Edge-coloring via fixable subgraphs

Daniel W. Cranston[*] Landon Rabern[†]

July 10, 2015

Abstract

We give a general framework for showing that graphs are reducible for edge-coloring. A particular form of reducibility, called *fixability*, can be considered without reference to a containing graph. This has two key benefits: (i) we can now formulate necessary conditions for fixability, and (ii) the problem of fixability is easy for a computer to solve. The necessary condition of *superabundance* is sufficient for multistars and we conjecture that it is sufficient for trees as well (this would generalize the technique of Tashkinov trees). Via computer, we can generate thousands of reducible configurations, but we have short proofs for only a small fraction of these. The computer is able to write LaTeX code for its proofs, but they are only marginally enlightening and can run thousands of pages long. We give examples of how to use some of these reducible configurations to prove conjectures on edge-coloring for small maximum degree. Our aim in writing this paper is to spur development of methods for humans to understand what the computer already knows.

1 Introduction

Suppose we want to k-color a graph G. If we already have a k-coloring of an induced subgraph H of G, we might try to extend this coloring to all of G. We can view this task as the problem of list coloring $G - H$, where each vertex v in $G - H$ gets a list of colors formed from $\{1, \ldots, k\}$ by removing all colors used on $N(v)$ in H. Such list coloring problems have proved interesting in their own right, outside the context of completing partial colorings. In many situations we cannot complete just any k-coloring of H to all of G. Instead, we may need to modify the k-coloring of H to get a coloring we can extend. Given rules for how we may modify the k-coloring of H, we can recast the task of modifying the k-coloring and then extending it to G as the problem of list coloring $G - H$, where each vertex gets a list as before, but now we may modify these lists in a prescribed manner. Studying such list coloring/modifying problems in their own right has also proved useful.

As an example of this paradigm, the second author proved [4] a common generalization of Hall's marriage theorem and Vizing's theorem on edge-coloring. The present paper generalizes a special case of this result and puts it into a broader context. An interesting caveat

[*]Department of Mathematics and Applied Mathematics, Viriginia Commonwealth University, Richmond, VA; dcranston@vcu.edu; Research of the first author is partially supported by NSA Grant 98230-15-1-0013.

[†]LBD Data Solutions, Lancaster, PA; landon.rabern@gmail.com

arises when investigating this list coloring/modifying paradigm. Since we often want to prove coloring results for all graphs having certain properties and not just some fixed graph, we only have partial control over the outcome of a recoloring of H. For example, if we swap colors red and green in a component C of the red-green subgraph (that is, we perform a Kempe change), we may succeed in making some desired vertex red, but if C is somewhat arbitrary, we cannot precisely control what happens to the colors of the other vertices. In the list modifying/coloring paradigm, we model this lack of control as a two-player game—we move by doing the part of the recoloring we desire and then the other player gets a turn to muck things up. In the original context where we want to color G, the opponent is the graph G; more precisely, the embedding of $G - H$ in G is one way to describe a strategy for the second player. The general paradigm that we described above is for vertex coloring. In the rest of the paper, we consider only the special case that is edge-coloring (or, equivalently, vertex coloring line graphs).

All multigraphs are loopless. Let G be a multigraph, L a list assignment on $V(G)$, and $\text{pot}(L) = \bigcup_{v \in V(G)} L(v)$. An *$L$-pot* is a set X containing $\text{pot}(L)$. An *L-edge-coloring* is an edge-coloring π of G such that $\pi(xy) \in L(x) \cap L(y)$ for all $xy \in E(G)$; furthermore, we require $\pi(xy) \neq \pi(xz)$ for each vertex x and distinct neighbors y and z of x. For the maximum degree in G we write $\Delta(G)$, or simply Δ, when G is clear from context. For the edge-chromatic number of G we write $\chi'(G)$. We often denote the set $\{1, \dots, k\}$ by $[k]$.

2 Completing edge-colorings

Our goal is to convert a partial k-edge-coloring of a multigraph M into a k-edge-coloring of (all of) M. For a partial k-edge-coloring π of M, let M_π be the subgraph of M induced by the uncolored edges and let L_π be the list assignment on the vertices of M_π given by $L_\pi(v) = [k] - \{\tau \mid \pi(vx) = \tau \text{ for some edge } vx \in E(M)\}$.

Kempe chains give a powerful technique for converting a partial k-edge-coloring into a k-edge-coloring of the whole graph. The idea is to repeatedly exchange colors on two-colored paths until M_π has an edge-coloring ζ such that $\zeta(xy) \in L_\zeta(x) \cap L_\zeta(y)$ for all $xy \in E(M_\pi)$. In this sense the original list assignment L_π on M_π is *fixable*. In the next section, we give an abstract definition of this notion that frees us from the embedding in the ambient graph M. As we will see, computers enjoy this new freedom.

Throughout this section, we let G be a multigraph, L a list assignment on $V(G)$, and P an arbitrary L-pot.

2.1 Fixable graphs

Thinking in terms of a two-player game is a good aid to intuition and we encourage the reader to continue doing so. However, a simple recursive definition is equivalent and has far less baggage. For distinct colors $a, b \in P$, let $S_{L,a,b}$ be all the vertices of G that have exactly one of a or b in their list; more precisely, $S_{L,a,b} = \{v \in V(G) \mid |\{a, b\} \cap L(v)| = 1\}$.

Definition 1. G is *(L, P)-fixable* if either

(1) G has an L-edge-coloring; or

(2) there are different colors $a, b \in P$ such that for every partition $X_1, \ldots, X_t$ of $S_{L,a,b}$ into sets of size at most two, there exists $J \subseteq [t]$ so that G is (L', P)-fixable, where L' is formed from L by swapping a and b in $L(v)$ for every $v \in \bigcup_{i \in J} X_i$.

The meaning of (1) is clear. Intuitively, (2) says the following. There is some pair of colors, a and b, such that regardless of how the vertices of $S_{L,a,b}$ are paired via Kempe chains for colors a and b (or not paired with any vertex of $S_{L,a,b}$), we can swap the colors on some subset J of the Kempe chains so that the resulting partial edge-coloring is fixable.

We write L-fixable as shorthand for $(L, \mathrm{pot}(L))$-fixable. When G is (L, P)-fixable, the choices of a, b, and J in each application of (2) determine a tree where all leaves have lists satisfying (1). The *height* of (L, P) is the minimum possible height of such a tree. We write $h_G(L, P)$ for this height and let $h_G(L, P) = \infty$ when G is not (L, P)-fixable.

Lemma 2.1. *If a multigraph M has a partial k-edge-coloring π such that M_π is $(L_\pi, [k])$-fixable, then M is k-edge-colorable.*

Proof. Our proof is by induction on the height of $(L_\pi, [k])$. Choose a partial k-edge-coloring π of M such that M_π is $(L_\pi, [k])$-fixable. If $h_{M_\pi}(L_\pi, [k]) = 0$, then (1) must hold for M_π and L_π; that is, M_π has an edge-coloring ζ such that $\zeta(x) \in L_\pi(x) \cap L_\pi(y)$ for all $xy \in E(M_\pi)$. This means that $\pi \cup \zeta$ is the desired k-edge-coloring of M.

So we may assume that $h_{M_\pi}(L_\pi, [k]) > 0$. Let $a, b \in [k]$ be a choice in (2) that leads to a tree of height $h_{M_\pi}(L_\pi, [k])$. Let H be the subgraph of M induced on all edges e with $\pi(e) \in \{a, b\}$. Let S be the vertices in M_π with degree exactly one in H. Consider the component C_x in H for each $x \in S$. We have $|V(C_x) \cap S| \in \{1, 2\}$ and hence the components of H give a partition $X_1, \ldots, X_t$ of S into sets of size at most two. Moreover, exchanging colors a and b on C_x has the effect of swapping a and b in $L_\pi(v)$ for each $v \in V(C_x) \cap S$. Hence we can achieve the needed swapping of colors in the lists in (2) by exchanging colors on the components of H.

By (2) there is $J \subseteq [t]$ so that M_π is $(L', [k])$-fixable, where L' is formed from L_π by swapping a and b in $L_\pi(v)$ for every $v \in \bigcup_{i \in J} X_i$. Furthermore, there is a J such that $(L', [k])$ has height less than that of $(L, [k])$. Let π' be the partial k-edge-coloring of M created from π by performing the color exchanges to create L' from L_π. By the induction hypothesis, M is k-edge-colorable. $\square$

The definition of L-fixable was originally motivated by generalizing the Fixer-Breaker game in [4] from complete graphs to arbitrary graphs. The direct generalization of that game gives us less power because it does not account for the fact that two-colored paths cannot cross (in particular, they cannot start at the same vertex). Interpreted in the Fixer-Breaker game, the choice of partition in (2) is forcing Breaker to choose two-colored paths in a way that is consistent with being embedded in *some* graph. For stars the two games have identical winning conditions because the obvious necessary condition is sufficient. It is more natural to phrase some proofs in terms of the following weakening of fixable. Basically, we only allow flipping one Kempe chain at a time. This weakening is still logically stronger than the direct generalization of the Fixer-Breaker game since it takes into account the fact that two-colored paths cannot start at the same vertex.

Definition 2. G is *weakly (L, P)-fixable* if either

(1) G has an L-edge-coloring; or

(2) there are different colors $a, b \in P$ and some vertex $v \in S_{L,a,b}$ such that for every $X \subseteq S_{L,a,b}$ with $|X| \leq 2$ and $v \in X$, it holds that G is (L', P)-fixable where L' is formed from L by swapping a and b in $L(v)$ for every $v \in X$.

