

most low Alon-Tarsi notes

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1 Introduction

We consider graphs with vertices labeled by natural numbers; that is, pairs (G, h) where G is a graph and $h: V(G) \rightarrow \mathbb{N}$. We say that (G, h) is AT if G is $(d_G - h)$ -AT. When H is an induced subgraph of G , we simplify notation by referring to the pair (H, h) where we really mean $(H, h|_{V(H)})$.

2 Subgraphs, subdivisions and cuts

Definition 1. A graph G is *h -minimal* if G is connected and (H, h) is not AT for every proper induced subgraph H of G . A graph G is *h -greedy-minimal* if G is connected and (H, h) is not AT for every proper induced subgraph H of G where $h(v) = 0$ for all $v \in V(G) \setminus V(H)$. Note that if G is h -minimal then it is also h -greedy-minimal.

Lemma 2.1. *If G is connected and (G, h) is not AT, then G is h -greedy-minimal.*

Proof. If there were a proper induced subgraph H such that $(H, h|_{V(H)})$ is AT, then by ordering the vertices of each component of $G - V(H)$ by increasing distance to H and directing all edges away from H in this order we conclude that (G, h) is AT. \square

Lemma 2.2. *If (G', h') is formed from (G, h) by subdividing an edge e of G twice and having h' give zero on the two new vertices, then*

1. *if (G, h) is AT, then (G', h') is AT; and*
2. *if (G', h') is AT, then either (G, h) is AT or $(G - e, h)$ is AT.*

Proof. Suppose $e = xy$ and call the new vertices x' and y' so that G' contains the induced path $xx'y'y$. For (1), let D be an orientation of G showing that (G, h) is AT. By symmetry we may assume $xy \in E(D)$. Make an orientation D' of G' from D by replacing xy with the directed path $xx'y'y$. There is a natural parity preserving bijection between the spanning Eulerian subgraphs of D and D' , so we conclude that (G', h') is AT.

For (2), let D' be an orientation of G' showing that (G', h') is AT. Suppose G' contains the directed path $xx'y'y$ or the directed path $yy'x'x$. By symmetry, we can assume it is $xx'y'y$. Then make an orientation D of G by replacing $xx'y'y$ with the directed edge xy . As

above, we have a parity preserving bijection between the spanning Eulerian subgraphs of D and D' , so we conclude that (G, h) is AT. Otherwise, no spanning Eulerian subgraph of D' contains a cycle passing through x' and y' . So, the spanning Eulerian subgraph counts of D' are the same as those of $D' - x' - y'$. But this gives an orientation of $G - e$ showing that $(G - e, h)$ is AT. \square

Lemma 2.3. *Let $\{A_1, A_2\}$ be a separation of G such that $A_1 \cap A_2 = \{x\}$. If $G[A_i]$ is f_i -AT for $i \in [2]$, then G is f -AT where $f(v) = f_i(v)$ for $v \in V(A_i - x)$ and $f(x) = f_1(x) + f_2(x) - 1$. Going the other direction, if G is f -AT, then $G[A_i]$ is f_i -AT for $i \in [2]$ where $f_i(v) = f(v)$ for $v \in V(A_i - x)$ and $f_1(x) + f_2(x) \leq f(x) + 1$.*

Proof. For $i \in [2]$, choose an orientation D_i of A_i showing that A_i is f_i -AT. Together these give an orientation D of G and since no cycle has vertices in both $A_1 - x$ and $A_2 - x$, we have

$$\begin{aligned} EE(D) - EO(D) &= EE(D_1)EE(D_2) + EO(D_1)EO(D_2) - (EE(D_1)EO(D_2) + EO(D_1)EE(D_2)) \\ &= (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2)) \\ &\neq 0. \end{aligned}$$

Hence G is f -AT.

