

## 1. OVERVIEW

The biggest open problem in edge-coloring is the Goldberg–Seymour conjecture. Over the past two decades, the main tool for attacking this problem has become Tashkinov trees, a vast generalization of Vizing fans and Kierstead paths. The second author proved that if  $G$  is a line graph, then  $\chi(G) \leq \max\{\omega(G), \frac{7\Delta(G)+10}{8}\}$ . In the same paper, he conjectured that  $\chi(G) \leq \max\{\omega(G), \frac{5\Delta(G)+8}{6}\}$ , which is best possible. We call the latter inequality the  $\frac{5}{6}$ -Conjecture, and in this paper we prove it. Along the way, we develop more general techniques and results that will likely be of independent interest, due to their use in approaching the Goldberg–Seymour conjecture.

A graph  $G$  is *elementary* if  $\chi'(G) = \mathcal{W}(G)$ ; such graphs satisfy the Goldberg–Seymour Conjecture. (We begin by proving that every minimal counterexample to the  $\frac{5}{6}$ -Conjecture is elementary. In Section 3, we conclude by also proving that every elementary graph satisfies the  $\frac{5}{6}$ -conjecture.) A *defective color* for a Tashkinov tree  $T$  is a color used on more than one edge from  $V(T)$  to  $V(G) - V(T)$ ; a Tashkinov tree is *strongly closed* if it has no defective color. Andersen [1] and Goldberg [2] showed that if  $G$  is critical, then  $G$  is elementary if there exists  $e \in E(G)$  and  $X \subseteq V(G)$  and a  $k$ -edge-coloring  $\varphi$  of  $G - e$  such that  $X$  contains the endpoints of  $e$  and  $X$  is elementary and strongly closed w.r.t.  $\varphi$ . Thus, to show that  $G$  is elementary, it suffices to show that if  $G$  is  $(k+1)$ -critical, then there exists an edge  $e \in E(G)$  and a  $k$ -coloring  $\varphi$  of  $G - e$  such that some maximal Tashkinov tree containing  $e$  is strongly closed. The following definition is useful. A vertex  $v \in V(G)$  is *special* if every Vizing fan rooted at  $v$  (taken over all  $k$ -colorings of  $G - e$ , over all edges  $e$  incident to  $v$ ) has at most 3 vertices, including  $v$ . As a warmup, in Section 2 we prove that if  $\chi'(G) \geq \Delta(G) + 2$  and every vertex of  $G$  is special, then  $G$  is elementary, i.e.,  $\chi'(G) = \mathcal{W}(G)$ . Next, we push our methods further, allowing our maximal Tashkinov tree to have at most 3 non-special vertices.

In Section 3, we show that if  $G$  is a minimal counterexample to the  $\frac{5}{6}$ -Conjecture, then every non-special vertex  $v$  has  $d_G(v) < \frac{3}{4}\Delta(G)$ . Since every maximal Tashkinov tree  $T$  is elementary, and every non-special vertex misses more than  $k/4$  colors, we conclude that  $T$  has at most 3 non-special vertices. Thus our results from Section 2 apply. As a consequence, every minimal counterexample to the  $\frac{5}{6}$ -Conjecture is elementary. To complete the proof of the  $\frac{5}{6}$ -conjecture, we prove that it follows from the Goldberg–Seymour Conjecture. More precisely, we show for each graph  $G$  that if  $\chi'(G) = \mathcal{W}(G)$ , then  $\chi'(G) \leq \max\{\omega(G), \frac{5\Delta(G)+8}{6}\}$ .

## 2. USEFUL LEMMAS

Throughout this section, let  $G$  be a  $(k+1)$ -edge-critical multigraph for some  $k \geq \Delta(G) + 1$ . We use the following notation. Let  $\varphi$  be a partial  $k$ -edge-coloring of  $G$ . For each vertex  $v \in V(G)$ , let  $\varphi(v)$  be the set of colors used in  $\varphi$  on edges incident to  $v$  and let  $\overline{\varphi}(v) = [k] \setminus \varphi(v)$ . For an uncolored edge  $e_0$ , a *Tashkinov tree* is a sequence  $v_0, e_1, v_1, e_2, \dots, v_{t-1}, e_t, v_t$  such that all  $v_i$  are distinct and  $e_i = v_j v_i$  for some  $j$  and  $\ell$  with  $0 \leq j < i$  and  $0 \leq \ell < i$  such that  $\varphi(e_i) \in \overline{\varphi}(v_\ell)$ .

