Edge Lower Bounds for List Critical Graphs, via Discharging

Dan and Landon

January 27, 2016

1 Introduction

A *proper coloring* of a graph G assigns colors to vertices so that adjacent vertices receive distinct colors. A graph G is k-*colorable* if G has a proper coloring using at most k colors. The *chromatic number* $\chi(G)$ of G is the least k for which G is k-colorable. A graph G is k-*chromatic* when $\chi(G) = k$, and G is k-*critical* when G is not $(k-1)$-colorable, but every proper subgraph of G is $(k-1)$-colorable. Every k-critical graph G is k-chromatic since, for any vertex v, we can extend a $(k-1)$-coloring of $G - v$ to a k-coloring of G, by giving v a new color. If G is k-chromatic, then any minimal k-chromatic subgraph of G is k-critical. As a result, many questions about k-chromatic graphs reduce to questions about k-critical graphs, which have more structure. Dirac [3] introduced critical graphs in 1951. Since every k-critical graph G has minimum degree at least $k-1$, clearly $2\|G\| \ge (k-1)|G|$. Much work has focused on determining the minimum number of edges in a k-critical graph. In 1957, Dirac [4] generalized Brooks' theorem [2] by showing that any k-critical graph G with $k \ge 4$ and $|G| \ge k+2$ must satisfy

$$2\|G\| \ge (k-1)|G| + k - 3.$$

In 1963, Gallai [6] strengthened this bound for large $|G|$. Let

$$g_k(n, c) := \left(k - 1 + \frac{k-3}{(k-c)(k-1)+k-3}\right) n.$$

Gallai showed that every k-critical graph G with $k \ge 4$ and $|G| \ge k+2$ satisfies $2\|G\| \ge g_k(|G|, 0)$. In 1997, Krivelevich [14] strengthened this lower bound to $g_k(|G|, 2)$. In 2003, Kostochka and Stiebitz [13] further strengthened this bound, for $k \ge 6$, to $2\|G\| \ge g_k(|G|, (k-5)\alpha_k)$, where

$$\alpha_k := \frac{1}{2} - \frac{1}{(k-1)(k-2)}.$$

Table 1 gives the values of these bounds for small k. In 2012, Kostochka and Yancey [11] had a remarkable breakthrough on this problem, showing that every k-critical graph G with $k \ge 4$ must satisfy

$$\|G\| \geq \left\lceil \frac{(k+1)(k-2)\,|G| - k(k-3)}{2(k-1)} \right\rceil.$$

Moreover, their bound is tight for $k = 4$ and $n \geq 6$ as well as for infinitely many values of $|G|$ for each $k \geq 5$. This bound has numerous applications to coloring problems, including a short proof of Grötsch's theorem that triangle-free planar graphs are 3-colorable [10] and short proofs of the results on coloring with respect to Ore degree in [8, 15, 12].

Given these applications to coloring problems, it is natural to study the same problem for more general types of coloring. In this article, we prove better lower bounds on the number of edges in a critical graph, for both the list coloring and online list coloring problems. To state our results we need some definitions.

List coloring was introduced by Vizing [18] and independently by Erdős, Rubin, and Taylor [5]. Let G be a graph. A list assignment on G is a function L from $V(G)$ to the subsets of $\mathbb{N}$. A graph G is *L-colorable* if G has a proper coloring π such that $\pi(v) \in L(v)$ for all v. A graph G is *L-critical* if G is not L-colorable, but every proper subgraph H of G is $L\big|_{V(H)}$-colorable. For $f\colon V(G) \to \mathbb{N}$, a list assignment L is an *f-assignment* if $|L(v)| = f(v)$ for each $v \in V(G)$. If $f(v) = k$ for all $v \in V(G)$, then we also call an f-assignment a k-assignment. We say that G is *f-choosable* if G is L-colorable for every f-assignment L. We say that G is k-list-critical if G is L-critical for some k-list assignment L. The best, known-lower bound on the number of edges in a k-list-critical graph, was given by Kostochka and Stiebitz [13] in 2003. It states that for $k \geq 9$ and every graph $G \neq K_k$ if G is a k-list-critical graph, then $2\,\|G\| \geq g_k(|G|\,, \frac{1}{3}(k-4)\alpha_k)$. We improve their bound to $2\,\|G\| \geq g_k(|G|\,, (k-3)\alpha_k)$ for $k \geq 7$ (see Table 1).

Online list coloring was independently introduced by Zhu [19] and Schauz [17] (Schauz called it *paintability*). Let G be a graph and $f\colon V(G) \to \mathbb{N}$. We say that G is *online f-choosable* if $f(v) \geq 1$ for all $v \in V(G)$ and for every $S \subseteq V(G)$ there is an independent set $I \subseteq S$ such that $G - I$ is online f'-choosable where $f'(v) := f(v)$ for $v \in V(G) - S$ and $f'(v) := f(v) - 1$ for $v \in S - I$. Observe that if a graph is online f-choosable then it is f-choosable. When $f(v) := k-1$ for all $v \in V(G)$, we say that G is *online k-list-critical* if G is not online f-choosable, but every proper subgraph H of G is online $f\big|_{V(H)}$-choosable. In 2012, Riasat and Schauz [16] showed that Gallai's bound $2\,\|G\| \geq g_k(|G|\,, 0)$ holds for online k-list-critical graphs. We improve this for $k \geq 7$ by proving the same bound as we have for list coloring: $2\,\|G\| \geq g_k(|G|\,, (k-3)\alpha_k)$.

For a graph G, we define $d_0\colon V(G) \to \mathbb{N}$ by $d_0(v) := d_G(v)$. The d_0-choosable graphs were first characterized by Borodin [1] and independently by Erdős, Rubin and Taylor [5]. The connected graphs which are not d_0-choosable are precisely the Gallai trees (connected graphs in which every block is complete or an odd cycle). The generalization to a characterization of d_0-AT graphs was first given in [7] by Hladkỳ, Král and Schauz.

