Beyond Degree Choosability

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Abstract

Let G be a connected graph with maximum degree Δ. Brooks' theorem states that G has a Δ-coloring unless G is a complete graph or an odd cycle. A graph G is *degree-choosable* if G can be properly colored from its lists whenever each vertex v gets a list of $d(v)$ colors. In the context of list coloring, Brooks' theorem can be strengthened to the following. Every connected graph G is degree-choosable unless each block of G is a complete graph or an odd cycle; such a graph G is a *Gallai tree*.

This degree-choosability result was further strengthened to Alon–Tarsi orientations; these are orientations of G in which the numbers of spanning Eulerian subgraphs with an even number of edges differs from the number with an odd number of edges. A graph G is *degree-AT* if G has an Alon–Tarsi orientation such that each vertex has indegree at least 1. Hladky, Kral, and Schauz showed that a connected graph is degree-AT if and only if it is not a Gallai tree. In this paper, we consider pairs (G, x) where G is a connected graph and x is some specified vertex in $V(G)$. We characterize pairs such that G has no Alon–Tarsi orientation in which each vertex has indegree at least 1 and x has indegree at least 2. When G is 2-connected, the characterization is simple to state.

1 Introduction

Brooks' theorem is one of the fundamental results in graph coloring. For every connected graph G, it says that G has a Δ-coloring unless G is a clique $K_{\Delta+1}$ or an odd cycle. When we seek to prove coloring results by induction, we often want to color a subgraph H where different vertices have different lists of allowable colors (those not already used on their neighbors in the coloring of $G - H$). This gives rise to *list coloring*. Vizing [13] and, independently, Erdős, Rubin, and Taylor [5] extended Brooks' theorem to list coloring. They proved an analogue of Brooks' theorem when each vertex v has Δ allowable colors (possibly different colors for different vertices). Erdős, Rubin, and Taylor [5] and Borodin [3] strengthened this Brooks' analogue to the following result, where a *Gallai tree* is a connected graph in which each block is a clique or an odd cycle.

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Theorem A. *If G is connected and not a Gallai tree, then for any list assignment L with $|L(v)| = d(v)$ for all $v \in V(G)$, graph G has a proper coloring φ with $\varphi(v) \in L(v)$ for all v.*

The graphs in Theorem A are *degree-choosable*. It is easy to check that every Gallai tree is not degree-choosable. So the set of all connected graphs that are not degree-choosable are precisely the Gallai trees. Hladky, Kral, and Schauz [6] extended this characterization to the setting of Alon–Tarsi orientations.

For any digraph D, a spanning Eulerian subgraph is one in which each vertex has indegree equal to outdegree. The *parity* of a spanning Eulerian subgraph is the parity of its number of edges. For an orientation of a graph G, let EE (resp. EO) denote the number of even (resp. odd) spanning Eulerian subgraphs. An orientation is *Alon–Tarsi* (or AT) if EE and EO differ. A graph G is f-AT if it has an Alon–Tarsi orientation D such that $d^+(v) \leq f(v)-1$ for each vertex v. In particular, G is *degree-AT* if it is f-AT, where $f(v) = d(v)$ for all v. Similarly, a graph G is *f-choosable* if G has a proper coloring φ from any list assignment L such that $|L(v)| = f(v)$ for all $v \in V(G)$. Alon and Tarsi [1] used algebraic methods to prove the following theorem for choosability. Later, Schauz strengthened the result to paintability, which we discuss briefly in Section 5.

Theorem B. *For a graph G and $f\colon V(G) \to \mathbb{N}$, if G is f-AT, then G is also f-*choosable.

In this paper we characterize those graphs G with a specified vertex x that are not f-AT, where $f(x) = d(x) - 1$ and $f(v) = d(v)$ for all other $v \in V(G)$. All such graphs are formed from a few 2-connected building blocks, by repeatedly applying two operations. Similar to that for degree-AT, our characterization remains unchanged in the contexts of list-coloring and paintability. We see a sharp contrast when we consider graphs G with two specified vertices x_1 and x_2 that are not f-AT, where $f(x_i) = d(x_i) - 1$ for each $i \in \{1, 2\}$ and $f(v) = d(v)$ for all other $v \in V(G)$. For Alon–Tarsi orientations, we have more than 50 exceptional graphs on seven vertices or fewer. Furthermore, the characterizations for list-coloring, paintability, and Alon–Tarsi orientations all differ.

We consider graphs with vertices labeled by natural numbers; that is, pairs (G, h) where G is a graph and $h\colon V(G) \to \mathbb{N}$. We focus on the case when $h(x) = 1$ for some x and $h(v) = 0$ for all other v; we denote this labeling as h_x. We say that (G, h) *is AT* if G is $(d_G - h)$-AT. When H is an induced subgraph of G, we simplify notation by referring to the pair (H, h) when we really mean $\big(H, h\restriction_{V(H)}\big)$.

