A common generalization of Hall's theorem and Vizing's edge-coloring theorem

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LBD Data

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Hall's theorem

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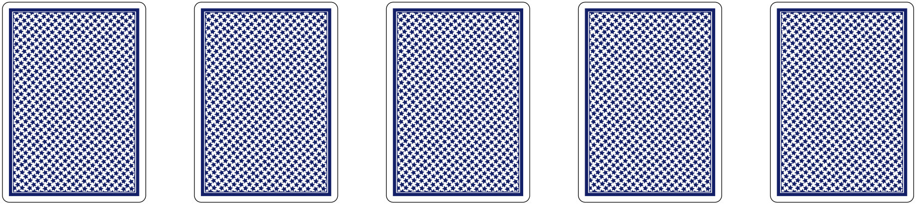
  - $A_1 = \{1, 2\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2\}$
- Hall's theorem: this is the only thing that can go wrong

$$\text{SDR exists} \;\Leftrightarrow \left|\bigcup_{i \in I} A_i\right| \geq |I| \text{ for all } I \subseteq \{1, \ldots, n\}$$

some card games

the simplest variation

- Dealer vs. Player



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- Player wins if he can pick a Royal Flush, one card from each stack

some card games

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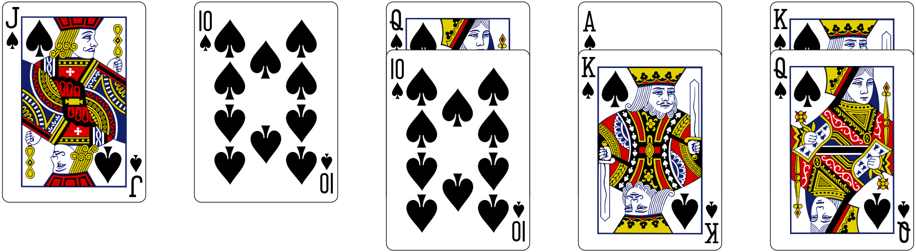

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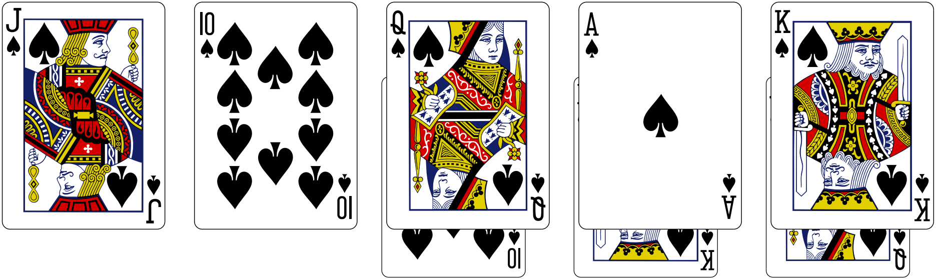
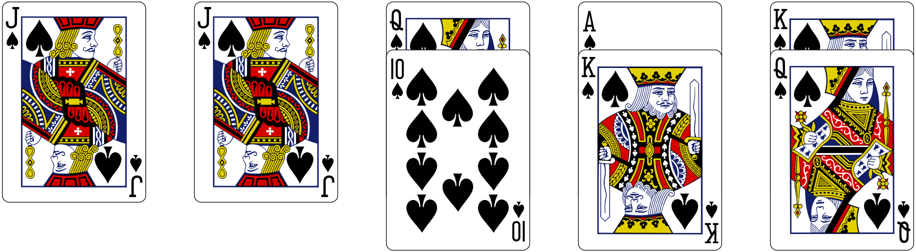


some card games

example, a Dealer win



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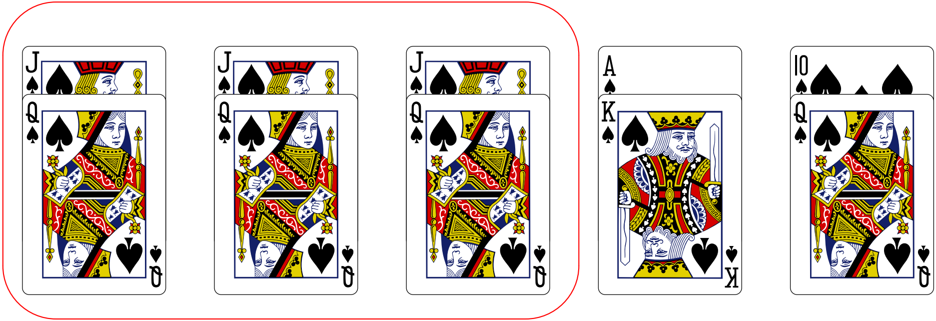