Corollary ??. *For $k \geq 5$ and $G \neq K_k$ a k-AT-critical graph, we have $2\,\|G\| \geq g_k(|G|\,, c)$ where $c := (k-3)\alpha_k$ when $k \geq 7$ and $c := (k-4)\alpha_k$ when $k \in \{5, 6\}$.*

	k-Critical G				k-ListCritical G		
	Gallai [6]	Kriv [14]	KS [13]	KY [11]	KS [13]	KR [9]	Here
k	$d(G) \geq$	$d(G) \geq$	$d(G) \geq$	$d(G) \geq$	$d(G) \geq$	$d(G) \geq$	$d(G) \geq$
4	3.0769	3.1429	—	3.3333	—	—	**??**
5	4.0909	4.1429	—	4.5000	—	4.0984	**??**
6	5.0909	5.1304	5.0976	5.6000	—	5.1053	**??**
7	6.0870	6.1176	6.0990	6.6667	—	6.1149	**6.1192**
8	7.0820	7.1064	7.0980	7.7143	—	7.1128	**7.1167**
9	8.0769	8.0968	8.0959	8.7500	8.0838	8.1094	**8.1130**
10	9.0722	9.0886	9.0932	9.7778	9.0793	9.1055	**9.1088**
15	14.0541	14.0618	14.0785	14.8571	14.0610	14.0864	**14.0884**
20	19.0428	19.0474	19.0666	19.8947	19.0490	19.0719	**19.0733**

Table 1: History of lower bounds on the average degree $d(G)$ of k-critical and k-list-critical graphs G.

2 Gallai's bound via discharging

One of the earliest results bounding the number of edges in a critical graph is the following theorem, due to Gallai. The key lemma he proved, Lemma 2.2, gives an upper bound on the number of edges in (what is now called) a Gallai tree. The rest of his proof is an easy counting argument. As a warmup, and to illustrate the approach that we take in Section 4, we rephrase this counting in terms of discharging.

Theorem 2.1 (Gallai). *For $k \geq 4$ and $G \neq K_k$ a k-AT-critical graph, we have*

$$d(G) > k - 1 + \frac{k-3}{k^2-3}.$$

Proof. We use the discharging method. Give each vertex v initial charge $d_G(v)$. First, each k^+-vertex gives charge $\frac{k-1}{k^2-3}$ to each of its $(k-1)$-neighbors. Now the vertices in each component of the low vertex subgraph share their total charge equally. Let $\text{ch}^*(v)$ denote resulting charge on v. We finish the proof by showing that $\text{ch}^*(v) \geq k - 1 + \frac{k-3}{k^2-3}$ for all $v \in V(G)$.

If v is a k^+-vertex, then $ch^*(v) \geq d_G(v) - \frac{k-1}{k^2-3} d_G(v) = \left(1 - \frac{k-1}{k^2-3}\right) d_G(v) \geq \left(1 - \frac{k-1}{k^2-3}\right) k = k - 1 + \frac{k-3}{k^2-3}$ as desired.

Instead, let T be a component of the low vertex subgraph. Now the vertices in T receive total charge

$$\frac{k-1}{k^2-3} \sum_{v \in V(T)} k - 1 - d_G(v) = \frac{k-1}{k^2-3} \left((k-1)|T| - 2\,\|T\|\right).$$

So, after distributing this charge out equally, each vertex in T receives charge

$$\frac{1}{|T|} \frac{k-1}{k^2-3} ((k-1)|T| - 2\,\|T\|) = \frac{k-1}{k^2-3} \left((k-1) - d(T)\right).$$

By Lemma 2.2, this is at least

$$\frac{k-1}{k^2-3}\left((k-1)-\left(k-2+\frac{2}{k-1}\right)\right)=\frac{k-1}{k^2-3}\left(\frac{k-3}{k-1}\right)=\frac{k-3}{k^2-3}.$$

Hence each low vertex ends with charge at least $k-1+\frac{k-3}{k^2-3}$ as desired. $\square$

Lemma 2.2 (Gallai)**.** *For $k \geq 4$ and $T \in \mathcal{T}_k$, we have $d(T) < k-2+\frac{2}{k-1}$.*

Proof. Suppose the lemma is false and choose a counterexample T minimizing $|T|$. Now T has at least two blocks. Let B be an endblock of T. If B is K_t for some $t \in \{2, \ldots, k-2\}$, then remove the non-cut vertices of B from T to get T'. By minimality of $|T|$, we have

$$2\|T\| - t(t-1) = 2\|T'\| < \left(k-2+\frac{2}{k-1}\right)|T'| = \left(k-2+\frac{2}{k-1}\right)(|T|-(t-1)).$$

Hence, we have the contradiction

$$2\|T\| < \left(k-2+\frac{2}{k-1}\right)|T| + (t+2-k-\frac{2}{k-1})(t-1) \leq \left(k-2+\frac{2}{k-1}\right)|T|.$$

The case when B is an odd cycle is similar to that above; a longer cycle just makes the inequality stronger. Finally, if $B = K_{k-1}$, remove all vertices of B from T to get T'. By minimality of $|T|$, we have

$$\begin{aligned}2\|T\| - (k-1)(k-2) - 2 = 2\|T'\| & \\ &< \left(k-2+\frac{2}{k-1}\right)|T'| \\ &= \left(k-2+\frac{2}{k-1}\right)|T| - \left(k-2+\frac{2}{k-1}\right)(k-1).\end{aligned}$$

Hence, $2\|T\| < \left(k-2+\frac{2}{k-1}\right)|T|$, a contradiction. $\square$

3 A refined bound on $||T||$

Lemma 2.2 is essentially best possible, as shown by a path of copies of K_{k-1}, with each successive pair of copies linked by a copy of K_2. When the path T has m copies of K_{k-1}, we get $2||T|| = m(k-1)(k-2)+2(m-1) = (k-2+\frac{2}{k-2})|T|-2$. And a small modification to the proof above yields $2||T|| \leq (k-2+\frac{2}{k-2})|T|-2$. Fortunately, this is not the end of the story.

