Edge lower bounds via discharging notes

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1 Introduction

For a graph G, let $d(G)$ be the average degree of G. Let $\mathcal{T}_k$ be the Gallai trees with maximum degree at most $k-1$, excepting K_k.

2 Gallai's bound via discharging

Theorem 2.1 (Gallai)**.** *For $k \geq 4$ and $G \neq K_k$ a k-AT-critical graph, we have*

$$d(G) > k - 1 + \frac{k-3}{k^2-3}.$$

Proof. Start with initial charge function $\mathrm{ch}(v) = d_G(v)$. Have each k^+-vertex give charge $\frac{k-1}{k^2-3}$ to each of its $(k-1)$-neighbors. Then let the vertices in each component of the low vertex subgraph share their total charge equally. Let $\mathrm{ch}^*(v)$ be the resulting charge function. We finish the proof by showing that $\mathrm{ch}^*(v) \geq k - 1 + \frac{k-3}{k^2-3}$ for all $v \in V(G)$.

If v is a k^+-vertex, then $ch^*(v) \geq d_G(v) - \frac{k-1}{k^2-3} d_G(v) = \left(1 - \frac{k-1}{k^2-3}\right) d_G(v) \geq \left(1 - \frac{k-1}{k^2-3}\right) k = k - 1 + \frac{k-3}{k^2-3}$ as desired.

Let T be a component of the low vertex subgraph. Then the vertices in T receive total charge

$$\frac{k-1}{k^2-3} \sum_{v \in V(T)} k - 1 - d_G(v) = \frac{k-1}{k^2-3} \left((k-1)|T| - 2\left\|T\right\|\right).$$

So, after distributing this charge out equally, each vertex in T receives charge

$$\frac{1}{|T|} \frac{k-1}{k^2-3} ((k-1)|T| - 2\left\|T\right\|) = \frac{k-1}{k^2-3} \left((k-1) - d(T)\right).$$

By Lemma 2.2, this is at least

$$\frac{k-1}{k^2-3} \left((k-1) - \left(k - 2 + \frac{2}{k-1}\right)\right) = \frac{k-1}{k^2-3} \left(\frac{k-3}{k-1}\right) = \frac{k-3}{k^2-3}.$$

Hence each low vertex ends with charge at least $k - 1 + \frac{k-3}{k^2-3}$ as desired. $\square$

Lemma 2.2 (Gallai)**.** *For $k \geq 4$ and $T \in \mathcal{T}_k$, we have $d(T) < k - 2 + \frac{2}{k-1}$.*

Proof. Suppose not and choose a counterexample T minimizing $|T|$. Then T has at least two blocks. Let B be an endblock of T. If B is K_t for $2 \le t \le k-2$, then remove the non-separating vertices of B from T to get T'. By minimality of $|T|$, we have

$$2\,\|T\| - t(t-1) = 2\,\|T'\| < \left(k-2+\frac{2}{k-1}\right)|T'| = \left(k-2+\frac{2}{k-1}\right)|T| - \left(k-2+\frac{2}{k-1}\right)(t-1).$$

Hence we have the contradiction

$$2\,\|T\| < \left(k-2+\frac{2}{k-1}\right)|T| + (t+2-k-\frac{2}{k-1})(t-1) \le \left(k-2+\frac{2}{k-1}\right)|T|.$$

The case when B is an odd cycle is the same as the above, a longer cycle just makes things better. Finally, if $B = K_{k-1}$, remove all vertices of B from T to get T'. By minimality of $|T|$, we have

$$\begin{aligned}
2\,\|T\| - (k-1)(k-2) - 2 = 2\,\|T'\| & \\
&< \left(k-2+\frac{2}{k-1}\right)|T'| \\
&= \left(k-2+\frac{2}{k-1}\right)|T| - \left(k-2+\frac{2}{k-1}\right)(k-1).
\end{aligned}$$

Hence $2\,\|T\| < \left(k-2+\frac{2}{k-1}\right)|T|$, a contradiction. □

3 An initial improved bound

Lemma 2.2 is best possible as can be seen by the family of graphs with blocks on a path alternating K_{k-1} and K_2. But we have reducible configurations (see the last section for the precise statements) that place restrictions on K_{k-1} blocks. To state these restrictions, we need the following auxiliary bipartite graph.

