A common generalization of Hall's theorem and Vizing's edge-coloring theorem

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LBD Data

Miami University Colloquium
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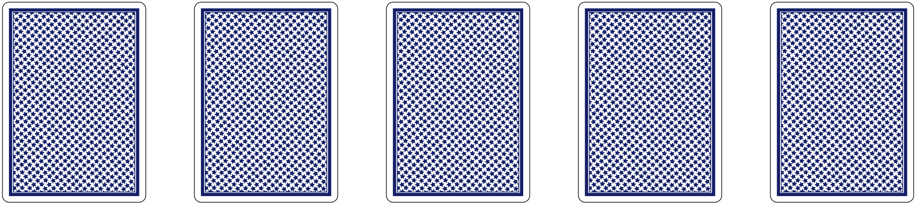
some card games

the simplest variation

- two players, Dealer and Player



some card games

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some card games

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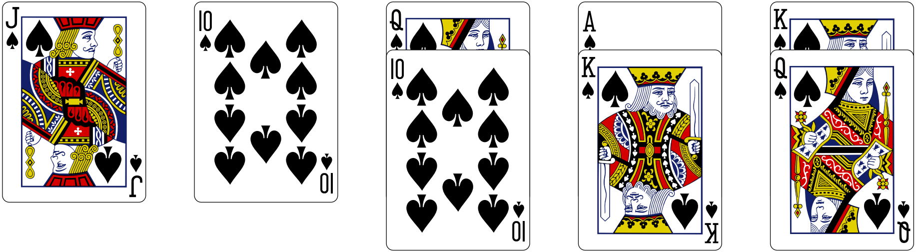

some card games

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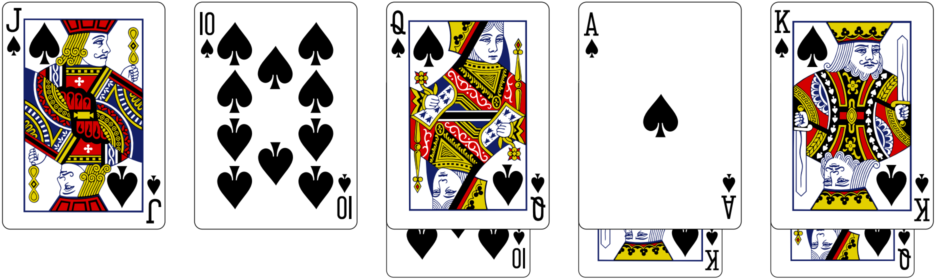
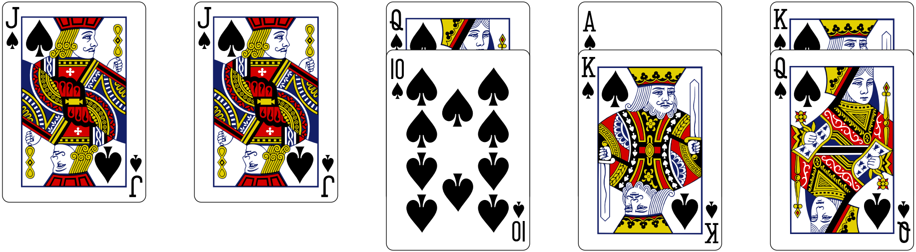


