1 Eulerian Circuits

Definition 1.1. A graph is *Eulerian* if it has a closed trail containing all the edges.

Lemma 1. *A connected graph is Eulerian iff every vertex has even degree.*

Proof. Let G be a connected graph.

If G has an eulerian tour C, then each passage of C through a vertex uses two edges and the first edge is paired with the last edge at the first vertex. Hence each vertex of G is even.

Assume every vertex of G has even degree. Let T be a trail in G with maximum number of edges. If T is not closed, then it contains an odd number of edges incident to its end vertex x. But x has even degree, so we can extend T to a larger trail. This contradiction shows that T must be closed.

Now suppose T misses some edge of G and pick $xy \in E(G) - E(T)$ such that the distance from x to $V(T)$ is minimized. Then $x \in V(T)$ since G is connected. T is closed, so we may rotate it so that it starts (and ends) at x. But then extending T along xy contradicts the maximality of T. Hence T is a closed trail containing every edge of G. $\square$

2 Connectivity

Lemma 2. *For any connected graph G and any $z \in V(G)$, there exists a total ordering $\leqslant$ of $V(G)$ with z minimum, such that $G[x \mid x \leqslant y]$ is connected for each $y \in V(G)$.*

Proof. Let G be a connected graph and $z \in V(G)$. Let H be a maximal induced subgraph of G which has such an ordering with z minimum. If $H \neq G$, then, since G is connected, some $w \in V(G) - V(H)$ has an edge into H. Thus we may add w to H as the last vertex in the ordering contradicting the maximality of H. Hence $H = G$ and we have our ordering. $\square$

Mader's Average Degree Theorem. *Fix $k \in \mathbb{N}_{\geqslant 1}$. Every graph G with $d(G) \geqslant 4k$ has a $(k+1)$-connected subgraph H such that $d(H) > d(G) - 2k$.*

Proof. Let G be a graph with $d(G) \geqslant 4k$. Let $t = \frac{d(G)}{2} \geqslant 2k$. Of all subgraphs G' of G satisfying

$$|G'| \geqslant 2k \text{ and } \|G'\| > t(|G'| - k), \tag{1}$$

choose H minimizing $|H|$.

If $|H| = 2k$, then $\|H\| > tk \geqslant 2k^2 > \binom{|H|}{2}$ which is impossible. Hence $|H| > 2k$. Then $\delta(H) > t$ for otherwise removing a vertex of degree at most t gets a smaller subgraph satisfying (1). In particular, $|H| \geqslant t$ and hence $d(H) = \frac{2\|H\|}{|H|} > d(G) - 2k$ as desired.

It remains to show that H is $(k+1)$-connected. Assume otherwise that H has a proper seperation $\{U_1, U_2\}$ of order at most k. Put $H_i = G[U_i]$. Each $x \in U_1 - U_2$ has edges only into H_1 and hence $|H_1| > d(x) > t \geqslant 2k$. Similarly, $|H_2| \geqslant 2k$. By the minimality of $|H|$, no H_i can satisfy (1) and hence $\|H_i\| \leqslant t(|H_i| - k)$. But then $\|H\| \leqslant \|H_1\| + \|H_2\| \leqslant t(|H_1| + |H_2| - 2k) \leqslant t(|H| + k - 2k) \leqslant t(|H| - k)$ condtradicting (1). $\square$

2.1 Connected and 2-connected graphs

Lemma 3. *A graph is 2-connected iff it can be constructed from a cycle by successively adding H-paths to graphs H already constructed.*

Proof. Plainly, any graph constructed thusly is 2-connected. Let G be a 2-connected graph. Then G contains a cycle and hence contains a maximal subgraph H constructed as described. Since any edge $xy \in E(G) - E(H)$ with $x, y \in H$ would define an H-path, H must be an induced subgraph of G. Assume $H \neq G$. Then, since G is connected, there is an edge $xy \in E(G)$ where $x \in V(H)$ and $y \in V(G - H)$. Since G is 2-connected, there must be a shortest path P from y to H in $G - x$. But then xyP is an H-path and $H \cup xyP$ is constructible, contradicting the maximality of H. □

Definition 2.1. A *block* in a graph G is a maximal connected subgraph without a cutvertex.

Thus a block is either a maximal 2-connected subgraph, an edge with its ends, or an isolated vertex. By their maximality, any two different blocks of a graph overlap in at most one vertex (which must be a cutvertex of G). Hence every edge of G lies in a unique block and G is the union of its blocks.

Lemma 4. *Let G be a graph.*

1. *The cycles of G are precisely the cycles of its blocks.*

2. *The bonds of G are precisely the minimal cuts of its blocks.*

Proof. Proof of (1): Any cycle of G is a connected subgraph without a cutvertex and hence is contained in a maximal such subgraph – a block. Proof of (2): Let F be a bond in G. Let $xy \in F$. Then xy is in a unique block B. By the maximality condition on blocks, G contains no B-path. In particular, any xy-path in G is contained in B. Hence x and y are seperated by $F \cap E(B)$, but then by minimality of F, we must have $F = F \cap E(B)$ and hence $F \subseteq E(B)$. As we saw above, any cut seperating x and y in B also seperates them in G, hence F is a bond in B. □

Lemma 5. *The following statements are equivalent for distinct edges e, f of a graph G.*

1. *e and f belong to a common block of G;*

2. *e and f belong to a common cycle of G;*

3. *e and f belong to a common bond of G;*

Proof. (1) ⇒ (2): Say e and f are in a block B. Since G is 2-connected, there are 2 disjoint $e - f$ paths by Menger's theorem (or induction on the construction above). These together with e and f give a cycle containing them.

(2) ⇒ (3): Let C be a cycle containing e and f. Then $\{e, f\}$ cuts C into two connected sets A and B. Let A' be a maximal connected subset of $V(G)$ containing A and disjoint from B. Let B' be a maximal connected subset of $V(G)$ containing B and disjoint from A'. Then $E(A', B')$ is a bond in G containing e and f.

(3) ⇒ (1): This is immediate from the previous lemma. □

2.2 Structure of 3-connected graphs

Given an edge e in a graph G, we write $G \dot{-} e$ for the multigraph obtained from $G - e$ by suppressing any end of e that has degree 2 in $G - e$.

Lemma 6. *Let e be an edge in a graph G. If $G \dot{-} e$ is 3-connected, then so is G.*

Proof. □

Lemma 7. *Every 3 connected graph $G \neq K^4$ has an edge e such that $G \dot{-} e$ is a 3-connected graph.*

Proof. □

Theorem 8. *A graph G is 3-connected iff there exists a sequence $G_0, \ldots, G_n$ of graphs such that*

1. *$G_0 = K^4$ and $G_n = G$;*
2. *G_{i+1} has an edge e such that $G_i = G_{i+1} \dot{-} e$ for every $i < n$.*

Moreover, the graphs in any such sequence are all 3-connected.

Proof. If G is 3-connected, use Lemma 7 to get the graphs $G_n, \ldots, G_0$ in turn. The moreover is immediate from Lemma 6. □

Lemma 9. *Every 3 connected graph $G \neq K^4$ has an edge e such that G/e is again 3-connected.*

Proof. □

Theorem 10. *A graph G is 3-connected iff there exists a sequence $G_0, \ldots, G_n$ of graphs with the following two properties*

1. *$G_0 = K^4$ and $G_n = G$;*
2. *G_{i+1} has an edge xy such that $d(x), d(y) \geqslant 3$ and $G_i = G_{i+1}/xy$, for every $i < n$.*

Proof. □

Theorem 11. *The cycle space of a 3-connected graph is generated by its non-separating induced cycles.*

2.3 Menger's Theorem

Menger's Theorem. *For any graph G and $A, B \subseteq V(G)$, the minimum number of vertices separating A from B in G is the maximum number of disjoint $A - B$ paths in G.*

Proof. Assume not and choose a counterexample G minimizing $\|G\|$. Let k be the minimum number of vertices that separate A from B in G. If $\|G\| = 0$, then $|A \cap B| = k$ and there are k disjoint $A - B$ paths of length one in G. Thus G has an edge $e = xy$. Let v_e be the vertex resulting from contracting e to form G/e. Put $A' = (A - \{x, y\}) \cup \{v_e\}$ if $\{x, y\} \cap A \neq \varnothing$ and $A' = A$ otherwise. Similarly for B'. Since a collection of k disjoint $A' - B'$ paths in G/e would induce a collection of k disjoint $A - B$ paths in G, the minimality of $\|G\|$ gives an $A' - B'$ separator Y in G/e with $|Y| \leqslant k - 1$. Then $v_e \in Y$ since otherwise Y would be too small of an $A - B$ separator in G. Hence $X = (Y - \{v_e\}) \cup \{x, y\}$ is an $A - B$ separator in G with $|X| = k$.

