

Coloring graphs with dense neighborhoods

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Abstract

It is shown that any graph with maximum degree Δ in which the average degree of the induced subgraph on the set of all neighbors of each vertex exceeds $\frac{6k^2}{6k^2+1}\Delta + k + 6$ is either $(\Delta - k)$ -colorable or contains a clique on more than $\Delta - 2k$ vertices. In the $k = 1$ case we improve the bound on the average degree to $\frac{2}{3}\Delta + 4$ and the bound on the clique number to $\Delta - 1$. As corollaries, we show that every graph satisfies $\chi \leq \max\{\omega, \Delta - 1, 4\alpha\}$ and every graph satisfies $\chi \leq \max\left\{\omega, \Delta - 1, \left\lceil \frac{15 + \sqrt{48n + 73}}{4} \right\rceil\right\}$.

1 Introduction

Using ideas developed for strong coloring by Haxell [7] and by Aharoni, Berger and Ziv [1], we make explicit a recoloring technique and apply it to coloring graphs with dense neighborhoods. Our notation follows Diestel [6] unless otherwise specified. One notable point of departure is our use of K_t instead of K^t for the complete graph on t vertices. Reed [15] has conjectured that every graph satisfies

$$\chi \leq \left\lceil \frac{\omega + \Delta + 1}{2} \right\rceil.$$

Our first result implies this conjecture without the round-up for graphs where every vertex is in a big clique.

Theorem 6.4. *Let $k \geq 1$. Every graph G with $\omega(G) \leq \Delta(G) - 2k$ such that every vertex is in a clique on $\frac{2k}{2k+1}\Delta(G) + 2k + 1$ vertices is $(\Delta(G) - k)$ -colorable.*

Using probabilistic methods, Reed [15] proved a similar-looking result that is much better for very large k and Δ . For comparison, we modify the statement to look as close to Theorem 6.4 as possible.

Theorem 1.1 (Reed [15]). *There exists Δ_0 such that for $k \geq 0$ every graph G with $\Delta(G) \geq \Delta_0$ and $\frac{69999999}{70000000}\Delta(G) \leq \omega(G) \leq \Delta(G) - 2k$ is $(\Delta(G) - k)$ -colorable.*

This implies Theorem 6.4 when k gets to be larger than around 35 million. In fact, Reed states that with some care the constant can be brought down to $\frac{9999}{10000}$ and so really his method starts implying Theorem 6.4 when k gets larger than 5000. Moreover, Theorem 1.1 just needs a large clique while Theorem 6.4 requires every vertex to be in a large clique.

It turns out that if every neighborhood has many edges, it is guaranteed that every vertex is in a large clique. This implies the following. Recall that the *average degree* of a graph G is $d(G) := \frac{2\|G\|}{|G|}$.

Theorem 6.6. *Let $k \geq 0$. Every graph G with $\omega(G) \leq \Delta(G) - 2k$ such that $d(G[N(v)]) \geq \frac{6k^2}{6k^2+1}\Delta(G) + k + 6$ for each $v \in V(G)$ is $(\Delta(G) - k)$ -colorable.*

To tighten these results up, further development of the theory of f -choosability where $f(v) = d(v) - k$ for $k \geq 2$ is needed. For $k = 1$ this theory was developed in [5] and using it in the case of $(\Delta - 1)$ coloring, we achieve tighter bounds which have bearing on the conjecture of Borodin and Kostochka [3].

Conjecture 1.2 (Borodin and Kostochka [3]). *Every graph G with $\chi(G) \geq \Delta(G) \geq 9$ contains $K_{\Delta(G)}$.*

Also using probabilistic methods, Reed [16] has proved this conjecture for very large Δ . Using d_1 -choosability theory, we prove the following.

Theorem 5.3. *Every graph G with $\chi(G) \geq \Delta(G) \geq 9$ such that every vertex is in a clique on $\frac{2}{3}\Delta(G) + 2$ vertices contains $K_{\Delta(G)}$.*

From this it follows that it would be enough to prove the Borodin-Kostochka conjecture for irregular graphs.

Theorem 5.4. *Every graph G satisfying $\chi(G) \geq \Delta(G) = k \geq 9$ either contains K_k or contains an irregular critical subgraph satisfying $\chi(G) = \Delta(G) = k - 1$.*

We also get a neighborhood density version.

Theorem 5.9. *Every graph G with $\omega(G) < \Delta(G)$ such that $d(G[N(v)]) \geq \frac{2}{3}\Delta(G) + 4$ for each $v \in V(G)$ is $(\Delta(G) - 1)$ -colorable.*

Finally, we use these ideas to prove the following bounds on the chromatic number. The first generalizes the result of Beutelspacher and Hering [2] that the Borodin-Kostochka conjecture holds for graphs with independence number at most two. This result was generalized in another direction in [4] where the conjecture was proved for claw-free graphs.

Theorem 5.11. *Every graph G satisfies $\chi(G) \leq \max\{\omega(G), \Delta(G) - 1, 4\alpha(G)\}$.*

The second bound shows that the Borodin-Kostochka conjecture holds for graphs with maximum degree on the order of the square root of their order. This improves on prior bounds of $\Delta > \frac{n+1}{2}$ from Beutelspacher and Hering [2] and $\Delta > \frac{n-6}{3}$ of Naserasr [14].

Theorem 5.12. *Every graph G satisfies $\chi(G) \leq \max\left\{\omega(G), \Delta(G) - 1, \left\lceil \frac{15 + \sqrt{48|G| + 73}}{4} \right\rceil\right\}$.*

2 Strong coloring

For a positive integer r , a graph G with $|G| = rk$ is called *strongly r -colorable* if for every partition of $V(G)$ into parts of size r there is a proper coloring of G that uses all r colors on each part. If $|G|$ is not a multiple of r , then G is strongly r -colorable iff the graph formed by adding $r \left\lceil \frac{|G|}{r} \right\rceil - |G|$ isolated vertices to G is strongly r -colorable. The *strong chromatic number* $s\chi(G)$ is the smallest r for which G is strongly r -colorable.

Note that a strong r -coloring of G with respect to a partition V_1, \dots, V_k of $V(G)$ with $|V_i| = r$ must partition $V(G)$ into r independent transversals of V_1, \dots, V_k . In [18], Szabó and Tardos constructed partitioned graphs with part sizes $2\Delta - 1$ that have no independent transversal. So we must have $s\chi(G) \geq 2\Delta(G)$.

Haxell [7] proved that $s\chi(G) \leq 3\Delta(G) - 1$. Aharoni, Berger and Ziv [1] gave a simple proof that $s\chi(G) \leq 3\Delta(G)$. It is this latter proof whose recoloring technique we use. First we need a lemma allowing us to pick an independent transversal when one of the sets has only one element. We'll skip the proof since it is in [1] and is a special case of Lemma 3.2 proved below.

Lemma 2.1. *Let H be a graph and $V_1 \cup \dots \cup V_r$ a partition of $V(H)$. Suppose that $|V_i| \geq 2\Delta(H)$ for each $i \in [r]$. If a graph G is formed by attaching a new vertex x to fewer than $2\Delta(H)$ vertices of H , then G has an independent set $\{x, v_1, \dots, v_r\}$ where $v_i \in V_i$ for each $i \in [r]$.*

Theorem 2.2. *Every graph G satisfies $s\chi(G) \leq 3\Delta(G)$.*

Proof. We only need to prove that any graph G with $n := 3\Delta(G)k$ vertices have a $3\Delta(G)$ coloring for each $k \geq 1$. Suppose not and choose a counterexample G minimizing $\|G\|$. Put $r := 3\Delta(G)$ and let V_1, \dots, V_k be a partition of G for which there is no acceptable coloring. Then the V_i are independent by minimality of $\|G\|$. By symmetry we may assume there are adjacent vertices $x \in V_1$ and $y \in V_2$. Apply minimality of $\|G\|$ to get an r -coloring π of $G - xy$ with $\pi(V_i) = [r]$ for each $i \in [k]$. We will modify π to get such a coloring of G .