some card games

example, a Dealer win



some card games

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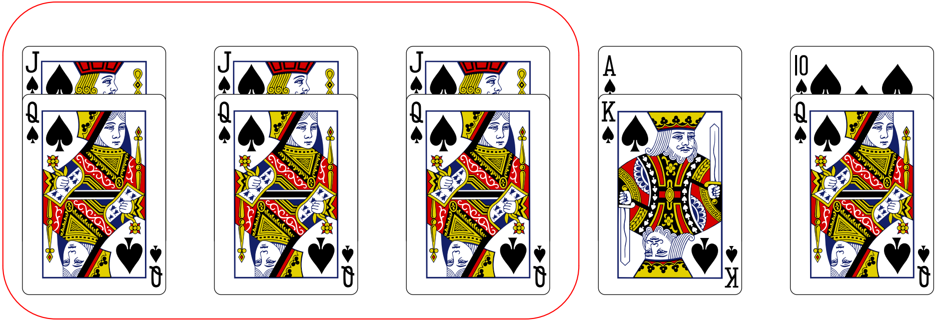
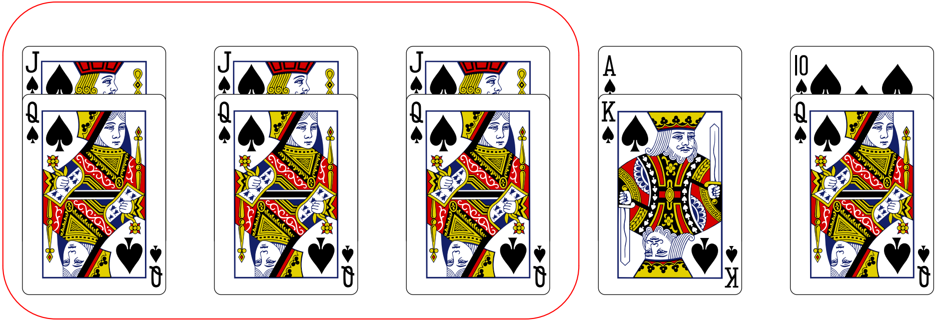


- Player cannot win if there is a set of k stacks that together have fewer than k different cards
- Hall's theorem says: **Player wins otherwise**



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some card games

making things harder for Dealer

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some card games

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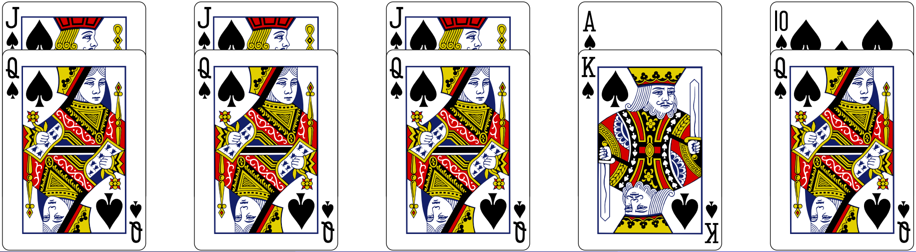
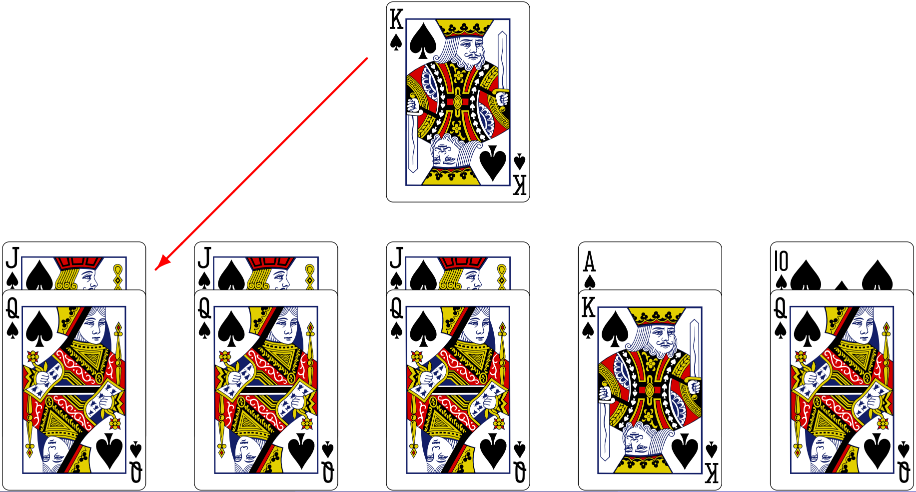


- Player picks a King from the deck and swaps it for a Queen in the first stack



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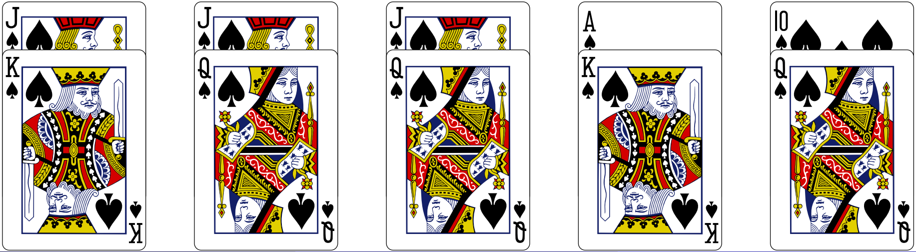

