A common generalization of Hall's theorem and Vizing's edge-coloring theorem

landon rabern

LBD Data

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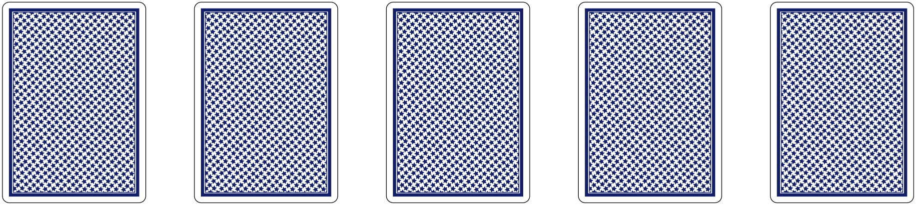
  - $A_1 = \{1, 2\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2\}$
- Hall's theorem: this is the only thing that can go wrong

$$\text{SDR exists} \Leftrightarrow \left|\bigcup_{i \in I} A_i\right| \geq |I| \text{ for all } I \subseteq \{1, \ldots, n\}$$

some card games

the simplest variation

- two players, Dealer and Player



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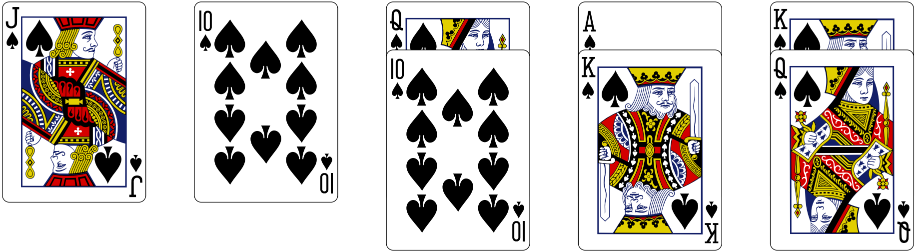
- Player wins if he can pick a Royal Flush, one card from each stack

some card games

example, a Player win



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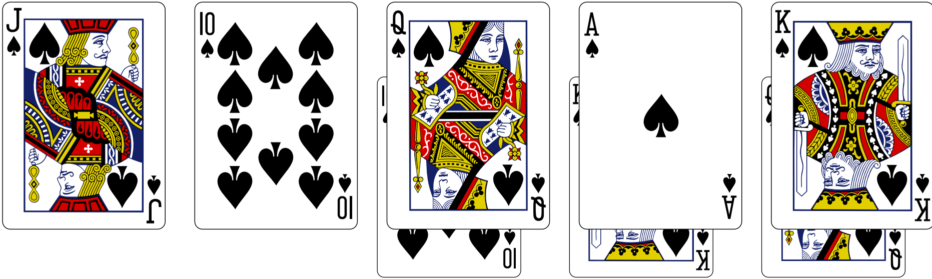
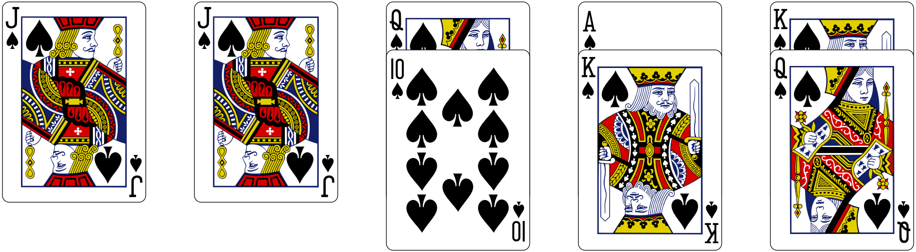


some card games

example, a Dealer win



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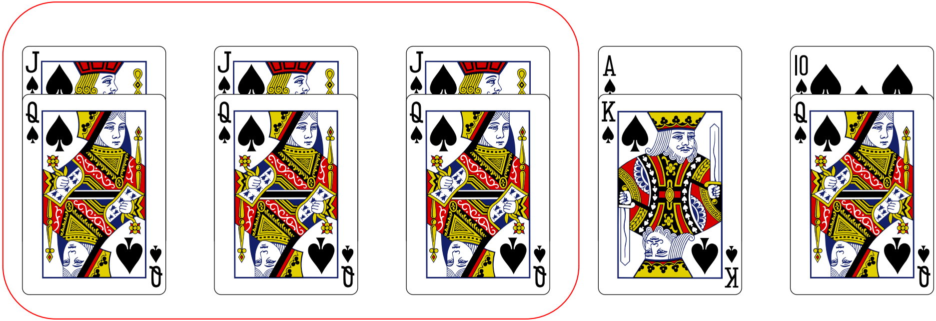

