

# most low Alon-Tarsi notes

September 19, 2015

## 1 Introduction

We consider graphs with vertices labeled by natural numbers; that is, pairs  $(G, h)$  where  $G$  is a graph and  $h: V(G) \rightarrow \mathbb{N}$ . We say that  $(G, h)$  is AT if  $G$  is  $(d_G - h)$ -AT. When  $H$  is an induced subgraph of  $G$ , we simplify notation by referring to the pair  $(H, h)$  where we really mean  $(H, h|_{V(H)})$ .

## 2 Subgraphs, subdivisions and cuts

**Definition 1.** A graph  $G$  is *h-minimal* if  $G$  is connected and  $(H, h)$  is not AT for every proper induced subgraph  $H$  of  $G$ . A graph  $G$  is *h-greedy-minimal* if  $G$  is connected and  $(H, h)$  is not AT for every proper induced subgraph  $H$  of  $G$  where  $h(v) = 0$  for all  $v \in V(G) \setminus V(H)$ . Note that if  $G$  is *h-minimal* then it is also *h-greedy-minimal*.

**Lemma 2.1.** *If  $G$  is connected and  $(G, h)$  is not AT, then  $G$  is h-greedy-minimal.*

*Proof.* If there were a proper induced subgraph  $H$  such that  $(H, h|_{V(H)})$  is AT, then by ordering the vertices of each component of  $G - V(H)$  by increasing distance to  $H$  and directing all edges away from  $H$  in this order we conclude that  $(G, h)$  is AT.  $\square$

**Lemma 2.2.** *If  $(G', h')$  is formed from  $(G, h)$  by subdividing an edge  $e$  of  $G$  twice and having  $h'$  give zero on the two new vertices, then*

1. *if  $(G, h)$  is AT, then  $(G', h')$  is AT; and*
2. *if  $(G', h')$  is AT, then either  $(G, h)$  is AT or  $(G - e, h)$  is AT.*

*Proof.* Suppose  $e = xy$  and call the new vertices  $x'$  and  $y'$  so that  $G'$  contains the induced path  $xx'y'y$ . For (1), let  $D$  be an orientation of  $G$  showing that  $(G, h)$  is AT. By symmetry we may assume  $xy \in E(D)$ . Make an orientation  $D'$  of  $G'$  from  $D$  by replacing  $xy$  with the directed path  $xx'y'y$ . There is a natural parity preserving bijection between the spanning Eulerian subgraphs of  $D$  and  $D'$ , so we conclude that  $(G', h')$  is AT.

For (2), let  $D'$  be an orientation of  $G'$  showing that  $(G', h')$  is AT. Suppose  $G'$  contains the directed path  $xx'y'y$  or the directed path  $yy'x'x$ . By symmetry, we can assume it is  $xx'y'y$ . Then make an orientation  $D$  of  $G$  by replacing  $xx'y'y$  with the directed edge  $xy$ . As

above, we have a parity preserving bijection between the spanning Eulerian subgraphs of  $D$  and  $D'$ , so we conclude that  $(G, h)$  is AT. Otherwise, no spanning Eulerian subgraph of  $D'$  contains a cycle passing through  $x'$  and  $y'$ . So, the spanning Eulerian subgraph counts of  $D'$  are the same as those of  $D' - x' - y'$ . But this gives an orientation of  $G - e$  showing that  $(G - e, h)$  is AT.  $\square$

**Lemma 2.3.** *Let  $\{A_1, A_2\}$  be a separation of  $G$  such that  $A_1 \cap A_2 = \{x\}$ . If  $G[A_i]$  is  $f_i$ -AT for  $i \in [2]$ , then  $G$  is  $f$ -AT where  $f(v) = f_i(v)$  for  $v \in V(A_i - x)$  and  $f(x) = f_1(x) + f_2(x) - 1$ . Going the other direction, if  $G$  is  $f$ -AT, then  $G[A_i]$  is  $f_i$ -AT for  $i \in [2]$  where  $f_i(v) = f(v)$  for  $v \in V(A_i - x)$  and  $f_1(x) + f_2(x) \leq f(x) + 1$ .*