**Lemma 0.** *Let  $\varphi$  be a  $k$ -edge-coloring of  $G - v_0 v_1$ . Suppose  $\alpha \in \overline{\varphi}(v_0)$  and  $\beta \in \overline{\varphi}(v_1)$ . Let  $P = v_1 v_2 \dots v_r$  be an  $\alpha - \beta$  path with edges  $e_i = v_i v_{i+1}$  for  $1 \leq i \leq r - 1$ . If  $v_i$  is special*

for all odd  $i$ , then for any  $\tau \in \overline{\varphi}(v_0)$  there are edges  $f_i = v_i v_{i+1}$  for  $1 \leq i \leq r-1$  such that  $f_i = e_i$  for  $i$  even and  $\varphi(f_i) = \tau$  for  $i$  odd.

*Proof.* Suppose not and choose a counterexample minimizing  $r$ . By minimality of  $r$ , we have  $\varphi(v_{r-1}v_r) = \alpha$  and we have  $f_i = v_i v_{i+1}$  for  $1 \leq i \leq r-2$  such that  $f_i = e_i$  for  $i$  even and  $\varphi(f_i) = \tau$  for  $i$  odd. Swap  $\alpha$  and  $\beta$  on  $e_i$  for  $1 \leq i \leq r-3$  and then color  $v_0 v_1$  (call this edge  $e_0$ ) with  $\alpha$  and uncolor  $e_{r-2}$ . Let  $\varphi'$  be the resulting coloring. Since  $k \geq \Delta(G) + 1$ , some color other than  $\alpha$  is missing at  $v_{r-2}$ ; let  $\gamma$  be such a color. Now  $v_{r-1}$  is special since  $r-1$  is odd (since  $P$  starts and ends with  $\alpha$ ), so there is an edge  $e = v_{r-1}v_r$  with  $\varphi'(e) = \gamma$ . Swap  $\tau$  and  $\alpha$  on  $e_i$  for  $0 \leq i \leq r-3$  to get a new coloring  $\varphi^*$ . Now  $\gamma$  and  $\tau$  are both missing at  $v_{r-2}$  in  $\varphi^*$ . Since  $v_{r-1}$  is special, the fan with  $v_{r-2}, v_{r-1}, v_r$  and  $e$  implies that there is an edge  $f_{r-1} = v_{r-1}v_r$  with  $\varphi^*(f_{r-1}) = \tau$ . But we have never recolored  $f_{r-1}$ , so  $\varphi(f_{r-1}) = \tau$ , a contradiction.  $\square$

Recall that for a  $k$ -edge-coloring  $\varphi$  of  $G - e$ , a maximal Tashkinov tree starting with  $e$  may not be unique. However, if  $T_1$  and  $T_2$  are both such trees, then it is easy to show that  $V(T_1) \subseteq V(T_2)$ ; by symmetry, also  $V(T_2) \subseteq V(T_1)$ , so  $V(T_1) = V(T_2)$ .

**Lemma 1.** *Let  $T$  be a maximal Tashkinov tree with respect to a  $k$ -edge-coloring  $\varphi$  of  $G - xy$ . If at most one  $v \in V(T)$  is non-special, then, for all  $\alpha \in \overline{\varphi}(x)$  and  $\beta \in \overline{\varphi}(y)$ , the  $\alpha - \beta$  path  $P$  from  $y$  to  $x$  has  $V(P) = V(T)$ .*

*Proof.* Let  $\varphi$  be a  $k$ -edge-coloring of  $G - xy$  and let  $T$  be a maximal Tashkinov tree containing  $xy$ . Choose  $\alpha \in \overline{\varphi}(x)$  and  $\beta \in \overline{\varphi}(y)$ . Note that the  $\alpha - \beta$  path  $P$  starting from  $y$  must end at  $x$  (or else we can perform a Kempe swap on  $P$  and color edge  $xy$  with color  $\alpha$ ). We show that  $P$  is a maximal Tashkinov tree; hence  $V(P) = V(T)$ . Say  $P = v_0 \cdots v_r$ , where  $v_0 = y$  and  $v_r = x$ . Suppose  $P$  is not maximal; so some color  $\tau$  is missing on  $P$  and some edge colored  $\tau$  leaves  $P$ . Note that  $E(P) \cup \{xy\}$  is an odd cycle. We can relabel  $V(P)$  so that  $\tau$  is missing at  $v_0$ , the edge colored  $\tau$  leaving  $V(P)$  is incident to  $v_i$  and all vertices  $v_j$  with  $0 < j < i$  are special. Further, if  $v_i$  is non-special, then we can choose  $i$  to be even (by possibly going around the cycle the other way). Also, we can recolor  $E(P) \cup \{xy\}$  with  $\alpha$  and  $\beta$  so that only  $v_0 v_1$  is uncolored and  $\alpha$  and  $\beta$  are missing at  $v_0$  and  $v_1$ , respectively. Now by Lemma 0, each of these edges colored  $\alpha$  has a parallel edge colored  $\tau$ . In particular, the edge colored  $\tau$  incident to  $v_i$  ends in  $P$ , a contradiction.  $\square$

