

Beyond Degree Choosability

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Abstract

Let G be a connected graph with maximum degree Δ . Brooks' theorem states that G has a Δ -coloring unless G is a complete graph or an odd cycle. A graph G is *degree-choosable* if G can be properly colored from its lists whenever each vertex v gets a list of $d(v)$ colors. In the context of list coloring, Brooks theorem can be strengthened to the following. Every connected graph G is degree-choosable unless each block of G is a complete graph or an odd cycle; such a graph G is a *Gallai tree*.

This degree-choosability result was further strengthened to Alon–Tarsi orientations; these are orientations of G in which the numbers of spanning Eulerian subgraphs with an even number of edges differs from the number with an odd number of edges. A graph G is *degree-AT* if G has an Alon–Tarsi orientation such that each vertex has indegree at least 1. Hladky, Kral, and Schauz showed that a connected graph is degree-AT if and only if it is not a Gallai tree. In this paper, we consider pairs (G, x) where G is a connected graph and x is some specified vertex in $V(G)$. We characterize pairs such that G has no Alon–Tarsi orientation in which each vertex has indegree at least 1 and x has indegree at least 2. When G is 2-connected, the characterization is simple to state.

1 Introduction

Brooks' theorem is one of the fundamental results in graph coloring. For every connected graph G , it says that G has a Δ -coloring unless G is a clique $K_{\Delta+1}$ or an odd cycle. When we seek to prove coloring results by induction, we often want to color a subgraph H where different vertices have different lists of allowable colors (those not already used on their neighbors in the coloring of $G - H$). This gives rise to *list coloring*. Vizing [12] and, independently, Erdős, Rubin, and Taylor [4] extended Brooks' theorem to list coloring. They proved an analogue of Brooks' theorem when each vertex v has Δ allowable colors (possibly different colors for different vertices). In fact, they proved the following strengthening, where a *Gallai tree* is a connected graph in which each block is a clique or an odd cycle.

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Theorem A. *If G is connected and not a Gallai tree, then for any list assignment L with $|L(v)| = d(v)$ for all $v \in V(G)$, graph G has a proper coloring φ with $\varphi(v) \in L(v)$ for all v .*

The graphs in Theorem A are *degree-choosable*. It is easy to check that every Gallai tree is not degree-choosable. So the set of all connected graphs that are not degree-choosable are precisely the Gallai trees. Hladky, Kral, and Schauz [5] extended this characterization to the setting of Alon–Tarsi orientations.

For any digraph D , a spanning Eulerian subgraph is one in which each vertex has indegree equal to outdegree. The *parity* of a spanning Eulerian subgraph is the parity of its number of edges. For an orientation of a graph G , let EE (resp. EO) denote the number of even (resp. odd) spanning Eulerian subgraphs. An orientation is *Alon–Tarsi* (or AT) if EE and EO differ. A graph G is f -AT if it has an Alon–Tarsi orientation D such that $d^+(v) \leq f(v) - 1$ for each vertex v . In particular, G is *degree-AT* if it is f -AT, where $f(v) = d(v)$ for all v . Alon and Tarsi [1] used algebraic methods to show that if G is f -AT, for any f , then G also has a proper coloring φ from any list assignment L such that $|L(v)| = f(v)$ for all $v \in V(G)$; such a graph G is *f-choosable*.

In this paper we characterize those graphs G with a specified vertex x that are not f -AT, where $f(x) = d(x) - 1$ and $f(v) = d(v)$ for all other $v \in V(G)$. All such graphs are formed from a few 2-connected building blocks, by repeatedly applying two operations. Similar to that for degree-AT, our characterization remains unchanged in the contexts of list-coloring and paintability. We see a sharp contrast when we consider graphs G with two specified vertices x_1 and x_2 that are not f -AT, where $f(x_i) = d(x_i) - 1$ for each $i \in \{1, 2\}$ and $f(v) = d(v)$ for all other $v \in V(G)$. For Alon–Tarsi orientations, we have more than 50 exceptional graphs on seven vertices or fewer. Furthermore, the characterizations for list-coloring, paintability, and Alon–Tarsi orientations all differ.

We consider graphs with vertices labeled by natural numbers; that is, pairs (G, h) where G is a graph and $h: V(G) \rightarrow \mathbb{N}$. We focus on the case when $h(x) = 1$ for some x and $h(v) = 0$ for all other v ; we denote this labeling as h_x . We say that (G, h) is AT if G is $(d_G - h)$ -AT. When H is an induced subgraph of G , we simplify notation by referring to the pair (H, h) when we really mean $(H, h|_{V(H)})$.

Given a pair (G, h) and a specified edge $e \in E(G)$, when we *stretch* e , we form (G', h') from (G, h) by subdividing e twice and setting $h'(v_i) = 0$ for each of the two new vertices, v_1 and v_2 (and $h'(v) = h(v)$ for all other vertices v). In Section 2, we prove a Stretching Lemma, which shows that if (G, h) is not AT and $e \in E(G)$, then stretching e often yields another pair (G', h') that is also not AT. Thus, stretching plays a key role in our main result.

