GOLDBERG'S CONJECTURE FROM A GAME

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Abstract. We characterize the initial positions from which the first player has a winning strategy in a certain two-player game. Goldberg's Conjecture on edge-coloring follows from a special case.

1. Introduction

The game is played on a multigraph G by *Fixer* and *Breaker*. To setup, we assign a list of colors $L(v)$ to each $v \in V(G)$. Put $\mathrm{Pot}(L) := \bigcup_{v \in V(G)} L(v)$. Fixer always gets the first move and he wins if and only if on some turn, before moving, he can construct a proper edge coloring of G, say $\pi \colon E(G) \to \mathrm{Pot}(L)$ such that $\pi(xy) \in L(x) \cap L(y)$ for each $xy \in E(G)$.

Fixer's turn. Pick $\alpha \in \mathrm{Pot}(L)$ and $v \in V(G)$ with $\alpha \notin L(v)$ and set $L(v) := L(v) \cup \{\alpha\} \smallsetminus \{\beta\}$ for some $\beta \in L(v)$.

Breaker's turn. If Fixer modified $L(v)$ by inserting α and removing β, Breaker can pick at most one vertex in $V(G) \smallsetminus \{v\}$ and modify its list by swapping α for β or β for α.

For $\alpha \in \mathrm{Pot}(L)$ and $H \subseteq G$, put $d_H(\alpha) := |\{v \in V(H) \mid \alpha \in L(v)\}|$.

Conjecture 1. *If $|L(v)| \geq d(v) + 1$ for all $v \in V(G)$, then Fixer has a winning strategy against Breaker if and only if for each $H \subseteq G$,*

$$\sum_{\alpha \in \mathrm{Pot}(L)} \left\lfloor \frac{d_H(\alpha)}{2} \right\rfloor \geq \|H\| .$$

Suppose G is not k-edge-colorable for some $k \geq \Delta(G)+1$. Let $L(v) = [k]$ for all $v \in V(G)$. Then Fixer has no moves, so Fixer has a winning strategy if and only if G is k-edge-colorable. By Conjecture 1, there must be $H \subseteq G$ with

$$k \left\lfloor \frac{|H|}{2} \right\rfloor = \sum_{\alpha \in \mathrm{Pot}(L)} \left\lfloor \frac{|H|}{2} \right\rfloor = \sum_{\alpha \in \mathrm{Pot}(L)} \left\lfloor \frac{d_H(\alpha)}{2} \right\rfloor < \|H\| .$$

That is, $k \leq \left\lfloor \frac{\|H\|-1}{\lfloor \frac{|H|}{2} \rfloor} \right\rfloor$. Hence G is $w(G)$-edge-colorable where $w(G) := \max_{\substack{H \subseteq G \\ |H| \geq 2}} \left\lceil \frac{\|H\|}{\lfloor \frac{|H|}{2} \rfloor} \right\rceil$ if it is not $(\Delta(G)+1)$-edge-colorable. This is Goldberg's Conjecture.