*Proof.* For  $i \in [2]$ , choose an orientation  $D_i$  of  $A_i$  showing that  $A_i$  is  $f_i$ -AT. Together these give an orientation  $D$  of  $G$  and since no cycle has vertices in both  $A_1 - x$  and  $A_2 - x$ , we have

$$\begin{aligned} EE(D) - EO(D) &= EE(D_1)EE(D_2) + EO(D_1)EO(D_2) - (EE(D_1)EO(D_2) + EO(D_1)EE(D_2)) \\ &= (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2)) \\ &\neq 0. \end{aligned}$$

Hence  $G$  is  $f$ -AT.

Now, suppose  $G$  is  $f$ -AT and choose an orientation  $D$  of  $G$  showing this. Put  $D_i = D[A_i]$  for  $i \in [2]$ . Then, as above, we have  $0 \neq EE(D) - EO(D) = (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2))$  and hence  $EE(D_1) - EO(D_1) \neq 0$  and  $EE(D_2) - EO(D_2) \neq 0$ . Since the in-degree of  $x$  in  $D$  is the sum of the in-degree of  $x$  in  $D_1$  and the in-degree of  $x$  in  $D_2$ , the lemma follows.  $\square$

**Corollary 2.4.** *Let  $G$  be an  $h$ -greedy-minimal graph. If  $(G, h)$  is AT and  $G$  has an induced path  $x_1x_2x_3x_4$  such that  $d_G(x_2) = d_G(x_3) = 2$  and  $h(x_2) = h(x_3) = 0$ , then*

$$((G - x_2 - x_3) + x_1x_4, h|_{V(G) \setminus \{x_2, x_3\}}) \text{ is AT.}$$

*Proof.* Suppose  $(G, h)$  is AT and  $G$  has such an induced path  $x_1x_2x_3x_4$ . Applying Lemma 2.2 part (2) shows that either  $(G - x_2 - x_3, h|_{V(G) \setminus \{x_2, x_3\}})$  is AT or  $((G - x_2 - x_3) + x_1x_4, h|_{V(G) \setminus \{x_2, x_3\}})$  is AT. But  $G - x_2 - x_3$  is a proper induced subgraph of  $G$ , so the former cannot happen since  $G$  is  $h$ -greedy-minimal and  $h(x_2) = h(x_3) = 0$ . Hence  $((G - x_2 - x_3) + x_1x_4, h|_{V(G) \setminus \{x_2, x_3\}})$  is AT.  $\square$

### 3 Extension lemma

This is a key lemma from [1], it generalizes a lemma from [2] from list coloring to Alon-Tarsi orientations. This is what I talked about in Baltimore. The basic idea is that in some cases we can pair off odd/even spanning Eulerian subgraphs via a parity reversing bijection.

**Lemma 3.1.** *Let  $G$  be a multigraph without loops and  $f: V(G) \rightarrow \mathbb{N}$ . If there are  $F \subseteq G$  and  $Y \subseteq V(G)$  such that:*

1. *any multiple edges in  $G$  are contained in  $G[Y]$ ; and*

2.  $f(v) \geq d_G(v)$  for all  $v \in V(G) \setminus Y$ ; and
3.  $f(v) \geq d_{G[Y]}(v) + d_F(v) + 1$  for all  $v \in Y$ ; and
4. For each component  $T$  of  $G - Y$  there are different  $x_1, x_2 \in V(T)$  where  $N_T[x_1] = N_T[x_2]$  and  $T - \{x_1, x_2\}$  is connected such that either:
  - (a) there are  $x_1y_1, x_2y_2 \in E(F)$  where  $y_1 \neq y_2$  and  $N(x_i) \cap Y = \{y_i\}$  for  $i \in [2]$ ; or
  - (b)  $|N(x_2) \cap Y| = 0$  and there is  $x_1y_1 \in E(F)$  where  $N(x_1) \cap Y = \{y_1\}$ ,

then  $G$  is  $f$ -AT.

