SPARSE GRAPHS ADMIT HOMOMORPHISMS INTO ODD CYCLES

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Abstract.

1. Introduction

All graphs under consideration are nonempty finite simple graphs. For graphs G and H, we indicate the existence of a homomorphism from G to H or lack thereof by $G \to H$ and $G \not\to H$, respectively. We write $H \trianglelefteq G$ to indicate that H is an induced subgraph of G, when we want the containment to be proper, we write $H \triangleleft G$.

2. Potential functions

Kostochka and Yancey [3] used "potential functions" to great effect in proving lower bounds on the number of edges in critical graphs. Here we generalize this idea and prove some basic facts.

Definition 1. For positive integers a and b, the (a, b)-*potential function* is the function from graphs to $\mathbb{Z}$ given by $\rho_{a,b}(G) := a\,|G| - b\,\|G\|$. Additionally, put

$$\hat{\rho}_{a,b}(G) := \min_{H \trianglelefteq G} \rho_{a,b}(H).$$

The invariant $\hat{\rho}_{a,b}(G)$ is a measure of the sparseness of G, the larger $\hat{\rho}_{a,b}(G)$ is, the sparser G is. For example, if $\hat{\rho}_{a,b}(G) \geq 0$, then $\text{mad}(G) \leq \frac{2a}{b}$ where $\text{mad}(G)$ is the maximum average degree of G.

For any fixed graph T, we are interested in proving results of the form: any sufficiently sparse graph admits a homomorphism into T. To do so, it will be useful to get the benefits of having a minimum counterexample without being bound to a fixed inductive context. To achieve this, we use *mules* as introduced in [2, 4].

2.1. Mules.

Definition 2. If G and H are graphs, an *epimorphism* is a graph homomorphism $f \colon G \twoheadrightarrow H$ such that $f(V(G)) = V(H)$. We indicate this with the arrow $\twoheadrightarrow$.

Definition 3. Let G be a graph. A graph A is called a *child* of G if $A \neq G$ and there exists $H \trianglelefteq G$ and an epimorphism $f \colon H \twoheadrightarrow A$.

Note that the child-of relation is a strict partial order on the set of (finite simple) graphs $\mathcal{G}$. We call this the *child order* on $\mathcal{G}$ and denote it by '$\prec$'. By definition, if $H \triangleleft G$ then $H \prec G$.

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$$\begin{tikzcd} H \arrow[r, hook, "\iota"] \arrow[d, two heads, "h"'] & G \arrow[d, two heads, "h'"] \\ Q \arrow[r, hook, "\iota"'] & G_h \end{tikzcd}$$

FIGURE 1. The commutative diagram for G_h.

Lemma 1. *The ordering $\prec$ is well-founded on $\mathcal{G}$; that is, every nonempty subset of $\mathcal{G}$ has a minimal element under $\prec$.*

Proof. Let $\mathcal{T}$ be a nonempty subset of $\mathcal{G}$. Pick $G \in \mathcal{T}$ minimizing $|V(G)|$ and then maximizing $|E(G)|$. Since any child of G must have fewer vertices or more edges (or both), we see that G is minimal in $\mathcal{T}$ with respect to $\prec$. □

Definition 4. Let $\mathcal{T}$ be a collection of graphs. A minimal graph in $\mathcal{T}$ under the child order is called a *$\mathcal{T}$-mule*.

2.2. **Basic facts.**

For a graph T together with positive integers a, b and c, let $\mathcal{C}_{T,a,b,c}$ be the set of graphs G such that $G \not\to T$ and $\hat{\rho}_{a,b}(G) \geq c$.

Lemma 2. *Let G be a $\mathcal{C}_{T,a,b,c}$-mule. If $H \lhd G$, then $H \to T$.*

Proof. Since $\hat{\rho}_{a,b}(H) \geq \hat{\rho}_{a,b}(G) \geq c$ and $H \prec G$, we must have $H \to T$ since G is a $\mathcal{C}_{T,a,b,c}$-mule. □

Definition 5. Let H be an induced subgraph of a graph G and $h\colon H \twoheadrightarrow Q$ an epimorphism onto some graph Q. Let G_h be the image of the natural extension of h to an epimorphism h' defined on G; that is, G_h and h' are such that the diagram in Figure 1 commutes (where ι indicates the inclusion map).

Lemma 3. *Let G be a $\mathcal{C}_{T,a,b,c}$-mule and Q an arbitrary graph. If $H \unlhd G$ with $H \neq Q$ such that $H \twoheadrightarrow Q$, then $\rho_{a,b}(H) > \hat{\rho}_{a,b}(Q)$.*