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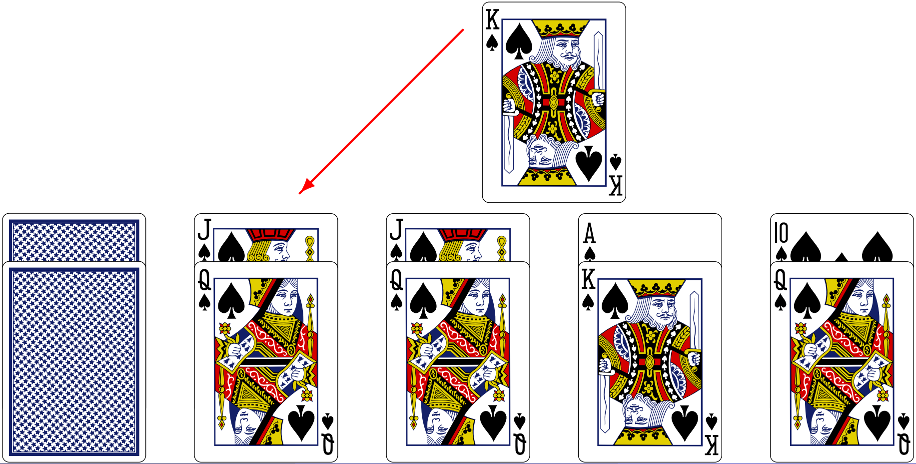

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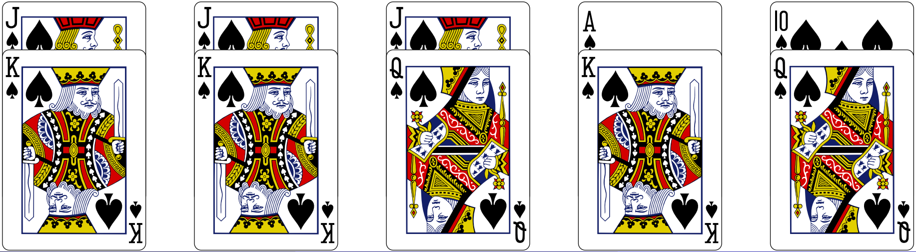

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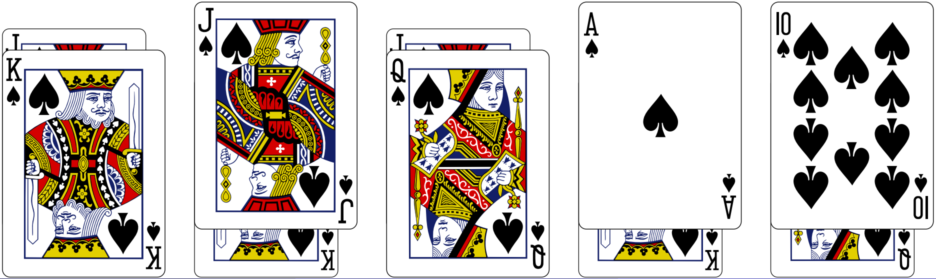
- Player picks a King from the deck and swaps it for a Queen in the first stack
- Dealer can swap a King and Queen in one of the other stacks
- Player wins no matter what Dealer does



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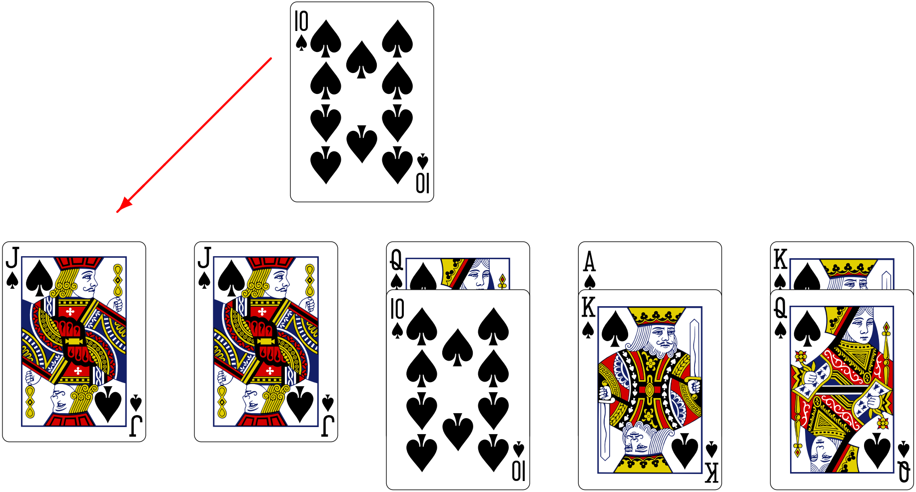