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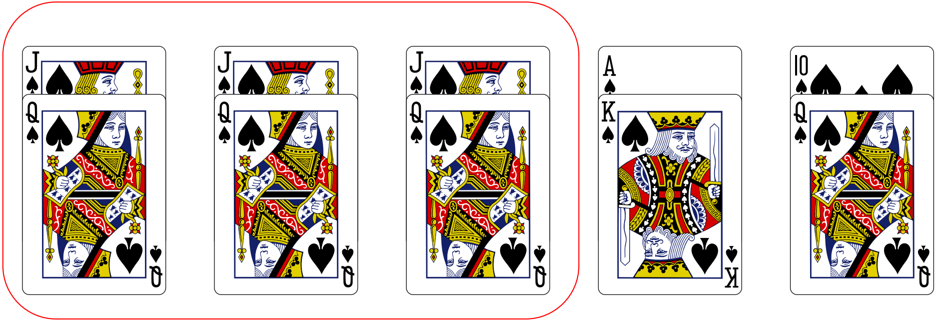
- Hall's theorem says: **Player wins otherwise**



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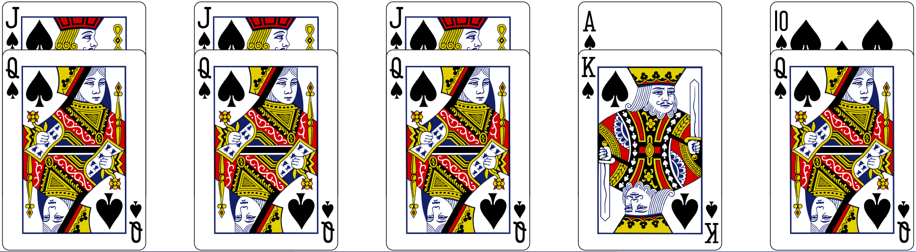
Winning

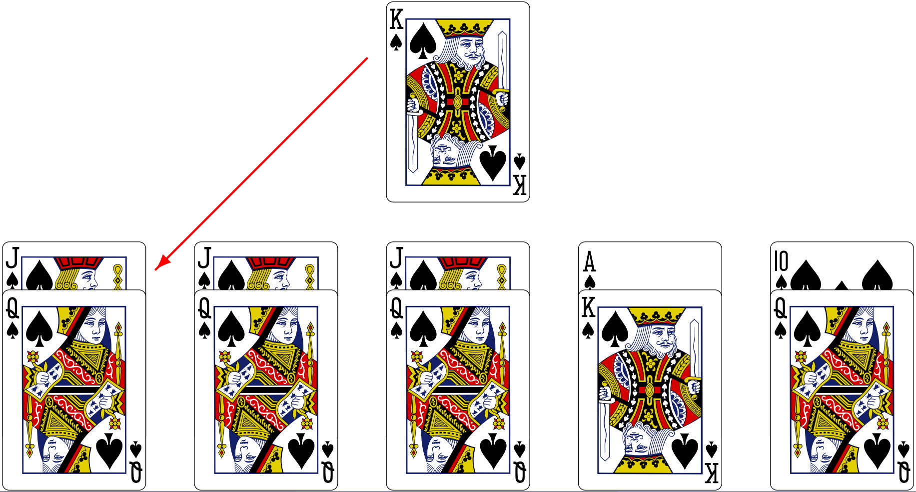
Player wins if he can pick a Royal Flush at the start of one of his turns, otherwise Dealer wins.

some card games

example, a Player win



- Player picks a King from the deck and swaps it for a Queen in the first stack



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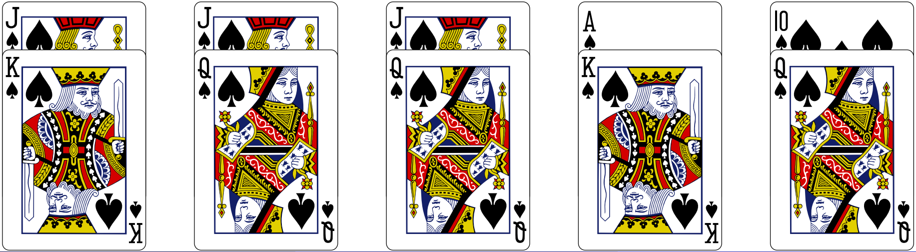