Given a pair (G, h) and a specified edge $e \in E(G)$, when we *stretch* e, we form (G', h') from (G, h) by subdividing e twice and setting $h'(v_i) = 0$ for each of the two new vertices, v_1 and v_2 (and $h'(v) = h(v)$ for all other vertices v). In Section 2, we prove a Stretching Lemma, which shows that if (G, h) is not AT and $e \in E(G)$, then stretching e often yields another pair (G', h') that is also not AT. Thus, stretching plays a key role in our main result.

It is easy to check that the three pairs (G, h) shown in Figure 1 are not AT (and we do this below, in Proposition 1.1). Let $\mathcal{D}$ be the collection of all pairs formed from the graphs in Figure 1 by stretching each bold edge 0 or more times. The Stretching Lemma implies that each pair in $\mathcal{D}$ is not AT. Our main result is that these are the only pairs (G, h_x), where G is 2-connected and neither a clique nor an odd cycle, such that the (G, h_x) is not AT (for some vertex $x \in V(G)$).

Main Theorem. *Let G be 2-connected and let $x \in V(G)$. Now (G, h_x) is AT if and only if*

1. *$d(x) = 2$ and $G - x$ is not a Gallai tree; or*
2. *$d(x) \geq 3$, G is not complete, and $(G, h_x) \notin \mathcal{D}$.*

Near the end of Section 4, with a little more work we extend our Main Theorem, by removing the hypothesis of 2-connectedness, to characterize all pairs (G, h_x) that are not AT. The characterization of degree-choosable graphs have been applied to prove a variety of graph coloring results [2, 4, 9, 10, 12]. Likewise, we think our main results in this paper may be helpful in proving other results for Alon–Tarsi orientations, such as giving better lower bounds on the number of edges in AT-critical graphs.

To conclude this section, we show that each pair in $\mathcal{D}$ is not AT.

Proposition 1.1. *If $(G, h_x) \in \mathcal{D}$, then (G, h_x) is not AT.*

Proof. For each pair $(G, h_x) \in \mathcal{D}$, we construct a list assignment L such that $|L(x)| = d(x)-1$ and $|L(v)| = d(v)$ for all other $v \in V(G)$, but G has no proper coloring from L. Now (G, h_x) is not AT, by the contrapositive of Theorem B.

Let (G, h_x) be some stretching of the leftmost pair in Figure 1. Assign the list $\{1, 2, 3\}$ to each of the vertices on the unbolded triangle and assign the list $\{1, 2\}$ to each other vertex. If G has some coloring from these lists, then vertex x, labeled 1 in the figure, must get color 1 or 2; by symmetry, assume it is 1. Along each of these paths, colors must alternate $2, 1, \ldots$. Each of the paths from x to the triangle has odd length; thus, color 1 is forbidden from appearing on the triangle. So G has no coloring from L. Now let (G, h_x) be some stretching of the center pair in Figure 1. The proof is identical to the first case, except that each path has odd length, so if x is color 1, then color 2 is forbidden on the triangle.

Finally, consider the rightmost pair in Figure 1. Here $d(x) = 4$ and $d(v) = 3$ for all other $v \in V(G)$. Thus, it suffices to show that G is not 3-colorable. Assume that G has a 3-coloring and, by symmetry, assume that x is colored 1. Now colors 2 and 3 must each appear on two neighbors of x. Thus, the two remaining vertices must be colored 1. Since they are adjacent, this is a contradiction, which proves that G is not 3-colorable. $\square$

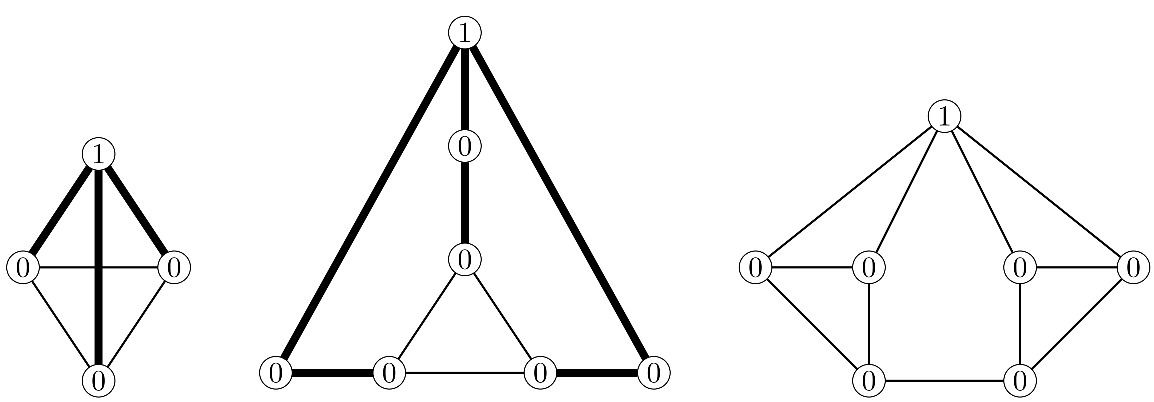

Figure 1: The seed blocks for $\mathcal{D}$. Each bold edge can be removed without making the figure AT.