Lemma 2.2. *If G is weakly (L, P)-fixable, then G is (L, P)-fixable.*

Proof. Clearly if (1) holds we are done. So assume we have different colors $a, b \in P$ and vertex $v \in S_{L,a,b}$ as in (2). Given a partition $X_1, \ldots, X_t$ of $S_{L,a,b}$ into sets of size at most two, let i be the index with $v \in X_i$. Then $J = \{i\}$ is the desired subset of $[t]$. $\square$

We have yet to find an example where weakly fixable is actually weaker.

Conjecture 2.3. *G is (L, P)-fixable if and only if G is weakly (L, P)-fixable.*

2.2 Some examples

A graph G is Δ-*edge-critical*, or simply *edge-critical*, if $\chi'(G) > \Delta$, but $\chi'(G - e) \leq \Delta$ for every edge e. A configuration H is *reducible* if there exists an edge $e \in E(H)$ such that whenever H appears as a subgraph (not necessarily induced) of a graph G, given any Δ-edge-coloring of $G - e$, there exists a Δ-edge-coloring of G. We'd like a way to talk about subgraphs being reducible for k-edge-coloring. Lemma 2.1 gives us this with respect to a fixed partial k-edge-coloring π, but we want a condition independent of the particular coloring. Note that we have a lower bound on the sizes of the lists in L_π; specifically, if π is a partial k-edge-coloring of a multigraph M, then $|L_\pi(v)| \geq k + d_{M_\pi}(v) - d_M(v)$ for every $v \in M_\pi$. Using this lower bound, we get our desired condition as follows.

Definition 3. If G is a graph and $f \colon V(G) \to \mathbb{N}$, then G is (f, k)-fixable if G is $(L, [k])$-fixable for every L with $|L(v)| \geq k + d_G(v) - f(v)$ for all $v \in V(G)$.

By Lemma 2.1, if G is (f, k)-fixable, then G cannot be a subgraph of a $(k + 1)$-edge-critical graph M where $d_M(v) \leq f(v)$ for all $v \in V(G)$. Now we can talk about a graph G with vertices labeled by f being k-fixable or not. The computer is extremely good at finding k-fixable graphs. Combined with discharging arguments, this gives a powerful method for proving (modulo trusting the computer) edge-coloring results for small Δ. We'll see some examples of such proofs later, for now Figure 1 shows some 3-fixable graphs. A gallery of hundreds more fixable graphs is available at `https://dl.dropboxusercontent.com/u/8609833/Web/GraphData/Fixable/index.html`.

The penultimate graph in Figure 1 is an example of the more general fact that a k-regular graph with $f(v) = k$ for all v is k-fixable precisely when it is k-edge-colorable. That the third graph in Figure 1 is reducible follows from Vizing's Adjacency Lemma.

2.3 A necessary condition

Since the edges incident to a given vertex must all get different colors, we have the following.

Lemma 2.4. *If G is (L, P)-fixable, then $|L(v)| \geq d_G(v)$ for all $v \in V(G)$.*

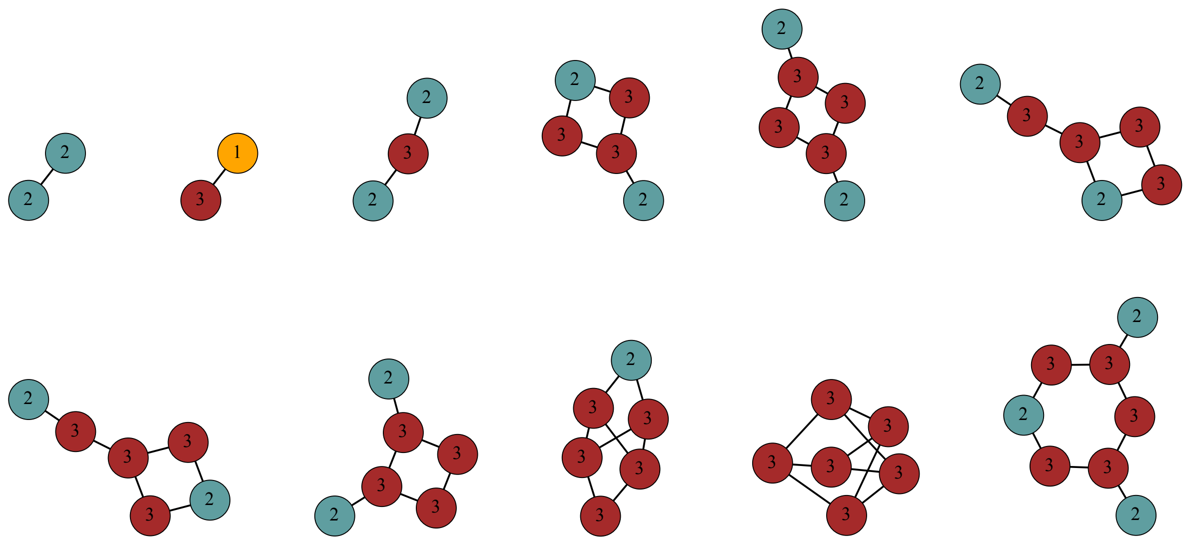

Figure 1: Some small 3-fixable graphs.

By considering the maximum size of matchings in each color, we get a more interesting necessary condition. For $C \subseteq \mathrm{pot}(L)$ and $H \subseteq G$, let $H_{L,C}$ be the subgraph of H induced by the vertices v with $L(v) \cap C \neq \emptyset$. When L is clear from context, we may write H_C for $H_{L,C}$. If $C = \{\alpha\}$, we may write H_α for H_C. For $H \subseteq G$, put

$$\psi_L(H) = \sum_{\alpha \in \mathrm{pot}(L)} \left\lfloor \frac{|H_{L,\alpha}|}{2} \right\rfloor.$$

Each term in the sum gives an upper bound on the size of a matching in color α. So $\psi_L(H)$ is an upper bound on the number of edges in a partial L-edge-coloring of H. We say that (H, L) is *abundant* if $\psi_L(H) \geq \|H\|$ and that (G, L) is *superabundant* if for every $H \subseteq G$, the pair (H, L) is abundant.

Lemma 2.5. *If G is (L, P)-fixable, then (G, L) is superabundant.*

Proof. Suppose to the contrary that G is (L, P)-fixable and there is $H \subseteq G$ such that (H, L) is not abundant. We show that for all distinct $a, b \in P$ there is a partition $X_1, \dots, X_t$ of $S_{a,b}$ into sets of size at most two, such that for all $J \subseteq [t]$, the pair (H, L') is not abundant where L' is formed from L by swapping a and b in $L(v)$ for every $v \in \bigcup_{i \in J} X_i$. Since G can never be edge-colored from a list assignment that is not superabundant, this contradicts the (L, P)-fixability of G.

Pick distinct colors $a, b \in P$. Let $S = S_{L,a,b} \cap V(H)$ and let S_a be the $v \in S$ with $a \in L(v)$. Put $S_b = S \setminus S_a$. Swapping a and b will only effect the terms $\left\lfloor \frac{|S_a|}{2} \right\rfloor$ and $\left\lfloor \frac{|S_b|}{2} \right\rfloor$ in $\psi_L(H)$. So, if $\psi_L(H)$ is increased by the swapping, it must be that both $|S_a|$ and $|S_b|$ are odd and after swapping they are both even. Say $S_a = \{a_1, \dots, a_p\}$ and $S_b = \{b_1, \dots, b_q\}$. By symmetry, we may assume $p \leq q$. For $i \in [p]$, let $X_i = \{a_i, b_i\}$. Since both p and q are odd, $q - p$ is even, so we get a partition by, for each $j \in \left[\frac{q-p}{2}\right]$, letting $X_{p+j} = \{b_{p+2j-1}, b_{p+2j}\}$.

For any $i \in [p]$, swapping a and b in $L(v)$ for every $v \in X_i$ maintains $|S_a|$ and $|S_b|$. For any $j \in \left[\frac{q-p}{2}\right]$, swapping a and b in $L(v)$ for every $v \in X_{p+j}$ maintains the parity of $|S_a|$ and $|S_b|$. So no choice of J can increase $\psi_L(H)$ and hence (H, L') is never abundant. $\square$

In particular, we have learned the following.

Corollary 2.6. *If G is (f,k)-fixable, then (G,L) is superabundant for every L with $L(v) \subseteq [k]$ and $|L(v)| \geq k + d_G(v) - f(v)$ for all $v \in V(G)$.*

Intuitively, superabundance requires the potential for a large enough matching in each color. If instead we require the existence of a large enough matching in each color, we get a stronger condition that has been studied before. For a multigraph H, let $\nu(H)$ be the number of edges in a maximum matching of H. For a list assignment L on H, let $\eta_L(H) = \sum_{\alpha \in \mathrm{pot}(L)} \nu(H_\alpha)$. Note that we always have $\psi_L(H) \geq \eta_L(H)$.

The following generalization of Hall's theorem was proved by Marcotte and Seymour [3] and independently by Cropper, Gyárfás, and Lehel [2]. By a *multitree* we mean a tree that possibly has edges of multiplicity greater than one.

Lemma 2.7 (Marcotte and Seymour)**.** *Let T be a multitree and L a list assignment on $V(T)$. If $\eta_L(H) \geq \|H\|$ for all $H \subseteq T$, then T has an L-edge-coloring.*

In [4], the second author proved that superabundance is also a sufficient condition for fixability when we restrict our graphs to be multistars. This immediately implies the fan equation (a definition is available in [5, p. 19ff]). The proof uses Hall's theorem to reduce to a smaller star and one might hope we could do the same for arbitrary trees with Lemma 2.7 in place of Hall's theorem (thus giving a short proof that Tashkinov trees are elementary), but we haven't yet been able to make this work.

2.4 Fixability of stars

When G is a star, the conjunction of our two necessary conditions is sufficient. This generalizes Vizing fans [6]; in the next section we will define "Kierstead-Tashkinov-Vizing assignments" and show that they are always superabundant. In [4], the second author proved a common generalization of Theorem 2.8 and Hall's theorem; we reproduce the proof for the special case of edge-coloring.