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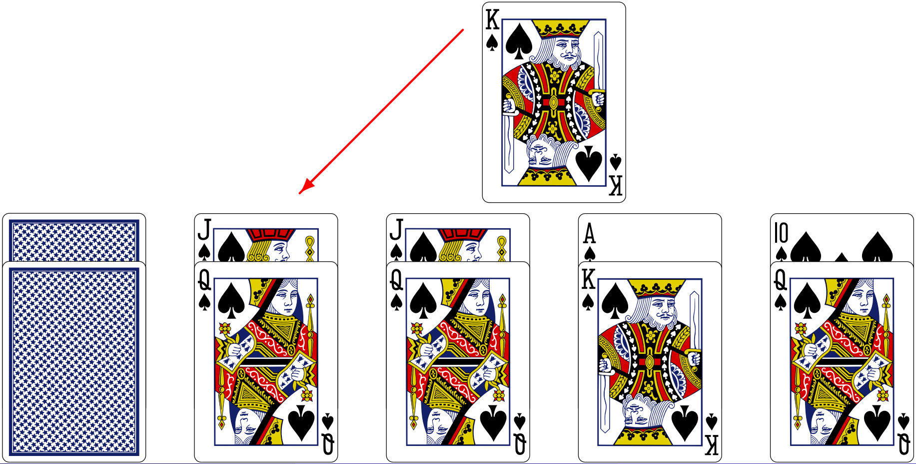

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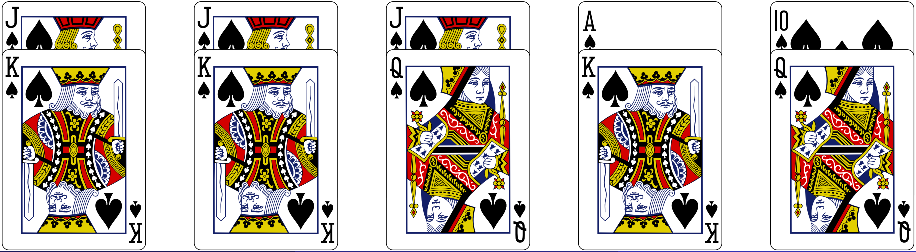

some card games

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- Player picks a King from the deck and swaps it for a Queen in the first stack
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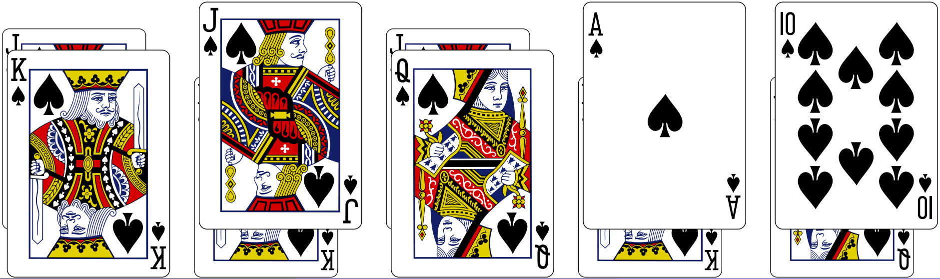
- Player wins no matter what Dealer does



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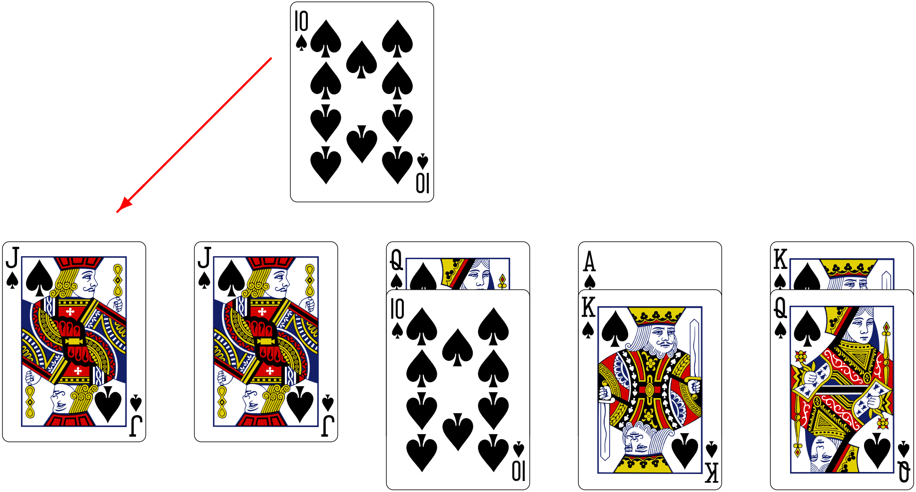