Then, any $A - X$ separator or $X - B$ separator in G is also an $A - B$ seperator in G. Since $x, y \in X$, the same goes for any $A - X$ separator or $X - B$ separator in $G - e$. In particular, any $A - X$ seperator in $G - e$ must have at least k vertices and thus by minimality of $\|G\|$ there must be k disjoint $A - X$ paths in $G - e$. Similarly, there are k disjoint $X - B$ paths in $G - e$. Since X separates A from B, these two path systems cannot meet outside of X and thus can be combined into k disjoint $A - B$ paths. □

2.4 Linking

Definition 2.2. Let G be a graph and $X \subseteq V(G)$. We call X *linked* in G if whenever we pick different vertices $s_1, \dots, s_l, t_1, \dots, t_l \in X$ we can find disjoint paths $P_1, \dots, P_l$ in G such that each P_i links s_i to t_i and has no inner vertex in X.

Definition 2.3. If $|G| \geqslant 2k$ and every set X with $|X| \leqslant 2k$ is linked in G, then G is *k-linked*.

Lemma 12. *There is a function $h\colon \mathbb{N} \to \mathbb{N}$ such that every graph of average degree at least $h(r)$ contains K^r as a topological minor, for every $r \in \mathbb{N}$.*

Proof. □

Theorem 13 (Jung, Larman and Mani)**.** *There is a function $f\colon \mathbb{N} \to \mathbb{N}$ such that every $f(k)$-connected graph is k-linked, for all $k \in \mathbb{N}$.*

Proof. □

3 Trees

Lemma 14. *The following assertions are equivalent for a graph T:*

1. *T is a tree (a connected acyclic graph);*

2. *there is a unique path between any $x, y \in V(T)$;*

3. *T is connected and the removal of any edge of T disconnects T;*

4. *T is acyclic and the addition of any edge of $\overline{T}$ creates a cycle.*

Proof. (1) $\Rightarrow$ (2): As T is connected, there is at least one path. Assume some pair of vertice has more than one path and choose $x, y \in V(T)$ such that there are at least two different paths xPy and xQy minimizing $d(x, y)$. Then P and Q are internally disjoint by minimality of $d(x, y)$. But then $xPyQx$ is a cycle in T.

(2) $\Rightarrow$ (3): By assumption, T is connected. Assume there is some $xy \in E(T)$ such that $T - xy$ is connected and let xPy be a path in $T - xy$. Then xy and xPy are different paths from x to y contradicting (2).

(3) $\Rightarrow$ (4): If T had a cycle, then removing an edge xy on the cycle would leave a connected graph (any path using xy can be rerouted around the cycle) contradicting (3). If there was some edge xy that could be added that did not create a cycle, then $T + xy$ would be a tree and hence the removal of xy must disconnected it, but it doesn't.

(4) $\Rightarrow$ (1): T is acyclic by assumption. If T were disconnected, then we could add an edge between two of its components without creating a cycle contradicting (4). □

Lemma 15. *Every connected graph contains a spanning tree.*

Proof. Let G be a connected graph. Let T be a minimal spanning connected subgraph of G. Plainly, any edge in T with an endpoint of degree one in T is not on a cycle. Let $xy \in E(T)$ with $d(x) \geqslant d(y) \geqslant 2$. Then $T - xy$ still spans G, so must be disconnected. Hence xy is not on a cycle in T. Since no edge of T is on a cycle, T is acyclic and hence a tree. □

Lemma 16. *A connected graph with n vertices is a tree iff it has $n - 1$ edges.*

Proof. Assume not and let n be minimal such that the lemma fails. Let G be a graph with n vertices. Let $v_1, \ldots, v_n$ be the vertices of G in the order guaranteed by Lemma 2 and put $H = [v_1, \ldots, v_{n-1}]$. Then H is connected and thus by minimality is a tree iff it has $n - 2$ edges. Now G is a tree iff H is a tree and v_n has exactly one edge into H. Hence G is a tree iff G has $n - 1$ edges. □

Lemma 17. *Let G be a graph. If T is a tree with $|T| \leqslant \delta(G) + 1$, then $T \subseteq G$.*

Proof. Assume not and let T be a tree with $|T| \leqslant \delta(G)+1$ such that $T \nsubseteq G$ and $|T|$ minimal. Then $|T| \geqslant 2$ and hence T has a leaf x. By minimality, G contains the tree $T - x$. Let y be x's neighbor in T. We have $d_G(y) \geqslant |T| - 1$, but y has at most $|T| - 2$ neighbors in $T - x$ and thus y has an unused neighbor for x to use. Hence $T \subseteq G$. □

3.1 Normal Trees

Given a tree T with root r, the *tree-order* on T with respect to r is given by $x \leqslant y$ iff $x \in rTy$.

Normal Tree. Let G be a graph. A rooted tree T contained in G is called *normal* if the ends of every T-path in G are comparable in the tree order of T.

Lemma 18. *For any connected graph G, any $r \in V(G)$ and any path P in G starting at r, there is a normal spanning tree of G with root r containing P.*

Proof. Assume not and let G be a counterexample minimizing $|G|$. Let $r \in V(G)$ and P a path in G starting at r. Let M be a maximal path starting at r and containing P. Say the other end of M is x. Then, as M is maximal, x is not a cut vertex in G. Hence, by minimality of $|G|$, $G-x$ has a normal spanning tree T with root r containing $M-x$. Extend T along M to a spanning tree T' in G. To see that T' is normal, let $ab \in E(G)$. If both a and b are in T then they are comparable in T and hence T'. Otherwise, without loss of generality, $a = x$. But M is a maximal path, so x only has neighbors in M and thus $b < x = a$. Thus T' is a normal spanning tree in G with root r containing P. □

Corollary 19. *Every bridgeless connected graph has a strong orientation.*

Proof. Let G be a bridgeless connected graph. Pick $r \in V(G)$. By Lemma 18, G has a normal spanning tree T with root r. Let $xy \in E(G)$ with $x \leqslant y$ in the tree-order on T. If $xy \in E(T)$ orient xy towards y otherwise orient xy towards x. We claim this is a strong orientation. It will be enough to show that for any $z \in V(G)$ there is a directed path from r to z and a directed path from z to r. Since T spans G, for any $z \in V(G)$ there is a directed path from r to z on T. Assume there is some vertex which has no directed path to r and let z be such a vertex minimal in the tree-order. Then $z \neq r$ and hence we have $w \in V(T)$ such that $wz \in E(T)$. Let $U = \{x \in V(G) \mid z \leqslant x\}$ and $B = \{x \in V(G) \mid x < z\}$. Since $T - wz$ is disconnected and $G - wz$ is connected there must be some $ub \in E(G) - E(T)$ from U to $G - U$. Since T is normal, ub must be from U to B. But then composing the directed path in T from z to u, the directed edge ub and the directed path from b to r guaranteed by minimality of z gives us a directed path from z to r. This contradiction completes the proof. □

4 Bipartite Coloring and Matching

Lemma 20. *A graph is bipartite iff it contains no odd cycle.*

Proof. The forward direction is plain. For the reverse, assume the lemma is false and let G be a graph containing no odd cycle which is not bipartite minimizing $|G|$. Plainly G is 3-critical. Pick $x \in V(G)$ and let $\{\{x\}, A, B\}$ be a proper coloring of G. For $z \in N(x) \cap A$, let C_z be the component of z in $G[A \cup B]$.

First, assume there is $z \in N(x) \cap A$ such that there exists $y \in N(x) \cap B \cap C_z$. Let P be a path from z to y in C_z. Then $|P|$ is even as P alternates between A and B. Thus xPx is an odd cycle in G giving a contradiction.

Put $C = \bigcup_{z \in N(x) \cap A} C_z$. Then $N(x) \cap B \cap C = \varnothing$. Move $A \cap C$ into B and $B \cap C$ into A to get a new 3-coloring $\{\{x\}, A', B'\}$ of G. Then, we have moved all of $N(x) \cap A$ into B and moved none of $N(x) \cap B$. Hence x has no neighbors in A' and we have the bipartition $\{A' \cup \{x\}, B'\}$ of G giving a contradiction. □

Lemma 21 (Lovász, proof by Gasparian)**.** *A graph G is perfect iff $\alpha(H)\omega(H) \geqslant |H|$ for each $H \trianglelefteq G$. In particular, a graph is perfect iff its complement is perfect.*

Proof. For the forward direction, just note that if G is perfect, then $\alpha(H)\omega(H) = \alpha(H)\chi(H) \geqslant |H|$ for every $H \trianglelefteq G$.