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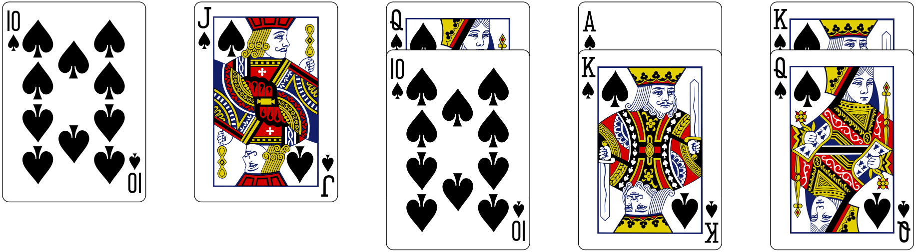

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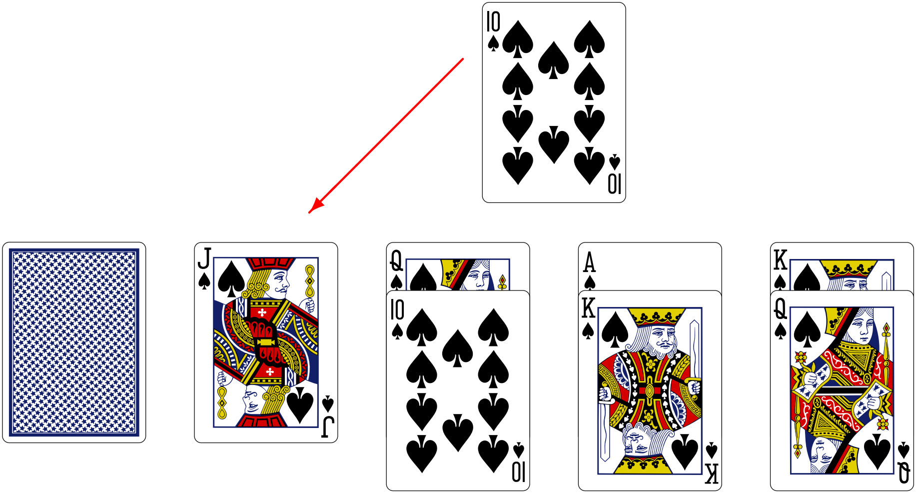

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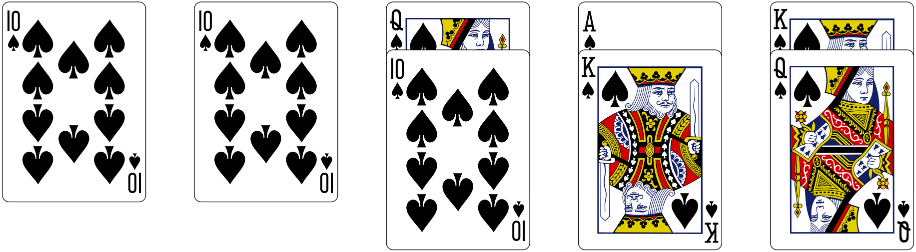
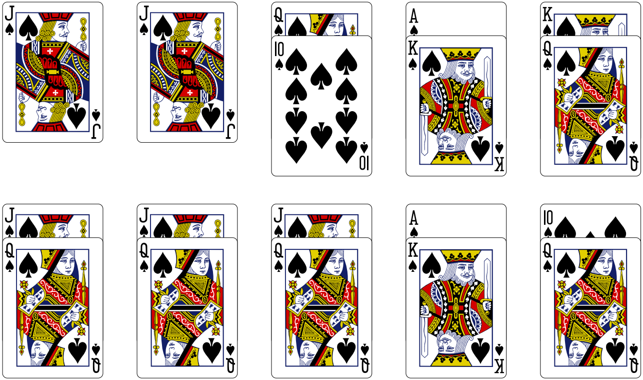
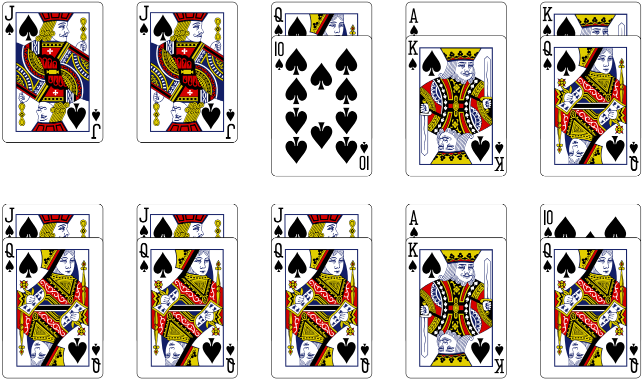