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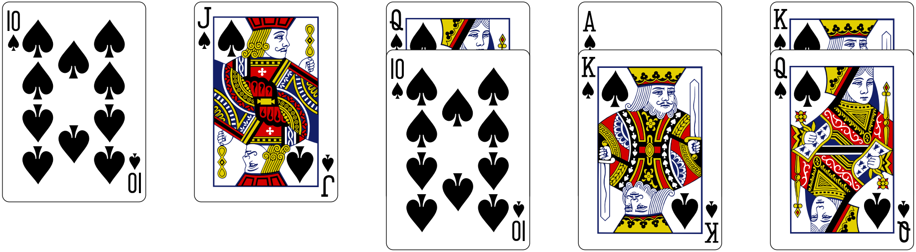

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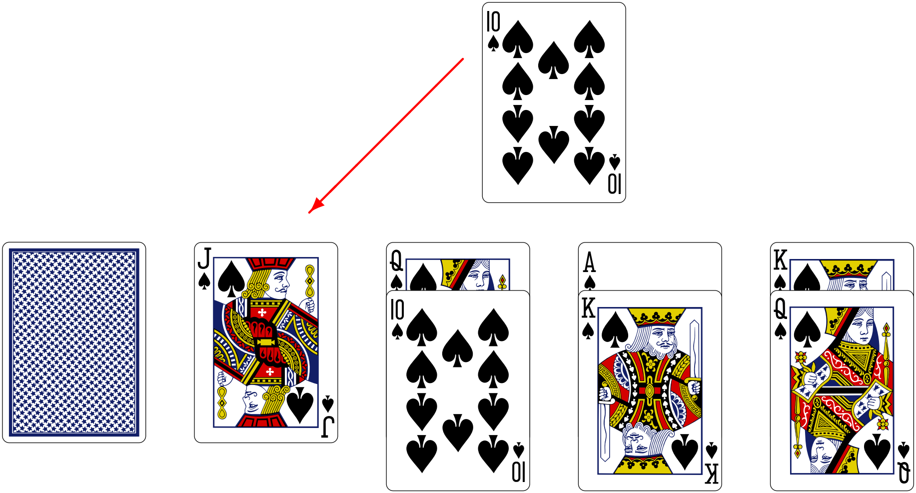

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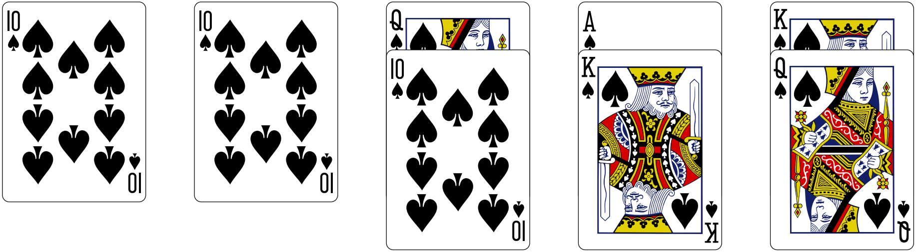
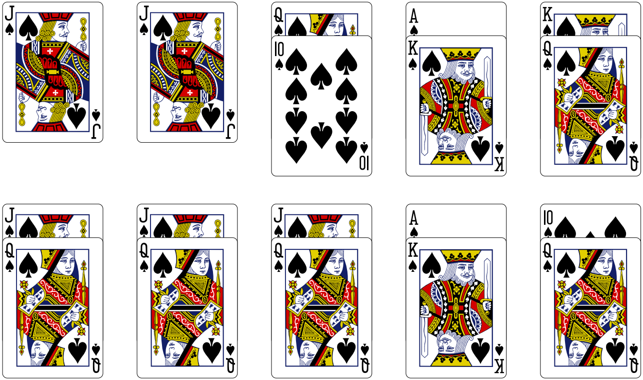
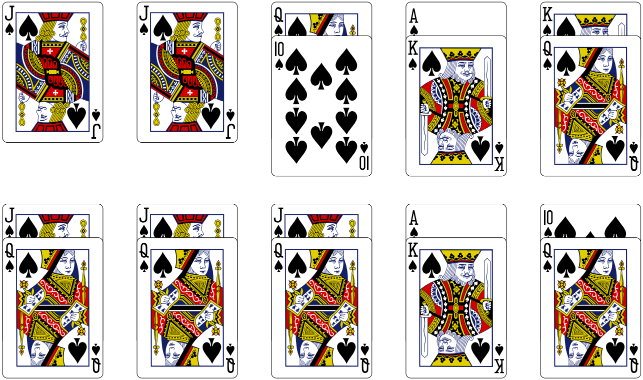


some card games

what was the difference?