Theorem 2.8. *If G is a multistar, then G is weakly L-fixable if and only if (G,L) is superabundant and $|L(v)| \geq d_G(v)$ for all $v \in V(G)$.*

Proof. Our strategy is simply to increase $\eta_L(G)$ if we can; if we cannot, then Hall's theorem allows us to reduce to a smaller graph. An easy way to describe this strategy is via a double induction as follows. Suppose the theorem is false and choose a counterexample (G,L) minimizing $\|G\|$ and subject to that maximizing $\eta_L(G)$.

Let z be the center of the multistar G. Create a bipartite graph B with parts U and Y, where $U = \{zw \in E(G) \mid L(z) \cap L(w) \neq \emptyset\}$ and $Y := \{\alpha \in \mathrm{pot}(L) \mid \nu(G_\alpha) = 1\}$, and $zw \in U$ is adjacent to $\alpha \in Y$ if and only if $\alpha \in L(z) \cap L(w)$. Informally, Y is the set of colors α that can be used on at least one edge, and U is the set of edges e with at least one color available on e, and a color α is adjacent to an edge e if α can be used on e.

First, suppose $|Y| < \|G\|$. Since $|L(z)| \geq d_G(z) = \|G\|$, we have a color $\tau \in L(z)$ that cannot be used on any edge. Suppose there is a color $\beta \in Y$ that can be used on at least three edges. Let zw be some edge where β can be used. Since G is not L-fixable, there is $X \subseteq S_{L,\tau,\beta}$ with $|X| \leq 2$ and $w \in X$ such that G is not L'-fixable, where L' is formed from L by swapping τ and β in $L(v)$ for every $v \in X$. Since β can be used on at least three edges (for L), it can be used on at least one edge for L'. Further, τ can also be used on at least one edge for L'. Thus $\eta_{L'}(G) > \eta_L(G)$. Since (G, L') is still superabundant, this violates maximality of $\eta_L(G)$. Hence, each color $\beta \in Y$ can be used on at most two edges. So, each color in Y contributes at most one to $\psi_L(G)$. Since $|Y| < \|G\| \leq \psi_L(G)$, there must be a color $\gamma \notin Y$ such that $|G_\gamma - z| \geq 2$. Since G is not L-fixable, there is $X \subseteq S_{L,\tau,\gamma}$ with $|X| \leq 2$ and $z \in X$ such that G is not L'-fixable where L' is formed from L by swapping τ and γ in $L(v)$ for every $v \in X$. Since $\nu(G_{L,\tau}) = 0$ and $\nu(G_{L,\gamma}) = 0$ and $\nu(G_{L',\gamma}) = 1$, we have $\eta_{L'}(G) > \eta_L(G)$. Since (G, L') is still superabundant, this violates maximality of $\eta_L(G)$.

Hence we must have $|Y| \geq \|G\|$. In particular, $|N_B(Y)| \leq |Y|$ so we may choose a set of colors $C \subseteq Y$ such that C is a minimal nonempty set satisfying $|N_B(C)| \leq |C|$. If $|C| \geq |N_B(C)|+1$, then, for any $\tau \in C$, we have $|C - \tau| = |C|-1 \geq |N_B(C)| \geq |N_B(C - \tau)|$, which contradicts the minimality of C. Thus, $|C| = |N_B(C)|$. Furthermore, by minimality of C, every nonempty $D \subset C$ satisfies $|N_B(D)| > |D|$, so Hall's Theorem yields a perfect matching M between C and $N_B(C)$.

For each color/edge pair $\{\alpha, zw\} \in M$, use color α on edge zw. Form G' from G by removing all the colored edges and then discarding any isolated vertices. Note that z lost exactly $|C|$ colors from its list and also $d_{G'}(z) = d_G(z) - |C|$, so $|L'(z)| = |L(z)| - |C| \geq d_G(z) - |C| = d_{G'}(z)$. Each other vertex $w \in V(G')$ satisifes $d_{G'}(w) = d_G(w)$ and $|L'(w)| = |L(w)|$, so $|L'(w)| \geq d_{G'}(w)$. Since G is not L-fixable and C and $\text{pot}(L')$ are disjoint it must be that G' is not L'-fixable. For each $H \subseteq G'$, we have $\psi_{L'}(H) = \psi_L(H)$. For each color $\alpha \in Y$, if $\alpha \in C$, then $\lfloor |H_{L,\alpha}|/2 \rfloor = 0$, since $E(H) \cap N_B(C) = \emptyset$. Similarly, if $\alpha \notin Y$, then each $v \in V(G')$ satisfies $\alpha \in L'(v)$ if and only if $\alpha \in L(v)$. Thus, H is abundant for L' precisely because H is abundant for L. But $\|G'\| < \|G\|$, so by minimality of $\|G\|$, G' is L'-fixable, a contradiction. $\square$

As shown in [4], the *fan equation* is a direct consequence of Theorem 2.8. This, in turn, implies most classical edge-coloring results including Vizing's Adjacency Lemma.

2.5 Kierstead-Tashkinov-Vizing assignments

Many edge-coloring results have been proved using a specific kind of superabundant pair (G, L) where superabundance can be proved via a special ordering. That is, the orderings given by the definition of Vizing fans, Kierstead paths, and Tashkinov trees (these structures are all standard tools in edge-coloring; defenitions and more background are available in [5]). In this section, we show how superabundance follows easily from these orderings.

A list assignment L on G is a *Kierstead-Tashkinov-Vizing* assignment (henceforth *KTV-assignment*) if for some $xy \in E(G)$, there is a total ordering '$<$' of $V(G)$ such that

1. there is an edge-coloring π of $G - xy$ such that $\pi(uv) \in L(u) \cap L(v)$ for each edge $uv \in E(G - xy)$;

2. $x < z$ for all $z \in V(G - x)$;

3. $G[w \mid w \leq z]$ is connected for all $z \in V(G)$;

4. for each edge $wz \in E(G - xy)$, there is a vertex $u < \max\{w, z\}$ such that $\pi(wz) \in L(u) - \{\pi(e) \mid e \in E(u)\}$;

5. there are distinct vertices $s, t \in V(G)$ with $L(s) \cap L(t) - \{\pi(e) \mid e \in E(s) \cup E(t)\} \neq \emptyset$.

Lemma 2.9. *If L is a KTV-assignment on G, then (G, L) is superabundant.*

Proof. Let L be a KTV-assignment on G, and let $H \subseteq G$. We will show that (H, L) is abundant. Clearly it suffices to consider the case when H is an induced subgraph, so we assume this. Property (1) gives that $G - xy$ has an edge-coloring π, so $\psi_L(H) \geq \|H\| - 1$; also $\psi_L(H) \geq \|H\|$ if $\{x, y\} \not\subseteq V(H)$. Furthermore $\psi_L(H) \geq \|H\|$ if s and t from property (5) are both in $V(H)$, since then $\psi_L(H)$ gains 1 over the naive lower bound, due to the color in $L(s) \cap L(t)$. So $V(G) - V(H) \neq \emptyset$.

Now choose a vertex $z \in V(G) - V(H)$ that is smallest under $<$. Let $H' = G[w \mid w \leq z]$. By the minimality of z, we have $H' - z \subseteq H$. By property (2), $|H'| \geq 2$. By property (3), H' is connected and thus there is $w \in V(H' - z)$ adjacent to z. So, we have $w < z$ and $wz \in E(G) - E(H)$. Now $\pi(wz) \in L(w)$, so property (4) implies that there exists a vertex u with $u < \max\{w, z\} = z$ and $\pi(wz) \in L(u) - \{\pi(e) | e \in E(u)\}$. Since $u \in V(H' - z) \subseteq V(H)$, we again gain 1 over the naive lower bound on $\psi_L(H)$, due to the color in $L(u) \cap L(w)$. So $\psi_L(H) \geq \|H\|$. $\square$

3 Applications of small k-fixable graphs

3.1 Impoved lower bound on the average degree of 3-critical graphs

Let P^* denote the Petersen graph with a vertex deleted (see Figure 2). Jakobsen [?, ?] noted that P^* is 3-critical and has average degree $2.\overline{6}$. He showed that every 3-critical graph has average degree at least $2.\overline{6}$, and he asked whether equality holds only for P^*. In [?], we answered his question affirmatively. More precisely, we showed that every 3-critical graph other than P^* has average degree at least $2 + \frac{26}{37} = 2.\overline{702}$. The proof crucially depends on the fact that the configurations in Figure 3 are reducible. As we noted in [?], by using the computer to prove reducibility of additional configurations, we can strengthen this result. Specifically, every 3-critical graph has average degree at least $2 + \frac{22}{31} \approx 2.7097$ unless it is P^* or one other exceptional graph, the Hajós join of two copies of P^*. (For comparison, there exists an infinite family of 3-critical graphs with average degree less than 2.75.) This strengthening relies primarily on the fact that the configuration in Figure 4 is reducible, even if one or more pairs of its 2-vertices are identified. However, the simplest proof we have of this fact is computer-generated and fills about 100 pages.

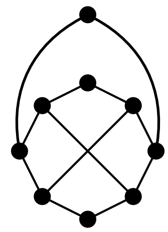

Figure 2: The Peterson graph with one vertex removed.

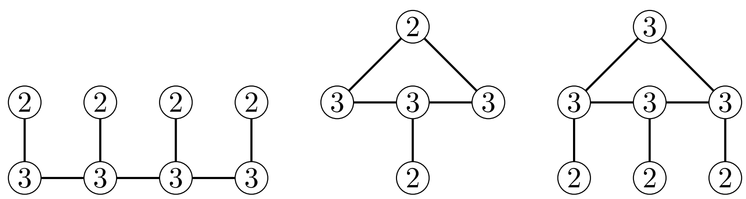

Figure 3: Three subgraphs forbidden from a 3-critical graph G. (The number at each vertex specifies its degree in G.)

