only 2 long vertices, $z_1$ and $z_2$. By Lemma 11, $z_1$ and $z_2$ are consecutive on $C$. Lemma 8 implies that $C$ has a subpath $Q$ with endpoints $z_1, z_2$ such that $x \in V(Q)$ and $y \notin V(Q - z_1 - z_2)$ and for each $i \in [2]$ there are no $\tau$-colored edges between $z_i$ and its neighbors on $C$. By rotating the $\alpha, \beta$ coloring of $C$, we can assume that $x = v_0$ and $\alpha, \tau \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Note that $P_{v_1}(\tau,\beta)$ must end at $v_0$ (since otherwise we can recolor the Kempe chain and color $v_0v_1$ with $\tau$). Let $C' = P_{v_1}(\tau,\beta) + v_0v_1$. Note that $C'$ must include $v_1Qz_1$ and also $v_0Qz_2$ (the $\beta$-colored edges are present by definition and the $\tau$-colored edges are present by the Parallel Edge Lemma). Thus, $z_1, z_2 \in V(C')$. Since $z_1$ and $z_2$ are not consecutive on $C'$ and $C'$ contains no other long vertices by the maximality condition on $C$, Lemma 11 gives a contradiction.

So instead $C$ contains exactly 3 long vertices, $z_1$, $z_2$, and $z_3$. By Lemma 10, $C = z_1z_2Az_3Bz_1$ where $A$ and $B$ are paths of even length. Also, $z_1$ is $z_2$-long and $z_2$ is $z_1$-long.

By Lemma 8, $C$ has a subpath $Q$ with endpoints $z_1, z_3$ and with $x \in V(Q)$ and $y \notin V(Q - z_1 - z_3)$ such that there are no $\tau$-colored edges between $z_i$ and its neighbors along $C$ for each $i \in \{1,3\}$ (it could happen that $z_3$ has a $\tau$-colored edge parallel to an edge of $C$, so the endpoints of $Q$ are $z_1, z_2$, but now we get a contradiction as in the previous case, by letting $C' = P_{v_1}(\tau,\beta) + v_0v_1$). By rotating the $\alpha, \beta$ coloring of $C$, we may assume that $x = v_0$. Again, let $C' = P_{v_1}(\tau,\beta) + v_0v_1$. We know that $C'$ contains $z_1$ and $z_3$ and that $z_1$ and $z_2$ are not consecutive on $C'$. Note also that all long vertices in $V(C')$ must be among $z_1, z_2, z_3$, since otherwise $\nu(T) \geq 4$, contradicting our hypothesis. So by Lemma 11, either $z_1$ and $z_3$ are consecutive on $C'$ or $z_2$ and $z_3$ are consecutive on $C'$.

Suppose that $z_2$ and $z_3$ are consecutive on $C'$, and thus connected by a $\tau$-colored edge. Now applying Lemma 10 shows that $z_2$ is $z_3$-long and $z_3$ is $z_2$-long, so we satisfy (2) in the conclusion of the lemma (by swapping the names of $z_1$ and $z_2$).

So instead $z_1$ and $z_3$ must be consecutive on $C'$, and thus connected by a $\tau$-colored edge. If $z_1 = v_1$, then we have a fan with an $\alpha$-colored edge from $z_1$ to $z_2$ and a $\tau$-colored edge from $z_1$ to $z_3$, so $z_1$ is $\{z_2, z_3\}$-long.

Now assume that $z_1 \neq v_1$. Let $z_1'$ be the predecessor of $z_1$ on the path from $v_0$ (through $v_1$) to $z_1$. We can shift the coloring so that $z_1'z_1$ is uncolored and $z_1z_2$ is colored $\alpha$ (as in the proof of the Parallel Edge Lemma). In fact, we can shift either the $\alpha, \beta$ edges or the $\tau, \beta$ edges. This gives the options that either $\alpha \in \overline{\varphi}(z_1')$ or $\tau \in \overline{\varphi}(z_1')$, whichever we prefer. Suppose we shift the $\tau, \beta$ edges. Now choose $\gamma \in \overline{\varphi}(z_1') - \alpha - \tau$. Consider the $\gamma$-colored edge $e$ incident to $z_1$. If $e$ goes to $z_2$, then we $z_1$ is $\{z_2, z_3\}$-long, by colors $\gamma$ and $\tau$; so we satisfy (1) in the conclusion of the lemma. If instead $e$ goes to $z_3$, then instead of shifting the $\tau, \beta$ edges we shift the $\alpha, \beta$ edges; note that this recoloring preserves the fact that $\gamma$ is missing at $z_1'$. Now again $z_1$ is $\{z_2, z_3\}$-long, this time by colors $\alpha$ and $\gamma$; so we again satisfy (1) in the conclusion of the lemma.