some card games

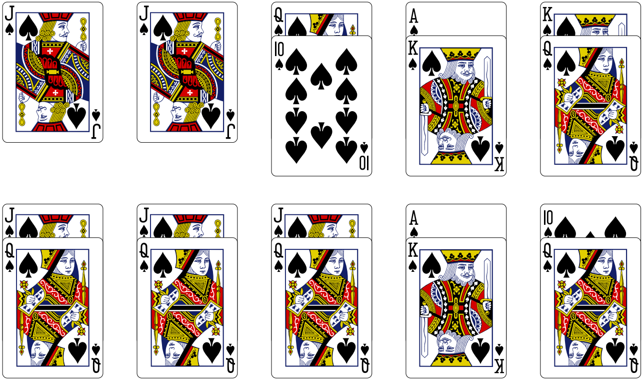
what was the difference?



- in the top game, Dealer can prevent Player from increasing the number of different cards in the first two stacks



- in the top game, Dealer can prevent Player from increasing the number of different cards in the first two stacks
- in the bottom game, Dealer cannot prevent prevent Player from increasing the number of different cards in the first three stacks



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some card games

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Degree

The *degree* of a card C in a set of stacks S is the number of times C appears in S. We write $d_S(C)$ for this quantity.

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Necessary Condition

If Player has a winning strategy, then for every set of stacks S we must have

$$\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil \geq |S|.$$

- **this necessary condition is also suffcient**

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Winning Condition

Player has a winning strategy if and only if for every set of stacks S we have

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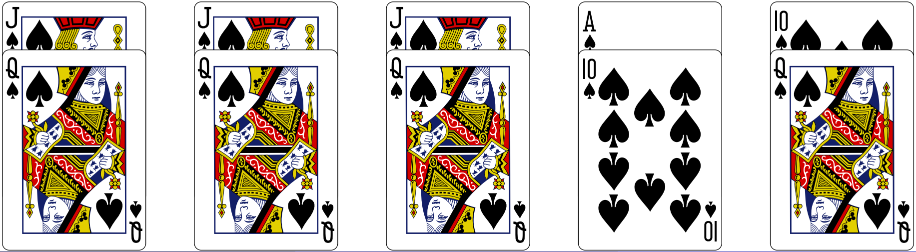
1. Player looks for a set of card types that give a system of distinct representatives of all the stacks containing them



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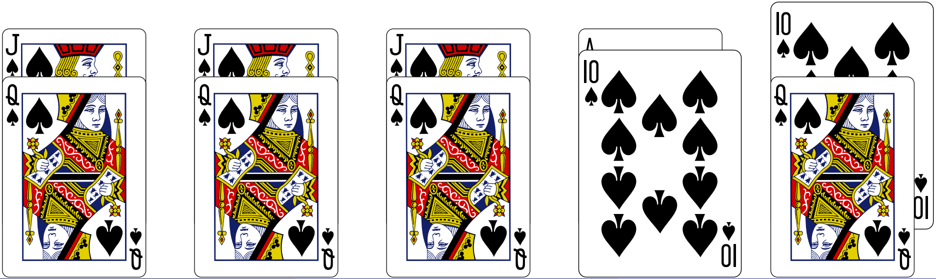