A *defective color* for a Tashkinov tree  $T$  in a critical graph  $G$  is a color used on more than one edge from  $V(T)$  to  $V(G) - V(T)$ . Let  $t(G)$  be the maximum size of a Tashkinov tree over all  $e \in E(G)$  and all  $k$ -edge-colorings  $\varphi$  of  $G - e$ . If a Tashkinov tree  $T$  has  $|T| = t(G)$ , then  $T$  is *maximum*.

**Lemma 2.** *Let  $T$  be a Tashkinov tree with respect to a  $k$ -edge-coloring  $\varphi$  of  $G - v_0 v_1$ . If  $T$  is maximum and at most one  $v \in V(T)$  is non-special, then  $V(T)$  has no defective colors.*

*Proof.* Use Lemma 1 to get an  $\alpha - \beta$  path  $P = v_1 \cdots v_r v_0$  with  $V(P) = V(T)$ ; recall that  $P$  is a maximum size Tashkinov tree. Suppose that  $P$  has a defective color  $\delta$  with respect to  $\varphi$ . Let  $\tau$  be missing at  $v_2$ . Consider a maximal  $\tau - \delta$  path  $Q$ , starting at  $v_2$ . Since  $P$  is maximal,  $\delta$  is not missing at any vertex of  $P$ ; since  $V(P)$  is elementary,  $\tau$  is not missing at any vertex of  $P$  other than  $v_2$ . As a result,  $Q$  ends outside  $V(P)$ . Now  $Q$  could leave  $V(T)$  and re-enter

it repeatedly, but  $Q$  ends outside  $V(P)$ , so there is a last vertex  $w \in V(Q) \cap V(P)$ ; say  $Q$  ends at  $z \in V(G) - V(P)$ . Let  $\pi \notin \{\alpha, \beta\}$  be a color missing at  $w$ . Since  $P$  is maximal, no edge colored  $\tau$  or  $\pi$  leaves  $V(P)$ . So, we can swap  $\tau$  and  $\pi$  on every edge in  $G - V(P)$  without changing the fact that  $P$  is a maximum size Tashkinov tree. Now swap  $\delta$  and  $\pi$  on the subpath of  $Q$  from  $w$  to  $z$ ; since  $\pi$  is missing at  $w$ , the  $\delta - \pi$  path does end at  $w$ . Now  $\delta$  is missing at  $w$ , but  $\delta$  was defective in  $\varphi$ , so there are still edges colored  $\delta$  leaving  $V(P)$ , adding such an edge gets a larger Tashkinov tree, a contradiction.  $\square$

**Theorem 3.** *Let  $G$  be a multigraph. If at most one  $v \in V(G)$  is non-special, then  $\chi'(G) \leq \max \{ \lceil \chi'_f(G) \rceil, \Delta(G) + 1 \}$ .*

*Proof.* We may assume that  $G$  is critical. If  $\chi'(G) \leq \Delta(G) + 1$ , then we are done. So instead, assume that  $\chi'(G) \geq \Delta(G) + 2$  and let  $k = \chi'(G) - 1$ . Choose  $e \in E(G)$ . Now by Lemma 2,  $G - e$  has a  $k$ -edge-coloring  $\varphi$  and a maximal Tashkinov tree  $T$  such that  $V(T)$  has no defective color, i.e.,  $V(T)$  is strongly closed. By a Theorem of Andersen and Goldberg (see also [Stiebitz, p. 8–9]), this implies that  $\chi'(G) = \lceil \chi'_f(G) \rceil$ .  $\square$

### 3. THE EASY BOUND

Let  $G$  be a multigraph. The *claw-degree* of  $x \in V(G)$  is

$$d_{\text{claw}}(x) := \max_{\substack{S \subseteq N(x) \\ |S|=3}} \frac{1}{4} \left( d(x) + \sum_{v \in S} d(v) \right).$$