It is easy to check that the three pairs (G, h) shown in Figure 1 are not AT. Let \mathcal{D} be the collection of all pairs formed from the graphs in Figure 1 by stretching each bold edge 0 or more times. The Stretching Lemma implies that each pair in \mathcal{D} is not AT. Our main result is that these are the only pairs (G, h_x) , where G is 2-connected and neither a clique nor an odd cycle, such that the (G, h_x) is not AT (for some vertex $x \in V(G)$).

Main Theorem. *Let G be 2-connected and let $x \in V(G)$. Now (G, h_x) is AT if and only if*

1. $d(x) = 2$ and $G - x$ is not a Gallai tree; or
2. $d(x) \geq 3$, G is not complete, and $(G, h_x) \notin \mathcal{D}$.

Near the end of Section 4, with a little more work we extend our Main Theorem, by removing the hypothesis of 2-connectedness, to characterize all pairs (G, h_x) that are not AT. The characterization of degree-choosable graphs have been applied to prove a variety of graph coloring results [2, 3, 8, 9, 11]. Likewise, we think our main results in this paper may be helpful in proving other results for Alon–Tarsi orientations, such as giving better lower bounds on the number of edges in AT-critical graphs.

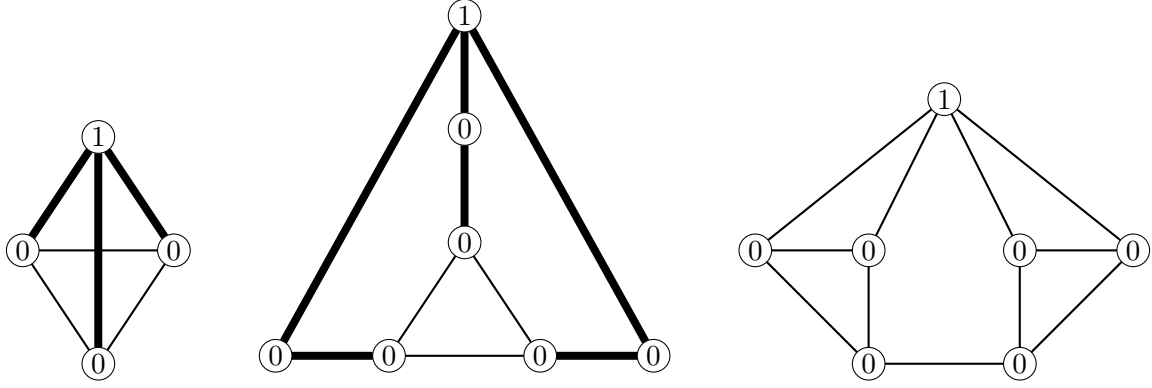


Figure 1: The seed blocks for \mathcal{D} . Each bold edge can be removed without making the figure AT.

2 Subgraphs, subdivisions, and cuts

When Hladky, Kral, and Schauz characterized degree-AT graphs, their proof relied heavily on the observation that a connected graph G is degree-AT if and only if G has some induced subgraph H such that H is degree-AT. Below, we reprove this easy lemma, and also extend it to our setting of pairs (G, h_x) .

Lemma 2.1. *Let G be a connected graph and let H be an induced subgraph of G . If H is degree-AT, then also G is degree-AT. Similarly, if $x \in V(H)$ and (H, h_x) is AT, then also (G, h_x) is AT. Further, if $x \notin V(H)$ and $d_G(x) \geq 2$, then (G, h_x) is AT.*

Proof. Suppose that H is degree-AT, and let D' be an orientation of H showing this. Extend D' to an orientation D of G by orienting all edges away from H , breaking ties arbitrarily, but consistently. Now every directed cycle in D is also a directed cycle in D' (and vice versa), so G is degree-AT. The proof of the second statement is identical. The proof of the third statement is similar, but now if some edge xy has endpoints equidistant from H , then xy should be oriented into x . \square

Recall that, given a pair (G, h) and a specified edge $e \in E(G)$, when we *stretch* e , we form (G', h') from (G, h) by subdividing e twice and setting $h'(v_i) = 0$ for each of the two new vertices, v_1 and v_2 (and $h'(v) = h(v)$ for all other vertices v). By repeatedly stretching edges, starting from the pairs in Figure 1, we form all pairs (G, h_x) , where G is 2-connected and (G, h_x) is not AT.

Stretching Lemma. *Form (G', h') from (G, h) by stretching some edge $e \in E(G)$. Now*

- (1) if (G, h) is AT, then (G', h') is AT; and
- (2) if (G', h') is AT, then either (G, h) is AT or $(G - e, h)$ is AT.

