

graph theory notes*

Stiebitz's proof of Gallai's conjecture on the number of components in the high and low vertex subgraphs of critical graphs

Tibor Gallai conjectured the following in 1963 [1, 2] and Michael Stiebitz proved it in 1982 [3]. For a graph G , let $\mathcal{L}(G)$ be the subgraph of G induced on the vertices of degree $\delta(G)$ and let $\mathcal{H}(G)$ be the subgraph of G induced on the vertices of degree larger than $\delta(G)$.

Theorem (Stiebitz). *If G is a color-critical graph with $\delta(G) = \chi(G) - 1$, then $\mathcal{H}(G)$ has at most as many components as $\mathcal{L}(G)$.*

In fact, Stiebitz proved a stronger statement. Theorem follows immediately from Lemma 4 using $X = V(\mathcal{L}(G))$.

A bipartite graph G with parts A and B has *positive surplus* with respect to A if $|N(S)| > |S|$ for all $S \subseteq A$. Note that A could be empty here, in which case the graph has positive surplus vacuously. We say that G is *2-forest* with respect to A if G is a forest where each vertex in A has degree 2 in G . Plainly, a 2-forest has positive surplus. We first need a couple lemmas about bipartite graphs with positive surplus. The following is well-known (see [?]).

Lemma 1. *Let G be a bipartite graph with parts $A \neq \emptyset$ and B such that G has positive surplus with respect to A . If $G - e$ does not have positive surplus for each $e \in E(G)$, then G is a 2-forest with respect to A .*

Lemma 2. *Let G be a bipartite graph with parts $A \neq \emptyset$ and B such that G has positive surplus with respect to A . Then there is $x \in A$ such that for any different $y, z \in N(x)$, the bipartite graph G' formed by contracting $\{y, z\}$ and removing x has positive surplus with respect to $A \setminus \{x\}$.*

Proof. We may assume $|A| \geq 2$ for otherwise the statement is trivially true. We show that there is $x \in A$ such that for any different $y, z \in N(x)$, G contains a 2-forest F with respect to A containing the edges xy and xz . Consider the graph F' formed from F by contracting $\{y, z\}$ and removing x . Then F' is a 2-forest with respect to A since the degree of vertices in A don't change and x is a separating vertex in F . Now G' contains F' and hence has positive surplus, so we are done.

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By Lemma 1, G contains a 2-forest with respect to A . So, we can choose a maximal subgraph H of G for which there is $x \in A$ such that for any different $y, z \in N_H(x)$, H contains a 2-forest with respect to A containing the edges xy and xz . If $H = G$ we are done, so suppose $H \neq G$. Then we have $y \in N_G(x)$ and $z \in N_G(x) \setminus N_H(x)$ such that H does not contain a 2-forest with respect to A containing the edges xy and xz . \square

Note that following lemma is trivially true in the $X = \emptyset$ case, we allow this to avoid having to handle a base case where $G[X]$ is connected separately.

Lemma 3 (Stiebitz). *Let G be a graph and $X \subseteq V(G)$ such that*

- $d_G(x) \leq k - 1$ for all $x \in X$; and
- $\chi(G - X) \leq k - 1$; and
- for each component C of $G[X]$, we have $\chi(G - V(C)) \leq k - 1$.

Suppose $G - X$ is the disjoint union of (possibly not connected) graphs $M_1, \dots, M_{\ell+1}$ such that the bipartite graph \mathcal{B} formed by contracting each M_i and each component of $G[X]$ has positive surplus with respect to the $G[X]$ side. If f_i is a $(k - 1)$ -coloring of M_i for each $i \in [\ell + 1]$, then there are permutations $\pi_1, \dots, \pi_{\ell+1}$ of $[k - 1]$ such that the $(k - 1)$ -coloring of $G - X$ given by $(\pi_1 \circ f_1) \cup \dots \cup (\pi_{\ell+1} \circ f_{\ell+1})$ extends to a $(k - 1)$ -coloring of G .

Proof. Suppose the lemma is false and choose a counterexample G and nonempty $X \subseteq V(G)$ so that $|X|$ is as small as possible. So, $G - X$ is the disjoint union of graphs $M_1, \dots, M_{\ell+1}$ and we have $(k - 1)$ -colorings f_i of M_i for each $i \in [\ell + 1]$ so that no permutations allow us to extend to a $(k - 1)$ -coloring of G .

Claim 1. *Each non-separating vertex in $G[X]$ has neighbors in at least two of the M_i .* Suppose to the contrary that we have a component C of $G[X]$ and $x \in V(C)$ a non-separating vertex that has neighbors in at most one of the M_i . Let $X' = X \setminus \{x\}$. The hypotheses of the lemma are satisfied with X' in place of X and $M'_i = G[V(M_i) \cup \{x\}]$ in place of M_i since \mathcal{B} remains the same and hence still has positive surplus. Since \mathcal{B} has positive surplus, we must have $|C| \geq 2$ and hence x has at most $k - 2$ neighbors in $G - X$, so we can greedily complete the given $(k - 1)$ -coloring of $G - X$ to $G - X'$. Applying minimality of $|X|$ to this coloring of $G - X'$, we get permutations that allow us to extend to a $(k - 1)$ -coloring of G . But these same permutations also allow us to extend the given $(k - 1)$ -coloring of $G - X$ to G , a contradiction.

