# Edge lower bounds via discharging notes

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## 1 Introduction

For a graph $G$, let $d(G)$ be the average degree of $G$. Let $\mathcal{T}_k$ be the Gallai trees with maximum degree at most $k-1$, excepting $K_k$.

## 2 Gallai's bound via discharging

**Theorem 2.1** (Gallai). *For $k \geq 4$ and $G \neq K_k$ a k-AT-critical graph, we have*

$$d(G) < k - 1 + \frac{k-3}{k^2-3}.$$

*Proof.* Start with initial charge function $\mathrm{ch}(v) = d_G(v)$. Have each $k^+$-vertex give charge $\frac{k-1}{k^2-3}$ to each of its $(k-1)$-neighbors. Then let the vertices in each component of the low vertex subgraph share their total charge equally. Let $\mathrm{ch}^*(v)$ be the resulting charge function. We finish the proof by showing that $\mathrm{ch}^*(v) \geq k - 1 + \frac{k-3}{k^2-3}$ for all $v \in V(G)$.

If $v$ is a $k^+$-vertex, then $ch^*(v) \geq d_G(v) - \frac{k-1}{k^2-3} d_G(v) = \left(1 - \frac{k-1}{k^2-3}\right) d_G(v) \geq \left(1 - \frac{k-1}{k^2-3}\right) k = k - 1 + \frac{k-3}{k^2-3}$ as desired.

Let $T$ be a component of the low vertex subgraph. Then the vertices in $T$ receive total charge

$$\frac{k-1}{k^2-3} \sum_{v \in V(T)} k - 1 - d_G(v) = \frac{k-1}{k^2-3} \left((k-1)|T| - 2\,\|T\|\right).$$

So, after distributing this charge out equally, each vertex in $T$ receives charge

$$\frac{1}{|T|} \frac{k-1}{k^2-3} ((k-1)|T| - 2\,\|T\|) = \frac{k-1}{k^2-3} \left((k-1) - d(T)\right).$$

By Lemma 2.2, this is at least

$$\frac{k-1}{k^2-3} \left((k-1) - \left(k - 2 + \frac{2}{k-1}\right)\right) = \frac{k-1}{k^2-3} \left(\frac{k-3}{k-1}\right) = \frac{k-3}{k^2-3}.$$

Hence each low vertex ends with charge at least $k - 1 + \frac{k-3}{k^2-3}$ as desired. $\square$

**Lemma 2.2** (Gallai). *For $k \geq 4$ and $T \in \mathcal{T}_k$, we have $d(T) < k - 2 + \frac{2}{k-1}$.*

*Proof.* Suppose not and choose a counterexample $T$ minimizing $|T|$. Then $T$ has at least two blocks. Let $B$ be an endblock of $T$. If $B$ is $K_t$ for $2 \leq t \leq k-2$, then remove the non-separating vertices of $B$ from $T$ to get $T'$. By minimality of $|T|$, we have

$$2\,\|T\| - t(t-1) = 2\,\|T'\| < \left(k - 2 + \frac{2}{k-1}\right)|T'| = \left(k - 2 + \frac{2}{k-1}\right)|T| - \left(k - 2 + \frac{2}{k-1}\right)(t-1).$$

Hence we have the contradiction

$$2\,\|T\| < \left(k - 2 + \frac{2}{k-1}\right)|T| + (t + 2 - k - \frac{2}{k-1})(t-1) \leq \left(k - 2 + \frac{2}{k-1}\right)|T|.$$

The case when $B$ is an odd cycle is the same as the above, a longer cycle just makes things better. Finally, if $B = K_{k-1}$, remove all vertices of $B$ from $T$ to get $T'$. By minimality of $|T|$, we have

$$
\begin{aligned}
2\,\|T\| - (k-1)(k-2) - 2 &= 2\,\|T'\| \\
&< \left(k - 2 + \frac{2}{k-1}\right)|T'| \\
&= \left(k - 2 + \frac{2}{k-1}\right)|T| - \left(k - 2 + \frac{2}{k-1}\right)(k-1).
\end{aligned}
$$

Hence $2\,\|T\| < \left(k - 2 + \frac{2}{k-1}\right)|T|$, a contradiction. $\square$

# 3    An initial improved bound

Lemma 2.2 is best possible as can be seen by the family of graphs with blocks on a path alternating $K_{k-1}$ and $K_2$. But we have reducible configurations (see the last section for the precise statements) that place restrictions on $K_{k-1}$ blocks. To state these restrictions, we need the following auxiliary bipartite graph.

For a $k$-AT-critical graph $G$, let $\mathcal{L}(G)$ be the subgraph of $G$ induced on the $(k-1)$-vertices and $\mathcal{H}(G)$ the subgraph of $G$ induced on the $k$-vertices. For $T \in \mathcal{T}_k$, let $W^k(T)$ be the set of vertices of $T$ that are contained in some $K_{k-1}$ in $T$. Let $\mathcal{B}_k(G)$ be the bipartite graph with one part $V(\mathcal{H}(G))$ and the other part the components of $\mathcal{L}(G)$. Put an edge between $y \in V(\mathcal{H}(G))$ and a component $T$ of $\mathcal{L}(G)$ if and only if $N(y) \cap W^k(T) \neq \emptyset$. Then Lemma 4.2 says that $\mathcal{B}_k(G)$ is 2-degenerate.

We can use this fact to refine our discharging argument. Let $\epsilon$ and $\gamma$ be parameters that we will determine where $\epsilon \leq \gamma < 2\epsilon$. Start with initial charge function $\mathrm{ch}(v) = d_G(v)$.