Finally, assume that the $\gamma$-colored edge incident to $z_1$ goes to some vertex other than $z_2$ and $z_3$. Now let $C'' = P_{z_1}(\gamma, \beta) + z_1z_1'$. Since $V(C'') \subseteq V(T)$, Lemmas 11 and 10 imply that $z_2$ and $z_3$ are adjacent on $C''$ and furthermore $z_2$ is $z_3$-long and $z_3$ is $z_2$-long; thus, we satisfy (2) in the conclusion of the lemma. □

We need the following result from [?], which we use to handle the case when $G$ is not $k$-thin.

**Theorem 13** ([?])**.** *If $Q$ is the line graph of a graph $G$ and $Q$ is vertex critical, then*

$$\chi(Q) \leq \max\left\{\omega(Q), \Delta(Q) + 1 - \frac{\mu(G) - 1}{2}\right\}.$$

Now we prove the main result of this section.

**Theorem 14.** *If $Q$ is the line graph of $G$, then*

$$\chi(Q) \leq \max\left\{\lceil \chi_f(Q) \rceil, \Delta(G) + 1, \left\lceil \frac{5\Delta(Q) + 3}{6} \right\rceil\right\}.$$

*Proof.* Suppose the theorem is false and choose a counterexample minimizing $|Q|$. Let $k = \max\left\{\lceil \chi_f(Q) \rceil, \Delta(G) + 1, \left\lceil \frac{5\Delta(Q)+3}{6} \right\rceil\right\}$. Say $Q = L(G)$ for a graph $G$. The minimality of $Q$ implies that $G$ is critical and $\chi(Q) = k + 1$, for some $k \geq \Delta(G) + 1$.

The heart of the proof is Claim 1, which roughly says that if $x$ is long, then $d(x) < \frac{3}{4}\Delta(G)$. Moreover, we can improve this bound further if $x$ is the root of a long fan $F$ such that either (i) $F$ has length more than 3 or (ii) some of the other vertices in $F$ have degree less than $\Delta(G)$. The claims thereafter are all essentially applications of Claim 1.

**Claim 1.** *Let $F$ be a fan rooted at $x$ with respect to a $k$-edge-coloring of $G - xy$. If $S \subseteq V(F) - x$ and $|S| \geq 3$, then*

$$d(x) \leq \frac{1}{5|S| - 11}\left(2|S| - 12 + \sum_{v \in S} d(v)\right).$$

*In particular, if $|S| = 3$, then $d(x) \leq \frac{1}{4}\left(-6 + \sum_{v\in S} d(v)\right)$.*

Proof: Since $F$ is elementary, we have
$$2 + k - d(x) + \sum_{v \in S} k - d(v) \leq k,$$
so
$$2 + |S|k \leq d(x) + \sum_{v \in S} d(v).$$
Using $k \geq \frac{5}{6}(\Delta(Q)+1) - \frac{1}{3} \geq \frac{5}{6}(d(x)+d(v)-\mu(xv)) - \frac{1}{3}$ for each $v \in S$, we get
$$2 + \sum_{v \in S} \left( \frac{5}{6}(d(x)+d(v)-\mu(xv)) - \frac{1}{3} \right) \leq d(x) + \sum_{v \in S} d(v),$$
so multiplying by 6 and rearranging terms gives
$$12 + (5|S|-6)\, d(x) - 2|S| \leq \sum_{v \in S} 5\mu(xv) + \sum_{v \in S} d(v).$$
Now $\sum_{v \in S} \mu(xv) \leq d(x)$, so this implies
$$12 + (5|S|-11)\, d(x) - 2|S| \leq \sum_{v \in S} d(v).$$
Solving for $d(x)$ gives
$$d(x) \leq \frac{1}{5|S|-11} \left( 2|S| - 12 + \sum_{v \in S} d(v) \right),$$
and when $|S| = 3$, we get $d(x) \leq \frac{1}{4}\left(-6 + \sum_{v \in S} d(v)\right)$. □

**Claim 2.** *If $x \in V(G)$ is long, then $d(x) \leq \frac{3}{4}\Delta(G) - 1$.*
Proof: This is immediate from Claim 1, since $d(v) \leq \Delta(G)$ for all $v \in S$. □

