1. Overview

The biggest open problem in edge-coloring is the Goldberg–Seymour conjecture. Over the past two decades, the main tool for attacking this problem has become Tashkinov trees, a vast generalization of Vizing fans and Kierstead paths. The second author proved that if G is a line graph, then $\chi(G) \leq \max\{\omega(G), \frac{7\Delta(G)+10}{8}\}$. In the same paper, he conjectured that $\chi(G) \leq \max\{\omega(G), \frac{5\Delta(G)+8}{6}\}$, which is best possible. We call the latter inequality the $\frac{5}{6}$-Conjecture, and in this paper we prove it. Along the way, we develop more general techniques and results that will likely be of independent interest, due to their use in approaching the Goldberg–Seymour conjecture.

A graph G is *elementary* if $\chi'(G) = \mathcal{W}(G)$; such graphs satisfy the Goldberg–Seymour Conjecture. (We begin by proving that every minimal counterexample to the $\frac{5}{6}$-Conjecture is elementary. In Section 3, we conclude by also proving that every elementary graph satisfies the $\frac{5}{6}$-conjecture.) A *defective color* for a Tashkinov tree T is a color used on more than one edge from $V(T)$ to $V(G) - V(T)$; a Tashkinov tree is *strongly closed* if it has no defective color. Andersen [] and Goldberg [] showed that if G is critical, then G is elementary if there exists $e \in E(G)$ and $X \subseteq V(G)$ and a k-edge-coloring φ of $G - e$ such that X contains the endpoints of e and X is elementary and strongly closed w.r.t. φ. Thus, to show that G is elementary, it suffices to show that if G is $(k+1)$-critical, then there exists an edge $e \in E(G)$ and a k-coloring φ of $G - e$ such that some maximal Tashkinov tree containing e is strongly closed. The following definition is useful. A vertex $v \in V(G)$ is *special* if every Vizing fan rooted at v (taken over all k-colorings of $G - e$, over all edges e incident to v) has at most 3 vertices, including v. As a warmup, in Section 2 we prove that if $\chi'(G) \geq \Delta(G)+2$ and every vertex of G is special, then G is elementary, i.e., $\chi'(G) = \mathcal{W}(G)$. Next, we push our methods further, allowing our maximal Tashkinov tree to have at most 3 non-special vertices.

In Section 3, we show that if G is a minimal counterexample to the $\frac{5}{6}$-Conjecture, then every non-special vertex v has $d_G(v) < \frac{3}{4}\Delta(G)$. Since every maximal Tashkinov tree T is elementary, and every non-special vertex misses more than $k/4$ colors, we conclude that T has at most 3 non-special vertices. Thus our results from Section 2 apply. As a consequence, every minimal countexample to the $\frac{5}{6}$-Conjecture is elementary. To complete the proof of the $\frac{5}{6}$-conjecture, we prove that it follows from the Goldberg–Seymour Conjecture. More precisely, we show for each graph G that if $\chi'(G) = \mathcal{W}(G)$, then $\chi'(G) \leq \max\{\omega(G), \frac{5\Delta(G)+8}{6}\}$.

2. Useful lemmas

Throughout this section, let G be a $(k+1)$-edge-critical multigraph for some $k \geq \Delta(G)+1$. We use the following notation. Let φ be a partial k-edge-coloring of G. For each vertex $v \in V(G)$, let $\varphi(v)$ be the set of colors used in φ on edges incident to v and let $\overline{\varphi}(v) = [k] \setminus \varphi(v)$. For an uncolored edge e_0, a *Tashkinov tree* is a sequence $v_0, e_1, v_1, e_2, \ldots, v_{t-1}, e_t, v_t$ such that all v_i are distinct and $e_i = v_j v_i$ for some j and ℓ with $0 \leq j < i$ and $0 \leq \ell < i$ such that $\varphi(e_i) \in \overline{\varphi}(v_\ell)$.