For a k-AT-critical graph G, let $\mathcal{L}(G)$ be the subgraph of G induced on the $(k-1)$-vertices and $\mathcal{H}(G)$ the subgraph of G induced on the k-vertices. For $T \in \mathcal{T}_k$, let $W^k(T)$ be the set of vertices of T that are contained in some K_{k-1} in T. Let $\mathcal{B}_k(G)$ be the bipartite graph with one part $V(\mathcal{H}(G))$ and the other part the components of $\mathcal{L}(G)$. Put an edge between $y \in V(\mathcal{H}(G))$ and a component T of $\mathcal{L}(G)$ if and only if $N(y) \cap W^k(T) \neq \emptyset$. Then Lemma 4.2 says that $\mathcal{B}_k(G)$ is 2-degenerate.

We can use this fact to refine our discharging argument. Let ϵ and γ be parameters that we will determine where $\epsilon \le \gamma < 2\epsilon$. Start with initial charge function $\mathrm{ch}(v) = d_G(v)$.

1. Each k^+-vertex gives charge ϵ to each of its $(k-1)$-neighbors not in a K_{k-1},

2. Each $(k+1)^+$-vertex give charge γ to each of its $(k-1)$-neighbors in a K_{k-1},

3. Let $Q = \mathcal{B}_k(G)$. Repeat the following steps until Q is empty.

   (a) For each component T of $\mathcal{L}(G)$ in Q that has degree at most two in Q do the following:

i. For each $v \in V(\mathcal{H}(G)) \cap V(Q)$ such that $\left|N_G(v) \cap W^k(T)\right| = 2$, pick one $x \in N_G(v) \cap W^k(T)$ and send charge γ from v to x,

ii. Remove T from Q.

(b) Pick $v \in V(\mathcal{H}(G)) \cap V(Q)$ with degree at most two in Q. Send charge γ from v to each $x \in N_G(v) \cap W^k(T)$ for each component T of $\mathcal{L}(G)$ where $vT \in E(Q)$.

(c) Remove v from Q.

4. Have the vertices in each component of $\mathcal{L}(G)$ share their total charge equally.

Let $\mathrm{ch}^*(v)$ be the resulting charge function. Here is some intuition for why this might be a useful refinement. In (3b), v sends charge to at most two different T and so, by Lemma 4.1 (or our 'beyond degree choosability' classification), v loses charge at most 3γ. On the other hand, from (3a) each component T of $\mathcal{L}(G)$ receives charge γ for all but at most two non-separating vertices in a K_{k-1} (the at most two is coming from Lemma 4.1 and the fact that we leave T in Q until it has degree at most two and when it does, we send up to two extra γ to T in (3ai) as needed). Note that (3ai) doesn't cause any $v \in V(\mathcal{H}(G))$ to lose more than 3γ, because it only gets enacted when the component T is about to be removed, after that v does not have two neighbors in another component. So, we can get each T almost as much charge as we could hope for without losing too much from the k-vertices. We don't have the same control over $(k+1)^+$-vertices, but it won't matter since they have extra charge to start with and sending γ to every $(k-1)$-neighbor will leave enough charge (we'll use $\gamma < 2\epsilon$ here).

To analyze this discharging procedure we need a bound like Lemma 2.2, but taking into account the number of edges in $\mathcal{B}_k(G)$. We can do this by taking into account the number of non-separating vertices in K_{k-1}'s in T. To this end, for $T \in \mathcal{T}_k$, let $q(T)$ be the number of non-separating vertices in a K_{k-1} in T. We give a family of such bounds. Without more reducible configurations we can't hope to do better than average degree $k-3$ because of K_{k-2} components, that is why the bound below has $(k-3+p(k))|T|$, a slight worsening of average degree $k-3$.

Lemma 3.1. *Let $K \geq 7$ and $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$, $h\colon \mathbb{N} \to \mathbb{R}$ be such that for all $k \geq K$ we have*

1. *$f(k) \geq t(t+2-k-p(k))$ for all $t \in [k-2]$; and*
2. *$f(k) \geq (5-k-p(k))s$ for all $s \geq 5$; and*
3. *$f(k) \geq (k-1)(1-p(k)-h(k))$; and*
4. *$p(k) \geq h(k)+5-k$; and*
5. *$p(k) \geq \frac{3}{k-2}$; and*
6. *$p(k) \geq \frac{2+h(k)}{k-2}$; and*
7. *$(k-1)p(k)+(k-3)h(k) \geq k+1$.*