*Proof.* Suppose not and pick a counterexample  $(G, f, F, Y)$  minimizing  $|G - Y|$ . If  $|G - Y| = 0$ , then  $Y = V(G)$  and thus  $f(v) \geq d_G(v) + 1$  for all  $v \in V(G)$  by (3). Pick an acyclic orientation  $D$  of  $G$ . Then  $EE(D) = 1$ ,  $EO(D) = 0$  and  $d_D^+(v) \leq d_G(v) \leq f(v) - 1$  for all  $v \in V(D)$ . Hence  $G$  is  $f$ -AT. So, we must have  $|G - Y| > 0$ .

Pick a component  $T$  of  $G - Y$  and pick  $x_1, x_2 \in V(T)$  as guaranteed by (4). First, suppose (4a) holds. Put  $G' := (G - T) + y_1y_2$ ,  $F' := F - T$ ,  $Y' := Y$  and let  $f'$  be  $f$  restricted to  $V(G')$ . Then  $G'$  has an orientation  $D'$  where  $f'(v) \geq d_{D'}^+(v) + 1$  for all  $v \in V(D')$  and  $EE(D') \neq EO(D')$ , for otherwise  $(G', f', F', Y')$  would contradict minimality. By symmetry we may assume that the new edge  $y_1y_2$  is directed toward  $y_2$ . Now we use the orientation of  $D'$  to construct the desired orientation of  $D$ . First, we use the orientation on  $D' - y_1y_2$  on  $G - T$ . Now, order the vertices of  $T$  as  $x_1, x_2, z_1, z_2, \dots$  so that every vertex has at least one neighbor to the right. Orient the edges of  $T$  left-to-right in this ordering. Finally, we use  $y_1x_1$  and  $x_2y_2$  and orient all other edges between  $T$  and  $G - T$  away from  $T$ . Plainly,  $f(v) \geq d_D^+(v) + 1$  for all  $v \in V(D)$ . Since  $y_1x_1$  is the only edge of  $D$  going into  $T$ , any Eulerian subgraph of  $D$  that contains a vertex of  $T$  must contain  $y_1x_1$ . So, any Eulerian subgraph of  $D$  either contains (i) neither  $y_1x_1$  nor  $x_2y_2$ , (ii) both  $y_1x_1$  and  $x_2y_2$ , or (iii)  $y_1x_1$  but not  $x_2y_2$ . We first handle (i) and (ii) together. Consider the function  $h$  that maps an Eulerian subgraph  $Q$  of  $D'$  to an Eulerian subgraph  $h(Q)$  of  $D$  as follows. If  $Q$  does not contain  $y_1y_2$ , let  $h(Q) = \iota(Q)$  where  $\iota(Q)$  is the natural embedding of  $D' - y_1y_2$  in  $D$ . Otherwise, let  $h(Q) = \iota(Q - y_1y_2) + \{y_1x_1, x_1x_2, x_2y_2\}$ . Then  $h$  is a parity-preserving injection with image precisely the union of those Eulerian subgraphs of  $D$  in (i) and (ii). Hence if we can show that exactly half of the Eulerian subgraphs of  $D$  in (iii) are even, we will conclude  $EE(D) \neq EO(D)$ , a contradiction. To do so, consider an Eulerian subgraph  $A$  of  $D$  containing  $y_1x_1$  and not  $x_2y_2$ . Since  $x_1$  must have in-degree 1 in  $A$ , it must also have out-degree 1 in  $A$ . We show that  $A$  has a mate  $A'$  of opposite parity. Suppose  $x_2 \notin A$  and  $x_1z_1 \in A$ ; then we make  $A'$  by removing  $x_1z_1$  from  $A$  and adding  $x_1x_2z_1$ . If  $x_2 \in A$  and  $x_1x_2z_1 \in A$ , we make  $A'$  by removing  $x_1x_2z_1$  and adding  $x_1z_1$ . Hence exactly half of the Eulerian subgraphs of  $D$  in (iii) are even and we conclude  $EE(D) \neq EO(D)$ , a contradiction.