Lemma 0. *Let φ be a k-edge-coloring of $G - v_0v_1$. Suppose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Let $P = v_1v_2\cdots v_r$ be an $\alpha - \beta$ path with edges $e_i = v_iv_{i+1}$ for $1 \leq i \leq r-1$. If v_i is special* *for all odd i, then for any $\tau \in \overline{\varphi}(v_0)$ there are edges $f_i = v_iv_{i+1}$ for $1 \le i \le r-1$ such that $f_i = e_i$ for i even and $\varphi(f_i) = \tau$ for i odd.*

Proof. Suppose not and choose a counterexample minimizing r. By minimality of r, we have $\varphi(v_{r-1}v_r) = \alpha$ and we have $f_i = v_iv_{i+1}$ for $1 \le i \le r-2$ such that $f_i = e_i$ for i even and $\varphi(f_i) = \tau$ for i odd. Swap α and β on e_i for $1 \le i \le r-3$ and then color v_0v_1 (call this edge e_0) with α and uncolor e_{r-2}. Let φ' be the resulting coloring. Since $k \ge \Delta(G)+1$, some color other than α is missing at v_{r-2}; let γ be such a color. Now v_{r-1} is special since $r-1$ is odd (since P starts and ends with α), so there is an edge $e = v_{r-1}v_r$ with $\varphi'(e) = \gamma$. Swap τ and α on e_i for $0 \le i \le r-3$ to get a new coloring φ^*. Now γ and τ are both missing at v_{r-2} in φ^*. Since v_{r-1} is special, the fan with v_{r-2}, v_{r-1}, v_r and e implies that there is an edge $f_{r-1} = v_{r-1}v_r$ with $\varphi^*(f_{r-1}) = \tau$. But we have never recolored f_{r-1}, so $\varphi(f_{r-1}) = \tau$, a contradiction. $\square$

Recall that for a k-edge-coloring φ of $G-e$, a maximal Tashkinov tree starting with e may not be unique. However, if T_1 and T_2 are both such trees, then it is easy to show that $V(T_1) \subseteq V(T_2)$; by symmetry, also $V(T_2) \subseteq V(T_1)$, so $V(T_1) = V(T_2)$.

Lemma 1. *Let T be a maximal Tashkinov tree with respect to a k-edge-coloring φ of $G-xy$. If at most one $v \in V(T)$ is non-special, then, for all $\alpha \in \overline{\varphi}(x)$ and $\beta \in \overline{\varphi}(y)$, the $\alpha-\beta$ path P from y to x has $V(P) = V(T)$.*

Proof. Let φ be a k-edge-coloring of $G-xy$ and let T be a maximal Tashkinov tree containing xy. Choose $\alpha \in \overline{\varphi}(x)$ and $\beta \in \overline{\varphi}(y)$. Note that the $\alpha-\beta$ path P starting from y must end at x (or else we can perform a Kempe swap on P and color edge xy with color α). We show that P is a maximal Tashkinov tree; hence $V(P) = V(T)$. Say $P = v_0 \cdots v_r$, where $v_0 = y$ and $v_r = x$. Suppose P is not maximal; so some color τ is missing on P and some edge colored τ leaves P. Note that $E(P) \cup \{xy\}$ is an odd cycle. We can relabel $V(P)$ so that τ is missing at v_0, the edge colored τ leaving $V(P)$ is incident to v_i and all vertices v_j with $0 < j < i$ are special. Further, if v_i is non-special, then we can choose i to be even (by possibly going around the cycle the other way). Also, we can recolor $E(P) \cup \{xy\}$ with α and β so that only v_0v_1 is uncolored and α and β are missing at v_0 and v_1, respectively. Now by Lemma 0, each of these edges colored α has a parallel edge colored τ. In particular, the edge colored τ incident to v_i ends in P, a contradiction. $\square$

A *defective color* for a Tashkinov tree T in a critical graph G is a color used on more than one edge from $V(T)$ to $V(G) \setminus V(T)$. Let $t(G)$ be the maximum size of a Tashkinov tree over all $e \in E(G)$ and all k-edge-colorings φ of $G-e$. Let $\mathcal{T}(G)$ be the set of all triples (T, e, φ) such that $e \in E(G)$, φ is a k-edge-coloring of $G-e$ and T is a Tashkinov tree with respect to e and φ with $|T| = t(G)$.