There are two potential places that we could improve the bound in Theorem 2.1. For each graph G, we could show that either (i) the bound in Lemma 2.2 is loose or (ii) many of the k^+-vertices finish with extra charge, because they have incident edges leading to other k^+-vertices (rather than only $(k-1)$-vertices, as allowed in the proof of Theorem 2.1). A good way to quantify this slackness is with the parameter $q(T)$, which denotes the number of non-cut vertices in T that appear in copies of K_{k-1}. When $q(T)$ is small relative to $|T|$, we can save as in (i) above. And when it is large, we can save as in (ii). In the direction of (i), we now prove a bound on $||T||$ akin to that in Lemma 2.2, but which is stronger when $q(T) \le |T|\frac{k-3}{k-1}$. In Section 4 we do the discharging; at that point we handle case (2), using a reducibility lemma proved in [9].

Without more reducible configurations we can't hope to bound $d(T)$ below $k-3$, due to disjoint copies of K_{k-2}. This is why our next bound on $2||T||$ has the form $(k-3+p(k))|T|$; since we will always have $p(k) > 0$, this is slightly worse than average degree $k-3$. To get the best edge bound we will take $p(k) = \frac{3}{k-2}$, but we prefer to prove the more general formulation, which shows that previous work of Gallai and Kostochka and Steibitz fits the same pattern. This general version will also be more convenient for the discharging. It is helpful to handle separately the cases $K_{k-1} \not\subseteq T$ and $K_{k-1} \subseteq T$. The former is simpler, since it implies $q(T) = 0$, so we start there.

Lemma 3.1. *Let $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$. For all $k \ge 5$ and $T \in \mathcal{T}_k$ with $K_{k-1} \not\subseteq T$, we have*

$$2\,\|T\| \le (k-3+p(k))\,|T| + f(k)$$

whenever the following conditions hold:

1. *$p(k) \ge \frac{-f(k)}{k-2}$; and*
2. *$p(k) \ge \frac{-f(k)}{5} + 5 - k$; and*
3. *$0 \ge f(k) \ge -k+2$; and*
4. *$p(k) \ge \frac{3}{k-2}$.*

Proof. Suppose not and choose a counterexample T minimizing $|T|$. If T is K_t for some $t \in [k-2]$, then $t(t-1) > (k-3+p(k))t + f(k)$. After substituting $p(k) \ge \frac{-f(k)}{k-2}$ from (1), this simplifies to $-t(k-2) > f(k)$, which contradicts (4). If T is C_{2r+1} for $r \ge 2$, then $2(2r+1) > (k-3+p(k))(2r+1) + f(k)$ and hence $(5-k-p(k))(2r+1) > f(k)$. Since $f(k) \le 0$, this contradicts (2). (Note that we only use (1), (2), and (3) when T has a single block; these are the base cases when the proof is phrased using induction.)

Let D be an induced subgraph such that $T \setminus D$ is connected. (We will choose D to be a connected subgraph contained in at most three blocks of T.) Let $T' = T \setminus D$. By the minimality of $|T|$, we have

$$2\,\|T'\| \le (k-3+p(k))\,|T'| + f(k).$$

Since T is a counterexample, subtracting this inequality from the inequality for $2||T||$ gives

$$2\,\|T\| - 2\,\|T'\| > (k-3+p(k))|D|. \tag{*}$$

Suppose T has an endblock B that is K_t for some $3 \le t \le k-3$; let x_B be a cut-vertex of B and let $D = B - x_B$. Now (*) gives $2\,\|T\| - 2\,\|T'\| = |B|\,(|B|-1) > (k-3+p(k))(|B|-1)$, which is a contradiction, since $|B| \le k-3$ and $p(k) > 0$. Suppose instead that T has an endblock B that is an odd cycle. Again, let $D = B - x_B$. Now we get $2|B| > (k-3+p(k))(|B|-1)$. This simplifies to $|B| < 1 + \frac{2}{k-5+p(k)}$, which is a contradiction, since the denominator is always at least 1 (using (4) when $k = 5$). Finally suppose that T has an endblock B that is K_2. Now (*) gives $2 > k - 3 + p(k)$, which is again a contradiction, since $k \geq 5$ and $p(k) > 0$.

To handle the case when B is K_{k-2} we need to remove x_B from T as well, so we simply let $D = B$. Since $B = K_{k-2}$, we have either $d_T(x_B) = k - 2$ or $d_T(x_B) = k - 1$. When $d_T(x_B) = k - 2$, we have

$$(k-2)(k-3) + 2 > (k - 3 + p(k))(k-2),$$

contradicting (4).

The only remaining case is when B is K_{k-2} and $d_T(x_B) = k-1$. Each case above applied when B was any endblock of T, so we may assume that every endblock of T is a K_{k-2} that shares a vertex with an odd cycle. Choose an endblock B that is the end of a longest path in the block-tree of T. Let C be the odd cycle sharing a vertex x_B with B. Consider a neighbor y of x_B on C that either (i) lies only in C or (ii) lies also in an endblock A that is a copy of K_{k-2} (such a neighbor exists because B is at the end of a longest path in the block-tree). In (i), let $D = B \cup \{y\} + yx_B$; in (ii), let $D = B \cup A + yx_B$.

In (i), equation (*) gives

$$(k-2)(k-3) + 2(3) > (k - 3 + p(k))(k-1).$$

This simplifies to $6 > k - 3 + (k-1)p(k)$, and eventually, by (4), to $6 > k + \frac{3}{k-2}$, which yields a contradiction.

In (ii), equation (*) gives

$$2(k-2)(k-3) + 2(3) > 2(k - 3 + p(k))(k-2),$$

which simplifies to

$$3 > (k-2)p(k),$$

again contradicting (4). $\square$

Lemma 3.1 gives the tightest bound on $||T||$ when $p(k) = \frac{3}{k-2}$ and $f(k) = -3$. However, for the discharging in Section 4, it will be convenient to apply Lemma 3.1 with a larger $p(k)$, to match the best value of $p(k)$ that works in the analogous lemma, when $K_{k-1} \subseteq T$. We now prove such a lemma, for $K_{k-1} \subseteq T$. Its statement is similar to the previous one, but with an extra term in the bound, as well as slightly different hypotheses.