Now suppose (4b) holds. Put  $G' := G - T$ ,  $F' := F - T$ ,  $Y' := Y$  and define  $f'$  by  $f'(v) = f(v)$  for all  $v \in V(G' - y_1)$  and  $f'(y_1) = f(y_1) - 1$ . Then  $G'$  has an orientation  $D'$  where  $f'(v) \geq d_{D'}^+(v) + 1$  for all  $v \in V(D')$  and  $EE(D') \neq EO(D')$ , for otherwise  $(G', f', F', Y')$  would contradict minimality. We orient  $G - T$  according to  $D$ , orient  $T$  as in the previous case, again use  $y_1x_1$  and orient all other edges between  $T$  and  $G - T$  away from  $T$ . Since we decreased  $f'(y_1)$  by 1, the extra out edge of  $y_1$  is accounted for and we have

$f(v) \geq d_D^+(v) + 1$  for all  $v \in V(D)$ . Again any additional Eulerian subgraph must contain  $y_1x_1$  and since  $x_2$  has no neighbor in  $G - T$  we can use  $x_2$  as before to build a mate of opposite parity for any additional Eulerian subgraph. Hence  $EE(D) \neq EO(D)$  giving our final contradiction.  $\square$

## 4 Degree-AT graphs

A graph  $G$  is called *degree-AT* if  $(G, h)$  is AT where  $h$  is the constant zero function.

**Lemma 4.1.** *A connected graph  $G$  is degree-AT if it is not a Gallai tree.*

*Proof.* Suppose there exists a connected graph that is not a Gallai tree, but is also not degree-AT. Let  $G$  be such a graph with as few vertices as possible. Since  $G$  is not degree-AT, no induced subgraph  $H$  of  $G$  is degree-AT by Lemma 2.1. Hence, for any  $v \in V(G)$  that is not a cutvertex,  $G - v$  must be a Gallai tree by minimality of  $|G|$ .

If  $G$  has more than one block, then for endblocks  $B_1$  and  $B_2$ , choose noncutvertices  $w \in B_1$  and  $x \in B_2$ . By the minimality of  $|G|$ , both  $G - w$  and  $G - x$  are Gallai trees. Since every block of  $G$  appears either as a block of  $G - w$  or as a block of  $G - x$ , every block of  $G$  is either complete or an odd cycle. Hence,  $G$  is a Gallai tree, a contradiction. So instead  $G$  has only one block, that is,  $G$  is 2-connected. Further,  $G - v$  is a Gallai tree for all  $v \in V(G)$ .

Let  $v$  be a vertex of minimum degree in  $G$ . Since  $G$  is 2-connected,  $d_G(v) \geq 2$  and  $v$  is adjacent to a noncutvertex in every endblock of  $G - v$ . If  $G - v$  has a complete block  $B$  with noncutvertices  $x_1, x_2$  where  $v \leftrightarrow x_1$  and  $v \not\leftrightarrow x_2$ , then we can apply Lemma 3.1 with  $Y = \{v\}$  and  $F = vx_1$  to conclude that  $G$  is degree-AT, a contradiction. So,  $v$  must be adjacent to every noncutvertex in every complete endblock of  $G - v$ .

Suppose  $d_G(v) \geq 3$ . Then no endblock of  $G - v$  can be an odd cycle of length at least 5 (there would be vertices of degree 3 but we'd have  $d_G(v) \geq 4$ ). Let  $B$  be a smallest complete endblock of  $G - v$ . Then for a noncutvertex  $x \in V(B)$ , we have  $d_G(x) = |B|$  and hence  $d_G(v) \leq |B|$ . If  $G - v$  has at least two endblocks, then  $2(|B| - 1) \leq |B|$  and hence  $d_G(v) \leq |B| = 2$ , a contradiction. Hence  $G - v = B$  and  $v$  is joined to  $B$ , so  $G$  is complete, a contradiction.

