most low Alon-Tarsi notes

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1 Introduction

We consider graphs with vertices labeled by natural numbers; that is, pairs (G, h) where G is a graph and $h\colon V(G) \to \mathbb{N}$. We say that (G, h) *is AT* if G is $(d_G - h)$-AT.

2 Subgraphs, subdivisions and cuts

Lemma 2.1. *If G is connected and (G, h) is not AT, then $\left(H, h\restriction_{V(H)}\right)$ is not AT for every induced subgraph H of G where $h(v) = 0$ for all $v \in V(G) \setminus V(H)$.*

Proof. If there were such an H such that $\left(H, h\restriction_{V(H)}\right)$ is AT, then by ordering the vertices of each component of $G - V(H)$ by increasing distance to H and directing all edges away from H in this order we conclude that (G, h) is AT. $\square$

Lemma 2.2. *For any (G', h') formed from (G, h) by subdividing an edge of G twice and letting h' give zero on the new vertices, (G, h) is AT if and only if (G', h') is AT.*

Proof. This is immediate since there is a parity preserving bijection between the spanning Eulerian subgraphs of G and G'. $\square$

Lemma 2.3. *Let $\{A_1, A_2\}$ be a separation of G such that $A_1 \cap A_2 = \{x\}$. If $G[A_i]$ is f_i-AT for $i \in [2]$, then G is f-AT where $f(v) = f_i(v)$ for $v \in V(A_i - x)$ and $f(x) = f_1(x) + f_2(x) - 1$. Going the other direction, if G is f-AT, then $G[A_i]$ is f_i-AT for $i \in [2]$ where $f_i(v) = f(v)$ for $v \in V(A_i - x)$ and $f_1(x) + f_2(x) \le f(x) + 1$.*

Proof. For $i \in [2]$, choose an orientation D_i of A_i showing that A_i is f_i-AT. Together these give an orientation D of G and since no cycle has vertices in both $A_1 - x$ and $A_2 - x$, we have

$$\begin{aligned}
EE(D) - EO(D) &= EE(D_1)EE(D_2) + EO(D_1)EO(D_2) - (EE(D_1)EO(D_2) + EO(D_1)EE(D_2)) \\
&= (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2)) \\
&\neq 0.
\end{aligned}$$

Hence G is f-AT.

Now, suppose G is f-AT and choose an orientation D of G showing this. Put $D_i = D[A_i]$ for $i \in [2]$. Then, as above, we have $0 \neq EE(D) - EO(D) = (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2))$ and hence $EE(D_1) - EO(D_1) \neq 0$ and $EE(D_2) - EO(D_2) \neq 0$. Since the in-degree of x in D is the sum of the in-degree of x in D_1 and the in-degree of x in D_2, the lemma follows. □

3 Extension lemma

This is a key lemma from [1], it generalizes a lemma from [2] from list coloring to Alon-Tarsi orientations. This is what i talked about in Baltimore. The basic idea is that in some cases we can pair off odd/even spanning Eulerian subgraphs via a parity reversing bijection.

Lemma 3.1. *Let G be a multigraph without loops and $f\colon V(G) \to \mathbb{N}$. If there are $F \subseteq G$ and $Y \subseteq V(G)$ such that:*

1. *any multiple edges in G are contained in $G[Y]$; and*
2. *$f(v) \geq d_G(v)$ for all $v \in V(G) \setminus Y$; and*
3. *$f(v) \geq d_{G[Y]}(v) + d_F(v) + 1$ for all $v \in Y$; and*
4. *For each component T of $G - Y$ there are different $x_1, x_2 \in V(T)$ where $N_T[x_1] = N_T[x_2]$ and $T - \{x_1, x_2\}$ is connected such that either:*
   (a) there are $x_1y_1, x_2y_2 \in E(F)$ where $y_1 \neq y_2$ and $N(x_i) \cap Y = \{y_i\}$ for $i \in [2]$; or
   (b) $|N(x_2) \cap Y| = 0$ and there is $x_1y_1 \in E(F)$ where $N(x_1) \cap Y = \{y_1\}$,

then G is f-AT.