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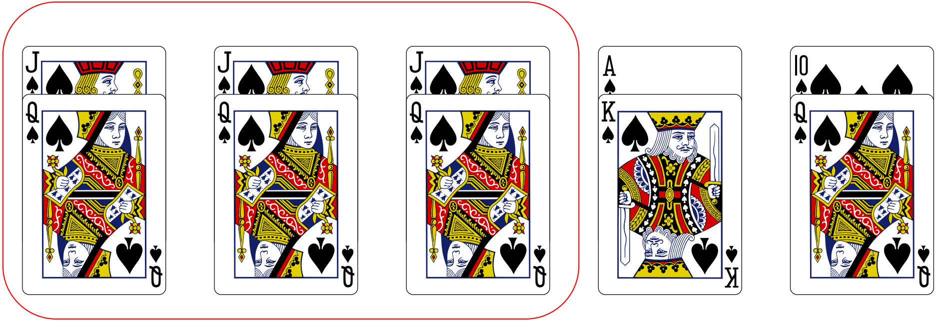
- Hall's theorem says: **Player wins otherwise**



some card games

making things harder for Dealer

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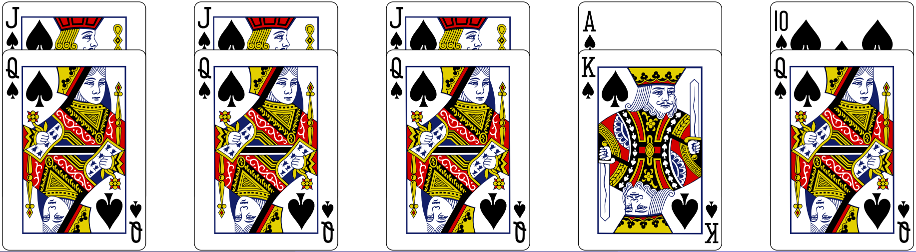
Winning

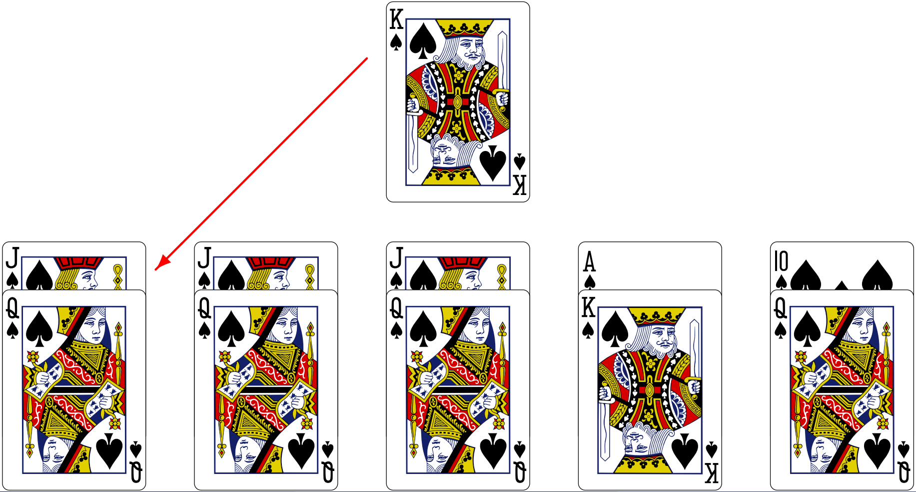
Player wins if he can pick a Royal Flush at the start of one of his turns, otherwise Dealer wins.