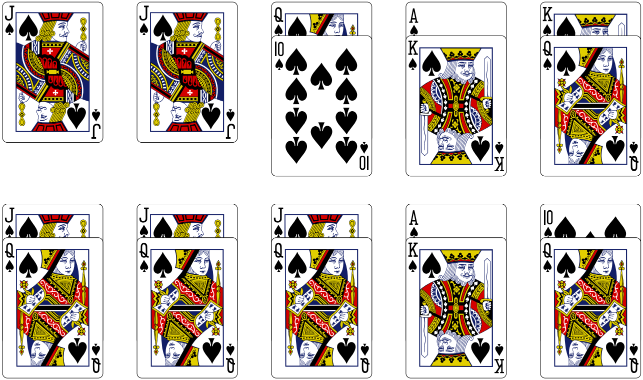
- in the top game, Dealer can prevent Player from increasing the number of different cards in the first two stacks



some card games

what was the difference?

- in the top game, Dealer can prevent Player from increasing the number of different cards in the first two stacks
- in the bottom game, Dealer cannot prevent prevent Player from increasing the number of different cards in the first three stacks



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- Player can turn $2t+1$ of the same card into $t+1$ different cards, so C is 'worth' $\left\lceil \frac{d_S(C)}{2} \right\rceil$

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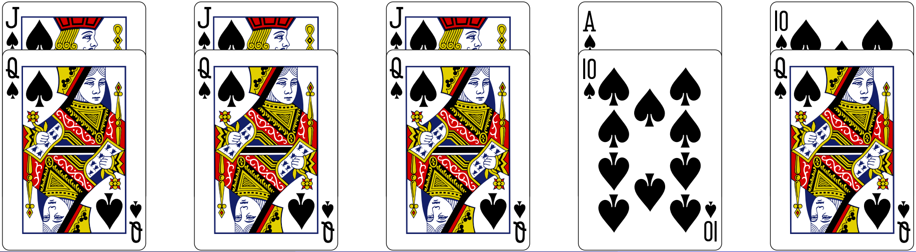
some card games

proof idea

1. Player looks for a set of card types that give a system of distinct representatives of all the stacks containing them



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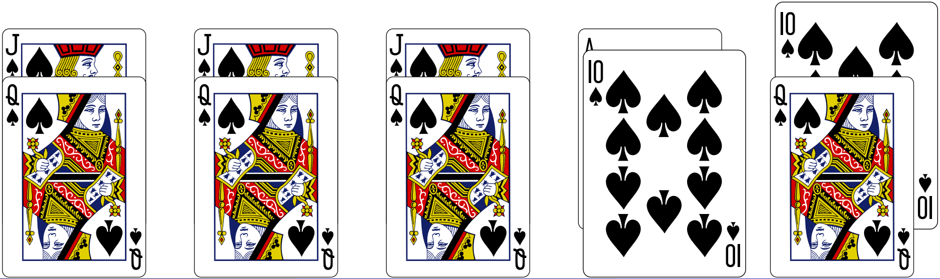

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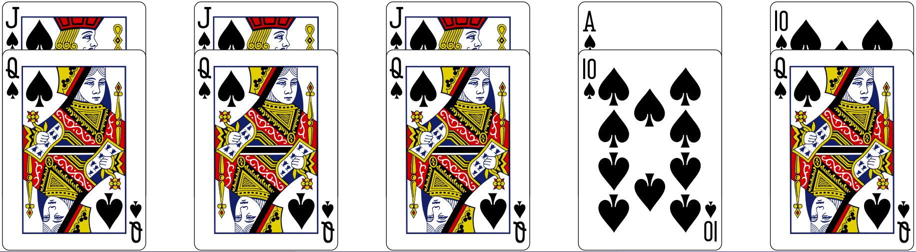

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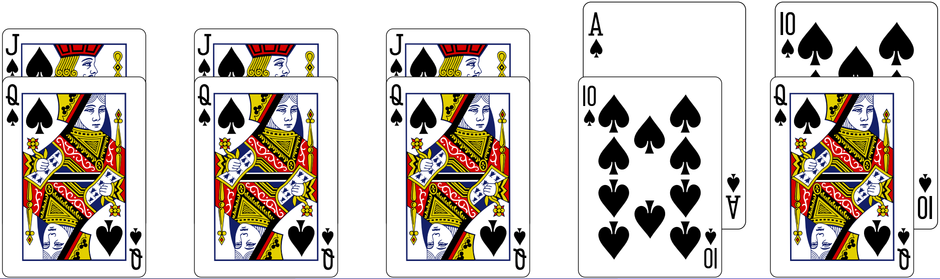