Lemma 2. *Let T be a Tashkinov tree with respect to a k-edge-coloring φ of $G - v_0v_1$. If $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$ and at most one $v \in V(T)$ is non-special, then T has no defective colors.*

Proof. Use Lemma 1 to get an $\alpha-\beta$ path $P = v_1 \dots v_rv_0$ with $V(P) = V(T)$; recall that P is a maximum size Tashkinov tree. Suppose that P has a defective color δ with respect to φ. Let τ be missing at v_2. Consider a maximal $\tau-\delta$ path Q, starting at v_2. Since P is maximal,

δ is not missing at any vertex of P; since $V(P)$ is elementary, τ is not missing at any vertex of P other than v_2. As a result, Q ends outside $V(P)$. Now Q could leave $V(T)$ and re-enter it repeatedly, but Q ends outside $V(P)$, so there is a last vertex $w \in V(Q) \cap V(P)$; say Q ends at $z \in V(G) - V(P)$. Let $\pi \notin \{\alpha, \beta\}$ be a color missing at w. Since P is maximal, no edge colored τ or π leaves $V(P)$. So, we can swap τ and π on every edge in $G - V(P)$ without changing the fact that P is a maximum size Tashkinov tree. Now swap δ and π on the subpath of Q from w to z; since π is missing at w, the $\delta - \pi$ path does end at w. Now δ is missing at w, but δ was defective in φ, so there are still edges colored δ leaving $V(P)$, adding such an edge gets a larger Tashkinov tree, a contradiction. □

Theorem 3. *Let G be a multigraph. If at most one $v \in V(G)$ is non-special, then* $\chi'(G) \leq \max\left\{\lceil \chi'_f(G) \rceil, \Delta(G)+1\right\}$.

Proof. We may assume that G is critical. If $\chi'(G) \leq \Delta(G) + 1$, then we are done. So instead, assume that $\chi'(G) \geq \Delta(G) + 2$ and let $k = \chi'(G) - 1$. Choose $e \in E(G)$. Now by Lemma 2, there is $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$ such that T has no defective color, i.e., $V(T)$ is strongly closed. By a Theorem of Andersen and Goldberg (see also [Stiebitz, p. 8–9]), this implies that $\chi'(G) = \lceil \chi'_f(G) \rceil$. □

3. The easy bound

Let G be a multigraph. The *claw-degree* of $x \in V(G)$ is

$$d_{\text{claw}}(x) := \max_{\substack{S \subseteq N(x) \\ |S|=3}} \frac{1}{4}\left(d(x) + \sum_{v \in S} d(v)\right).$$

The *claw-degree* of G is

$$d_{\text{claw}}(G) := \max_{x \in V(G)} d_{\text{claw}}(x).$$

Theorem 4. *If G is a multigraph, then*

$$\chi'(G) \leq \max\left\{\lceil \chi'_f(G) \rceil, \Delta(G)+1, \left\lceil \frac{4}{3} d_{claw}(G) \right\rceil\right\}.$$

Proof. Suppose not and choose a counterexample G minimizing $\|G\|$; note that G is edge-critical. Let $k = \chi'(G) - 1$, so $k \geq \left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil$. By Theorem 3, G has a non-special vertex x. Choose $xy_1 \in E(G)$ and a k-edge-coloring φ of $G - xy_1$ such that φ has a fan F of length 3 rooted at x with leaves y_1, y_2, y_3. Since $V(F)$ is elementary,

$$2 + k - d(x) + \sum_{i \in [3]} k - d(y_i) \leq k,$$

and hence

$$d_{\text{claw}}(x) \geq \frac{1}{4}\left(d(x) + \sum_{i \in [3]} d(y_i)\right) \geq \frac{3k+2}{4}.$$

This gives the contradiction

$$\left\lceil \frac{4}{3} d_{\text{claw}}\left(G\right) \right\rceil \leq k \leq \frac{4}{3} d_{\text{claw}}\left(G\right) - \frac{2}{3}.$$

□

4. A stronger bound

Throughout this section, let G be a non-elementary critical multigraph with $\chi'(G) = k + 1 \geq \Delta(G) + 2$.

For vertices $x, y \in V(G)$, we say that x is *y-special* if every Vizing fan rooted at x, with respect to any k-edge-coloring of $G - xy$, has at most 3 vertices.