Lemma 3.2. *Let $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$, $h\colon \mathbb{N} \to \mathbb{R}$. For all $k \geq 5$ and $T \in \mathcal{T}_k$ with $K_{k-1} \subseteq T$, we have*

$$2\,\|T\| \leq (k - 3 + p(k))\,|T| + f(k) + h(k)q(T)$$

whenever the following conditions hold:

1. *$f(k) \geq (k-1)(1 - p(k) - h(k))$; and*
2. *$p(k) \geq \frac{3}{k-2}$; and*

3. $p(k) \geq h(k) + 5 - k$; *and*

4. $p(k) \geq \frac{2+h(k)}{k-2}$; *and*

5. $(k-1)p(k) + (k-3)h(k) \geq k+1$.

Proof. The proof is similar to that of Lemma 3.1. The main difference is that now our only base case is $T = K_{k-1}$. For this reason, we replace hypotheses (1), (2), and (3) of Lemma 3.1, which we used only for the base cases of that proof, with our new hypothesis (1), which we use for the current base case. When some endblock B is an odd cycle or K_t, with $3 \leq t \leq k-3$, the induction step is identical to that in Lemma 3.1, since deleting D does not change $q(T)$. It is easy to check that, as needed, $K_{k-1} \subseteq T'$. Thus, we need to consider the induction step when T has an endblock B that is K_2, K_{k-2}, or K_{k-1}. As we will see, these three cases require hypotheses (3), (4), and (5), respectively.

Let T be a counterexample minimizing $|T|$. Let D be an induced subgraph such that $T \setminus D$ is connected, and let $T' = T \setminus D$. The same argument as in Lemma 3.1 now gives

$$2||T|| - 2||T'|| > (k - 3 + p(k))\,|D| + h(k)\,(q(T) - q(T'))\,. \qquad (*)$$

If B is K_2, then $q(T') \leq q(T) + 1$ and (*) gives $2 > k - 3 + p(k) - h(k)$, contradicting (2). So every endblock of B is K_{k-2} or K_{k-1}. To handle these cases, we will need to remove x_B from T as well. Suppose some endblock B is K_{k-1} and $K_{k-1} \subseteq T \setminus B$. Let $T' = T \setminus B$. Now $q(T) \geq q(T') - (k-2) + 1$. So (*) gives

$$(k-1)(k-2) + 2 > (k - 3 + p(k))(k-1) + h(k)(k-3).$$

This simplifies to $k + 1 > (k-1)p(k) + (k-3)h(k)$, which contradicts (5). Thus, at most one endblock of T is a copy of K_{k-1}. Since the cases above apply when B is any endblock, each other endblock must be K_{k-2}. Let B be such an endblock, and x_B its cut vertex. So $d_T(x_B) = k-2$ or $d_T(x_B) = k-1$. In the former case, $q(T_i) \leq q(T) + 1$, and in the latter, $q(T) = q(T_i)$. If $d_T(x_B) = k-2$, then (*) gives

$$(k-2)(k-3) + 2 > (k - 3 + p(k))(k-2) - h(k),$$

which simplifies to $2 + h(k) > \frac{p(k)}{k-2}$, and contradicts (4).

Hence, all but at most one endblock of T is a copy of K_{k-2} with a cut vertex that is also in an odd cycle. Let B be such an endblock at the end of a longest path in the block-tree of T, and let C be the odd cycle sharing a vertex x_B with B. Consider a neighbor y of x_B on C that either (i) lies only in block C or (ii) lies also in an endblock A that is a copy of K_{k-2} (such a neighbor exists because B is at the end of a longest path in the block-tree). In (i), let $D = B \cup \{y\} + yx_B$; in (ii), let $D = B \cup A + yx_B$. Let $T' = T - V(D)$. In each case, we have $q(T') = q(T)$, so the analysis is identical to that in the proof of Lemma 3.1. $\square$

Now some examples of using Lemma 3.1 and Lemma 3.2. What happens if we take $h(k) = 0$ in Lemma 3.2? Now by (5), we need $(k-1)p(k) \geq k+1$ and hence $p(k) \geq 1 + \frac{2}{k-1}$. Taking $p(k) = 1 + \frac{2}{k-1}$, (3) requires $f(k) \geq -2$. Using $h(k) = 0$, $p(k) = 1 + \frac{2}{k-1}$, and $f(k) = -2$, all of the conditions are satisfied in both of Lemmas 3.1 and 3.2, so we conclude

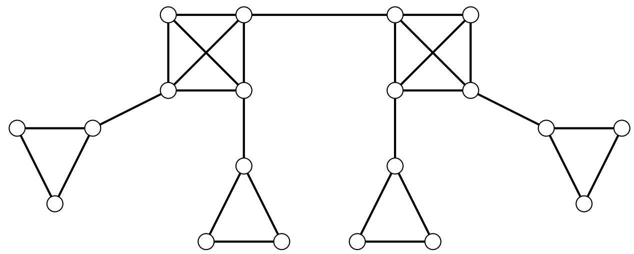

Figure 1: The $m = 2$ case of the construction.

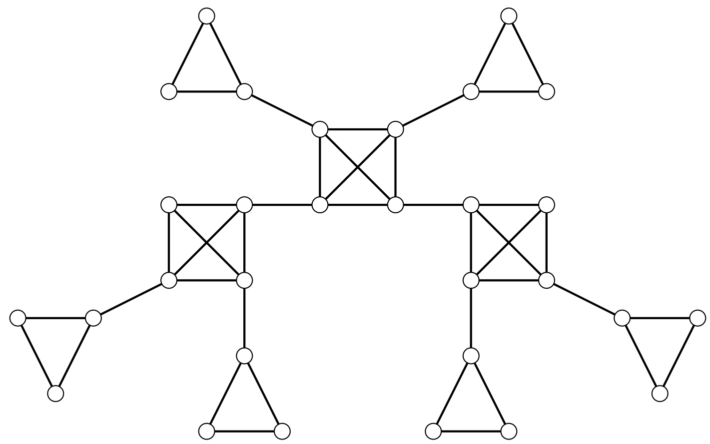

Figure 2: The $m = 3$ case of the construction.

$2\,\|T\| \leq \left(k - 2 + \frac{2}{k-1}\right)|T| - 2$ for every $T \in \mathcal{T}_k$ when $k \geq 5$. This is the previously mentioned slight refinement of Gallai's Lemma 2.2.