Now, suppose G is f -AT and choose an orientation D of G showing this. Put $D_i = D[A_i]$ for $i \in [2]$. Then, as above, we have $0 \neq EE(D) - EO(D) = (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2))$ and hence $EE(D_1) - EO(D_1) \neq 0$ and $EE(D_2) - EO(D_2) \neq 0$. Since the in-degree of x in D is the sum of the in-degree of x in D_1 and the in-degree of x in D_2 , the lemma follows. \square

Corollary 2.4. *Let G be an h -greedy-minimal graph. If (G, h) is AT and G has an induced path $x_1x_2x_3x_4$ such that $d_G(x_2) = d_G(x_3) = 2$ and $h(x_2) = h(x_3) = 0$, then*

$$((G - x_2 - x_3) + x_1x_4, h|_{V(G) \setminus \{x_2, x_3\}}) \text{ is AT.}$$

Proof. Suppose (G, h) is AT and G has such an induced path $x_1x_2x_3x_4$. Applying Lemma 2.2 part (2) shows that either $(G - x_2 - x_3, h|_{V(G) \setminus \{x_2, x_3\}})$ is AT or $((G - x_2 - x_3) + x_1x_4, h|_{V(G) \setminus \{x_2, x_3\}})$ is AT. But $G - x_2 - x_3$ is a proper induced subgraph of G , so the former cannot happen since G is h -greedy-minimal and $h(x_2) = h(x_3) = 0$. Hence $((G - x_2 - x_3) + x_1x_4, h|_{V(G) \setminus \{x_2, x_3\}})$ is AT. \square

3 Extension lemma

This is a key lemma from [1], it generalizes a lemma from [2] from list coloring to Alon-Tarsi orientations. This is what I talked about in Baltimore. The basic idea is that in some cases we can pair off odd/even spanning Eulerian subgraphs via a parity reversing bijection.

Lemma 3.1. *Let G be a multigraph without loops and $f: V(G) \rightarrow \mathbb{N}$. If there are $F \subseteq G$ and $Y \subseteq V(G)$ such that:*

1. *any multiple edges in G are contained in $G[Y]$; and*

2. $f(v) \geq d_G(v)$ for all $v \in V(G) \setminus Y$; and
3. $f(v) \geq d_{G[Y]}(v) + d_F(v) + 1$ for all $v \in Y$; and
4. For each component T of $G - Y$ there are different $x_1, x_2 \in V(T)$ where $N_T[x_1] = N_T[x_2]$ and $T - \{x_1, x_2\}$ is connected such that either:
 - (a) there are $x_1y_1, x_2y_2 \in E(F)$ where $y_1 \neq y_2$ and $N(x_i) \cap Y = \{y_i\}$ for $i \in [2]$; or
 - (b) $|N(x_2) \cap Y| = 0$ and there is $x_1y_1 \in E(F)$ where $N(x_1) \cap Y = \{y_1\}$,

then G is f -AT.

Proof. Suppose not and pick a counterexample (G, f, F, Y) minimizing $|G - Y|$. If $|G - Y| = 0$, then $Y = V(G)$ and thus $f(v) \geq d_G(v) + 1$ for all $v \in V(G)$ by (3). Pick an acyclic orientation D of G . Then $EE(D) = 1$, $EO(D) = 0$ and $d_D^+(v) \leq d_G(v) \leq f(v) - 1$ for all $v \in V(D)$. Hence G is f -AT. So, we must have $|G - Y| > 0$.