The *claw-degree* of  $G$  is

$$d_{\text{claw}}(G) := \max_{x \in V(G)} d_{\text{claw}}(x).$$

**Theorem 4.** *If  $G$  is a multigraph, then*

$$\chi'(G) \leq \max \left\{ \lceil \chi'_f(G) \rceil, \Delta(G) + 1, \left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil \right\}.$$

*Proof.* Suppose not and choose a counterexample  $G$  minimizing  $\|G\|$ ; note that  $G$  is edge-critical. Let  $k = \chi'(G) - 1$ , so  $k \geq \lceil \frac{4}{3} d_{\text{claw}}(G) \rceil$ . By Theorem 3,  $G$  has a non-special vertex  $x$ . Choose  $xy_1 \in E(G)$  and a  $k$ -edge-coloring  $\varphi$  of  $G - xy_1$  such that  $\varphi$  has a fan  $F$  of length 3 rooted at  $x$  with leaves  $y_1, y_2, y_3$ . Since  $V(F)$  is elementary,

$$2 + k - d(x) + \sum_{i \in [3]} k - d(y_i) \leq k,$$

and hence

$$d_{\text{claw}}(x) \geq \frac{1}{4} \left( d(x) + \sum_{i \in [3]} d(y_i) \right) \geq \frac{3k + 2}{4}.$$

This gives the contradiction

$$\left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil \leq k \leq \frac{4}{3} d_{\text{claw}}(G) - \frac{2}{3}.$$

$\square$

## 4. A STRONGER BOUND

An edge  $x_1x_2 \in E(G)$  is *special* if for at least one  $i \in [2]$ , every Vizing fan containing  $x_1x_2$ , rooted at  $x_i$  (taken over all  $k$ -colorings of  $G - x_1x_2$ ), has at most 3 vertices. First, some helper lemmas.

**Lemma 5.** *Let  $\varphi$  be a  $k$ -edge-coloring of  $G - v_0v_1$ . Suppose  $\alpha \in \overline{\varphi}(v_0)$  and  $\beta \in \overline{\varphi}(v_1)$ . Let  $P = v_1v_2 \cdots v_r$  be an  $\alpha - \beta$  path. Let  $i, j \in [r]$  with  $i + 3 \leq j$  such that  $j - i$  is odd. If  $v_t$  is special for all  $i < t < j$ , then  $\mu(G) \geq 2k - d(v_i) - d(v_j)$ .*

**Corollary 6.** *Let  $\varphi$  be a  $k$ -edge-coloring of  $G - v_0v_1$ . Suppose  $\alpha \in \overline{\varphi}(v_0)$  and  $\beta \in \overline{\varphi}(v_1)$ . Let  $P = v_1v_2 \cdots v_r$  be an  $\alpha - \beta$  path. Let  $i, a, j \in [r]$  with  $i + 2 \leq a \leq j - 2$  such that  $j - i$  is odd. If  $v_t$  is special for all  $i < t < a$  and  $a < t < j$ , then  $\mu(G) \geq 2k - d(v_i) - d(v_a)$  or  $\mu(G) \geq 2k - d(v_a) - d(v_j)$ .*

**Lemma 7.** *Let  $T$  be a maximum size Tashkinov tree with respect to a  $k$ -edge-coloring  $\varphi$  of  $G - v_0v_1$  in  $G$ . If all but at most three  $v \in V(T)$  is special, then*

- (1)  $V(T)$  has no defective colors; or
- (2) there are non-special vertices  $x_1, x_2 \in V(T)$  such that  $\mu(G) \geq 2k - d(x_1) - d(x_2)$ ; or
- (3)  $E(T)$  contains a non-special edge and  $T$  has three non-special vertices.

*Proof.* Choose  $\alpha \in \overline{\varphi}(v_0)$  and  $\beta \in \overline{\varphi}(v_1)$  so that the length of the  $\alpha - \beta$  path  $P = v_1v_2 \cdots v_rv_0$  is maximized. Suppose both (2) and (3) do not hold. To prove (1), it will suffice to show that  $P$  is a maximal Tashkinov tree. Suppose  $P$  is not maximal. By Lemma 2 there must be non-special vertices  $v_i, v_j$ , where  $i < j$ . Without loss of generality, there is  $\tau \in \overline{\varphi}(v_0)$  a  $\tau$ -colored edge leaving  $P$  from  $v_b$ . Then, by Lemma 0,  $i$  is odd,  $j$  is even and  $i \leq b \leq j$ . By shifting the coloring, we may assume  $i = 1$ .

