SOME NOTES

1. About Lemma 4.6

Lemma 4.6 can be improved, see Lemma 1.6 below. You don't need the $\Delta_0 \geq 10^{20}$ condition here. If you are willing to use existing theory (from [**?**]), the proof is much shorter also, it doesn't really have much to do with the particular problem, really just d_1-choosability stuff.

The following are either lifted straight out of [**?**] or we include their short proof. None of the proofs are difficult and the development is natural and reusable.

Corollary 1.1. *For $t \geq 4$, $K_t * B$ is not d_1-choosable iff B is almost complete; or $t = 4$ and B is E_3 or a claw; or $t = 5$ and B is E_3.*

Lemma 1.2. *Let A and B be graphs such that $G := A * B$ is not d_1-choosable. If either $|A| \geq 2$ or B is d_0-choosable and L is a bad d_1-assignment on G, then*

1. *for any independent set $I \subseteq V(B)$ with $|I| = 3$, we have $\bigcap_{v \in I} L(v) = \emptyset$; and*
2. *for disjoint nonadjacent pairs $\{x_1, y_1\}$ and $\{x_2, y_2\}$ at least one of the following holds*
   (a) *$L(x_1) \cap L(y_1) = \emptyset$;*
   (b) *$L(x_2) \cap L(y_2) = \emptyset$;*
   (c) *$|L(x_1) \cap L(y_1)| = 1$ and $L(x_1) \cap L(y_1) = L(x_2) \cap L(y_2)$.*

Lemma 1.3. *Let H be a d_0-choosable graph such that $G := K_1 * H$ is not d_1-choosable and L a minimal bad d_1-assignment on G. If some nonadjacent pair in H have intersecting lists, then $|Pot(L)| \leq |H| - 1$.*

Lemma 1.4. *If B is a graph with $\delta(B) \geq \frac{|B|+1}{2}$ such that $K_1 * B$ is not d_1-choosable, then $\omega(B) \geq |B| - 1$ or $B = E_3 * K_4$.*

Proof. Suppose the lemma is false and let L be a minimal bad d_1-assignment on B. First note that if B does not contain disjoint nonadjacent pairs x_1, y_1 and x_2, y_2, then $\omega(B) \geq |B| - 1$ or $B = E_3 * K_4$ by Corollary 1.1.

By Dirac's theorem, B is hamiltonian and in particular 2-connected. Since B cannot be an odd cycle or complete, B is d_0-choosable.

By the Small Pot Lemma, $|Pot(L)| \leq |B|$. Since $|L(x_1)| + |L(x_2)| \geq |B| + 1$, the lists intersect and thus Lemma 1.3 shows that $|Pot(L)| \leq |B| - 1$. But then $|L(x_i) \cap L(y_i)| \geq 2$ for each i and Lemma 1.2 gives a contradiction. □

Note that the neighborhoods we will be looking at are huge, so the $B = E_3 * K_4$ case will never happen here.

End of stuff from [**?**].

Let $\mathcal{D}_1$ be the collection of graphs without induced d_1-choosable subgraphs. Plainly, $\mathcal{D}_1$ is hereditary. For a graph G and $t \in \mathbb{N}$, let $\mathcal{C}_t$ be the maximal cliques in G having at least

t vertices. We prove the following decomposition result for graphs in $\mathcal{D}_1$ which generalizes Reed's decomposition in [**?**].

Lemma 1.5. *Suppose $G \in \mathcal{D}_1$ has $\Delta(G) \geq 8$ and contains no $K_{\Delta(G)}$. If $\frac{\Delta(G)+5}{2} \leq t \leq \Delta(G)-1$, then $\bigcup \mathcal{C}_t$ can be partitioned into sets $D_1, \dots, D_r$ such that for each $i \in [r]$ at least one of the following holds:*

- *$D_i \in \mathcal{C}_t$,*
- *$D_i = C_i \cup \{x_i\}$ where $C_i \in \mathcal{C}_t$ and $|N(x_i) \cap C_i| \geq t-1$,*
- *each $v \in V(G) - D_i$ has at most $t-2$ neighbors in C_i.*

Proof. Suppose $|C_i| \leq |C_j|$ and $C_i \cap C_j \neq \emptyset$. Then $|C_i \cap C_j| \geq |C_i| + |C_j| - (\Delta + 1) \geq 4$. It follows from Corollary 1.1 that $|C_i - C_j| \leq 1$.

Now suppose C_i intersects C_j and C_k. By the above, $|C_i \cap C_j| \geq \frac{\Delta(G)+3}{2}$ and similarly $|C_i \cap C_k| \geq \frac{\Delta(G)+3}{2}$. Hence $|C_i \cap C_j \cap C_k| \geq \Delta(G)+3-(\Delta(G)-1) = 4$. Put $I := C_i \cap C_j \cap C_k$ and $U := C_i \cup C_j \cup C_k$. By maximality of C_i, C_j, C_k, U cannot induce an almost complete graph. Thus, by Corollary 1.1, $|U| \in \{4, 5\}$ and the graph induced on $U - I$ is E_3. But then $t \leq 6$ and hence $\Delta(G) \leq 7$, a contradiction.