Assume the reverse direction is false and let G be a counterexample with the minimum number of vertices. Then, by minimality, every proper induced subgraph of G is perfect and $\omega(G) < \chi(G)$. Thus, for each independent $I \subseteq V(G)$, we must have $\chi(G-I) = \omega(G-I) = \omega(G) = \chi(G) - 1$.

Put $n = |G|$, $\alpha = \alpha(G)$ and $\omega = \omega(G)$. Let $A_0 = \{v_1, \dots, v_\alpha\}$ be a maximum independent set in G. For $1 \leqslant i \leqslant \alpha$, let $\left\{A_{(i-1)\omega+1}, \dots, A_{i\omega}\right\}$ be a proper ω-coloring of $G - v_i$.

Let K be an ω-clique in G. We claim that $A_i \cap K = \varnothing$ for at most one $0 \leqslant i \leqslant \alpha\omega$. To prove the claim, first assume $A_0 \cap K = \varnothing$. Then $K \subseteq V(G - v)$ for each $v \in A_0$. Hence, for each $v \in A_0$, K intersects every color class in any ω-coloring of $G - v$ and in particular, $A_t \cap K \neq \varnothing$ for all $t \geqslant 1$. Now assume $A_0 \cap K = \{w\}$. For each $v \in A_0 - \{w\}$, K intersects every color class in any ω-coloring of $G - v$. Also, K intersects all but one color class in any ω-coloring of $G - w$. In particular, K intersects all but one A_t for $0 \leqslant t \leqslant \alpha\omega$. This proves the claim.

Since $\omega(G - A_i) = \omega$, we have an ω-clique K_i in $G - A_i$ for each $0 \leqslant i \leqslant \alpha\omega$. We know that $|A_i \cap K_j| \leqslant 1$ for each i, j, since A_i is independent and K_j is complete. Since $A_i \cap K_i = \varnothing$, by the above claim, we have $|A_i \cap K_j| = \delta_{ij}$.

Let A be the $(\alpha\omega + 1) \times n$ matrix whose i-th row is the incidence vector of A_i. Let B be the $n \times (\alpha\omega+1)$ matrix whose i-th column is the incidence vector of K_i. Put $X = AB$. Then, since $|A_i \cap K_j| = \delta_{ij}$, we see that X is the $(\alpha\omega+1) \times (\alpha\omega+1)$ matrix with $X_{ij} = \delta_{ij}$. Plainly, $\det X \neq 0$ and hence $X \colon \mathbb{R}^{\alpha\omega+1} \to \mathbb{R}^{\alpha\omega+1}$ is injective. Thus, $B \colon \mathbb{R}^{\alpha\omega+1} \to \mathbb{R}^n$ is injective and in particular, $n \geqslant \alpha\omega + 1$. But this contradicts our assumption that $\alpha\omega \geqslant n$. □

Lemma 22. *The complement of any bipartite graph is perfect.*

Proof. Plainly bipartite graphs are perfect. Hence so are their complements by Lemma 21. □

The König-Egerváry Theorem. *Every bipartite graph satisfies $\tau = \nu$.*

Proof. Let G be a bipartite graph. By Lemma 22, we have

$$|G| - \tau(G) = \alpha(G) = \omega(\overline{G}) = \chi(\overline{G}) = |G| - \nu(G).$$

Hence $\tau(G) = \nu(G)$. □

Hall's Theorem. *A bipartite graph with parts A and B has a matching of A into B iff $|N(X)| \geqslant |X|$ for all $X \subseteq A$.*

Proof. Let G be a bipartite graph with parts A and B. The reverse implication is plain. For the forward implication, assume $|N(X)| \geqslant |X|$ for all $X \subseteq A$. Then, in $\overline{G}$, each $X \subseteq A$ is joined to at most $|B| - |X|$ vertices in B. Hence $\omega(\overline{G}) \leqslant |B|$. By Lemma 22, we have $|G| - \nu(G) = \chi(\overline{G}) = \omega(\overline{G}) \leqslant |B|$. Hence $\nu(G) \geqslant |A|$. Since A and B are independent, any matching of size $|A|$ is a matching of A into B. This completes the proof. □

König's Theorem. *Every bipartite graph satisfies $\chi' = \Delta$.*

Proof. Let H be the line graph of a bipartite graph G. By Lemma 22, $\overline{G}$ is perfect. Hence $\omega(\overline{H}) = \alpha(H) = \nu(G) = |G| - \chi(\overline{G}) = |G| - \omega(\overline{G}) = \tau(G) = \chi(\overline{H})$.

Since removing edges from a bipartite graph leaves a bipartite graph, being the complement of the line graph of a bipartite graph is a hereditary property. Hence the above shows that the complement of the line graph of a bipartite graph is perfect.

Now, let G be a bipartite graph. Then $\overline{L(G)}$ is perfect and hence $L(G)$ is perfect by Lemma 21. But G has no triangles, so $\Delta(G) = \omega(L(G)) = \chi(L(G)) = \chi'(G)$. □

Dilworth's Theorem. *In any poset $(P, <)$, the maximum size of an antichain in P equals the minimum size of a chain parition of P.*

Proof. Let $(P, <)$ be an arbitrary poset. For $i \geqslant 1$, let A_i be the $x \in P$ such that the longest chain in P ending in x has i elements. Then, each A_i is an antichain since if $y, z \in A_i$ with $y < z$, then the union of $\{z\}$ with any length i chain ending in y gives the contradiction $z \notin A_i$. Let $c(P)$ be the length of the longest chain in P. Then $A_i = \varnothing$ for $i > c(P)$. Hence $A_1, \ldots, A_{c(P)}$ is a partition of P into $c(P)$ antichains.

Let G_P be the graph with $V(G_P) = P$ and $E(G_P) = \{uv \mid u < v \text{ or } v < u\}$. Then, from the above, $\chi(G_P) \leqslant c(P) = \omega(G_P)$ and hence $\chi(G_P) = \omega(G_P)$. Call such a G_P a *comparability graph.*

Since the class of comparability graphs is plainly hereditary, the above proves that they are perfect. Hence by Lemma 21, so are their complements. Thus, if $(P, <)$ is a poset, we have $\chi(\overline{G_P}) = \omega(\overline{G_P}) = \alpha(G_P)$. But a clique in G_P is precisely a chain in P and an independent set is precisely an antichain in P. Hence, the maximum size of an antichain in P equals the minimum number of chains into which P can be partitioned. □

4.1 Stable Matchings

Definition 4.1. Let G be a graph. A *set of preferences* for G is a collection of total orderings $\{\leqslant_v\}_{v \in V(G)}$ where $\leqslant_v$ orders $E(v)$. A matching M in G is called *stable* if for every $e \in E(G) - M$, there exists $f \in M$ such that e and f have a common vertex v with $e <_v f$.

Stable Matching Lemma. *Let G be a bipartite graph. For every set of preferences, G has a stable matching.*

Proof. Say G has parts A and B. Let $\{\leqslant_v\}_{v \in V(G)}$ be a set of preferences on G. Define a partial order $\leqslant$ on the matchings of G by $M' \leqslant M$ iff for every $b \in B$ and $ab \in M'$, there exists $cb \in M$ such that $ab \leqslant_b cb$.

Given a matching M, call a vertex $a \in A$ *acceptable* to $b \in B$ if $ab \in E(G) - M$ and any $cb \in M$ satisfies $cb <_b ab$. Call $a \in A$ *happy with* M if either a is unmatched in M or its matching edge $f \in M$ satisfies $f >_a e$ for all edges $e = ab$ such that a is acceptable to b. If every vertex in A is happy with M, call M *happy.*

Note that the empty matching is happy. Thus, since G is finite, we may choose a maximal (under $\leqslant$) happy matching M. Assume there is some $a \in A$ which is unmatched in M but acceptable to some vertex in B. Choose $b \in B$ such that a is acceptable to b maximizing ab under $\leqslant_a$. Remove any edge incident to b in M and then add ab to get a new matching M'. By our choice of ab, M' is happy. Clearly, $M \leqslant M'$ and since $M \neq M'$ we have $M < M'$ contradicting the maximality of M.

