fractional BK

August 24, 2014

King, Lu and Peng [1] showed that $\chi_f(G) \leq 4 - \frac{2}{67}$ when $\Delta(G) \leq 4$ and G does not contain K_4 or C_8^2. Moreover, they showed that this bound of $4 - \frac{2}{67}$ lifts to larger Δ by hitting maximum cliques (and a few other structures). Edwards and King improved this bound for $\Delta \geq 6$ using a probabilistic method, getting an upper bound of $6 - \frac{2}{45}$ for $\Delta = 6$. The goal is to improve these bounds by just improving the $\Delta = 4$ case.

Here is the basic form of the argument. I am going to loosen the bounds for simplicity. Suppose $\Delta(G) = 4$ and G doesn't contain K_4 or C_8^2. Take a 161-coloring of G^4 and let X be a color class. In G, blow up each vertex of X to K_2. The resulting graph Q has $\Delta(Q) \leq 5$, where the vertices of degree 5 are exactly the blown-up vertices and their neighbors. So, by our choice of X, the high vertex subgraph of Q is the disjoint union of graphs of the form $K_2 * T$ where $|T| = 4$ (or $|T| = 3$ if the blown-up vertex was low, but that case is easier). Call these components of the high vertex subgraph $H_1, \ldots, H_k$.

Lemma 0.1. *Q is 4-colorable.*

First, let's see what Lemma 0.1 gets us.

Theorem 0.2. $\chi_f(G) \leq 4 - \frac{2}{81}$.

Proof. For each of the 161 color classes of G^4, Lemma 0.1 gives a 4-coloring of G where each vertex in X gets 2 colors. Putting together such colorings for each of the 161 color classes gives a 162-fold coloring of G from a pot of $4 * 161$ colors. □

Proof of Lemma 0.1. Suppose not and let P be a 5-critical subgraph of Q. Let L be the low vertex subgraph of P and H the high vertex subgraph. Since X is independent in G^4, for each $v \in L$, there is a j such that $N(v) - L \subseteq V(H_j)$. Note that there may be vertices from some H_i that are now in L, if so they must have all their neighbors in L. Also note that for the blown-up vertices that are still high, all their neighbors are high.

By renumbering if necessary, the components of H are $H_1, \ldots, H_s$ for some $s \leq k$. Say $H_i = K_2 * T_i$ for each i. We are going to color the K_2 from each H_i in such a way that we can complete the coloring on L. There are 6 subsets of $[4]$ of size 2, assign one $\{a, b\}$ to each H_i, color the K_2 with a and b and then color T_i with $[4] - \{a, b\}$ so that the color classes are as imbalanced as possible (so if T_i is E_4, only one color is used, if $\alpha(T_i) = 3$, one color is used on three vertices, the other on one). We will show that there is a way to make such an assignment of 2-sets to the H_i so that the coloring is completable on L.

Remember that each low vertex has neighbors in at most one H_i, for $v \in V(L)$, let $h(v)$ be the i such that v has a neighbor in H_i (or 0 if v has no high neighbors). Given such an assignment to the H_i, a component A of L is *good* if

1. A has a noncutvertex v with at least two high neighbors colored the same; or

2. A has adjacent noncutvertices v, w such that $H_{h(v)}$ and $H_{h(w)}$ are assigned different 2-sets.

Such a component is good because we can color greedily towards v or v, w and finish. Note that if A has only one vertex, say v, then it is good for every assignment to the H_i since v will always satisfy (1).

Take an assignment of 2-sets to the H_i giving the maximum number of good components in L. If every component is good, then we are good. So suppose we have a component A of L that is not good.

First, suppose A has a block B with adjacent noncutvertices v and w such that $h(v) \neq h(w)$.

□

References

[1] Andrew D King, Linyuan Lu, and Xing Peng, *A fractional analogue of brooks' theorem*, SIAM Journal on Discrete Mathematics **26** (2012), no. 2, 452–471.