some card games

example, a Player win



- Player picks a King from the deck and swaps it for a Queen in the first stack



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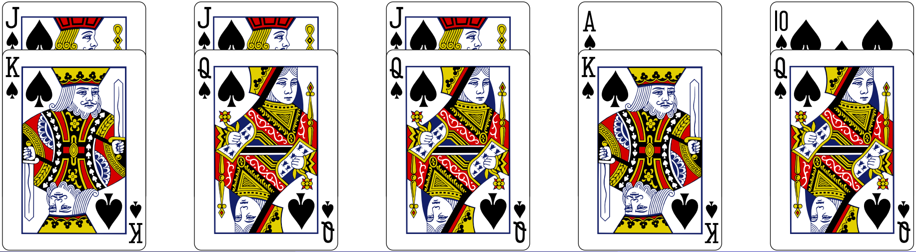

- Player picks a King from the deck and swaps it for a Queen in the first stack
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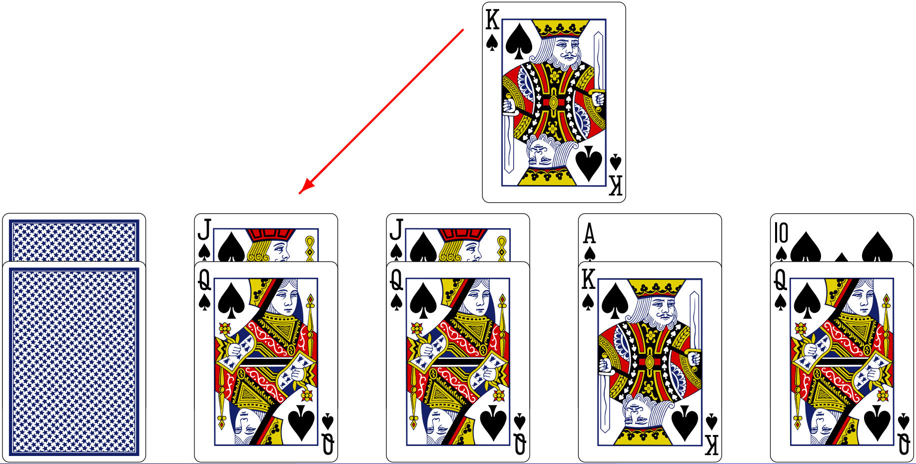

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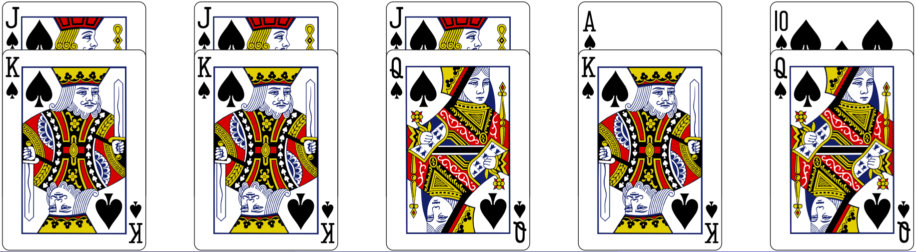

some card games

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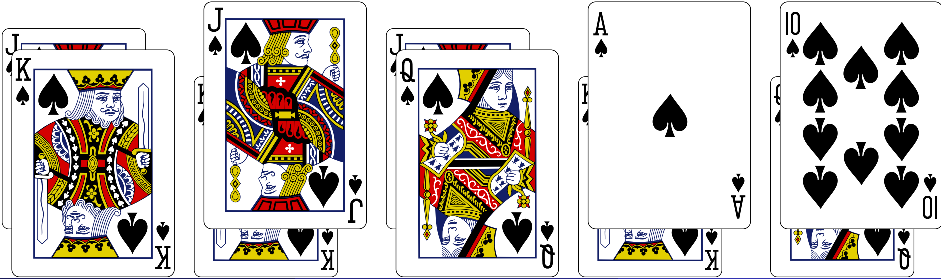
- Player picks a King from the deck and swaps it for a Queen in the first stack
- Dealer can swap a King and Queen in one of the other stacks
- Player wins no matter what Dealer does