2 Subgraphs, subdivisions, and cuts

When Hladky, Kral, and Schauz characterized degree-AT graphs, their proof relied heavily on the observation that a connected graph G is degree-AT if and only if G has some induced subgraph H such that H is degree-AT. Below, we reprove this easy lemma, and also extend it to our setting of pairs (G, h_x).

Lemma 2.1. *Let G be a connected graph and let H be an induced subgraph of G. If H is degree-AT, then also G is degree-AT. Similarly, if $x \in V(H)$ and (H, h_x) is AT, then also (G, h_x) is AT. Further, if $x \notin V(H)$ and $d_G(x) \geq 2$, then (G, h_x) is AT.*

Proof. Suppose that H is degree-AT, and let D' be an orientation of H showing this. Extend D' to an orientation D of G by orienting all edges away from H, breaking ties arbitrarily, but consistently. Now every directed cycle in D is also a directed cycle in D' (and vice versa), so G is degree-AT. The proof of the second statement is identical. The proof of the third statement is similar, but now if some edge xy has endpoints equidistant from H, then xy should be oriented into x. □

Recall that, given a pair (G, h) and a specified edge $e \in E(G)$, when we *stretch* e, we form (G', h') from (G, h) by subdividing e twice and setting $h'(v_i) = 0$ for each of the two new vertices, v_1 and v_2 (and $h'(v) = h(v)$ for all other vertices v). By repeatedly stretching edges, starting from the pairs in Figure 1, we form all pairs (G, h_x), where G is 2-connected and (G, h_x) is not AT.

Stretching Lemma. *Form (G', h') from (G, h) by stretching some edge $e \in E(G)$. Now*

(1) if (G, h) is AT, then (G', h') is AT; and

(2) if (G', h') is AT, then either (G, h) is AT or $(G - e, h)$ is AT.

Proof. Suppose $e = u_1u_2$ and call the new vertices v_1 and v_2 so that G' contains the induced path $u_1v_1v_2u_2$. For (1), let D be an orientation of G showing that (G, h) is AT. By symmetry we may assume $u_1u_2 \in E(D)$. Form an orientation D' of G' from D by replacing u_1u_2 with the directed path $u_1v_1v_2u_2$. We have a natural parity preserving bijection between the spanning Eulerian subgraphs of D and D', so we conclude that (G', h') is AT.

For (2), let D' be an orientation of G' showing that (G', h') is AT. Suppose G' contains the directed path $u_1v_1v_2u_2$ or the directed path $u_2v_2v_1u_1$. By symmetry, we can assume it is $u_1v_1v_2u_2$. Now form an orientation D of G by replacing $u_1v_1v_2u_2$ with the directed edge u_1u_2. As above, we have a parity preserving bijection between the spanning Eulerian subgraphs of D and D', so we conclude that (G, h) is AT. Otherwise, no spanning Eulerian subgraph of D' contains a cycle passing through v_1 and v_2. So, the spanning Eulerian subgraph counts of D' are the same as those of $D' - v_1 - v_2$. However, this gives an orientation of $G - e$ showing that $(G - e, h)$ is AT. □

Given a pair (G, h) that is not AT, the tretching Lemma] suggests a way to construct a larger graph G' such that (G', h') is not AT. Specifically, we have the following.

Corollary 2.2. *If e is an edge in G such that (G, h) is not AT and $(G - e, h)$ is not AT, then stretching e gives a pair (G', h') that is not AT.*

In some cases, we can also use the Stretching Lemma to construct a smaller graph $\widehat{G}$ such that $(\widehat{G}, h)$ is not AT.

Corollary 2.3. *Let G be a graph with an induced path $u_1v_1v_2u_2$ such that $d_G(v_1) = d_G(v_2) = 2$. If (G, h) is AT, where $h(v_1) = h(v_2) = 0$, and $(G - v_1 - v_2, h)$ is not AT, then*

$$\left((G - v_1 - v_2) + u_1u_2, h\restriction_{V(G)\setminus\{v_1,v_2\}}\right) \text{ is AT.}$$

Proof. Suppose the pair (G, h) satisfies the hypotheses. Applying part (2) of the Stretching Lemma shows that either $\left(G - v_1 - v_2, h\restriction_{V(G)\setminus\{v_1,v_2\}}\right)$ is AT or $\left((G - v_1 - v_2) + u_1u_2, h\restriction_{V(G)\setminus\{v_1,v_2\}}\right)$ is AT. By hypothesis, the former is false. Thus, the latter is true. □

With standard vertex coloring, we can easily reduce to the case where G is 2-connected. If G is a connected graph with two blocks, B_1 and B_2, meeting at a cutvertex x, then we can color each of B_1 and B_2 independently, and afterward we can permute colorings to match at x. For Alon–Tarsi orientations, the situation is not quite as simple. However, the following lemma plays a similar role for us.

