Recoloring trees

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1 The trees

For a coloring π of a graph G, put $I_\gamma := \pi^{-1}(\gamma)$ for each $\gamma \in \mathrm{im}(\pi)$. Also, for different $\alpha, \beta \in \mathrm{im}(\pi)$, put $G_{\alpha,\beta} := G[I_\alpha \cup I_\beta]$.

Let (T, r) be a tree in with root r. We think of the edges of T as directed away from the root. We write $C(v)$ for the children (out-neighbors) of $v \in V(T)$ and for $v \neq r$ we write $P(v)$ for the parent (unique in-neigbor) of v in T. To enable uniform statements we set $P(r) := \perp$ and extend all functions f to have $f(\perp) = \perp$ where $\perp$ is outside the universe of our objects. Let G be a graph and (T, r) an induced tree in G. For a coloring π of G, (T, r) is π-normal if

1. $N_T(v) = \pi^{-1}(\pi(C(v))) \cap N_G(v)$ for each $v \in V(T)$; and
2. for different $\alpha, \beta \in \mathrm{im}(\pi)$ and any maximal directed path $x_1 x_2 \cdots x_s$ with $s \geq 3$ in $T \cap G_{\alpha,\beta}$, we have $|C(x_i) \cap G_{\alpha,\beta}| = 1$ for $i \in [s-2]$.

This definition needs some explanation. Our aim is to recolor the vertices of T so that $\left|I_{\pi(r)}\right|$ decreases without using more colors. To do this, we will try to repeatedly recolor leaves that have no neighbor in some color class until we have recolored enough of T to recolor r. Condition (1) means that if v has a child in T of color γ, then all of v's neighbors in G of color γ are also neighbors of v in T. That is, T encodes all the information about what vertices need to be recolored for v to be colored γ. Condition (2) prevents us from getting in our own way as we recolor vertices having the same color as their grandparents.

1.1 An ordering on rooted trees

Let (T, r) be a tree with root r. A *build order* of (T, r) is a tuple $(v_1, A_1), \dots (v_s, A_s)$ where:

1. $v_1 = r$; and
2. $A_i \subseteq C(v_i)$ for each $i \in [s]$; and
3. $A_i \neq \emptyset$ for $i \in [s]$ and $A_i \cap A_j = \emptyset$ for $j \neq i$; and
4. $T\left[\{r\} \cup \bigcup_{i \in [k]} A_i\right]$ is connected for each $k \in [s]$; and

5. $V(T) = \{r\} \cup \bigcup_{i\in[s]} A_i$.

The *profile* $p(B)$ of a build order $B := (v_1, A_1), \dots (v_s, A_s)$ is $(|A_1|, \dots, |A_s|)$. We use the partial order on all triples (T, r, B) induced by the Kleene-Brouwer order on the profiles of their build orders; that is, $(T, r, B) < (T', r', B')$ iff $p(B')$ is a proper prefix of $p(B)$ or there is k such that $(a_1, \dots, a_{k-1}, a_k)$ is a prefix of $p(B)$, $(a_1, \dots, a_{k-1}, a'_k)$ is a prefix of B' and $a_k < a'_k$.

In our applications all of our trees will be induced subtrees of a fixed finite graph and so there will be only finitely many triples (T, r, B) under consideration; in particular, the ordering is well-founded and we can do induction on the triples.

1.2 Recoloring

Let G be a graph, π a coloring of G and and (T, r) a π-normal induced tree in G. A build order $B := (v_1, A_1), \dots (v_s, A_s)$ of (T, r) *respects* π if for each $i \in [s]$ we have $A_i = C(v_i) \cap I_c \neq \emptyset$ for some $c \in \operatorname{im}(\pi)$. Call a triple (T, r, B) π-normal if (T, r) is π-normal and B is a build order of (T, r) respecting π. Fix a π-normal triple (T, r, B).

Suppose we have $v \in V(T)$ and $c_v \in \operatorname{im}(\pi) - \{\pi(r), \pi(v)\}$ such that $N_G(v) \cap I_{c_v} \subseteq \{P(v)\}$. We call the following operation $(v \Rightarrow c_v)$-recoloring. Let $x_1 x_2 \cdots x_s$ be the maximal path in T where $x_s = v$ and $C(x_i) \cap I_{\pi(x_{i+1})} = \{x_{i+1}\}$ for $2 \leq i \leq s-1$. Note that, by maximality, if $C(x_1) \cap I_{\pi(x_2)} = \{x_2\}$, then $x_1 = r$; when this happens we say we have *finished.* Let π' be the coloring obtained from π by coloring v with c_v and x_i with $\pi(x_{i+1})$ for $2 \leq i \leq s-1$ and if we have finished, coloring x_1 with $\pi(x_2)$. Since T is induced, it is clear that π' is a proper coloring using at most as many colors as π. When we have finished we succeed in recoloring r since $x_1 = r$. Otherwise, let T' be the component of x_1 in $T - x_2$. Again since T is induced, it is clear that T' is π'-normal. Let k be such that $x_2 \in A_k$. Then $B' := ((A_1, v_1), \dots, (A_k - x_2, v_k)$ is a build order of (T', r) since $|A_k| = \left|C(x_1) \cap I_{\pi(x_2)}\right| \geq 2$. Then (T', r, B') is π-normal and $(T', r, B') < (T, r, B)$.

Definition 1. For each graph G, $r \in V(G)$ and $k \in \mathbb{N}^+$, let $\Gamma(G, r, k)$ be all triples (T, r, B) that are π-normal where π is a k-coloring of G.

Lemma 1.1. *Let G be a graph, fix $r \in V(G)$ and $k \in \mathbb{N}^+$. If $(T, r, B) \in \Gamma(G, r, k)$ is minimal, then either:*

1. *there is $v \in V(T)$ and $c_v \in \operatorname{im}(\pi) - \{\pi(r), \pi(v)\}$ with $N_G(v) \cap I_{c_v} \subseteq \{P(v)\}$ such that $(v \Rightarrow c_v)$-recoloring finishes; or*

2. *for every $v \in V(T)$ and $c_v \in \operatorname{im}(\pi) - \{\pi(r), \pi(v)\}$ we have $N_G(v) \cap I_{c_v} \not\subseteq \{P(v)\}$; moreover, if $c_v \notin \pi(C(v))$ then either:*

   (a) $V(T) \cup N_G(v) \cap I_{c_v}$ does not induce a tree in G; or

   (b) $c_v = \pi(P(v))$ and $|N_G(v) \cap I_{c_v}| \geq 3$.