Suppose  $v_t$  is special for  $1 < t < j$ . Then, since (2) does not hold, Lemma 5 implies  $j = 2$ . By symmetry, we may assume  $b = 1$ . Consider the  $\tau - \beta$  path  $Q$  starting at  $v_1$ . Plainly,  $Q$  must end at  $v_0$ . Since  $P$  is an  $\alpha - \beta$  path,  $Q$  must re-enter  $P$  along a  $\tau$ -edge. But we just showed that  $\tau$  edges can only leave at  $v_1$  or  $v_2$ . So, the  $\tau - \beta$  path re-enters  $P$  at  $v_2$ . But then we replaced a single edge  $v_1v_2$  with a path of length at least three, so the  $\tau - \beta$  path is longer than the  $\alpha - \beta$  path, contradicting our maximality condition on  $P$ .

Hence there is  $1 < a < j$  such that  $v_a$  is non-special. Since (2) does not hold and we are assuming there are at most three non-special vertices, Lemma 6 implies that  $a = 2$  or  $a = j - 1$ . By symmetry, we may assume  $a = 2$ . Uncolor  $v_1v_2$  and color  $v_0v_1$  with  $\alpha$ . Let  $F_i$  be a maximal fan rooted at  $v_i$  for  $i \in [2]$ . If  $|F_i| \leq 3$ , then all the colors in  $\overline{\varphi}(v_{3-i})$  must appear on edges between  $v_i$  and  $v_{3(i-1)}$ . But then (2) is violated. Hence  $|F_i| \geq 4$  for  $i \in [2]$ . So,  $v_1v_2$  is a non-special edge, violating (3).  $\square$

The *claw-degree* of  $x_1x_2 \in E(G)$  is

$$d_{\text{claw}}(x_1x_2) := \frac{1}{4} \min_{i \in [2]} \max_{\substack{S \subseteq N(x_i) \\ |S|=3 \\ S \ni x_{3-i}}} \left( d(x_i) + \sum_{v \in S} d(v) \right).$$

For  $q \in \mathbb{N}$ , put  $G_q := \{v \in V(G) : d(v) \geq q\}$ . Put

$$d_q(G) := \max_{x \in G_q} d_{\text{claw}}(x),$$

and

$$d_q^e(G) := \max_{\substack{xy \in E(G) \\ \{x,y\} \cap G_q \neq \emptyset}} d_{\text{claw}}(xy),$$

**Theorem 8.** *If  $G$  is a multigraph and  $q \in \mathbb{N}$ , then*

$$\chi'(G) \leq \max \left\{ \lceil \chi'_f(G) \rceil, \Delta(G) + 1, \left\lceil \frac{4}{3} d_q(G) \right\rceil, \left\lceil \frac{4}{3} d_{\lceil \frac{2}{3} q \rceil}^e(G) \right\rceil, \left\lceil \frac{4}{3} q \right\rceil, q + \left\lceil \frac{\mu(G)}{2} \right\rceil \right\}.$$

*Proof.* Let  $G$  be a minimal counterexample. Then  $G$  is edge-critical. Let  $T$  be a maximum size Tashkinov tree. We are good if  $T$  has one or fewer non-special vertices. Let  $x$  be a non-special vertex in  $T$ . As in the proof of Theorem 4, we get  $d_{\text{claw}}(x) \geq \frac{3k+2}{4}$ . Hence if any vertex in  $G_q$  is non-special we are done as in Theorem 4. Hence every non-special vertex  $x$  has  $d(x) \leq q - 1$ . If  $T$  has four or more non-special vertices, then  $4(q - 1) \geq 3k + 2$ . But then  $k > \lceil \frac{4}{3} q \rceil - 1 \geq k + 1$ , a contradiction. Hence  $T$  has two or three non-special vertices. If Lemma 7(2) holds, we have  $\mu(G) \geq 2k - 2(q - 1)$  which gives  $q \geq k + 1 - \frac{1}{2}\mu(G)$ . Hence  $k > q + \frac{1}{2}\mu(G) - 1 \geq k$ , a contradiction. So, by Lemma 7(3),  $E(T)$  contains a non-special edge  $x_1x_2$  there is a non-special  $x_3 \in V(T) \setminus \{x_1, x_2\}$ .