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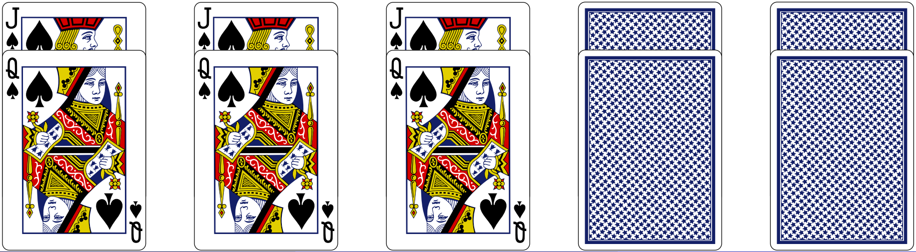

some card games
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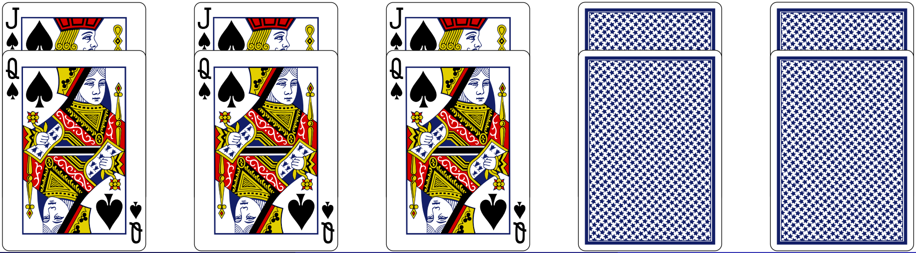

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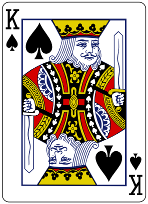

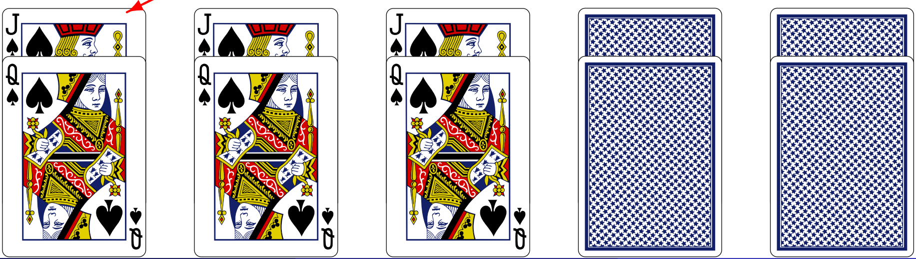

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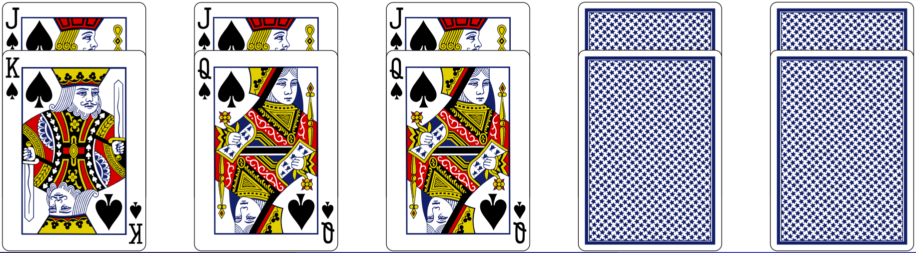

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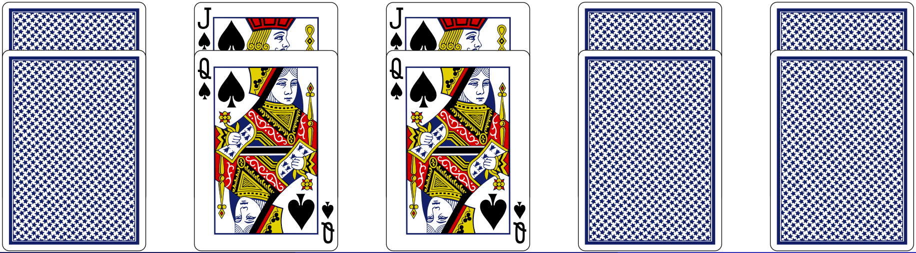