Then for $k \geq K$ and $T \in \mathcal{T}_k$, we have

$$2\|T\| \leq (k-3+p(k))\,|T| + f(k) + h(k)q(T).$$

Proof. Suppose not and choose a counterexample T minimizing $|T|$. First, suppose T is K_t for $t \in [k-2]$. Then $t(t-1) > (k-3+p(k))t + f(k)$ contradicting (1). If T is C_{2r+1} for $r \geq 2$, then $2(2r+1) > (k-3+p(k))(2r+1)+f(k)$ and hence $f(k) < (5-k-p(k))(2r+1)$ contradicting (2). If T is K_{k-1}, then $(k-1)(k-2) > (k-3+p(k))(k-1)+f(k)+h(k)(k-1)$ contradicting (3).

Hence T has at least two blocks. Let B be an endblock of T and x_B the cutvertex of T contained in B. Let $T' = T - (V(B) \setminus \{x_B\})$. Then, by minimality of $|T|$, we have

$$2\|T'\| \leq (k-3+p(k))\,|T'| + f(k) + h(k)q(T').$$

Hence

$$2\|T\| - 2\|B\| \leq (k-3+p(k))\,(|T| - (|B|-1)) + f(k) + h(k)q(T').$$

Since T is a counterexample, this gives

$$2\|B\| > (k-3+p(k))(|B|-1) + h(k)\left(q(T) - q(T')\right). \tag{*}$$

Suppose B is K_t for $3 \leq t \leq k-3$ or B is an odd cycle. Then $q(T') = q(T)$, $2\|B\| \leq |B|\,(|B|-1)$ and $2\|B\| = 2\,|B|$ if $|B| > k-3$. Since $p(k) \geq \frac{4}{k-2}$ by (5), this contradicts *.

If B is K_2, then $q(T') \leq q(T)+1$ and * gives $2 > k-3+p(k)-h(k)$ contradicting (4).

To handle the cases when B is K_{k-2} or K_{k-1} we need to remove x_B from T as well. Let $T^* = T - V(B)$. Then, by minimality of $|T|$, we have

$$2\|T^*\| \leq (k-3+p(k))\,|T^*| + f(k) + h(k)q(T^*).$$

Hence

$$2\|T\| - 2\|B\| - 2(d_T(x_B) - d_B(x_B)) \leq (k-3+p(k))\,(|T| - |B|) + f(k) + h(k)q(T^*).$$

Since T is a counterexample and B is complete, this gives

$$2\|B\| > (k-3+p(k))\,|B| - 2(d_T(x_B) + 1 - |B|) + h(k)\left(q(T) - q(T^*)\right),$$

which is

$$2\|B\| > (k-1+p(k))\,|B| - 2d_T(x_B) - 2 + h(k)\left(q(T) - q(T^*)\right). \tag{**}$$

Suppose B is K_{k-1}. Then $d_T(x_B) = k-1$ and $q(T^*) \leq q(T) - (k-2) + 1 = q(T) - (k-3)$. From **, we have

$$\begin{aligned}(k-1)(k-2) &> (k-1+p(k))(k-1) - 2(k-1) - 2 + h(k)(k-3)\\ &= (k-1)(k-2) + p(k)(k-1) - (k+1) + h(k)(k-3),\end{aligned}$$

contradicting (7).

Finally, suppose B is K_{k-2}. Then $d_T(x_B) = k-1$ or $d_T(x_B) = k-2$. In the former case, $q(T) = q(T^*)$ and in the latter $q(T^*) \leq q(T)+1$. If $d_T(x_B) = k-2$, we have

$$(k-2)(k-3) > (k-1+p(k))(k-2) - 2(k-2) - 2 - h(k) = (k-2)(k-3) - 2 + (k-2)p(k) - h(k),$$

contradicting (6).