some card games

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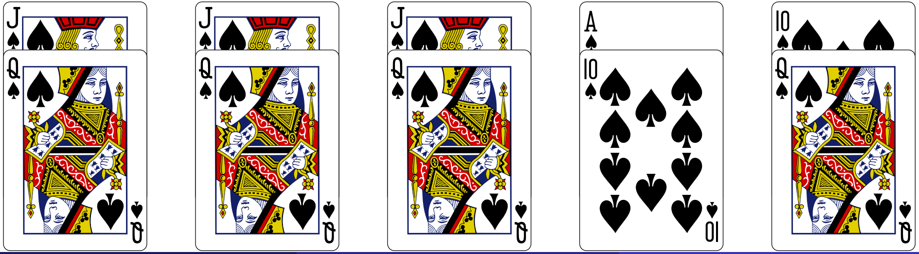

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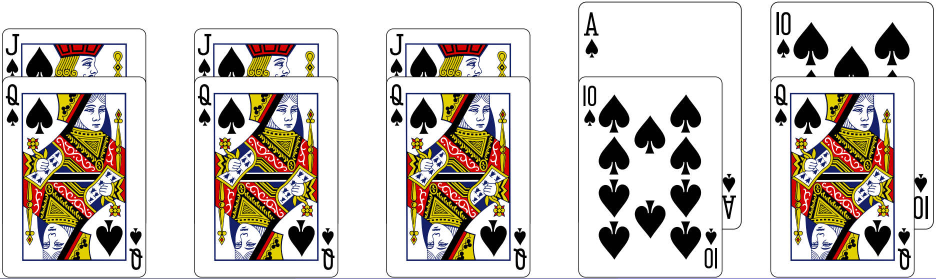

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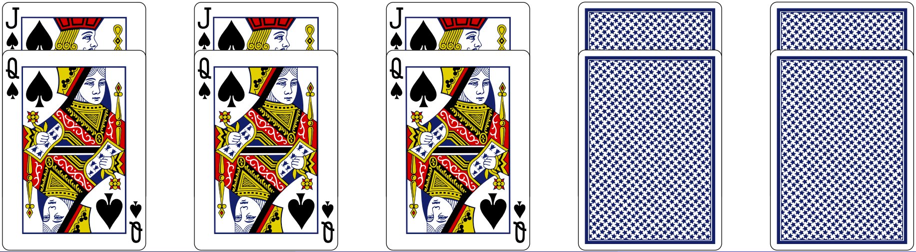
2. Player calls those stacks done and never plays with those card types again



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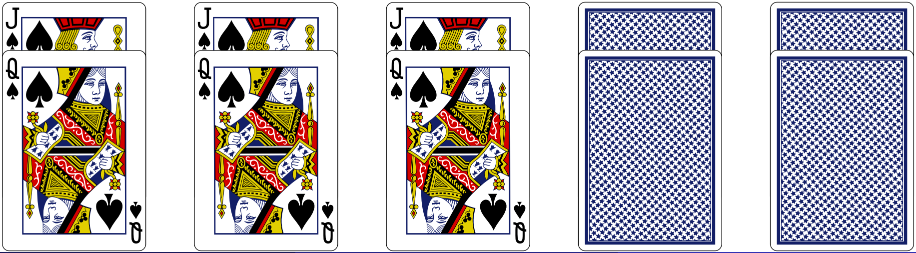

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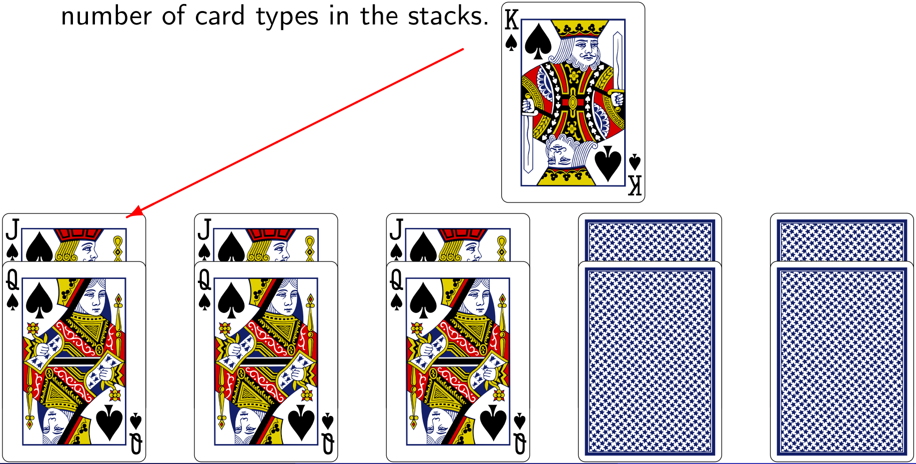