**Claim 3.** *If $x_1x_2 \in E(G)$ such that $x_1$ is $x_2$-long and $x_2$ is $x_1$-long, then*
$$d(x_i) \leq \frac{2}{3}\Delta(G) - 2 \text{ for all } i \in [2].$$

Proof: By Claim 1, for each $i \in [2]$,
$$d(x_i) \leq \frac{1}{4}\left(-6 + \sum_{v \in S} d(v)\right) \leq \frac{1}{4}\left(-6 + d(x_{3-i}) + 2\Delta(G)\right),$$
Substituting the bound on $d(x_{3-i})$ into that on $d(x_i)$ and simplifying gives for each $i \in [2]$,
$$d(x_i) \leq -2 + \frac{2}{3}\Delta(G).$$
□

**Claim 4.** *If $x_1x_2, x_1x_3 \in E(G)$ such that $x_1$ is $\{x_2, x_3\}$-long, $x_2$ is $x_1$-long and $x_3$ is long, then*
$$d(x_1) \leq -\frac{8}{5} + \frac{3}{5}\Delta(G),$$

$$d(x_2) \le -\frac{7}{5} + \frac{13}{20}\Delta(G).$$

Proof: By Claim 1, we have

$$d(x_1) \le \frac{1}{4}\left(-6 + \sum_{v\in S} d(v)\right) \le \frac{1}{4}\left(-6 + d(x_2) + d(x_3) + \Delta(G)\right),$$

$$d(x_2) \le \frac{1}{4}\left(-6 + \sum_{v\in S} d(v)\right) \le \frac{1}{4}\left(-6 + d(x_1) + 2\Delta(G)\right).$$

By the same calculation as in Claim 3, these together imply

$$d(x_1) \le -2 + \frac{2}{5}\Delta(G) + \frac{4}{15}d(x_3).$$

Since $x_3$ is long, using Claim 2, we get

$$d(x_1) \le -\frac{34}{15} + \frac{3}{5}\Delta(G),$$

and hence

$$d(x_2) \le -\frac{61}{15} + \frac{13}{20}\Delta(G).$$

$\square$

**Claim 5.** *The theorem is true.*

Proof: Let $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$. By Lemma 12, one of the following holds:

(1) $G$ is elementary; or
(2) $G$ is not $k$-thin; or
(3) $\nu(T) = 3$ and $V(T)$ contains vertices $x_1, x_2, x_3$ such that $x_1$ is $x_2$-long, $x_2$ is $x_1$-long, $x_2$ is $x_3$-long, and $x_3$ is $x_2$-long; or
(4) $\nu(T) = 3$ and $V(T)$ contains vertices $x_1, x_2, x_3$ such that $x_1$ is $\{x_2, x_3\}$-long, $x_2$ is $x_1$-long, and $x_3$ is long; or
(5) $V(T)$ contains four long vertices $x_1, x_2, x_3, x_4$.

If (1) holds, then $\chi(Q) = \lceil \chi_f(Q) \rceil$, which contradicts our choice of $Q$ as a counterexample.

If (2) holds, then Claim 2 implies that $\mu(G) \ge 2k - \frac{3}{2}\Delta(G) + 2$. Now Theorem 13 gives

$$\begin{aligned} k+1 &\le \Delta(Q) + 1 - \frac{2k - \frac{3}{2}\Delta(G) + 2}{2} \\ &= \Delta(Q) + 1 - k + \frac{3}{4}\Delta(G) - 1, \end{aligned}$$

so

$$2(k+1) \le \Delta(Q) + 1 + \frac{3}{4}\Delta(G).$$

Substituting $\Delta(G) \le k$ and solving for $k$ gives

$$k \le \frac{4}{5}\Delta(Q) - \frac{4}{5} < \frac{5}{6}\Delta(Q) + \frac{1}{2} \le k,$$

which is a contradiction.

Suppose (3) holds. Now

$$2 + \sum_{i \in [3]} k - d(x_i) \leq k,$$

so Claim 3 implies

$$3\left(\frac{2}{3}\Delta(G) - 2\right) \geq 2k + 2,$$

which is a contradiction, since $\Delta(G) \leq k$.