Hence, in M, every unmatched $a \in A$ is unacceptable to all $b \in B$ and every $a \in A$ is happy with M. Let $xy \in E(G) - M$ with $x \in A$. First assume x is unmatched in M. Then x is unacceptable to y and hence there is $zy \in M$ such that $xy <_y zy$. Otherwise, we have $xw \in M$ and $xw >_x xt$ for all t such that x is acceptable to t. Hence, either $xt <_x xw$ or x is unacceptable to t and so we have $ft \in M$ such that $xt <_t ft$. Thus M is stable. $\square$

4.2 Applications

Corollary 23. *For every $k \geqslant 1$, every k-regular bipartite graph has a perfect matching.*

Proof. Let G be a k-regular bipartite graph with parts A and B. For, $X \subseteq A$ we have $k|X| = |E(X, N(X))| \leqslant k|N(X)|$ and thus $|X| \leqslant |N(X)|$. By Lemma 4, there exists a matching of A into B. By symmetry we also have a matching of B into A, thus $|A| = |B|$ and either of these matchings will do for the desired perfect matching. $\square$

Corollary 24 (Peterson 1891)**.** *Every regular graph of positive even degree has a 2-factor (a 2-regular spanning subgraph).*

Proof. Let $k \geqslant 1$ and let G be a $2k$-regular graph. Without loss of generality, assume G is connected. By Lemma 1, G has an Euler tour $v_0e_0 \cdots e_{r-1}v_r$ with $v_r = v_0$. Create a graph H by replacing every $v \in V(G)$ by a pair (v^-, v^+) and each edge v_iv_{i+1} by the edge $v_i^+v_{i+1}^-$. Then H is bipartite and k-regular and hence by Corollary 23 has a perfect matching. Now collapsing each pair (v_i^-, v_i^+) back into v_i gives a 2-factor in G. $\square$

5 General Matching

Given a graph G, let $q(G)$ be the number of odd components of G. A graph G is called *factor-critical* if $G \neq \varnothing$ and $G - v$ has a perfect matching for each $v \in V(G)$. For $S \subseteq V(G)$, put $\operatorname{def}(S) = q(G - S) - |S|$. Let $\operatorname{def}(G) = \max_{S \subseteq V(G)} \operatorname{def}(S)$.

Lemma 25. *For every graph G and any maximal $S \subseteq V(G)$ maximizing def(S) we have:*

1. *Each component of $G - S$ is factor-critical.*
2. *S is matchable into the components of $G - S$; in particular, $|S| \leqslant q(G - S)$;*
3. *G has a perfect matching iff $q(G - S) = |S|$.*

Proof. Assume not and choose a counterexample minimizing $|G|$. Let $S \subseteq V(G)$ be a maximal set maximizing $\operatorname{def}(S)$.

Let C be a component of $G-S$. First assume $|C|$ is even. Pick $c \in C$ and put $T = S \cup \{c\}$. Then some component of $C - c$ is odd and hence $q(G - T) \geqslant q(G - S) + 1$, but then $\mathrm{def}(T) \geqslant \mathrm{def}(S)$ contradicting the maximality of S. Hence $|C|$ is odd.

Now assume C is not factor critical. Then we have $c \in V(C)$ such that $C' = C - c$ has no perfect matching. By the minimality of $|G|$ we have $S' \subseteq V(C')$ satisfying the statement of the lemma. Since C' has no perfect matching, $C' - S'$ has more than $|S'|$ components all of which are factor-critical and hence odd. Thus $q(C' - S') > |S'|$. Now $|C|$ is odd, so $|C'|$ is even. Hence $q(C' - S')$ and $|S'|$ have the same parity. In particular, we must have $q(C' - S') \geqslant |S'| + 2$. Put $T = S \cup \{c\} \cup S'$. Then $q(G - T) = q(G - S) - 1 + q(C' - S')$ giving $\mathrm{def}(T) = q(G - S) - |S| + q(C' - S') - |S'| - 2 \geqslant \mathrm{def}(S) + 2 - 2 = \mathrm{def}(S)$. But this contradicts the maximality of S. Hence C is factor critical. This completes the proof of (1).

To prove (2), assume otherwise that S is not matchable into the components of $G - S$. Then, by Hall's theorem, there is $B \subseteq S$ which has edges into fewer than $|B|$ components of $G - S$. Put $T = S - B$. Then $\mathrm{def}(T) = q(G - T) - |T| = q(G - T) - |S| + |B| > q(G - S) - |B| - |S| + |B| = \mathrm{def}(S)$ where the penultimate inequality follows since B connects up fewer than $|B|$ components. This contradicts the maximality of $\mathrm{def}(S)$.

To prove (3), first assume that G has a perfect matching. By (2),$|S| \leqslant q(G - S)$. Also plainly to have a perfect matching we must have $q(G - S) \leqslant |S|$. For the reverse, assume $q(G - S) = |S|$. Now (1) and (2) together easily give us a perfect matching in G. □

Tutte-Berge Matching Formula. *A maximum matching in a graph G has size $\frac{1}{2}(|G| - \mathit{def}(G))$.*

Proof. Let $S \subseteq V(G)$ be a maximal set maximizing $\mathrm{def}(S)$. By Lemma 25 we have a matching from S into the components of $G - S$. Let $C_1, \ldots, C_{|S|}$ be the components in this matching and $C_{|S|+1}, \ldots, C_k$ be the other components. Then as each C_i is factor critical, we have a matching in G of size $\frac{1}{2}(|G| - (k - |S|)) = \frac{1}{2}(|G| - \mathrm{def}(G))$ since each C_i is odd. Hence any maximum matching in G has at least $\frac{1}{2}(|G| - \mathrm{def}(G))$ edges.

Now let M be a maximum matching in G. For $T \subseteq V(G)$, let $M_T \subseteq M$ be the edges with at least one end in T. Then $|M| \leqslant |M_T| + |M - M_T| \leqslant |T| + \frac{1}{2}(|G| - |T| - q(G - T)) = \frac{1}{2}(|G| - \mathrm{def}(T))$. Hence $|M| \leqslant \frac{1}{2}(|G| - \mathrm{def}(G))$. □

An immediate consequence is Tutte's matching theorem.

Tutte's Matching Theorem. *A graph G has a perfect matching iff $q(G - S) \leqslant |S|$ for every $S \subseteq V(G)$.*

Gallai-Edmonds Decomposition. *Let G be a graph and let $D \subseteq V(G)$ be the vertices which are missed by some maximum matching of G. Let A be the vertices of $G - D$ which are adjacent to at least one vertex of D. Finally, let $C = V(G - A - D)$. The following statements hold.*

1. *the components of $G[D]$ are factor-critical;*

2. *$G[C]$ has a perfect matching;*

3. *the bipartite graph obtained from $G-C$ by removing the edges of $G[A]$ and contracting each component of $G[D]$ to a single vertex has positive surplus (as viewed from A);*

4. *if M is a maximum matching in G, it contains a near-perfect matching of each component of $G[D]$, a perfect matching of each component of $G[C]$ and matches all vertices of A with vertices in distinct components of $G[D]$;*

5. $\nu(G) = \frac{1}{2}\left(|G| - c(G[D]) + |A|\right)$.

Proof. Let $T \subseteq V(G)$ be a maximal set maximizing $\mathrm{def}(T)$. By Lemma 25, we have a matching from T into the components of $G-T$. Hence every $S \subseteq T$ must have neighbors in at least $|S|$ components of $G-T$. Since $\varnothing \subseteq T$ has neighbors in zero components of $G-T$, we can choose a maximal $R \subseteq T$ such that R has neighbors in exactly $|R|$ components of $G-T$. Let R' be the vertices in the components of $G-T$ in which R has a neighbor.

Let M be a maximum matching in G. Since each component of $G-T$ is factor critical, M must contain a near perfect matching in each component of $G-T$. But then since M is maximum, the rest of the edges of M must be a matching of T into the components of $G-T$. In particular, the vertices of T are in every maximum matching. Since R has neighbors in only $|R|$ components of $G-T$, R must be matched with these components in every maximum matching. Hence the vertices of $R' \cup T$ are in every maximum matching.

Let $D' = V(G-T-R')$. If $R = T$, then $D' = D = \varnothing$. So, let's assume that $R \neq T$. By maximality of R, each $\varnothing \neq S \subseteq T - R$ must have neighbors in at least $|S|+1$ components of $G[D']$. Thus, by Hall's theorem, we have a matching of $T-R$ into any set of all but one component of $G[D']$. In particular, each vertex of D' is missed by some maximum matching. Since the vertices of $R' \cup T$ are in every maximum matching we conclude that $D' = D$. Also, $T-R$ is precisely the set of vertices not in D that have an edge into D; that is, $A = T-R$. This leaves $C = R \cup R'$.