Lemma 5. *Let $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$. Suppose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $P = v_1v_2 \cdots v_rv_0$ be the $\alpha - \beta$ path from v_1 to v_0. Then there are $a, b \in [r]$ such that both v_a, v_b are non-special and $b - a$ is odd.*

Proof. Let $v_{i_1}, \dots v_{i_p}$ be the non-special vertices in P. By Lemma 2, $p \geq 2$. Suppose $i_a - i_b$ is even for all $a, b \in [p]$. Now $(P, v_0v_1, \varphi) \notin \mathcal{T}(G)$, for otherwise G would be elementary. Hence there are $j, \ell \in [r]$ and $\tau \in \overline{\varphi}(v_j)$ such that $\varphi(v_\ell z) = \tau$ for some $z \in V(T) \setminus V(P)$. By rotating the $\alpha - \beta$ coloring on P, we may assume that $j = 0$.

Let m be the least odd integer such that v_m is non-special (if it exists, otherwise, let $m = r$). Let m' be the largest even integer such that $v_{m'}$ is non-special (if it exists, otherwise, let $m' = 0$). By Lemma 0, there is a τ-colored edge between v_t and v_{t+1} for all odd $t \in [m-1]$. By uncoloring v_rv_0, coloring v_0v_1 with β and then running the same argument going the opposite direction on P, we conclude that there is a τ-colored edge between v_t and v_{t+1} for all odd t with $m' + 1 \leq t \leq r - 1$.

But $i_a - i_b$ is even for all $a, b \in [p]$, so either $m = r$ or $m' = 0$. Hence there is a τ-colored edge between $v_{\ell-1}v_\ell$ or $v_\ell v_{\ell+1}$ contradicting $\varphi(v_\ell z) = \tau$. □

We say that G is *k-thin* if $\mu(G) < 2k - d(x) - d(y)$ for all non-special $x, y \in V(G)$.

Lemma 6. *Suppose G is k-thin. Let $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$. Suppose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $P = v_1v_2 \cdots v_rv_0$ be the $\alpha - \beta$ path from v_1 to v_0. Then there is $i \in [r-1]$ such that both v_i and v_{i+1} are non-special.*

Proof. Let $v_{i_1}, \dots v_{i_p}$ be the non-special vertices in P. Suppose $i_{j+1} - i_j > 1$ for all $j \in [p-1]$. Since G is thin, Lemma 0 implies that $i_{j+1} - i_j$ is even for all $j \in [p-1]$. Hence $i_a - i_b$ is even for all $a, b \in [p]$ violating Lemma 5. □

For a Tashkinov tree T, let $\nu(T)$ be the number of non-special vertices in T.

Lemma 7. *Suppose G is k-thin. If $(T, e, \varphi) \in \mathcal{T}(G)$ for $e \in E(G)$, then $\nu(T) \geq 3$.*

Proof. Lemma 2 gives $\nu(T) \geq 2$. Suppose $\nu(T) = 2$. Let $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Let $P = v_1v_2 \cdots v_rv_0$ be the $\alpha - \beta$ path from v_1 to v_0. By Lemma 6, there is $i \in [r-1]$ such that both v_i and v_{i+1} are non-special. By rotating the $\alpha - \beta$ coloring on P, we may assume that $i = 1$.

Since G is not elementary, we have $a, b \in [r]$ and $\tau \in \overline{\varphi}(v_a)$ such that $\varphi(v_b z) = \tau$ for some $z \in V(T) \setminus V(P)$. Consider the $\tau - \beta$ path Q starting at v_b. By Lemma 6, v_1 and v_2 (the only non-special vertices in T) must be contiguous on Q. But there cannot be a τ or β edge between v_1 and v_2, a contradiction. □

Lemma 8. *Suppose G is k-thin. If $(T, e, \varphi) \in \mathcal{T}(G)$ for $e \in E(G)$ and $\nu(T) = 3$, then there are non-special $x_1, x_2, x_3 \in V(T)$ such that $x_i x_j \in E(G)$ and x_i is not x_j-special for all $i, j \in [3]$ with $i \neq j$.*

Proof. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ so that the $\alpha - \beta$ path $P = v_1 v_2 \cdots v_r v_0$ from v_1 to v_0 so as to maximize the number of non-special vertices in P. Then P contains 3 non-special vertices since otherwise we can use the argument in Lemma 7. By symmetry, we may assume that v_1 and v_2 are non-special. Let v_j be the other non-special vertex.