Proof. Suppose $e = u_1u_2$ and call the new vertices v_1 and v_2 so that G' contains the induced path $u_1v_1v_2u_2$. For (1), let D be an orientation of G showing that (G, h) is AT. By symmetry we may assume $u_1u_2 \in E(D)$. Form an orientation D' of G' from D by replacing u_1u_2 with the directed path $u_1v_1v_2u_2$. We have a natural parity preserving bijection between the spanning Eulerian subgraphs of D and D' , so we conclude that (G', h') is AT.

For (2), let D' be an orientation of G' showing that (G', h') is AT. Suppose G' contains the directed path $u_1v_1v_2u_2$ or the directed path $u_2v_2v_1u_1$. By symmetry, we can assume it is $u_1v_1v_2u_2$. Now form an orientation D of G by replacing $u_1v_1v_2u_2$ with the directed edge u_1u_2 . As above, we have a parity preserving bijection between the spanning Eulerian subgraphs of D and D' , so we conclude that (G, h) is AT. Otherwise, no spanning Eulerian subgraph of D' contains a cycle passing through v_1 and v_2 . So, the spanning Eulerian subgraph counts of D' are the same as those of $D' - v_1 - v_2$. However, this gives an orientation of $G - e$ showing that $(G - e, h)$ is AT. \square

Given a pair (G, h) that is not AT, the stretching Lemma] suggests a way to construct a larger graph G' such that (G', h') is not AT. Specifically, we have the following.

Corollary 2.2. *If e is an edge in G such that (G, h) is not AT and $(G - e, h)$ is not AT, then stretching e gives a pair (G', h') that is not AT.*

In some cases, we can also use the Stretching Lemma to construct a smaller graph \widehat{G} such that (\widehat{G}, h) is not AT.

Corollary 2.3. *Let G be a graph with an induced path $u_1v_1v_2u_2$ such that $d_G(v_1) = d_G(v_2) = 2$. If (G, h) is AT, where $h(v_1) = h(v_2) = 0$, and $(G - v_1 - v_2, h)$ is not AT, then*

$$((G - v_1 - v_2) + u_1u_2, h|_{V(G) \setminus \{v_1, v_2\}}) \text{ is AT.}$$

Proof. Suppose the pair (G, h) satisfies the hypotheses. Applying part (2) of the Stretching Lemma shows that either $(G - v_1 - v_2, h|_{V(G) \setminus \{v_1, v_2\}})$ is AT or $((G - v_1 - v_2) + u_1u_2, h|_{V(G) \setminus \{v_1, v_2\}})$ is AT. By hypothesis, the former is false. Thus, the latter is true. \square

With standard vertex coloring, we can easily reduce to the case where G is 2-connected. If G is a connected graph with two blocks, B_1 and B_2 , meeting at a cutvertex x , then we can color each of B_1 and B_2 independently, and afterward we can permute colorings to match at x . For Alon–Tarsi orientations, the situation is not quite as simple. However, the following lemma plays a similar role for us.

Lemma 2.4. *Let A_1, A_2 , and $x \in V(G)$ be such that $A_1 \cup A_2 = V(G)$ and $A_1 \cap A_2 = \{x\}$. If $G[A_i]$ is f_i -AT for each $i \in \{1, 2\}$, then G is f -AT, where $f(v) = f_i(v)$ for each $v \in V(A_i - x)$ and $f(x) = f_1(x) + f_2(x) - 1$. Going the other direction, if G is f -AT, then $G[A_i]$ is f_i -AT for each $i \in \{1, 2\}$, where $f_i(v) = f(v)$ for each $v \in V(A_i - x)$ and $f_1(x) + f_2(x) \leq f(x) + 1$.*

Proof. We begin with the first statement. For each $i \in \{1, 2\}$, choose an orientation D_i of A_i showing that A_i is f_i -AT. Together these D_i give an orientation D of G . Since no cycle has vertices in both $A_1 - x$ and $A_2 - x$, we have

$$\begin{aligned} EE(D) - EO(D) &= EE(D_1)EE(D_2) + EO(D_1)EO(D_2) - EE(D_1)EO(D_2) - EO(D_1)EE(D_2) \\ &= (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2)) \\ &\neq 0. \end{aligned}$$

Hence G is f -AT.

Now we prove the second statement. Suppose that G is f -AT and choose an orientation D of G showing this. Let $D_i = D[A_i]$ for each $i \in \{1, 2\}$. As above, we have $0 \neq EE(D) - EO(D) = (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2))$. Hence, $EE(D_1) - EO(D_1) \neq 0$ and $EE(D_2) - EO(D_2) \neq 0$. Since the indegree of x in D is the sum of the indegree of x in D_1 and the indegree of x in D_2 , the lemma follows. \square

3 Degree-AT graphs and an Extension Lemma

Recall that our Main Theorem relies on a characterization of degree-AT graphs. As we mentioned in the introduction, a description of degree-choosable graphs was first given by Vizing [12] and Erdős, Rubin, and Taylor [4]. Hladky, Kral, and Schaud [5] later extended the proof from [4] to Alon–Tarsi orientations. This proof relies on Rubin’s Block lemma, which states that every 2-connected graph G contains an induced even cycle with at most one chord, unless G is a clique or an odd cycle. For variety, and completeness, we include a new proof; it extends ideas of Kostochka, Stiebitz, and Wirth [7] from list-coloring to Alon–Tarsi orientations. For this proof we need the following very special case of a key lemma in [6]. When vertices x and y are adjacent, we write $x \leftrightarrow y$; otherwise $x \nleftrightarrow y$.