Claim 2. *There is a component C of $G[X]$ and $i \neq j$ such that some non-separating $x \in V(C)$ has neighbors in M_i and M_j and the bipartite graph formed from \mathcal{B} by contracting $\{M_i, M_j\}$ and removing C has positive surplus.*

By Lemma 2, $G[X]$ has a component C such that for any neighbors M_i, M_j of C in \mathcal{B} , the bipartite graph formed from \mathcal{B} by contracting $\{M_i, M_j\}$ and removing C has positive surplus. Let x be a non-separating vertex in C . By Claim 1, there are different M_i, M_j in which x has neighbors, so C with x and i, j works.

Claim 3. *The lemma is true.* Let C, x, i, j be as in Claim 2, by symmetry we may assume $i = 1$ and $j = 2$. Let $G' = G - V(C)$ and $X' = X \setminus V(C)$. Then $G' - X'$ is the disjoint union of the ℓ graphs $M_1 \cup M_2, M_3, \dots, M_{\ell+1}$. Pick $y_1 \in N(x) \cap V(M_1)$ and $y_2 \in N(x) \cap V(M_2)$.

We permute the colors in the coloring of f_2 so that y_1 and y_2 get the same color. This will save one color for x so that we can greedily color C , ending at x . Formally, let τ be a permutation of $[k-1]$ such that $(\tau \circ f_2)(y_2) = f_1(y_1)$ and let $f_* = f_1 \cup (\tau \circ f_2)$. By Claim 2 and minimality of $|X|$, we can apply the lemma to G' with $M_1 \cup M_2, M_3, \dots, M_{\ell+1}$ and colorings $f_*, f_3, \dots, f_{\ell+1}$ to get permutations $\pi_*, \pi_3, \dots, \pi_{\ell+1}$ such that the $(k-1)$ -coloring of $G' - X'$ given by $(\pi_* \circ f_*) \cup (\pi_3 \circ f_3) \cup \dots \cup (\pi_{\ell+1} \circ f_{\ell+1})$ extends to a $(k-1)$ -coloring of G' . But this is the same as the $(k-1)$ -coloring $(\pi_* \circ f_1) \cup (\pi_* \circ \tau \circ f_2) \cup (\pi_3 \circ f_3) \cup \dots \cup (\pi_{\ell+1} \circ f_{\ell+1})$, so using the permutations $\pi_*, \pi_* \circ \tau, \pi_3, \dots, \pi_{\ell+1}$ we get a coloring of $G - X$ that extends to $G - V(C)$. In these colorings, y_1 and y_2 receive the same color. This means that x has $k-1-(d_G(x)-d_C(x))+1 \geq d_C(x)+1$ colors available and each other vertex v in C has $k-1-(d_G(v)-d_C(v))+1 \geq d_C(v) \geq d_C(v)$ colors available. So, coloring C greedily in order of decreasing distance from x gives an extension to a $(k-1)$ -coloring of G , a contradiction. \square

Note that, like the previous one, the following lemma is trivially true in the $X = \emptyset$ case, again we allow this to avoid having to handle a base case where $G[X]$ is connected separately.

Lemma 4 (Stiebitz). *Let G be a connected graph and $X \subseteq V(G)$ such that*

- $d_G(x) \leq k-1$ for all $x \in X$; and
- $\chi(G-X) \leq k-1$; and
- for each component C of $G[X]$, we have $\chi(G-V(C)) \leq k-1$; and
- $G[X]$ has ℓ components and $G-X$ has at least $\ell+1$ components.

Then G is $(k-1)$ -colorable.

Proof. Suppose not and choose a counterexample minimizing $|X|$. If $|X| = 0$, the lemma is trivially true, so we must have $|X| \geq 1$. Let \mathcal{B} be the bipartite graph formed by contracting each component of $G-X$ and each component of $G[X]$. If \mathcal{B} has positive surplus with respect to the $G[X]$ side, then applying Lemma 3 to any $(k-1)$ -coloring of $G-X$ gives a $(k-1)$ -coloring of G . So, we may assume that \mathcal{B} does not have positive surplus. That is, for some t , $G[X]$ has a set of t components $\{C_1, \dots, C_t\}$ which together have neighbors in at most t components of $G-X$. But then the other $\ell-t$ components of $G[X]$ together have neighbors in at least $\ell+1-t$ components of $G-X$ since G is connected. Let $X' = X \setminus \bigcup_{i \in [t]} V(C_i)$. Then the hypotheses of the lemma are satisfied with X' in place of X , so minimality of $|X|$ shows that G is $(k-1)$ -colorable, a contradiction. \square

References

- [1] T. Gallai, *Kritische graphen I.*, Math. Inst. Hungar. Acad. Sci **8** (1963), 165–192 (in German).
- [2] ———, *Kritische graphen II.*, Math. Inst. Hungar. Acad. Sci **8** (1963), 373–395 (in German).
- [3] M. Stiebitz, *Proof of a conjecture of T. Gallai concerning connectivity properties of colour-critical graphs*, Combinatorica **2** (1982), no. 3, 315–323.