By symmetry, we may assume that $\pi(x) = \pi(y) = 1$. For $2 \leq i \leq k$, let z_i be the unique element of $\pi^{-1}(1) \cap V_i$ and put $W_i := V_i - \{v \in V_i \mid \pi(v) = \pi(w) \text{ for some } w \in N(z_i)\}$. Then $|W_i| \geq 2\Delta(G)$ and we may apply Lemma 2.1 to get a G -independent transversal w_1, w_2, \dots, w_k of $\{x\}, W_2, W_3, \dots, W_k$. Define a new coloring ζ of G by

$$\zeta(v) := \begin{cases} 1 & \text{if } v = w_i \\ \pi(w_i) & \text{if } v = z_i \\ \pi(v) & \text{otherwise.} \end{cases}$$

Then ζ is a proper coloring of G with $\zeta(V_i) = [r]$ for each $i \in [k]$, a contradiction. \square

3 The recoloring technique

We can extract the idea in the proof of Theorem 2.2 to get a general recoloring technique. Suppose G is a k -vertex-critical graph and pick $x \in V(G)$ and $(k-1)$ -coloring

π of $H := G - x$. Let Z be a color class of π , say $Z = \pi^{-1}(1)$. For each $z \in Z$, let O_z be the neighbors of z which get a color that no other neighbor of z gets; that is, put $O_z := \{v \in N_H(z) \mid \pi(v) \notin \pi(N_H(z) - v)\}$. Suppose the O_z are pairwise disjoint. If we could find an independent transversal $\{x\} \cup \{v_z\}_{z \in Z}$ of $\{x\}$ together with the O_z , then recoloring each $z \in Z$ with $\pi(v_z)$ and coloring each vertex in $\{x\} \cup \{v_z\}_{z \in Z}$ with 1 gives a proper $(k-1)$ -coloring of G . This is exactly what happens in the above proof of the strong coloring result. To make this work more generally, we need to find situations where each $G[O_z]$ has high minimum degree. Also, intuitively, the O_z intersecting each other should make things easier since recoloring a vertex in the intersection of O_{z_1} and O_{z_2} works for both z_1 and z_2 . In our applications we will allow some restricted intersections.

3.1 A lopsided transversal lemma

We will need a lopsided version of Lemma 2.1 generalizing King's [11] lopsided version of Haxell's lemma. In [9], Haxell and Szabó developed a technique for dealing with independent transversals. In [8], Haxell used this technique to give simpler proof of her transversal lemma. The proof gives a bit more and we record that here. This is just slightly more general than the extension given in [1] by Aharoni, Berger and Ziv. We write $f: A \twoheadrightarrow B$ for a surjective function from A to B . Let G be a graph. For a k -coloring $\pi: V(G) \twoheadrightarrow [k]$ of G and a subgraph H of G we say that $I := \{x_1, \dots, x_k\} \subseteq V(H)$ is an H -independent transversal of π if I is an independent set in H and $\pi(x_i) = i$ for all $i \in [k]$.

Lemma 3.1. *Let G be a graph and $\pi: V(G) \twoheadrightarrow [k]$ a proper k -coloring of G . Suppose that π has no G -independent transversal, but for every $e \in E(G)$, π has a $(G - e)$ -independent transversal. Then for every $xy \in E(G)$ there is $J \subseteq [k]$ with $\pi(x), \pi(y) \in J$ and an induced matching M of $G[\pi^{-1}(J)]$ with $xy \in M$ such that:*

1. $\bigcup M$ totally dominates $G[\pi^{-1}(J)]$,
2. the multigraph with vertex set J and an edge between $a, b \in J$ for each $uv \in M$ with $\pi(u) = a$ and $\pi(v) = b$ is a (simple) tree. In particular $|M| = |J| - 1$.

Proof. Suppose the lemma is false and choose a counterexample G with $\pi: V(G) \twoheadrightarrow [k]$ so as to minimize k . Let $xy \in E(G)$. By assumption π has a $(G - xy)$ -independent transversal T . Note that we must have $x, y \in T$ lest T be a G -independent transversal of π .

By symmetry we may assume that $\pi(x) = k-1$ and $\pi(y) = k$. Put $X := \pi^{-1}(k-1)$, $Y := \pi^{-1}(k)$ and $H := G - N(\{x, y\}) - E(X, Y)$. Define $\zeta: V(H) \rightarrow [k-1]$ by $\zeta(v) := \min\{\pi(v), k-1\}$. Note that since $x, y \in T$, we have $|\zeta^{-1}(i)| \geq 1$ for each $i \in [k-2]$. Put $Z := \zeta^{-1}(k-1)$. Then $Z \neq \emptyset$ for otherwise $M := \{xy\}$ totally dominates $G[X \cup Y]$ giving a contradiction.

Suppose ζ has an H -independent transversal S . Then we have $z \in S \cap Z$ and by symmetry we may assume $z \in X$. But then $S \cup \{y\}$ is a G -independent transversal of π , a contradiction.

Let $H' \subseteq H$ be a minimal spanning subgraph such that ζ has no H' -independent transversal. Now $d(z) \geq 1$ for each $z \in Z$ for otherwise $T - \{x, y\} \cup \{z\}$ would be an H' -independent transversal of ζ . Pick $zw \in E(H')$. By minimality of k , we have $J \subseteq [k-1]$ with $\zeta(z), \zeta(w) \in J$ and an induced matching M of $H'[\zeta^{-1}(J)]$ with $zw \in M$ such that

1. $\bigcup M$ totally dominates $H'[\zeta^{-1}(J)]$,
2. the multigraph with vertex set J and an edge between $a, b \in J$ for each $uv \in M$ with $\zeta(u) = a$ and $\zeta(v) = b$ is a (simple) tree.

Put $M' := M \cup \{xy\}$ and $J' := J \cup \{k\}$. Since H' is a spanning subgraph of H , $\bigcup M$ totally dominates $H[\zeta^{-1}(J)]$ and hence $\bigcup M'$ totally dominates $G[\pi^{-1}(J')]$. Moreover, the multigraph in (2) for M' and J' is formed by splitting the vertex $k-1 \in J$ into two vertices and adding an edge between them and hence it is still a tree. This final contradiction proves the lemma. \square

Now the needed lopsided transversal lemma follows quickly.

Lemma 3.2. *Let H be a graph and $V_1 \cup \dots \cup V_r$ a partition of $V(H)$. Suppose there exists $t \geq 1$ such that for each $i \in [r]$ and each $v \in V_i$ we have $d(v) \leq \min\{t, |V_i| - t\}$. For any $S \subseteq V(H)$ with $|S| < \min\{|V_1|, \dots, |V_r|\}$, there is an independent transversal I of V_1, \dots, V_r with $I \cap S = \emptyset$.*

Proof. Suppose the lemma fails for such an $S \subseteq V(H)$. Put $H' := H - S$ and let V'_1, \dots, V'_r be the induced partition of H' . Then there is no independent transversal of V'_1, \dots, V'_r and $|V'_i| \geq 1$ for each $i \in [r]$. Create a graph Q by removing edges from H' until it is edge minimal without an independent transversal. Pick $yz \in E(Q)$ and apply Lemma 3.1 on yz with the induced partition to get the guaranteed $J \subseteq [r]$ and the tree T with vertex set J and an edge between $a, b \in J$ for each $uv \in M$ with $u \in V'_a$ and $v \in V'_b$. By our condition, for each $uv \in E(V_i, V_j)$, we have $|N_H(u) \cup N_H(v)| \leq \min\{|V_i|, |V_j|\}$.

Choose a root c of T . Traversing T in leaf-first order and for each leaf a with parent b picking $|V_a|$ from $\min\{|V_a|, |V_b|\}$ we get that the vertices in M together dominate at most $\sum_{i \in J-c} |V_i|$ vertices in H . Since $|S| < |V_c|$, M cannot totally dominate $\bigcup_{i \in J} V'_i$, a contradiction. \square

We note that the condition on S can be weakened slightly. Suppose we have ordered the V_i so that $|V_1| \leq |V_2| \leq \dots \leq |V_r|$. Then for any $S \subseteq V(H)$ with $|S| < |V_2|$ such that $V_1 \not\subseteq S$, there is an independent transversal I of V_1, \dots, V_r with $I \cap S = \emptyset$. The proof is the same except when we choose our root c , choose it so as to maximize $|V_c|$. Since $|J| \geq 2$, we get $|V_c| \geq |V_2| > |S|$ at the end.

4 List coloring lemmas

To apply the recoloring technique to a graph G , we need a decomposition of G into subgraphs that are nearly complete. To achieve this decomposition, we exclude many induced subgraphs using the list coloring lemmas proved in [5]. In this section we state these lemmas and prove two additional list coloring results. The reader can safely skip this section and return as needed.

Many of the list coloring results concern the *join* graphs A and B ; that is, the graph formed from the disjoint union of A and B by adding all possible edges between them. We write $A * B$ for the join of A and B .

