1. Useful lemmas

Let G be $(k+1)$-edge-critical for some $k \geq \Delta(G)+1$. Call $v \in V(G)$ *special* if every fan rooted at v has at most 3 vertices (including v).

Lemma 0. *Let G be $(k+1)$-edge-critical for some $k \geq \Delta(G)+1$. Let ϕ be a k-edge-coloring of $G - v_0v_1$. Suppose α is missing at v_0 and β is missing at v_1. Let $P = v_1v_2 ... v_r$ be an $\alpha - \beta$ path with edges $e_i = v_iv_{i+1}$ for $1 \leq i \leq r-1$. If v_i is special for all odd i, then for any τ that is missing at v_0 there are edges $f_i = v_iv_{i+1}$ for $1 \leq i \leq r-1$ such that $f_i = e_i$ for i even and $\phi(f_i) = \tau$ for i odd.*

Proof. Suppose not and choose a counterexample minimizing r. Then, by minimality of r, we must have $\phi(v_{r-1}v_r) = \alpha$ and we have $f_i = v_iv_{i+1}$ for $1 \leq i \leq r-2$ such that $f_i = e_i$ for i even and $\phi(f_i) = \tau$ for i odd. Swap α and β on e_i for $1 \leq i \leq r-3$ and then color v_0v_1 (call this edge e_0) with α and uncolor e_{r-2}. Let ϕ' be the resulting coloring. Since $k \geq \Delta(G)+1$, there is another color missing at v_{r-2} besides α, let γ be such a color. Now v_{r-1} is special since $r-1$ is odd (since P starts and ends with α), so there is an edge $e = v_{r-1}v_r$ with $\phi'(e) = \gamma$. Now swap τ and α on e_i for $0 \leq i \leq r-3$ to get a new coloring ϕ^*. Then γ and τ are both missing at v_{r-2} in ϕ^*. Since v_{r-1} is special, the fan with v_{r-2}, v_{r-1}, v_r and e implies that there is an edge $f_{r-1} = v_{r-1}v_r$ with $\phi^*(f_{r-1}) = \tau$. Now swap α and τ back on e_i for $0 \leq i \leq r-3$ and then shift the $\alpha - \beta$ coloring one to the right to get back to ϕ. We have all the desired f_i, a contradiction. □

Lemma 1. *Let T be a maximal Tashkinov tree with respect to a k-edge-coloring ϕ of $G - xy$. If every $v \in V(T)$ is special, then for all α missing at x and β missing at y, the $\alpha - \beta$ path P from y to x has $V(P) = V(T)$.*

Proof. We show that P is a maximal Tashkinov tree, then we must have $V(P) = V(T)$. Say $P = v_0v_1 \dots v_r$ where $v_0 = y$ and v_r is the vertex right before x. Suppose P is not maximal. Then there is some color δ missing on P (say at v_i) and an edge colored δ leaving P (say from v_j). We have a 2-colored cycle, so by symmetry we may assume that $i < j$. Using Lemma 0, we can walk from i to j showing that every other edge on the path has a parallel edge colored δ. When we get to v_j, this means the δ edge ends in P, a contradiction. □

A *defective color* for a Tashkinov tree T in a critical graph G is a color used on more than one edge from $V(T)$ to $V(G) - V(T)$.

Lemma 2. *Let T be a maximum size Tashkinov tree with respect to a k-edge-coloring ϕ of $G - v_0v_1$ in G. If every $v \in V(T)$ is special, then $V(T)$ has no defective colors.*

Proof. Use Lemma 1 to get an $\alpha - \beta$ path $P = v_0v_1 \dots v_r$ with $V(P) = V(T)$. Suppose the maximum size Tashkinov tree P has a defective color δ with respect to ϕ. Let τ be missing at v_2. Consider a maximal $\tau - \delta$ path Q. Since $V(P)$ is elementary, δ is not missing at any vertex of P and τ is not missing at any other vertex of P besides v_2. In particular Q ends outside $V(T)$. Now Q could leave $V(T)$ and re-enter and bounce around inside a bunch (in fact Q must contain every δ-colored edge leaving $V(T)$, but we don't need that), but Q ends outside $V(T)$, so there is a last vertex $w \in V(Q) \cap V(P)$ (this is what the Stiebitz book calls an "exit vertex"), say Q ends at $z \in V(G) - V(T)$. Let $\pi \notin \{\alpha, \beta\}$ be a color missing at w.