Now we need to handle the remaining case when B is K_{k-2} and $d_T(x_B) = k-1$. All of the above cases were for when B was any endblock of T, so we may assume that every endblock of T is a K_{k-2} that shares a vertex with an odd cycle. Choose an endblock B that is the end of a longest path in the block-tree of T. Let C be the odd cycle sharing a vertex x_B with B. We claim that all but one vertex of C is in an endblock. Since B is the end of a longest path, C cannot have two non-cut-vertices that are both not in endblocks, for then we could get a longer path. So, to prove our claim, it will suffice to show that every vertex of C is a cut-vertex. Suppose $v \in V(C)$ is not a cut-vertex. Then $d_T(v) = 2$ and hence by minimality of $|T|$

$$2\|T\| - 4 \leq (k-3+p(k))(|T|-1) + f(k) + h(k)q(T-v),$$

Since $q(T-v) = q(T)$, the fact that T is a counterexample implies

$$4 > k-3+p(k),$$

a contradiction since $k \geq K \geq 7$ and $p(k) > 0$. So, we have shown that all but one vertex of C is in an endblock. Hence there are endblocks A and B such that $x_A, x_B \in V(C)$ and x_A is adjacent to x_B. let $\hat{T} = T - (V(A) \cup V(B))$. Then $q(\hat{T}) = q(T)$. Since the edge $x_A x_B$ is shared, by minimality of $|T|$, we have

$$2\|T\| - 2\|A\| - 2\|B\| - 6 \leq (k-3+p(k))(|T| - |A| - |B|) + f(k) + h(k)q(T).$$

Since T is a counterexample, this gives

$$2\|A\| + 2\|B\| + 6 > (k-3+p(k))(|A| + |B|),$$

which is

$$2(k-2)(k-3) + 6 > 2(k-3+p(k))(k-2),$$

giving

$$3 > (k-2)p(k),$$

which contradicts (5). $\square$

Lemma 3.2. *Let $K \geq 7$ and $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$, $h\colon \mathbb{N} \to \mathbb{R}$ be such that for all $k \geq K$ we have*

1. *$f(k) \geq (k-1)(1-p(k)-h(k))$; and*
2. *$p(k) \geq h(k) + 5 - k$; and*
3. *$p(k) \geq \frac{3}{k-2}$; and*
4. *$p(k) \geq \frac{2+h(k)}{k-2}$; and*
5. *$(k-1)p(k) + (k-3)h(k) \geq k+1$.*

Then for $k \geq K$ and $T \in \mathcal{T}_k$ with $K_{k-1} \subseteq T$, we have

$$2\|T\| \leq (k-3+p(k))\,|T| + f(k) + h(k)q(T).$$

Proof. Suppose not and choose a counterexample T minimizing $|T|$. If T has only one block, then $T = K_{k-1}$ and hence $(k-1)(k-2) > (k-3+p(k))(k-1) + f(k) + h(k)(k-1)$ contradicting (1).

Hence T has at least two blocks. Suppose T has an endblock B that is an odd cycle or K_t for $3 \leq t \leq k-3$. Let x_B the cutvertex of T contained in B. Let $T' = T - (V(B) \setminus \{x_B\})$. Then $K_{k-1} \subseteq T'$, so by minimality of $|T|$, we have

$$2\|T'\| \leq (k-3+p(k))\,|T'| + f(k) + h(k)q(T').$$

Hence

$$2\|T\| - 2\|B\| \leq (k-3+p(k))\,(|T| - (|B|-1)) + f(k) + h(k)q(T').$$

Since T is a counterexample, this gives

$$2\|B\| > (k-3+p(k))(|B|-1) + h(k)\left(q(T) - q(T')\right). \tag{*}$$

Then $q(T') = q(T)$, $2\|B\| \leq |B|\,(|B|-1)$ and $2\|B\| = 2\,|B|$ if $|B| > k-3$. Since $p(k) \geq \frac{3}{k-2}$ by (3), this contradicts *.

If B is K_2, then $q(T') \leq q(T) + 1$ and * gives $2 > k - 3 + p(k) - h(k)$ contradicting (2).

Hence every endblock of B is K_{k-2} or K_{k-1}. To handle these cases, we will need to remove x_B from T as well. Let $\{B_1, \ldots, B_r\}$ be the endblocks B_i of T such that $K_{k-1} \subseteq T - V(B_i)$. Then $r \geq 1$ since two K_{k-1} blocks cannot share a vertex. For $i \in [r]$, let $T_i = T - V(B_i)$.