Let G be a graph. A *list assignment* to the vertices of G is a function from $V(G)$ to the finite subsets of \mathbb{N} . A list assignment L to G is *good* if G has a coloring c where $c(v) \in L(v)$ for each $v \in V(G)$. It is *bad* otherwise. We call the collection of all colors that appear in L , the *pot* of L . That is $Pot(L) := \bigcup_{v \in V(G)} L(v)$. For a subgraph H of G we write $Pot_H(L) := \bigcup_{v \in V(H)} L(v)$. For $S \subseteq Pot(L)$, let G_S be the graph $G[\{v \in V(G) \mid L(v) \cap S \neq \emptyset\}]$. We also write G_c for $G_{\{c\}}$. For $f: V(G) \rightarrow \mathbb{N}$, an f -assignment on G is an assignment L of lists to the vertices of G such that $|L(v)| = f(v)$ for each $v \in V(G)$. We say that G is f -choosable if every f -assignment on G is good. Given $f: V(G) \rightarrow \mathbb{N}$, we have a partial order on the f -assignments to G given by $L < L'$ iff $|Pot(L)| < |Pot(L')|$. When we talk of *minimal* f -assignments, we mean minimal with respect to this partial order.

We'll need a lemma about bad list assignments with minimum pot size proved in [5]. Some form of this lemma which we call the *Small Pot Lemma* has appeared independently in at least two places we know of—Kierstead [10] and Reed and Sudakov [17]. We also use the following precursor to this lemma.

Lemma 4.1. *Let G be a graph and $f: V(G) \rightarrow \mathbb{N}$. Assume G is not f -choosable and let L be a minimal bad f -assignment. Assume $L(v) \neq Pot(L)$ for each $v \in V(G)$. Then, for each nonempty $S \subseteq Pot(L)$, any coloring of G_S from L uses some color not in S .*

Small Pot Lemma. *Let G be a graph and $f: V(G) \rightarrow \mathbb{N}$ with $f(v) < |G|$ for all $v \in V(G)$. If G is not f -choosable, then G has a minimal bad f -assignment L such that $|Pot(L)| < |G|$.*

We also need the notion of d_k -choosability from [5].

Definition 1. Let G be a graph and $r \in \mathbb{Z}$. Then G is d_k -choosable if G is f -choosable where $f(v) = d(v) - k$.

An in-depth study of the d_1 -choosable graphs was performed in [5]. We only need a small portion of those results here and their generalization to d_k -choosability. We use the following results on d_1 -choosability.

Lemma 4.2. *For $t \geq 4$, $K_t * B$ is not d_1 -choosable iff $\omega(B) \geq |B| - 1$; or $t = 4$ and B is E_3 or a claw; or $t = 5$ and B is E_3 .*

Lemma 4.3. *Let A be a graph with $|A| \geq 4$. Let L be a list assignment on $G := E_2 * A$ such that $|L(v)| \geq d(v) - 1$ for all $v \in V(G)$ and each component D of A has a vertex v such that $|L(v)| \geq d(v)$. Then L is good on G .*

Lemma 4.4. *Let A and B be graphs such that $G := A * B$ is not d_1 -choosable. If either $|A| \geq 2$ or B is d_0 -choosable and L is a bad d_1 -assignment on G , then*

1. *for any independent set $I \subseteq V(B)$ with $|I| = 3$, we have $\bigcap_{v \in I} L(v) = \emptyset$; and*
2. *for disjoint nonadjacent pairs $\{x_1, y_1\}$ and $\{x_2, y_2\}$ at least one of the following holds*
 - (a) $L(x_1) \cap L(y_1) = \emptyset$;
 - (b) $L(x_2) \cap L(y_2) = \emptyset$;
 - (c) $|L(x_1) \cap L(y_1)| = 1$ and $L(x_1) \cap L(y_1) = L(x_2) \cap L(y_2)$.

Lemma 4.5. *Let H be a d_0 -choosable graph such that $G := K_1 * H$ is not d_1 -choosable and L a minimal bad d_1 -assignment on G . If some nonadjacent pair in H have intersecting lists, then $|Pot(L)| \leq |H| - 1$.*

Lemma 4.6. *Let H be a d_0 -choosable graph such that $G := K_1 * H$ is not f -choosable where $f(v) \geq d(v)$ for the v in the K_1 and $f(x) \geq d(x) - 1$ for $x \in V(H)$. If L is a minimal bad f -assignment on G , then all nonadjacent pairs in H have disjoint lists.*

Lemma 4.7. *If B is a graph with $\delta(B) \geq \frac{|B|+1}{2}$ such that $K_1 * B$ is not d_1 -choosable, then $\omega(B) \geq |B| - 1$ or $B = E_3 * K_4$.*

Proof. Suppose the lemma is false and let L be a minimal bad d_1 -assignment on B . First note that if B does not contain disjoint nonadjacent pairs x_1, y_1 and x_2, y_2 , then $\omega(B) \geq |B| - 1$ or $B = E_3 * K_4$ by Lemma 4.2.

By Dirac's theorem, B is hamiltonian and in particular 2-connected. Since B cannot be an odd cycle or complete, B is d_0 -choosable.

By the Small Pot Lemma, $|Pot(L)| \leq |B|$. Since $|L(x_1)| + |L(x_2)| \geq |B| + 1$, their lists intersect and thus Lemma 4.5 shows that $|Pot(L)| \leq |B| - 1$. But then $|L(x_i) \cap L(y_i)| \geq 2$ for each i and Lemma 4.4 gives a contradiction. \square

We'll need the following simple consequence of the pigeonhole principle.

Lemma 4.8. *Let $r \in \mathbb{N}_{\geq 1}$. If S_1, \dots, S_m are subsets of a finite set T with $|S_i| \geq r$ for each $i \in [r]$ and $\sum_{i \in [m]} |S_i| \geq (m-1)|T| + r$, then $\left| \bigcap_{i \in [m]} S_i \right| \geq r$.*

Lemma 4.9. *$K_r * E_{k+2}$ is d_k -choosable when $k \geq 0$ and $r \geq (k+1)(k+2)$.*

Proof. Suppose not and let L be a minimal bad d_k -assignment on $K_r * E_{k+2}$ where $r = (k+1)(k+2)$. By the Small Pot Lemma, we have $|Pot(L)| \leq r + k + 1$. For each v in the K_r we have $|L(v)| = r + 1$ and for each v in the E_{k+2} we have $|L(v)| = r - k$. Let A be the colors appearing on the K_r and B the colors appearing on the E_{k+2} . By Lemma 4.1 we must have $B \subseteq A$. Let $\{X, Y\}$ be a partition of E_{k+2} with $|X| \geq |Y|$ and $|X| - |Y| \leq 1$. Now $\sum_{x \in X} |L(x)| = |X|(r - k) \geq |X|(k^2 + 2k + 2) = (|X| - 1)(k^2 + 4k + 3) - |X|(2k + 1) + k^2 + 4k + 3 \geq (|X| - 1)|Pot(L)| - \frac{k+3}{2}(2k + 1) + k^2 + 4k + 3 > (|X| - 1)|Pot(L)|$. Hence Lemma 4.8 gives $c_1 \in \bigcap_{x \in X} L(x)$. Similarly, we have $c_2 \in \bigcap_{x \in Y} L(x)$. Color all of X with c_1 and all of Y with c_2 . This leaves each vertex in K_r with a list of size at least $r - 1$. We could complete the coloring if two of these lists were different. Hence they are all the same and thus $L(x) = L(y)$ for all $x, y \in K_r$. Since $B \subseteq A$, we must have $|Pot(L)| = r + 1$. But then $\sum_{x \in E_{k+2}} |L(x)| = (k+2)(r - k) \geq (k+2)(k^2 + 2k + 2) = (k+1)(k^2 + 3k + 3) + 1 > (k+1)|Pot(L)|$ and hence we have $c \in \bigcap_{x \in E_{k+2}} L(x)$. Now we may color all of E_{k+2} with c and finish the coloring on K_r , a contradiction. \square

The following lemma was proved in [5].

Lemma 4.10. *Let $k \geq 1$. If B is a graph such that $K_{3k+1} * B$ is not d_k -choosable, then $\omega(B) \geq |B| - 2k$.*

With almost an identical proof we get the following extension that allows us to handle vertices of less than maximum degree more efficiently.

Lemma 4.11. *Let $1 \leq j \leq k$. Let B be a graph and L a list assignment on $G := K_{3k+1} * B$ such that $|L(v)| \geq d(v) - j$ for $v \in V(K_{3k+1})$ and $|L(v)| \geq d(v) - k$ for $v \in V(B)$. If L is bad on G , then $\omega(B) \geq |B| - 2j$.*

To prove our final list coloring lemma, we need another tool from [5].

Lemma 4.12. *Fix $k \geq 1$. Let A be a connected graph and B an arbitrary graph such that $A * B$ is not d_k -choosable. Let L be a minimal bad d_k -assignment on $A * B$. If B is colorable from L using at most $|B| - k$ colors, then $|Pot(L)| \leq |A| + |B| - 2$.*

Lemma 4.13. *Let $k \geq 1$. If B is the complement of a bipartite graph and $K_{3k+1} * B$ is not d_k -choosable, then $\omega(B) \geq |B| - k$.*

Proof. Suppose not, let B be such a graph and L a minimal bad d_k -assignment on $K_{3k+1} * B$. Let A be the K_{3k+1} . Let $M := \{\{x_1, y_1\}, \dots, \{x_t, y_t\}\}$ be a maximum matching in the complement of B . If $t \leq k$, then since B is perfect we have $\omega(B) = \chi(B) \geq |B| - k$ giving a contradiction.