SKETCH: Use the fact that the $\tau - \beta$ path starting at v_1 must have contiguous non-special vertices to conclude that there is a τ edge from v_2 to v_j. Repeat the argument, now using the $\tau - \beta$ path instead of the $\alpha - \beta$ path and using v_2 and v_j in place of v_1 and v_2. This gets an edge from v_j to v_1. □

Lemma 9. *Suppose G is k-thin. If $(T, e, \varphi) \in \mathcal{T}(G)$ for $e \in E(G)$ and $\nu(T) \leq 4$, then there are non-special $x_1, x_2 \in V(T)$ such that $x_1 x_2 \in E(G)$ and x_1 is not x_2-special.*

Proof. By Lemma 7 and Lemma 8 we may assume $\nu(T) = 4$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ so that the $\alpha - \beta$ path $P = v_1 v_2 \cdots v_r v_0$ from v_1 to v_0 so as to maximize the number of non-special vertices in P. Then P contains 4 non-special vertices since otherwise we can use the arguments in Lemma 7 and in Lemma 8.

By Lemma 6 there are contiguous non-special vertices in P. By symmetry, we may assume that v_1 and v_2 are non-special. Let v_a and v_b with $2 < a < b$ be the other two non-special vertices in P. Suppose v_i is v_{3-i}-special for $i \in [2]$.

SKETCH: Look at the fans both ways, we violate thinness on either $v_0 v_1$ or $v_2 v_3$ depending on the parity of b. □

Theorem 10 (from strengthening Brooks paper)**.** *If Q is the line graph of a multigraph G and Q is vertex critical, then*

$$\chi(Q) \leq \max\left\{\omega(Q), \Delta(Q) + 1 - \frac{\mu(G) - 1}{2}\right\}.$$

Theorem 11. *If Q is the line graph of a multigraph, then*

$$\chi(Q) \leq \max\left\{\lceil \chi_f(Q) \rceil, \lceil \epsilon(\Delta(Q) + 1) \rceil\right\}.$$

Proof. Suppose the theorem is false and choose a counterexample minimizing $|Q|$. Put $k = \max\{\lceil \chi_f(Q) \rceil, \lceil \epsilon(\Delta(Q) + 1) \rceil\}$. Say $Q = L(G)$ for a multigraph G. Then G is k-edge-critical.

Claim 0. *Let F be a fan rooted at x with respect to a k-edge-coloring of $G - xy$. If $|F| = 4$, then*

$$d(x) < \frac{1 - \epsilon}{2\epsilon - 1} \sum_{v \in V(F - x)} d(v).$$

Since F is elementary, we have

$$2 + k - d(x) + \sum_{v \in V(F-x)} k - d(v) \leq k,$$

so

$$2 + (|F| - 1)k \leq d(x) + \sum_{v \in V(F-x)} d(v).$$

Using $k \geq \epsilon(\Delta(Q) + 1) \geq \epsilon(d(x) + d(v) - \mu(xv))$ for each $v \in V(F - x)$, we get

$$2 + \sum_{v \in V(F-x)} \epsilon(d(x) + d(v) - \mu(xv)) \leq d(x) + \sum_{v \in V(F-x)} d(v),$$

so

$$2 + (\epsilon|F| - 1 - \epsilon)\, d(x) \leq \sum_{v \in V(F-x)} \epsilon\mu(xv) + \sum_{v \in V(F-x)} (1 - \epsilon)d(v).$$

Now $\sum_{v \in V(F-x)} \mu(xv) \leq d(x)$, so this becomes

$$2 + (\epsilon|F| - 1 - 2\epsilon)\, d(x) \leq \sum_{v \in V(F-x)} (1 - \epsilon)d(v).$$

Using $|F| = 4$ gives

$$d(x) < \frac{1 - \epsilon}{2\epsilon - 1} \sum_{v \in V(F-x)} d(v).$$

Claim 1. *If $x \in V(G)$ with $d(x) \geq \frac{3(1-\epsilon)}{2\epsilon - 1}\Delta(G)$, then x is special.*

Immediate from Claim 0.