Lemma 2.4. *Let $A_1, A_2 \subseteq V(G)$, and $x \in V(G)$ be such that $A_1 \cup A_2 = V(G)$ and $A_1 \cap A_2 = \{x\}$. If $G[A_i]$ is f_i-AT for each $i \in \{1, 2\}$, then G is f-AT, where $f(v) = f_i(v)$ for each $v \in V(A_i - x)$ and $f(x) = f_1(x) + f_2(x) - 1$. Going the other direction, if G is f-AT, then $G[A_i]$ is f_i-AT for each $i \in \{1, 2\}$, where $f_i(v) = f(v)$ for each $v \in V(A_i - x)$ and $f_1(x) + f_2(x) \leq f(x) + 1$.*

Proof. We begin with the first statement. For each $i \in \{1, 2\}$, choose an orientation D_i of A_i showing that A_i is f_i-AT. Together these D_i give an orientation D of G. Since no cycle has vertices in both $A_1 - x$ and $A_2 - x$, we have

$$\begin{aligned} EE(D) - EO(D) &= EE(D_1)EE(D_2) + EO(D_1)EO(D_2) - EE(D_1)EO(D_2) - EO(D_1)EE(D_2) \\ &= (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2)) \\ &\neq 0. \end{aligned}$$

Hence G is f-AT.

Now we prove the second statement. Suppose that G is f-AT and choose an orientation D of G showing this. Let $D_i = D[A_i]$ for each $i \in \{1, 2\}$. As above, we have $0 \neq EE(D) - EO(D) = (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2))$. Hence, $EE(D_1) - EO(D_1) \neq 0$ and $EE(D_2) - EO(D_2) \neq 0$. Since the indegree of x in D is the sum of the indegree of x in D_1 and the indegree of x in D_2, the lemma follows. □

3 Degree-AT graphs and an Extension Lemma

Recall that our Main Theorem relies on a characterization of degree-AT graphs. As we mentioned in the introduction, a description of degree-choosable graphs was first given by Borodin [3] and Erdős, Rubin, and Taylor [5]. Hladky, Kral, and Schauz [6] later extended the proof from [5] to Alon–Tarsi orientations. This proof relies on Rubin's Block lemma, which states that every 2-connected graph G contains an induced even cycle with at most one chord, unless G is a clique or an odd cycle. For variety, and completeness, we include a new proof; it extends ideas of Kostochka, Stiebitz, and Wirth [8] from list-coloring to Alon–Tarsi orientations. For this proof we need the following very special case of a key lemma in [7]. When vertices x and y are adjacent, we write $x \leftrightarrow y$; otherwise $x \not\leftrightarrow y$.

Lemma 3.1. *Let G be a graph and $x \in V(G)$ such that H is connected, where $H := G - x$. If there exist $z_1, z_2 \in V(H)$ with $N_H[z_1] = N_H[z_2]$ such that $x \leftrightarrow z_1$ and $x \not\leftrightarrow z_2$, then G is f-AT where $f(x) = 2$ and $f(v) = d_G(v)$ for all $v \in V(H)$.*

Proof. Order the vertices of H with z_1 first and z_2 second so that every vertex, other than z_1, has at least one neighbor preceding it. Orient each edge of H from its earlier endpoint toward its later endpoint. Orient xz_1 into z_1 and orient all other edges incident to x into x. Let D be the resulting orientation. Clearly, $d_D^+(v) \leq f(v) - 1$ for all $v \in V(D)$. So, we just need to check that $EE(D) \neq EO(D)$.

Since xz_1 is the only edge of D leaving x, and $D - x$ is acyclic, every spanning Eulerian subgraph of D that has edges must have edge xz_1. Consider an Eulerian subgraph A of D containing xz_1. Since z_1 has indegree 1 in A, it must also have outdegree 1 in A. We show that A has a mate A' of opposite parity. If $z_2 \in A$ then $z_1z_2w \in A$, for some w, so we form A' from A by removing z_1z_2w and adding z_1w. If instead $z_1z_2 \notin A$, then $z_2 \notin A$ and $z_1w \in A$ for some $w \in N_H[z_1] - z_2$, so we form A' from A by removing z_1w and adding z_1z_2w. Hence exactly half of the Eulerian subgraphs of D that contain edges are even. Since the edgeless spanning subgraph of D is an even Eulerian subgraph, we conclude that $EE(D) = EO(D) + 1$. Hence G is f-AT. □

We use the previous lemma to give a new proof of the characterization of degree-AT graphs.