Hence $t \geq k + 1$. For $i \in [t]$ we have $|L(x_i)| + |L(y_i)| \geq d_B(x_i) + d_B(y_i) + 4k + 2 \geq |B| - 2 + 4k + 2 = |B| + 4k$ since $\alpha(B) \leq 2$. By the Small Pot Lemma, we have $|Pot(L)| \leq |A| + |B| - 1 = |B| + 3k$. Hence we have different colors c_1, \dots, c_k such that $c_i \in L(x_i) \cap L(y_i)$ for each $i \in [k]$. For each such i , color each of x_i, y_i with c_i . Then we can complete the coloring on B since $|A| \geq k + 1$. We just colored B with at most $|B| - k$ colors and hence applying Lemma 4.12 gives $|Pot(L)| \leq |B| + 3k - 1$. So, now we can pick $c_{k+1} \in L(x_{k+1}) \cap L(y_{k+1}) - \{c_1, \dots, c_k\}$, color each of x_i, y_i with c_i for $i \in [k + 1]$, then complete the coloring to B . Now each vertex in A has at least $k + 1$ colors used twice on its neighborhood, so we can complete the coloring, a contradiction. \square

5 Borodin-Kostochka when every vertex is in a big clique

The case of $(\Delta - 1)$ -coloring is easier and provides a good warm-up for general coloring. Also, we achieve tighter bounds in this case because the list coloring theory is more developed.

5.1 A general decomposition

Let \mathcal{D}_1 be the collection of graphs without induced d_1 -choosable subgraphs. Plainly, \mathcal{D}_1 is hereditary. For a graph G and $t \in \mathbb{N}$, let \mathcal{C}_t be the maximal cliques in G having at least t vertices. We prove the following decomposition result for graphs in \mathcal{D}_1 which generalizes Reed's decomposition in [16].

Lemma 5.1. *Suppose $G \in \mathcal{D}_1$ has $\Delta(G) \geq 8$ and contains no $K_{\Delta(G)}$. If $\frac{\Delta(G)+5}{2} \leq t \leq \Delta(G) - 1$, then $\bigcup \mathcal{C}_t$ can be partitioned into sets D_1, \dots, D_r such that for each $i \in [r]$ at least one of the following holds:*

- $D_i = C_i \in \mathcal{C}_t$,
- $D_i = C_i \cup \{x_i\}$ where $C_i \in \mathcal{C}_t$ and $|N(x_i) \cap C_i| \geq t - 1$.

Moreover, each $v \in V(G) - D_i$ has at most $t - 2$ neighbors in C_i for each $i \in [r]$.

Proof. Suppose $|C_i| \leq |C_j|$ and $C_i \cap C_j \neq \emptyset$. Then $|C_i \cap C_j| \geq |C_i| + |C_j| - (\Delta + 1) \geq 4$. It follows from Lemma 4.2 that $|C_i - C_j| \leq 1$.

Now suppose C_i intersects C_j and C_k . By the above, $|C_i \cap C_j| \geq \frac{\Delta(G)+3}{2}$ and similarly $|C_i \cap C_k| \geq \frac{\Delta(G)+3}{2}$. Hence $|C_i \cap C_j \cap C_k| \geq \Delta(G) + 3 - (\Delta(G) - 1) = 4$. Put $I := C_i \cap C_j \cap C_k$ and $U := C_i \cup C_j \cup C_k$. By maximality of C_i, C_j, C_k , U cannot induce an almost complete graph. Thus, by Lemma 4.2, $|U| \in \{4, 5\}$ and the graph induced on $U - I$ is E_3 . But then $t \leq 6$ and hence $\Delta(G) \leq 7$, a contradiction.

The existence of the required partition is immediate. \square

When $D_i \in \mathcal{C}_t$, we put $K_i := C_i := D_i$ and when $D_i = C_i \cup \{x_i\}$, we put $K_i := N(x_i) \cap C_i$.

5.2 Doing the recoloring

Let G be a graph. For $v \in V(G)$, we let $\omega(v)$ be the size of a largest clique in G containing v . The proofs of the results in this section go more smoothly when we strengthen the induction in terms of the parameter $\rho(G) := \max_{v \in V(G)} d(v) - \omega(v)$.

Lemma 5.2. *For $k \geq 9$, every graph G satisfying $\Delta(G) \leq k$, $\omega(G) < k$ and $\rho(G) \leq \frac{k}{3} - 2$ is $(k - 1)$ -colorable.*

Proof. Suppose the theorem fails for some $k \geq 9$ and choose a counterexample G minimizing $|G| + \|G\|$. Put $\Delta := \Delta(G)$. If $\Delta < k$, then $\Delta = k - 1$ and by Brooks' theorem G contains K_k , a contradiction. Thus $\chi(G) = k = \Delta$. Also, for any $v \in V(G)$ we have $\rho(G - v) \leq \rho(G)$, applying our minimality condition on G implies that G is vertex critical.

Therefore $\delta(G) \geq \Delta - 1$ and $G \in \mathcal{D}_1$. For any $v \in V(G)$, we have $\Delta - 1 - \omega(v) \leq d(v) - \omega(v) \leq \frac{\Delta}{3} - 2$ and hence $\omega(v) \geq \frac{2}{3}\Delta + 1$. Applying Lemma 5.1 with $t := \frac{2}{3}\Delta + 1$ we get a partition D_1, \dots, D_r of $\bigcup \mathcal{C}_t = V(G)$. Note that for $i \in [r]$, if $K_i \neq D_i$ then all vertices in K_i are high by Lemma 4.3. Pick $x \in K_1$. Then x has $|C_1| - 1 \leq \Delta - 2$ neighbors in D_1 if $K_i = D_i$ and $|C_1| \leq \Delta - 1$ if $K_i \neq D_i$. Hence, by our note, x has a neighbor $w \in V(G) - D_1$.

We now claim that xw is a critical edge in G . Suppose otherwise that $\chi(G - xw) = \Delta$. Then by minimality of G we must have $\rho(G - xw) > \rho(G)$. Hence there is some vertex $v \in N(x) \cap N(w)$ so that every largest clique containing v contains xw . But v is in some D_j and all largest cliques containing v are contained in D_j and hence do not contain xw , a contradiction.

Let π be a $(\Delta - 1)$ -coloring of $G - xw$ chosen so that $\pi(x) = 1$ and so as to minimize $|\pi^{-1}(1)|$. Consider π as a coloring of $G - x$. One key property of π we will use is that since x got 1 in the coloring of $G - xw$ and $x \in K_1$, no vertex of $D_1 - x$ gets colored 1 by π .

Now put $Z := \pi^{-1}(1)$ and for $z \in Z$, let O_z be as defined in Section 3. By minimality of $|Z|$, each $z \in Z$ has at least one neighbor in every color class of π . Hence z has two or more neighbors in at most $2 + d(z) - \Delta$ of π 's color classes. For each $z \in Z$ we have $i(z)$ such that $z \in D_{i(z)}$. For $z \in Z$ such that $i(z) \notin i(Z - z)$, put $V_z := O_z \cap C_{i(z)}$. We have $|V_z| \geq \omega(z) - 1 - (2 + d(z) - \Delta)$. Since $\omega(z) \geq d(z) - \frac{1}{3}\Delta + 2$, we have $|V_z| \geq \frac{2}{3}\Delta - 1$. Each $y \in V_z$ is adjacent to all of $C_{i(z)} - \{y\}$ and hence has at most $d(y) + 1 - |C_{i(z)}|$

neighbors outside $D_{i(z)}$. Since $\omega(y) \geq d(y) + 2 - \frac{1}{3}\Delta$, we conclude that y has at most $d(y) + 1 - (d(y) + 2 - \frac{1}{3}\Delta) = \frac{1}{3}\Delta - 1$ neighbors outside $D_{i(z)}$.

Now let Z' be the $z \in Z$ with $i(z) \in i(Z - z)$. Then Z' can be partitioned into pairs $\{z, z'\}$ such that $i(z) = i(z')$. For such a pair, one of z, z' is $x_{i(z)}$ and the other is in $C_{i(z)} - K_{i(z)}$. Put $V_z := O_z \cap O_{z'} \cap K_{i(z)}$ and don't define $V_{z'}$. We have $|V_z| \geq \min\{\omega(z), \omega(z')\} - 1 - (2 + d(z) - \Delta) - (2 + d(z') - \Delta) \geq -\frac{1}{3}\Delta + 2 - 1 - 2(2 - \Delta) - \max\{d(z), d(z')\} = \frac{5}{3}\Delta - \max\{d(z), d(z')\} - 3 \geq \frac{2}{3}\Delta - 3$. Each $y \in V_z$ is adjacent to all of $D_{i(z)} - \{y\}$ and hence has at most $d(y) + 1 - |D_{i(z)}|$ neighbors outside $D_{i(z)}$. Since $|D_{i(z)}| = \omega(y) + 1 \geq d(y) + 3 - \frac{1}{3}\Delta$, we conclude that y has at most $\frac{1}{3}\Delta - 2$ neighbors outside $D_{i(z)}$.

