An improvement on Brooks' theorem

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Outline

Introduction

Theorem (Brooks 1941)

Every graph with $\Delta \geq 3$ *satisfies* $\chi \leq \max\{\omega, \Delta\}$.

Definition

The *Ore-degree* of an edge xy in a graph G is $\theta(xy) = d(x) + d(y)$. The *Ore-degree* of a graph G is $\theta(G) = \max_{xy \in E(G)} \theta(xy)$.

- every graph satisfies $\left\lfloor \frac{\theta}{2} \right\rfloor \leq \Delta$
- greedy coloring (in any order) shows that every graph satisfies $\chi \leq \left\lfloor \frac{\theta}{2} \right\rfloor + 1$

Theorem (Kierstead and Kostochka 2009)

Every graph with $\theta \geq 12$ *satisfies* $\chi \leq \max\left\{\omega, \left\lfloor\frac{\theta}{2}\right\rfloor\right\}$.

Kierstead and Kostochka [2] conjectured that the 12 could be reduced to 10. That this would be best possible can be seen from the following example which has $\theta = 9$, $\omega = 4$ and $\chi = 5$.

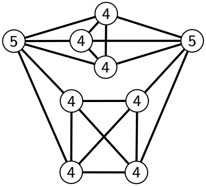

Figure: O_5, a counterexample with $\theta = 9$.

Rephrasing the problem

- let G be a critical graph with $\chi = \left\lfloor \frac{\theta}{2} \right\rfloor + 1$
- it follows that G must satisfy $\theta \leq 2\chi - 1$
- if $\Delta < \chi$ we are done by Brooks' theorem
- otherwise we have $\theta \geq \delta + \Delta \geq 2\chi - 1$ giving $\theta = 2\chi - 1$
- thus, $\chi = \Delta$ and no two vertices of max degree in G can be adjacent

Definition

Let G be a graph. The low vertex subgraph $\mathcal{L}(G)$ is the graph induced on the vertices of degree $\chi(G) - 1$. The high vertex subgraph $\mathcal{H}(G)$ is the graph induced on the vertices of degree at least $\chi(G)$.

Problem

Prove that $K_{\Delta(G)+1}$ is the only critical graph G with $\chi(G) \geq \Delta(G) \geq 6$ such that $\mathcal{H}(G)$ is edgeless.

Kierstead and Kostochka's proof

- take a minimal counterexample G and use minimality to prove some structural properties
- $\mathcal{H}(G)$ has at most as many components as $\mathcal{L}(G)$ by a result of Stiebitz [7]
- since $\mathcal{H}(G)$ is edgeless it has at most as many vertices as $\mathcal{L}(G)$ has components
- apply Alon and Tarsi's algebraic list coloring theorem to an auxilliary bipartite graph
- do some counting and get a contradiction
- it only works for $\theta \geq 12$

In [5] we solved the problem in a more general fashion.

Theorem (Rabern 2010)

$K_{\Delta(G)+1}$ is the only critical graph G with $\chi(G) \geq \Delta(G) \geq 6$ and $\omega(\mathcal{H}(G)) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2$.

Setting $\omega(\mathcal{H}(G)) = 1$ proves the conjecture of Kierstead and Kostochka.

Proof outline

- take a minimal counterexample G and use minimality to prove some structural properties
- run a carefully chosen recoloring algorithm to construct a large "dense" subgraph H
- inductively $\Delta - 1$ color $G - H$
- use minimality of G to show that the $\Delta - 1$ coloring can be completed to H

Partitioned colorings

Definition

Let G be a vertex critical graph. Let $a \geq 1$ and $r_1, \ldots, r_a$ be such that $1 + \sum_i r_i = \chi(G)$. By a $(r_1, \ldots, r_a)$-*partitioned coloring* of G we mean a proper coloring of G of the form

$$\{\{x\}, L_{11}, L_{12}, \ldots, L_{1r_1}, L_{21}, L_{22}, \ldots, L_{2r_2}, \ldots, L_{a1}, L_{a2}, \ldots, L_{ar_a}\}.$$

Here $\{x\}$ is a singleton color class and each L_{ij} is a color class.

Mozhan's Lemma

Lemma (Mozhan 1983)

Let G be a vertex critical graph. Let $a \geq 1$ and $r_1, \ldots, r_a$ be such that $1 + \sum_i r_i = \chi(G)$. Of all $(r_1, \ldots, r_a)$-partitioned colorings of G pick one minimizing

$$\sum_{i=1}^{a} \left| E\left(G\left[\bigcup_{j=1}^{r_i} L_{ij} \right] \right) \right|.$$

Remember that $\{x\}$ is a singleton color class in the coloring. Put $U_i = \bigcup_{j=1}^{r_i} L_{ij}$ and let $Z_i(x)$ be the component of x in $G[\{x\} \cup U_i]$. If $d_{Z_i(x)}(x) = r_i$, then $Z_i(x)$ is complete if $r_i \geq 3$ and $Z_i(x)$ is an odd cycle if $r_i = 2$.