Hence, we must have  $d_G(v) = 2$ . Suppose  $G - v$  has at least 2 endblocks. Then, it has exactly 2 and  $v$  is adjacent to one noncutvertex in each. Neither of the endblocks can be odd cycles of length at least 5 since then we could get a smaller counterexample by Lemma 2.2. Since  $v$  is adjacent to every noncutvertex in every complete endblock of  $G - v$ , both endblocks must be  $K_2$ . But then either  $G = C_4$  (which is trivially degree-AT) or we can get a smaller counterexample by Lemma 2.2. So,  $G - v$  must be 2-connected. Since  $G - v$  is a Gallai tree, it is either complete or an odd cycle. If  $G - v$  is not complete, we can get a smaller counterexample by Lemma 2.2. So,  $G - v$  is complete and  $v$  is adjacent to every noncutvertex of  $G - v$ ; that is,  $G$  is complete, a contradiction.  $\square$

## 5 When $h$ is 1 for at most one vertex

For a graph  $G$  and  $x \in V(G)$  let  $h_x: V(G) \rightarrow \mathbb{N}$  be defined by  $h_x(x) = 1$  and  $h_x(v) = 0$  for all  $v \in V(G - x)$ . We classify the connected  $h_x$ -minimal graphs  $G$  such that  $(G, h_x)$  is AT

for some  $x \in V(G)$ .

To start we will reduce to the case when  $G$  is 2-connected.

**Lemma 5.1.** *Let  $G$  be  $h_x$ -minimal for  $x \in V(G)$  and let  $\mathcal{B}$  be the set of blocks of  $G$  containing  $x$ . Then  $(G, h_x)$  is AT if and only if*

1.  $\mathcal{B}$  contains at least two degree-AT graphs; or
2.  $G$  is 2-connected and  $(G, h_x)$  is AT.

*Proof.* Since  $G$  is  $h_x$ -minimal, no block outside of  $\mathcal{B}$  is degree-AT. The lemma follows since if  $G$  is not 2-connected, then  $(G, h_x)$  is AT if and only if (1) holds by Lemma 2.3.  $\square$

**Lemma 5.2.** *If  $G$  is a connected graph and  $x \in V(G)$  with  $d_G(x) = 2$ , then  $(G, h_x)$  is AT if and only if  $G - x$  is degree-AT.*

*Proof.* Let  $D$  be an orientation of  $G$  showing that  $(G, h_x)$  is AT. Then  $d_D^-(x) = 2$  and hence no spanning Eulerian subgraph contains a cycle passing through  $x$ . Therefore, the Eulerian subgraph counts in  $G - x$  are different and  $G - x$  is degree-AT. The other direction is immediate from Lemma 2.1.  $\square$

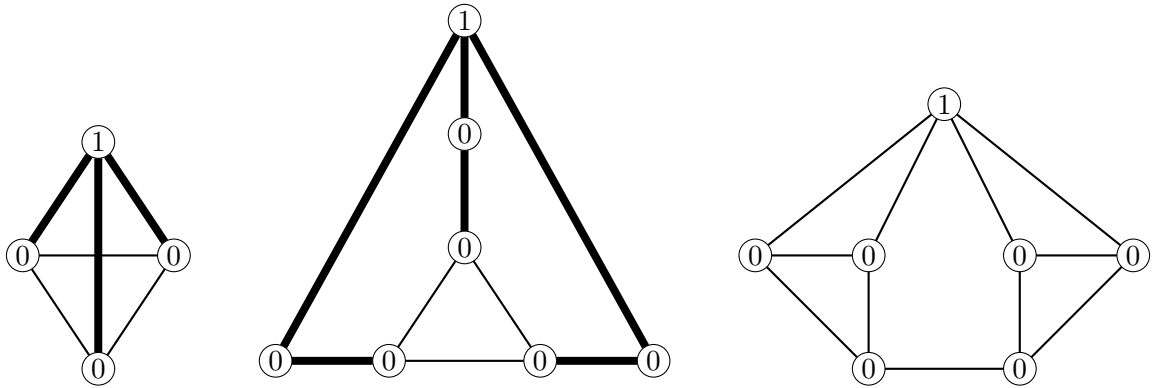


Figure 1: The seed blocks. Each bold edge can be removed without making the figure AT.

Lemma 2.2 part (2) suggests a way to construct  $G$  such that  $(G, h)$  is not AT from smaller graphs. Specifically, we have the following.