With these facts the proof is easy. By Lemma 25, the components of $G[D] = G-T-R'$ are factor-critical. As we saw above, every maximum matching of G induces a perfect matching of $G[C] = G[R \cup R']$. This proves (1) and (2). Now (3) follows from maximality of R as above. Finally, (4) and (5) are immediate. □

5.1 Applications

Corollary 26 (Peterson 1891)**.** *Every bridgeless cubic graph has a perfect matching.*

Proof. Let G be a bridgeless cubic graph. Let $S \subseteq V(G)$ and C an odd component of $G-S$. Then as $\sum_{v \in V(C)} d_C(v) = 2\,\|C\|$ is even and $\sum_{v \in V(C)} d_G(v) = 3\,|C|$ is odd, there must be an odd number of edges from C to S. Since G has no bridge, there must be at least 3 edges from C to S. Hence there are at least $3q(G-S)$ edges from S to $G-S$. But also, there are at most $3\,|S|$ such edges. Hence $q(G-S) \leqslant |S|$. Thus G has a perfect matching by Tutte's theorem. □

5.2 Augmenting Paths

Augmenting Path. Given a matching M, an *M-alternating path* is a path that alternates between edges in M and edges not in M. An M-alternating path whose endpoints are not indicent with M is called an *M-augmenting path*.

Berge's Theorem. *A matching M in a graph G is a maximum matching in G iff G has no M-augmenting path.*

Proof. The forward direction is plain since replacing the M-edges in an M-augmenting path with the non-M-edges yields a larger matching.

For the reverse direction we prove the contrapositive. Assume M' and M are matchings in a graph G with $|M'| > |M|$. We will construct an M-augmenting path in G. Consider the symmetric difference $F = M \Delta M'$. Then we have $\Delta(F) \leqslant 2$ and hence F is a disjoint union of paths and cycles. Moreover, the edges of any cycle in F must alternate between M and M' and thus the number from M equals the number from M'. But $|M'| > |M|$, so some component of F must have more edges from M' than M. The only possibility for such a component is a path that both starts and ends with an edge from M'. But such a path must be M-augmenting. □

5.3 Packing and Covering

Skipped Erdos-Posa for now.

Lemma 27. *Let G be a multigraph. Let $\{F_1, \ldots, F_k\}$ be a set of edge-disjoint spanning forests in G maximizing $|E(F_1 \cup \cdots \cup F_k)|$. Then for every edge $xy \in E(G) - E(F_1 \cup \cdots \cup F_k)$ there exists $U \subseteq V(G)$ such that $x, y \in U$ and $F_i[U]$ is connected for each $i \in [k]$.*

Proof. Long. Assume we will be given this as an assumption like before. □

Nash-Williams and Tutte Theorem. *A multigraph contains k edge-disjoint spanning trees iff for every partition P of its vertex set it has at least $k(|P| - 1)$ cross-edges.*

Proof. The forward implication is plain since collapsing each part to a vertex we get a connected graph with $|P|$ vertices and hence each spanning tree must have at least $|P| - 1$ cross-edges.

For the forward direction, let G be a counterexample minimizing $|G|$. Let $\{F_1, \ldots, F_k\}$ be a set of edge-disjoint spanning forests in G maximizing $|E(F_1 \cup \cdots \cup F_k)|$. If all the F_i are trees, we are done, so assume some F_i is not a tree. Then

$$\sum_{i \in [k]} \|F_i\| < k(|G| - 1).$$

The parition of G into singletons together with our assumption shows that $\|G\| \geqslant k(|G| - 1)$. Hence there exists an edge $xy \in E(G) - E(F_1 \cup \cdots \cup F_k)$. By Lemma 27, we have $U \subseteq V(G)$ such that $x, y \in U$ and $F_i[U]$ is connected for each $i \in [k]$.

Let $H = G/U$; that is, the graph formed from G by collapsing U to a single vertex v_U. Since v_U is in a single part in any partition P of H, P has the same number of cross edges as the partition of G formed by expanding v_U back to U. In particular, P has at least $k(|P| - 1)$ cross edges. Since $x, y \in U$, $|H| < |G|$ and hence by the minimality of $|G|$, H has k edge-disjoint spanning trees $T_1, \ldots, T_k$. Now $F_i[U]$ is connected for each i and hence is a spanning tree of U. Thus replacing v_U in T_i with $F_i[U]$ gives k edge-disjoint spanning trees in G. □

Corollary 28. *A $2k$-edge-connected multigraph contains k edge-disjoint spanning trees.*

Proof. Let G be a $2k$-edge-connected multigraph. Let P be a partition of $V(G)$. Since G is $2k$-edge-connected, there must be at least $2k$ edges from any part to the rest of the graph. Adding these up for each part, we count each edge twice and thus there are at least $\frac{1}{2}(2k\,|P|) = k\,|P| \geqslant k(|P| - 1)$ cross-edges. Hence G has k edge-disjoint spanning trees by the Nash-Williams Tutte Theorem. □

Nash-Williams Theorem. *A multigraph G can be partitioned into at most k forests iff $\|G[U]\| \leqslant k(|U| - 1)$ for each $\varnothing \neq U \subseteq V(G)$.*

Proof. For the forward implication, just note that a forest on $|U|$ vertices has at most $|U| - 1$ edges.

Let $\{F_1, \ldots, F_k\}$ be a set of edge-disjoint spanning forests in G maximizing $|E(F_1 \cup \cdots \cup F_k)|$. If the F_i don't partition G, then pick some $xy \in E(G) - E(F_1 \cup \cdots \cup F_k)$. By Lemma 27, we have $U \subseteq V(G)$ such that $x, y \in U$ and $F_i[U]$ is connected for each $i \in [k]$. In particular, $\|F_i[U]\| \geqslant |U| - 1$ for each $i \in [k]$. But xy is an edge in U as well, so $\|G[U]\| > k(|U| - 1)$ contradicting our assumption. □

6 General Coloring

6.1 Vertex Coloring

Theorem 29. *For every graph G we have $\chi(G) \leqslant \frac{1}{2} + \sqrt{2\,\|G\| + \frac{1}{4}}$.*

Proof. In any $\chi(G)$ coloring of G, there must be at least one edge between any two color classes and thus $\|G\| \geqslant \binom{\chi(G)}{2}$. Solving for $\chi(G)$ proves the theorem. □

Lemma 30. *For every graph G we have $\chi(G) \leqslant col(G) = \max_{H \subseteq G} \delta(H) + 1$.*

Proof. Let G be a graph and F a $\chi(G)$-critical subgraph of G. Then $\delta(F) \geqslant \chi(G) - 1$ and hence $\chi(G) \leqslant \delta(F) + 1 \leqslant \max_{H \subseteq G} \delta(H) + 1$. □

Lemma 31. *Let G be a non-complete 2-connected graph with $\delta(G) \geqslant 3$. Then G contains an induced P_3, say abc, such that $G - a - c$ is connected.*

Proof. Since G is connected and not complete, it contains induced P_3's. If G is 3-connected, any induced P_3 will do. Otherwise, let $\{b, x\} \subseteq V(G)$ be a cutset. Since $G - b$ is not 2-connected, it has at least two endblocks B_1, B_2. But G is 2-connected, so b must be adjacent to non cut vertices $a \in B_1$ and $c \in B_2$. Thus $G - a - c$ is connected since $d(b) \geqslant 3$. Whence abc is our desired P_3. □

Brooks' Theorem. *Every graph with* $\Delta \geqslant 3$ *satisfies* $\chi \leqslant \max\{\omega, \Delta\}$.

Proof. Assume not and let G be a counterexample minimizing $|G|$. Plainly, G must be regular, 2-connected and not complete. Let abc be the induced P_3 guaranteed by Lemma 31. By Lemma 2, we have an ordering $b, x_1, x_2, \ldots, x_k$ of $V(G-a-c)$ such that $G[b, x_1, \ldots, x_i]$ is connected for each $1 \leqslant i \leqslant k$. Thus, greedily coloring with $V(G)$ ordered $a, c, x_k, x_{k-1}, \ldots, x_1, b$ uses only $\Delta(G)$ colors. □

Definition 6.1. For each $k \in \mathbb{N}$, define the class of *k-constructible* graphs as follows:

1. K^k is k-constructible.
2. If G is k-constructible and $xy \in E(\overline{G})$, then $(G + xy)/xy$ is k-constructible.
3. If G_1, G_2 are k-constructible and there are vertices x, y_1, y_2 such that $G_1 \cap G_2 = \{x\}$ and $xy_1 \in E(G_1)$ and $xy_2 \in E(G_2)$, then also $(G_1 \cup G_2) - xy_1 - xy_2 + y_1y_2$ is k-constructible.