Let H be the subgraph of G induced on the union of the V_z . Put $S := N(x) \cap V(H)$. Since $Z \cap D_1 = \emptyset$, x has at least $|D_1| - 1$ neighbors in D_1 none of which are in S . Hence $|S| \leq d(x) + 1 - |D_1| \leq d(x) + 1 - \omega(x) \leq \frac{\Delta}{3} - 1 < |V_z|$ for all V_z since $\Delta \geq 7$. Hence we may apply Lemma 3.2 on H with $t := \frac{1}{3}\Delta - 1$ to get an independent set $\{v_z\}_{z \in Z}$ disjoint from S where $v_z \in V_z$. Recoloring each $z \in Z$ with $\pi(z)$ and coloring $x \cup \{v_z\}_{z \in Z}$ with 1 gives a $(\Delta - 1)$ -coloring of G , a contradiction. \square

The following special case is a bit easier to digest.

Theorem 5.3. *Every graph G with $\chi(G) \geq \Delta(G) \geq 9$ such that every vertex is in a clique on $\frac{2}{3}\Delta(G) + 2$ vertices contains $K_{\Delta(G)}$.*

5.3 Reducing to the irregular case

It is easy to see that if there are irregular counterexamples to the Borodin-Kostochka conjecture, then there are regular examples as well: take an irregular counterexample G , clone it, add an edge between each vertex with degree less than $\Delta(G)$ and its clone; repeat until you have a regular graph (from [13]).

But what about the converse? If there are regular examples, must there be (connected) irregular examples? We'll see that the answer is yes, but we need to decrease the maximum degree by one.

Theorem 5.4. *Every graph G satisfying $\chi(G) \geq \Delta(G) = k \geq 9$ either contains K_k or contains an irregular critical subgraph satisfying $\chi(G) = \Delta(G) = k - 1$.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Then G is vertex critical. If every vertex in G were contained in a $(k - 1)$ -clique, then Theorem 5.3 would give a K_k in G , impossible. Hence we may pick $v \in V(G)$ not in a $(k - 1)$ -clique. If v is high, choose a $(k - 1)$ -coloring π of $G - v$ so that the color class T of π where v has two neighbors is as large as possible; if v is low, let π be a $(k - 1)$ -coloring of $G - v$ where some color class T of π is as large as possible. By symmetry, we may assume that $\pi(T) = k - 1$.

Now we have a $(k - 1)$ -coloring ζ of $H := G - T$ given by $\zeta(x) = \pi(x)$ for $x \neq v$ and $\zeta(v) = k - 1$. Since $\chi(H) = k - 1$, the maximality condition on T together with Brooks' theorem gives $\Delta(H) = k - 1$. Note that $d_H(v) = k - 2$. Let H' be a $(k - 1)$ -critical subgraph of H . Then H' must contain v and hence is not K_{k-1} . Since $d_{H'}(v) = k - 2$ and $\Delta(H') = k - 1$ (by Brooks' theorem), H' is an irregular critical subgraph of G satisfying $\chi = \Delta = k - 1$, a contradiction. \square

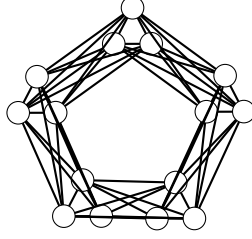


Figure 1: M_8 : A C_5 with vertices blown-up to triangles.

Since the only known critical (or connected even) counterexample to Borodin-Kostochka for $\Delta = 8$ is regular (see Figure 1) we might hope that the following strengthened conjecture is true.

Conjecture 5.5. *Every critical graph G with $\chi(G) \geq \Delta(G) = 8$ is regular.*

5.4 Dense neighborhoods

Here we show that the Borodin-Kostochka conjecture holds for graphs where each neighborhood has “most” of its possible edges. First, we need to convert high average degree in a neighborhood into a large clique in the neighborhood. We need the following extension of a fundamental result of Mader [12] (see Diestel [6] for some history of this result). We will also need d_1 -choosability results from [5] as well as some ideas for dealing with average degree in neighborhoods used in [4].

Lemma 5.6. *For $k \geq 1$, every graph G with $d(G) \geq 4k$ has a $(k + 1)$ -connected induced subgraph H such that $d(H) > d(G) - 2k$.*

Lemma 5.7. *If B is a graph with $d(B) \geq \omega(B) + 2$, then B has an induced subgraph H such that $K_1 * H$ is f -choosable where $f(v) \geq d(v)$ for the v in the K_1 and $f(x) \geq d(x) - 1$ for $x \in V(H)$.*

Proof. Let B be such a graph. Applying Lemma 5.6 with $k := 1$, we get a 2-connected subgraph H of B with $d(H) > d(B) - 2 \geq \omega(B)$. Since H is 2-connected, if it is not d_0 -choosable, then it is either an odd cycle or complete. The former is impossible since $d(H) \geq 3$, hence H would be complete and we’d have the contradiction $\omega(H) > \omega(B)$. Hence H is d_0 -choosable.

Suppose $K_1 * H$ isn’t f -choosable and let L be a minimal bad f -assignment on $K_1 * B$. By Lemma 4.6, no nonadjacent pair in H have intersecting lists and hence we must have $\sum_{v \in V(H)} |L(v)| \leq |Pot(L)| \omega(H)$. Since for each $v \in V(H)$ we have $|L(v)| \geq d_H(v)$ and by the Small Pot Lemma we have $|Pot(L)| \leq |H|$, we must have $d(H) \leq \omega(H) \leq \omega(B) < d(H)$, a contradiction. \square

Lemma 5.8. *If B is a graph with $d(B) \geq \omega(B) + 3$, then B has an induced subgraph H such that $K_1 * H$ is d_1 -choosable.*

Proof. Let B be such a graph. Applying Lemma 5.6 with $k := 1$, we get a 2-connected subgraph H of B with $d(H) > d(B) - 2 \geq \omega(B) + 1$. As in the proof of Lemma 5.7, we

see that H is d_0 -choosable. Suppose $K_1 * H$ is not d_1 -choosable and let L be a minimal bad d_1 -assignment on $K_1 * H$. Combining Lemma 4.5 with the same argument as in the proof of Lemma 5.7 shows that $|Pot(L)| \leq |H| - 1$.

Now, for $c \in Pot(L)$, we consider how big the color graphs H_c can be. All of the information comes from Lemma 4.4. We have $\alpha(G_c) \leq 2$ for all $c \in Pot(L)$. First, suppose we have $c \in Pot(L)$ such that $|H_c| \geq \omega(H) + 3$. Then, using Lemma 4.4, we see that $|H_{c'}| \leq \omega(H)$ for all $c' \in Pot(L) - c$ and hence $\sum_{\gamma \in Pot(L)} |H_\gamma| \leq |H| + (|Pot(L)| - 1)\omega(H) \leq |H|\omega(H) + |H| - 2\omega(H)$. Now suppose we have $c \in Pot(L)$ such that $|H_c| = \omega(H) + 2$. Then, using Lemma 4.4 again, we see that $|H_{c'}| \leq \omega(H) + 1$ for all $c' \in Pot(L) - c$ and hence $\sum_{\gamma \in Pot(L)} |H_\gamma| \leq 1 + |Pot(L)|(\omega(H) + 1) \leq |H|\omega(H) + |H| - \omega(H)$.

Therefore we must have $2\|H\| \leq |H|(\omega(H) + 1) - \omega(H)$ and hence $d(H) \leq \omega(H) + 1 < d(H)$, a contradiction. \square

Theorem 5.9. *Every graph G with $\omega(G) < \Delta(G)$ such that $d(G[N(v)]) \geq \frac{2}{3}\Delta(G) + 4$ for each $v \in V(G)$ is $(\Delta(G) - 1)$ -colorable.*

Proof. Suppose not and let G be a counterexample. Put $\Delta := \Delta(G)$. Let H be a Δ -vertex-critical induced subgraph of G . Then $\delta(H) \geq \Delta - 1$ and H has no d_1 -choosable induced subgraphs. By Theorem 5.3, we must have $v \in V(H)$ with $\omega(v) < \frac{2}{3}\Delta + 2$. Suppose $d(H_v) < d(G[N(v)])$. Then $d_H(v) = \Delta - 1$ and $\|H_v\| \geq \|G[N(v)]\| - (\Delta - 1)$; therefore, $d(H_v) > d(G[N(v)]) - 1 \geq \frac{2}{3}\Delta + 3$. Applying Lemma 5.7 gives $\omega(v) > d(H_v) - 1 \geq \frac{2}{3}\Delta + 2$, a contradiction.