Then, by minimality of $|T|$, we have

$$2\|T_i\| \leq (k-3+p(k))\,|T_i| + f(k) + h(k)q(T_i).$$

Hence

$$2\|T\| - 2\|B_i\| - 2(d_T(x_{B_i}) - d_{B_i}(x_{B_i})) \leq (k-3+p(k))\,(|T| - |B_i|) + f(k) + h(k)q(T_i).$$

Since T is a counterexample and B_i is complete, this gives

$$2\|B_i\| > (k-3+p(k))\,|B_i| - 2(d_T(x_{B_i}) + 1 - |B_i|) + h(k)\left(q(T) - q(T_i)\right),$$

which is

$$2\|B_i\| > (k-1+p(k))\,|B_i| - 2d_T(x_{B_i}) - 2 + h(k)\left(q(T) - q(T_i)\right). \tag{**}$$

If B_i is K_{k-1}, we have $d_T(x_{B_i}) = k-1$ and $q(T_i) \leq q(T) - (k-2) + 1 = q(T) - (k-3)$. From **, we have

$$\begin{aligned}(k-1)(k-2) &> (k-1+p(k))(k-1) - 2(k-1) - 2 + h(k)(k-3) \\ &= (k-1)(k-2) + p(k)(k-1) - (k+1) + h(k)(k-3),\end{aligned}$$

contradicting (5).

Hence $B_i = K_{k-2}$ for all $i \in [r]$. So $d_T(x_{B_i}) = k-1$ or $d_T(x_{B_i}) = k-2$. In the former case, $q(T) = q(T_i)$ and in the latter $q(T_i) \leq q(T) + 1$. If $d_T(x_{B_i}) = k-2$, we have

$$(k-2)(k-3) > (k-1+p(k))(k-2)-2(k-2)-2-h(k) = (k-2)(k-3)-2+(k-2)p(k)-h(k),$$

contradicting (4).

Hence $d_T(x_{B_i}) = k-1$ for all $i \in [r]$. If B and B' are blocks that are the ends of a longest path in the block-tree of T, then $\{B, B'\} \cap \{B_1, \dots, B_r\} \neq \emptyset$. By symmetry, we may assume that $B = B_1$. Let C be the odd cycle sharing a vertex x_B with B. We claim that all but one vertex of C is in an endblock. Since B is the end of a longest path, C cannot have two non-cut-vertices that are both not in endblocks, for then we could get a longer path. So, to prove our claim, it will suffice to show that every vertex of C is a cut-vertex. Suppose $v \in V(C)$ is not a cut-vertex. Then $d_T(v) = 2$ and hence by minimality of $|T|$

$$2\,\|T\| - 4 \leq (k-3+p(k))\,(|T|-1) + f(k) + h(k)q(T-v),$$

Since $q(T-v) = q(T)$, the fact that T is a counterexample implies

$$4 > k-3+p(k),$$

a contradiction since $k \geq K \geq 7$ and $p(k) > 0$ by (3). So, we have shown that all but one vertex of C is in an endblock. Hence there are endblocks A and B such that $x_A, x_B \in V(C)$ and x_A is adjacent to x_B. let $\hat{T} = T - (V(A) \cup V(B))$. Then $K_{k-1} \subseteq \hat{T}$ and $q(\hat{T}) = q(T)$. Since the edge $x_A x_B$ is shared, by minimality of $|T|$, we have

$$2\,\|T\| - 2\,\|A\| - 2\,\|B\| - 6 \leq (k-3+p(k))\,(|T| - |A| - |B|) + f(k) + h(k)q(T).$$

Since T is a counterexample, this gives

$$2\,\|A\| + 2\,\|B\| + 6 > (k-3+p(k))(|A| + |B|),$$

which is

$$2(k-2)(k-3) + 6 > 2(k-3+p(k))(k-2),$$

giving

$$3 > (k-2)p(k),$$

which contradicts (3). □

Lemma 3.3. *Let $K \geq 7$ and $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$ be such that for all $k \geq K$ we have*

1. $p(k) \geq \frac{-f(k)}{k-2}$*; and*
2. $p(k) \geq \frac{-f(k)}{5} + 5 - k$*; and*
3. $p(k) \geq \frac{3}{k-2}$.

Then for $k \geq K$ and $T \in \mathcal{T}_k$ with $K_{k-1} \not\subseteq T$, we have

$$2\,\|T\| \leq (k-3+p(k))\,|T| + f(k).$$

Proof. Suppose not and choose a counterexample T minimizing $|T|$. First, suppose T is K_t for $t \in [k-2]$. Then $t(t-1) > (k-3+p(k))t + f(k)$ contradicting (1). If T is C_{2r+1} for $r \geq 2$, then $2(2r+1) > (k-3+p(k))(2r+1)+f(k)$ and hence $f(k) < (5-k-p(k))(2r+1)$ contradicting (2).

