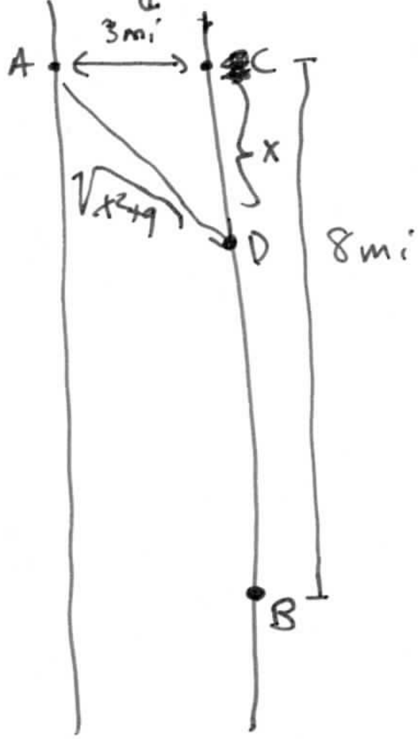
basically still river



row at 6 mi/h
run at 8 mi/h

where should he hit land to get there fastest?

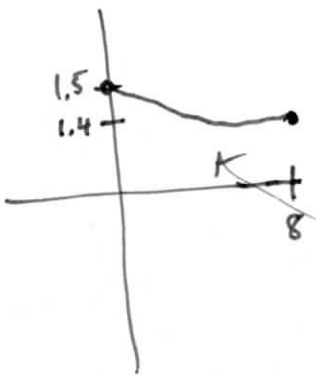
row distance: $\sqrt{x^2+9}$

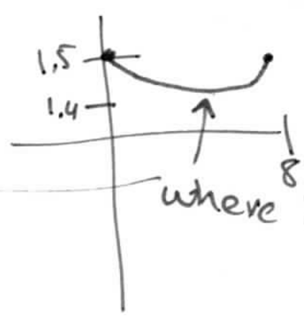
run distance: $8-x$

$$T(x) = \frac{\sqrt{x^2+9}}{6} + \frac{8-x}{8}$$

$$T(0) = \frac{\sqrt{9}}{6} + \frac{8}{8} = 1.5\text{ h}$$

$$T(8) = \frac{\sqrt{64+9}}{6} + \frac{8-8}{8} = \frac{\sqrt{73}}{6} \approx 1.42\text{ h}$$





where is this valley?

tangent line is flat
has slope 0
derivative is 0

Want all x values where
$f'(x)=0$

$T'(x) = 0$

how

$T(x) = \frac{1}{6}(x^2+9)^{1/2} + \frac{1}{8}(8-x)$

$T(x) = \frac{1}{6}(x^2+9)^{1/2} + 1 - \frac{1}{8}x$ using $(x^n)' = nx^{n-1}$

$T'(x) = \left(\frac{1}{6}(x^2+9)^{1/2}\right)' + 0 - \frac{1}{8}$

$T'(x) = \frac{1}{6}\left((x^2+9)^{1/2}\right)' - \frac{1}{8}$

how?

$f(x) = x^{1/2}$, what is $f'(x)$? only shown for $n = 1, 2, 3, 4, \ldots$

need more tools

Derivatives of compositions of functions (aka The Chain Rule)

Say $f(x) = g(h(x))$

then for any a,

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x-a} = \lim_{x \to a} \frac{g(h(x)) - g(h(a))}{x-a}$$

ok? $$= \lim_{x \to a} \frac{g(h(x)) - g(h(a))}{h(x) - h(a)} \cdot \frac{h(x) - h(a)}{x-a}$$

ok? $$= \left(\lim_{x \to a} \frac{g(h(x)) - g(h(a))}{h(x) - h(a)}\right)\left(\lim_{x \to a} \frac{h(x) - h(a)}{x-a}\right)$$