Theorem 32 (Hajós 1961)**.** *Let G be a graph and $k \in \mathbb{N}$. Then $\chi(G) \geqslant k$ iff G has a k-constructible subgraph.*

Proof. We first prove that any k-constructible graph (and hence any supergraph) satisfies $\chi \geqslant k$. Operation (2) cannot decrease the chromatic number of G since any coloring of $(G + xy)/xy$ gives a coloring of G where x and y are colored the same. If a graph resulting from operation (3) had a $(k-1)$-coloring π, then $\pi(y_1) \neq \pi(y_2)$ and hence without loss of generality $\pi(x) \neq \pi(y_1)$. But then π is a proper $(k-1)$-coloring of G_1 which is impossible by induction.

Now, assume the other direction is false and choose a graph G with $\chi(G) \geqslant k$ having no k-constructible subgraph first minimizing $|G|$ and then maximizing $\|G\|$.

Note that G cannot be a complete multipartite graph since then it would contain K^k. Hence $\overline{G}$ contains an induced P_3, say y_1xy_2. By maximality of $\|G\|$, for $i = 1, 2$ the edge xy_i lies in a k-constructible subgraph H_i of $G + xy_i$.

Let H_2' be a copy of H_2 such that $V(H_2') \cap V(G) = \{x\} \cup V(H_2 - H_1)$ and there is an isomorphism $\phi\colon V(H_2) \to V(H_2')$ with $\phi(z) = z$ for $z \in V(H_2) \cap V(H_2')$. Then $H_1 \cap H_2' = \{x\}$ and hence using operation (3) we see that $H = (H_1 \cup H_2') - xy_1 - xy_2' + y_1y_2'$ is k-constructible. Identifying in H each vertex in $H_2' - G$ with the vertex it is a copy of in H_2 is a sequence of applications of operation (2) and hence $(H_1 \cup H_2) - xy_1 - xy_2 + y_1y_2$ is the desired k-constructible subgraph of G.

□

6.2 Edge Coloring

For a k-edge-coloring π of a graph G, let $\pi(x) = \{\pi(xy) \mid xy \in E(G)\}$ for $x \in V(G)$ and for $i \in [k]$ put $\pi_i = \{x \in N(v) \mid i \notin \pi(x)\}$.

Lemma 33. *If G is a simple graph and there exists $k \in \mathbb{N}$ and $v \in V(G)$ such that each of the following hold:*

1. *$\chi'(G - v) \leqslant k$;*
2. *$d(v) \leqslant k$;*
3. *$d(x) \leqslant k$ for all $x \in N(v)$;*
4. *$d(x) = k$ for at most one $x \in N(v)$.*

Then $\chi'(G) \leqslant k$.

Proof. Assume not and choose a counterexample G, vertex $v \in V(G)$ and $k \in \mathbb{N}$ minimizing k. Then v satisfies each of (1), (2), (3) and (4). By adding dummy pendant edges to v and its neighbors if necessary, we may assume that $d(v) = k$, $d(x) = k$ for exactly one $x \in N(v)$ and $d(y) = k - 1$ for $y \in N(v) - \{x\}$.

Choose a k-edge-coloring π of $G - v$ minimizing $\sum_{i \in [k]} |\pi_i|^2$. First, assume $|\pi_i| \neq 1$ for all $i \in [k]$. Then, we have $\sum_{i \in [k]} |\pi_i| = |\{(i, x) \in [k] \times N(v) \mid i \notin \pi(x)\}| = \sum_{x \in N(v)} (k - d_{G-v}(x)) = 2d(v) - 1 < 2k$. Hence there exists $a \in [k]$ such that $|\pi_a| = 0$. Also, since $2d(v) - 1$ is odd, there must be $b \in [k]$ such that $|\pi_b|$ is odd and hence at least 3. Pick $z \in \pi_b$ and consider a maximum length path zPw with edges alternating between color a and color b starting at z. Exchange colors a and b on P to get a new k-edge-coloring π' of $G - v$. Note that for any internal vertex x of P we have $\pi'(x) = \pi(x)$. Since every vertex in $N(v)$ is incident with color a, if $w \in N(v)$, then by maximality of P, the last edge of P must be colored a. Hence, in any case, $|\pi'_a|^2 + |\pi'_b|^2 < |\pi_a|^2 + |\pi_b|^2$ contradicting our minimality assumption on π.

Hence, we may assume $\pi_i = \{z\}$ for some $z \in N(v)$ and $i \in [k]$. Make a graph H by removing vz as well as all $e \in E(G)$ with $\pi(e) = i$ from G. Then $H - v$ is $(k-1)$-edge-colored and we have removed exactly one neighbor from v and each of its neighbors. Hence, by minimality of k, we must have $\chi'(H) \leqslant k - 1$. But then adding back in the edges we removed all colored with the same new color gives a k-edge-coloring of G. This final contradiction completes the proof. □

Vizing's Simple Theorem. *Every simple graph satisfies $\Delta \leqslant \chi' \leqslant \Delta + 1$.*

Proof. Let G be a simple graph. Plainly, $\chi'(G) \geqslant \Delta(G)$. Applying Lemma 33 inductively with $k = \Delta(G) + 1$ proves that $\chi'(G) \leqslant \Delta(G) + 1$. □

Lemma 34. *If G is a multigraph and there exists $k \in \mathbb{N}$ and $v \in V(G)$ such that each of the following hold:*

1. *$\chi'(G - v) \leqslant k$;*

2. $d(v) \leqslant k$;

3. $d(x) + \mu(vx) \leqslant k + 1$ *for all* $x \in N(v)$;

4. $d(x) + \mu(vx) = k + 1$ *for at most one* $x \in N(v)$.

Then $\chi'(G) \leqslant k$.

Proof. Assume not and choose a counterexample G, vertex $v \in V(G)$ and $k \in \mathbb{N}$ minimizing k. Then v satisfies each of (1), (2), (3) and (4). By adding dummy pendant edges to v and its neighbors if necessary, we may assume that $d(v) = k$, $d(x) + \mu(vx) = k + 1$ for exactly one $x \in N(v)$ and $d(y) + \mu(vy) = k$ for $y \in N(v) - \{x\}$.

Choose a k-edge-coloring π of $G - v$ minimizing $\sum_{i \in [k]} |\pi_i|^2$. First, assume $|\pi_i| \neq 1$ for all $i \in [k]$. Then, we have $\sum_{i \in [k]} |\pi_i| = |\{(i, x) \in [k] \times N(v) \mid i \notin \pi(x)\}| = \sum_{x \in N(v)} (k - d_{G-v}(x)) = -1 + \sum_{x \in N(v)} 2\mu(vx) = 2d(v) - 1 < 2k$. Hence there exists $a \in [k]$ such that $|\pi_a| = 0$. Also, since $2d(v) - 1$ is odd, there must be $b \in [k]$ such that $|\pi_b|$ is odd and hence at least 3. Pick $z \in \pi_b$ and consider a maximum length path zPw with edges alternating between color a and color b starting at z. Exchange colors a and b on P to get a new k-edge-coloring π' of $G - v$. Note that for any internal vertex x of P we have $\pi'(x) = \pi(x)$. Since every vertex in $N(v)$ is incident with color a, if $w \in N(v)$, then by maximality of P, the last edge of P must be colored a. Hence, in any case, $|\pi'_a|^2 + |\pi'_b|^2 < |\pi_a|^2 + |\pi_b|^2$ contradicting our minimality assumption on π.

Hence, we may assume $\pi_i = \{z\}$ for some $z \in N(v)$ and $i \in [k]$. Make a multigraph H by removing one edge between v and z as well as all $e \in E(G)$ with $\pi(e) = i$ from G. Then $H - v$ is $(k-1)$-edge-colored and we have removed exactly one neighbor from v and each of its neighbors. Hence, by minimality of k, we must have $\chi'(H) \leqslant k - 1$. But then adding back in the edges we removed all colored with the same new color gives a k-edge-coloring of G. This final contradiction completes the proof. □

Vizing's Theorem. *Every multigraph satisfies* $\Delta \leqslant \chi' \leqslant \Delta + \mu$.