Proof. Suppose not and pick a counterexample (G, f, F, Y) minimizing $|G - Y|$. If $|G - Y| = 0$, then $Y = V(G)$ and thus $f(v) \geq d_G(v) + 1$ for all $v \in V(G)$ by (3). Pick an acyclic orientation D of G. Then $EE(D) = 1$, $EO(D) = 0$ and $d^+_D(v) \leq d_G(v) \leq f(v) - 1$ for all $v \in V(D)$. Hence G is f-AT. So, we must have $|G - Y| > 0$.

Pick a component T of $G - Y$ and pick $x_1, x_2 \in V(T)$ as guaranteed by (4). First, suppose (4a) holds. Put $G' := (G - T) + y_1y_2$, $F' := F - T$, $Y' := Y$ and let f' be f restricted to $V(G')$. Then G' has an orientation D' where $f'(v) \geq d^+_{D'}(v) + 1$ for all $v \in V(D')$ and $EE(D') \neq EO(D')$, for otherwise (G', f', F', Y') would contradict minimality. By symmetry we may assume that the new edge y_1y_2 is directed toward y_2. Now we use the orientation of D' to construct the desired orientation of D. First, we use the orientation on $D' - y_1y_2$ on $G - T$. Now, order the vertices of T as $x_1, x_2, z_1, z_2, \ldots$ so that every vertex has at least one neighbor to the right. Orient the edges of T left-to-right in this ordering. Finally, we use y_1x_1 and x_2y_2 and orient all other edges between T and $G - T$ away from T. Plainly, $f(v) \geq d^+_D(v) + 1$ for all $v \in V(D)$. Since y_1x_1 is the only edge of D going into T, any Eulerian subgraph of D that contains a vertex of T must contain y_1x_1. So, any Eulerian subgraph of D either contains (i) neither y_1x_1 nor x_2y_2, (ii) both y_1x_1 and x_2y_2, or (iii) y_1x_1 but not x_2y_2. We first handle (i) and (ii) together. Consider the function h that maps an Eulerian subgraph Q of D' to an Eulerian subgraph $h(Q)$ of D as follows. If Q does not contain y_1y_2, let $h(Q) = \iota(Q)$ where $\iota(Q)$ is the natural embedding of $D' - y_1y_2$ in D. Otherwise, let $h(Q) = \iota(Q - y_1y_2) + \{y_1x_1, x_1x_2, x_2y_2\}$. Then h is a parity-preserving injection with image precisely the union of those Eulerian subgraphs of D in (i) and (ii). Hence if we can show that exactly half of the Eulerian subgraphs of D in (iii) are even, we will conclude $EE(D) \neq EO(D)$, a contradiction. To do so, consider an Eulerian subgraph A of D containing y_1x_1 and not x_2y_2. Since x_1 must have in-degree 1 in A, it must also have out-degree 1 in A. We show that A has a mate A' of opposite parity. Suppose $x_2 \notin A$ and $x_1z_1 \in A$; then we make A' by removing x_1z_1 from A and adding $x_1x_2z_1$. If $x_2 \in A$ and $x_1x_2z_1 \in A$, we make A' by removing $x_1x_2z_1$ and adding x_1z_1. Hence exactly half of the Eulerian subgraphs of D in (iii) are even and we conclude $EE(D) \neq EO(D)$, a contradiction.