Lemma 3.1. *Let G be a graph and $x \in V(G)$ such that H is connected, where $H := G - x$. If there exist $z_1, z_2 \in V(H)$ with $N_H[z_1] = N_H[z_2]$ such that $x \leftrightarrow z_1$ and $x \nleftrightarrow z_2$, then G is f -AT where $f(x) = 2$ and $f(v) = d_G(v)$ for all $v \in V(H)$.*

Proof. Order the vertices of H with z_1 first and z_2 second so that every vertex, other than z_1 , has at least one neighbor preceding it. Orient each edge of H from its earlier endpoint toward its later endpoint. Orient xz_1 into z_1 and orient all other edges incident to x into x . Let D be the resulting orientation. Clearly, $d_D^+(v) \leq f(v) - 1$ for all $v \in V(D)$. So, we just need to check that $EE(D) \neq EO(D)$.

Since xz_1 is the only edge of D leaving x , and $D - x$ is acyclic, every spanning Eulerian subgraph of D that has edges must have edge xz_1 . Consider an Eulerian subgraph A of D containing xz_1 . Since z_1 has indegree 1 in A , it must also have outdegree 1 in A . We show that A has a mate A' of opposite parity. If $z_2 \in A$ then $z_1z_2w \in A$, for some w , so we form A' from A by removing z_1z_2w and adding z_1w . If instead $z_1z_2 \notin A$, then $z_2 \notin A$ and $z_1w \in A$ for some $w \in N_H[z_1] - z_2$, so we form A' from A by removing z_1w and adding z_1z_2w . Hence exactly half of the Eulerian subgraphs of D that contain edges are even. Since the edgeless spanning subgraph of D is an even Eulerian subgraph, we conclude that $EE(D) = EO(D) + 1$. Hence G is f -AT. \square

We use the previous lemma to give a new proof of the characterization of degree-AT graphs.

Lemma 3.2. *A connected graph G is degree-AT if it is not a Gallai tree.*

Proof. Suppose there exists a connected graph that is not a Gallai tree, but is also not degree-AT. Let G be such a graph with as few vertices as possible. Since G is not degree-AT, no induced subgraph H of G is degree-AT by Lemma 2.1. Hence, for any $v \in V(G)$ that is not a cutvertex, $G - v$ must be a Gallai tree by minimality of $|G|$.

If G has more than one block, then for endblocks B_1 and B_2 , choose noncutvertices $w \in B_1$ and $x \in B_2$. By the minimality of $|G|$, both $G - w$ and $G - x$ are Gallai trees. Since every block of G appears either as a block of $G - w$ or as a block of $G - x$, every block of G is either complete or an odd cycle. Hence, G is a Gallai tree, a contradiction. So instead G has only one block, that is, G is 2-connected. Further, $G - v$ is a Gallai tree for all $v \in V(G)$.

Let v be a vertex of minimum degree in G . Since G is 2-connected, $d_G(v) \geq 2$ and v is adjacent to a noncutvertex in every endblock of $G - v$. If $G - v$ has a complete block B with noncutvertices x_1, x_2 where $v \leftrightarrow x_1$ and $v \not\leftrightarrow x_2$, then we can apply Lemma 3.1 to conclude that G is degree-AT, a contradiction. So, v must be adjacent to every noncutvertex in every complete endblock of $G - v$.

Suppose $d_G(v) \geq 3$. Now no endblock of $G - v$ can be an odd cycle of length at least 5 (G would have vertices of degree 3 and also $d_G(v) \geq 4$, contradicting the minimality of $d_G(v)$). Let B be a smallest complete endblock of $G - v$. Now for a noncutvertex $x \in V(B)$, we have $d_G(x) = |B|$ and hence $d_G(v) \leq |B|$. If $G - v$ has at least two endblocks, then $2(|B| - 1) \leq |B|$, so $d_G(v) \leq |B| = 2$, a contradiction. Hence, $G - v = B$ and v is joined to B , so G is complete, which is a contradiction.