The existence of the required partition is immediate. $\square$

This can quickly be turned into a decomposition for d-dense graphs. Let G be a minimum counterexample. Then $G \in \mathcal{D}_1$. Call $v \in V(G)$ d-sparse if it has more than $d\Delta$ non-edges in its neighborhood. The $3d$ in the following isn't optimal.

Lemma 1.6. *Let $0 \leq d \leq \frac{\Delta}{10} - \frac{3}{2}$. We can partition $V(G)$ into $S, D_1, \dots, D_r$ so that*

1. *each vertex in S is d-sparse,*
2. *each D_i contains a vertex w_i such that $D_i - w_i$ is a clique of size at least $\Delta - 3d + 1$,*
3. *no vertex outside of D_i has more than $\frac{3\Delta}{4}$ neighbors in D_i and w_i has at least $\frac{3\Delta}{4}$ neighbors in D_i.*

Proof. Put $t := \frac{3}{4}\Delta + 1$, $B := \bigcup \mathcal{C}_t$ and $S := V(G) - B$. Apply Lemma 1.5 to get $D_1, \dots, D_r$ partitioning B. We claim that some subset of $\{D_1, \dots, D_r\}$ works. For item (i), we need to check that each $v \in S$ is d-sparse. We know (Lemma 9.2.2 in the other write-up) that each $v \in S$ has more than $\binom{\Delta-1}{2} - \frac{2}{5}\Delta^2 \geq (\frac{\Delta}{10} - \frac{3}{2})\Delta$ non-edges in its neighborhood, so v is d-sparse.

Item (iii) follows by the definition of the D_i. Now item (ii). If for any i, all vertices of D_i are d-sparse, then just move all of D_i into S. So now we may assume that each D_i contains a non-sparse vertex v_i. Clearly, the largest clique in G containing v_i is contained in D_i. Hence it will be enough to show that v_i is in a $\Delta - 3d + 1$ clique. We can do this with the same computation in the proof of Lemma 9.2.2 before. Let x be some v_i. Suppose x is in no $\Delta - 3d + 1$ clique, then using Lemma 1.4, we get a sequence $y_1, \dots, y_{3d} \in N(x)$ such that

$$|N(y_i) \cap (N(x) - \{y_1, \dots, y_{i-1}\})| \leq \frac{1}{2}(\Delta + 1 - i).$$

Hence x is d-sparse since it has at least

$$\frac{1}{2}\sum_{i=1}^{3d}(\Delta - i) > d\Delta.$$

non-edges in its neighborhood. $\square$

2. About Lemma 5.3

We actually need the C to be one of the C_i, not just maximal, otherwise, for some i where $D_i = C_i \cup \{w_i\}$, we could choose C to be the maximal clique containing w_i that intersects C_i in $\frac{3}{4}\Delta$ vertices. If C_i were bigger than C, the lemma fails for C. The lemma isn't used for anything but the C_i, so this doesn't change anything. Here is the statement and proof.

Lemma 2.1. *Each $v \in C_i$ has at most one neighbor outside of C_i with more than 4 neighbors in C_i and no such neighbor if v is low.*

Proof. Suppose otherwise that we have $v \in C_i$ with two neighbors $w_1, w_2 \in V(G) - C_i$ each with 5 or more neighbors in C_i. Put $Q := G[\{w_1, w_2\} \cup C_i - v]$, then v is joined to Q and hence $K_1 * Q \trianglelefteq G$. We show that $K_1 * Q$ must be d_1-choosable.

First, suppose there are different $z_1, z_2 \in C_i$ such that $\{w_1, z_1\}$ and $\{w_2, z_2\}$ are independent. Since Q contains an induced diamond, it is d_0-choosable. Let L be a minimal bad d_1-assignment on $K_1 * Q$. Then $|L(w_i)| + |L(z_i)| \geq 4 + |Q| - 3 = |Q| + 1$. By the Small Pot Lemma, $|Pot(L)| \leq |Q|$. Hence $L(w_1) \cap L(z_1) \neq \emptyset$ and Lemma 1.3 shows that $|Pot(L)| \leq |Q| - 1$, but then $|L(w_i) \cap L(z_i)| \geq 2$ and Lemma 1.2 gives a contradiction.

By maximality of C_i, neither w_1 nor w_2 can be adjacent to all of C_i hence it must be the case that there is $y \in C_i$ such that w_1 and w_2 are joined to $C_i - y$. If w_1 and w_2 aren't adjacent, then G contains $K_6 * E_3$ contradicting Corollary 1.1. Hence C_i intersects the larger clique $\{w_1, w_2\} \cup C_i - \{y\}$, this is impossible by the definition of C_i.

When v is low, an argument similar to the above shows that there can be no z_1 in C_i so that $\{w_1, z_1\}$ is independent, and hence $C_i \cup \{w_1\}$ is a clique contradicting maximality of C_i. $\square$