**Corollary 5.3.** *If  $e$  is an edge in  $G$  such that  $(G, h)$  is not AT and  $(G - e, h)$  is not AT, then  $(G', h')$  is not AT where  $(G', h')$  is formed from  $(G, h)$  by subdividing  $e$  twice and having  $h'$  give zero on the two new vertices.*

Let  $\mathcal{D}$  be the smallest collection of pairs  $(G, h)$  containing the pairs in Figure 1 that is closed under the operation in Corollary 5.3. Explicitly,  $\mathcal{D}$  is all graphs that can be created from the graphs in Figure 1 by replacing each bold edge with an odd length path. The rightmost graph in Figure 1 (the Moser Spindle) has no bold edges.

For a connected graph  $G$  and endblock  $B$  of  $G$ , let  $x_B$  be the cutvertex of  $G$  contained in  $B$ .

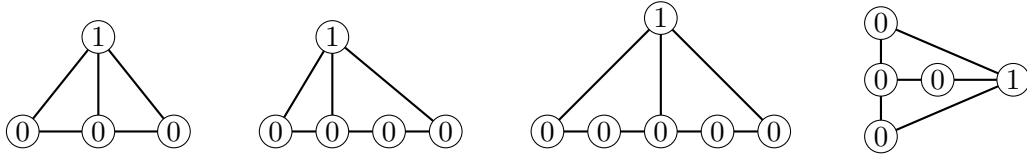


Figure 2: These are AT.

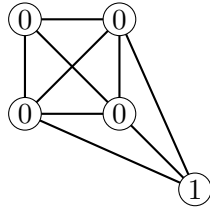


Figure 3: This is AT.

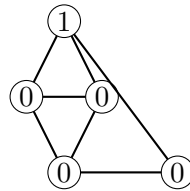


Figure 4: This is AT.

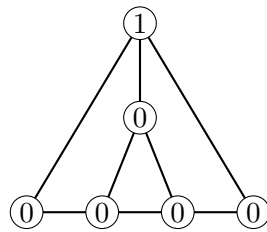


Figure 5: This is AT.

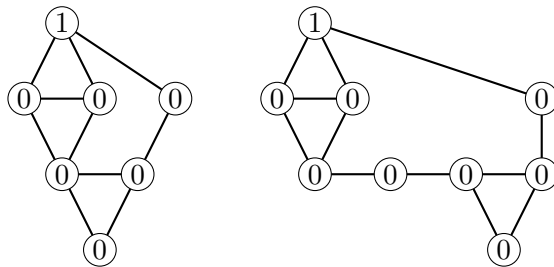


Figure 6: These are AT.

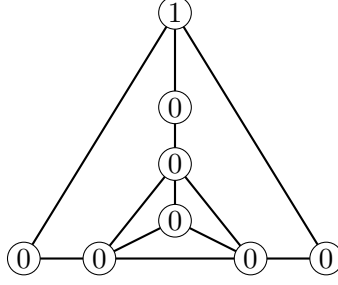


Figure 7: This is AT.

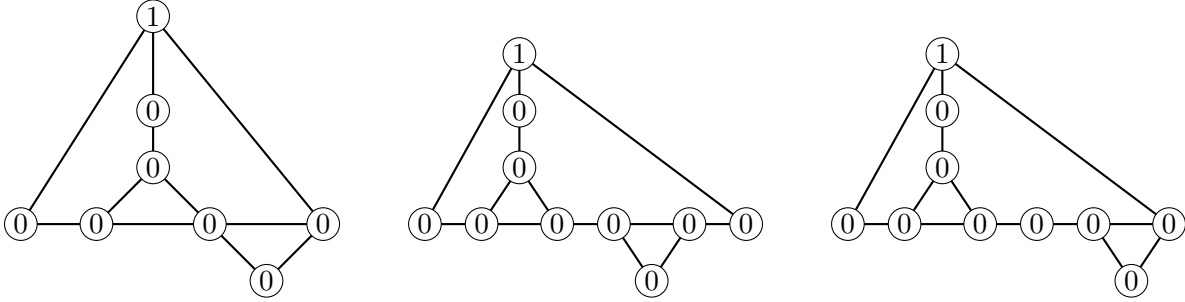


Figure 8: These are AT.