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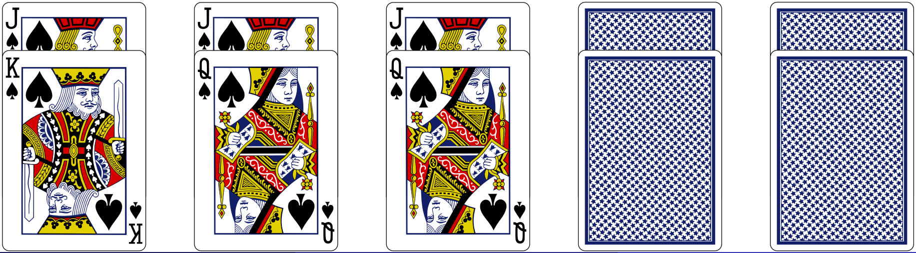

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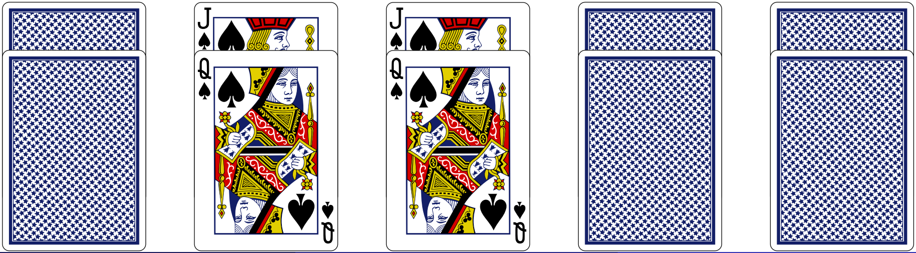

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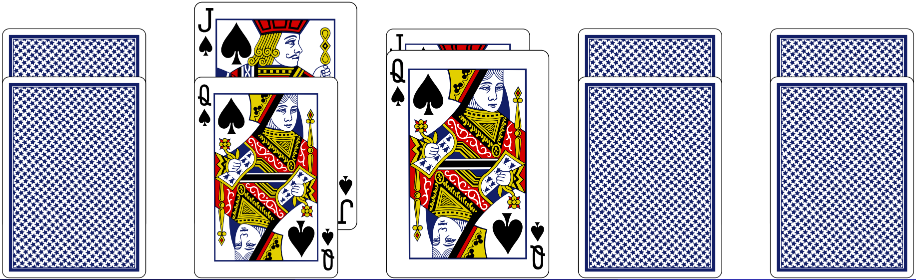

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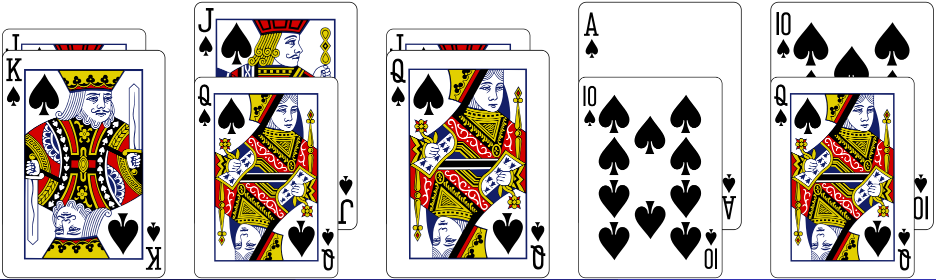

A generalization of Hall's theorem
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Player has a winning strategy in the t-game if and only if for every set of stacks S we have

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- Hall's theorem is the winning condition in the $(k-1)$-game when there are k total stacks