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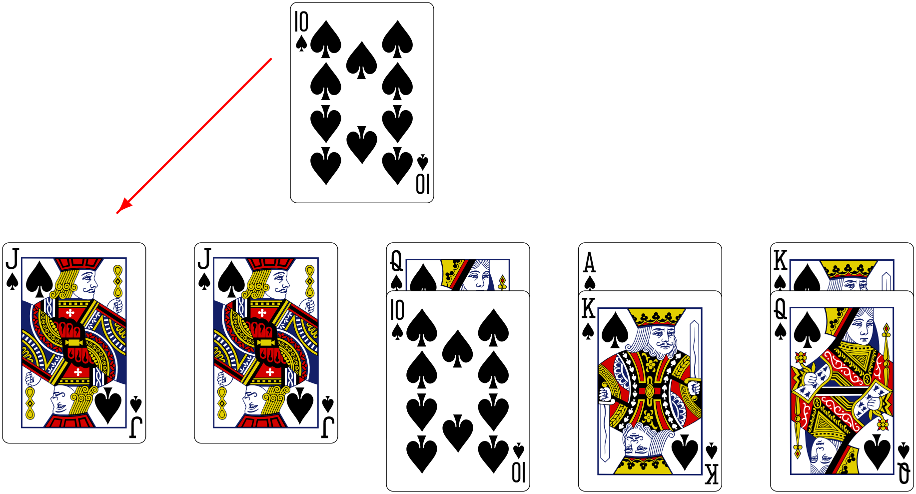

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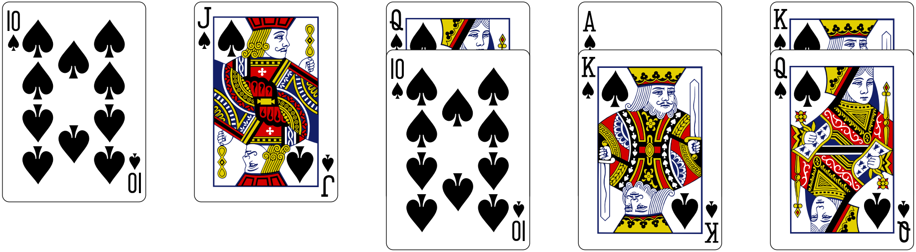

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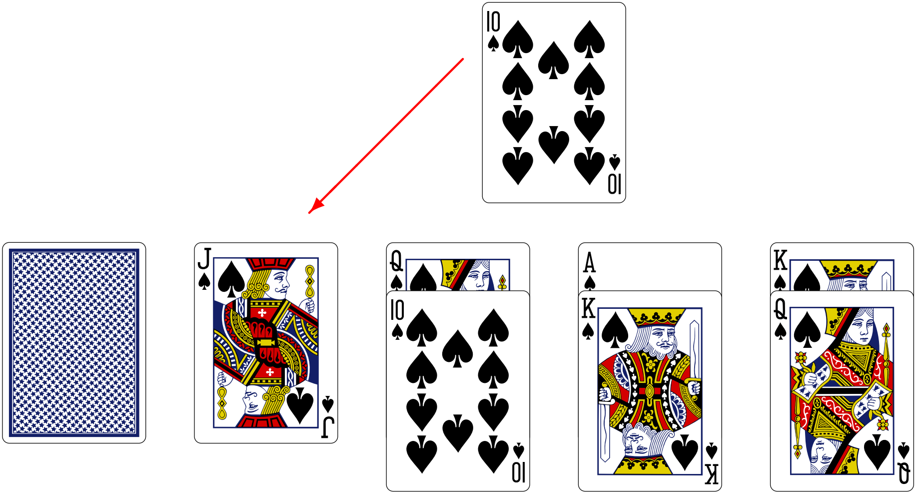

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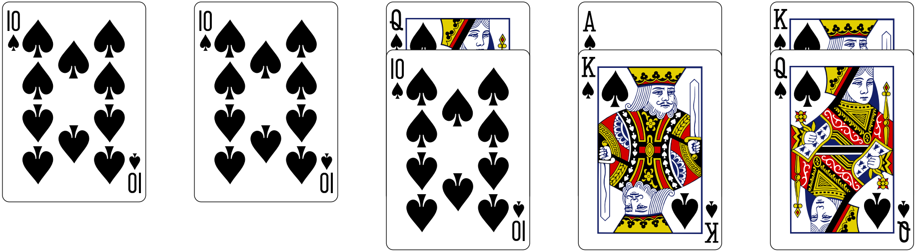

