

December 31, 2016

Let $c_k^*(\mathcal{L})$ be the number of components of \mathcal{L} containing a copy of K_{k-1} . Let $q_k(\mathcal{L})$ be the number of non-cut vertices in \mathcal{L} that appear in copies of K_{k-1} . Let $\beta_k(\mathcal{L})$ be the independence number of the subgraph of \mathcal{L} induced on the vertices of degree $k-1$. When k is defined in context, we just write $c^*(\mathcal{L})$, $q(\mathcal{L})$ and $\beta(\mathcal{L})$. Let $\mathcal{H}(G)$ be the subgraph of G induced on vertices of degree greater than $\delta(G)$. Let $\mathcal{L}(G)$ be the subgraph of G induced on vertices of degree $\delta(G)$.

Definition 1. The *maximum independent cover number* of a graph G is the maximum $\text{mic}(G)$ of $\|I, V(G) \setminus I\|$ over all independent sets I of G .

Definition 2. A graph G is *OC-reducible* to H if H is a nonempty induced subgraph of G which is online f_H -choosable where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not OC-reducible to any nonempty induced subgraph, then it is *OC-irreducible*.

Lemma 1. Every OC-irreducible graph G satisfies

$$2 \|G\| > (\delta(G) - 1) |G| + \text{mic}(G).$$

Lemma 2. If G is an OC-irreducible graph where $\mathcal{H}(G)$ is edgeless, $\Delta := \Delta(G) = \delta(G) + 1$ and $\mathcal{L} := \mathcal{L}(G)$, then

$$2 \|\mathcal{L}\| > \left(\Delta - 2 - \frac{2}{\Delta - 2} \right) |\mathcal{L}| + \frac{\Delta(\Delta - 1)}{\Delta - 2} \beta_\Delta(\mathcal{L}).$$

Proof. Let G be such a graph. Put $\mathcal{H} := \mathcal{H}(G)$ and $\mathcal{L} := \mathcal{L}(G)$. Since \mathcal{H} is edgeless,

$$\begin{aligned} \Delta |\mathcal{H}| &= \|\mathcal{H}, \mathcal{L}\| \\ &= (\Delta - 1) |\mathcal{L}| - 2 \|\mathcal{L}\|, \end{aligned} \tag{1}$$

so, by Lemma 1,

$$\begin{aligned} (\Delta - 1) |\mathcal{L}| + \Delta |\mathcal{H}| &= 2 \|G\| \\ &> (\Delta - 2) |G| + \text{mic}(G) \\ &\geq (\Delta - 2) |G| + \Delta |\mathcal{H}| + (\Delta - 1) \beta_\Delta(\mathcal{L}) \\ &= (\Delta - 2) |\mathcal{L}| + (2\Delta - 2) |\mathcal{H}| + (\Delta - 1) \beta_\Delta(\mathcal{L}), \end{aligned}$$

so simplifying and using (1) again gives

$$\begin{aligned} |\mathcal{L}| &> (\Delta - 2) |\mathcal{H}| + (\Delta - 1) \beta_{\Delta}(\mathcal{L}) \\ &= \frac{\Delta - 2}{\Delta} ((\Delta - 1) |\mathcal{L}| - 2 \|\mathcal{L}\|) + (\Delta - 1) \beta_{\Delta}(\mathcal{L}), \end{aligned}$$

now some mild manipulation yields the desired bound. \square

Lemma 3. $\left(\frac{3k-7}{k^2-4k+5}, \frac{(k-1)(k-4)}{k^2-4k+5}, 2, \frac{-2(k-1)(k-4)}{k^2-4k+5} \right)$ is 5-Gallai.

Lemma 4. Let G be a non-complete AT-irreducible graph with $\delta(G) = k - 1$ where $k \geq 5$. Let \mathcal{L} be the subgraph of G induced on $(k - 1)$ -vertices, \mathcal{H}^- the subgraph of G induced on k -vertices and \mathcal{H}^+ the subgraph of G induced on $(k + 1)^+$ -vertices. Then

$$q(\mathcal{L}) \leq c^*(\mathcal{L}) + 4 |\mathcal{H}^-| + \|\mathcal{H}^+, \mathcal{L}\|,$$

and if $k \geq 7$, then

$$q(\mathcal{L}) \leq 2c^*(\mathcal{L}) + 3 |\mathcal{H}^-| + \|\mathcal{H}^+, \mathcal{L}\|.$$

Lemma 5. If G is an OC-irreducible graph where $\mathcal{H}(G)$ is edgeless, $\Delta := \Delta(G) = \delta(G) + 1 \geq 7$, $\mathcal{L} := \mathcal{L}(G)$ and $\mathcal{H} := \mathcal{H}(G)$, then

$$2 \|\mathcal{L}\| \leq \left(\Delta - 3 + \frac{3\Delta - 7}{\Delta^2 - 4\Delta + 5} \right) |\mathcal{L}| + \frac{3(\Delta - 1)(\Delta - 4)}{\Delta^2 - 4\Delta + 5} |\mathcal{H}| + 2\beta_{\Delta}(\mathcal{L}).$$

Proof. Combine the second inequality in Lemma 4 with Lemma 3. \square

Lemma 6. If G is an OC-irreducible graph where $\mathcal{H}(G)$ is edgeless, $\Delta := \Delta(G) = \delta(G) + 1 \geq 7$, $\mathcal{L} := \mathcal{L}(G)$ and $\mathcal{H} := \mathcal{H}(G)$, then

$$\left(\Delta - 10 + \frac{4(\Delta + 1)}{\Delta^2 - 4\Delta + 5} \right) |\mathcal{L}| + (\Delta^2 - 3\Delta + 4) \beta_{\Delta}(\mathcal{L}) < 0,$$

in particular, $\Delta \leq 9$.

Proof. Combine Lemma 2 with Lemma 5 and the fact that $|\mathcal{H}| < \frac{|L|}{\Delta - 2}$. \square