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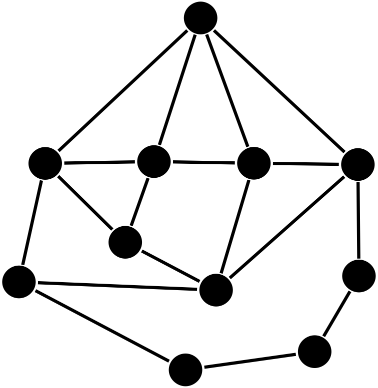
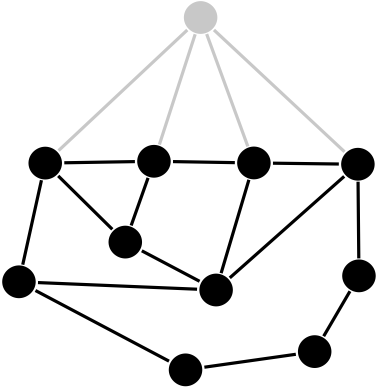
Vizing's theorem

Any simple graph can be edge-colored using at most one more color than its maximum degree.

- take a minimum counterexample to Vizing's theorem, say it has maximum degree Δ



- take a minimum counterexample to Vizing's theorem, say it has maximum degree Δ
- remove a vertex of degree Δ and edge-color the rest with $\Delta + 1$ colors



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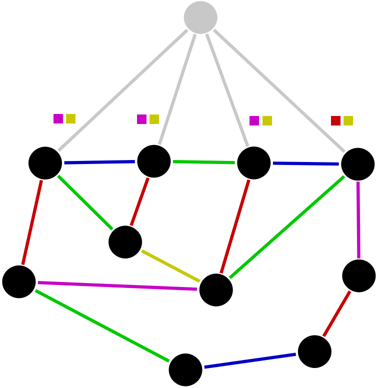

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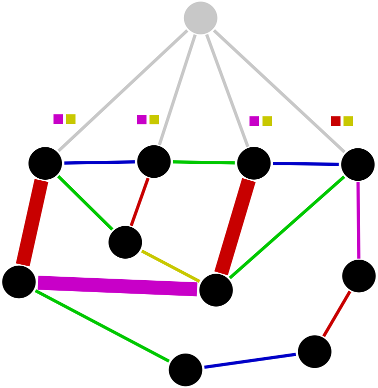

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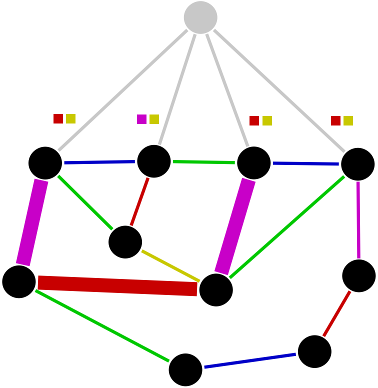

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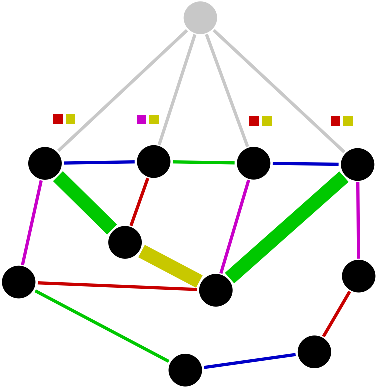

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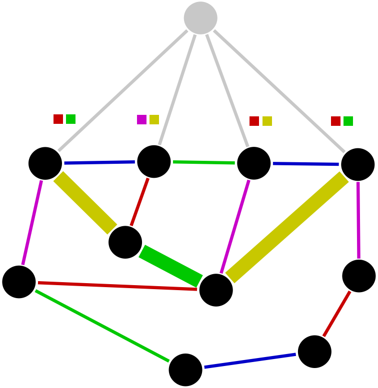

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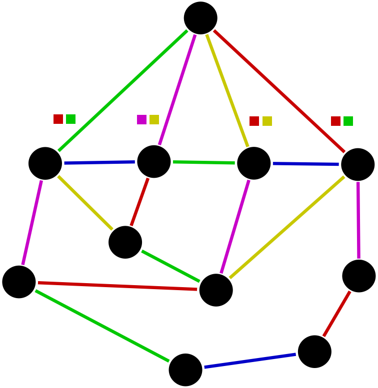

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- so, we have the desired winning condition

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- there is a much more general game that unifies a large chunk of edge-coloring theory