Hence T has at least two blocks. Let B be an endblock of T and x_B the cutvertex of T contained in B. Let $T' = T - (V(B) \setminus \{x_B\})$. Then, by minimality of $|T|$, we have

$$2\,\|T'\| \leq (k-3+p(k))\,|T'| + f(k).$$

Hence

$$2\,\|T\| - 2\,\|B\| \leq (k-3+p(k))\,(|T| - (|B|-1)) + f(k).$$

Since T is a counterexample, this gives

$$2\,\|B\| > (k-3+p(k))(|B|-1). \tag{*}$$

Suppose B is K_t for $3 \leq t \leq k-3$ or B is an odd cycle. Then $2\,\|B\| \leq |B|\,(|B|-1)$ and $2\,\|B\| = 2\,|B|$ if $|B| > k-3$. Since $p(k) \geq \frac{3}{k-2}$ by (3), this contradicts *.

If B is K_2, then * gives $2 > k-3+p(k)$, a contradiction since $k \geq 5$ and $p(k) > 0$ by (3).

To handle the case when B is K_{k-2} we need to remove x_B from T as well. Let $T^* = T - V(B)$. Then, by minimality of $|T|$, we have

$$2\,\|T^*\| \leq (k-3+p(k))\,|T^*| + f(k).$$

Hence

$$2\,\|T\| - 2\,\|B\| - 2(d_T(x_B) - d_B(x_B)) \leq (k-3+p(k))\,(|T| - |B|) + f(k).$$

Since T is a counterexample and B is complete, this gives

$$2\,\|B\| > (k-3+p(k))\,|B| - 2(d_T(x_B) + 1 - |B|),$$

which is

$$2\,\|B\| > (k-1+p(k))\,|B| - 2d_T(x_B) - 2. \tag{**}$$

Since $B = K_{k-2}$, we have either $d_T(x_B) = k-1$ or $d_T(x_B) = k-2$. If $d_T(x_B) = k-2$, we have

$$(k-2)(k-3) > (k-1+p(k))(k-2) - 2(k-2) - 2 = (k-2)(k-3) - 2 + (k-2)p(k),$$

contradicting (3).

Now we need to handle the remaining case when B is K_{k-2} and $d_T(x_B) = k-1$. All of the above cases were for when B was any endblock of T, so we may assume that every endblock of T is a K_{k-2} that shares a vertex with an odd cycle. Choose an endblock B that is the end of a longest path in the block-tree of T. Let C be the odd cycle sharing a vertex x_B with B. We claim that all but one vertex of C is in an endblock. Since B is the end of a longest path, C cannot have two non-cut-vertices that are both not in endblocks, for then we could get a longer path. So, to prove our claim, it will suffice to show that every vertex of C is a cut-vertex. Suppose $v \in V(C)$ is not a cut-vertex. Then $d_T(v) = 2$ and hence by minimality of $|T|$

$$2\,\|T\| - 4 \leq (k - 3 + p(k))\,(|T| - 1) + f(k),$$

The fact that T is a counterexample implies

$$4 > k - 3 + p(k),$$

a contradiction since $k \geq K \geq 7$ and $p(k) > 0$. So, we have shown that all but one vertex of C is in an endblock. Hence there are endblocks A and B such that $x_A, x_B \in V(C)$ and x_A is adjacent to x_B. let $\hat{T} = T - (V(A) \cup V(B))$. Since the edge $x_A x_B$ is shared, by minimality of $|T|$, we have

$$2\,\|T\| - 2\,\|A\| - 2\,\|B\| - 6 \leq (k - 3 + p(k))\,(|T| - |A| - |B|) + f(k).$$

Since T is a counterexample, this gives

$$2\,\|A\| + 2\,\|B\| + 6 > (k - 3 + p(k))(|A| + |B|),$$

which is

$$2(k-2)(k-3) + 6 > 2(k - 3 + p(k))(k-2),$$

giving

$$3 > (k-2)p(k),$$

which contradicts (3). $\square$

Lemma 3.3 works with $p(k) = \frac{3}{k-2}$ and $f(k) = -3$. We probably get (2) for free, clean up later. In the discharging it will be convenient to apply Lemma 3.3 with a larger $p(k)$ to match the one we get when we have K_{k-1} blocks (we can even be wasteful with the charge and use $f(k) = 0$).