If  $\{x_1, x_2\} \cap G_{\lceil \frac{2}{3} q \rceil} = \emptyset$ , then

$$2 + k - d(x_1) + k - d(x_2) + k - d(x_3) \leq k,$$

so

$$\left\lceil \frac{2}{3} q \right\rceil - 1 + 2(q - 1) \leq 2k + 2,$$

so  $k > \lceil \frac{4}{3} q \rceil - 1 \geq k + 1$ , a contradiction.

Hence  $\{x_1, x_2\} \cap G_{\lceil \frac{2}{3} q \rceil} \neq \emptyset$ . That implies  $d_{\lceil \frac{2}{3} q \rceil}^e(G) \geq d_{\text{claw}}(x_1x_2)$ . Since  $x_1x_2$  is non-special, we have for  $i \in [2]$  a fan  $F_i$  rooted at  $x_i$  such that  $x_{3-i} \in V(F_i)$  and  $|F_i| = 4$ . Hence

$$2 + k - d(x_i) + k - d(x_{3-i}) + \sum_{v \in F_i \setminus \{x_i, x_{3-i}\}} d(v) \leq k,$$

so,

$$d(x_i) + d(x_{3-i}) + \sum_{v \in F_i \setminus \{x_i, x_{3-i}\}} d(v) \geq 3k + 2,$$

so,  $d_{\lceil \frac{2}{3} q \rceil}^e(G) \geq d_{\text{claw}}(x_1x_2) \geq \frac{3}{4}k + \frac{1}{2}$ , a contradiction.  $\square$

**Corollary 9.** *If  $Q$  is the line graph of a multigraph, then*

$$\chi(Q) \leq \max \left\{ \omega(Q), \frac{5}{6} (\Delta(Q) + 1) + \frac{1}{2} \right\}.$$

*sketch.* Assume the result holds fractionally. Suppose the result is false. Say  $Q = L(G)$ . Apply Theorem 8 with  $q = \frac{3}{4}\Delta(G)$ . Then at least one of the bounds in the max exceed  $\frac{5}{6}(\Delta(Q) + 1) + \frac{1}{2}$ . Suppose  $\frac{4}{3}d_q(G) > \frac{5}{6}(\Delta(Q) + 1) + \frac{1}{2}$ , we have  $x \in G_q$  with neighbors  $v_1, v_2, v_3$  such that

$$(d(x) + \sum_i d(v_i))/3 > \frac{5}{6}(\Delta(Q) + 1) + \frac{1}{2},$$

this is now just our previous computation showing that vertices of degree at least  $\frac{3}{4}\Delta(G)$  are special, so not possible.

Now suppose  $q + \frac{\mu(G)}{2} > (\frac{5}{6}(\Delta(Q) + 1) + \frac{1}{2})$ . Then  $\frac{\mu(G)-1}{2} \geq \frac{5}{6}(\Delta(Q) + 1) - q$ , plugging into brooks strengthening, we get

$$k + 1 \leq (\Delta(G) + 1) - \frac{5}{6}(\Delta(Q) + 1) - q = \frac{1}{6}(\Delta(Q) + 1) + q = \frac{1}{6}(\Delta(Q) + 1) + \frac{3}{4}\Delta(G).$$

Hence  $\Delta(G) > \frac{8}{9}(\Delta(Q) + 1) > k + 1$ , a contradiction.

So, we must have  $\frac{4}{3}d_{\lceil \frac{2}{3}q \rceil}^e(G) > \frac{5}{6}(\Delta(Q) + 1) + \frac{1}{2}$ . We have  $\lceil \frac{2}{3}q \rceil \geq \frac{1}{2}\Delta(G)$ . So, there is  $x_1 x_2 \in E(G)$  where  $d(x_1) \geq \frac{1}{2}\Delta(G)$  (say, we know it for at least one of them) and there is  $S_i \subseteq N(x_i)$  with  $|S| = 3$  and  $x_{3-i} \in S$  such that

$$\left(\frac{4}{3}\right) \left(\frac{1}{4}\right) \left(d(x_i) + \sum_{v \in S} d(v)\right) \geq \frac{5}{6}(\Delta(Q) + 1) + \frac{1}{2},$$

so

$$d(x_i) + d(x_{3-i}) + \sum_{v \in S \setminus \{x_{3-i}\}} d(v) \geq \frac{5}{2}(\Delta(Q) + 1) + \frac{3}{2},$$

□

## 5. THE $\frac{5}{6}$ -CONJECTURE

**Lemma 10.** *If  $H$  is a connected multigraph and  $G = L(H)$ , then  $\mathcal{W}(H) \leq \max\{\omega(G), 5(\Delta(G) + 1)/6 + 3/6\}$ .*