1. Each $k^+$-vertex gives charge $\epsilon$ to each of its $(k-1)$-neighbors not in a $K_{k-1}$,

2. Each $(k+1)^+$-vertex give charge $\gamma$ to each of its $(k-1)$-neighbors in a $K_{k-1}$,

3. Let $Q = \mathcal{B}_k(G)$. Repeat the following steps until $Q$ is empty.

    (a) Remove all components $T$ of $\mathcal{L}(G)$ in $Q$ that have degree at most two in $Q$.

(b) Pick $v \in V(\mathcal{H}(G)) \cap V(Q)$. Send charge $\gamma$ from $v$ to each $x \in N_G(v) \cap W^k(T)$ for each component $T$ of $\mathcal{L}(G)$ where $vT \in E(Q)$.

(c) Remove $v$ from $Q$.

4. Have the vertices in each component of $\mathcal{L}(G)$ share their total charge equally.

Let $\mathrm{ch}^*(v)$ be the resulting charge function. Here is some intuition for why this might be a useful refinement. In (3b), $v$ sends charge to at most two different $T$ and so, by Lemma 4.1 (or our 'beyond degree choosability' classification), $v$ loses charge at most $3\gamma$. On the other hand, from (3a) each component $T$ of $\mathcal{L}(G)$ receives charge $\gamma$ for all but at most four non-separating vertices in a $K_{k-1}$ (the at most four again coming from Lemma 4.1 and the fact that we leave $T$ in $Q$ until it has degree at most two). So, we can get each $T$ almost as much charge as we could hope for without losing too much from the $k$-vertices. We don't have the same control over $(k+1)^+$-vertices, but it won't matter since they have extra charge to start with and sending $\gamma$ to every $(k-1)$-neighbor will leave enough charge (we'll use $\gamma < 2\epsilon$ here).

To analyze this discharging procedure we need a bound like Lemma 2.2, but taking into account the number of edges in $\mathcal{B}_k(G)$. We can do this by taking into account the number of non-separating vertices in $K_{k-1}$'s in $T$. To this end, for $T \in \mathcal{T}_k$, let $q(T)$ be the number of non-separating vertices in a $K_{k-1}$ in $T$. We give a family of such bounds.

**Lemma 3.1.** *Let $K \in \mathbb{N}$ and $p \colon \mathbb{N} \to \mathbb{R}$, $f \colon \mathbb{N} \to \mathbb{R}$, $h \colon \mathbb{N} \to \mathbb{R}$ be such that for all $k \geq K$ we have*

1. *$f(k) \geq (k-1)(1 - p(k) - h(k))$; and*

2. *$f(k) \geq t(t + 2 - k - p(k))$ for all $t \in [k-2]$; and*

3. *$p(k) \geq h(k) + 5 - k$; and*

4. *$p(k) \geq \frac{4}{k-2}$; and*

5. *$p(k) \geq \frac{2+h(k)}{k-2}$; and*

6. *$(k-1)p(k) + (k-3)h(k) \geq k+1$.*

*Then for $k \geq K$ and $T \in \mathcal{T}_k$, we have*

$$2\,\|T\| \leq (k - 3 + p(k))\,|T| + f(k) + h(k)q(T).$$

# 4   Reducible Configurations

**Definition 1.** A graph $G$ is *AT-reducible* to $H$ if $H$ is a nonempty induced subgraph of $G$ which is $f_H$-AT where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If $G$ is not AT-reducible to any nonempty induced subgraph, then it is *AT-irreducible*.

This first lemma tells us how a single high vertex can interact with the low vertex subgraph. This is the version Hal and i used, it (and more) follows from the classification in "mostlow".

**Lemma 4.1.** *Let $k \geq 5$ and let $G$ be a graph with $x \in V(G)$ such that:*

1. *$K_k \nsubseteq G$; and*

2. *$G - x$ has $t$ components $H_1, H_2, \ldots, H_t$, and all are in $\mathcal{T}_k$; and*

3. *$d_G(v) \leq k - 1$ for all $v \in V(G - x)$; and*

4. *$\left| N(x) \cap W^k(H_i) \right| \geq 1$ for $i \in [t]$; and*

5. *$d_G(x) \geq t + 2$.*

*Then $G$ is $f$-AT where $f(x) = d_G(x) - 1$ and $f(v) = d_G(v)$ for all $v \in V(G - x)$.*

To deal with more than one high vertex we need the following auxiliary bipartite graph. For a graph $G$, $\{X, Y\}$ a partition of $V(G)$ and $k \geq 4$, let $\mathcal{B}_k(X, Y)$ be the bipartite graph with one part $Y$ and the other part the components of $G[X]$. Put an edge between $y \in Y$ and a component $T$ of $G[X]$ if and only if $N(y) \cap W^k(T) \neq \emptyset$. The next lemma tells us that we have a reducible configuration if this bipartite graph has minimum degree at least three.

**Lemma 4.2.** *Let $k \geq 7$ and let $G$ be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \nsubseteq G$; and*

2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*

3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*

4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $\delta(\mathcal{B}) \geq 3$.*

*Then $G$ has an induced subgraph $G'$ that is $f$-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.*

We also have the following version with asymmetric degree condition on $\mathcal{B}$. The point here is that this works for $k \geq 5$. As we'll see in the next section, the consequence is that we trade a bit in our size bound for the proof to go through with $k \in \{5, 6\}$.

**Lemma 4.3.** *Let $k \geq 5$ and let $G$ be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \nsubseteq G$; and*

2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*

3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*

4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $d_{\mathcal{B}}(y) \geq 4$ for all $y \in Y$ and $d_{\mathcal{B}}(T) \geq 2$ for all components $T$ of $G - Y$.*

*Then $G$ has an induced subgraph $G'$ that is $f$-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.*