

# graph theory notes\*

## Stiebitz's proof of Gallai's conjecture on the number of components in the high and low vertex subgraphs of critical graphs

Tibor Gallai conjectured the following in 1963 [1, 2] and Michael Stiebitz proved it in 1982 [3]. For a graph  $G$ , let  $\mathcal{L}(G)$  be the subgraph of  $G$  induced on the vertices of degree  $\delta(G)$  and let  $\mathcal{H}(G)$  be the subgraph of  $G$  induced on the vertices of degree larger than  $\delta(G)$ .

**Theorem** (Stiebitz). *If  $G$  is a color-critical graph with  $\delta(G) = \chi(G) - 1$ , then  $\mathcal{H}(G)$  has at most as many components as  $\mathcal{L}(G)$ .*

In fact, Stiebitz proved a stronger statement. Theorem follows immediately from Lemma using  $X = V(\mathcal{L}(G))$ .

**Lemma** (Stiebitz). *Let  $G$  be a connected graph and  $\emptyset \neq X \subseteq V(G)$  such that*

- $d_G(x) \leq k - 1$  for all  $x \in X$ ; and
- for each component  $C$  of  $G[X]$ , we have  $\chi(G - V(C)) \leq k - 1$ ; and
- $G[X]$  has  $\ell$  components and  $G - X$  has at least  $\ell + 1$  components.

*If  $G - X$  is the disjoint union of (possibly not connected) graphs  $M_1, \dots, M_{\ell+1}$  and  $f_i$  is a  $(k - 1)$ -coloring of  $M_i$  for each  $i \in [\ell + 1]$ , then there are permutations  $\pi_1, \dots, \pi_{\ell+1}$  of  $[k - 1]$  such that the  $(k - 1)$ -coloring of  $G - X$  given by  $(\pi_1 \circ f_1) \cup \dots \cup (\pi_{\ell+1} \circ f_{\ell+1})$  extends to a  $(k - 1)$ -coloring of  $G$ .*

*Proof.* Suppose the lemma is false and choose a counterexample  $G$  and nonempty  $X \subseteq V(G)$  so that  $|X|$  is as small as possible. So,  $G - X$  is the disjoint union of graphs  $M_1, \dots, M_{\ell+1}$  and we have  $(k - 1)$ -colorings  $f_i$  of  $M_i$  for each  $i \in [\ell + 1]$  so that no permutations allow us to extend to a  $(k - 1)$ -coloring of  $G$ .

**Claim 1.** *Each component of  $G[X]$  has edges to at least two of the  $M_i$ .* Suppose to the contrary that we have a component  $C$  of  $G[X]$  that has edges to at most one of the  $M_i$ . Then, since  $G$  is connected, we must have  $\ell \geq 2$ . But now the hypotheses of the lemma are satisfied with  $X' = X \setminus V(C)$  in place of  $X$ , so by minimality of  $|X|$  we get permutations that allow us to extend to a  $(k - 1)$ -coloring of  $G$ , a contradiction.

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**Claim 2.** *Each non-separating vertex in  $G[X]$  has neighbors in at least two of the  $M_i$ .* Suppose to the contrary that we have a component  $C$  of  $G[X]$  and  $x \in V(C)$  a non-separating vertex that has neighbors in at most one of the  $M_i$ . Then, by Claim 1, we must have  $|C| \geq 2$ . But then  $x$  has at most  $k - 2$  neighbors in  $G - X$ , so we can greedily complete any  $(k - 1)$ -coloring of  $G - X$  to  $G - X'$  where  $X' = X \setminus \{x\}$ . So, the hypotheses of the lemma are satisfied with  $X'$  in place of  $X$ . Again, by minimality of  $|X|$ , we get permutations that allow us to extend to a  $(k - 1)$ -coloring of  $G$ , a contradiction.

**Claim 3.** *If  $G[X]$  has at least two components, then  $G - V(C)$  is connected for every component  $C$  of  $G[X]$ .* Suppose  $G[X]$  has at least two components and let  $C$  be a component of  $G[X]$ . If  $G - V(C)$  is disconnected, then with  $X' = V(C)$  in place of  $X$ , the hypotheses of the lemma are satisfied. By minimality of  $|X|$ , we get permutations that allow us to extend to a  $(k - 1)$ -coloring of  $G$ , a contradiction.

**Claim 4.** *The lemma is true.* Pick a component  $C$  in  $G[X]$  and a non-separating vertex  $x \in V(C)$ . By Claim 2 and symmetry, we may assume that  $x$  has neighbors  $y_1, y_2$  in  $M_1, M_2$  respectively. Let  $G' = G - V(C)$  and  $X' = X \setminus V(C)$ . Then  $G' - X'$  is the disjoint union of the  $\ell$  graphs  $M_1 \cup M_2, M_3, \dots, M_{\ell+1}$ . Let  $\tau$  be a permutation of  $[k - 1]$  such that  $(\tau \circ f_2)(y_2) = f_1(y_1)$  and let  $f_* = f_1 \cup (\tau \circ f_2)$ .

Suppose  $G'$  is connected. By minimality of  $|X|$ , we can apply the lemma to  $G'$  with  $M_1 \cup M_2, M_3, \dots, M_{\ell+1}$  and colorings  $f_*, f_3, \dots, f_{\ell+1}$  to get permutations  $\pi_*, \pi_3, \dots, \pi_{\ell+1}$  such that the  $(k - 1)$ -coloring of  $G' - X'$  given by  $(\pi_* \circ f_*) \cup (\pi_3 \circ f_3) \cup \dots \cup (\pi_{\ell+1} \circ f_{\ell+1})$  extends to a  $(k - 1)$ -coloring of  $G'$ . But this is the same as the  $(k - 1)$ -coloring  $(\pi_* \circ f_1) \cup (\pi_* \circ \tau \circ f_2) \cup (\pi_3 \circ f_3) \cup \dots \cup (\pi_{\ell+1} \circ f_{\ell+1})$ , so using the permutations  $\pi_*, \pi_* \circ \tau, \pi_3, \dots, \pi_{\ell+1}$  we get a coloring of  $G - X$  that extends to  $G - V(C)$ .

If  $G'$  is not connected, then  $X = V(C)$  by Claim 3. So,  $\ell = 1$  and  $f_*$  is a  $(k - 1)$ -coloring of  $G - V(C)$ .

In these colorings,  $y_1$  and  $y_2$  receive the same color. This means that  $x$  has  $k - 1 - (d_G(x) - d_C(x)) + 1 \geq d_C(x) + 1$  colors available and each other vertex  $v$  in  $C$  has  $k - 1 - (d_G(v) - d_C(v)) + 1 \geq d_C(v) \geq d_C(v)$  colors available. So, coloring  $C$  greedily in order of decreasing distance from  $x$  gives an extension to a  $(k - 1)$ -coloring of  $G$ , a contradiction.  $\square$

## References

- [1] T. Gallai, *Kritische graphen I.*, Math. Inst. Hungar. Acad. Sci **8** (1963), 165–192 (in German).
- [2] ———, *Kritische graphen II.*, Math. Inst. Hungar. Acad. Sci **8** (1963), 373–395 (in German).
- [3] M. Stiebitz, *Proof of a conjecture of T. Gallai concerning connectivity properties of colour-critical graphs*, Combinatorica **2** (1982), no. 3, 315–323.