Pick a component T of $G - Y$ and pick $x_1, x_2 \in V(T)$ as guaranteed by (4). First, suppose (4a) holds. Put $G' := (G - T) + y_1y_2$, $F' := F - T$, $Y' := Y$ and let f' be f restricted to $V(G')$. Then G' has an orientation D' where $f'(v) \geq d_{D'}^+(v) + 1$ for all $v \in V(D')$ and $EE(D') \neq EO(D')$, for otherwise (G', f', F', Y') would contradict minimality. By symmetry we may assume that the new edge y_1y_2 is directed toward y_2 . Now we use the orientation of D' to construct the desired orientation of D . First, we use the orientation on $D' - y_1y_2$ on $G - T$. Now, order the vertices of T as $x_1, x_2, z_1, z_2, \dots$ so that every vertex has at least one neighbor to the right. Orient the edges of T left-to-right in this ordering. Finally, we use y_1x_1 and x_2y_2 and orient all other edges between T and $G - T$ away from T . Plainly, $f(v) \geq d_D^+(v) + 1$ for all $v \in V(D)$. Since y_1x_1 is the only edge of D going into T , any Eulerian subgraph of D that contains a vertex of T must contain y_1x_1 . So, any Eulerian subgraph of D either contains (i) neither y_1x_1 nor x_2y_2 , (ii) both y_1x_1 and x_2y_2 , or (iii) y_1x_1 but not x_2y_2 . We first handle (i) and (ii) together. Consider the function h that maps an Eulerian subgraph Q of D' to an Eulerian subgraph $h(Q)$ of D as follows. If Q does not contain y_1y_2 , let $h(Q) = \iota(Q)$ where $\iota(Q)$ is the natural embedding of $D' - y_1y_2$ in D . Otherwise, let $h(Q) = \iota(Q - y_1y_2) + \{y_1x_1, x_1x_2, x_2y_2\}$. Then h is a parity-preserving injection with image precisely the union of those Eulerian subgraphs of D in (i) and (ii). Hence if we can show that exactly half of the Eulerian subgraphs of D in (iii) are even, we will conclude $EE(D) \neq EO(D)$, a contradiction. To do so, consider an Eulerian subgraph A of D containing y_1x_1 and not x_2y_2 . Since x_1 must have in-degree 1 in A , it must also have out-degree 1 in A . We show that A has a mate A' of opposite parity. Suppose $x_2 \notin A$ and $x_1z_1 \in A$; then we make A' by removing x_1z_1 from A and adding $x_1x_2z_1$. If $x_2 \in A$ and $x_1x_2z_1 \in A$, we make A' by removing $x_1x_2z_1$ and adding x_1z_1 . Hence exactly half of the Eulerian subgraphs of D in (iii) are even and we conclude $EE(D) \neq EO(D)$, a contradiction.

Now suppose (4b) holds. Put $G' := G - T$, $F' := F - T$, $Y' := Y$ and define f' by $f'(v) = f(v)$ for all $v \in V(G' - y_1)$ and $f'(y_1) = f(y_1) - 1$. Then G' has an orientation D' where $f'(v) \geq d_{D'}^+(v) + 1$ for all $v \in V(D')$ and $EE(D') \neq EO(D')$, for otherwise (G', f', F', Y') would contradict minimality. We orient $G - T$ according to D , orient T as in the previous case, again use y_1x_1 and orient all other edges between T and $G - T$ away from T . Since we decreased $f'(y_1)$ by 1, the extra out edge of y_1 is accounted for and we have

$f(v) \geq d_D^+(v) + 1$ for all $v \in V(D)$. Again any additional Eulerian subgraph must contain y_1x_1 and since x_2 has no neighbor in $G - T$ we can use x_2 as before to build a mate of opposite parity for any additional Eulerian subgraph. Hence $EE(D) \neq EO(D)$ giving our final contradiction. \square

4 Degree-AT graphs

A graph G is called *degree-AT* if (G, h) is AT where h is the constant zero function.

Lemma 4.1. *A connected graph G is degree-AT if it is not a Gallai tree.*

Proof. Suppose there exists a connected graph that is not a Gallai tree, but is also not degree-AT. Let G be such a graph with as few vertices as possible. Since G is not degree-AT, no induced subgraph H of G is degree-AT by Lemma 2.1. Hence, for any $v \in V(G)$ that is not a cutvertex, $G - v$ must be a Gallai tree by minimality of $|G|$.

If G has more than one block, then for endblocks B_1 and B_2 , choose noncutvertices $w \in B_1$ and $x \in B_2$. By the minimality of $|G|$, both $G - w$ and $G - x$ are Gallai trees. Since every block of G appears either as a block of $G - w$ or as a block of $G - x$, every block of G is either complete or an odd cycle. Hence, G is a Gallai tree, a contradiction. So instead G has only one block, that is, G is 2-connected. Further, $G - v$ is a Gallai tree for all $v \in V(G)$.