Thus, we have $d_G(v) = 2$. Suppose $G - v$ has at least two endblocks. Now it has exactly two and v is adjacent to one noncutvertex in each. Neither of the endblocks can be odd cycles of length at least five, since then we can get a smaller counterexample by the Stretching Lemma. Since v is adjacent to every noncutvertex in every complete endblock of $G - v$, both endblocks must be K_2 . But now either $G = C_4$ (which is trivially degree-AT) or we can get a smaller counterexample by the Stretching Lemma. So, $G - v$ must be 2-connected. Since $G - v$ is a Gallai tree, it is either complete or an odd cycle. If $G - v$ is not complete, then we can get a smaller counterexample by the Stretching Lemma. So, $G - v$ is complete and v is adjacent to every noncutvertex of $G - v$; that is, G is complete, a contradiction. \square

4 When h is 1 for at most one vertex

For a graph G and $x \in V(G)$ recall that $h_x: V(G) \rightarrow \mathbb{N}$ is defined as $h_x(x) = 1$ and $h_x(v) = 0$ for all $v \in V(G - x)$. We classify the connected graphs G such that (G, h_x) is AT for some $x \in V(G)$. We begin with the case when G is 2-connected, which takes most of the work. At the end of the section, we extend this to all connected graphs.

We will show that for most 2-connected graphs G and vertices $x \in V(G)$, the pair (G, h_x) is AT. Specifically, this is true for all pairs except those in \mathcal{D} , defined in the introduction. In view of Lemma 2.1, for a 2-connected graph G and $x \in V(G)$, to show (G, h_x) is AT it

suffices to find some induced subgraph H , with $x \in V(H)$, such that (H, h_x) is AT. The subgraphs H that we consider all have $d_H(x) \geq 3$. This motivates the next lemma, which allows us to reduce to the case $d_G(x) \geq 3$.

Lemma 4.1. *If G is a connected graph and $x \in V(G)$ with $d_G(x) = 2$, then (G, h_x) is AT if and only if $G - x$ is degree-AT.*

Proof. Let D be an orientation of G showing that (G, h_x) is AT. Then $d_D^-(x) = 2$ and hence no spanning Eulerian subgraph contains a cycle passing through x . Therefore, the Eulerian subgraph counts in $G - x$ are different and $G - x$ is degree-AT. The other direction is immediate from Lemma 2.1. \square

A θ -graph, $\Theta_{a,b,c}$, consists of two 3-vertices joined by three internally disjoint paths, with lengths a , b , and c . We will see shortly that if H is a θ -graph with $d_H(x) = 3$, then (H, h_x) is AT. Thus, we can assume that G has no induced θ -graph H with $d_H(x) = 3$. All of the other forbidden subgraphs H that we consider (and show that (H, h_x) is AT) can be viewed as θ -graphs with at most three extra edges. We show if G is 2-connected with $x \in V(G)$ and $d(x) \geq 3$, then either (i) $G \in \mathcal{D}$ or (ii) G contains one of these forbidden “nearly θ ” graphs.

Lemma 4.2. *If H is a θ -graph, $\Theta_{a,b,c}$, with $d_H(x) = 3$, then (H, h_x) is AT (regardless of a , b , and c).*

Proof. By the Stretching Lemma, this follows directly from the fact that the four pairs shown in Figure 2 are AT. \square

The simplest of our nearly θ -graphs are called T graphs (they have a single extra edge). For positive integers a , b , and c , let $T_{a,b,c}$ be the graph consisting of (i) a triangle $z_1 z_2 z_3$ and (ii) disjoint paths $z_i P_i w_i$ where P_i has length a_i for $i \in \{1, 2, 3\}$ and (iii) a vertex x adjacent to w_1, w_2, w_3 .

Lemma 4.3. *$(T_{a,b,c}, h_x)$ is AT whenever exactly one or two of a, b , and c is even.*

Proof. By the Stretching Lemma, this follows from the fact that Figure 4 and Figure 5 are AT. \square

The graph we consider next can be viewed as a θ -graph, with the three extra edges among z_1, z_2 , and z_3 .

Lemma 4.4. *Form H from $T_{a,b,c}$ by adding a new vertex with neighborhood $\{z_1, z_2, z_3\}$. Now (H, h_x) is AT.*

Proof. This follows from Lemma 4.3 and Lemma 2.1 using Lemma 2 and the fact that Figure 3 is AT and Figure 7 is AT. \square

Lemma 4.5. *Let G be a T -graph. Let P be a path of G where all internal vertices of P have degree 2 in G and one endvertex of P has degree 2 in G . Form G' from G by adding a path P' (of length at least 2) joining the endvertices of P . Now (G', h_x) is AT.*

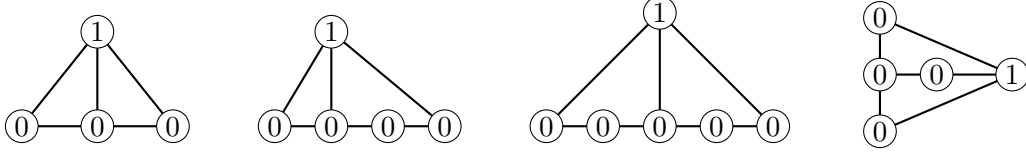


Figure 2: These are AT.