Instead, let's make $p(k)$ as small as Lemma 3.2 allows. By (4), $h(k) \leq (k-2)p(k) - 2$. Plugging this into (5) and solving, we get $p(k) \geq \frac{3k-5}{k^2-4k+5}$. Now $\frac{3k-5}{k^2-4k+5} \geq \frac{3}{k-2}$ for $k \geq 5$, so $p(k) = \frac{3k-5}{k^2-4k+5}$ satisfies (3). With $h(k) = \frac{k(k-3)}{k^2-4k+5}$, (4) and (5) are also satisfied. Now with $f(k) = -\frac{2(k-1)(2k-5)}{k^2-4k+5}$, condition (1) is satisfied, so by Lemma 3.2 we have the following.

Corollary 3.3. *For $k \geq 5$ and $T \in \mathcal{T}_k$ with $K_{k-1} \subseteq T$, we have*

$$2\,\|T\| \leq \left(k - 3 + \frac{3k-5}{k^2-4k+5}\right)|T| - \frac{2(k-1)(2k-5)}{k^2-4k+5} + \frac{k(k-3)}{k^2-4k+5}q(T).$$

If we put a similar bound of Kostochka and Stiebitz into this form, we get the following.

Lemma 3.4 (Kostochka-Stiebitz)**.** *For $k \geq 7$ and $T \in \mathcal{T}_k$, we have*

$$2\,\|T\| \leq \left(k - 3 + \frac{4(k-1)}{k^2-3k+4}\right)|T| - \frac{4(k^2-3k+2)}{k^2-3k+4} + \frac{k^2-3k}{k^2-3k+4}q(T).$$

Note that $\frac{3k-5}{(k-5)(k-1)} < \frac{4(k-1)}{k^2-3k+4}$ for $k \geq 7$.

In Section 4, we will see that the bound we get on $d(G)$ is primarily a function of the $p(k)$ with which we apply Lemma 3.2: the smaller $p(k)$ is, the better bound we get on $d(G)$.

So it useful to note that the choice of $p(k)$ in Corollary 3.3 is best possible. We now give a construction to prove this. Let X be a K_{k-1} with $k-3$ pendant edges. At the end of each pendant edge, put a K_{k-2}. Make a path of copies of X by adding one edge between the K_{k-1} in each copy of X (in the only way possible to keep the degrees at most $k-1$). Let T be the path made like this from m copies of X. Now $q(T) = 2$ (from the end blocks), so if T satisfies the bound in Lemma 3.2, then we must have

$$\begin{aligned} &m((k-1)(k-2) + (k-3)(k-2)(k-3) + 2(k-3)) + 2(m-1) \\ \le &(k-3+p(k))(m)(k-1+(k-2)(k-3)) + f(k) + 2h(k), \end{aligned}$$

which reduces to

$$m(k-1) + 2m(k-3) + 2(m-1) \le (m)(k-1+(k-2)(k-3))p(k) + f(k) + 2h(k).$$

Now solving for $p(k)$ gives

$$p(k) \ge \frac{m(k-1) + 2m(k-3) + 2(m-1) - f(k) - 2h(k)}{m((k-1) + (k-2)(k-3))},$$

which simplifies to

$$p(k) \ge \frac{3k-5}{k^2-4k+5} - \frac{2 + f(k) + 2h(k)}{m(k^2-4k+5}.$$

Since we can make m arbitrarily large, this implies $p(k) \ge \frac{3k-5}{k^2-4k+5}$, as desired.

4 Discharging

4.1 Overview and Discharging Rules

Now we use the discharging method, together with the edge bound lemmas of the previous section, to give an improved bound on $d(G)$ for every k-critical graph G. It is helpful to view our proof here as a refinement and strengthening of the proof in Section 2 of Gallai's bound. For $T \in \mathcal{T}_k$, let $W^k(T)$ be the set of vertices of T that are contained in some K_{k-1} in T. For a k-AT-critical graph G, let $\mathcal{L}(G)$ denote the subgraph of G induced on the $(k-1)$-vertices and $\mathcal{H}(G)$ the subgraph of G induced on the k-vertices.

Note that in the proof of Gallai's bound, all $(k+1)^+$-vertices finish with extra charge; $(k+1)$-vertices have extra charge almost 1 and vertices of higher degree have even more. Our idea to improve the bound on $d(G)$ is to have the k-vertices give slightly less charge, ϵ, to their $(k-1)$-nbrs. Now all k^+-vertices finish with extra charge. But components of $L(G)$ have less charge, so we need to give them more charge from $(k+1)^+$-neighbors. How much charge will each component T of $L(G)$ receive? This depends on $||T||$. If $||T||$ is small, then T has many external neighbors, so T will receive lots of charge. If $||T||$ is large, then Lemma 3.2 implies that $q(T)$ is also large. So our plan is to send charge γ to T via each edge incident to a vertex in $W^k(T)$, i.e., one counted by $q(T)$. (For comparison with Gallai's bound, we will have $\epsilon < \frac{k-3}{k^2-3} < \gamma$.) If such an incident edge ends at a $(k+1)^+$-vertex v, then v will still finish with sufficient charge. Our concern, of course, is that a k-vertex will give charge

γ to too many vertices in $W^k(T)$. We would like to prove that each k-vertex has only a few neighbors in $W^k(T)$. Unfortunately, we can not (and we believe this is false). However, we can prove that something similar is true. We can assign each k-vertex to "sponsor" some adjacent vertices in $W^k(T)$, so that each k-vertex sponsors at most 3 such neighbors, and in each component T of $L(G)$ at most two vertices in $W^k(T)$ go unsponsored. This is an immediate consequence of Lemma 5.2, which says that the auxiliary bipartite graph $\mathcal{B}_k(G)$, defined in the next paragraph, is 2-degenerate. And now, the details.