Now some examples of using Lemma 3.1 and Lemma 3.2. What happens if we take $h(k) = 0$ in Lemma 3.1? Then, by (7), we need $(k-1)p(k) \geq k+1$ and hence $p(k) \geq 1 + \frac{2}{k-1}$. Taking $p(k) = 1 + \frac{2}{k-1}$, (3) requires $f(k) \geq -2$. Using $f(k) = -2$, all of the other conditions are satisfied and we conclude $2\,\|T\| \leq \left(k - 2 + \frac{2}{k-1}\right)|T| - 2$ for every $T \in \mathcal{T}_k$ when $k \geq 4$. This is a slight refinement of Gallai's Lemma 2.2.

Instead, let's make $p(k)$ as small as Lemma 3.2 will let us. By (4), $h(k) \leq (k-2)p(k) - 2$, plugging this in to (5) and solving we get $p(k) \geq \frac{3k-5}{k^2-4k+5}$. Now $\frac{3k-5}{k^2-4k+5} \geq \frac{3}{k-2}$ for $k \geq 7$, so $p(k) = \frac{3k-5}{k^2-4k+5}$ satisfies (3). With $h(k) = \frac{k(k-3)}{k^2-4k+5}$, (4) and (5) are also satisfied. Now with $f(k) = -\frac{2(k-1)(2k-5)}{k^2-4k+5}$, condition (1) is satisfied and hence by Lemma we have the following.

Corollary 3.4. *For $k \geq 7$ and $T \in \mathcal{T}_k$ with $K_{k-1} \subseteq T$, we have*

$$2\,\|T\| \leq \left(k - 3 + \frac{3k-5}{k^2-4k+5}\right)|T| - \frac{2(k-1)(2k-5)}{k^2-4k+5} + \frac{k(k-3)}{k^2-4k+5}q(T).$$

If we put the Kostochka-Stiebitz bound on $\sigma(T)$ into this form we get the following.

Lemma 3.5 (Kostochka-Stiebitz)**.** *For $k \geq 7$ and $T \in \mathcal{T}_k$, we have*

$$2\,\|T\| \leq \left(k - 3 + \frac{4(k-1)}{k^2-3k+4}\right)|T| - \frac{4(k^2-3k+2)}{k^2-3k+4} + \frac{k^2-3k}{k^2-3k+4}q(T).$$

Note that $\frac{3k-5}{(k-5)(k-1)} < \frac{4(k-1)}{k^2-3k+4}$ for $k \geq 7$.

3.1 Analyzing the discharging

Our discharging procedure gives charge ϵ to a component T for every incident edge not ending in a K_{k-1}. The number of such edges is exactly

$$A(T) := -q(T) + \sum_{v \in V(T)} k - 1 - d_T(v) = (k-1)\,|T| - 2\,\|T\| - q(T).$$

Suppose we have a bound when $K_{k-1} \subseteq T$ of the form

$$2\,\|T\| \leq (k - 3 + p(k))\,|T| + f(k) + h(k)q(T) + a(k, |T|).$$

So, when $K_{k-1} \subseteq T$ we get

$$A(T) \geq (2 - p(k))\,|T| - f(k) - (h(k) + 1)q(T) - a(k, |T|).$$

We will use $\gamma = (h(k)+1)\epsilon$ in order to make the $q(T)$ term cancel. That happens because T receives charge on all but at most two of its non-separating vertices in a K_{k-1}; that is, in discharging steps 2 and 3, T receives charge at least $\gamma \max\{0, q(G) - 2\}$. Hence in total T receives charge at least

$$\epsilon A(T) + \gamma(q(G) - 2) = \epsilon\,(2 - p(k))\,|T| - \epsilon\,(a(k, |T|) + f(k) + 2(h(k) + 1))\,.$$

Then each vertex of T receives charge at least

$$\epsilon \left(2 - p(k) - \frac{a(k, |T|) + f(k) + 2(h(k) + 1)}{|T|} \right).$$

Put

$$\zeta(p, f, h, a) := \min_{K_{k-1} \subseteq T \in \mathcal{T}_k} \left(2 - p(k) - \frac{a(k, |T|) + f(k) + 2(h(k) + 1)}{|T|} \right).$$