The recoloring algorithm

- take a $(\lfloor \frac{\Delta-1}{2} \rfloor, \lceil \frac{\Delta-1}{2} \rceil)$-partitioned coloring minimizing the above function
- prove that we may assume that x is a low vertex
- by Mozhan's lemma, the neighborhood of x in each part induces a clique or an odd cycle
- swap x with a low vertex x_1 in the right part
- swap x_1 with a low vertex x_2 in the left part
- continue swapping back and forth until you wrap around
- use the fact that you wrapped around to show that there are many edges between the two induced cliques (odd cycles)
- we have now constructed the desired large "dense" subgraph

Generalizing maximum degree

Definition

For $0 \leq \epsilon \leq 1$, define $\Delta_\epsilon(G)$ as

$$\left\lfloor \max_{xy \in E(G)} (1-\epsilon)\min\{d(x), d(y)\} + \epsilon \max\{d(x), d(y)\} \right\rfloor.$$

Note that $\Delta_1 = \Delta$, $\Delta_{\frac{1}{2}} = \left\lfloor \frac{\theta}{2} \right\rfloor$.

The generalized bound

Theorem (Rabern 2010)

For every $0 < \epsilon \leq 1$*, there exists* t_ϵ *such that every graph with* $\Delta_\epsilon \geq t_\epsilon$ *satisfies*

$$\chi \leq \max\{\omega, \Delta_\epsilon\}.$$

- the proof uses a recoloring algorithm similar to the above
- it would be interesting to determine, for each ϵ, the smallest t_ϵ that works
- that $t_1 = 3$ is smallest is Brooks' theorem
- the graph O_5 shows that $t_\epsilon = 6$ is smallest for $\frac{1}{2} \leq \epsilon < 1$
- we will see below that if $P \neq NP$, then t_0 does not exist and hence $t_\epsilon \to \infty$ as $\epsilon \to 0$

What about Δ_0?

- the above proofs only work for $\epsilon > 0$
- what happens when $\epsilon = 0$?
- the parameter Δ_0 has already been investigated by Stacho [6] under the name Δ_2

Definition (Stacho 2001)

For a graph G define

$$\Delta_2(G) = \max_{xy \in E(G)} \min\{d(x), d(y)\}.$$

Facts about Δ_2

- $\Delta_2 = \Delta_0$
- greedy coloring (in any order) shows that every graph satisfies $\chi \leq \Delta_2 + 1$
- for any fixed $t \geq 3$, the problem of determining whether or not $\chi(G) \leq \Delta_2(G)$ for graphs with $\Delta_2(G) = t$ is *NP*-complete (see [6])

A tempting thought

A tempting thought

There exists t such that every graph with $\Delta_2 \geq t$ satisfies $\chi \leq \max\{\omega, \Delta_2\}$.

- unfortunately, the tempting thought cannot hold for any t if $P \neq NP$
- to show this, we use Lovász's ϑ parameter [1] which can be appoximated in polynomial time and has the property that $\omega(G) \leq \vartheta(G) \leq \chi(G)$

A polynomial-time algorithm

- assume the tempting thought holds for some $t \geq 3$
- take any arbitrary graph with $\Delta_2 \geq t$
- first, compute Δ_2 in polynomial time
- second, compute x such that $x - \frac{1}{2} < \vartheta < x + \frac{1}{2}$ in polynomial time
- if $x \geq \Delta_2 + \frac{1}{2}$, then $\chi \geq \vartheta > \Delta_2$ and hence $\chi = \Delta_2 + 1$
- if $x < \Delta_2 + \frac{1}{2}$, then $\omega \leq \vartheta < \Delta_2 + 1$, and hence $\omega \leq \Delta_2$
- now, $\chi \leq \max\{\omega, \Delta_2\} \leq \Delta_2$
- we just gave a polynomial time algorithm to determine whether or not $\chi \leq \Delta_2$ for graphs with $\Delta_2 \geq t$
- this is impossible unless $P{=}NP$

What we can prove about Δ_0 (aka Δ_2)

Theorem (Rabern 2010)

Every graph with $\Delta \geq 3$ satisfies

$$\chi \leq \max\left\{\omega, \Delta_2, \frac{5}{6}(\Delta + 1)\right\}.$$

- the proof uses a recoloring algorithm similar to the above
- actually, all the above results about Δ_ϵ follow from this result

In joint work with Kostochka and Stiebitz [3] similar techniques were used to improve the bounds further. We give some highlights.

Theorem (Kostochka, Rabern and Stiebitz 2010)

Every graph with $\theta \geq 8$, except O_5, satisfies $\chi \leq \max\left\{\omega, \left\lfloor\frac{\theta}{2}\right\rfloor\right\}$.

Theorem (Kostochka, Rabern and Stiebitz 2010)

Every graph satisfies

$$\chi \leq \max\left\{\omega, \Delta_2, \frac{3}{4}(\Delta + 2)\right\}.$$

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