Since T is maximum size, no edge colored τ or π leaves $V(T)$. So, we can swap τ and π on every edge in $G - V(T)$ without changing the fact that T is a maximum size Tashkinov tree. Now swap δ and π on wQz (since π is missing at w, the $\delta - \pi$ path does end at w). Now δ is missing at w, but δ was defective in ϕ, so there are still edges colored δ leaving $V(T)$, adding such an edge gets a larger Tashkinov tree, a contradiction. □

Theorem 3. *If every $v \in V(G)$ is special, then $\chi'(G) \leq \max\left\{\left\lceil \chi'_f(G) \right\rceil, \Delta(G) + 1\right\}$.*

Proof. Immediate by Lemma 2 since strongly closed Tashkinov tree implies elementary. This is implied by Theorem 1.4 (p. 8–9) of [Stiebitz]. □

2. The easy bound

Let G be a multigraph. The *claw-degree* of $x \in V(G)$ is

$$d_{\text{claw}}(x) := \max_{\substack{S \subseteq N(x) \\ |S| = 3}} \frac{1}{4}\left(d(x) + \sum_{v \in S} d(v)\right).$$

The *claw-degree* of G is

$$d_{\text{claw}}(G) := \max_{x \in V(G)} d_{\text{claw}}(x).$$

Theorem 4. *If G is a multigraph, then*

$$\chi'(G) \leq \max\left\{\left\lceil \chi'_f(G) \right\rceil, \Delta(G) + 1, \left\lceil \frac{4}{3} d_{claw}(G) \right\rceil\right\}.$$

Proof. Suppose not and choose a counterexample G minimizing $\|G\|$. Then G is edge-critical with $\chi'(G) = \left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil + 1$. By Theorem 3 there is a non-special $x \in V(G)$. Let $k = \left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil$. Let $xy_1 \in E(G)$ and ϕ a k-edge-coloring of $G - xy_1$ such that there is a fan F of length 3 rooted at x with leaves y_1, y_2, y_3. Since $V(F)$ is elementary,

$$2 + k - d(x) + \sum_{i \in [3]} k - d(y_i) \leq k,$$

and hence

$$d_{\text{claw}}(x) \geq \frac{1}{4}\left(d(x) + \sum_{i \in [3]} d(y_i)\right) \geq \frac{3k + 2}{4}.$$

Hence, the contradiction

$$\left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil = k \leq \frac{4}{3} d_{\text{claw}}(G) - \frac{2}{3}.$$

□

3. A stronger bound

For $q \in \mathbb{N}$, put $G_q := \{v \in V(G) : d(v) \geq q\}$. Put
$$d_q(G) := \max_{x \in G_q} d_{\text{claw}}(x) .$$

Theorem 5. *If G is a multigraph and $q \in \mathbb{N}$, then*
$$\chi'(G) \leq \max\left\{\lceil \chi'_f(G) \rceil, \Delta(G)+1, \left\lceil \frac{4}{3} d_q(G) \right\rceil, \left\lceil \frac{4}{3} q \right\rceil, q + \frac{1}{2}\mu(G)\right\}.$$

To prove Theorem 5, we need to analyze Tashkinov trees that have up to three non-special vertices.

Lemma 6. *Let T be a maximum size Tashkinov tree with respect to a k-edge-coloring ϕ of $G - v_0v_1$ in G. If all but at most one $v \in V(T)$ is special, then $V(T)$ has no defective colors.*

Proof. One special vertex can't block the parallel edge making machine since we can just go the other way around the cycle. $\square$

Lemma 7. *Let T be a maximum size Tashkinov tree with respect to a k-edge-coloring ϕ of $G - v_0v_1$ in G. If all but at most two $v \in V(T)$ is special, then either $V(T)$ has no defective colors or there are non-special vertices $x_1, x_2 \in V(T)$ such that $\mu(G) \geq 2k - d(x_1) - d(x_2)$.*

Proof. If T has one or fewer non-special vertices, we are done. So suppose T has two. Choose α missing at v_0 and β missing at v_1 so that the length of the $\alpha - \beta$ path $P = v_1v_2 \cdots v_rv_0$ from v_1 to v_0 is maximized. It will suffice to show that P is a maximal Tashkinov tree. If not, then here must be non-special vertices v_i, v_j, where $i < j$. Without loss of generality, suppose there is τ missing at v_0 and a τ-colored edge leaving P from v_a. Then, by Lemma 0, i is odd, j is even and $i \leq a \leq j$.

