Let $c_k^*(\mathcal{L})$ be the number of components of $\mathcal{L}$ containing a copy of $K_{k-1}$. Let $q_k(\mathcal{L})$ be the number of non-cut vertices in $\mathcal{L}$ that appear in copies of $K_{k-1}$. Let $\beta_k(\mathcal{L})$ be the independence number of the subgraph of $\mathcal{L}$ induced on the vertices of degree $k-1$. When $k$ is defined in context, we just write $c^*(\mathcal{L})$, $q(\mathcal{L})$ and $\beta(\mathcal{L})$. Let $\mathcal{H}(G)$ be the subgraph of $G$ induced on vertices of degree greater than $\delta(G)$. Let $\mathcal{L}(G)$ be the subgraph of $G$ induced on vertices of degree $\delta(G)$.

**Definition 1.** The *maximum independent cover number* of a graph $G$ is the maximum $\mathrm{mic}(G)$ of $\|I, V(G) \setminus I\|$ over all independent sets $I$ of $G$.

**Definition 2.** A graph $G$ is *OC-reducible* to $H$ if $H$ is a nonempty induced subgraph of $G$ which is online $f_H$-choosable where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If $G$ is not OC-reducible to any nonempty induced subgraph, then it is *OC-irreducible*.

**Lemma 1.** *Every OC-irreducible graph $G$ satisfies*

$$2\,\|G\| > (\delta(G) - 1)\,|G| + \mathrm{mic}(G).$$

**Lemma 2.** *If $G$ is an OC-irreducible graph where $\mathcal{H}(G)$ is edgeless, $\Delta := \Delta(G) = \delta(G) + 1$ and $\mathcal{L} := \mathcal{L}(G)$, then*

$$2\,\|\mathcal{L}\| > \left(\Delta - 2 - \frac{2}{\Delta - 2}\right)|\mathcal{L}| + \frac{\Delta(\Delta - 1)}{\Delta - 2}\beta_\Delta(\mathcal{L}).$$

*Proof.* Let $G$ be such a graph. Put $\mathcal{H} := \mathcal{H}(G)$ and $\mathcal{L} := \mathcal{L}(G)$. Since $\mathcal{H}$ is edgeless,

$$\begin{aligned}
\Delta\,|\mathcal{H}| &= \|\mathcal{H}, \mathcal{L}\| \\
&= (\Delta - 1)\,|\mathcal{L}| - 2\,\|\mathcal{L}\|,
\end{aligned} \tag{1}$$

so, by Lemma 1,

$$\begin{aligned}
(\Delta - 1)\,|\mathcal{L}| + \Delta\,|\mathcal{H}| &= 2\,\|G\| \\
&> (\Delta - 2)\,|G| + \mathrm{mic}(G) \\
&\geq (\Delta - 2)\,|G| + \Delta\,|\mathcal{H}| + (\Delta - 1)\beta_\Delta(\mathcal{L}) \\
&= (\Delta - 2)\,|\mathcal{L}| + (2\Delta - 2)\,|\mathcal{H}| + (\Delta - 1)\beta_\Delta(\mathcal{L}),
\end{aligned}$$

so simplifying and using (1) again gives

$$|\mathcal{L}| > (\Delta - 2)|\mathcal{H}| + (\Delta - 1)\beta_\Delta(\mathcal{L})$$
$$= \frac{\Delta - 2}{\Delta}\left((\Delta - 1)|\mathcal{L}| - 2\|\mathcal{L}\|\right) + (\Delta - 1)\beta_\Delta(\mathcal{L}),$$

now some mild manipulation yields the desired bound. □

**Lemma 3.** $\left(\frac{3k-7}{k^2-4k+5}, \frac{(k-1)(k-4)}{k^2-4k+5}, 2, \frac{-2(k-1)(k-4)}{k^2-4k+5}\right)$ *is 5-Gallai.*

**Lemma 4.** *Let $G$ be a non-complete AT-irreducible graph with $\delta(G) = k - 1$ where $k \geq 5$. Let $\mathcal{L}$ be the subgraph of $G$ induced on $(k-1)$-vertices, $\mathcal{H}^-$ the subgraph of $G$ induced on $k$-vertices and $\mathcal{H}^+$ the subgraph of $G$ induced on $(k+1)^+$-vertices. Then*

$$q(\mathcal{L}) \leq c^*(\mathcal{L}) + 4\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\|,$$

*and if $k \geq 7$, then*

$$q(\mathcal{L}) \leq 2c^*(\mathcal{L}) + 3\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\|.$$

**Lemma 5.** *If $G$ is an OC-irreducible graph where $\mathcal{H}(G)$ is edgeless, $\Delta := \Delta(G) = \delta(G) + 1 \geq 7$, $\mathcal{L} := \mathcal{L}(G)$ and $\mathcal{H} := \mathcal{H}(G)$, then*

$$2\|\mathcal{L}\| \leq \left(\Delta - 3 + \frac{3\Delta - 7}{\Delta^2 - 4\Delta + 5}\right)|\mathcal{L}| + \frac{3(\Delta - 1)(\Delta - 4)}{\Delta^2 - 4\Delta + 5}|\mathcal{H}| + 2\beta_\Delta(\mathcal{L}).$$

*Proof.* Combine the second inequality in Lemma 4 with Lemma 3. □

**Lemma 6.** *If $G$ is an OC-irreducible graph where $\mathcal{H}(G)$ is edgeless, $\Delta := \Delta(G) = \delta(G) + 1 \geq 7$, $\mathcal{L} := \mathcal{L}(G)$ and $\mathcal{H} := \mathcal{H}(G)$, then*

$$\left(\Delta - 10 + \frac{4(\Delta + 1)}{\Delta^2 - 4\Delta + 5}\right)|\mathcal{L}| + \left(\Delta^2 - 3\Delta + 4\right)\beta_\Delta(\mathcal{L}) < 0,$$

*in particular, $\Delta \leq 9$.*

*Proof.* Combine Lemma 2 with Lemma 5 and the fact that $|\mathcal{H}| < \frac{|L|}{\Delta - 2}$. □

# 1   Further Reducibility Lemmas

**Lemma 7.** *Let $G$ be a directed graph and $x_1 x_2 \in E(G)$ such that $d^-(x_i) = 1$ for all $i \in [2]$. If $y \in V(G) \setminus \{x_1, x_2\}$, then*

$$EE(G + x_1 y + x_2 y) - EO(G + x_1 y + x_2 y) = EE(G) - EO(G).$$

## 2   Using Minimality

**Definition 3.** A graph $A$ is a *child* of a graph $G$ if there exists $H \lhd G$ and an epimorphism $f \colon H \twoheadrightarrow A$.

Note that the child-of relation is a strict partial order on the set of (finite simple) graphs $\mathcal{G}$. We call this the *child order* on $\mathcal{G}$ and denote it by '$\prec$'.

**Lemma 8.** *The ordering $\prec$ is well-founded on $\mathcal{G}$. That is, every non-empty subset of $\mathcal{G}$ has a minimal element under $\prec$.*

*Proof.* Let $\mathcal{T}$ be a non-empty subset of $\mathcal{G}$. Pick $G \in \mathcal{T}$ with the minimum number of vertices. Since any child of $G$ must have fewer vertices, we see that $G$ is minimal in $\mathcal{T}$ with respect to $\prec$. $\qquad\square$

**Definition 4.** Let $\mathcal{T}$ be a collection of graphs. A minimal graph in $\mathcal{T}$ under the child order is called a $\mathcal{T}$-*mule*.