*Proof.* Let  $d = d_H(x)$ ,  $\Delta = \Delta(H)$ , and  $h = |H|$ . Also, let  $p = \sum_{v \in N(x)} d_H(v)$  and let  $t = \Delta h - 2||H||$ . Note that  $0 < t \leq \Delta$ . Also  $p \geq Md - t$ . Now summing over  $N_H(x)$  gives

$$|N(x)|(\Delta h - t)/(h - 1) > 5/6((|N(x)| - 1)d + |N(x)|\Delta - t) + |N(x)|/2$$

Solving for  $|N(x)|$  gives

$$|N(x)| < (5d + 5t)/(3 + 5d + 5\Delta - 6(\Delta h - t)/(h - 1)).$$

Since the numerator and denominator are linear in  $t$ , the right side is maximized at one end of the interval  $1 \leq t \leq D$ . Letting  $t = D$ , gives  $|N(x)| < (5d + 5\Delta)/(3 + 5d - \Delta)$ , like you had originally. Letting  $t = 1$ , gives  $|N(x)| < (5d + 5)/(3 + 5d + 5\Delta - 6(\Delta h - 1)/(h - 1))$ , which requires a little more analysis, akin to what you wrote in your most recent email.

Does that look right to you?

I did the analysis a little differently, but I got to the same conclusion: Substituting  $d \geq 4D/5$  gives that if  $M \geq 3$ , then we must have  $h \leq 4$ , which implies  $h \leq 3$ , which contradicts  $M \geq 3$ .

So, I think I believe it. I also agree there must be an easier way. One thing that seems a little magical is that when  $5/6 - M/(h - 1) \geq 0$  all of the  $h$ 's go away.

$w(H)$  really has a ceiling in its definition, not sure how much that changes things. without, it is the fractional chromatic index.

i think we get some gain as well from the  $\Delta(H) + 2$  in place of  $\Delta(H)$  we get as i wrote in the previous emails. Maybe this helps with the ceiling.

We can use  $|H|$  odd to get a bit better on the ceiling in what you wrote since the top is even (divide both by two before doing ceiling approximation).

Thinking about your comment that we can assume  $H$  is critical, we can, but not how i was setting it up. Probably you are already thinking something like this:

Assume Goldberg. Take minimum counterexample to  $5/6$  conjecture, say  $G = L(H)$ . The  $H$  is critical. From the argument like in strengthening of Brooks, we get  $\chi(G) \geq \Delta(H) + 2$ . By Goldberg this implies

$$\chi(G) = \max_{Q \subseteq H \text{ s.t. } |Q| \geq 3 \text{ and odd}} \left\lceil \frac{2||Q||}{|Q| - 1} \right\rceil$$

If the max is achieved at a proper subgraph of  $H$ , then there is an edge we can remove without decreasing the max, but this decreases the chromatic number by criticality and the max is a lower bound, so impossible. Therefore,  $|H|$  is odd and

$$\chi(G) = \left\lceil \frac{2||H||}{|H| - 1} \right\rceil$$

so,

$$\left\lceil \frac{2||H||}{|H| - 1} \right\rceil \geq \Delta(H) + 2$$

$$2||H||/(|H| - 1) \geq \Delta(H) + 1$$

using

$$\Delta(H)|H| \geq 2||H||,$$

using  $\Delta(H)|H| \geq 2||H||$ , I get

$$\Delta(H) \geq |H| - 1,$$

□

I think we should be able to prove that the conjecture follows from Goldberg–Seymour. That lemma you proved is pretty useful. We can assume that  $H$  is critical, which implies that  $|N(x)| \geq 2$  for all  $x$  in  $H$ . Now let  $J$  be the simple graph underlying  $H$ . We know that  $\delta(J) \geq 2$ . Let  $B = \{x \in H \text{ s.t. } d_J(x) \geq 3\}$ . That lemma implies that  $|B| \leq 4$ . Further, if  $|B| = 4$ , then each vertex of  $B$  has degree 3 in  $J$ . If  $|B| = 3$ , then two vertices of  $B$  have degree 3 in  $J$  and one has degree 4 in  $J$ . Otherwise  $|B| \leq 2$ . Now if  $J$  has a vertex  $x$  of degree at least 5, and  $|B| = 2$ , then the other vertex in  $B$  has degree 3 in  $J$ . Now  $x$  must be a cut-vertex (since  $J$  is formed by identifying one vertex in multiple disjoint cycles, exactly one of which has a chord). But a cut-vertex in  $J$  is also a cut-vertex in  $H$ , which is a contradiction. Thus, we only need consider the cases when  $|B| = 3$  and  $|B| = 4$ , which have degree sequences  $3, 3, 3, 3, 2, \dots, 2$ . and  $4, 3, 3, 2, \dots, 2$ .  $|B| = 4$  is a subdivided  $K_4$  or a subdivision of a 4-cycle where one matching has multiplicity 2.  $|B| = 3$  is a subdivision of a triangulated 5-cycle. I haven't worked out those cases, but I don't think they should be too hard.