Let v be a vertex of minimum degree in G . Since G is 2-connected, $d_G(v) \geq 2$ and v is adjacent to a noncutvertex in every endblock of $G - v$. If $G - v$ has a complete block B with noncutvertices x_1, x_2 where $v \leftrightarrow x_1$ and $v \not\leftrightarrow x_2$, then we can apply Lemma 3.1 with $Y = \{v\}$ and $F = vx_1$ to conclude that G is degree-AT, a contradiction. So, v must be adjacent to every noncutvertex in every complete endblock of $G - v$.

Suppose $d_G(v) \geq 3$. Then no endblock of $G - v$ can be an odd cycle of length at least 5 (there would be vertices of degree 3 but we'd have $d_G(v) \geq 4$). Let B be a smallest complete endblock of $G - v$. Then for a noncutvertex $x \in V(B)$, we have $d_G(x) = |B|$ and hence $d_G(v) \leq |B|$. If $G - v$ has at least two endblocks, then $2(|B| - 1) \leq |B|$ and hence $d_G(v) \leq |B| = 2$, a contradiction. Hence $G - v = B$ and v is joined to B , so G is complete, a contradiction.

Hence, we must have $d_G(v) = 2$. Suppose $G - v$ has at least 2 endblocks. Then, it has exactly 2 and v is adjacent to one noncutvertex in each. Neither of the endblocks can be odd cycles of length at least 5 since then we could get a smaller counterexample by Lemma 2.2. Since v is adjacent to every noncutvertex in every complete endblock of $G - v$, both endblocks must be K_2 . But then either $G = C_4$ (which is trivially degree-AT) or we can get a smaller counterexample by Lemma 2.2. So, $G - v$ must be 2-connected. Since $G - v$ is a Gallai tree, it is either complete or an odd cycle. If $G - v$ is not complete, we can get a smaller counterexample by Lemma 2.2. So, $G - v$ is complete and v is adjacent to every noncutvertex of $G - v$; that is, G is complete, a contradiction. \square

5 When h is 1 for at most one vertex

For a graph G and $x \in V(G)$ let $h_x: V(G) \rightarrow \mathbb{N}$ be defined by $h_x(x) = 1$ and $h_x(v) = 0$ for all $v \in V(G - x)$. We classify the connected h_x -minimal graphs G such that (G, h_x) is AT

for some $x \in V(G)$.

To start we will reduce to the case when G is 2-connected.

Lemma 5.1. *Let G be h_x -minimal for $x \in V(G)$ and let \mathcal{B} be the set of blocks of G containing x . Then (G, h_x) is AT if and only if*

1. \mathcal{B} contains at least two degree-AT graphs; or
2. G is 2-connected and (G, h_x) is AT.

Proof. Since G is h_x -minimal, no block outside of \mathcal{B} is degree-AT. The lemma follows since if G is not 2-connected, then (G, h_x) is AT if and only if (1) holds by Lemma 2.3. \square

Lemma 5.2. *If G is a connected graph and $x \in V(G)$ with $d_G(x) = 2$, then (G, h_x) is AT if and only if $G - x$ is degree-AT.*

Proof. Let D be an orientation of G showing that (G, h_x) is AT. Then $d_D^-(x) = 2$ and hence no spanning Eulerian subgraph contains a cycle passing through x . Therefore, the Eulerian subgraph counts in $G - x$ are different and $G - x$ is degree-AT. The other direction is immediate from Lemma 2.1. \square