some card games

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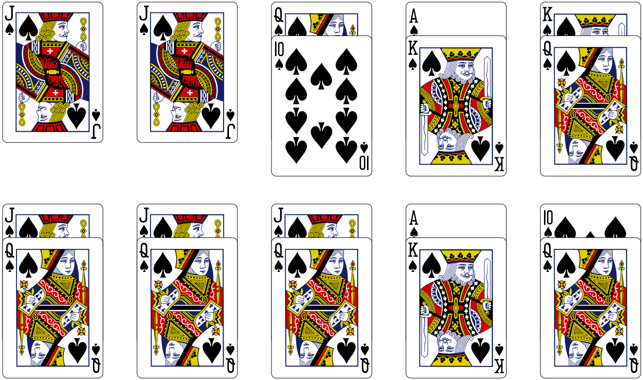

some card games

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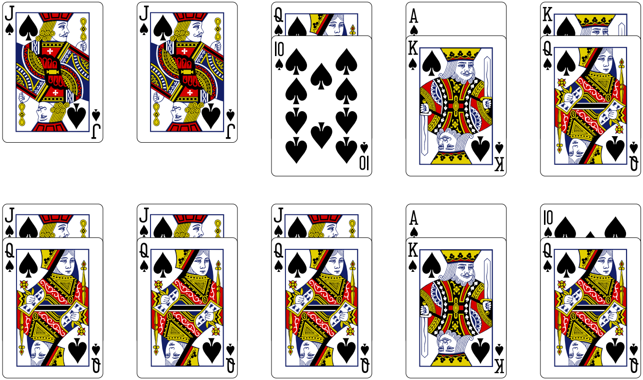
- in the top game, Dealer can prevent Player from increasing the number of different cards in the first two stacks



some card games

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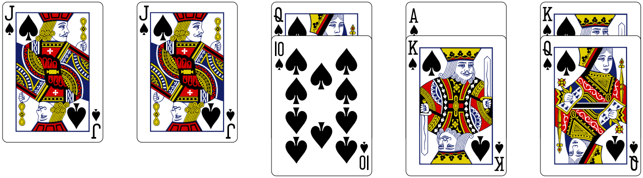
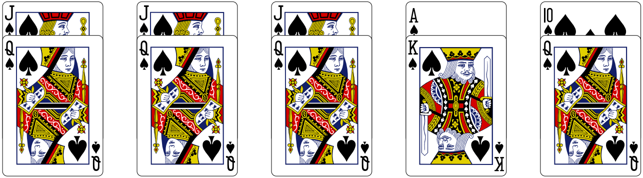
- in the top game, Dealer can prevent Player from increasing the number of different cards in the first two stacks
- in the bottom game, Dealer cannot prevent prevent Player from increasing the number of different cards in the first three stacks





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- Player can turn $2t+1$ of the same card into $t+1$ different cards, so C is 'worth' $\left\lceil \frac{d_S(C)}{2} \right\rceil$

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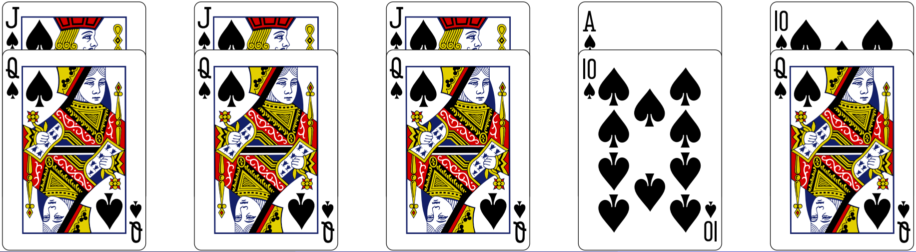