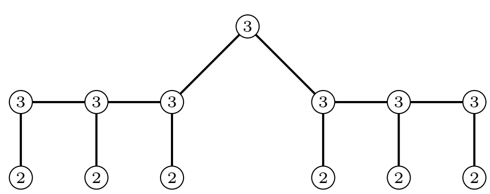

Figure 4: The big tree.

3.2 Impoved lower bound on the average degree of 4-critical graphs

Here we prove the $\Delta = 4$ case of Woodall's conjecture [7] on the average degree of a critical graph (modulo computer proofs of reducibility).

Theorem 3.1. *If G is an edge-critical graph with maximum degree 4, then G has average degree at least 3.6. This is best possible, as shown by $K_5 - e$.*

Proof. We use discharging with initial charge $\text{ch}(v) = d(v)$ and the following rules.

(R1) Each 2-vertex takes .8 from each 4-neighbor.

(R2) Each 3-vertex with three 4-neighbors takes .2 from each 4-neighbor. Each 3-vertex with two 4-neighbors takes .3 from each 4-neighbor.

(R3) Each 4-vertex with charge in excess of 3.6 after (R2) splits this excess evenly among its 4-neighbors with charge less than 3.6.

Now we show that every vertex v finishes with $\text{ch}^*(v) \geq 3.6$.

By VAL, each neigbhor of a 2-vertex v is a 4-neighbor. Thus, $\text{ch}^*(v) = 2 + 2(.8) = 3.6$.

By VAL, each 3-vertex v has at least two 4-neighbors, so $\text{ch}^*(v) = 3 + 3(.2)$ or $\text{ch}^*(v) = 3 + 2(.3)$. In either case, $\text{ch}^*(v) = 3.6$.

Now we consider a 4-vertex v. Note that (R3) will never drop the charge of a 4-vertex below 3.6; thus, in showing that $\text{ch}^*(v) \geq 3.6$, we need not consider (R3).

Claim 1. *Let v be a 4-vertex with no 2-neighbor. If v is not on a triangle with degrees 3,3,4, then v finishes (R2) with charge at least 3.6; otherwise v finishes (R2) with charge 3.4.*

If v has at most one 3-neighbor, then v finishes (R2) with charge at least 3.7. By VAL, v has at most two 3-neighbors, so assume exactly two. If each receives charge .2 from v, then v finishes (R3) with charge 3.6, as desired. Otherwise, some 3-neighbor of v has its own 3-neighbor. Now G has a path with degrees 3,4,3,3, which is (C6), and hence reducible. Here we use that v does not lie on a triangle with degrees 3,3,4.

Claim 2. *Let v be a 4-vertex. If v has only 4-neighbors, then v splits its excess charge of .4 at most 2 ways in (R3)* unless *every 3^--vertex within distance two of v is a 2-vertex; in that case v may split its excess at most 3 ways in (R3).*

From the previous claim, we see that a 4-vertex needs charge after (R2) only if it has a 2-neighbor or if it lies on a triangle with degrees 3,3,4. To prove the claim, we consider a 4-vertex v with 4-neighbors $v_1, \ldots, v_4$ such that at least three v_i either have 2-neighbors or lie on triangles with degrees 3,3,4.

First suppose that at least two v_i lie on triangles with degrees 3,3,4. If two of these triangles are vertex disjoint, then we have (C9); otherwise, we have (C10). So we conclude that at most one v_i lies on a triangle with degrees 3,3,4. Assume that we have exactly one. Now we have either (C11) or (C12). Thus, we conclude that no v_i lies on a triangle with degree 3,3,4. If each v_i has a 2-neighbor, then we have (C13), (C14), or (C15). Thus, the claim is true.

Claim 3. *Every vertex other than a 4-vertex with a 2-neighbor finishes (R3) with at least 3.6.*

By Claim 1, we need only consider a 4-vertex v on a triangle with degrees 3,3,4; call its 3-neighbors u_1 and u_2. Since v finishes (R2) with charge 3.4, we must show that in (R3) v receives charge at least .2. By Claim 2, this is true if v has any 4-neighbor with no 3-neighbors (note that its 4-neighbors cannot have 2-neighbors, since that yields (C16)). Thus, we assume that each 4-neighbor of v, call them u_3 and u_4, has a 3-neighbor. If u_3 or u_4 has a 3-neighbor other than u_1 or u_2, then the configuration is (C7), which is reducible. Thus, we may assume that u_3 is adjacent to u_1 and u_4 is adjacent to u_2. Hence, each of u_3 and u_4 finishes (R2) with charge .1. It suffices to show that all of this charge goes to v in (R3). Thus, we need only show that no other neighbor of u_3 or u_4 needs charge after (R2). If it does, then we have the reducible configuration (C8), where possibly the rightmost 3 is a 2. Thus, v finishes with charge $4 - 2(.3) + 2(.1) = 3.6$, as desired.

Claim 4. *Every 4-vertex on a triangle with degrees 2,4,4 finishes (R3) with at least 3.6.*

Let v be a 4-vertex on a triangle with degrees 2,4,4, and let v_1 and v_2 be its 2-neighbor and 4-neighbor on the triangle. Let v_3 and v_4 be its other neighbors. By VAL, $d(v_3) = d(v_4) = 4$. We will show that each of v_3 and v_4 has only 4-neighbors and that each gives at least .2 to v in (R3). If v_3 or v_4 has a 3^--neighbor, then we have (C17) or (C5), which are reducible; thus, each of v_3 and v_4 has only 4-neighbors. By Claim 2, vertex v_3 splits its charge of .4 at most two ways (thus, giving v at least .2) unless it gives charge to exactly three of its neighbors, each of which has a 2-neighbor. If this is the case, then we have (C18), (C19), or (C20), each of which is reducible. Thus, v_3 splits its charge at most two ways, and so gives v charge at least .2. By the same argument, v_4 gives v charge at least .2. Thus, v finishes (R3) with charge at least $4 - .8 + 2(.2) = 3.6$.

Now all that remains to consider is a 4-vertex v with a 2-neighbor and three 4-neighbors v_1, v_2, v_3. Further, we may assume that v does not lie on a triangle with degrees 2,4,4. Also, we may assume that each v_i has no 2-neighbor; since v lies on no 2,4,4 triangle, this would yield a copy of (C5), which is reducible.

Claim 5. *If any v_i has two or more 3-neighbors, then v finishes with at least* 3.6.

Suppose that v_1 has two 3-neighbors (by VAL it can have no more). First, we note that neither v_2 nor v_3 has a 3-neighbor. If it's distinct from those of v_1, then we have (C27); otherwise, we have (C28). Thus, each of v_2 and v_3 finishes (R2) with excess charge .4. So, it suffices to show that each of v_2 and v_3 splits its excess charge at most two ways. By Claim 2, this is true unless v_2 (say) splits its charge among three 4-neighbors, each of which has a 2-neighbor. If v_2 does so, then we have (C29), (C30), or (C31), each of which is reducible. Thus, v finishes (R3) with charge at least $4 - .8 + 2(.2) = 3.6$.

Claim 6. *If any v_i has a 3-neighbor, which itself has a 3-neighbor, then v finishes with at least* 3.6.

Assume that v_1 has a 3-neighbor, which itself has a 3-neighbor. First, we note that neither v_2 nor v_3 has a 3-neighbor. If so, then we have one of (C38), (C39), or (C40). Thus, each of v_2 and v_3 finishes (R2) with excess charge .4. So, it suffices to show that each of v_2 and v_3 splits its excess charge at most two ways. By Claim 2, this is true unless v_2 (say) splits its charge among three 4-neighbors, each of which has a 2-neighbor. If v_2 does so, then we have (C41), (C42), or (C43), each of which is reducible. Thus, v finishes (R3) with charge at least $4 - .8 + 2(.2) = 3.6$.

Claim 7. *If no v_i has a 3-neighbor, then v finishes with at least* 3.6.

Since each v_i has only 4-neighbors, it finishes (R2) with excess charge .4. By Claim 2, it splits this charge at most 3 ways. Thus, v finishes with charge at least $4 - .8 + 3(.4/3) = 3.6$.

Claim 8. *If at least two vertices v_i have 3-neighbors, then v finishes with at least* 3.6.

Assume that v_1 and v_2 each have 3-neighbors. By Claim 5, each has exactly one 3-neighbor. By Claim 6, the 3-neighbors of v_1 and v_2 do not themselves have 3-neighbors. Thus, each of v_1 and v_2 finishes (R2) with charge exactly 3.8. Hence, in (R3), each splits its excess charge of .2 among its 4-neighbors that need charge. We show that in (R3) all of this charge goes to v. Suppose, to the contrary, that v_1 sends some of its charge elsewhere; this could be to (i) a 4-vertex with a 2-neighbor or (ii) a 4-vertex on a triangle with degrees 3,3,4. In (i), we have one of (C21)–(C24). In (ii), we have one of (C25) and (C26). All such configurations are reducible, which proves the claim.

Claim 9. *If exactly one vertex v_i has a 3-neighbor, then v finishes with at least* 3.6.

Assume that v_1 has exactly one 3-neighbor (and v_2 and v_3 have no 3-neighbors), so v_2 and v_3 each finish (R2) with excess charge .4. It suffices to show that either v_2 and v_3 each give v at least half of their charge in (R3) or else at least one of them gives v all its charge in (R3); assume not. By symmetry, we assume that v_2 splits its charge at least three ways in (R3) and v_3 splits its charge at least two ways. If v_3 gives charge to a 4-vertex with a 2-neighbor (in addition to v), then we have one of (C32)–(C36). So assume that v_3 gives charge to a 4-vertex on a triangle with degrees 3,3,4. By Claim 6, neither of these 3-vertices on the triangle is the 3-neighbor of v_1. If we do not have (C33) or (C34), then we must have (C37), which is reducible. This proves the claim. □

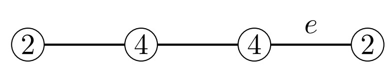

Figure 5: (C1) 18 total boards: In increasing depths (14, 1, 1, 1, 1).