Suppose (4) holds. Now

$$2 + \sum_{i \in [3]} k - d(x_i) \leq k,$$

so Claims 2 and 4 give

$$\left(\frac{3}{5} + \frac{13}{20} + \frac{3}{4}\right)\Delta(G) - \left(\frac{34}{15} + \frac{16}{15} + 1\right) \geq 2k + 2,$$

which is

$$2\Delta(G) - \frac{13}{3} \geq 2k + 2,$$

again a contradiction, since $\Delta(G) \leq k$.

So (5) must hold. But now

$$2 + \sum_{i \in [4]} k - d(x_i) \leq k,$$

so using Claim 2 gives

$$4\left(\frac{3}{4}\Delta(G) - 1\right) \geq 3k + 2,$$

a contradiction since $\Delta(G) \leq k$. □

This finishes the final case of Claim 5, which proves the theorem. □

In the previous theorem, we showed that $\chi(Q) \leq \max\left\{\lceil \chi_f(Q) \rceil, \Delta(G) + 1, \left\lceil \frac{5\Delta(Q)+3}{6} \right\rceil\right\}$. Now we show that if the maximum is attained by the second argument, then $G$ satisfies the $\frac{5}{6}$-Conjecture. We use the following lemma, which is implicit in [?].

**Lemma 15.** *If $Q$ is the line graph of a graph $G$ and $Q$ is vertex critical, then*

$$\chi(Q) \leq \max\left\{\Delta(G), \Delta(Q) + 1 + 2\mu(G) - \Delta(G)\right\}.$$

*Proof.* The fan equation implies this (see the proof in strengthening Brooks paper). □

**Corollary 16.** *If $Q$ is the line graph of a critical graph $G$ and $\chi(Q) \leq \Delta(G) + 1$, then*

$$\chi(Q) \leq \max\left\{\omega(Q), \left\lceil \frac{5\Delta(Q) + 3}{6} \right\rceil\right\}.$$

*Proof.* Let $k+1 = \chi(Q) \le \Delta(G)+1$. Suppose $\chi(Q) > \omega(Q)$. Then Lemma 15 gives

$$k+1 = \chi(Q) \le \Delta(Q) + 1 + 2\mu(G) - k,$$

so solving for $\mu(G)$ gives

$$\mu(G) \ge k - \frac{\Delta(Q)}{2}.$$

Applying Theorem 13 gives

$$k+1 = \chi(Q) \le \Delta(Q) + 1 - \frac{k - \frac{\Delta(Q)}{2} - 1}{2},$$

and solving for $k+1$ yields

$$\chi(Q) = k+1 \le \frac{5}{6}\Delta(Q) + \frac{4}{3} \le \left\lceil \frac{5\Delta(Q)+3}{6} \right\rceil.$$

□

Since $\omega(Q) \le \lceil \chi_f(Q) \rceil$, Theorem 14 and Corollary 16 together imply the following.

**Corollary 17.** *If $Q$ is the line graph of a multigraph, then*

$$\chi(Q) \le \max\left\{ \lceil \chi_f(Q) \rceil, \left\lceil \frac{5\Delta(Q)+3}{6} \right\rceil \right\}.$$

## 7. The $\frac{5}{6}$-Conjecture

**Lemma 18.** *Let $G$ be a critical, elementary graph with $\chi'(G) = k+1$ where $k \ge \Delta(G)+1$. Put $Q := L(G)$. If $k = \epsilon(\Delta(Q)+1) + \beta$, then for all $x \in V(G)$,*

$$|N(x)| = \frac{\epsilon\left(|G| - \Delta(G) - d_G(x) - 1 + S_1 + S_2 + S_3\left(|G|-1\right)\right)}{(1-\epsilon)\Delta(G) - \epsilon d_G(x) + 1 - \beta + S_3},$$

*where*

$$S_1 := \sum_{v \in N(x)} \Delta(Q) - d_Q(xv),$$

$$S_2 := 2 + \sum_{v \in V(G) \setminus N(x)} \Delta(G) - d_G(v),$$

$$S_3 := k - (\Delta(G)+1).$$

*Proof.* Since $G$ is critical and elementary, $|G|$ is odd and

$$k = \frac{2(\|G\|-1)}{|G|-1}. \tag{1}$$

Let $x \in V(G)$, put $M := |N(x)|$ and

$$P := \sum_{v \in N(x)} d_G(v).$$

Then

$$2(\|G\|-1) = \Delta(G)(|G|-M) - S_2 + P. \tag{2}$$

Since

$$\frac{2(\|G\|-1)}{|G|-1} = k = \Delta(G) + 1 + S_3,$$

using (2), we get

$$P = (|G|-1)(\Delta(G) + 1 + S_3) - \Delta(G)(|G| - M) + S_2,$$

which is

$$P = \Delta(G)(M-1) + |G| - 1 + S_2 + S_3(|G|-1). \tag{3}$$

Also, using $k = \epsilon\,(\Delta(Q)+1) + \beta$, we get

$$kM = \beta M + \epsilon S_1 + \epsilon \sum_{v \in N(x)} d_G(x) + d_G(v) - \mu(xv),$$