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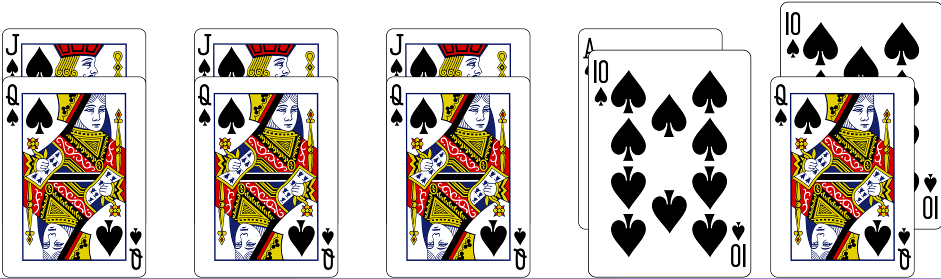

some card games

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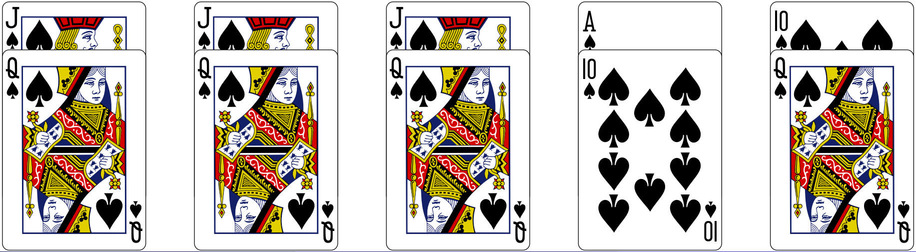

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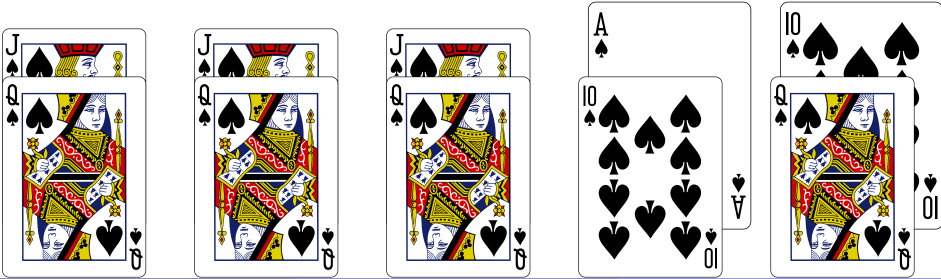

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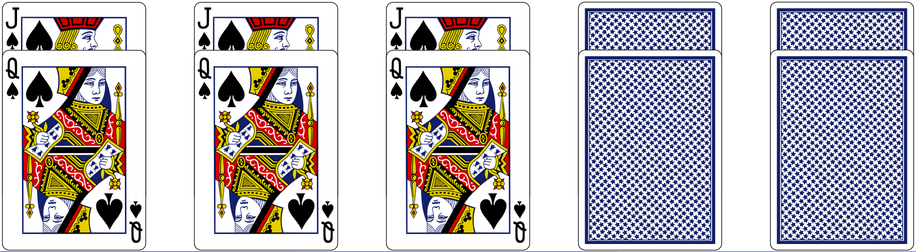
2. Player calls those stacks done and never plays with those card types again



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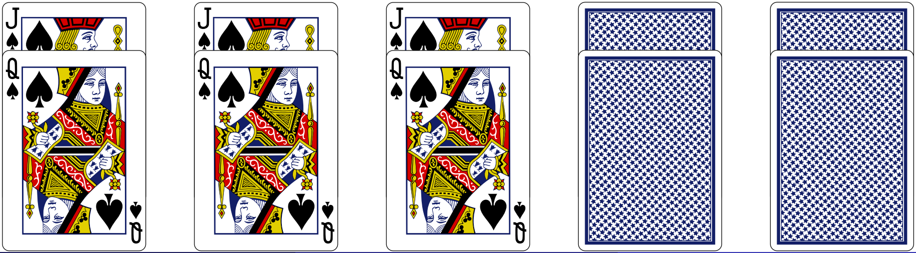