$h'(a)$

$$= \left(\lim_{x \to a} \frac{g(h(x)) - g(h(a))}{h(x) - h(a)}\right) h'(a)$$

ok? $$= \left(\lim_{h(x) \to h(a)} \frac{g(h(x)) - g(h(a))}{h(x) - h(a)}\right) h'(a)$$

ok? $$= g'(h(a)) \cdot h'(a).$$

Up to caveats

Chain rule

$$\boxed{(g(h(x)))' = g'(h(x)) \cdot h'(x)}$$

$$((g \circ h)(x))' = (g' \circ h)(x) \cdot h'(x)$$

$$(g \circ h)' = (g' \circ h) \cdot h$$

ex. $f(x) = (x^2+1)^3$

$f'(x) = ?$

write $f(x) = g(h(x))$

$g(x) = x^3 \rightarrow g'(x) = 3x^2$

$h(x) = x^2+1 \rightarrow h'(x) = 2x$

$$\begin{aligned} f'(x) &= g'(h(x)) \cdot h'(x) \\ &= 3(h(x))^2 \cdot h'(x) \\ &= 3(x^2+1)^2 \cdot h'(x) \\ &= 3(x^2+1)^2 \cdot (2x) \\ &= 6x(x^2+1)^2. \end{aligned}$$

does it help with $f(x) = x^{\frac{1}{2}}$?

yes, with a little cleverness.

$$(x^{1/2})^2 = x$$

$$((x^{1/2})^2)' = x' = 1$$

$$\parallel$$

$2(x^{1/2}) \cdot (x^{1/2})'$, so $(x^{1/2})' = \dfrac{1}{2x^{1/2}} = \frac{1}{2} x^{-\frac{1}{2}}$

back to the river

$$T'(x) = \underbrace{\frac{1}{6}\left((x^2+9)^{1/2}\right)'} - \frac{1}{8}$$

$$f(x) = \frac{1}{6}(x^2+9)^{1/2}$$

$$g(x) = x^2+1 \longrightarrow g'(x) = 2x$$

$$h(x) = \frac{1}{6}x^{1/2} \rightarrow h'(x) = \frac{1}{6}\left(\frac{1}{2}x^{-1/2}\right) = \frac{1}{12}x^{-\frac{1}{2}}$$

$f(x) = h(g(x))$

$$\cancel{f'(x) = g'(h(x))\cdot h'(x)}$$
$$\cancel{= 2(h(x))\cdot h'(x)}$$
$$\cancel{= 2\left(\frac{1}{6}x^{1/2}\right)\cdot h'(x)}$$
$$\cancel{= 2\left(\frac{1}{6}x^{1/2}\right)\cdot\left(\frac{1}{12}x^{-1/2}\right)}$$

$$f'(x) = h'(g(x))\cdot g'(x)$$
$$= \left(\frac{1}{12}\left(g(x)^{-\frac{1}{2}}\right)\right)\cdot(2x)$$
$$= \frac{1}{12}(x^2+9)^{-1/2}(2x)$$
$$= \frac{x}{6}(x^2+9)^{-1/2}.$$

$$0 = T'(x) = \frac{x}{6\sqrt{x^2+9}} - \frac{1}{8}$$

$$\frac{6}{8} = \frac{x}{\sqrt{x^2+9}}$$

$$\frac{6}{8}\sqrt{x^2+9} = x$$

$$\frac{3}{4}\sqrt{x^2+9} = x$$

$$\frac{9}{16}(x^2+9) = x^2$$

$$\frac{81}{16} = \frac{7}{16}x^2 \Rightarrow x^2 = \frac{81}{7}$$

$$x = \frac{9}{\sqrt{7}} \approx 3.4 \text{ mi}$$

$$T\left(\frac{9}{\sqrt{7}}\right) = \frac{\sqrt{\left(\frac{9}{\sqrt{7}}\right)^2+9}}{6} + \frac{8-\frac{9}{\sqrt{7}}}{8} \approx 1.33\text{ h}.$$