graph coloring tools

Contents

Preface

this comes prior to the face.

graphs

A *graph* is a collection of dots we call *vertices* some of which are connected by curves we call *edges*. The relative location of the dots and the shape of the curves are not relevant, we are only concerned with whether or not a given pair of dots is connected by a curve. Initially, we forbid edges from a vertex to itself and multiple edges between two vertices. If G is a graph, then $V(G)$ is its set of vertices and $E(G)$ its set of edges. We write $|G|$ for the number of vertices in $V(G)$ and $\|G\|$ for the number of edges in $E(G)$. Two vertices are *adjacent* if they are connected by an edge. The set of vertices to which v is adjacent is its *neighborhood*, written $N(v)$. For the size of v's neighborhood $|N(v)|$, we write $d(v)$ and call this the *degree* of v.

vertices
edges
$V(G)$, $E(G)$
$|G|$, $\|G\|$
adjacent
neighborhood
$N(v)$

[ADD PICTURES]

$d(v)$, degree

coloring vertices

The entire book concerns one simple task: we want to color the vertices of a given graph so that adjacent vertices receive different colors. With sufficiently many crayons and no preferences about what the coloring should look like, this is easy, we just use a different crayon for each vertex. Things get interesting when we ask how few different crayons we can use. We are definitely going to need an empty box of crayons and that will only do for the graph with no vertices at all. Given one crayon, we can handle all graphs with no edges. With two crayons, we can do any path and any cycle with an even number of vertices [PICTURE]. But, we can't handle a triangle or any other cycle with an odd number of vertices [PICTURE]. In fact, odd cycles are really the only thing that will prevent us from using just two crayons. A graph H is a *subgraph* of a graph G, written $H \subseteq G$ if $V(H) \subseteq V(G)$ and $E(H) \subseteq E(G)$. When $H \subseteq G$, we say that G *contains* H. If $v \in V(G)$, then $G - v$ is the graph we get by removing v and all edges incident to v from G. A graph is k-colorable if we can color its vertices with (at most) k colors such that adjacent vertices receive different colors. A 0-colorable graph is *empty*, a 1-colorable graph is *edgeless* and a 2-colorable graph is *bipartite*.

subgraph, $\subseteq$
contains
$G - v$

k-colorable

empty
edgeless
bipartite

Theorem 1. *A graph is 2-colorable just in case it contains no odd cycle.*

Proof. A graph containing an odd cycle clearly can't be 2-colored. For the other implication, suppose there is a graph that is not 2-colorable and doesn't contain an odd cycle. Then we may pick such a graph G with $|G|$ as small as possible. Surely, $|G| > 0$, so we may pick $v \in V(G)$. If $x, y \in N(v)$, then x is not adjacent to y since then xyz would be an odd cycle. So we can construct a graph H from G by removing v and identifying all of $N(v)$ to a new vertex x_v. Any odd cycle in H would contain x_v and hence give rise to an odd cycle in G passing through v. So H contains no odd cycle. Since $|H| < |G|$, appplying the theorem to H gives a 2-coloring of H, say with red and blue where x_v gets colored red. But this gives a 2-coloring of G by coloring all vertices in $N(v)$ red and v blue, a contradiction. $\square$

Well, this is embarrassing, coloring appears to be easy. Fortunately, things get more interesting when we move up to three colors.

Theorem 2. *3-coloring is hard supposing other things we think are hard are actually hard.*

Proof. reduce 3-SAT to 3-coloring. $\square$

basic estimates

Even though finding the minimum number of colors needed to color a graph is hard in general (supposing it is), we can still look for lower and upper bounds on this value. The *chromatic number* $\chi(G)$ of a graph G is the smallest k for which G is k-colorable. The simplest thing we can do is give each vertex a different color.

chromatic number
$\chi(G)$

THEOREM 3. *If G is a graph, then $\chi(G) \leq |G|$.*

The only graphs that attain the upper bound in Theorem 3 are the *complete* graphs; those in which any two vertices are adjacent. We can usually do much better by just arbitrarily coloring vertices, reusing colors when we can. The *maximum degree* $\Delta(G)$ of a graph G is the largest degree of any vertex in G; that is

complete
maximum degree
$\Delta(G)$

$$\Delta(G) := \max_{v \in V(G)} d(v).$$

THEOREM 4. *If G is a graph, then $\chi(G) \leq \Delta(G) + 1$.*

PROOF. Suppose there is a graph G that is not $(\Delta(G)+1)$-colorable. Then we may pick such a graph G with $|G|$ as small as possible. Surely, $|G| > 0$, so we may pick $v \in V(G)$. Then $|G - v| < |G|$ and $\Delta(G - v) \leq \Delta(G)$, so applying the theorem to $G - v$ gives a $(\Delta(G - v) + 1)$-coloring of $G - v$. But v has at most $\Delta(G)$ neighbors, so there is some color, say red, not used on $N(v)$, coloring v red gives a $(\Delta(G) + 1)$-coloring of G, a contradiction. □