Lemma 3.2. *A connected graph G is degree-AT if it is not a Gallai tree.*

Proof. Suppose there exists a connected graph that is not a Gallai tree, but is also not degree-AT. Let G be such a graph with as few vertices as possible. Since G is not degree-AT, no induced subgraph H of G is degree-AT by the Subgraph Lemma. Hence, for any $v \in V(G)$ that is not a cutvertex, $G - v$ must be a Gallai tree by minimality of $|G|$.

If G has more than one block, then for endblocks B_1 and B_2, choose noncutvertices $w \in B_1$ and $x \in B_2$. By the minimality of $|G|$, both $G - w$ and $G - x$ are Gallai trees. Since every block of G appears either as a block of $G - w$ or as a block of $G - x$, every block of G is either complete or an odd cycle. Hence, G is a Gallai tree, a contradiction. So instead G has only one block, that is, G is 2-connected. Further, $G - v$ is a Gallai tree for all $v \in V(G)$.

Let v be a vertex of minimum degree in G. Since G is 2-connected, $d_G(v) \geq 2$ and v is adjacent to a noncutvertex in every endblock of $G - v$. If $G - v$ has a complete block B with noncutvertices x_1, x_2 where $v \leftrightarrow x_1$ and $v \not\leftrightarrow x_2$, then we can apply Lemma 3.1 to conclude that G is degree-AT, a contradiction. So, v must be adjacent to every noncutvertex in every complete endblock of $G - v$.

Suppose $d_G(v) \geq 3$. Now no endblock of $G - v$ can be an odd cycle of length at least 5 (G would have vertices of degree 3 and also $d_G(v) \geq 4$, contradicting the minimality of $d_G(v)$). Let B be a smallest complete endblock of $G - v$. Now for a noncutvertex $x \in V(B)$, we have $d_G(x) = |B|$ and hence $d_G(v) \leq |B|$. If $G - v$ has at least two endblocks, then

$2(|B|-1) \leq |B|$, so $d_G(v) \leq |B| = 2$, a contradiction. Hence, $G - v = B$ and v is joined to B, so G is complete, which is a contradiction.

Thus, we have $d_G(v) = 2$. Suppose $G - v$ has at least two endblocks. Now it has exactly two and v is adjacent to one noncutvertex in each. Neither of the endblocks can be odd cycles of length at least five, since then we can get a smaller counterexample by the Stretching Lemma. Since v is adjacent to every noncutvertex in every complete endblock of $G - v$, both endblocks must be K_2. But now either $G = C_4$ (which is trivially degree-AT) or we can get a smaller counterexample by the Stretching Lemma. So, $G - v$ must be 2-connected. Since $G - v$ is a Gallai tree, it is either complete or an odd cycle. If $G - v$ is not complete, then we can get a smaller counterexample by the Stretching Lemma. So, $G - v$ is complete and v is adjacent to every noncutvertex of $G - v$; that is, G is complete, a contradiction. □

4 When h is 1 for at most one vertex

For a graph G and $x \in V(G)$ recall that $h_x \colon V(G) \to \mathbb{N}$ is defined as $h_x(x) = 1$ and $h_x(v) = 0$ for all $v \in V(G - x)$. We classify the connected graphs G such that (G, h_x) is AT for some $x \in V(G)$. We begin with the case when G is 2-connected, which takes most of the work. At the end of the section, we extend this to all connected graphs.

We will show that for most 2-connected graphs G and vertices $x \in V(G)$, the pair (G, h_x) is AT. Specifically, this is true for all pairs except those in $\mathcal{D}$, defined in the introduction. In view of the Subgraph Lemma, for a 2-connected graph G and $x \in V(G)$, to show (G, h_x) is AT it suffices to find some induced subgraph H, with $x \in V(H)$, such that (H, h_x) is AT. The subgraphs H that we consider all have $d_H(x) \geq 3$. This motivates the next lemma, which allows us to reduce to the case $d_G(x) \geq 3$.

Lemma 4.1. *If G is a connected graph and $x \in V(G)$ with $d_G(x) = 2$, then (G, h_x) is AT if and only if $G - x$ is degree-AT.*

Proof. Let D be an orientation of G showing that (G, h_x) is AT. Then $d^-_D(x) = 2$ and hence no spanning Eulerian subgraph contains a cycle passing through x. Therefore, the Eulerian subgraph counts in $G - x$ are different and $G - x$ is degree-AT. The other direction is immediate from the Subgraph Lemma □

A *θ-graph*, $\Theta_{a,b,c}$, consists of two vertices joined by three internally disjoint paths, with lengths a, b, and c. We will see shortly that if H is a θ-graph with $d_H(x) = 3$, then (H, h_x) is AT. Thus, we can assume that G has no induced θ-graph H with $d_H(x) = 3$. All of the other forbidden subgraphs H that we consider (and show that (H, h_x) is AT) can be viewed as θ-graphs with at most three extra edges. We show if G is 2-connected with $x \in V(G)$ and $d(x) \geq 3$, then either (i) $G \in \mathcal{D}$ or (ii) G contains one of these forbidden "nearly θ-' graphs.