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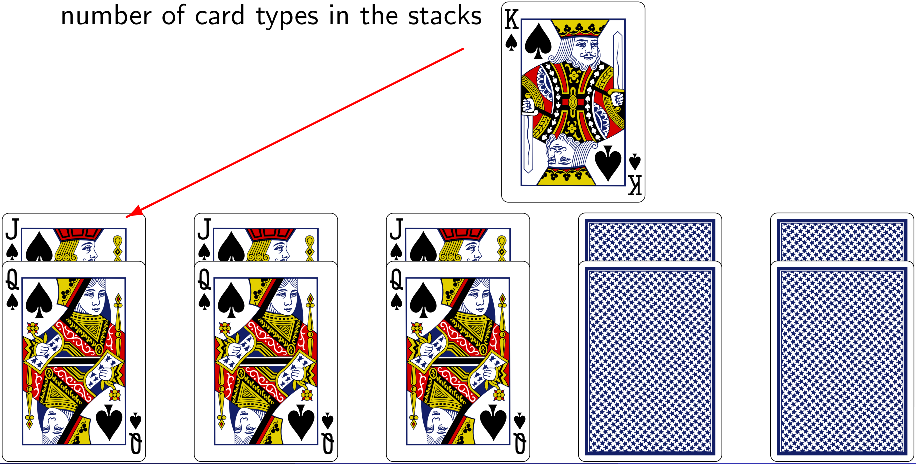

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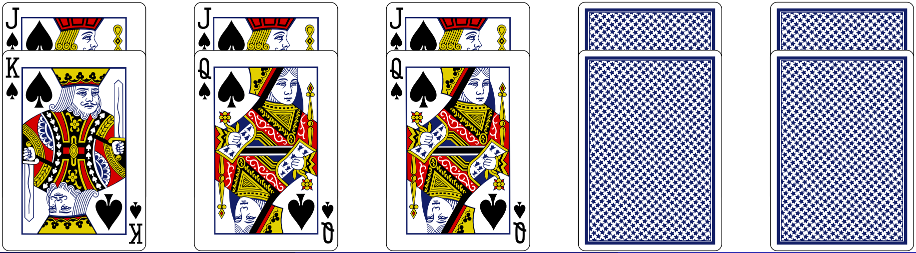

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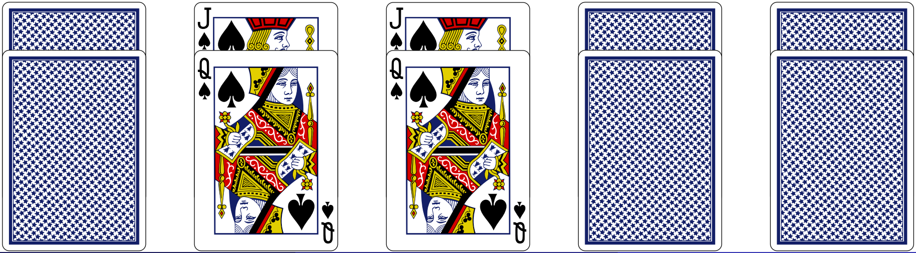

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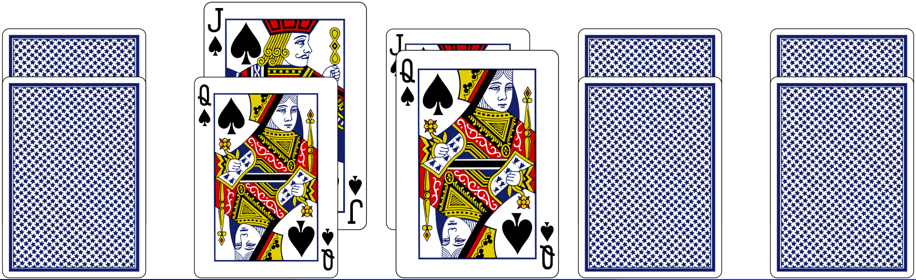

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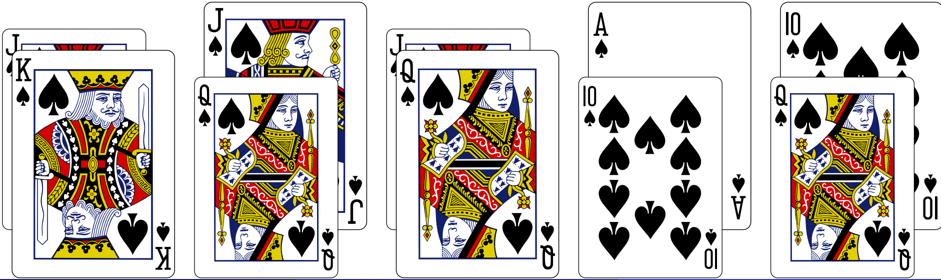

A generalization of Hall's theorem

making it harder for Player

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  - Player's moves are useless