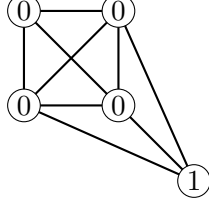


Figure 3: This is AT.

Proof. We can assume that G is not AT; otherwise, we are done by Lemma 2.1. By symmetry, assume P is a subpath P_3 . First, we get an orientation of G with indegree at least 1 for all vertices and $d^-(x) = 2$. Orient P_1 from z_1 to x , P_2 from z_2 to x , P_3 from x to z_3 , and the triangle as z_3z_2 , z_3z_1 , z_1z_2 . To get an orientation of G' , orient the new path P' consistently, and opposite of P . Now the only directed cycle containing edges of P' is $P'P$. Since the Eulerian subgraph counts are equal for G , they differ by 1 for G' . \square

Lemma 4.6. *If there exists $H \subsetneq G$ such that $d_H(x) = 4$ and H is the Moser spindle, shown in Figure 1c, then (G, h_x) is AT.*

Lemma 4.7. *Let G be 2-connected, and choose $x \in V(G)$ with $d(x) \geq 3$. Now (G, h_x) is AT if and only if G is not complete and $(G, h_x) \notin \mathcal{D}$.*

Proof. First suppose that $(G, h_x) \in \mathcal{D}$. Add details.

Now let G be 2-connected, choose $x \in V(G)$ with $d(x) \geq 3$, and suppose that $(G, h_x) \notin \mathcal{D}$. Since $G - x$ is connected, let H' be a smallest connected subgraph of $G - x$ containing three neighbors of x ; call these neighbors w , y , and z . Consider a spanning tree T of H' . Since H' is minimum, each leaf of T is among $\{w, y, z\}$. If T is a path, then H' is also a path. Otherwise, T is a subdivision of $K_{1,3}$. Let s be the vertex with $d_T(s) = 3$. If $E(G) - E(T)$ has any edge with both ends outside of $N(s)$, then we can delete some vertex in $N(s)$ and remain connected, contradicting the minimality of H' . Similarly, if $N(s)$ contains at least two edges, then $H' - s$ still connects, w , y , and z . Now let H be the subgraph of G induced

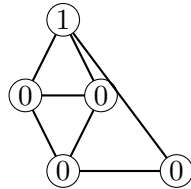


Figure 4: This is AT.

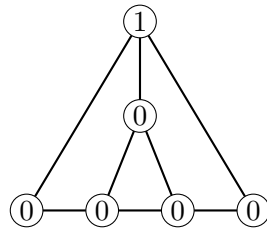


Figure 5: This is AT.

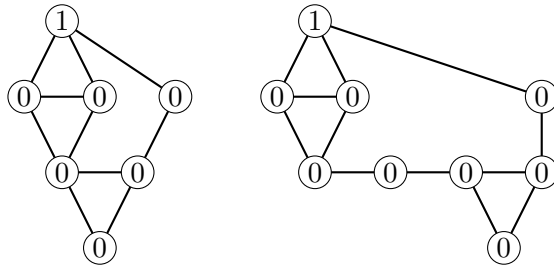


Figure 6: These are AT.

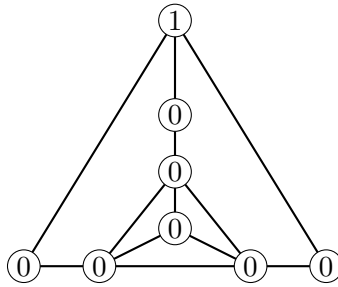


Figure 7: This is AT.

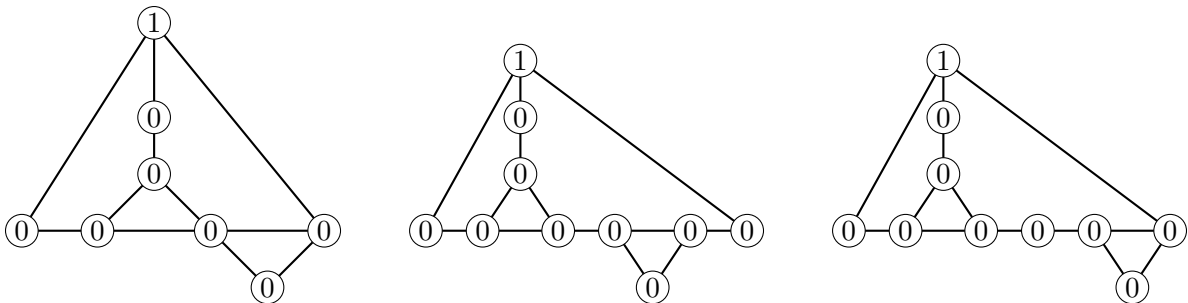


Figure 8: These are AT.

by $V(H') \cup \{x\}$. Note that H is either a θ -graph (if H' is a tree) or a T -graph (if H' has one extra edge in $N(s)$).