**Lemma 11.** *Suppose  $G = L(H)$  and  $G$  is a minimal counterexample to the  $\frac{5}{6}$ -Conjecture. Let  $k = \frac{5}{6}(\Delta(G) + 1)$ . If  $T$  is a Tashkinov tree w.r.t. a  $k$ -edge-coloring  $\varphi$  of  $H - e$ , then*

$$\sum_{v \in V(T)} d_H(v)(5d_T(v) - 6) \leq -12 + 5 \sum_{e \in E(T)} \mu_H(e)$$

*Proof.* Since  $T$  is elementary, the sets of colors missing at vertices of  $T$  are disjoint, so  $2 + \sum_{v \in V(T)} (k - d_H(v)) \leq k$ . Rewriting this gives  $k(|V(T)| - 1) \leq -2 + \sum_{v \in V(T)} d_H(v)$ . For each edge  $xy \in E(T)$ , we have  $k = \frac{5}{6}(\Delta(G) + 1) \geq \frac{5}{6}(d_H(x) + d_H(y) - \mu_H(xy))$ . Summing over all  $|T| - 1$  edges gives

$$\begin{aligned} -2 + \sum_{v \in V(T)} d_H(v) &\geq k(|V(T)| - 1) \\ &\geq \frac{5}{6}(\Delta(G) + 1)(|T| - 1) \\ &\geq \frac{5}{6} \sum_{uv \in E(T)} (d_H(u) + d_H(v) - \mu_H(uv)) \\ &= \frac{5}{6} \sum_{v \in V(T)} d_H(v)d_T(v) - \frac{5}{6} \sum_{uv \in E(T)} \mu_H(uv) \end{aligned}$$

To prove the lemma, we take the first and last expressions in the inequality chain, multiply by 6, then rearrange terms.  $\square$

**Corollary 12.** *If  $G = L(H)$  and  $G$  is a minimal counterexample to the  $\frac{5}{6}$ -Conjecture, then each  $x \in V(H)$  is special if  $d_H(x) > \frac{3}{4}\Delta(H) - 3$ .*

*Proof.* Suppose that  $x$  is a non-special vertex. Choose  $e$  incident to  $x$  and a  $k$ -edge-coloring  $\varphi$  of  $G - e$  such that there exists a Vizing fan  $T$  rooted at  $x$  with  $|T| \geq 4$ . Since every edge in  $F$  is incident to  $x$ , we have  $\sum_{e \in E(T)} \mu_H(e) \leq d_H(x)$ . From Lemma 11, we have

$$\begin{aligned} -12 + 5d_H(x) &\geq -12 + 5 \sum_{e \in E(T)} \mu_H(e) \\ &\geq \sum_{v \in T} (5d_T(v) - 6)d_H(v) \\ &\geq (5d_T(x) - 6)d_H(x) + \sum_{v \in T-x} (5d_T(v) - 6)d_H(v) \\ &= (5(|T| - 1) - 6)d_H(x) - \sum_{v \in V(T-x)} d_H(v), \end{aligned}$$

where the final equality holds because each vertex  $v \in T - x$  is a leaf. Now rearranging terms gives

$$\begin{aligned}
-12 + \sum_{v \in V(T-x)} d_H(v) &\geq (5(|T| - 1) - 11)d_H(x) \\
-12 + (|T| - 1)\Delta(H) &\geq (5(|T| - 16)d_H(x) \\
d_H(x) &\leq \frac{-12 + (|T| - 1)\Delta(H)}{5|T| - 16} \\
d_H(x) &\leq \frac{-12 + 3\Delta(H)}{4} = \frac{3}{4}\Delta(H) - 3,
\end{aligned}$$

where the final inequality holds because  $|T| \geq 4$  and the right side decreases as a function of  $|T|$ .  $\square$