Is there a function that is its own derivative?

- yes $f(x)=0$.

an interesting one?

We really only know how do differentiate polynomials (and fractions of them)

$f(x)=1+x+\frac{1}{2}x^2$ here

$f'(x)=1+x$

need x here, so

but then need $\frac{1}{2}x^2$

$f(x)=1+x+\frac{1}{2}x^2+\frac{1}{3}\cdot\frac{1}{2}x^3$

$f'(x)=1+x+\frac{1}{2}x^2$

Can we keep going?

$$f(x)=1+x+\frac{1}{2}x^2+\frac{1}{3\cdot 2}x^3+\frac{1}{4\cdot 3\cdot 2}x^4+\frac{1}{5\cdot 4\cdot 3\cdot 2}x^5+\cdots$$

<u>notation:</u> $n! = n(n-1)(n-2)(n-3)\cdots(1)$

$0! = 1$

$$f(x)=\frac{1}{0!}x^0+\frac{1}{1!}x^1+\frac{1}{2!}x^2+\frac{1}{3!}x^3+\frac{1}{4!}x^4\cdots$$

what is this f?

$f(0)=1$

$$f(1)=1+1+\frac{1}{2}+\frac{1}{3\cdot 2}+\frac{1}{4\cdot 3\cdot 2}+\frac{1}{5\cdot 4\cdot 3\cdot 2}+\cdots$$

$$=2+\frac{1}{2}+\frac{1}{6}+\frac{1}{24}+\frac{1}{120}+\frac{1}{720}+\frac{1}{5040}+\cdots$$

≈ 2.7182

Let's call this number e.

$$\boxed{e=1+\frac{1}{2}+\frac{1}{3\cdot 2}+\frac{1}{4\cdot 3\cdot 2}+\frac{1}{5\cdot 4\cdot 3\cdot 2}+\frac{1}{6\cdot 5\cdot 4\cdot 3\cdot 2}+\cdots}$$

exponential functions

$2^n = \underbrace{2\cdot 2\cdot 2\cdots 2}_{n}$

$2^{1/n} = \sqrt[n]{2}$

$2^{a/b} = \sqrt[b]{2^a}$

want to define 2^x for all x.

$2^a\cdot 2^b = 2^{a+b}$

$(2^a)^b = 2^{ab}$ etc. ← proofs

2^x when x is not rational?

2^x = approximate x by better and better rationals $\frac{a}{b}$ and compute $2^{a/b}\cdots$

Can change the base to whatever, say a.

$f(x) = a^x$.

what is $f'(x)$?

$$f'(x) = \lim_{h\to 0}\frac{a^{x+h} - a^x}{h} = \lim_{h\to 0}\frac{a^x\cdot a^h - a^x}{h}$$

$$= \lim_{h\to 0}\frac{a^x(a^h-1)}{h}$$

$$= a^x\cdot\lim_{h\to 0}\frac{a^h-1}{h}.$$

So, if $\lim_{h\to 0}\frac{a^h-1}{h} = 1$, then

$$f'(x) = f(x).$$

if you try to find the a for which this holds, you get $a \approx 2.7182$.

$a = 1 + \frac{1}{2} + \frac{1}{3\cdot 2} + \frac{1}{4\cdot 3\cdot 2} + \cdots$ works!

Can prove that that is, $a = e$.

two interesting functions that are their own derivative

$f(x) = e^x$

and $g(x) = 1 + x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 + \cdots$

$f(1) = e$ $\quad f'(x) = f(x)$ $\quad$ so $\quad f'(1) = f(1) = e$

$g(1) = e$ $\quad g'(x) = g(x)$, $\quad g'(1) = g(1) = e$

$f''(1) = e$ $\quad f'''(1) = e$

$g''(1) = e$ $\quad g'''(1) = e$

at $x=1$ f and g have same value and same values for all derivatives

change, change in change, ... all same must be $f=g$.

$$\boxed{e^x = 1 + x + \frac{1}{2}x^2 + \frac{1}{3\cdot 2}x^3 + \frac{1}{4\cdot 3\cdot 2}x^4 + \cdots}$$