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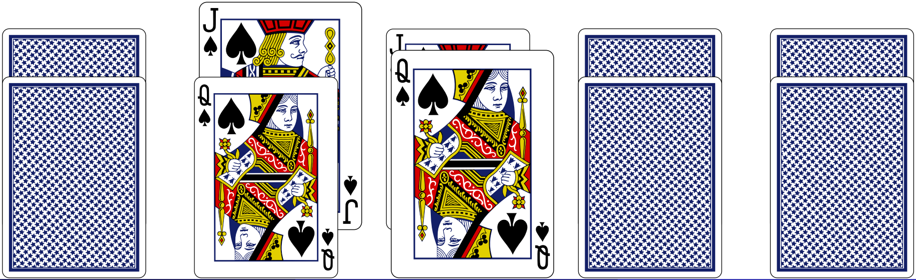

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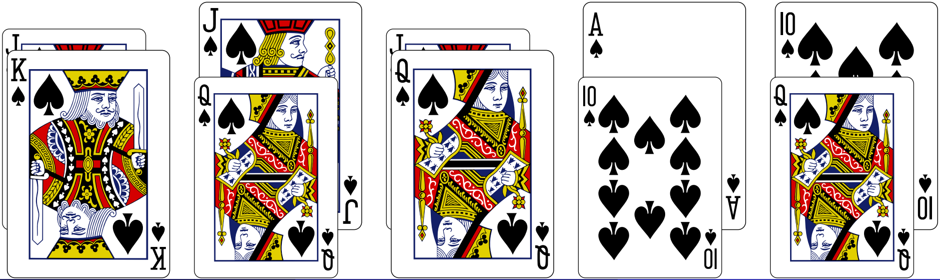

A generalization of Hall's theorem

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- for each $t \geq 1$, the *t-game* allows Dealer to make up to t swaps

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Player can win in the t-game if and only if for every set of stacks S we have

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  - Player's moves are useless

edge coloring

setup

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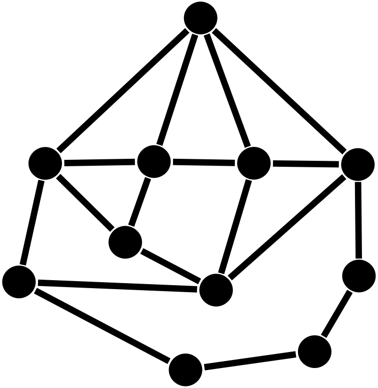
Vizing's theorem

Any simple graph can be edge-colored using at most one more color than its maximum degree.

- proceed by induction on the number of vertices



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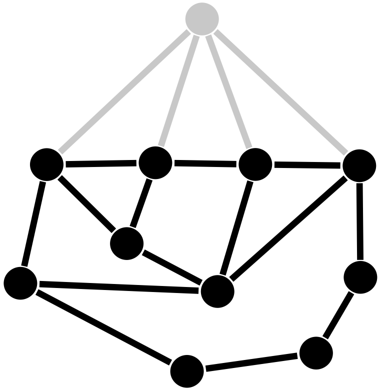

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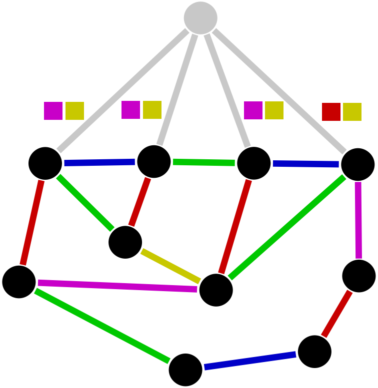

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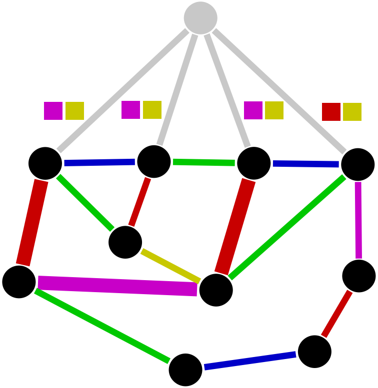

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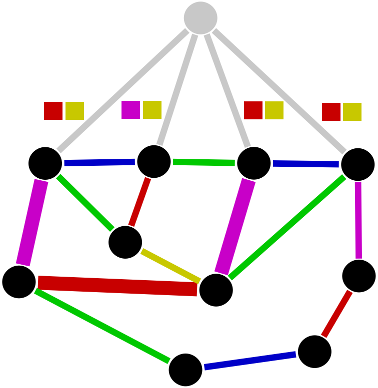