Lemma 4.2. *The pair (G, h_x) is AT whenever (i) G is a θ-graph, (ii) G is a T-graph and two paths P_i have lengths of opposite parities, or (iii) G is formed from a T-graph by adding an extra vertex with neighborhood $\{z_1, z_2, z_3\}$.*

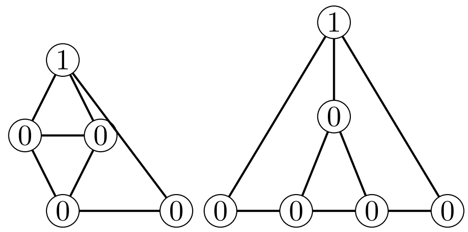

Figure 2: These are AT.

Proof. In each case, we give an AT orientation D of G such that $d_D^-(v) \geq h_x(v) + 1$ for each $v \in V(G)$.

Case (i). Orient the edges of each path P_i consistently, with P_1 and P_2 into x and P_3 out of x; this orientation satisfies the degree requirements. Further, it has exactly three spanning Eulerian subgraphs, including the empty subgraph. Thus, $EE + EO$ is odd, so $EE \neq EO$.

Case (ii). Let P_1 and P_2 be two paths with opposite parities. As before, orient the edges of each path consistently, with P_1 and P_2 into x and P_3 out of x. Orient the three additional edges as $\overrightarrow{z_1z_2}$, $\overrightarrow{z_2z_3}$, and $\overrightarrow{z_3z_1}$. The resulting digraph D has four spanning Eulerian subgraphs, 3 of one parity and 1 of the other. Note that the empty subgraph and the subgraph $\{\overrightarrow{z_1z_2}, \overrightarrow{z_2z_3}, \overrightarrow{z_3z_1}\}$ have opposite parities. Further, the parities are the same for the two subgraphs consisting of the directed cycles $xP_3z_3z_1P_1$ and $xP_3z_3z_1z_2P_2$. So, $EE \neq EO$.

Case (iii). The simplest instance of this case is when $G = K_5 - e$. Now (G, h_x) is AT by Lemma 2.4. In fact, that proof gives the stronger statement that there exists an orientation D satisfying the degree requirements such that $EE(D) = EO(D)+1$. In particular, $EE+EO$ is odd. To handle larger instances of this case, we repeatedly subdivide edges incident to x and orient each of the resulting paths consistently, and in the direction of the corresponding edge in D. The resulting orientation satisfies the degree requirements. Further, the sum $EE + EO$ remains unchanged, and thus odd. Hence, still $EE \neq EO$. □

Lemma 4.3. *Let G be a T-graph. Let P be a path of G where all internal vertices of P have degree 2 in G and one endvertex of P has degree 2 in G. Form G' from G by adding a path P' (of length at least 2) joining the endvertices of P. Now (G', h_x) is AT.*

Proof. We can assume that G is not AT; otherwise, we are done by Lemma 2.1. By symmetry, assume P is a subpath P_3. First, we get an orientation of G with indegree at least 1 for all vertices and $d^-(x) = 2$. Orient P_1 from z_1 to x, P_2 from z_2 to x, P_3 from x to z_3, and the triangle as z_3z_2, z_3z_1, z_1z_2. To get an orientation of G', orient the new path P' consistently, and opposite of P. Now the only directed cycle containing edges of P' is $P'P$. Since the Eulerian subgraph counts are equal for G, they differ by 1 for G'. □

Lemma 4.4. *Let G be 2-connected, and choose $x \in V(G)$ with $d(x) \geq 3$. Now (G, h_x) is AT if and only if G is not complete and $(G, h_x) \notin \mathcal{D}$.*

Proof. First suppose that $(G, h_x) \in \mathcal{D}$. Add details.

Now let G be 2-connected, choose $x \in V(G)$ with $d(x) \geq 3$, and suppose that $(G, h_x) \notin \mathcal{D}$. Since $G - x$ is connected, let H' be a smallest connected subgraph of $G - x$ containing three neighbors of x; call these neighbors w, y, and z. Consider a spanning tree T of H'. Since

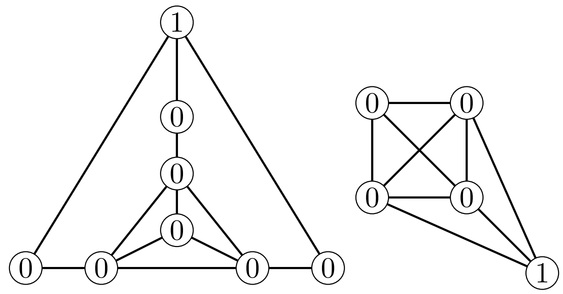

Figure 3: These are AT.