If H is a θ -graph, then (G, h_x) is AT, by Lemma 4.2 and Lemma 2.1. So assume H is a T -graph. Let z_1, z_2, z_3 be the vertices of degree 3 (other than x), and let P_1, P_2 , and P_3 denote the paths from x to z_1, z_2 , and z_3 ; when we write $V(P_i)$, we exclude x and z_i , so possibly $V(P_i)$ is empty for one or more $i \in \{1, 2, 3\}$. If any two of P_1, P_2 , and P_3 have lengths with opposite parities, then we are done by Lemma 4.3; so assume not.

Now $(H, h_x) \in \mathcal{D}$, so we can assume there exists $u \in V(G - H)$. Since G is 2-connected, consider how u can attach to H . In particular, u must have at least two disjoint paths (possibly single edges), P_4 and P_5 , to H . Choose u to minimize the sum of the lengths of P_4 and P_5 . We consider the following cases.

First suppose $|N_H(u)| \leq 2$. Consider the subgraph induced by H, u , and the vertices of P_4 and P_5 . First suppose that P_4 and P_5 both end at vertices in $\{x, z_1, z_2, z_3\}$. If one of these endpoints is x , then we have a θ -graph J with $d_J(x) = 3$, so we are done by Lemma 4.2 and Lemma 2.1. If neither path ends at x , then we have a θ -subgraph J with $x \notin J$, so we are again done by the same two lemmas. Otherwise, at least one of P_4 and P_5 ends at a vertex of degree 2 in H . Now we are done by Lemma 4.5.

Instead assume $|N_H(u)| \geq 3$. First suppose that $u \nleftrightarrow x$. Suppose further that $\{z_1, z_2, z_3\} \subseteq N_H(u)$. If u has only these three neighbors, then we are done by Figure 3 or Figure 7 (and the stretching lemma). So suppose u has another neighbor, w . Choose such a w as close as possible to z_1, z_2, z_3 ; by symmetry, say that w is on P_1 . Now $H - x$ has a θ -graph induced by u, w, z_1, z_2 , and the portion of P_1 from w to z_1 . Thus, we conclude that $\{z_1, z_2, z_3\} \not\subseteq N_H(u)$. If u has exactly two neighbors in $\{z_1, z_2, z_3\}$, then u, z_1, z_2, z_3 induces $K_4 - e$, which is a θ -graph; so assume not. Suppose u has neighbors in $V(P_i) \cup \{z_i\}$ for at least two $i \in \{1, 2, 3\}$, say $i = 1$ and $i = 2$; by symmetry, we can assume that $u \nleftrightarrow z_3$. Consider the graph induced by u, z_1, z_2, z_3 , and the paths from the neighbor of u on P_1 nearest to z_1 and the neighbor of u on P_2 nearest to z_2 . This subgraph is a θ -graph. Thus, there must exist some i such that $N_H(u) \subseteq z_i \cup V(P_i)$. Since $d_H(u) \geq 3$, $H - x$ contains a θ -graph, so we are done.

Instead, assume that $u \leftrightarrow x$. First suppose that u has a neighbor in $V(P_1) \cup \{z_1\}$ and u also has a neighbor in $V(P_2)$. Let w_1 be the neighbor of u nearest to x along P_1 and let w_2 be the neighbor of u nearest to x along P_2 . Consider the subgraph J induced by x, u, w_1, w_2 , and the paths along P_1 and P_2 from x to w_1 and w_2 . Since J is a θ -graph with $d_J(x) = 3$, we are done. So u must not have neighbors in these two sets. Thus, by symmetry, we can assume either (i) $N(u) \subseteq V(P_1) \cup \{z_1\}$ or (ii) $N(u) \subseteq \{x, z_1, z_2, z_3\}$.

Suppose we are in case (i). Since $d_H(u) \geq 3$ (and by the minimality of H' , and thus H), vertex u is adjacent to precisely x and the first two vertices along P_1 (one of which is possibly z_1). If P_1 or P_2 has vertices, then let J be the subgraph induced by u and $V(H) \setminus V(P_3)$. Note that $J - z_3$ is a θ -graph, so J is reducible by Lemma 4.5. Thus, we assume that $V(P_2) = V(P_3) = \emptyset$. This implies that u is not adjacent to z_1 , since all $V(P_i)$ have the same parity, by Lemma 4.3. So $|V(P_1)|$ is even. If $|V(P_1)| \geq 4$, then consider the subgraph J induced by $\{u\} \cup N_H(u) \cup (V(H) \setminus V(P_1))$. Note that J has two blocks and x is its cutvertex. Since neither block is a clique or an odd cycle, (J, h_x) is AT by Lemma 2.4. Thus, we conclude that $|V(P_1)| = 2$, so H is the Moser Spindle, shown in Figure 1c. Since $(H, h_x) \in \mathcal{D}$, we can assume that $H \subsetneq G$. However, now we are done by Lemma 4.6.