Hence we must have $d(H_v) = d(G[N(v)]) \geq \frac{2}{3}\Delta + 4$. Applying Lemma 5.8 gives $\omega(v) > d(H_v) - 2 \geq \frac{2}{3}\Delta + 2$, a contradiction. \square

5.5 Bounding the order and independence number

Lemma 5.10. *Let G be a vertex critical graph with $\chi(G) = \Delta(G) + 1 - k$. For every $v \in V(G)$, $G[N(v)]$ has an induced subgraph H_v with:*

1. $|H_v| \geq \Delta(G) - 2k$; and
2. $\delta(H_v) \geq |H_v| - (k + 1)(\alpha(G) - 1) - 1$; and
3. $\|H_v\| \geq |H_v|(|H_v| - (k + 2)) - (k + 1)(|G| + 2k - (\Delta(G) + 1))$.

Proof. Put $\Delta := \Delta(G)$. Pick $v \in V(G)$ and let π be a $(\Delta - k)$ -coloring of $G - v$. Let H_v be the subgraph of $G[N(v)]$ induced on $\{x \in N(v) \mid \pi(x) \notin \pi(N(v) - x)\}$. Plainly, $|H_v| \geq \Delta - 2k$.

By the usual Kempe chain argument, any $x, y \in V(H_v)$ must be in the same component of $C_{x,y} := G[\pi^{-1}(\pi(x)) \cup \pi^{-1}(\pi(y))]$. Thus if $xy \notin E(G)$, there must be a path of length at least 3 in $C_{x,y}$ from x to y and hence some vertex of color $\pi(x)$ other than x must have at least two neighbors of color $\pi(y)$ and some vertex of color $\pi(y)$ other than y must have at least two neighbors of color $\pi(x)$. We say that such an intermediate vertex *proxies* for xy . Each xy with $y \in V(H_v)$ must have some proxy $z_{xy} \in \pi^{-1}(\pi(x)) - x$ such that z_{xy} proxies for at most $k + 1$ total xw with $w \in V(H_v)$, for otherwise we could recolor all of xy 's proxies, swap $\pi(x)$ and $\pi(y)$ in x 's component of $C_{x,y}$ and then color v with $\pi(x)$ to get a

$(\Delta - k)$ -coloring of G . We conclude that x has at most $(k+1)(|\pi^{-1}(\pi(x))| - 1)$ non-neighbors in H_v . This gives (2) immediately.

For (3), note that $|\pi(i)| \geq 2$ for each $i \in [\Delta - k] - \pi(V(H_v))$ and hence $\sum_{j \in \pi(V(H_v))} |\pi^{-1}(j)| \leq |G| - 1 - 2(\Delta - k - |H_v|)$. Since $\|H_v\| \geq \sum_{j \in \pi(V(H_v))} (|H_v| - 1 - (k+1)(|\pi^{-1}(j)| - 1))$, (3) follows. \square

Theorem 5.11. *Every graph G satisfies $\chi(G) \leq \max \{\omega(G), \Delta(G) - 1, 4\alpha(G)\}$.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Since none of the terms on the right side increase when we remove a vertex, G is vertex critical. Since the Borodin-Kostochka conjecture holds for graphs with $\alpha = 2$ and $\Delta \geq 9$, we must have $\alpha(G) \geq 3$ and hence $\Delta(G) \geq 13$. By Lemma 5.3, there must be $v \in V(G)$ with $\omega(v) < \frac{2}{3}\Delta(G) + 2$. Applying (2) of Lemma 5.10, we get $H_v \subseteq G[N(v)]$ with $|H_v| \geq \Delta(G) - 2$ and $\delta(H_v) \geq |H_v| - 2\alpha(G) + 1$. Since $\Delta(G) \geq \chi(G) \geq 4\alpha(G) + 1$, we have $\delta(H_v) \geq |H_v| - \frac{\Delta(G)-1}{2} + 1 \geq \frac{|H_v|+1}{2}$. Applying Lemma 4.7 shows that either $H_v = K_3 * E_4$ or $\omega(H_v) \geq |H_v| - 1$. The former is impossible since $\Delta(G) > 9$. Therefore $\omega(v) \geq \omega(H_v) + 1 \geq \Delta(G) - 2 \geq \frac{2}{3}\Delta(G) + 2$ since $\Delta(G) \geq 12$, a contradiction. \square

Theorem 5.12. *Every graph satisfies $\chi(G) \leq \max \left\{ \omega(G), \Delta(G) - 1, \left\lceil \frac{15 + \sqrt{48|G| + 73}}{4} \right\rceil \right\}$.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Put $\Delta := \Delta(G)$ and $n := |G|$. Since none of the terms on the right side increase when we remove a vertex, G is vertex critical. By Lemma 5.3, there must be $v \in V(G)$ with $\omega(v) < \frac{2}{3}\Delta + 2$. Applying (3) of Lemma 5.10, we get $H_v \subseteq G[N(v)]$ with $|H_v| \geq \Delta - 2$ and $\|H_v\| \geq |H_v|(|H_v| - 3) - 2(n + 1 - \Delta)$. By Lemma 5.8, we must have $d(H_v) < \frac{2}{3}\Delta + 4$ and hence we have

$$\begin{aligned} \frac{2}{3}\Delta + 4 &> 2(|H_v| - 3) - \frac{4(n + 1 - \Delta)}{|H_v|} \\ &\geq 2(\Delta - 5) - \frac{4(n + 1 - \Delta)}{\Delta - 2}. \end{aligned}$$

Simplifying a bit, we get $6(n - 1) > (2\Delta - 15)(\Delta - 2)$. Since $\Delta \geq \chi(G) \geq \frac{19 + \sqrt{48n + 73}}{4}$, we have $6(n - 1) > \left(\frac{-11 + \sqrt{48n + 73}}{2}\right)\left(\frac{11 + \sqrt{48n + 73}}{4}\right) = \frac{48n - 48}{8} = 6(n - 1)$, a contradiction. \square

6 Coloring graphs when every vertex is in a big clique

6.1 The decomposition

We need a partitioning result similar to Lemma 5.1 in the general case. We deal with a set of pairwise intersecting O_z by only using vertices in their intersection. Since we need this intersection to be big in order to apply the independent transversal lemma, we need to limit the number of O_z that can pairwise intersect. For $k \geq 0$, let \mathcal{D}_k be the collection of graphs

without induced d_k -choosable subgraphs. Again, for a graph G and $t \in \mathbb{N}$, we let \mathcal{C}_t be the maximal cliques in G having at least t vertices.

We put off as much computation as possible until later, to this end, define $\mathcal{U}(k, \omega, \Delta) := \max \left\{ \frac{2}{3}(\Delta + 1), \frac{1}{2}(\Delta + 3k + 2), \frac{2k}{2k+1}(\omega + k) - 1, \frac{k+1}{k+2}\omega + 2k + 1 \right\}$.

Lemma 6.1. *Let $k \geq 1$ and suppose $G \in \mathcal{D}_k$. If $t \geq \mathcal{U}(k, \omega(G), \Delta(G))$, then $\bigcup \mathcal{C}_t$ can be partitioned into sets D_1, \dots, D_r so that for each $i \in [r]$ each of the following holds:*

1. $|D_i| \leq \omega(G[D_i]) + 2k$; and
2. $G[D_i]$ has at least $3k + 1$ universal vertices; and
3. If L is a maximum clique in $G[D_i]$, then for independent $I \subseteq D_i$ we have $|L \cap \bigcap_{v \in I} N(v)| \geq |L| - |I| (|L| + k - t)$; and
4. $G[D_i]$ has independence number at most $k + 1$.

Proof. Put $\Delta := \Delta(G)$ and $\omega := \omega(G)$. If $\omega < t$, then $\bigcup \mathcal{C}_t$ is empty and the lemma holds vacuously. Hence we may assume $\omega \geq t$. Let X_t be the intersection graph of \mathcal{C}_t . Since $t \geq \frac{2}{3}(\Delta + 1)$, X_t is a disjoint union of complete graphs. Let $\mathcal{F}_1, \dots, \mathcal{F}_r$ be the components of X_t and put $D_i := \bigcup \mathcal{F}_i$ for $i \in [r]$.

Fix $i \in [r]$. Choose $L \in \mathcal{F}_i$ with $|L| = \omega(G[D_i])$. Put $A := D_i - L$.