**Lemma 5.4.** *Let  $G$  be a connected graph and  $v \in V(G)$  a cutvertex of  $G$ . If  $G - v$  has  $t$  components, then there are endblocks  $B_1, \dots, B_t$  and an induced subdivision of  $K_{1,t}$  where the root is  $v$  and the leaves are  $x_{B_1}, \dots, x_{B_t}$ .*

*Proof.* Pick endblocks  $B_1, \dots, B_t$ , one in each component of  $G - v$ . Now the desired induced subdivision of  $K_{1,t}$  is the union of shortest paths from  $x_{B_1}$  to  $x_{B_i}$  for  $2 \leq i \leq t$ .  $\square$

Lemma 5.4 will be really useful in applying the following lemma. Note that we can always extend the induced subdivision of  $K_{1,3}$  or induced path we get one vertex into each endblock.

**Lemma 5.5.** *Let  $G$  be  $h_x$ -minimal for  $x \in V(G)$  with  $d_G(x) \geq 3$ . If  $(G, h_x)$  is not AT, then every induced subdivision of  $K_{1,3}$  in  $G$  contains at most two vertices in  $N(x)$ . In particular, every induced path in  $G$  contains at most two vertices in  $N(x)$ .*

*Proof.* This is immediate from Lemma 2.2 and the graphs in Figure 2.  $\square$

**Definition 2.** For  $a_1, a_2, a_3 \in \mathbb{N}$ , let  $T_{a_1, a_2, a_3}$  be the graph consisting of

- a triangle  $z_1 z_2 z_3$ ; and
- disjoint paths  $z_i P_i w_i$  where  $P_i$  has length  $a_i$  for  $i \in [3]$ ; and
- a vertex  $x$  adjacent to  $w_1, w_2, w_3$ .

**Lemma 5.6.**  *$(T_{a_1, a_2, a_3}, h_x)$  is AT when  $|\{i \in [3] \mid a_i \text{ is even}\}| \in \{1, 2\}$ .*

*Proof.* This follows from Lemma 2.2 and the fact that Figure 4 and Figure 5 are AT.  $\square$

**Lemma 5.7.** *Let  $H$  be formed from  $T_{a_1, a_2, a_3}$  by adding a new vertex with neighborhood  $\{z_1, z_2, z_3\}$ . Then  $(H, h_x)$  is AT.*

*Proof.* This follows from Lemma 5.6 and Lemma 2.1 using Lemma 2.2 and the fact that Figure 3 is AT and Figure 7 is AT.  $\square$

**Lemma 5.8.** *Let  $H$  be formed from  $T_{a_1, a_2, a_3}$  by adding a new vertex adjacent to two consecutive vertices on  $P_1$ . Then  $(H, h_x)$  is AT.*

*Proof.* This follows from Lemma 5.6 and Lemma 2.1, using Lemma 2.2 and the fact that the graphs in Figure 6 are AT and the graphs in Figure 8 are AT.  $\square$

**Lemma 5.9.** *Let  $G$  be  $h_x$ -minimal for  $x \in V(G)$ . If  $G$  is 2-connected, then  $(G, h_x)$  is AT if and only if*

1.  $d_G(x) \geq 3$ ; and
2.  $G$  is not complete; and
3.  $(G, h_x) \notin \mathcal{D}$ .

*Proof.* Suppose the lemma is false and choose a counterexample  $G$  minimizing  $|G|$ . If  $d_G(x) \leq 2$ , then  $(G, h_x)$  is not AT by Lemma 5.2 since  $G$  is  $h_x$ -minimal. So, we must have  $d_G(x) \geq 3$ . Since  $(G, h_x)$  is not AT if  $(G, h_x) \in \mathcal{D}$  by construction, it must be that  $(G, h_x) \notin \mathcal{D}$  and  $(G, h_x)$  is not AT.

**Claim 0.**  $G - x$  is a Gallai tree and  $x$  is adjacent to a noncutvertex in every endblock of  $G - x$ . This follows since  $G$  is  $h_x$ -minimal and 2-connected.

