graph theory notes*

Stiebitz's proof of Gallai's conjecture on the number of components in the high and low vertex subgraphs of critical graphs

Tibor Gallai conjectured the following in 1963 [1, 2] and Michael Stiebitz proved it in 1982 [3]. For a graph G, let $\mathcal{L}(G)$ be the subgraph of G induced on the vertices of degree $\delta(G)$ and let $\mathcal{H}(G)$ be the subgraph of G induced on the vertices of degree larger than $\delta(G)$.

Theorem (Stiebitz)**.** *If G is a color-critical graph with $\delta(G) = \chi(G) - 1$, then $\mathcal{H}(G)$ has at most as many components as $\mathcal{L}(G)$.*

In fact, Stiebitz proved a stronger statement.

Lemma. *If G is a connected graph, then for every nonempty $X \subseteq V(G)$, at least one of the following holds:*

1. *$G - X$ has at most as many components as $G[X]$; or*
2. *X contains a vertex of degree at least $\chi(G)$; or*
3. *$G[X]$ has a component C such that $\chi(G - V(C)) = \chi(G)$.*

Applying the Lemma with $X = V(\mathcal{L}(G))$ yields the Theorem since neither (2) nor (3) can occur in a color-critical graph.

Proof of Lemma. Suppose the Lemma is false and choose a counterexample G and nonempty $X \subseteq V(G)$ minimizing $|X|$. □

References

[1] T. Gallai, *Kritische graphen I.*, Math. Inst. Hungar. Acad. Sci **8** (1963), 165–192 (in German).

[2] ———, *Kritische graphen II.*, Math. Inst. Hungar. Acad. Sci **8** (1963), 373–395 (in German).

[3] M. Stiebitz, *Proof of a conjecture of T. Gallai concerning connectivity properties of colour-critical graphs*, Combinatorica **2** (1982), no. 3, 315–323.

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