H' is minimum, each leaf of T is among $\{w, y, z\}$. If T is a path, then H' is also a path. Otherwise, T is a subdivision of $K_{1,3}$. Let s be the vertex with $d_T(s) = 3$. If $E(G) - E(T)$ has any edge with both ends outside of $N(s)$, then we can delete some vertex in $N(s)$ and remain connected, contradicting the minimality of H'. Similarly, if $N(s)$ contains at least two edges, then $H' - s$ still connects, w, y, and z. Now let H be the subgraph of G induced by $V(H') \cup \{x\}$. Note that H is either a θ-graph (if H' is a tree) or a T-graph (if H' has one extra edge in $N(s)$).

If H is a θ-graph, then (G, h_x) is AT, by Lemma 4.2 and the Subgraph Lemma. So assume H is a T-graph. Let z_1, z_2, z_3 be the vertices of degree 3 (other than x), and let P_1, P_2, and P_3 denote the paths from x to z_1, z_2, and z_3; when we write $V(P_i)$, we exclude x and z_i, so possibly $V(P_i)$ is empty for one or more $i \in \{1, 2, 3\}$. If any two of P_1, P_2, and P_3 have lengths with opposite parities, then we are done by Lemma 4.2; so assume not.

Now $(H, h_x) \in \mathcal{D}$, so we can assume that $V(G - H) \neq \emptyset$. Choose $u \in V(G - H)$, and let H_u be a minimal 2-connected induced subgraph of G that contains $V(H) \cup \{u\}$. By the Subgraph Lemma and Lemma 3.2, $G - x$ is a Gallai tree. Thus, so is $H_u - x$; in particular, the block B_u of $H_u - x$ containing u is complete or an odd cycle. Therefore, we either have (i) $V(B_u) \cap V(H) = \{z_1, z_2, z_3\}$ or (ii) $V(B_u) \cap V(H) \subseteq P_i \cup \{z_i\}$ for some $i \in \{1, 2, 3\}$.

Suppose (i) happens. Now $N_G(u) \cap V(H_u - x) = \{z_1, z_2, z_3\}$. If $x \not\leftrightarrow u$, then (G, h_x) is AT by the Subgraph Lemma and Lemma 4.2. If $x \leftrightarrow u$, then x must have odd length paths to each z_i, by Lemma 4.2, with u in the role of some z_i. Further, $x \leftrightarrow z_i$ for all $i \in \{1, 2, 3\}$, since otherwise (G, h_x) is AT by the Subgraph Lemma, Lemma 4.2, and the Stretching Lemma. So, $H = K_4$ and H_u is K_5. This implies that (ii) cannot happen for any vertex in $V(G - H)$ (if $V(B_u) \cap V(H) = \{z_i\}$ for some i, then (G, h_x) is AT by Lemma 4.2 and the Subgraph Lemma). So (i) happens for every vertex in $V(G - H)$; in particular, $V(G - H)$ is joined to $\{x, z_1, z_2, z_3\}$. Since G is not complete, it must contain an induced copy of Figure 3; hence, (G, h_x) is AT by the Subgraph Lemma.

Assume instead that (ii) happens for every vertex in $V(G - H)$. Pick $u \in V(G - H)$. By symmetry, assume that $V(B_u) \cap V(H) \subseteq P_1$. Let $z_1 P_1 = v_1 v_2 \cdots v_\ell$, where $v_\ell \leftrightarrow x$. First, assume that B_u is an odd cycle of length at least 5. If there is $u' \in V(B_u) \setminus V(H)$ with $u' \leftrightarrow x$, then G contains a θ-graph and (G, h_x) is AT, by Lemma 4.2 and Subgraph Lemma. So, we may assume that $u' \not\leftrightarrow x$ for all $u' \in V(B_u) \setminus V(H)$. Now we are done by Lemma 4.3 and the Subgraph Lemma.

So, we may assume that B_u is complete. If $V(B_u) \cap V(H) = \{v_\ell\}$, then G has an induced θ-graph J, where $d_J(x) = d_J(v_\ell) = 3$, so we are done by Lemma 4.2 and the Subgraph

Lemma. Thus, we must have $V(B_u) \cap V(H) = \{v_j, v_{j+1}\}$ for some $j \in \{1, \ldots, \ell\}$. In particular, B_u is a triangle. If $j \neq \ell - 1$, then $u \not\leftrightarrow x$, by the minimality of H; so (G, h_x) is AT by the Subgraph Lemma and Lemma 4.3. Thus, we conclude that $u \leftrightarrow x$, so $j = \ell - 1$. Hence, H_u is formed from T-graph by adding a vertex u added that is adjacent to x and to the vertices of a K_2 endblock D_u of $H - x$. Suppose there are distinct vertices $u_1, u_2 \in V(G - H)$ such that $D_{u_1} = D_{u_2}$. Now G contains an induced copy of Figure 3(b), so (G, h_x) is AT by Lemma 4.2 and the Subgraph Lemma. Thus, each K_2 endblock has at most one such u.