We want the k-vertices to end with enough charge, the worst case is when

$$1 - (3\gamma + (k - 3)\epsilon) = \epsilon\zeta(p, f, h, a),$$

and thus

$$\epsilon = \frac{1}{k + 3h(k) + \zeta(p, f, h, a)},$$

$$\gamma = \frac{h(k) + 1}{k + 3h(k) + \zeta(p, f, h, a)}.$$

When $K_{k-1} \not\subseteq T$, we have $q(T) = 0$. By Lemma 3.3 with $f(k) = 0$, we get

$$2\,\|T\| \leq (k - 3 + p(k))\,|T|$$

and hence

$$A(T) \geq (2 - p(k))\,|T|\,,$$

which is sufficient charge when $\zeta(p, f, h, a) \leq 2 - p(k)$.

It remains to check that the $(k+1)^+$-vertices don't give away too much charge. Let v be a $(k+1)^+$-vertex, then v ends with charge at least

$$d(v) - \gamma d(v) = (1-\gamma)d(v) \geq (1-\gamma)(k+1) = (k+1)\frac{k-1+2h(k)+\zeta(p,f,h,a)}{k+3h(k)+\zeta(p,f,h,a)},$$

so we need

$$(k+1)\frac{k-1+2h(k)+\zeta(p,f,h,a)}{k+3h(k)+\zeta(p,f,h,a)} \geq k-1+\frac{\zeta(p,f,h,a)}{k+3h(k)+\zeta(p,f,h,a)},$$

simplifying, we get that we need

$$\text{some gross thing that should be easy to satisfy} \tag{*}$$

Let's just add this as another requirement, it will be easily satisfied by the functions we want to use. We have proved the following.

Theorem 3.6. *Let $K \geq 7$ and $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$, $h\colon \mathbb{N} \to \mathbb{R}$, $a\colon \mathbb{N}\times\mathbb{N} \to \mathbb{R}$ be functions satisfying *. If for all $k \geq K$ and $T \in \mathcal{T}_k$ with $K_{k-1} \subseteq T$ we have*

$$2\,\|T\| \leq (k-3+p(k))\,|T| + f(k) + h(k)q(T) + a(|T|,k),$$

then for $k \geq K$ and $G \neq K_k$ a k-AT-critical graph, we have

$$d(G) \geq k-1+\frac{\zeta(p,f,h,a)}{k+3h(k)+\zeta(p,f,h,a)}.$$

4 Reducible Configurations

Definition 1. A graph G is *AT-reducible* to H if H is a nonempty induced subgraph of G which is f_H-AT where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not AT-reducible to any nonempty induced subgraph, then it is *AT-irreducible*.

This first lemma tells us how a single high vertex can interact with the low vertex subgraph. This is the version Hal and i used, it (and more) follows from the classification in "mostlow".

Lemma 4.1. *Let $k \geq 5$ and let G be a graph with $x \in V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *$G - x$ has t components $H_1, H_2, \ldots, H_t$, and all are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k-1$ for all $v \in V(G-x)$; and*
4. *$\left|N(x) \cap W^k(H_i)\right| \geq 1$ for $i \in [t]$; and*
5. *$d_G(x) \geq t+2$.*

Then G is f-AT where $f(x) = d_G(x) - 1$ and $f(v) = d_G(v)$ for all $v \in V(G - x)$.

To deal with more than one high vertex we need the following auxiliary bipartite graph. For a graph G, $\{X, Y\}$ a partition of $V(G)$ and $k \geq 4$, let $\mathcal{B}_k(X, Y)$ be the bipartite graph with one part Y and the other part the components of $G[X]$. Put an edge between $y \in Y$ and a component T of $G[X]$ if and only if $N(y) \cap W^k(T) \neq \emptyset$. The next lemma tells us that we have a reducible configuration if this bipartite graph has minimum degree at least three.

Lemma 4.2. *Let $k \geq 7$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*
4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $\delta(\mathcal{B}) \geq 3$.*

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.

We also have the following version with asymmetric degree condition on $\mathcal{B}$. The point here is that this works for $k \geq 5$. As we'll see in the next section, the consequence is that we trade a bit in our size bound for the proof to go through with $k \in \{5, 6\}$.

Lemma 4.3. *Let $k \geq 5$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*
4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $d_{\mathcal{B}}(y) \geq 4$ for all $y \in Y$ and $d_{\mathcal{B}}(T) \geq 2$ for all components T of $G - Y$.*

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.