Claim 2. *If $x_1 x_2 \in E(G)$ with*

$$d(x_i) \geq \frac{2(1 - \epsilon)}{3\epsilon - 2}\Delta(G),$$

for at least one $i \in [2]$, then x_1 is x_2-special or x_2 is x_1-special.

Suppose x_1 is not x_2-special and x_2 is not x_1-special. Then by Claim 0, we have for $i \in [2]$,

$$d(x_i) < \frac{1 - \epsilon}{2\epsilon - 1} \sum_{v \in V(F-x)} d(v) \leq \frac{1 - \epsilon}{2\epsilon - 1} \left(d(3 - x_i) + 2\Delta(G) \right),$$

Solving the system gives for $i \in [2]$,

$$d(x_i) < \frac{2(1 - \epsilon)}{3\epsilon - 2}\Delta(G).$$

Claim 3. *If $xy \in E(G)$ with*

$$d(x) \geq \frac{1 - \epsilon^2}{(2\epsilon - 1)^2}\Delta(G),$$

then y is special or x is y-special.

Suppose y is not special and x is not y-special. Then by Claim 0, we have

$$d(x) < \frac{1 - \epsilon}{2\epsilon - 1} \sum_{v \in V(F-x)} d(v) \leq \frac{1 - \epsilon}{2\epsilon - 1} \left(d(y) + 2\Delta(G) \right),$$

Since y is not special, Claim 1 gives $d(y) < \frac{3(1-\epsilon)}{2\epsilon-1}\Delta(G)$ and thus

$$d(x) < \frac{1-\epsilon^2}{(2\epsilon-1)^2}\Delta(G).$$

Claim 4. *The theorem is true.*

Let $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$. By Lemmas 7, 8 and 9 one of the following holds:

(1) G is elementary; or
(2) G is not thin; or
(3) $\nu(T) = 3$ and $E(T)$ contains non-special $x_1, x_2, x_3 \in V(T)$ such that $x_ix_j \in E(G)$ and x_i is not x_j-special for all $i, j \in [3]$ with $i \neq j$; or
(4) $\nu(T) = 4$ and $E(T)$ contains an edge x_1x_2 where x_2 is not special and x_1 is not x_2-special; or
(5) $V(T)$ contains five non-special vertices x_1, x_2, x_3, x_4, x_5.

If (1) holds, then $k + 1 = \lceil \chi_f(Q) \rceil \leq k$, a contradiction.

If (2) holds, then by Claim 1 we have $\mu(G) \geq 2k - 2\frac{3(1-\epsilon)}{2\epsilon-1}\Delta(G)$. Hence Theorem 10 gives

$$\leq k + 1 \leq \Delta(Q) + 1 - k - \frac{3(1-\epsilon)}{2\epsilon-1}\Delta(G) + \frac{1}{2},$$

so

$$2(k+1) \leq \Delta(Q) + \frac{5}{2} - \frac{3(1-\epsilon)}{2\epsilon-1}\Delta(G).$$

Since $k \geq \Delta(G) + 1$, this gives

$$k + 1 < \frac{\Delta(Q) + \frac{5}{2}}{2 + \frac{3(1-\epsilon)}{2\epsilon-1}},$$

which is a contradiction when $\epsilon > \frac{1}{2}$.

Suppose (3) holds. So

$$2 + \sum_{i\in[3]} k - d(x_i) \leq k,$$

using Claim 2, this gives

$$3\left(\frac{2(1-\epsilon)}{3\epsilon-2}\right)\Delta(G) \geq 2k + 2,$$

which is a contradiction when $\epsilon \geq \frac{5}{6}$.

Suppose (4) holds. Let x_3, x_4 be non-special vertices in $V(T) \setminus \{x_1, x_2\}$.

$$2 + \sum_{i\in[4]} k - d(x_i) \leq k,$$

using Claim 1 and Claim 3 gives

$$\left(3\frac{3(1-\epsilon)}{2\epsilon-1} + \frac{1-\epsilon^2}{(2\epsilon-1)^2}\right)\Delta(G) \geq 3k + 2,$$

which is a contradiction when $\epsilon \geq \frac{39+\sqrt{157}}{62} \approx 0.831$.

So (5) must hold. But then

$$2 + \sum_{i \in [5]} k - d(x_i) \leq k,$$

using Claim 1 gives

$$\frac{15(1-\epsilon)}{2\epsilon - 1} \Delta(G) \geq 4k + 2,$$

which is a contradiction when $\epsilon \geq \frac{19}{23} \approx 0.826$.