Let $\mathcal{B}_k(G)$ be the bipartite graph with one part $V(\mathcal{H}(G))$ and the other part the components of $\mathcal{L}(G)$. Put an edge between $y \in V(\mathcal{H}(G))$ and a component T of $\mathcal{L}(G)$ if and only if $N(y) \cap W^k(T) \neq \emptyset$. Now Lemma 5.2 says that $\mathcal{B}_k(G)$ is 2-degenerate. Let ϵ and γ be parameters, to be chosen, such that $\epsilon < \gamma < 2\epsilon$. Our initial charge function is $\mathrm{ch}(v) = d_G(v)$. We redistribute charge according to the following rules, applied successively.

1. Each k^+-vertex gives charge ϵ to each of its $(k-1)$-neighbors not in a K_{k-1},
2. Each $(k+1)^+$-vertex give charge γ to each of its $(k-1)$-neighbors in a K_{k-1},
3. Let $Q = \mathcal{B}_k(G)$. Repeat the following steps until Q is empty.
   (a) For each component T of $\mathcal{L}(G)$ in Q with degree at most two in Q do the following:
      i. For each $v \in V(\mathcal{H}(G)) \cap V(Q)$ such that $\left|N_G(v) \cap W^k(T)\right| = 2$, pick one $x \in N_G(v) \cap W^k(T)$ and send charge γ from v to x,
      ii. Remove T from Q.
   (b) For each vertex v of $\mathcal{H}(G)$ in Q with degree at most two in Q do the following:
      i. Send charge γ from v to each $x \in N_G(v) \cap W^k(T)$ for each component T of $\mathcal{L}(G)$ where $vT \in E(Q)$.
      ii. Remove v from Q.
4. Have the vertices in each component of $\mathcal{L}(G)$ share their total charge equally.

First, note that Step 3 will eventually result in Q being empty. This is because $\mathcal{B}_k(G)$ is 2-degenerate, as shown in Lemma 5.2. Next, consider a k-vertex v. In (3bi) v gives away γ to each neighbor in at most two components of $\mathcal{L}(G)$. So it is important that v have few neighbors in these components. Fortunately, this is true. By Lemma 5.1, v has at most 2 neighbors in any component of $\mathcal{L}(G)$. Further, v has at most one component where it has 2 neighbors. Thus, in (3ai) and (3bi), v gives away a total of at most 3γ. Finally, consider a component T. In (3bi), T receives charge γ via every edge incident in $\mathcal{B}_k(G)$, except possibly two (that are still present when v is deleted in (3aii)). Again, by Lemma 5.2, no such v has three neighbors in T. Further, combining this with Steps (2) and (3ai), T receives γ along all but at most two incident edges leading to k-vertices. Thus, T receives charge at least $\gamma(q(T) - 2)$ in Steps (2) and (3).

4.2 Analyzing the Discharging and the Main Result

Finally, we analyze the charge received by each component T of $\mathcal{L}(G)$. We choose ϵ and γ to maximize the minimum, over all vertices, of the final charge. The following theorem is the main result of this paper.

Theorem 4.1. *Let $k \geq 7$ and $p \colon \mathbb{N} \to \mathbb{R}$, $f \colon \mathbb{N} \to \mathbb{R}$, $h \colon \mathbb{N} \to \mathbb{R}$. If G is a k-AT-critical graph, and $G \neq K_k$, then*

$$d(G) \geq k - 1 + \frac{2 - p(k)}{k + 2 + 3h(k) - p(k)},$$

whenever p, f, and k satisfy:

1. $f(k) \geq (k-1)(1 - p(k) - h(k))$; *and*
2. $p(k) \geq \frac{3}{k-2}$; *and*
3. $p(k) \geq h(k) + 5 - k$; *and*
4. $p(k) \geq \frac{2+h(k)}{k-2}$; *and*
5. $(k-1)p(k) + (k-3)h(k) \geq k + 1$; *and*
6. $2(h(k) + 1) + f(k) \leq 0$; *and*
7. $p(k) + (k-5)h(k) \leq k + 1$.

Before we prove Theorem 4.1, we show that two previous results on this problem follow immediately from this theorem. Note that (1)–(5) are the hypotheses of Lemma 3.2. As a first test, let $p(k) = 1 - \frac{2}{k-1}$, $f(k) = -2$ and $h(k) = 0$. Now the hypotheses of Theorem 4.1 are satisfied when $k \geq 7$, and we get Gallai's bound: $d(G) \geq k - 1 + \frac{k-3}{k^2-3}$. Next, let's use the Kostochka-Stiebitz bound, that is, $p(k) = \frac{4(k-1)}{k^2-3k+4}$, $f(k) = -\frac{4(k^2-3k+2)}{k^2-3k+4}$ and $h(k) = \frac{k^2-3k}{k^2-3k+4}$. Again, the hypotheses of Theorem 4.1 are satisfied when $k \geq 7$ and we get

$$d(G) \geq k - 1 + \frac{2(k-2)(k-3)}{(k-1)(k^2+3k-12)}.$$

This is exactly the bound in the paper of Kierstead and the second author [9].

Finally, to get our sharpest bound on $d(G)$, we use the bound in Corollary 3.3, that is, $p(k) = \frac{3k-5}{k^2-4k+5}$, $f(k) = -\frac{2(k-1)(2k-5)}{k^2-4k+5}$, and $h(k) = \frac{k(k-3)}{k^2-4k+5}$. The hypotheses of Theorem 4.1 are satisfied when $k \geq 7$ and we get $d(G) \geq k - 1 + \frac{(k-3)(2k-5)}{k^3+k^2-15k+15}$. This is better than the bound in [9] for $k \geq 7$. We record this as our main corollary.