Now suppose (4b) holds. Put $G' := G - T$, $F' := F - T$, $Y' := Y$ and define f' by $f'(v) = f(v)$ for all $v \in V(G' - y_1)$ and $f'(y_1) = f(y_1) - 1$. Then G' has an orientation D' where $f'(v) \geq d^+_{D'}(v) + 1$ for all $v \in V(D')$ and $EE(D') \neq EO(D')$, for otherwise (G', f', F', Y') would contradict minimality. We orient $G - T$ according to D, orient T as in the previous case, again use y_1x_1 and orient all other edges between T and $G - T$ away from T. Since we decreased $f'(y_1)$ by 1, the extra out edge of y_1 is accounted for and we have $f(v) \geq d^+_D(v) + 1$ for all $v \in V(D)$. Again any additional Eulerian subgraph must contain y_1x_1 and since x_2 has no neighbor in $G - T$ we can use x_2 as before to build a mate of opposite parity for any additional Eulerian subgraph. Hence $EE(D) \neq EO(D)$ giving our final contradiction. $\square$

4 Degree-AT graphs

A graph G is called *degree-AT* if (G, h) is *AT* where h is the constant zero function.

Lemma 4.1. *A connected graph G is degree-AT if it is not a Gallai tree.*

Proof. Suppose there exists a connected graph that is not a Gallai tree, but is also not degree-AT. Let G be such a graph with as few vertices as possible. Since G is not degree-AT, no induced subgraph H of G is degree-AT by Lemma 2.1. Hence, for any $v \in V(G)$ that is not a cutvertex, $G - v$ must be a Gallai tree by minimality of $|G|$.

If G has more than one block, then for endblocks B_1 and B_2, choose noncutvertices $w \in B_1$ and $x \in B_2$. By the minimality of $|G|$, both $G - w$ and $G - x$ are Gallai trees. Since every block of G appears either as a block of $G - w$ or as a block of $G - x$, every block of G is either complete or an odd cycle. Hence, G is a Gallai tree, a contradiction. So instead G has only one block, that is, G is 2-connected. Further, $G - v$ is a Gallai tree for all $v \in V(G)$.

Let v be a vertex of minimum degree in G. Since G is 2-connected, $d_G(v) \geq 2$. By Lemma 2.3, v is adjacent to a noncutvertex in every endblock of $G - v$. If $G - v$ has a complete block B with noncutvertices x_1, x_2 where $v \leftrightarrow x_1$ and $v \not\leftrightarrow x_2$, then we can apply Lemma 3.1 with $Y = \{v\}$ and $F = vx_1$ to conclude that G is degree-AT, a contradiction. So, v must be adjacent to every noncutvertex in every complete endblock of $G - v$.

Suppose $d_G(v) \geq 3$. Then no endblock of $G - v$ can be an odd cycle of length at least 5 (there would be vertices of degree 3 but we'd have $d_G(v) \geq 4$). Let B be a smallest complete endblock of $G - v$. Then for a noncutvertex $x \in V(B)$, we have $d_G(x) = |B|$ and hence $d_G(v) \leq |B|$. If $G - v$ has at least two endblocks, then $2(|B| - 1) \leq |B|$ and hence $d_G(v) \leq |B| = 2$, a contradiction. Hence $G - v = B$ and v is joined to B, so G is complete, a contradiction.

Hence, we must have $d_G(v) = 2$. Suppose $G - v$ has at least 2 endblocks. Then, it has exactly 2 and v is adjacent to one noncutvertex in each. Neither of the endblocks can be odd cycles of length at least 5 since then we could get a smaller counterexample by Lemma 2.2. Since v is adjacent to every noncutvertex in every complete endblock of $G - v$, both endblocks must be K_2. But then either $G = C_4$ (which is trivially degree-AT) or we can get a smaller counterexample by Lemma 2.2. So, $G - v$ must be 2-connected. Since $G - v$ is a Gallai tree, it is either complete or an odd cycle. If $G - v$ is not complete, we can get a smaller counterexample by Lemma 2.2. So, $G - v$ is complete and v is adjacent to every noncutvertex of $G - v$; that is, G is complete, a contradiction. □

References

[1] Hal Kierstead and Landon Rabern, *Improved lower bounds on the number of edges in list critical and online list critical graphs*, arXiv preprint arXiv:1406.7355 (2014).

[2] A.V. Kostochka and M. Stiebitz, *A new lower bound on the number of edges in colour-critical graphs and hypergraphs*, Journal of Combinatorial Theory, Series B **87** (2003), no. 2, 374–402.