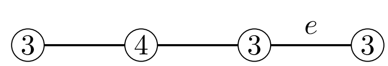

Figure 6: (C2) 26 total boards: In increasing depths (22, 1, 1, 1, 1).

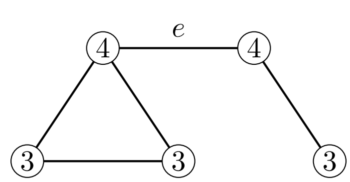

Figure 7: (C3) 84 total boards: In increasing depths (52, 20, 6, 4, 2).

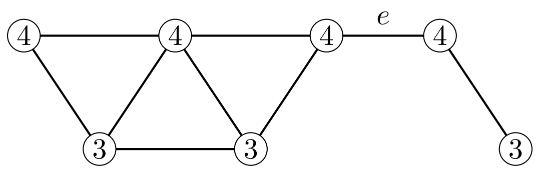

Figure 8: (C4) 32 total boards: In increasing depths (22, 5, 2, 2, 1).

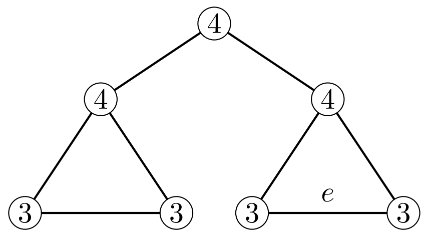

Figure 9: (C5) 652 total boards: In increasing depths (578, 36, 32, 6).

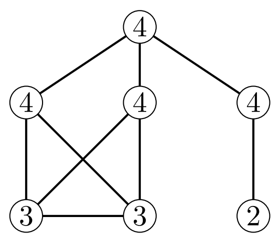

Figure 10: (C6) 72 total boards: In increasing depths (49, 20, 3).

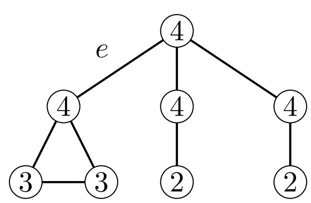

Figure 11: (C7) 3936 total boards: In increasing depths (2492, 750, 310, 266, 96, 14, 8).

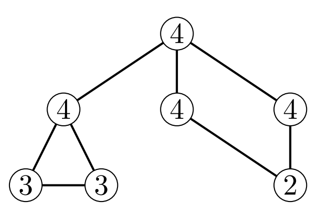

Figure 12: (C8) 391 total boards: In increasing depths (205, 163, 23).

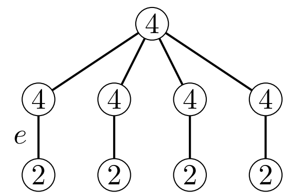

Figure 13: (C9) 6546 total boards: In increasing depths (5277, 279, 282, 285, 186, 135, 81, 21).

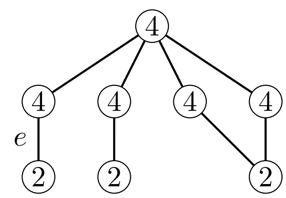

Figure 14: (C10) 526 total boards: In increasing depths (429, 51, 28, 16, 2).

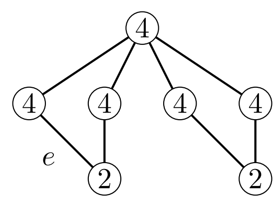

Figure 15: (C11) 41 total boards: In increasing depths (36, 5).

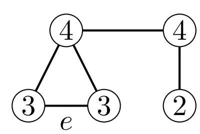

Figure 16: (C12) 55 total boards: In increasing depths (49, 4, 2).

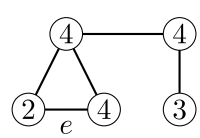

Figure 17: (C13) 22 total boards: In increasing depths (18, 1, 1, 1, 1).

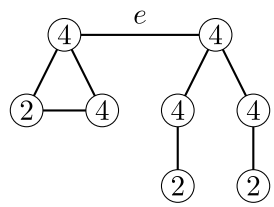

Figure 18: (C14) 1480 total boards: In increasing depths (868, 178, 140, 160, 92, 14, 18, 10).

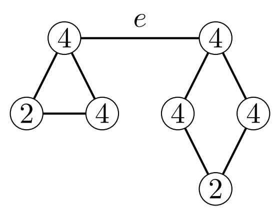

Figure 19: (C15) 114 total boards: In increasing depths (74, 28, 12).

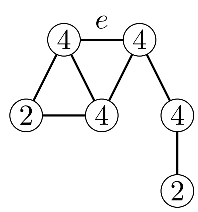

Figure 20: (C16) 62 total boards: In increasing depths (44, 15, 3).

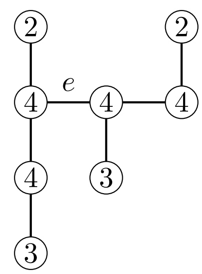

Figure 21: (C17) 7124 total boards In increasing depths (3648, 1081, 551, 579, 606, 448, 196, 15).

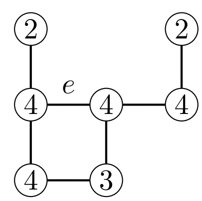

Figure 22: (C18) 1094 total boards In increasing depths (570, 248, 131, 96, 43, 6).

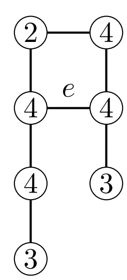

Figure 23: (C19) 534 total boards In increasing depths (297, 162, 19, 15, 17, 9, 2, 2, 2, 2, 2, 2, 3).

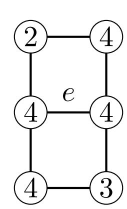

Figure 24: (C20) 85 total boards In increasing depths: (47, 27, 4, 4, 3).

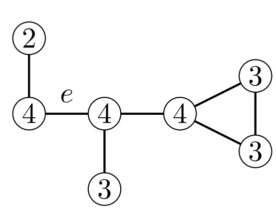

Figure 25: (C21) 1406 total boards: In increasing depths (790, 374, 188, 50, 4).

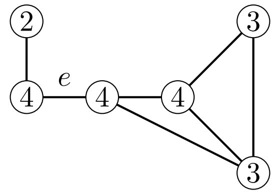

Figure 26: (C22) 72 total boards: In increasing depths (57, 14, 1).

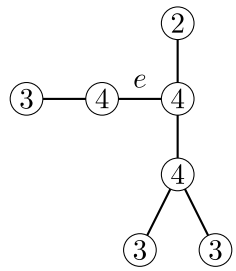

Figure 27: (C23) 2230 total boards In increasing depths (1158, 426, 267, 225, 100, 48, 6).

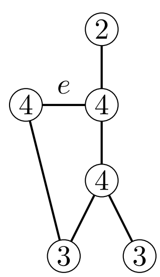

Figure 28: (C24) 335 total boards In increasing depths (182, 96, 44, 10, 2, 1).

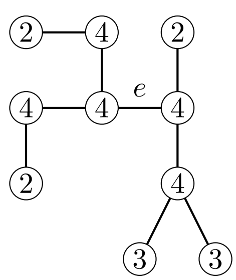

Figure 29: (C25) 108416 total boards: In increasing depths (58218, 17840, 12265, 10729, 5935, 2981, 422, 26).

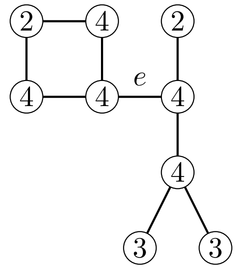

Figure 30: (C26) 8022 total boards: In increasing depths (4703, 1877, 1022, 420).

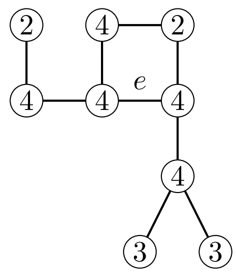

Figure 31: (C27) 8239 total boards: In increasing depths (4666, 2373, 922, 270, 8).

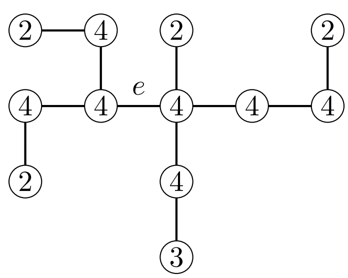

Figure 32: (C28) 579808 total boards: In increasing depths (292466, 64554, 28706, 24886, 24340, 23199, 23908, 26010, 25574, 23929, 15613, 5823, 792, 8).

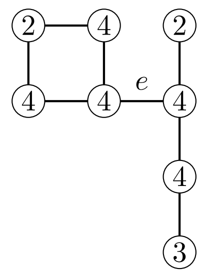

Figure 33: (C29) 1740 total boards: In increasing depths (929, 301, 132, 152, 146, 58, 18, 4).

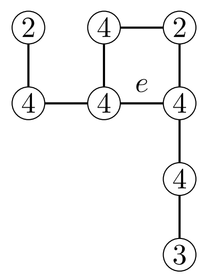

Figure 34: (C30) 1786 total boards: In increasing depths (925, 422, 107, 111, 99, 53, 24, 8, 6, 7, 7, 8, 9).