Both complete graphs and odd cycles attain the upper bound in Theorem 4. Theorem 1 says we can do better for graphs that don't contain odd cycles. We can also do better for graphs that don't contain large complete subgraphs. A set of vertices S in a graph G is a *clique* if the vertices in S are pairwise adjacent. The *clique number* of a graph G, written $\omega(G)$, is the number of vertices in a largest clique in G.

clique
$\omega(G)$

THEOREM 5. *If G is a graph, then $\chi(G) \geq \omega(G)$.*

A set of vertices S in a graph G is *independent* if the vertices in S are pairwise non-adjacent. The *independence number* of a graph G, written $\alpha(G)$, is the number of vertices in a largest independent set in G.

independent
$\alpha(G)$

THEOREM 6. *If G is a graph with $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$, then $\chi(G) \leq \Delta(G)$.*

PROOF. Suppose there is a graph G with $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$ that is not $\Delta(G)$-colorable. Then we may pick such a graph G with $|G|$ as small as possible. Let S be a maximal independent set in G. Since S is maximal, every vertex in $G - S$ has a neighbor in S, so $\Delta(G) > \Delta(G - S)$. If red is an unused color in a $\chi(G - S)$-coloring of $G - S$, then by coloring all vertices in S red we get a $(\chi(G - S) + 1)$-coloring of G. So, $\Delta(G) + 1 \leq \chi(G) \leq \chi(G - S) + 1$. We conclude $\chi(G - S) > \Delta(G - S)$ and thus $\Delta(G) = \chi(G - S) = \Delta(G - S) + 1$ by Theorem 4. Since $|G - S| < |G|$, applying the theorem to $G - S$ shows that $\Delta(G - S) < 3$ or $\Delta(G - S) < \omega(G - S)$. So, either $\chi(G - S) = \Delta(G) = 3$ or $\omega(G - S) \geq \Delta(G)$. In the former case, let X be the vertex set of an odd cycle in $G - S$ guaranteed by Theorem 1. In the latter case, let X be a $\Delta(G)$-clique in $G - S$.

Since S is maximal and $\omega(G) \leq \Delta(G)$, there are $x_1, x_2 \in X$ and $y_1, y_2 \in S$ such that x_1 is adjacent to y_1 and x_2 is adjacent to y_2. Construct a graph H from $G - X$ by adding the edge y_1y_2. Since $|H| < |G|$, applying the theorem to H shows that $\omega(H) > \Delta(G)$ or $\chi(H) \leq \Delta(G)$. Suppose $\chi(H) \leq \Delta(G)$. Then there is a $\Delta(G)$-coloring of $G - X$ where y_1 and y_2 receive different colors, say red and blue respectively. Pick the first vertex z in a shortest path P from x_1 to x_2 in X that has a blue colored neighbor in $V(H)$. Each vertex in X has $\Delta(G) - 1$ neighbors in X and hence at most one neighbor in $V(H)$. So, $z \neq x_1$ since x_1 already has a red colored neighbor in $V(H)$. Let w be be the vertex preceding z on P. Then w has no blue colored neighbor. Since X is the vertex set of a cycle or a complete graph, there is a path Q from w to z passing through every vertex of X. Color w blue and then proceed along Q, coloring one vertex at a time. Since each vertex we encounter before we get to z has at most $\Delta(G) - 1$ colored neighbors, we always have an available color to use. But, z is adjacent to both w and another blue colored vertex in $V(H)$, so there is an available color for z as well. This gives a $\Delta(G)$-coloring of G, a contradiction.