Proof. Suppose to the contrary that there is $H \unlhd G$ with $H \neq Q$ such that $H \twoheadrightarrow Q$ and $\rho_{a,b}(H) \leq \hat{\rho}_{a,b}(Q)$. Let h be an epimorphism from H onto Q. Since G is a $\mathcal{C}_{T,a,b,c}$-mule, G_h cannot be a child of G. But we have an epimorphism h' from G onto G_h and $G_h \neq G$ since $H \neq Q$, so it must be that $G_h \notin \mathcal{C}_{T,a,b,c}$. Since $G \to G_h$ and $G \not\to T$, we must have $G_h \not\to T$. Therefore $\hat{\rho}_{a,b}(G_h) < c$. Pick $W \unlhd G_h$ with $\rho_{a,b}(W) < c$. Since $W \not\subseteq G$, we must have $V(W) \cap V(Q) \neq \emptyset$. Hence $\rho_{a,b}\left(G\left[(V(W) - V(Q)) \cup V(H)\right]\right) \leq \rho_{a,b}(W) - \hat{\rho}_{a,b}(Q) + \rho_{a,b}(H) \leq \rho_{a,b}(W) - \hat{\rho}_{a,b}(Q) + \hat{\rho}_{a,b}(Q) = \rho_{a,b}(W) < c$, a contradiction since $\hat{\rho}_{a,b}(G) \geq c$. □

We have the following basic bound on the potential of non-complete subgraphs of G.

Corollary 4. *Let G be a $\mathcal{C}_{T,a,b,c}$-mule. If $H \unlhd G$ is not complete and $\chi(H) \leq \frac{2a}{b}$, then $\rho_{a,b}(H) > a$.*

Proof. Suppose $\chi(H) = k \leq \frac{2a}{b}$. Then there is an epimorphism from H onto K_k given by contracting all color classes in a k-coloring of H. Since $H \neq K_k$, Lemma 3 gives $\rho_{a,b}(H) > \hat{\rho}_{a,b}(K_k)$. But $\hat{\rho}_{a,b}(K_k) = \min_{t \in [k]} at - b\binom{t}{2} = a$ since $k \leq \frac{2a}{b}$, so we have the desired bound. □

3. The proof

For a graph G, positive integer k and $S \subseteq V(G)$, the *k-potential* of S in G is

$$\rho^k_G(S) := (4k+1)|S| - (4k-1)\left\|G[S]\right\|.$$

When G is clear from context, we drop the subscript and just write $\rho^k(S)$. Kostochka and Yancey [3] proved that if $\rho^1_G(S) \geq 3$ for all $\emptyset \neq S \subseteq V(G)$, then G is 3-colorable. We generalize this as follows.

Theorem 5. *Let G be a graph and k a positive integer. If $\rho^k_G(S) \geq 4k-1$ for all $\emptyset \neq S \subseteq V(G)$, then G has a homomorphism into C_{2k+1}.*

Suppose Theorem 5 is false for some k and choose a counterexample G minimizing $|G|$. We will prove a sequence of structural properties of G and ultimately derive a contradiction. First, we need a basic fact about the potential of induced subgraphs of C_{2k+1}.

Lemma 6. $\rho^k_{C_{2k+1}}(S) \geq 4k+1$ *for all* $\emptyset \neq S \subseteq C_{2k+1}$*. Moreover,* $\rho^k_{C_{2k+1}}(V(C_{2k+1})) = 4k+2$.

Proof. □

Lemma 7. $\rho^k_G(S) \geq 4k+2$ *for every* $S \subsetneq V(G)$ *with* $|S| \geq 2k+2$.

Proof. Suppose to the contrary that there is $S \subsetneq V(G)$ with $|S| \geq 2k+2$ and $\rho^k_G(S) \leq 4k+1$. Since $S \neq V(G)$, by minimality of $|G|$ we have a homomorphism $h\colon G[S] \to C_{2k+1}$ which gives rise to a homomorphism $h'\colon G \to G_h$. As noted in Definition 5, $G_h \not\to C_{2k+1}$ since $G \not\to C_{2k+1}$. Since $|S| \geq 2k+2$, we have $|G_h| < |G|$. So, minimality of $|G|$ gives $W \subseteq V(G_h)$ with $\rho^k_{G_h}(W) \leq 4k-2$. Put $X := h'(S)$. Since $W \not\subseteq V(G)$, we must have $|W \cap X| \neq \emptyset$. Since every non-empty subset of $h'(S)$ has potential at least $4k+1$ (by Lemma 6), $\rho^k_G((W - X) \cup S) \leq \rho^k_{G_h}(W) - (4k+1) + \rho^k_G(S) \leq \rho^k_{G_h}(W) \leq 4k-2$, a contradiction since $\emptyset \neq S \subseteq (W - X) \cup S$. □

We'll also need a basic fact about homomorphisms between odd cycles (see Albertson and Collins [1]).

Lemma 8. *If* $C_{2j+1} \to C_{2k+1}$*, then* $j \geq k$.

References

[1] M.O. Albertson and K. Collins. Homomorphisms of 3-chromatic graphs. *Discrete Mathematics*, 54(2):127–132, 1985.

[2] Daniel W. Cranston and Landon Rabern. Conjectures equivalent to the Borodin-Kostochka conjecture that appear weaker. *Arxiv preprint, `http://arxiv.org/abs/1203.5380`*, 2012.

[3] A. Kostochka and M. Yancey. Ore's Conjecture on color-critical graphs is almost true. *Arxiv preprint, `http://arxiv.org/abs/1209.1050`*, 2012.

[4] L. Rabern. *Coloring graphs from almost maximum degree sized palettes.* PhD thesis, Arizona State University, 2013.