Since $\sum_{v \in N(x)} \mu(xv) = d_G(x)$,we have

$$kM = \beta M + \epsilon S_1 + \epsilon d_G(x)(M-1) + \epsilon P. \tag{4}$$

Plugging (3) into (4) and solving for $M$ gives

$$M = \frac{\epsilon\,(|G| - \Delta(G) - d_G(x) - 1 + S_1 + S_2 + S_3\,(|G|-1))}{(1-\epsilon)\Delta(G) - \epsilon d_G(x) + 1 - \beta + S_3},$$

as desired. □

Using $\epsilon = \frac{5}{6}$, we get the following.

**Lemma 19.** *Let $G$ be a critical, elementary graph with $\chi'(G) = k+1$ where $k \geq \Delta(G)+1$. Put $Q := L(G)$. If $k = \frac{5}{6}\,(\Delta(Q)+1) + \beta$, then for all $x \in V(G)$,*

$$|N(x)| = \frac{5\,(|G| - \Delta(G) - d_G(x) - 1 + S_1 + S_2 + S_3\,(|G|-1))}{\Delta(G) - 5d_G(x) + 6(1 - \beta + S_3)},$$

*where*

$$S_1 := \sum_{v \in N(x)} \Delta(Q) - d_Q(xv),$$

$$S_2 := 2 + \sum_{v \in V(G) \setminus N(x)} \Delta(G) - d_G(v),$$

$$S_3 := k - (\Delta(G)+1).$$

**Lemma 20.** *Let $G$ be a critical, elementary graph with $\chi'(G) = k+1$ where $k \geq \Delta(G)+1$. Put $Q := L(G)$. If $k = \frac{5}{6}\,(\Delta(Q)+1) + \beta$ where $\beta \geq -\frac{1}{3}$, then for all $x \in V(G)$ with $|N(x)| \geq 3$,*

$$d_G(x) \leq \frac{|N(x)|}{5\,(|N(x)|-2)}\Delta(G) - \frac{1}{|N(x)|-2} \sum_{v \in V(G) \setminus N[x]} \Delta(G) - d_G(v).$$

*Proof.* Say $|N(x)| = 2 + S_4$ for some $S_4 \geq 1$. Applying Lemma 19 and simplifying using $S_1 \geq 0$ and $\beta \geq -\frac{1}{3}$ gives

$$(5 + 5S_4)d_G(x) \leq (7 + S_4)\Delta(G) - 5|G| + 21 + S_3(-5|G| + 17 + 6S_4) + 8S_4 - 5S_2. \tag{5}$$

Put

$$t := \sum_{v \in V(G) \setminus N[x]} \Delta(G) - d_G(v).$$

Then $S_2 = t + 2 + \Delta(G) - d_G(x)$. Using this in (5), we get

$$5S_4 d_G(x) \leq (2 + S_4)\Delta(G) - 5|G| + 11 + S_3(-5|G| + 17 + 6S_4) + 8S_4 - 5t. \tag{6}$$

The desired bound follows when $S_4 \leq \frac{5}{8}|G| - 2$, since then

$$-5|G| + 11 + S_3(-5|G| + 17 + 6S_4) + 8S_4 \leq 0.$$

So, suppose $S_4 > \frac{5}{8}|G| - 2$. Rearranging, we get

$$5S_4 d_G(x) \leq 3S_4\Delta(G) - 5|G| + 11 + S_3(-5|G| + 17 + 6S_4) + 8S_4 - (2S_4 - 2)\Delta(G) \tag{7}$$

We know that

$$-5|G| + 10 + 5S_4 + S_3(-5|G| + 10 + 5S_4) \leq 0,$$

so

$$5S_4 d_G(x) \leq 3S_4\Delta(G) + 1 + 7S_3 + (S_3 + 3)S_4 - (2S_4 - 2)\Delta(G) \tag{8}$$

