Edge lower bounds via discharging notes

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1 Introduction

For a graph G, let $d(G)$ be the average degree of G. Let $\mathcal{T}_k$ be the Gallai trees with maximum degree at most $k-1$, excepting K_k.

2 Gallai's bound via discharging

Theorem 2.1 (Gallai)**.** *For $k \geq 4$ and $G \neq K_k$ a k-AT-critical graph, we have*

$$d(G) < k - 1 + \frac{k-3}{k^2-3}.$$

Proof. Start with initial charge function $\mathrm{ch}(v) = d_G(v)$. Have each k^+-vertex give charge $\frac{k-1}{k^2-3}$ to each of its $(k-1)$-neighbors. Then let the vertices in each component of the low vertex subgraph share their total charge equally. Let $\mathrm{ch}^*(v)$ be the resulting charge function. We finish the proof by showing that $\mathrm{ch}^*(v) \geq k - 1 + \frac{k-3}{k^2-3}$ for all $v \in V(G)$.

If v is a k^+-vertex, then $ch^*(v) \geq d_G(v) - \frac{k-1}{k^2-3} d_G(v) = \left(1 - \frac{k-1}{k^2-3}\right) d_G(v) \geq \left(1 - \frac{k-1}{k^2-3}\right) k = k - 1 + \frac{k-3}{k^2-3}$ as desired.

Let T be a component of the low vertex subgraph. Then the vertices in T receive total charge

$$\frac{k-1}{k^2-3} \sum_{v \in V(T)} k - 1 - d_G(v) = \frac{k-1}{k^2-3} \left((k-1)|T| - 2\left\|T\right\|\right).$$

So, after distributing this charge out equally, each vertex in T receives charge

$$\frac{1}{|T|} \frac{k-1}{k^2-3} ((k-1)|T| - 2\left\|T\right\|) = \frac{k-1}{k^2-3} \left((k-1) - d(T)\right).$$

By Lemma 2.2, this is at least

$$\frac{k-1}{k^2-3} \left((k-1) - \left(k - 2 + \frac{2}{k-1}\right)\right) = \frac{k-1}{k^2-3} \left(\frac{k-3}{k-1}\right) = \frac{k-3}{k^2-3}.$$

Hence each low vertex ends with charge at least $k - 1 + \frac{k-3}{k^2-3}$ as desired. $\square$

Lemma 2.2 (Gallai)**.** *For $k \geq 4$ and $T \in \mathcal{T}_k$, we have $d(T) < k - 2 + \frac{2}{k-1}$.*

Proof. Suppose not and choose a counterexample T minimizing $|T|$. Then T has at least two blocks. Let B be an endblock of T. If B is K_t for $2 \leq t \leq k-2$, then remove the non-separating vertices of B from T to get T'. By minimality of $|T|$, we have

$$2\|T\|-t(t-1) = 2\|T'\| < \left(k-2+\frac{2}{k-1}\right)|T'| = \left(k-2+\frac{2}{k-1}\right)|T|-\left(k-2+\frac{2}{k-1}\right)(t-1).$$

Hence we have the contradiction

$$2\|T\| < \left(k-2+\frac{2}{k-1}\right)|T| + (t+2-k-\frac{2}{k-1})(t-1) \leq \left(k-2+\frac{2}{k-1}\right)|T|.$$

The case when B is an odd cycle is the same as the above, a longer cycle just makes things better. Finally, if $B = K_{k-1}$, remove all vertices of B from T to get T'. By minimality of $|T|$, we have

$$\begin{aligned} 2\|T\| - (k-1)(k-2) - 2 = 2\|T'\| & \\ &< \left(k-2+\frac{2}{k-1}\right)|T'| \\ &= \left(k-2+\frac{2}{k-1}\right)|T| - \left(k-2+\frac{2}{k-1}\right)(k-1). \end{aligned}$$

Hence $2\|T\| < \left(k-2+\frac{2}{k-1}\right)|T|$, a contradiction. □

3 An initial improved bound

Lemma 2.2 is best possible as can be seen by the family of graphs with blocks on a path alternating K_{k-1} and K_2. But we have reducible configurations (see the last section for the precise statements) that place restrictions on K_{k-1} blocks. To state these restrictions, we need the following auxiliary bipartite graph.

For a k-AT-critical graph G, let $\mathcal{L}(G)$ be the subgraph of G induced on the $(k-1)$-vertices and $\mathcal{H}(G)$ the subgraph of G induced on the k-vertices. For $T \in \mathcal{T}_k$, let $W^k(T)$ be the set of vertices of T that are contained in some K_{k-1} in T. Let $\mathcal{B}_k(G)$ be the bipartite graph with one part $V(\mathcal{H}(G))$ and the other part the components of $\mathcal{L}(G)$. Put an edge between $y \in V(\mathcal{H}(G))$ and a component T of $\mathcal{L}(G)$ if and only if $N(y) \cap W^k(T) \neq \emptyset$. Then Lemma 4.2 says that $\mathcal{B}_k(G)$ is 2-degenerate.