Claim 1. $|A| \leq 2k$ and $|\bigcap \mathcal{F}_i| \geq 3k + 1$. Choose $S \subseteq A$ with $|S| = \min \{2k + 1, |A|\}$. Choose $\mathcal{Q} \subseteq \mathcal{F}_i$ such that $S \subseteq \bigcup \mathcal{Q}$ so as to minimize $|\mathcal{Q}|$. Then for any $Q \in \mathcal{Q}$ there exists $v_Q \in S \cap (Q - \bigcup (\mathcal{Q} - \{Q\}))$. In particular, $|\mathcal{Q}| \leq |S| = 2k + 1$. Suppose $|L \cap \bigcap \mathcal{Q}| \geq 3k + 1$. Then by Lemma 4.10 there is a clique with at least $|S| + |L| - 2k$ vertices intersecting L . Since L is a maximum size clique in D_i we must have $|A| \leq |S| \leq 2k$. But then $A = S$ and $\bigcap \mathcal{F}_i = L \cap \bigcap \mathcal{Q}$ since $A \subseteq \bigcup \mathcal{Q}$.

Therefore to prove the claim it suffices to show that $|L \cap \bigcap \mathcal{Q}| \geq 3k + 1$. If $\mathcal{Q} = \emptyset$, then we are done since $|L| \geq 3k + 1$. Hence $|\mathcal{Q}| \geq 1$. For any $Q \in \mathcal{Q}$, we have $|Q \cap L| \geq 3k + 1$ since $t \geq \frac{1}{2}(\Delta + 3k + 2)$. Hence we may apply Lemma 4.13 to get $|Q - L| + |L| - k \leq |L|$ and hence $|Q - L| \leq k$. So we have $|Q \cap L| \geq |Q| - k$ for all $Q \in \mathcal{Q}$. If $\sum_{Q \in \mathcal{Q}} |Q \cap L| \geq (|\mathcal{Q}| - 1)|L| + 3k + 1$, then applying Lemma 4.8 gives the desired conclusion $|L \cap \bigcap \mathcal{Q}| \geq 3k + 1$. Hence, if the claim fails, we must have

$$\begin{aligned} (|\mathcal{Q}| - 1)\omega + 3k + 1 &> \sum_{Q \in \mathcal{Q}} |Q \cap L| \\ &\geq \sum_{Q \in \mathcal{Q}} (|Q| - k) \\ &\geq |\mathcal{Q}|(t - k). \end{aligned}$$

Hence $t < \omega + k - \frac{\omega + 3k + 1}{|\mathcal{Q}|} \leq \frac{2k}{2k+1}(\omega + k) - 1$ since $|\mathcal{Q}| \leq 2k + 1$, a contradiction. This proves (1) and (2).

Claim 2. *for independent $I \subseteq D_i$ we have $|L \cap \bigcap_{v \in I} N(v)| \geq \omega(G[D_i]) - |I|(\omega(G[D_i]) + k - t)$. For each $v \in I$ pick $Q_v \in \mathcal{F}_i$ containing v and put $\mathcal{Q} := \{Q_v \mid v \in I\}$. Note that $L \cap \bigcap_{v \in I} N(v) = L \cap \bigcap \mathcal{Q}$. As in the proof of Claim 1, we see that $|Q \cap L| \geq |Q| - k$ for*

each $Q \in \mathcal{Q}$. Therefore, if the claim fails, Lemma 4.8 shows that we must have $|\mathcal{Q}|(t - k) < (|\mathcal{Q}| - 1)|L| + |L| - |I|(|L| + k - t)$ and hence $t < k + |L| - |I| \frac{(|L| + k - t)}{|I|} = t$, a contradiction.

Claim 3. $G[D_i]$ has independence number at most $k + 1$. Suppose not and pick independent $I \subseteq D_i$ with $|I| = k + 2$. By Claim 2, $|D_i \cap \bigcap_{v \in I} N(v)| \geq |L| - (k + 2)(|L| + k - t)$. Since $t \geq \frac{k+1}{k+2}\omega + 2k + 1$, we have $|D_i \cap \bigcap_{v \in I} N(v)| \geq (k + 1)\omega - (k + 1)|L| + (k + 2)(2k + 1) - k(k + 2) \geq (k + 1)(k + 2)$. Now Lemma 4.9 gives a contradiction. \square

6.2 Doing the recoloring

Again, to eliminate tiresome computation, put $\mathcal{U}'(k, \omega, \Delta) := \max \left\{ \frac{k+2}{k+3}\Delta + 1, \mathcal{U}(k, \omega, \Delta) \right\}$.

Lemma 6.2. For $k \geq 1$ and $\gamma \in \mathbb{N}$, every graph G with $\Delta(G) \leq \gamma$, $\omega(G) \leq \gamma - 2k$ and $\rho(G) \leq \gamma - k - \mathcal{U}'(k, \omega(G), \gamma)$ is $(\gamma - k)$ -colorable.

Proof. Suppose the theorem fails and choose a counterexample first minimizing γ and subject to that minimizing $|G| + \|G\|$. Put $\Delta := \Delta(G)$. Plainly, we must have $\gamma > 2k$. Suppose $\Delta < \gamma$. If $k' \geq 1$, then G satisfies the hypotheses of the theorem with $\gamma' := \gamma - 1$ and $k' := k - 1$. By minimality of γ , $\chi(G) \leq \gamma' - k' = \gamma - k$, a contradiction. If $k' = 0$, then Brooks' theorem gives a contradiction. Hence $\Delta = \gamma$. For any $v \in V(G)$ we have $\rho(G - v) \leq \rho(G)$ and hence the second minimality condition on G implies that G is $(\Delta + 1 - k)$ -vertex-critical.

Therefore $\delta(G) \geq \Delta - k$ and $G \in \mathcal{D}_k$. For any $v \in V(G)$, we have $\Delta - k - \omega(v) \leq d(v) - \omega(v) \leq \Delta - k - \mathcal{U}'(k, \omega, \Delta)$ and hence $\omega(v) \geq \mathcal{U}'(k, \omega, \Delta)$. Applying Lemma 6.1 with $t := \mathcal{U}'(k, \omega, \Delta)$ we get a partition D_1, \dots, D_r of $\bigcup \mathcal{C}_t = V(G)$. For $i \in [r]$, let K_i be the universal vertices in $G[D_i]$, we know $|K_i| \geq 3k + 1$. Suppose every $x \in K_1$ has $N(x) \subseteq D_1$. Put $j := \min_{x \in K_1} \Delta - d(x)$. Since $|K_1| \geq 3k + 1$, applying Lemma 4.11, for every $x \in K_1$ we have $\Delta - 2j \geq \omega(G[D_i]) + 2(k - j) \geq |D_i| \geq d(x) + 1$, this contradicts the definition of j . Thus we have $x \in K_1$ that has a neighbor $w \in V(G) - D_1$.

We claim that xw is a critical edge in G . Suppose otherwise that $\chi(G - xw) = \Delta + 1 - k$. Then by minimality of G we must have $\rho(G - xw) > \rho(G)$. Hence there is some $v \in N(x) \cap N(w)$ so that every largest clique containing v contains xw . But v is in some D_j and all largest cliques containing v are contained in D_j and hence do not contain xw , a contradiction.

Let π be a $(\Delta - k)$ -coloring of $G - xw$ chosen so that $\pi(x) = 1$ and so as to minimize $|\pi^{-1}(1)|$. Consider π as a coloring of $G - x$. One key property of π we will use is that since x got 1 in the coloring of $G - xw$ and $x \in K_1$, no vertex of $D_1 - x$ gets colored 1 by π .