Since $d_G(x) \geq |N(x)| \geq \frac{5}{8}|G| \geq \frac{5}{8}\Delta(G)$, we have a contradiction unless

$$1 + 7S_3 + (S_3 + 3)S_4 - (2S_4 - 2)\Delta(G) > 0$$

By Shannon's bound $S_3 \leq \frac{\Delta(G)}{2}$, so

$$1 + \left(\frac{7}{2} + 2\right)\Delta(G) + \frac{\Delta(G) + 6}{2}S_4 - 2S_4\Delta(G) > 0,$$

which is

$$1 + \left(\frac{7}{2} + 2\right)\Delta(G) > \frac{3\Delta(G) - 6}{2}S_4,$$

so

$$\frac{5}{8}|G| - 2 < S_4 < \frac{11\Delta(G) + 2}{3\Delta(G) - 6}.$$

□

**Corollary 21.** *Let $G$ be a critical, elementary graph with $\chi'(G) = k+1$ where $k \geq \Delta(G)+1$. Put $Q := L(G)$. If $k = \frac{5}{6}(\Delta(Q) + 1) + \beta$ where $\beta \geq -\frac{1}{3}$, then there are at most two $x \in V(G)$ with $|N(x)| \geq 3$ and if there are two such $x_1, x_2$, then $|N(x_1)| = |N(x_2)| = 3$ and $x_1 \leftrightarrow x_2$.*

*Proof.* Since $G$ is critical and elementary, $|G|$ is odd and

$$\frac{2(\|G\| - 1)}{|G| - 1} = k \geq \Delta(G) + 1,$$

so

$$2\,\|G\| \geq \Delta(G)\,|G| + |G| - \Delta(G) + 1.$$

In particular,

$$\sum_{v \in V(G)} \Delta(G) - d_G(v) \leq \Delta(G) - 1 - |G|\,.$$

By Lemma 20, every $x \in V(G)$ with $3 \leq |N(x)| \leq \frac{5}{8}|G|$ has $d_G(x) \leq \frac{3}{5}\Delta(G)$, so there are at most two such $x$ since $\frac{2}{5} + \frac{2}{5} + \frac{2}{5} > 1$.

Suppose there are two such $x_1, x_2$ with $x_1 \not\leftrightarrow x_2$. If $|N(x_1)| \geq 4$, then Lemma 20 gives $d_G(x_1) \leq \frac{2}{5}\Delta(G)$ which is impossible because $d_G(x_2) \leq \frac{3}{5}\Delta(G)$. So, we must have $|N(x_1)| = |N(x_2)| = 3$. Since $x_1 \not\leftrightarrow x_2$, Lemma 20 gives for $i \in [2]$,

$$d_G(x_i) \leq \frac{3}{5}\Delta(G) - (\Delta(G) - d_G(x_{3-i})),$$

so

$$d_G(x_i) - d_G(x_{3-i}) \leq -\frac{2}{5}\Delta(G),$$

so $d_G(x_1) < d_G(x_2) < d_G(x_1)$, a contradiction. □

**Theorem 22.** *If $Q$ is the line graph of a multigraph, then $\chi(Q) \leq \max\left\{\omega(Q), \left\lceil \frac{5\Delta(Q)+3}{6} \right\rceil\right\}$.*

*Proof.* Suppose the theorem is false and choose a counterexample $Q$ minimizing $|Q|$. Then $Q = L(G)$ for a critical multigraph $G$. Say $\chi(Q) = \chi'(G) = k+1$. Then $k = \max\left\{\omega(Q), \left\lceil \frac{5\Delta(Q)+3}{6} \right\rceil\right\}$ by minimality of $|Q|$. By Corollary 17,

$$k + 1 \leq \max\left\{\lceil \chi_f(Q) \rceil, \left\lceil \frac{5\Delta(Q)+3}{6} \right\rceil\right\},$$

so

$$\lceil \chi_f(Q) \rceil = \left\lceil \frac{5\Delta(Q)+3}{6} \right\rceil + 1 = \chi(Q).$$

Therefore $G$ is elementary and $k = \frac{5}{6}(\Delta(Q)+1) + \beta$ for some $\beta \geq -\frac{1}{3}$. By Corollary 16, $k \geq \Delta(G) + 1$. Hence we may apply Corollary 21 to conclude that there are at most two $x \in V(G)$ with $|N(x)| \geq 3$ and if there are two such $x_1, x_2$, then $|N(x_1)| = |N(x_2)| = 3$ and $x_1 \leftrightarrow x_2$.

□

## References