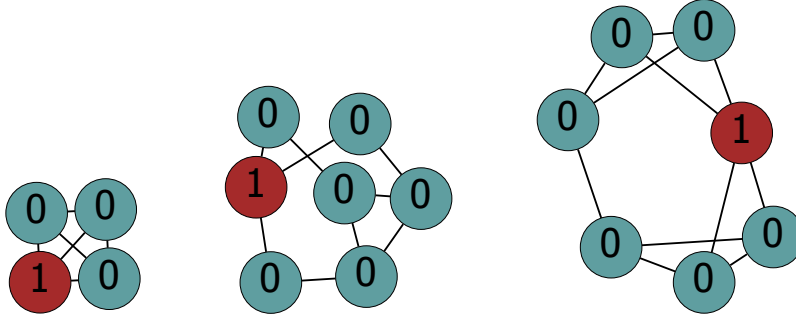


Figure 1: The seed blocks.

Lemma 2.2 part (2) suggests a way to construct G such that (G, h) is not AT from smaller graphs. Specifically, we have the following.

Corollary 5.3. *If e is an edge in G such that (G, h) is not AT and $(G - e, h)$ is not AT, then (G', h') is not AT where (G', h') is formed from (G, h) by subdividing e twice and having h' give zero on the two new vertices.*

Let \mathcal{D} be the smallest collection of pairs (G, h) containing the pairs in Figure 1 that is closed under the operation in Corollary 5.3.

Lemma 5.4. *Let G be h_x -minimal for $x \in V(G)$ with $d_G(x) \geq 3$. If (G, h_x) is not AT, then every induced subdivision of $K_{1,3}$ in G contains at most two vertices in $N(x)$. In particular, every induced path in G contains at most two vertices in $N(x)$.*

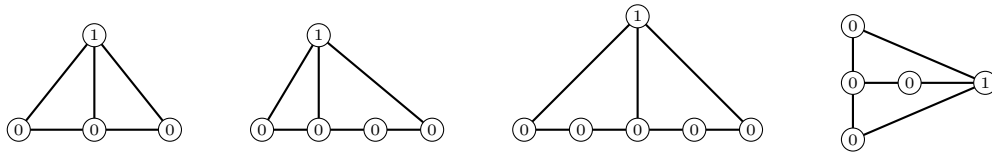


Figure 2: These are AT.

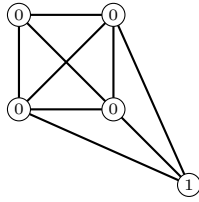


Figure 3: This is AT.

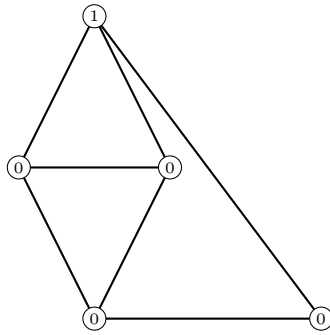


Figure 4: This is AT.

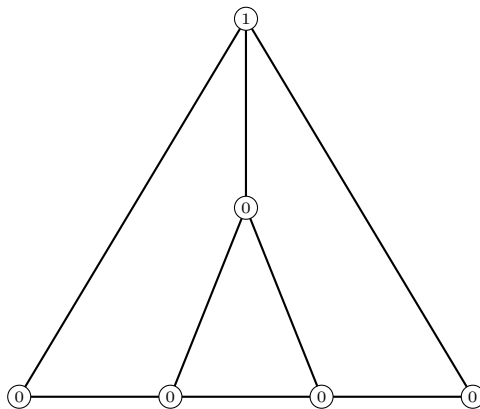


Figure 5: This is AT.

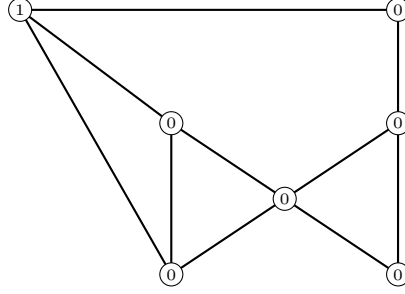


Figure 6: This is AT.

Proof. This is immediate from Lemma 2.2 and the graphs in Figure 2. \square

Lemma 5.5. *Let G be h_x -minimal for $x \in V(G)$. If G is 2-connected, then (G, h_x) is AT if and only if*

1. $d_G(x) \geq 3$; and
2. G is not complete and not an odd cycle; and
3. $(G, h_x) \notin \mathcal{D}$.