Proof. Let G be a multigraph. Plainly, $\chi'(G) \geqslant \Delta(G)$. Applying Lemma 34 inductively with $k = \Delta(G) + \mu(G)$ proves that $\chi'(G) \leqslant \Delta(G) + \mu(G)$. □

6.3 List Coloring

Lemma 35. *Let G be a plane graph with $|G| \geqslant 3$. Suppose that every inner face of G is bounded by a triangle and its outer face by a cycle $C = v_1 \dots v_k v_1$. Let L be a list assignment on $V(G)$ such that $|L(v_1)| = |L(v_2)| = 1$, $L(v_1) \neq L(v_2)$, $|L(x)| \geqslant 3$ for each $x \in V(C - v_1 - v_2)$, and finally $|L(x)| \geqslant 5$ for each $x \in V(G - C)$. Then G can be colored from the L.*

Proof. Assume not and let G be a counterexample minimizing $|G|$. If $|G| = 3$, then G is a triangle and the result follows. Hence $|G| \geqslant 4$.

First assume C has a chord vw. Then $C + vw$ breaks into two cycles C_1 and C_2 with v_1v_2 in exactly one of them. Without loss of generality, assume $v_1v_2 \in E(C_1)$. For $i = 1, 3$, let G_i be the subgraph of $G + vw$ induced on the vertices on and inside C_i. Then, by minimality of $|G|$, we can color G_1 from its lists. Since $vw \in E(G_1)$, v and w get different colors in this coloring, say c_v and c_w respectively. Define a list assignment L' on G_2 by setting $L'(v) = \{c_v\}$, $L'(w) = \{c_w\}$ and $L'(x) = L(x)$ for each $x \in V(G_2 - v - w)$. Then again by minimality of $|G|$, we can color G_2 from L'. But these colorings together give a coloring of G from the L, contradiction.

Thus we may assume that C has no chord. Let $v_1, u_1, \ldots, u_m, v_{k-1}$ be the neighbors of v_k in their natural cyclic order around v_k. By assumption, the inner faces of C are bounded by triangles. In particular, $v_1u_1 \cdots u_m v_{k-1}$ is a path P in G. Let C' be the cycle $P \cup (C - V_k)$. Pick different $a, b \in L(v_k) - L(v_1)$ and remove them from $L(u_i)$ for each $i \in [m]$ to get a new list assignment L' on $G - v_k$. By minimality of $|G|$, $G - v_k$ has a coloring from L'. Since v_{k-1} used at most one of a or b, we have a color left to use to complete the coloring to v_k. This contradiction completes the proof. □

Theorem 36 (Thomassen 1994)**.** *Every planar graph is* 5*-choosable.*

Proof. Let G be a plane graph and L a 5-assignment on $V(G)$. Add edges to G until it is a maximal plane graph H. Then, by maximality, H is a plane triangulation with boundary $v_1v_2v_3v_1$. Pick different colors, $c_1 \in L(v_1)$ and $c_2 \in L(v_2)$ and set $L(v_1) = \{c_1\}$, $L(v_2) = \{v_2\}$. Then we have a coloring of H (and hence G) from L by Lemma 35. □

Lemma 37. *Let D be a kernel-perfect digraph and L a list assignment on $V(D)$. If $d^+_D(v) < |L(v)|$ for every $v \in V(D)$, then D can be colored from the lists.*

Proof. Assume not and choose a counterexample D minimizing $|D|$. Pick some $a \in \bigcup_{v \in V(D)} L(v)$ and let $U = \{v \in V(D) \mid a \in L(v)\}$. By assumption, $D[U]$ has a kernel K. Color the vertices of K with a to get a list assignment L' on $D - K$. Since $|L'(v)| < |L(v)|$ implies that $v \in U$ and hence has an edge into K we see that $d^+_{D-K}(v) < |L'(v)|$ for each $v \in V(D - K)$. Since $D - K$ is again kernel-perfect we can complete the coloring by minimality of $|D|$. □

Galvin's Theorem. *Every bipartite graph satisfies $ch' = \chi'$.*

Proof. By definition, every graph satisfies $ch' \geqslant \chi'$. To prove the reverse inequality, let G be a bipartite graph with parts A and B and let c be a $k = \chi'(G)$ edge-coloring of G. Put $H = L(G)$ and define a partial order $<$ on $V(H)$ by $e < f$ iff $e \cap f \subseteq A$ and $c(e) < c(f)$ or $e \cap f \subseteq B$ and $c(e) > c(f)$. For each $v \in V(G)$, the restriction of $<$ to the edges incident with v is then a total order $<_v$.

Now, $<$ defines an orientation D of H by directing e to f iff $e < f$. Let L be a list assignment on $V(D)$ with $|L(v)| = k$ for each $v \in V(D)$. For $e \in V(D)$ we have

$$\begin{aligned} d^+(e) &= |\{f \in V(D) \mid e < f\}| \\ &= |\{f \in V(D) \mid e \cap f \subseteq A \text{ and } c(e) < c(f)\}| + |\{f \in V(D) \mid e \cap f \subseteq B \text{ and } c(e) > c(f)\}| \\ &\leqslant |\{c(e)+1, \ldots, k\}| + |\{1, \ldots, c(e)-1\}| \leqslant k - 1. \end{aligned}$$

For any $F \trianglelefteq H$, we have $R \subseteq G$ such that $F = L(R)$ and the $<_v$ for $v \in V(R)$ give a set of preferences for R and hence by the Stable Matching lemma, R has a stable matching M. But then M is independent in F and for any $xy \in E(F) - M$, either there exists $xz \in M$ with $xy <_x xz$ or $wy \in M$ with $xy <_y wy$ – thus xy has an edge into M in F. Whence D is kernel-perfect and $d_D^+(v) < |L(v)|$ for every $v \in V(D)$ and is therefore colorable from the lists by Lemma 37. □

6.4 Perfect Graphs

Theorem 38. *Chordal graphs are perfect.*

Proof. Assume not and choose a chordal non-perfect graph G minimizing $|G|$. Then every induced subgraph of G is again chordal and hence perfect. Thus for any $v \in V(G)$ we must have $\chi(G-v) = \omega(G-v) \leqslant \omega(G) < \chi(G)$. That is, G is vertex critical and hence G has no clique cutset.

Let S be a minimal cutset in G. Since S is not a clique, we have non-adjacent $x, y \in S$. Let C_1, C_2 be components of $G - S$. By minimality of S, both x and y must have neighbors in both C_1 and C_2. But then putting together a shortest path from x to y through C_1 with one through C_2 gives an induced cycle in G of length at least 4 contradicting the chordality of G. This contradiction completes the proof. □

Theorem 39. *The graph resulting from replacing all vertices of a perfect graph with perfect graphs is perfect.*

Proof. Clearly it is enough to show that replacing a single vertex of a perfect graph by a perfect graph gives a perfect graph. Assume this is not the case and choose a perfect graph G, $v \in V(G)$ and a perfect graph F first minimizing $|F|$ and then minimizing $|G|$ such that replacing v by F in G yields an imperfect graph. Let D be G with v replaced by F. Then any induced subgraph of D is perfect by minimality of $|F|$ and $|G|$. Hence we must have $\omega(D) < \chi(D)$. Thus $\omega(D) = \omega(D-y) = \chi(D-y) = \chi(D) - 1$ for any $y \in V(D)$.

Pick $x \in V(F)$ and let π be an $\omega(D)$-coloring of $D - x$. Let $C_1, \ldots, C_k$ be the color classes of π that contain a vertex of $F - x$. Then each $y \in V(F)$ is non-adjacent to all of $\bigcup_i C_i - V(F)$. Hence we must have $\omega(F) = \chi(F) \geqslant k+1$ and hence $\omega(F) = \chi(F) = k+1$. Note that x must be in every $(k+1)$-clique in F. But this was for any $x \in V(F)$, thus every vertex of F is in every $(k+1)$-clique in F showing that $F = K^{k+1}$. If $k > 1$, then by minimality of $|F|$ replacing v with K^k and then one of the vertices of the K^k with K^2 shows that D is perfect. Hence $k = 1$.