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edge coloring
setup

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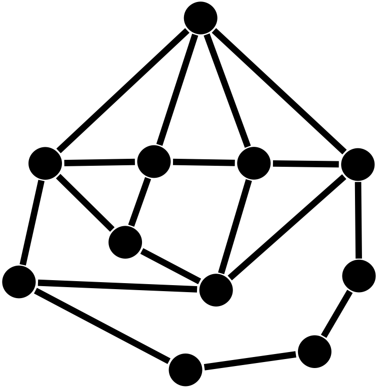
Vizing's theorem

Any simple graph can be edge-colored using at most one more color than its maximum degree.

- proceed by induction on the number of vertices



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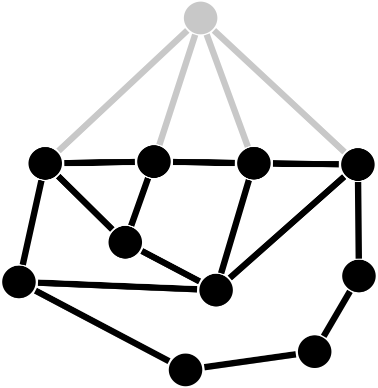

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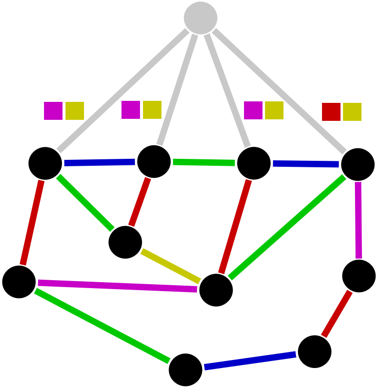

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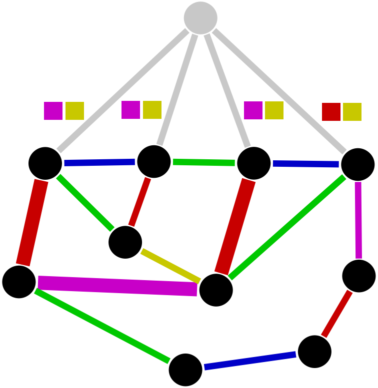

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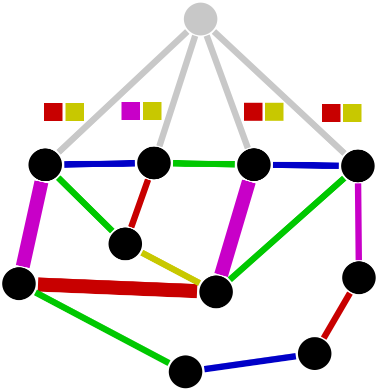

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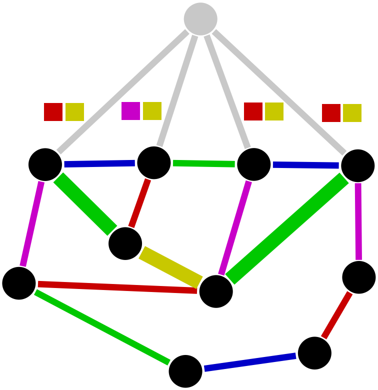

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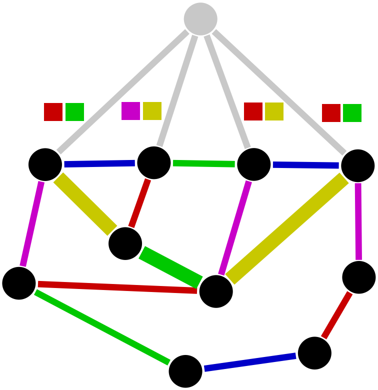

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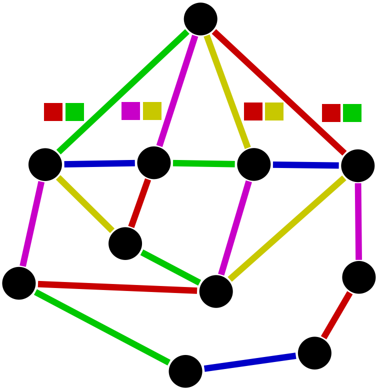

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