Suppose $j - i > 1$. Since there a no non-special vertices between v_i and v_j, using Lemma 0 on the path $v_i \cdots v_j$ we see that there are edges on that path that must have parallel edges of all colors missing at v_i as well as all colors missing at v_j. Since these color sets are disjoint, we have $\mu(G) \geq 2k - d(v_i) - d(v_j)$.

So, $j = i + 1$. By symmetry, we may assume $a = i$. Consider the $\tau - \beta$ path starting at v_i. If this path never returns to P, then the $\tau - \beta$ cycle contains only one non-special vertex on it (since it doesn't contain v_j) and so we can win as in Lemma 6. So, the $\tau - \beta$ path does return to P. It must enter along a τ-edge (since P is an $\alpha - \beta$ path). But we just showed that τ edges can only leave at v_i or v_j. So, the $\tau - \beta$ path re-enters P at v_j. But then we replaced a single edge v_iv_j with a path of length at least three, so the $\tau - \beta$ path is longer than the $\alpha - \beta$ path, contradicting our maximality condition on P. $\square$

Lemma 8. *Let T be a maximum size Tashkinov tree with respect to a k-edge-coloring ϕ of $G - v_0v_1$ in G. If all but at most three $v \in V(T)$ is special, then either $V(T)$ has no defective colors or there are non-special vertices $x_1, x_2 \in V(T)$ such that $\mu(G) \geq 2k - d(x_1) - d(x_2)$.*

Proof. Similar to the previous, we don't even need to take a maximal path though. Say we get special vertices v_i, v_b, v_j with $i < b < j$. By looking at parities, it becomes evident that there is no way to avoid getting $\mu(G) \geq 2k - d(v_i) - d(v_j)$ or $\mu(G) \geq 2k - d(v_i) - d(v_b)$ or $\mu(G) \geq 2k - d(v_b) - d(v_j)$ $\square$

The above should all be unified, like pull out a lemma dealing with the parities and when we are guaranteed $\mu(G) \geq 2k - d(v_i) - d(v_j)$.

Lemma 9. *Let T be a maximum size Tashkinov tree with respect to a k-edge-coloring ϕ of $G - v_0v_1$ in G. If all but at most four $v \in V(T)$ is special, then*

- *$V(T)$ has no defective colors; or*
- *there are non-special vertices $x_1, x_2 \in V(T)$ such that $\mu(G) \geq 2k - d(x_1) - d(x_2)$; or*
- *there are non-special vertices $x_1, x_2, x_3, x_4 \in V(T)$ and hence $\sum_{i\in[4]} d(x_i) \geq 3k + 2$*

Proof. Immediate from Lemma 8. i think we can get a bit better than $3k + 2$ because there is another vertex in T since $|T|$ is odd. □

Proof of Theorem 5. Let G be a minimal counterexample. Then G is edge-critical. Let T be a maximum size Tashkinov tree. We are good if T has one or fewer non-special vertices. Let x be a non-special vertex in T. As in the proof of Theorem 4, we get $d_{\text{claw}}(x) \geq \frac{3k+2}{4}$. Hence if any vertex in G_q is non-special we are done as in Theorem 4. Hence every non-special vertex x has $d(x) \leq q - 1$. If T has four or more non-special vertices, then $4(q - 1) \geq 3k + 2$ by the third bullet of Lemma 9. But then $k > \left\lceil \frac{4}{3}q \right\rceil - 1 \geq k + 1$, a contradiction. Hence T has two or three non-special vertices. By Lemma 9, we have $\mu(G) \geq 2k - 2(q - 1)$ which gives $q \geq k + 1 - \frac{1}{2}\mu(G)$. Hence $k > q + \frac{1}{2}\mu(G) - 1 \geq k$, a contradiction. □