Say $V(F) = \{x, y\}$ with $y \in C_1$. Then y cannot be in an $\omega(D)$-clique K in $D - x$ since then $K \cup \{x\}$ would be an $(\omega(D)+1)$-clique in D. Hence $\omega(D) - 1 = \omega(D - x - (C_1 - \{y\})) = \chi(D - x - (C_1 - \{y\}))$. Putting this coloring together with the color class $(C_1 - \{y\}) \cup \{x\}$ gives an $\omega(D)$-coloring of D. This final contradiction completes the proof. □

7 Extremal Graphs

Turán Graph. Let $r \leqslant n$ be positive integers. We write $T_{n,r}$ for the complete r-partite graph $K_{n_1,\ldots,n_r}$ where $\sum_i n_i = n$ and $|n_i - n_j| \leqslant 1$ for all i, j.

Turán's Theorem. *Let $r \leqslant n$ be positive integers. If G is a K_{r+1}-free graph with n vertices and the maximum number of edges, then $G = T_{n,r}$.*

Proof. Let G be a K_{r+1}-free graph with n vertices and the maximum number of edges.

First, assume G is a complete multipartite graph $K_{n_1,\ldots,n_s}$ with $n_i \geqslant n_j$ for $i \leqslant j$. Then $s \leqslant r$ since G is K_{r+1}-free. If $s < r$, then $n_1 \geqslant 2$ and $K_{1,n_1-1,n_2,\ldots,n_s}$ is K_{r+1}-free and has more edges. Thus $s = r$. If $n_1 - n_s \geqslant 2$, then $K_{n_1-1,n_2,\ldots,n_{s-1},n_s+1}$ is K_{r+1}-free and has more edges. Thus $G = T_{n,r}$ and we are done.

Therefore, we may assume that $\overline{G}$ is not a disjoint union of cliques. Hence G contains an induced $\overline{P_3}$, say with vertices x, y, z where $yz \in E(G)$ and $xy, xz \notin E(G)$.

First, assume $d(x) \geqslant d(y)$ and $d(x) \geqslant d(z)$. Create a new graph H by adding two copies of x to G and removing y and z. Plainly, H is K_{r+1}-free and $|E(H)| = |E(G)| + 2d(x) - (d(y) + d(z) - 1) > |E(G)|$. This is a contradiction.

Hence, without loss of generality, we may assume that $d(x) < d(y)$. Now create a new graph F by adding a copy of y to G and removing x. Plainly, F is K_{r+1}-free and $|E(F)| = |E(G)| + d(y) - d(x) > |E(G)|$. This final contradiction completes the proof. □

8 Directed Graphs

Definition 8.1. A *path cover* of a directed graph G is a set of vertex disjoint directed paths in G which together cover all the vertices of G. If P is a path cover of G, we let $\mathrm{ter}(P)$ be the set of endpoints of the paths in P.

Gallai-Milgram Theorem. *For any directed graph G, every path cover P of G with $ter(P)$ minimal has an independent transversal.*

Proof. Assume the theorem is false and let G be a counterexample with $|G|$ minimal. Let $P = \{P_1, \ldots, P_k\}$ be a path cover of G with $\mathrm{ter}(P)$ minimal. For $1 \leqslant i \leqslant k$, let x_i be the endpoint of P_i. If $\{x_1, \ldots, x_k\}$ is independent, then we have the desired transversal. Thus we may assume that $x_2x_1 \in E(G)$. If P_1 has length zero, then removing P_1 from the cover and replacing P_2 with $P_2x_2x_1$ gives a path cover P' with $\mathrm{ter}(P') \subset \mathrm{ter}(P)$ contradicting the minimality of $\mathrm{ter}(P)$. Hence we may let y be the second to last vertex on P_1.

Now $Q = \{P_1y, P_2, \ldots, P_k\}$ is a path cover of $G - x_1$. Assume there is some path cover Q' of $G - x_1$ with $\mathrm{ter}(Q') \subset \mathrm{ter}(Q)$. If $y \in \mathrm{ter}(Q')$, then we may extend the path ending in y by yx_1 to get a path cover of G violating the minimality of $\mathrm{ter}(P)$. Now, if $x_2 \in \mathrm{ter}(Q')$, then we may extend the path ending in x_2 by x_2x_1 to again get a path cover of G violating the minimality of $\mathrm{ter}(P)$. Hence $\mathrm{ter}(Q') \subseteq \{x_3, x_4, \ldots, x_k\}$. But then adding x_1 as a path of length zero to the cover again contradicts the minimality of $\mathrm{ter}(P)$.

Hence $\mathrm{ter}(Q)$ is minimal among path covers of $G - x_1$. Now, by the minimality of $|G|$, we get an independent transversal in $\{P_1y, P_2, \ldots, P_k\}$ which is also an independent transversal in P. $\square$

Gallai-Roy Theorem. *Every directed graph G contains a directed path of length $\chi(G)$.*

Proof. Let G be a directed graph and G' a maximal acyclic induced subgraph of G. Define a coloring π on $V(G')$ by letting $\pi(x)$ be the length of the longest directed path in G' starting at x. Then π is proper since if $xy \in E(G')$, then tacking xy onto the front of a longest path starting at y (which cannot end at x since G' is acyclic) shows that $\pi(x) > \pi(y)$. By maximality of G', $G' + wz$ must contain a cycle for any edge $wz \in E(G) - E(G')$. Hence G' contains a directed path from z to w in G' and therefore $\pi(z) > \pi(w)$ as above. Thus π is a proper coloring of G as well. But then G contains a directed path of length at least $|im(\pi)| \geqslant \chi(G)$. $\square$

Richardson's Theorem. *Any directed graph without odd directed cycles has a kernel.*

Proof. Assume not and let G be a kernel-less directed graph without odd directed cycles minimizing $|G|$. Then G is connected. First assume G is not strongly connected and let A be a a sink the the finite acyclic graph formed by collapsing each strong component of G to a single vertex. Then A has a kernel U by minimality of $|G|$. Let T be the vertices in G that have an edge into U. Put $H = G - (U \cup T)$. Then, by minimality, H has a kernel V. Put $W = U \cup V$. Plainly, W is a kernel in G.

Hence we may assume that G is strongly connected. If G is biparite, then each part is a kernel in G. Hence we may assume that the underlying undirected graph of G contains an odd cycle $v_0v_2 \cdots v_rv_0$. We construct an odd closed directed walk in G starting and ending at v_0. Consider our indices modulo r. If $v_iv_{i+1} \in E(G)$, let $P_i = v_iv_{i+1}$; otherwise let P_i be a shortest directed path from v_i to v_{i+1}. Then P_i has odd length for each i since otherwise $P_iv_{i+1}v_i$ would be an odd directed cycle in G. Joining the P_i end-to-end in order gives the desired odd closed directed walk in G.

Hence we may let Z be a minimal length odd closed directed walk in G. Since Z is not a directed cycle, it hits some vertex more than once. Pick such a $v \in Z$ minimizing the length of the walk L between v and itself. Then L must be an even directed cycle. But then removing L from Z gives a shorter odd closed directed walk. This final contradiction completes the proof. $\square$

9 Constructions

Triangle free graphs with large chromatic number, etc. Mycielski graphs and shift graphs.

10 Problems

Problem 40. *Any tree with an even number of vertices contains a unique spanning subgraph in which every vertex has odd degree.*

Proof. First we prove that such a spanning subgraph exists and then we prove uniqueness.

To get a contradiction, assume there is some tree with an even number of vertices that does not contain a spanning subgraph in which every vertex has odd degree. Let T be a such a tree with the minimum number of vertices. If every vertex of T has odd degree, then T itself is the desired spanning subgraph. Hence we may assume that we have $v \in V(T)$ such that $d(v)$ is even. Assume some component A of $T - v$ has an even number of vertices. Then, since $|T|$ is even, $|T - A|$ is even as well. Both A an $T - A$ are connected, so we may apply minimality of T to get spanning subgraphs in each of them in which every vertex has odd degree. The union of these spanning subgraphs is a spanning subgraph of T in which every vertex has odd degree. Hence it must be that each component of $T - v$ has an odd number of vertices. Now, $T - v$ has $d(v)$ components and since $d(v)$ is even, we conclude that $|T - v|$ is even and hence $|T|$ is odd. This contradiction completes the proof.

Now we prove uniqueness. Assume S_1 and S_2 are distinct spanning subgraphs of T in which each vertex has odd degree. Consider the symmetric difference $F = S_1 \Delta S_2$. Clearly, every vertex of F has even degree. Since S_1 and S_2 are distinct, some vertex $v \in V(F)$ has positive degree. Hence every vertex in the component of v in F has degree at least 2. Since T is finite, the component of v in F must contain a cycle contradicting the fact that T is a tree. □