We can use this fact to refine our discharging argument. Let ϵ and γ be parameters that we will determine where $\epsilon \leq \gamma < 2\epsilon$. Start with initial charge function $\mathrm{ch}(v) = d_G(v)$.

1. Each k^+-vertex gives charge ϵ to each of its $(k-1)$-neighbors not in a K_{k-1},

2. Each $(k+1)^+$-vertex give charge γ to each of its $(k-1)$-neighbors in a K_{k-1},

3. Let $Q = \mathcal{B}_k(G)$. Repeat the following steps until Q is empty.

    (a) Remove all components T of $\mathcal{L}(G)$ in Q that have degree at most two in Q.

(b) Pick $v \in V(\mathcal{H}(G)) \cap V(Q)$. Send charge γ from v to each $x \in N_G(v) \cap W^k(T)$ for each component T of $\mathcal{L}(G)$ where $vT \in E(Q)$.

(c) Remove v from Q.

4. Have the vertices in each component of $\mathcal{L}(G)$ share their total charge equally.

Let $\text{ch}^*(v)$ be the resulting charge function. Here is some intuition for why this might be a useful refinement. In (3b), v sends charge to at most two different T and so, by Lemma 4.1 (or our 'beyond degree choosability' classification), v loses charge at most 3γ. On the other hand, from (3a) each component T of $\mathcal{L}(G)$ receives charge γ for all but at most four non-separating vertices in a K_{k-1} (the at most four again coming from Lemma 4.1 and the fact that we leave T in Q until it has degree at most two). So, we can get each T almost as much charge as we could hope for without losing too much from the k-vertices. We don't have the same control over $(k+1)^+$-vertices, but it won't matter since they have extra charge to start with and sending γ to every $(k-1)$-neighbor will leave enough charge (we'll use $\gamma < 2\epsilon$ here).

To analyze this discharging procedure we need a bound like Lemma 2.2, but taking into account the number of edges in $\mathcal{B}_k(G)$. We can do this by taking into account the number of non-separating vertices in K_{k-1}'s in T. To this end, for $T \in \mathcal{T}_k$, let $q(T)$ be the number of non-separating vertices in a K_{k-1} in T. We give a family of such bounds.

Lemma 3.1. *Let $K \in \mathbb{N}$ and $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$, $h\colon \mathbb{N} \to \mathbb{R}$ be such that for all $k \geq K \geq 4$ we have*

1. *$f(k) \geq t(t+2-k-p(k))$ for all $t \in [k-2]$; and*
2. *$f(k) \geq (5-k-p(k))s$ for all $s \geq 5$; and*
3. *$f(k) \geq (k-1)(1-p(k)-h(k))$; and*
4. *$p(k) \geq h(k)+5-k$; and*
5. *$p(k) \geq \frac{4}{k-2}$; and*
6. *$p(k) \geq \frac{2+h(k)}{k-2}$; and*
7. *$(k-1)p(k)+(k-3)h(k) \geq k+1$.*

Then for $k \geq K$ and $T \in \mathcal{T}_k$, we have

$$2\,\|T\| \leq (k-3+p(k))\,|T| + f(k) + h(k)q(T).$$

Proof. Suppose not and choose a counterexample T minimizing $|T|$. First, suppose T is K_t for $t \in [k-2]$. Then $t(t-1) > (k-3+p(k))t + f(k)$ contradicting (1). If T is C_{2r+1} for $r \geq 2$, then $2(2r+1) > (k-3+p(k))(2r+1) + f(k)$ and hence $f(k) < (5-k-p(k))(2r+1)$ contradicting (2). If T is K_{k-1}, then $(k-1)(k-2) > (k-3+p(k))(k-1)+f(k)+h(k)(k-1)$ contradicting (3).

Hence T has at least two blocks. Let B be an endblock of T and x_B the cutvertex of T contained in B. Let $T' = T - (V(B) \setminus \{x_B\})$. Then, by minimality of $|T|$, we have

$$2\,\|T'\| \leq (k-3+p(k))\,|T'| + f(k) + h(k)q(T').$$

Hence

$$2\,\|T\| - 2\,\|B\| \leq (k-3+p(k))\,(|T| - (|B|-1)) + f(k) + h(k)q(T').$$

Since T is a counterexample, this gives

$$2\,\|B\| > (k-3+p(k))(|B|-1) + h(k)\,(q(T) - q(T'))\,. \tag{*}$$

Suppose B is K_t for $3 \leq t \leq k-3$ or B is an odd cycle. Then $q(T') = q(T)$, $2\,\|B\| \leq |B|\,(|B|-1)$ and $2\,\|B\| = 2\,|B|$ if $|B| > k-3$. Since $p(k) \geq \frac{4}{k-2}$ by (5), this contradicts *.

If B is K_2, then $q(T') \leq q(T) + 1$ and * gives $2 > k-3+p(k)-h(k)$ contradicting (4).