Corollary 4.2. *If G is a k-AT-critical graph, with $k \geq 7$, and $G \neq K_k$, then*

$$d(G) \geq k - 1 + \frac{(k-3)(2k-5)}{k^3 + k^2 - 15k + 15}.$$

Now we prove Theorem 4.1

Proof of Theorem 4.1. Our discharging procedure in the previous section gives charge ϵ to a component T for every incident edge not ending in a K_{k-1}. The number of such edges is exactly

$$-q(T) + \sum_{v \in V(T)} (k - 1 - d_T(v)) = (k-1)\left|T\right| - 2\left\|T\right\| - q(T),$$

so let $A(T)$ denote this quantity. When $K_{k-1} \subseteq T$, since (1)–(5) hold, Lemma 3.2 implies that

$$2\|T\| \leq (k-3+p(k))\,|T| + f(k) + h(k)q(T).$$

So, when $K_{k-1} \subseteq T$ we get

$$\begin{aligned} A(T) &\geq (k-1)\,|T| - q(T) - ((k-3+p(k))\,|T| + f(k) + h(k)q(T)) \\ &= (2-p(k))\,|T| - f(k) - (h(k)+1)q(T). \end{aligned}$$

Hence, in total T receives charge at least

$$\begin{aligned} \epsilon A(T) + \gamma(q(T)-2) &\geq \epsilon(2-p(k))|T| - \epsilon f(k) - \epsilon(h(k)+1)q(T) + \gamma q(T) - 2\gamma \\ &= \epsilon(2-p(k))|T| + q(T)(\gamma - \epsilon(h(k)+1)) - (2\gamma + \epsilon f(k)) \end{aligned}$$

Our goal is to make $\epsilon(2-p(k))$ as large as possible, while ensuring that the final two terms are nonnegative. To make the second term 0, we let $\gamma = \epsilon(h(k)+1)$. Now the final term becomes $-\epsilon(2(h(k)+1)+f(k))$. For simplicity, we have added, as (6), that $2(h(k)+1)+f(k) \leq 0$. (Since we typically take $h(k) > 0$, as in Corollary 4.2, it is precisely this requirement that necessitates the use of $f(k)$ in Lemma 3.2.) Thus, T receives charge at least

$$\epsilon\,(2-p(k))\,|T|\,,$$

so each of its vertices gets at least $\epsilon(2-p(k))$. We also need each k-vertex to end with enough charge, and each of these loses at most $3\gamma + (k-3)\epsilon$. So we take

$$1 - (3\gamma + (k-3)\epsilon) = \epsilon\,(2-p(k))\,,$$

which gives

$$\epsilon = \frac{1}{k+2+3h(k)-p(k)},$$

$$\gamma = \frac{h(k)+1}{k+2+3h(k)-p(k)}.$$

Thus, after discharging, each k-vertex finishes with charge at least $k-1+\epsilon(2-p(k))$. The same bound holds for each $(k-1)$-vertex in a component T with a K_{k-1}.

When $K_{k-1} \not\subseteq T$, we have $q(T) = 0$. Applying Lemma 3.1 with $f(k) = 0$ and $p(k)$ as in the present, we get

$$2\|T\| \leq (k-3+p(k))\,|T|\,,$$

and hence

$$A(T) \geq (2-p(k))\,|T|\,,$$

which is sufficient charge.

It remains to check that the $(k+1)^+$-vertices don't give away too much charge. Let v be a $(k+1)^+$-vertex. Now v ends with charge at least

$$d(v) - \gamma d(v) = (1-\gamma)d(v) \geq (1-\gamma)(k+1) = (k+1)\frac{k+1+2h(k)-p(k)}{k+2+3h(k)-p(k)},$$

so we need that the inequality

$$(k+1)\frac{k+1+2h(k)-p(k)}{k+2+3h(k)-p(k)} \geq k-1+\frac{2-p(k)}{k+2+3h(k)-p(k)}$$

holds. This inequality reduces to

$$p(k)+(k-5)h(k) \leq k+1.$$

For simplicity, we have added this as (7), since it is easily satisfied by the p, f, and h we want to use. $\square$

The reason that we require $k \geq 7$ in Theorem 4.1 (and Corollary 4.2) is that the proof uses Lemma 5.2. However, for $k \in \{5,6\}$, Lemma 5.3 can play an analogous role. For $k \geq 7$, Lemma 5.2 implies that if G has no reducible configuration, then $B_k(G)$ is 2-degenerate. For $k \in \{5,6\}$, Lemma 5.3 implies that we can reduce $\mathcal{B}_k(G)$ to the empty graph by repeatedly deleting either a tree component vertex v with $d_{\mathcal{B}_k(G)}(v) \leq 1$ or else a vertex w in $V(\mathcal{B}_k(G)) \cap V(\mathcal{H}(G))$ with $d_{\mathcal{B}_k(G)}(v) \leq 3$. Thus, in the discharging, the tree corresponding to v receives charge at least $\gamma(q(T)-1)$ on edges ending at vertices in $W^k(T)$. Similarly, each k-vertex gives away charge at most $4\gamma+(k-4)\epsilon$. Now, to find the optimal value of ϵ, as in the proof of Theorem 4.1, we solve $(1-(4\gamma+\epsilon(k-4))=(2-p(k))\epsilon$. This gives $\epsilon = \frac{1}{k+2+4h(k)-p(k)}$ and, again, $\gamma = \epsilon(h(k)+1)$. In place of hypothesis (6), we have the slightly weaker requirement $h(k)+1+f(k) \leq 0$. The result is the following theorem and corollary, for $k \in \{5,6\}$.

Theorem 4.3. *Let $k \in \{5,6\}$ and $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$, $h\colon \mathbb{N} \to \mathbb{R}$. If G is a k-AT-critical graph, and $G \neq K_k$, then*

$$d(G) \geq k-1+\frac{2-p(k)}{k+2+4h(k)-p(k)},$$

whenever p, f, and k satisfy:

1. *$f(k) \geq (k-1)(1-p(k)-h(k))$; and*
2. *$p(k) \geq \frac{3}{k-2}$; and*
3. *$p(k) \geq h(k)+5-k$; and*
4. *$p(k) \geq \frac{2+h(k)}{k-2}$; and*
5. *$(k-1)p(k)+(k-3)h(k) \geq k+1$; and*
6. *$h(k)+1+f(k) \leq 0$; and*
7. *$p(k)+(k-5)h(k) \leq k+1$.*

To get the best bound on $d(G)$, as in Theorem 4.1, we use $p(k) = \frac{3k-5}{k^2-4k+5}$, $f(k) = -\frac{2(k-1)(2k-5)}{k^2-4k+5}$, and $h(k) = \frac{k(k-3)}{k^2-4k+5}$.