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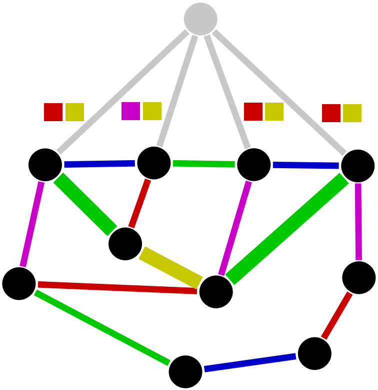

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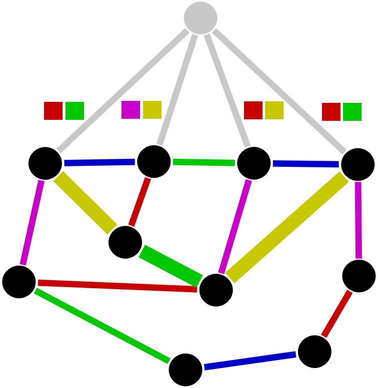

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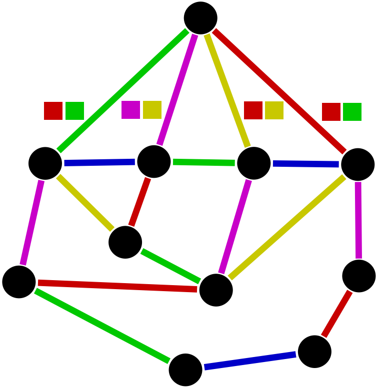

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- so, we have the desired winning condition

$$\sum_{C \in \bigcup S} \frac{d_S(C)}{2} \geq |S|$$

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  - generalizes easily to multigraphs
  - a more general game unifies much of edge-coloring theory

the more general game

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Pick α in the pot and $v \in V(G)$ with $\alpha \notin L(v)$ and set $L(v) := L(v) \cup \{\alpha\} - \beta$ for some $\beta \in L(v)$.

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Breaker's turn

If Fixer modified $L(v)$ by inserting α and removing β, then Breaker can either do nothing or pick $w \in V(G - v)$ and modify its list by swapping α for β or β for α.

Definition

For $C \subseteq \text{Pot}(L)$ and $H \subseteq G$, let $H_{L,C}$ be the subgraph of H induced on the vertices v with $L(v) \cap C \neq \emptyset$. For $H \subseteq G$, put

$$\psi_L(H) = \sum_{\alpha \in \text{Pot}(L)} \left\lfloor \frac{|H_{L,\alpha}|}{2} \right\rfloor.$$

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Superabundance

We say that (H, L) is abundant if $\psi_L(H) \geq \|H\|$ and that (H, L) is superabundant if for every $H' \subseteq H$, the pair (H', L) is abundant.

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Necessary Condition

If Fixer can win, then (G, L) is superabundant.

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- the chronicle C is a multigraph with vertex set $V(G) \cup \{\infty\}$ that will be updated as the game progresses. Each edge of C will be labeled with a doubleton of colors $\{\alpha, \beta\} \subseteq \text{Pot}(L)$. At the start of the game C is edgeless.

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Breaker's turn

If there is a $vx \in E(C - \infty)$ labeled $\{\alpha, \beta\}$, then Breaker swaps α and β at x. If instead $v\infty \in E(C)$, Breaker does nothing. Otherwise, Breaker can do nothing, or pick $w \in V(G - v)$ with $|\{\alpha, \beta\} \cap L(w)| = 1$ such that no edge incident to w in C has label $\{\alpha, \beta\}$, and swap α and β at w.

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Chronicle update

Remove all edges of C whose label intersects $\{\alpha, \beta\}$ in exactly one color. If Breaker swapped α and β at z and there is no vz edge in C labeled $\{\alpha, \beta\}$, then add one. Otherwise, if Breaker did nothing and there is no $v\infty$ edge in C labeled $\{\alpha, \beta\}$, then add one.

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Equivalent game

Fixer picks different colors $\alpha, \beta \in \mathrm{Pot}(L)$. Let S be the $w \in V(G)$ with $|\{\alpha, \beta\} \cap L(w)| = 1$. Breaker picks a partition $P_1, \ldots, P_k$ of S where $|P_i| \leq 2$ for all i. For each i, Fixer either chooses to swap α and β on all vertices in P_i or on no vertices in P_i.