To handle the cases when B is K_{k-2} or K_{k-1} we need to remove x_B from T as well. Let $T^* = T - V(B)$. Then, by minimality of $|T|$, we have

$$2\,\|T^*\| \leq (k-3+p(k))\,|T^*| + f(k) + h(k)q(T^*).$$

Hence

$$2\,\|T\| - 2\,\|B\| - 2(d_T(x_B) - d_B(x_B)) \leq (k-3+p(k))\,(|T| - |B|) + f(k) + h(k)q(T^*).$$

Since T is a counterexample and B is complete, this gives

$$2\,\|B\| > (k-3+p(k))\,|B| - 2(d_T(x_B) + 1 - |B|) + h(k)\,(q(T) - q(T^*))\,,$$

which is

$$2\,\|B\| > (k-1+p(k))\,|B| - 2d_T(x_B) - 2 + h(k)\,(q(T) - q(T^*))\,. \tag{**}$$

Suppose B is K_{k-2}. Then $d_T(x_B) = k-1$ or $d_T(x_B) = k-2$. In the former case, $q(T) = q(T^*)$ and in the latter $q(T^*) \leq q(T) + 1$. If $d_T(x_B) = k-1$, we have

$$(k-2)(k-3) > (k-1+p(k))(k-2) - 2(k-1) - 2 = (k-2)(k-3) - 4 + (k-2)p(k),$$

contradicting (5). If instead $d_T(x_B) = k-2$, we have

$$(k-2)(k-3) > (k-1+p(k))(k-2) - 2(k-2) - 2 - h(k) = (k-2)(k-3) - 2 + (k-2)p(k) - h(k),$$

contradicting (6).

Finally, suppose B is K_{k-1}. Then $d_T(x_B) = k-1$ and $q(T^*) \leq q(T) - (k-2) + 1 = q(T) - (k-3)$. From **, we have

$$\begin{aligned}(k-1)(k-2) &> (k-1+p(k))(k-1) - 2(k-1) - 2 + h(k)(k-3)\\ &= (k-1)(k-2) + p(k)(k-1) - (k+1) + h(k)(k-3),\end{aligned}$$

contradicting (7). □

Now some examples of using Lemma 3.1. What happens if we take $h(k) = 0$? Then, by (7), we need $(k-1)p(k) \geq k+1$ and hence $p(k) \geq 1 + \frac{2}{k-1}$. Taking $p(k) = 1 + \frac{2}{k-1}$, (3) requires $f(k) \geq -2$. Using $f(k) = -2$, all of the other conditions are satisfied and we conclude $2\,\|T\| \leq \left(k - 2 + \frac{2}{k-1}\right)|T| - 2$ for every $T \in \mathcal{T}_k$ when $k \geq 4$. This is a slight refinement of Gallai's Lemma 2.2.

4 Reducible Configurations

Definition 1. A graph G is *AT-reducible* to H if H is a nonempty induced subgraph of G which is f_H-AT where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not AT-reducible to any nonempty induced subgraph, then it is *AT-irreducible*.

This first lemma tells us how a single high vertex can interact with the low vertex subgraph. This is the version Hal and i used, it (and more) follows from the classification in "mostlow".

Lemma 4.1. *Let $k \geq 5$ and let G be a graph with $x \in V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *$G - x$ has t components $H_1, H_2, \ldots, H_t$, and all are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k-1$ for all $v \in V(G-x)$; and*
4. *$\left|N(x) \cap W^k(H_i)\right| \geq 1$ for $i \in [t]$; and*
5. *$d_G(x) \geq t+2$.*

Then G is f-AT where $f(x) = d_G(x) - 1$ and $f(v) = d_G(v)$ for all $v \in V(G-x)$.

To deal with more than one high vertex we need the following auxiliary bipartite graph. For a graph G, $\{X, Y\}$ a partition of $V(G)$ and $k \geq 4$, let $\mathcal{B}_k(X, Y)$ be the bipartite graph with one part Y and the other part the components of $G[X]$. Put an edge between $y \in Y$ and a component T of $G[X]$ if and only if $N(y) \cap W^k(T) \neq \emptyset$. The next lemma tells us that we have a reducible configuration if this bipartite graph has minimum degree at least three.

Lemma 4.2. *Let $k \geq 7$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k-1$ for all $v \in V(G-Y)$; and*
4. *with $\mathcal{B} := \mathcal{B}_k(V(G-Y), Y)$ we have $\delta(\mathcal{B}) \geq 3$.*

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.

We also have the following version with asymmetric degree condition on $\mathcal{B}$. The point here is that this works for $k \geq 5$. As we'll see in the next section, the consequence is that we trade a bit in our size bound for the proof to go through with $k \in \{5, 6\}$.

Lemma 4.3. *Let $k \geq 5$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*

3. $d_G(v) \leq k - 1$ *for all* $v \in V(G - Y)$; *and*

4. *with* $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ *we have* $d_{\mathcal{B}}(y) \geq 4$ *for all* $y \in Y$ *and* $d_{\mathcal{B}}(T) \geq 2$ *for all components* T *of* $G - Y$.

Then G *has an induced subgraph* G' *that is* f-*AT where* $f(y) = d_{G'}(y) - 1$ *for* $y \in Y$ *and* $f(v) = d_{G'}(v)$ *for all* $v \in V(G' - Y)$.