Corollary 4.4. *If G is a k-AT-critical graph, with $k \in \{5,6\}$, and $G \neq K_k$, then*

$$d(G) \geq k - 1 + \frac{(k-3)(2k-5)}{k^3 + 2k^2 - 18k + 15}.$$

Possible improvements:

1. Use a better bound on average degree of Gallai trees. i would like to find the best possible family in the form here. How does this bound compare to the hand waiving one in the other document?

2. In the discharging, the k-vertices lost 3γ even though they had degree two in Q because of the possibility of two edges into one component. Can we get this to 2γ somehow, like maybe we can order our picking so that no vertex is picked before the component where it has two edges has been removed.

3. Related to the previous item, improved reducible configurations, a less restrictive condition in Lemma 5.2 taking into account the two edges to a component issue.

5 Reducible Configurations

Definition 1. A graph G is *AT-reducible* to H if H is a nonempty induced subgraph of G which is f_H-AT where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not AT-reducible to any nonempty induced subgraph, then it is *AT-irreducible.*

This first lemma tells us how a single high vertex can interact with the low vertex subgraph. This is the version Hal and i used, it (and more) follows from the classification in "mostlow".

Lemma 5.1. *Let $k \geq 5$ and let G be a graph with $x \in V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*

2. *$G - x$ has t components $H_1, H_2, \ldots, H_t$, and all are in $\mathcal{T}_k$; and*

3. *$d_G(v) \leq k-1$ for all $v \in V(G-x)$; and*

4. *$\left|N(x) \cap W^k(H_i)\right| \geq 1$ for $i \in [t]$; and*

5. *$d_G(x) \geq t + 2$.*

Then G is f-AT where $f(x) = d_G(x) - 1$ and $f(v) = d_G(v)$ for all $v \in V(G-x)$.

To deal with more than one high vertex we need the following auxiliary bipartite graph. For a graph G, $\{X, Y\}$ a partition of $V(G)$ and $k \geq 4$, let $\mathcal{B}_k(X,Y)$ be the bipartite graph with one part Y and the other part the components of $G[X]$. Put an edge between $y \in Y$ and a component T of $G[X]$ if and only if $N(y) \cap W^k(T) \neq \emptyset$. The next lemma tells us that we have a reducible configuration if this bipartite graph has minimum degree at least three.

Lemma 5.2. *Let $k \geq 7$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*
4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $\delta(\mathcal{B}) \geq 3$.*

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.

We also have the following version with asymmetric degree condition on $\mathcal{B}$. The point here is that this works for $k \geq 5$. As we'll see in the next section, the consequence is that we trade a bit in our size bound for the proof to go through with $k \in \{5, 6\}$.

Lemma 5.3. *Let $k \geq 5$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*
4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $d_{\mathcal{B}}(y) \geq 4$ for all $y \in Y$ and $d_{\mathcal{B}}(T) \geq 2$ for all components T of $G - Y$.*

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.

References

[1] O.V. Borodin, *Criterion of chromaticity of a degree prescription*, Abstracts of IV All-Union Conf. on Th. Cybernetics, 1977, pp. 127–128.

[2] R.L. Brooks, *On colouring the nodes of a network*, Mathematical Proceedings of the Cambridge Philosophical Society, vol. 37, Cambridge Univ Press, 1941, pp. 194–197.

[3] G.A. Dirac, *Note on the colouring of graphs*, Mathematische Zeitschrift **54** (1951), no. 4, 347–353.

[4] ______, *A theorem of R.L. Brooks and a conjecture of H. Hadwiger*, Proceedings of the London Mathematical Society **3** (1957), no. 1, 161–195.

[5] P. Erdős, A.L. Rubin, and H. Taylor, *Choosability in graphs*, Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium, vol. 26, 1979, pp. 125–157.

[6] T. Gallai, *Kritische graphen i.*, Math. Inst. Hungar. Acad. Sci **8** (1963), 165–192 (in German).

[7] Jan Hladký, Daniel Král, and Uwe Schauz, *Brooks' theorem via the Alon-Tarsi theorem*, Discrete Math. **310** (2010), no. 23, 3426–3428. MR 2721105 (2012a:05115)

[8] H.A. Kierstead and A.V. Kostochka, *Ore-type versions of Brooks' theorem*, Journal of Combinatorial Theory, Series B **99** (2009), no. 2, 298–305.

[9] H.A. Kierstead and L. Rabern, *Improved lower bounds on the number of edges in list critical and online list critical graphs*, arXiv preprint arXiv:1406.7355 (2014).

[10] Alexandr Kostochka and Matthew Yancey, *Ore's conjecture for $k = 4$ and Grötzsch's theorem*, Combinatorica **34** (2014), no. 3, 323–329. MR 3223967

[11] ———, *Ore's conjecture on color-critical graphs is almost true*, J. Combin. Theory Ser. B **109** (2014), 73–101. MR 3269903

[12] A.V. Kostochka, L. Rabern, and M. Stiebitz, *Graphs with chromatic number close to maximum degree*, Discrete Mathematics **312** (2012), no. 6, 1273–1281.

[13] A.V. Kostochka and M. Stiebitz, *A new lower bound on the number of edges in colour-critical graphs and hypergraphs*, Journal of Combinatorial Theory, Series B **87** (2003), no. 2, 374–402.

[14] M. Krivelevich, *On the minimal number of edges in color-critical graphs*, Combinatorica **17** (1997), no. 3, 401–426.

[15] L. Rabern, *Δ-critical graphs with small high vertex cliques*, Journal of Combinatorial Theory, Series B **102** (2012), no. 1, 126–130.

[16] A. Riasat and U. Schauz, *Critically paintable, choosable or colorable graphs*, Discrete Mathematics **312** (2012), no. 22, 3373–3383.

[17] U. Schauz, *Mr. Paint and Mrs. Correct*, The Electronic Journal of Combinatorics **16** (2009), no. 1, R77.

[18] V.G. Vizing, *Vextex coloring with given colors*, Metody Diskretn. Anal. **29** (1976), 3–10 (in Russian).

[19] X. Zhu, *On-line list colouring of graphs*, The Electronic Journal of Combinatorics **16** (2009), no. 1, R127.