$\square$

5. The $\frac{5}{6}$-Conjecture

Lemma 12. *If H is a connected multigraph and $G = L(H)$, then $\mathcal{W}(H) \leq \max\{\omega(G), 5(\Delta(G)+1)/6 + 3/6\}$.*

Proof. Let $d = d_H(x)$, $\Delta = \Delta(H)$, and $h = |H|$. Also, let $p = \sum_{v \in N(x)} d_H(v)$ and let $t = \Delta h - 2||H||$. Note that $0 < t \leq \Delta$. Also $p \geq Md - t$. Now summing over $N_H(x)$ gives

$$|N(x)|(\Delta h - t)/(h-1) > 5/6((|N(x)|-1)d + |N(x)|\Delta - t) + |N(x)|/2$$

Solving for $|N(x)|$ gives

$$|N(x)| < (5d + 5t)/(3 + 5d + 5\Delta - 6(\Delta h - t)/(h-1)).$$

Since the numerator and denominator are linear in t, the right side is maximized at one end of the interval $1 \leq t \leq D$. Letting $t = D$, gives $|N(x)| < (5d + 5\Delta)/(3 + 5d - \Delta)$, like you had originally. Letting $t = 1$, gives $|N(x)| < (5d+5)/(3+5d+5\Delta-6(\Delta h-1)/(h-1))$, which requires a little more analysis, akin to what you wrote in your most recent email.

Does that look right to you?

I did the analysis a little differently, but I got to the same conclusion: Substituting $d \geq 4D/5$ gives that if $M \geq 3$, then we must have $h \leq 4$, which implies $h \leq 3$, which contradicts $M \geq 3$.

So, I think I believe it. I also agree there must be an easier way. One thing that seems a little magical is that when $5/6 - M/(h-1) \geq 0$ all of the h's go away.

$w(H)$ really has a ceiling in its definition, not sure how much that changes things. without, it is the fractional chromatic index.

i think we get some gain as well from the $\Delta(H) + 2$ in place of $\Delta(H)$ we get as i wrote in the previous emails. Maybe this helps with the ceiling.

We can use $|H|$ odd to get a bit better on the ceiling in what you wrote since the top is even (divide both by two before doing ceiling approximation).

Thinking about your comment that we can assume H is critical, we can, but not how i was setting it up. Probably you are already thinking something like this:

Assume Goldberg. Take minimum counterexample to 5/6 conjecture, say $G = L(H)$. The H is critical. From the argument like in strengthening of Brooks, we get $\chi(G) \geq \Delta(H) + 2$. By Goldberg this implies

$$\chi(G) = \max_{Q \subseteq H \text{ s.t. } |Q| \geq 3 \text{ and odd}} \left\lceil \frac{2||Q||}{|Q|-1} \right\rceil$$

If the max is achieved at a proper subgraph of H, then there is an edge we can remove without decreasing the max, but this decreases the chromatic number by criticality and the max is a lower bound, so impossible. Therefore, $|H|$ is odd and