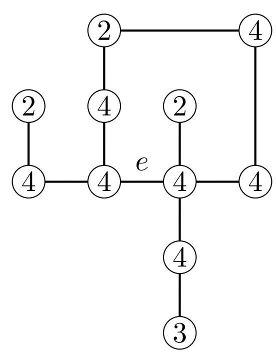

Figure 35: (C31) 47540 total boards: In increasing depths (23949, 6503, 3553, 3446, 3507, 3148, 2287, 1007, 140).

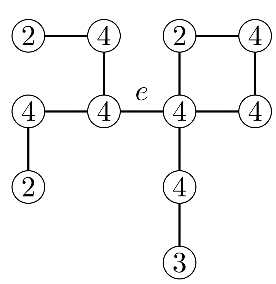

Figure 36: (C32) 46132 total boards: In increasing depths (23677, 8019, 3850, 3921, 3096, 2377, 998, 166, 28).

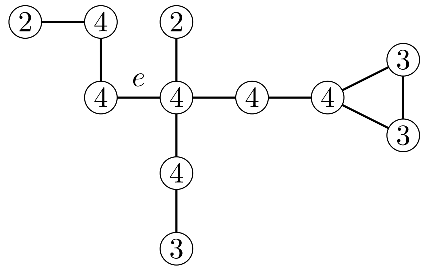

Figure 37: (C33) 134824 total boards: In increasing depths (66938, 19107, 11046, 9113, 8738, 8349, 7140, 3636, 737, 20).

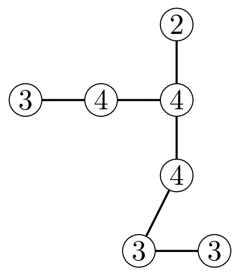

Figure 38: (C34) 3026 total boards: In increasing depth (1085, 380, 204, 147, 127, 143, 163, 145, 88, 90, 103, 28). (This configuration is weird because there is no edge e such that it works to consider only near colorings for $G - e$; we need to go through a board that is a near coloring for a different missing edge.)

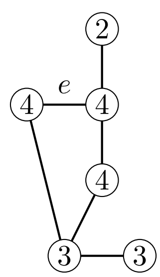

Figure 39: (C35) 142 total boards: In increasing depth (86, 49, 7).

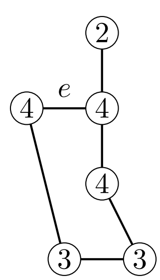

Figure 40: (C36) 328 total boards: In increasing depths (177, 50, 40, 41, 20).

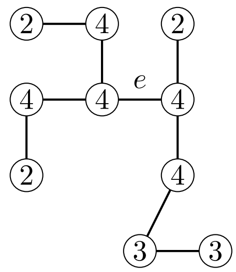

Figure 41: (C37) 108416 total boards: In increasing depths (54806, 12342, 6708, 6450, 6737, 6436, 5975, 5109, 2990, 697, 114, 35, 17).

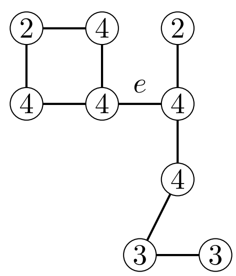

Figure 42: (C38) 8022 total boards: In increasing depths (4449, 1573, 796, 790, 388, 26).

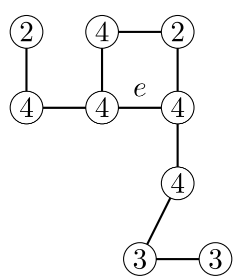

Figure 43: (C39) 8239 total boards: In increasing depths (4424, 2063, 653, 628, 400, 69, 2).

3.3 The conjecture of Hilton and Zhao for $\Delta = 4$

For a graph G, let G_Δ be the subgraph of G induced on vertices of degree $\Delta(G)$. Vizing's Adjacency Lemma implies that $\delta(G_\Delta) \geq 2$ in a critical graph G. A natural question is whether or not this is best possible. For example, can we have $\Delta(G_\Delta) = 2$ in a critical graph G? In fact, Hilton and Zhao have conjectured exactly when this can happen. A graph G is *overfull* if $||G|| > \left\lfloor \frac{|G|}{2} \right\rfloor \Delta(G)$. Let P^* be the Peterson graph with one vertex removed (see Figure 2).

Conjecture 3.2 (Hilton and Zhao)**.** *A connected graph G with $\Delta(G_\Delta) \leq 2$ is class 2 if and only if G is P^* or G is overfull.*

David and Gianfranco Cariolaro [1] proved this conjecture when $\Delta = 3$. Here we prove it when $\Delta = 4$, but we omit the very long computer-generated proofs of the reducibility of the graphs in Figure 44. Not all of the graphs in Figure 44 are 4-fixable; to prove reducibility of a graph H, we restrict the allowed list assignments on vertices to those from which all but one specified edge e of H can be colored. Such a list assignment is called *nearly-colorable*. Checking only the nearly-colorable assignments clearly suffices for reducibility.

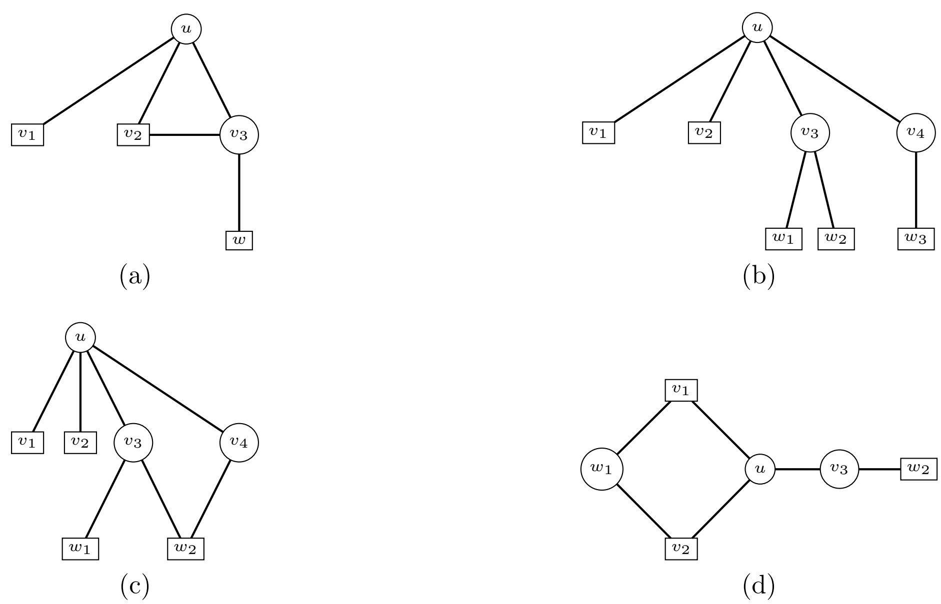

Figure 44: Vertices drawn as circles have degree 4 in G and vertices drawn as rectangles have degree 3 in G.

Since we do not include the reducibility proofs, we separate the proof into two parts. The first does not use the computer at all. Let $\mathcal{H}_4$ be the class of connected graphs with maximum degree 4, minimum degree 3, each vertex adjacent to at least two 4-vertices, and each 4-vertex adjacent to exactly two 4-vertices.

Lemma 3.3. *If G is a graph in $\mathcal{H}_4$ and G contains none of the four configurations in Figure 44 (not necessarily induced), then G is $K_5 - e$.*

Proof. Let G be a graph in $\mathcal{H}_4$. Note that every 4-vertex in G has exactly two 3-neighbors and two 4-neighbors. Let u denote a 4-vertex and let $v_1, \ldots, v_4$ denote its neighbors, where $d(v_1) = d(v_2) = 3$ and $d(v_3) = d(v_4) = 4$. When vertices x and y are adjacent, we write $x \leftrightarrow y$. We assume that G contains none of the configurations in Figure 44 and show that G must be $K_5 - e$.

First suppose that u has a 3-neighbor and a 4-neighbor that are adjacent. By symmetry, assume that $v_2 \leftrightarrow v_3$. Since Figure 44(a) is forbidden, we have $v_3 \leftrightarrow v_1$. Now consider v_4. If v_4 has a 3-neighbor distinct from v_1 and v_2, then we have a copy of Figure 44(d). Hence $v_4 \leftrightarrow v_1$ and $v_4 \leftrightarrow v_2$. If $v_3 \leftrightarrow v_4$, then G is $K_5 - e$. Suppose not, and let x be a 4-neighbor of v_4. Since G has no copy of Figure 44(d), x must be adjacent to v_1 and v_2. This is a contradiction, since v_1 and v_2 are 3-vertices, but now each has at least four neighbors. Hence, we conclude that each of v_1 and v_2 is non-adjacent to each of v_3 and v_4.

Now consider the 3-neighbors of v_3 and v_4. If they have zero 3-neighbors in common, then we have a copy of Figure 44(b). If they have one 3-neighbor in common, then we have a copy of Figure 44(c). Otherwise they have two 3-neighbors in common, so we have a copy of Figure 44(d). □

Since $K_5 - e$ is overfull, the next theorem implies Hilton and Zhao's conjecture for $\Delta = 4$. Given a class 2 graph $G \in \mathcal{H}_4$, we would like to consider a 4-edge-critical subgraph H, apply the previous lemma to find in H some copy of a configuration in Figure 44, and reach a contradiction, since each of these configurations is reducible. This is the main idea in the next proof. However, in order to apply Lemma 3.3, we must first show that $H \in \mathcal{H}_4$.