Now put $Z := \pi^{-1}(1)$ and for $z \in Z$, let O_z be as defined in Section 3. By minimality of $|Z|$, each $z \in Z$ has at least one neighbor in every color class of π . Hence z has two or more neighbors in at most $k + 1$ of π 's color classes. For each $z \in Z$ we have $i(z)$ such that $z \in D_{i(z)}$. For each $a \in i(Z)$, let L_a be a maximum clique in $G[D_a]$ and put $V_a := L_a \cap \bigcap_{z \in i^{-1}(a)} O_z$. By Lemma 6.1, we have $|i^{-1}(a)| \leq k + 1$ and

$$\begin{aligned}
|V_a| &\geq |L_a| - |i^{-1}(a)| (|L_a| + k - t) - |i^{-1}(a)| (k + 1) \\
&= |L_a| (1 - |i^{-1}(a)|) + |i^{-1}(a)| (t - 2k - 1) \\
&\geq \omega(1 - |i^{-1}(a)|) + |i^{-1}(a)| (t - 2k - 1) \\
&= \omega + |i^{-1}(a)| (t - 2k - 1 - \omega) \\
&\geq \omega + (k + 1)(t - 2k - 1 - \omega) \\
&\geq (k + 1) \left(\frac{k + 2}{k + 3} \Delta - 2k \right) - k\omega \\
&\geq (k + 1) \left(\frac{k + 2}{k + 3} \Delta - 2k \right) - k(\Delta - 2k) \\
&= \frac{(k + 1)(k + 2) - k(k + 3)}{k + 3} \Delta - 2k \\
&= 2 \left(\frac{\Delta}{k + 3} - k \right).
\end{aligned}$$

Also $|D_a| \geq |L_a| + |i^{-1}(a)| - 1$. So, each $y \in V_a$ has at most $d(y) - (|D_a| - 1) \leq d(y) - \omega(y) - |i^{-1}(a)| + 2 \leq d(y) - \omega(y) + 1 \leq \Delta - k - \frac{k+2}{k+3}\Delta = \frac{\Delta}{k+3} - k$ neighbors outside D_a . Now $|D_1| \geq t \geq \frac{k+2}{k+3}\Delta + 1$ and hence x has at most $\frac{\Delta}{k+3}$ neighbors outside D_1 . Let H be the subgraph of G induced on the union of the V_a . Then, we may apply Lemma 2.1 on H to get an independent set $\{x\} \cup \{v_z\}_{z \in Z}$ where $v_z \in V_z$. Recoloring each $z \in Z$ with $\pi(z)$ and coloring $x \cup \{v_z\}_{z \in Z}$ with 1 gives a $(\Delta - k)$ -coloring of G , a contradiction. \square

A little computation gives the following two more parsable results.

Lemma 6.3. *For $k \geq 1$ and $\gamma \in \mathbb{N}$, every graph G with $\Delta(G) \leq \gamma$, $\omega(G) \leq \gamma - 2k$ and $\rho(G) \leq \frac{\gamma}{2k+1} - (2k + 1)$ is $(\gamma - k)$ -colorable.*

Theorem 6.4. *Let $k \geq 1$. Every graph G with $\omega(G) \leq \Delta(G) - 2k$ such that every vertex is in a clique on $\frac{2k}{2k+1}\Delta(G) + 2k + 1$ vertices is $(\Delta(G) - k)$ -colorable.*

6.3 Dense neighborhoods

Lemma 6.5. *Let $k \geq 1$. If B is a graph with $\delta(B) \geq \frac{2k+1}{2k+2}|B| + k - 1$ such that $K_1 * B$ is not d_k -choosable, then $\omega(B) \geq |B| - 2k$.*

Proof. Let L be a minimal bad d_k -assignment on $K_1 * B$. By the Small Pot Lemma, we have $|Pot(L)| \leq |B|$. Let $X := \{\{x_1, y_1\}, \dots, \{x_s, y_s\}\}$ be a maximal set of pairwise disjoint independent sets of size 2 in B . Note that $s \leq \frac{|B|}{2}$. Put $K := B - \cup X$. Plainly, K is complete. Thus we are done if $s \leq k$.

Suppose $s \geq k + 1$. Put $\epsilon := \frac{1}{2k+2}$. For $i \in [s]$ we have $|L(x_i)| + |L(y_i)| \geq d_B(x_i) + d_B(y_i) - 2k + 2 \geq 2(1 - \epsilon)|B| \geq |B| + s$ and hence we may pick s different colors c_1, \dots, c_s where $c_i \in L(x_i) \cap L(y_i)$. Color both x_i and y_i with c_i to get a list assignment L' on K .

Each $v \in V(K)$ has at least $\delta(B) - (|K| - 1) \geq (1 - \epsilon)|B| + k - |K| = 2s + k - \epsilon|B|$ neighbors in $\cup X$ and hence is joined to at least $s + k - \epsilon|B|$ pairs $\{x_i, y_i\}$. Since L is bad, we must have $s < \epsilon|B|$.

Now consider a pair $\{x_i, y_i\}$. Vertices in $N(x_i) \cap N(y_i) \cap K$ have a color saved in L' so we wish to show this set is big. We have $|N(x_i) \cap N(y_i) \cap K| \geq 2(\delta(B) - (2s - 2)) - |K| = 2\delta(B) + 4 - |B| - 2s > (1 - 2\epsilon)|B| + 2k + 2 - 2s$.

For $v \in V(K)$, let $l(v)$ count the number of $i \in [s]$ such that v is joined to $\{x_i, y_i\}$. Then $|L'(v)| \geq |K| + l(v) - k$ for each $v \in V(K)$. By Hall's theorem, we can complete the coloring if for all $0 \leq a \leq k$ we have $|\{v \in V(K) \mid l(v) \geq k - a\}| \geq a + 1$. Thus it will suffice to show that for any $k - a$ indices $i_1, \dots, i_{k-a} \in [s]$ we have $\left| \bigcap_{j \in [k-a]} N(x_{i_j}) \cap N(y_{i_j}) \cap K \right| \geq a + 1$. If this were not the case for some a , then by Lemma 4.8 we would have

$$(k - a) \left(\frac{k}{k+1} |B| + 2(k+1) - 2s \right) < (k - a - 1)(|B| - 2s) + a + 1.$$

A simple computation shows that this is impossible. Hence we can complete the coloring, contradicting the fact that L is bad. \square

We can use this to turn Theorem 6.4 into a theorem about graphs with dense neighborhoods as follows. Little effort was put into optimizing the bound $d(G[N(v)])$ that we can get using Lemma 6.5 since with a more developed d_k -choosability theory any such bound would be easily defeated (as in the $k = 1$ case).

Theorem 6.6. *Let $k \geq 0$. Every graph G with $\omega(G) \leq \Delta(G) - 2k$ such that $d(G[N(v)]) \geq \frac{6k^2}{6k^2+1}\Delta(G) + k + 6$ for each $v \in V(G)$ is $(\Delta(G) - k)$ -colorable.*

Proof. Let G be such a graph and put $\Delta := \Delta(G)$. Suppose G is not $(\Delta - k)$ -colorable. If $k = 0$, then Brooks' theorem gives a contradiction. If $k = 1$, Theorem 5.9 gives a contradiction.

So, we must have $k \geq 2$. Let H be a $(\Delta + 1 - k)$ -vertex-critical subgraph of G . Then $\delta(H) \geq \Delta - k$ and hence $d(H_v) \geq d(G[N(v)]) - k$. By Theorem 6.4, there must be $v \in V(G)$ such that $\omega(v) < \frac{2k}{2k+1}\Delta + 2k + 1$.

Put $s := \frac{\Delta - k}{2k+1} - 6k$. Applying Lemma 6.5 repeatedly on $K_1 * H_v$, gives a sequence $y_1, \dots, y_s \in N_H(v)$ such that for each $i \in [s]$ we have

$$|N_H(y_i) \cap (N_H(v) - \{y_1, \dots, y_{i-1}\})| < \frac{2k+1}{2k+2}(|H_v| - i) + k - 1.$$

Hence the number of edges missing in H_v is at least

$$-(k-1)s + \frac{1}{2(k+1)} \sum_{i \in [s]} (\Delta - k - i).$$

On the other hand we have $d(H_v) \geq d(G[N(v)]) - k \geq \frac{6k^2}{6k^2+1}\Delta + 6$ and hence the number of edges missing in H_v is at most $\binom{\Delta-k}{2} - \frac{3k^2}{6k^2+1}\Delta(\Delta-k) - 3(\Delta-k) < \frac{1}{2(6k^2+1)}(\Delta-k)^2 - \frac{7}{2}(\Delta-k)$. Therefore, multiplying through by $2(k+1)(6k^2+1)$, we must have

$$-2(k^2-1)(6k^2+1)s + (6k^2+1)(s(\Delta-k) - s(s+1)) \leq (k+1)(\Delta-k)^2 - 7(k+1)(6k^2+1)(\Delta-k).$$

Let's collect everything to the left side and look at the coefficients of the powers of $\Delta - k$ individually. Plugging in for s , we have $\frac{6k^2+1}{2k+1} - \frac{6k^2+1}{(2k+1)^2} - (k+1) = \frac{8k^3-8k^2-3k-1}{(2k+1)^2}$ for $(\Delta - k)^2$. For $(\Delta - k)^1$ we get $\frac{-2(k^2-1)(6k^2+1)}{2k+1} - (6k-1)(6k^2+1) + 7(k+1)(6k^2+1) > 8(6k^2+1)$. Finally, for $(\Delta - k)^0$ we have $12k(k^2-1)(6k^2+1) - 36k^2 + 6k$. Thus all of the coefficients are positive for $k \geq 2$, a contradiction. \square

Problem. Develop d_k -choosability theory and improve the bound $d(G[N(v)]) \geq \frac{6k^2}{6k^2+1}\Delta(G) + k + 6$. In particular, can the dependence on k in $\frac{6k^2}{6k^2+1}$ be made linear?

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