So, $\omega(H) > \Delta(G)$. In particular, y_1 and y_2 each have exactly one neighbor in X and $\Delta(G) - 1$ neighbors in the same $\Delta(G) - 1$ clique A in $G - X$. Since S is maximal and $|X| \geq 3$, there must be adjacent $x_3 \in X \setminus \{x_1, x_2\}$ and $y_3 \in S \setminus \{y_1, y_2\}$. Applying the same argument with x_3, y_3 in place of x_2, y_2 shows that y_1 and y_3 each have exactly one neighbor in X and $\Delta(G) - 1$ neighbors in the same $\Delta(G) - 1$ clique B in $G - X$. Now $|A \cap B| = |A| + |B| - |A \cup B| \geq 2(\Delta(G) - 1) - d(y_1) \geq \Delta(G) - 2 > 0$. But there can't be a vertex in $A \cap B$ since it would be adjacent to y_1, y_2, y_3 as well as $\Delta(G) - 2$ vertices in A and thus have degree greater than $\Delta(G)$, a contradiction.

edge coloring

It is also useful to consider coloring the edges of a graph so that incident edges receive different colors. This appears to be at odds with our previous claim that this book was only about coloring vertices of graph; fortunately, edge coloring is just a special case of vertex coloring. If G is a graph, the *line graph* of G, written $L(G)$ is the graph with vertex set $E(G)$ where two edges of G are adjacent in $L(G)$ if they are incident in G. Coloring the edges of G is equivalent to coloring the vertices of $L(G)$.

For graphs with maximum degree zero (that is, no edges at all), we can get by with zero colors. With just one color we can edge color any graph with maximum degree at most one. We will definitely always need at least $\Delta(G)$ colors to edge color a graph G. Could we be so fortunate that the pattern continues and we can edge color any graph G with only $\Delta(G)$-colors? Not quite, but we can do so for bipartite (2-colorable) graphs. A graph is k-edge-colorable if we can color its edges with (at most) k colors such that incident edges receive different colors. A color c us *used* at a vertex v of G if an edge incident to v in G is colored with c. Otherwise, c is *available* at v. A *path* in G is a sequence of pairwise distinct vertices $x_1x_2\cdots x_k$ such that x_i is adjacent to x_{i+1} for $i \in [k-1]$.

used

available

path

Theorem 7. *If G is a bipartite graph, then G is $\Delta(G)$-edge-colorable.*

Proof. Suppose there is a graph G that is not $\Delta(G)$-edge-colorable. Then we may pick such a graph G with $\|G\|$ as small as possible. Now $\|G\| > 0$, since we can surely color a graph with zero edges using zero colors. Let xy be an edge in G. Since $\|G - xy\| < \|G\|$, applying the theorem to $G - xy$ gives an edge coloring of $G - xy$ using at most $\Delta(G)$ colors. Now each of x and y are incident to at most $\Delta(G) - 1$ edges in $G - xy$ and G has no $\Delta(G)$-edge-coloring, so there is a color red available at x and a different color blue available at y. There is a unique maximal length path P starting at x with edges alternately colored blue and red. Let z be the last vertex of P. Since P alternates between two colors, P has even length. In particular, $z \neq y$ since G does not contain an odd cycle by Theorem 1. But then we get a $\Delta(G)$-edge-coloring of G by swapping the colors red and blue along P and coloring xy blue, a contradiction. □

It may come as a surpise that even though we might need more than $\Delta(G)$ colors to edge color a graph G, we will only ever need at most one extra color. For bipartite graphs we were able to repair an almost correct coloring by swapping colors along a path because we had control over where this path ended. In the general case we don't have the same control over a path between two vertices, but we can exert some measure of control over paths leaving and entering a larger structure. The larger structure we use here is the whole neighborhood of a vertex.

Theorem 8. *If G is a graph, then G is $(\Delta(G)+1)$-edge-colorable.*

Proof. $\square$

hardness

We now know that every graph G can be edge colored with either $\Delta(G)$ or $\Delta(G)+1$ colors. So, edge coloring is basically trivial, right? Furtunately, no it isn't, the collection of graphs requiring $\Delta(G)+1$ colors is very rich. Another way to say this, is that it is a hard problem to decide whether or not edge coloring a given graph G requires $\Delta(G)+1$ colors.

Theorem 9. *Deciding whether or not edge coloring a given graph G requires $\Delta(G)+1$ colors is hard supposing other things we think are hard are actually hard.*

Proof. $\square$

vertex coloring, again

list coloring

online list coloring

kernel tools

degree again.

maximum independent covers.

polynomial tools

combinatorial nullstellensatz.

coefficient formulae.

a combinatorial interpretation.

edge coloring, again

fans as a greedy strategy

acceptable paths

acceptable trees

edge list coloring

more kernel method.

2-edge-coloring.

improved degree theorem.

a hint of quasiline and claw-free graphs.

shuffle tool

destroying non-complete

shuffling with more rules

looking at the entire recoloring digraph

independent transversalls

randomly

Haxell's tool

triangles within triangles are neat.

big maximum cliques intersect in simple ways

going lopsided

vertex transitive graphs

strong coloring

medium clique implies big clique

potential potential tool