Suppose instead that we are in case (ii). First, suppose there exists z_i such that $u \nleftrightarrow z_i$.

Now u, z_1, z_2, z_3 induce a θ -graph in $H - x$, so we are done by Lemmas 4.2 and 2.1. Thus, we assume that $N_H(u) = \{x, z_1, z_2, z_3\}$. Let J be a maximum clique containing x . We can assume that G is not complete, so there exists some vertex w outside of J . Since G is 2-connected, w has two vertex disjoint paths to J , not both ending at the same vertex; let J' denote the subgraph induced by x, J , and the vertices of these two paths. If neither path ends at x , then $J' - x$ is 2-connected and not a clique or an odd cycle. Now Lemmas 3.2 and 2.1 imply that (G, h_x) is AT.

Suppose instead that one path from w to J ends at x ; by symmetry, assume the other ends at z_1 . Since $w \notin V(J)$, vertex w has some non-neighbor in J ; by symmetry, say it is z_2 . Now let J'' be the subgraph induced by x, z_1, z_2, w , and the paths from w to x and z_1 . Note that J'' is a θ -graph with $d_{J''}(x) = 3$, so Lemmas 4.2 and 2.1 imply that (G, h_x) is AT. \square

Theorem 4.8. *If G is connected and $x \in V(G)$, then (G, h_x) is not AT if and only if*

- (1) $d(x) = 1$; or
- (2) G is a Gallai tree; or
- (3) $d(x) = 2$ and $G - x$ has a component that is a Gallai tree; or
- (4) x is a cutvertex, all but at most one x -lobe of G is a Gallai tree and for every x -lobe H of G , we have (G, h_x) is not AT; or
- (5) x is not a cutvertex and for the block B of G containing x , we have $(B, h_x) \in \mathcal{D}$ and all B -lobes of G are Gallai trees.

Proof. First, let's check that if any of (1)–(5) happen, then (G, h_x) is not AT. 1 and 2 are immediate. 3 follows from Lemma 4.2. 4 follows from Lemma 2.2. 5 follows from Lemma 4.9 and Lemma 2.2.

Now, for the other direction, suppose (G, h_x) is not AT and none of (1)–(5) happen. By Lemma 4.2 (and not 1 and not 3) we must have $d(x) \geq 3$. Suppose x is a cutvertex. Then (by not 4), at least two x -lobes of G are not Gallai trees or (H, h_x) is AT for some x -lobe H of G . But then (G, h_x) is AT by Lemma 2.2, a contradiction. So, x is not a cutvertex. Suppose the block B of G containing x is complete or $(B, h_x) \in \mathcal{D}$. Now (by not 2 and not 5) some B -lobe H of G is not a Gallai tree. Since $G - x$ is connected, we conclude that $G - x$ is degree-AT and by directing all edges incident to x into x , we get that (G, h_x) is AT, a contradiction. By Lemma 2.2, we know that (B, h_x) is not AT. Since B is 2-connected, the argument in the Lemma in the last email shows that B is h_x -**minimal**. We just learned that B is not complete and $(B, h_x) \notin \mathcal{D}$, so applying Lemma 4.9 shows that (B, h_x) is AT, a contradiction. \square

5 Choosability and Paintability

As we mentioned in the introduction, Alon and Tarsi showed that if a graph G is f -AT, then G is also f -choosable. *Online list coloring*, also called *painting* is similar to list coloring, but now the list for each vertex is progressively revealed, as the graph is colored. Schauz [10] extended

the Alon–Tarsi theorem, to show that if G is f -AT, then G is also f -paintable. In this section, we use our characterization of pairs (G, h_x) that are not AT to prove characterizations of pairs (G, h_x) that are not paintable and that are not choosable. More precisely, a pair (G, h_x) is *choosable* if G has a proper coloring from its lists L whenever L is such that $|L(x)| = d(x) - 1$ and $|L(v)| = d(v)$ for all other v ; otherwise (G, h_x) is *not choosable*. The definition of a pair being paintable is analogous. We characterize all pairs (G, h_x) , where G is connected and (G, h_x) is not choosable (resp. paintable). We will see that, in fact, these characterizations (for both choosability and paintability) are identical to that for pairs that are not AT.

Since all pairs (G, h_x) that are AT are also both paintable and choosable, it suffices to show that every pair (G, h_x) that is not AT is also not choosable (here we use that if a pair is paintable, then it is also choosable).

Theorem 5.1. *For every connected graph G , the pair (G, h_x) is not choosable if and only if (G, h_x) is not AT. Thus, the same characterization holds for pairs that are not paintable.*

Also, show that when we move up to two vertices x, y with $h(x) = h(y) = 1$, AT, choosability and paintability all separate.

6 Applications

When this is in a critical graph, we get h_x -minimality for free. If x has $t \geq 5$ neighbors in one component of the low vertex subgraph, then those neighbors are all in one K_t block. If $t = 4$, then the neighbors are either a K_4 block or we create the Moser Spindle.

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