Theorem 3.4. *A connected graph G with $\Delta(G) = 4$ and $\Delta(G_\Delta) \leq 2$ is class 2 if and only if G is $K_5 - e$.*

Proof. Let G be as stated in the theorem. If G is class 2, then G has a 4-critical subgraph H. Since H is 4-critical, it is connected, and every vertex has at least two neighbors of degree 4, by VAL. Further, since $\Delta(H_\Delta) \leq \Delta(G_\Delta) \leq 2$, VAL implies that H has minimum degree 3. Thus, $H \in \mathcal{H}_4$. By Lemma 3.3, either H is $K_5 - e$ or H contains one of the configurations in Figure 44. By computer, each of these configurations is reducible and hence cannot be a subgraph of the 4-critical graph H. Thus H is $K_5 - e$. Let x_1, x_2 be the degree 3 vertices in H. Then x_i has three degree 4 neighbors in H and hence $d_G(x_i) \leq 3$ since $\Delta(G_\Delta) \leq 2$. That is, x_i has no neighbors outside H. Since G is connected we must have $G = H = K_5 - e$. □

4 Superabundance sufficiency and adjacency lemmas

In the previous sections, we studied k-fixable graphs which are reducible configurations for graphs with fixed maximum degree. Here we study a more general notion that behaves similarly to Vizing Fans, Kierstead Paths and Tashkinov Trees. Specifically, we consider graphs that are fixable for all superabundant list assignments.

Definition 4. If G is a graph and $f\colon V(G) \to \mathbb{N}$ with $f(v) \geq d_G(v)$ for all $v \in V(G)$, then G is f-fixable if G is (L, P)-fixable for every L with $|L(v)| \geq f(v)$ for all $v \in V(G)$ and every L-pot P such that (G, L) is superabundant. If a graph G is f-fixable when $f(v) = d_G(v)$ for each v, then G is *degree-fixable*.

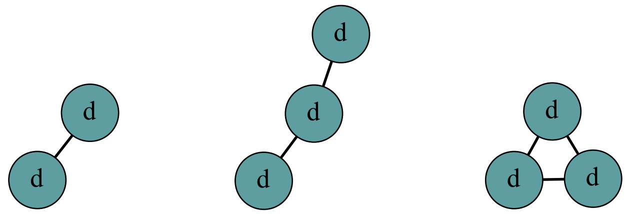

Figure 45: The fixable graphs on at most 3 vertices.

For example, Lemma 2.8 shows that multistars are degree-fixable. We have also found that the 4-cycle is degree-fixable.

Problem. *Classify the degree-fixable multigraphs (specifically, those that are containment minimal).*

Since $f(v) \geq d_G(v)$, it is convenient to express the values of f as $d + k$ for a non-negative integer k; this means $f(v) = d_G(v) + k$. When $k = 0$, we just write d as in Figure 45.

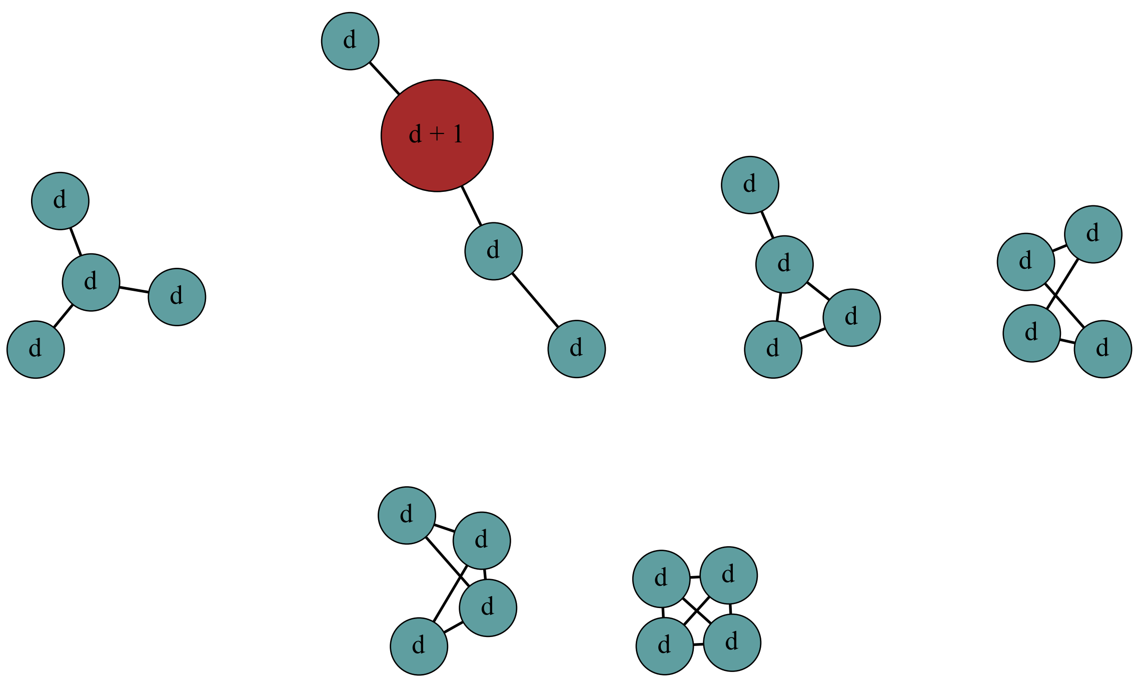

Figure 46: The fixable graphs on 4 vertices.

Looking at the trees in Figures 46, 47, and 48 we might conjecture that a tree is f-fixable as long as there is at most one internal vertex labeled "d". This conjecture continues to hold for many more examples.

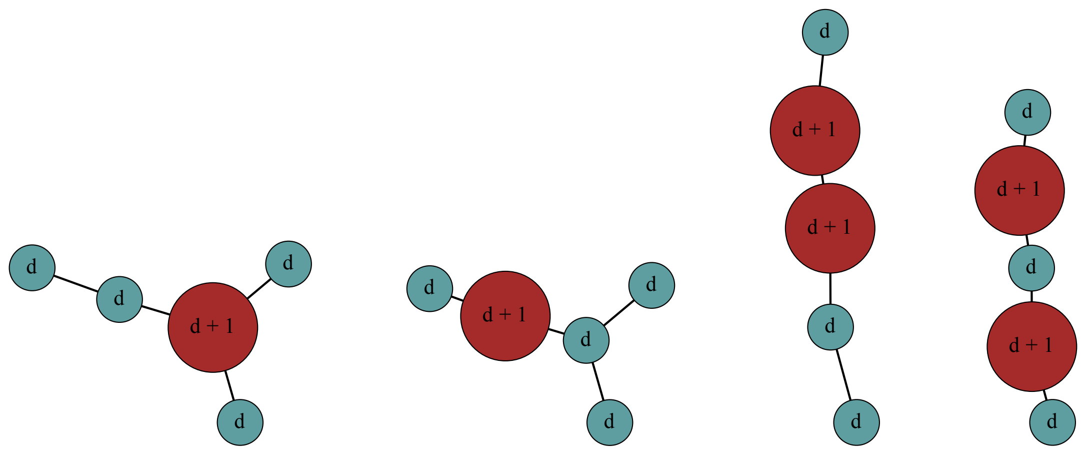

Figure 47: The fixable trees with maximum degree at most 3 on 5 vertices.

Conjecture 4.1. *A tree T is f-fixable if $f(v) = d_T(v)$ for at most one non-leaf v of T.*

Note that by Lemma 2.9, this would imply that Tashkinov trees are elementary under the same degree constraints. Can this be proved in the simpler case of Kierstead paths? For paths of length 4, this was done by Kostochka and Stiebitz, in the next section we present a generalization of this result to stars with one edge subdivided. One nice feature of the superabundance formulation is that since there is no need for an ordering like with Tashkinov trees, we can easily formulate results about graphs with cycles. The most general thing we might think is true is the following.

Conjecture 4.2 (false)**.** *A multigraph G is f-fixable if $f(v) > d_G(v)$ for all $v \in V(G)$.*

This is very strong and implies Goldberg's conjecture. Unfortunately, it is false. We can make counterexamples on a 5-cycle as in Figure 49.

Problem. *Show that C_5 is not f-fixable for any f.*

We have more questions than answers around these conjectures. For instance, what if we only look at nearly-colorable superabundant list assignments in Conjecture 4.2? The resulting conjecture is also stronger than Goldberg's conjecture and at present, we have no counterexamples.

4.1 Stars with one edge subdivided

The following conjecture would generalize the "Short Kierstead Paths" of Kostochka and Stiebitz (see [5]). Parts (a) and (b) are special cases of Conjecture 4.1. We have a rough draft of a proof for part (a) and we suspect parts (b) and (c) will be similar, but our draft is long and detailed, and we are still hoping to find a clean proof, like that for stars. Using this we show how to obtain Conjecture 4.3 a formula similar to the fan equation.

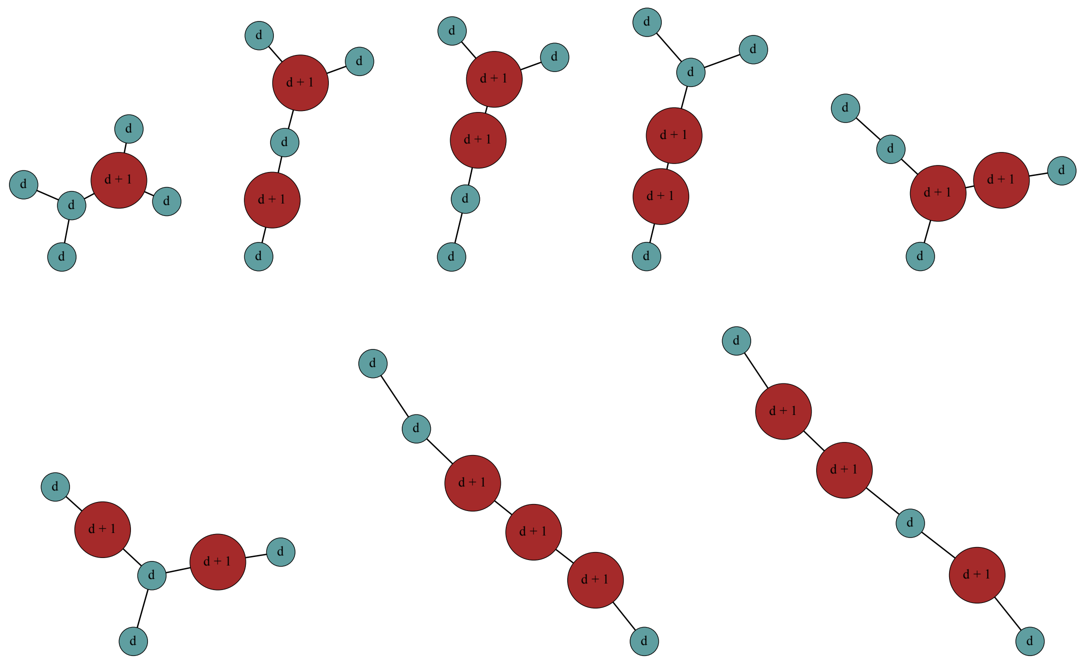

Figure 48: The fixable trees with maximum degree at most 3 on 6 vertices.