$$\chi(G) = \left\lceil \frac{2||H||}{|H|-1} \right\rceil$$

so,

$$\left\lceil \frac{2||H||}{|H|-1} \right\rceil \geq \Delta(H) + 2$$

$$2||H||/(|H|-1) \geq \Delta(H) + 1$$

using

$$\Delta(H)|H| \geq 2||H||,$$

using $\Delta(H)|H| \geq 2||H||$, I get

$$\Delta(H) \geq |H| - 1,$$

□

I think we should be able to prove that the conjecture follows from Goldberg–Seymour. That lemma you proved is pretty useful. We can assume that H is critical, which implies that $|N(x)| \geq 2$ for all x in H. Now let J be the simple graph underlying H. We know that $\delta(J) \geq 2$. Let $B = \{x \in H s.t. d_J(x) \geq 3\}$. That lemma implies that $|B| \leq 4$. Further, if $|B| = 4$, then each vertex of B has degree 3 in J. If $|B| = 3$, then two vertices of B have degree 3 in J and one has degree 4 in J. Otherwise $|B| \leq 2$. Now if J has a vertex x of degree at least 5, and $|B| = 2$, then the other vertex in B has degree 3 in J. Now x must be a cut-vertex (since J is formed by identifying one vertex in multiple disjoint cycles, exactly one of which has a chord). But a cut-vertex in J is also a cut-vertex in H, which is a contradiction. Thus, we only need consider the cases when $|B| = 3$ and $|B| = 4$, which have degree sequences $3, 3, 3, 3, 2, \ldots 2.$ and $4, 3, 3, 2, \ldots, 2$. $|B| = 4$ is a subdivided K_4 or a subdivision of a 4-cycle where one matching has multiplicity 2. $|B| = 3$ is a subdivision of a triangulated 5-cycle. I haven't worked out those cases, but I don't think they should be too hard.

Lemma 13. *Suppose $G = L(H)$ and G is a minimal counterexample to the $\frac{5}{6}$-Conjecture. Let $k = \frac{5}{6}(\Delta(G)+1)$. If T is a Tashkinov tree w.r.t. a k-edge-coloring φ of $H - e$, then*

$$\sum_{v\in V(T)} d_H(v)(5d_T(v) - 6) \leq -12 + 5 \sum_{e\in E(T)} \mu_H(e)$$

Proof. Since T is elementary, the sets of colors missing at vertices of T are disjoint, so $2 + \sum_{v\in V(T)}(k - d_H(v)) \leq k$. Rewriting this gives $k(|V(T)| - 1) \leq -2 + \sum_{v\in V(T)} d_H(v)$. For each edge $xy \in E(T)$, we have $k = \frac{5}{6}(\Delta(G)+1) \geq \frac{5}{6}(d_H(x) + d_H(y) - \mu_H(xy))$. Summing over all $|T| - 1$ edges gives

$$\begin{aligned}
-2 + \sum_{v\in V(T)} d_H(v) &\geq k(|V(T)| - 1) \\
&\geq \frac{5}{6}(\Delta(G)+1)(|T| - 1) \\
&\geq \frac{5}{6} \sum_{uv\in E(T)} (d_H(u) + d_H(v) - \mu_H(uv)) \\
&= \frac{5}{6} \sum_{v\in V(T)} d_H(v) d_T(v) - \frac{5}{6} \sum_{uv\in E(T)} \mu_H(uv)
\end{aligned}$$

To prove the lemma, we take the first and last expressions in the inequality chain, multiply by 6, then rearrange terms. □

Corollary 14. *If $G = L(H)$ and G is a minimal counterexample to the $\frac{5}{6}$-Conjecture, then each $x \in V(H)$ is special if $d_H(x) > \frac{3}{4}\Delta(H) - 3$.*

Proof. Suppose that x is a non-special vertex. Choose e incident to x and a k-edge-coloring φ of $G - e$ such that there exists a Vizing fan T rooted at x with $|T| \geq 4$. Since every edge in F is incident to x, we have $\sum_{e\in E(T)} \mu_H(e) \leq d_H(x)$. From Lemma 13, we have

$$\begin{aligned}
-12 + 5d_H(x) &\geq -12 + 5 \sum_{e\in E(T)} \mu_H(e) \\
&\geq \sum_{v\in T} (5d_T(v) - 6) d_H(v) \\
&\geq (5d_T(x) - 6) d_H(x) + \sum_{v\in T-x} (5d_T(v) - 6) d_H(v) \\
&= (5(|T| - 1) - 6) d_H(x) - \sum_{v\in V(T-x)} d_H(v),
\end{aligned}$$

where the final equality holds because each vertex $v \in T - x$ is a leaf. Now rearranging terms gives

$$\begin{aligned}
-12 + \sum_{v \in V(T-x)} d_H(v) &\geq (5(|T|-1) - 11)d_H(x) \\
-12 + (|T|-1)\Delta(H) &\geq (5(|T|-16)d_H(x) \\
d_H(x) &\leq \frac{-12 + (|T|-1)\Delta(H)}{5|T|-16} \\
d_H(x) &\leq \frac{-12 + 3\Delta(H)}{4} = \frac{3}{4}\Delta(H) - 3,
\end{aligned}$$

where the final inequality holds because $|T| \geq 4$ and the right side decreases as a function of $|T|$. $\square$