Let t be the number of K_2 endblocks in $H - x$. By construction, $t \leq 3$; this implies that $|V(G - H)| \leq t$. If $t = 0$, then $G = H = K_4$, which contradicts that G is not complete. If $t = 1$, then $G = H_u$ is the Moser spindle, for the unique $u \in V(G - H)$. So, assume that $t \in \{2, 3\}$. Now the subgraph induced by $\{u\} \cup V(H - P_1)$ is reducible by Lemma 4.3. So, again we are done by the Subgraph Lemma. □

Theorem 4.5. *If G is connected and $x \in V(G)$, then (G, h_x) is not AT if and only if*

(1) $d(x) = 1$; or

(2) G is a Gallai tree; or

(3) $d(x) = 2$ and $G - x$ has a component that is a Gallai tree; or

(4) x is a cutvertex, all but at most one x-lobe of G is a Gallai tree and for every x-lobe H of G, we have (G, h_x) is not AT; or

(5) x is not a cutvertex and for the block B of G containing x, we have $(B, h_x) \in \mathcal{D}$ and all B-lobes of G are Gallai trees.

Proof. First, lets check that if any of (1)–(5) happen, then (G, h_x) is not AT. 1 and 2 are immediate. 3 follows from Lemma 4.2. 4 follows from Lemma 2.2. 5 follows from Lemma 4.9 and Lemma 2.2.

Now, for the other direction, suppose (G, h_x) is not AT and none of (1)–(5) happen. By Lemma 4.2 (and not 1 and not 3) we must have $d(x) \geq 3$. Suppose x is a cutvertex. Then (by not 4), at least two x-lobes of G are not Gallai trees or (H, h_x) is AT for some x-lobe H of G. But then (G, h_x) is AT by Lemma 2.2, a contradiction. So, x is not a cutvertex. Suppose the block B of G containing x is complete or $(B, h_x) \in \mathcal{D}$. Now (by not 2 and not 5) some B-lobe H of G is not a Gallai tree. Since $G - x$ is connected, we conclude that $G - x$ is degree-AT and by directing all edges incident to x into x, we get that (G, h_x) is AT, a contradiction. By Lemma 2.2, we know that (B, h_x) is not AT. Since B is 2-connected, the argument in the Lemma in the last email shows that B is h_x**-minimal**. We just learned that B is not complete and $(B, h_x) \notin \mathcal{D}$, so applying Lemma 4.9 shows that (B, h_x) is AT, a contradiction. □

5 Choosability and Paintability

As we mentioned in the introduction, Alon and Tarsi showed that if a graph G is f-AT, then G is also f-choosable. *Online list coloring*, also called *painting* is similar to list coloring, but now the list for each vertex is progressively revealed, as the graph is colored. Schauz [11] extended the Alon–Tarsi theorem, to show that if G is f-AT, then G is also f-paintable. In this section, we use our characterization of pairs (G, h_x) that are not AT to prove characterizations of pairs (G, h_x) that are not paintable and that are not choosable. More precisely, a pair (G, h_x) is *choosable* if G has a proper coloring from its lists L whenever L is such that $|L(x)| = d(x)-1$ and $|L(v)| = d(v)$ for all other v; otherwise (G, h_x) is *not choosable*. The definition of a pair being paintable is analogous. We characterize all pairs (G, h_x), where G is connected and (G, h_x) is not choosable (resp. paintable). We will see that, in fact, these characterizations (for both choosability and paintability) are identical to that for pairs that are not AT.

For completeness, we include the following definition of f-paintable. Schauz gave a more intuitive (yet equivalent) definition, in terms of a two player game. We say that G is *f-paintable* if either (i) G is empty or (ii) $f(v) \geq 1$ for all $v \in V(G)$ and for every $S \subseteq V(G)$ there is an independent set $I \subseteq S$ such that $G - I$ is f'-paintable where $f'(v) := f(v)$ for $v \in V(G) - S$ and $f'(v) := f(v) - 1$ for $v \in S - I$.

Since all pairs (G, h_x) that are AT are also both paintable and choosable, it suffices to show that every pair (G, h_x) that is not AT is also not choosable (here we use that if a pair is paintable, then it is also choosable).

Theorem 5.1. *For every connected graph G, the pair (G, h_x) is not choosable if and only if (G, h_x) is not AT. Thus, the same characterization holds for pairs that are not paintable.*

Also, show that when we move up to two vertices x, y with $h(x) = h(y) = 1$, AT, choosability and paintability all separate.

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