Proof. Suppose the lemma is false and choose a counterexample G minimizing $|G|$. If $d_G(x) \leq 2$, then (G, h_x) is not AT by Lemma 5.2 since G is h_x -minimal. So, we must have $d_G(x) \geq 3$. Since (G, h_x) is not AT if $(G, h_x) \in \mathcal{D}$ by construction, it must be that $(G, h_x) \notin \mathcal{D}$ and (G, h_x) is not AT. Since G is h_x -minimal and 2-connected, $G - x$ is a Gallai tree.

Since G is 2-connected, x is adjacent to a noncutvertex in every endblock of $G - x$. If $G - x$ has a complete block B with noncutvertices v_1, v_2 where $x \leftrightarrow v_1$ and $x \not\leftrightarrow v_2$, then we can apply Lemma 3.1 with $Y = \{x\}$ and $F = xv_1$ to conclude that (G, h_x) is AT, a contradiction. So, x must be adjacent to every noncutvertex in every complete endblock of $G - x$.

If $G - x$ contains an induced path $v_1v_2v_3v_4$ such that $d_G(v_2) = d_G(v_3) = 2$, then we get a smaller counterexample by applying Lemma 2.2 part (1). So, $G - x$ has no such induced path.

Suppose G has an endblock B that is an odd cycle of length $\ell \geq 5$. Then x is adjacent to at least $\lfloor \frac{\ell}{2} \rfloor \geq 2$ noncutvertices in B . If $G - x$ had another endblock B' , then we can pick two neighbors of x in B and one neighbor of x in B' all on an induced path in $G - x$, violating Lemma 5.4. But if $G - x = B$, we can again violate Lemma 5.4. Hence every endblock of $G - x$ is complete.

Since $G - x$ is not complete (otherwise G would be complete), this also implies that $G - x$ has at least two blocks.

Suppose $G - x$ has a complete endblock $B = K_t$ for $t \geq 4$. Let $v \in V(B)$ be a noncutvertex in $G - x$. Then $G - v$ is 2-connected and h_x -minimal, so by minimality of $|G|$, we conclude that $d_{G-v}(x) \leq 2$, $G - v$ is complete or an odd cycle, or $(G - v, h_x) \in \mathcal{D}$. First, suppose $d_{G-v}(x) \leq 2$. Then $d_G(v) = 3$ and hence $G - x$ has only one endblock, impossible. If $G - v$ is complete, then so is G . Also, $G - v$ cannot be a noncomplete odd cycle since it contains K_3 . Hence, we must have $(G - v, h_x) \in \mathcal{D}$. Since all of v 's neighbors in $G - x$ have degree at least 3

in $G - v$, removing v cannot create an induced path $v_1v_2v_3v_4$ such that $d_G(v_2) = d_G(v_3) = 2$. Hence $G - v$ must be one of the three graphs in Figure 1. The leftmost graph is complete and every endblock of the middle graph is K_2 , so $G - v$ must be the rightmost graph in Figure 1. But then G has the graph in Figure 3 as an induced subgraph, impossible. Hence, every complete endblock of $G - x$ has at most three vertices.

At this point, we know that every endblock of $G - x$ is K_2 or K_3 and x is adjacent to all noncutvertices in each endblock. Suppose $G - x$ has two endblocks B_1 and B_2 . Since $d_G(x) \geq 3$, we violate Lemma 5.4 unless ... \square

References

- [1] Hal Kierstead and Landon Rabern, *Improved lower bounds on the number of edges in list critical and online list critical graphs*, arXiv preprint arXiv:1406.7355 (2014).
- [2] A.V. Kostochka and M. Stiebitz, *A new lower bound on the number of edges in colour-critical graphs and hypergraphs*, Journal of Combinatorial Theory, Series B **87** (2003), no. 2, 374–402.