Conjecture 4.3. *Let G be a star with one edge subdivided, where r is the center of the star, t the vertex at distance two from r, and s the intervening vertex. If L is superabundant and $|L(v)| \geq d_G(v)$ for all $v \in V(G)$, then G is L-fixable if at least one of the following holds:*

(a) $|L(r)| > d_G(r)$; *or*

(b) $|L(s)| > d_G(s)$; *or*

(c) $\psi_L(G) > \|G\|$.

For a graph H and $v \in V(H)$, let $E_H(v)$ be the edges incident to v in H. Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q) + 1$ and $G \subseteq Q$. For a $\Delta(Q)$-edge-coloring π of

Figure 49: Counterexamples to Conjecture 4.2.

$Q-E(G)$, let $L_\pi(v) = [\Delta(Q)] - \pi\,(E_Q(v) - E(G))$ for all $v \in V(G)$. Graph G is a Ψ-*subgraph* of Q if there is a $\Delta(Q)$-edge-coloring π of $Q - E(G)$ such that each $H \subsetneq G$ is abundant. Let $E_L(H) = |\{\alpha \in \text{pot}(L) \mid |H_{L,\alpha}| \text{ is even}\}|$ and $O_L(H) = |\{\alpha \in \text{pot}(L) \mid |H_{L,\alpha}| \text{ is odd}\}|$. Note that $\text{pot}(L) = E_L(G) + O_L(G)$.

Lemma 4.4. *Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q) + 1$. If $G \subseteq Q$ and π is a $\Delta(Q)$-edge-coloring of $Q - E(G)$ such that $\|G\| \ge \psi_L(G)$, then $|O_{L_\pi}(G)| \ge \sum_{v\in V(G)} \Delta(Q) - d_Q(v)$. Furthermore, if $\|G\| > \psi_L(G)$, then $|O_{L_\pi}(G)| > \sum_{v\in V(G)} \Delta(Q) - d_Q(v)$.*

Proof. The proof is a straightforward counting argument. For fixed degrees and list sizes, as $|O_L(G)|$ gets larger, $\psi_L(G)$ gets smaller (half as quickly). The details forthwith. Let $L = L_\pi$.

Since $\|G\| \ge \psi_L(G)$, we have

$$\|G\| \ge \sum_{\alpha\in\text{pot}(L)} \left\lfloor \frac{|G_{L,\alpha}|}{2} \right\rfloor = \sum_{\alpha\in\text{pot}(L)} \frac{|G_{L,\alpha}|}{2} - \sum_{\alpha\in O_L(H)} \frac{1}{2}. \tag{1}$$

Also,

$$\begin{aligned}
\sum_{\alpha\in\text{pot}(L)} \frac{|G_{L,\alpha}|}{2} &= \sum_{v\in V(G)} \frac{\Delta(Q) - (d_Q(v) - d_G(v))}{2} \\
&= \sum_{v\in V(G)} \frac{d_G(v)}{2} + \sum_{v\in V(G)} \frac{\Delta(Q) - d_Q(v)}{2} \\
&= \|G\| + \sum_{v\in V(G)} \frac{\Delta(Q) - d_Q(v)}{2}. \qquad (2)
\end{aligned}$$

Now we solve for $\|G\| - \sum_{\alpha\in\text{pot}(L)} \frac{|G_{L,\alpha}|}{2}$ in (1) and (2), set the expressions equal, and then simplify. The result is (3).

$$|O_L(G)| \ge \sum_{v\in V(G)} \Delta(Q) - d_Q(v). \tag{3}$$

Finally, if the inequality in (1) is strict, then the inequality in (3) is also strict. □

Again, let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q) + 1$ and $G \subseteq Q$. If there is a $\Delta(Q)$-edge-coloring π of $Q - E(G)$ such that each $H \subsetneq G$ is abundant, then G is a Ψ-*subgraph* of Q.

Conjecture 4.5. *Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q) + 1$. Let H be a star with one edge subdivided; let r be the center of the star, t the vertex at distance two from r, and s the intervening vertex. If H is a Ψ-subgraph of Q, then there exists $X \subseteq N(r)$ with $V(H - r - t) \subseteq X$ such that*

$$\sum_{v\in X\cup\{t\}} (d_Q(v) + 1 - \Delta(Q)) \ge 0.$$

Moreover, if $\{r, s, t\}$ does not induce a triangle in Q, then

$$\sum_{v \in X \cup \{t\}} (d_Q(v) + 1 - \Delta(Q)) \geq 1.$$

Furthermore, if $d_Q(r) < \Delta(Q)$ or $d_Q(s) < \Delta(Q)$, then both lower bounds improve by 1.

Proof. Let G be a maximal Ψ-subgraph of Q containing H such that G is a star with one edge subdivided. Let π be a coloring of $Q - E(G)$ showing that G is a Ψ-subgraph and let $L = L_\pi$.

We first show that $|E_L(G)| \geq d_Q(r) - d_G(r) - 1$ if rst induces a triangle; otherwise, $|E_L(G)| \geq d_Q(r) - d_G(r)$. Suppose rst does not induce a triangle; for an arbitrary $x \in N_Q(r) - V(G)$, let $\alpha = \pi(rx)$. If $\alpha \in O_L(G)$, then adding x to G gives a larger Ψ-subgraph of the required form; this contradicts the maximality of G. Hence $\alpha \in E_L(G)$. Therefore, $|E_L(G)| \geq d_Q(r) - d_G(r)$ as desired. If rst induces a triangle, then we lose one off this bound from the edge rt.

By Conjecture 4.3(c), we have $\psi_L(G) \leq \|G\|$. By Lemma 4.4, we have $|O_L(G)| \geq \sum_{v \in V(G)} \Delta(Q) - d_Q(v)$. If rst does not induce a triangle, then

$$\begin{aligned}
\Delta(Q) &\geq \operatorname{pot}(L) \\
&= |E_L(G)| + |O_L(G)| \\
&\geq d_Q(r) - d_G(r) + \sum_{v \in V(G)} \Delta(Q) - d_Q(v) \qquad (4) \\
&= \Delta(Q) - d_G(r) + \sum_{v \in V(G-r)} \Delta(Q) - d_Q(v) \\
&= \Delta(Q) + 1 + \sum_{v \in V(G-r)} \Delta(Q) - 1 - d_Q(v).
\end{aligned}$$

Therefore, $\sum_{v \in V(G-r)} \Delta(Q) - 1 - d_Q(v) \leq -1$. Negating gives the desired inequality. If rst induces a triangle, we lose one off the bound. Conjecture 4.3(a,b) gives the final statement. $\square$

5 The gap between fixability and reducibility

By abstracting away the containing graph, we may have lost some power in proving reducibility results. Surely we have when we only care about a certain class of graphs. For example, with planar graphs, not all Kempe path pairings are possible (if we add an edge for each pair, the resulting graph has to be planar). But, possibly there are graphs that are reducible for all containing graphs but are not fixable. We could strengthen "fixable" in various ways, but we have not yet found any need to do so. There is one strengthening we should mention because it makes fixable more induction friendly.

Definition 5. G is (L, P)-*subfixable* if either

(1) G is (L, P)-fixable; or

(2) there is $xy \in E(G)$ and $\tau \in L(x) \cap L(y)$ such that $G - xy$ is L'-subfixable where L' is formed from L by removing τ from $L(x)$ and $L(y)$.

Superabundance is a necessary condition for subfixability because coloring an edge cannot make a non-abundant subgraph abundant. The conjectures in this paper may be easier to prove with subfixable in place of fixable. That would really be just as good since it would give the exact same results for edge coloring.

References

[1] David Cariolaro and Gianfranco Cariolaro, *Colouring the petals of a graph*, Electronic Journal of Combinatorics **10** (2003), R6.

[2] Cropper, M. and Gyárfás, A. and Lehel, J., *Edge list multicoloring trees: An extension of Hall's theorem*, Journal of Graph Theory **42** (2003), no. 3, 246–255.

[3] Marcotte, O. and Seymour, P., *Extending an edge-coloring*, Journal of Graph Theory **14** (1990), no. 5, 565–573.

[4] Rabern, L., *A game generalizing Hall's Theorem*, Discrete Mathematics **320** (2014), no. 6, 87–91.

[5] Stiebitz, Michael and Scheide, Diego and Toft, Bjarne and Favrholdt, Lene M, *Graph Edge Coloring: Vizing's Theorem and Goldberg's Conjecture*, vol. 75, Wiley. com, 2012.

[6] V.G. Vizing, *Vextex coloring with given colors*, Metody Diskretn. Anal. **29** (1976), 3–10 (in Russian).

[7] Douglas R Woodall, *The average degree of an edge-chromatic critical graph*, Discrete Mathematics **308** (2008), no. 5, 803–819.