**Claim 1.**  $G - x - v$  has at most two components for any  $v \in V(G - x)$ . Suppose  $G - x$  has a cutvertex  $v$  such that  $G - x - v$  has at least three components. Then, by Lemma 5.4,  $G - x$  contains an induced  $K_{1,3}$  violating Lemma 5.5.

**Claim 2.**  $x$  is not adjacent to any cutvertex  $v$  of  $G - x$ . Using Lemma 5.4, we get an induced path from  $x_B$  to  $x_D$  containing  $v$ , where  $B$  and  $D$  are different endblocks violating Lemma 5.5.

**Claim 3.**  $G - x$  does not contain an induced path  $v_1 v_2 v_3 v_4$  such that  $d_G(v_2) = d_G(v_3) = 2$ . If it did, then we could get a smaller counterexample by applying Lemma 2.2 part (1).

**Claim 4.** Every block of  $G - x$  is complete. Suppose  $G - x$  has a block  $B$  that is an odd cycle  $v_1 v_2 \dots v_t v_1$  with  $t \geq 5$ .

**Subclaim 4a.**  $B$  contains at most two cutvertices of  $G - x$ . Otherwise there are  $a, b, c \in [t]$  such that  $v_a, \dots, v_b, \dots, v_c$  contains exactly three cutvertices  $v_a, v_b$  and  $v_c$ . Apply Lemma 5.4 to the component of  $G - \{x, v_1, \dots, v_{a-1}, v_{b+1}, \dots, v_t\}$  containing  $v_a, v_b, v_c$  with  $v = v_b$  to get an induced  $K_{1,3}$  violating Lemma 5.5.

**Subclaim 4b.**  $B$  contains at most one cutvertex of  $G - x$ . Otherwise, by Subclaim 4a,  $B$  has exactly two cutvertices  $v_a$  and  $v_b$ . By Claim 3,  $x$  is adjacent to a noncutvertex  $v \in V(B)$ . Consider the induced path given by applying Lemma 5.4 to  $v_a$ . If this path does not contain  $v$ , then have it go the other way around  $B$ . Now we have an induced path violating Lemma 5.5.

**Subclaim 4c.** Claim 4 is true. By Claim 3,  $x$  must be adjacent to at least every other noncutvertex of  $B$ . So, if  $G - x = B$ , we immediately violate Lemma 5.5. If instead,  $G - x$



has another endblock  $B'$  then we can pick two neighbors of  $x$  in  $B$  and one neighbor of  $x$  in  $B'$  all on an induced path in  $G - x$ , violating Lemma 5.5.

**Claim 5.** *If  $x$  is adjacent to a noncutvertex in a block, then  $x$  is adjacent to all noncutvertices in that block. In particular,  $x$  is adjacent to every noncutvertex in every endblock of  $G - x$ .* Suppose  $G - x$  has a block  $B$  with noncutvertices  $v_1, v_2$  where  $x \leftrightarrow v_1$  and  $x \nleftrightarrow v_2$ . By Claim 4,  $B$  is complete, so we can apply Lemma 3.1 with  $Y = \{x\}$  and  $F = xv_1$  to conclude that  $(G, h_x)$  is AT, a contradiction.

**Claim 6.**  *$G - x$  has at least two endblocks.* If not, then  $G - x$  is complete by Claim 0 and Claim 4. But then  $G$  is complete by Claim 5, a contradiction.

**Claim 7.** *The endblocks of  $G - x$  are all  $K_2$ , except possibly one  $K_3$ .* By Claim 4, every endblock is complete. Suppose  $G - x$  has an endblock  $B = K_t$  for  $t \geq 4$ . Then by Claim 2, Claim 5 and Claim 6,  $G$  has an induced Figure 3, impossible. So every endblock of  $G - x$  is  $K_2$  or  $K_3$ . Suppose  $G - x$  has two  $K_3$  endblocks  $B_1$  and  $B_2$ . Then  $G[\{x\} \cup V(B_i)]$  is degree-AT for  $i \in [2]$ . If there is no edge between  $B_1$  and  $B_2$ , then, by Lemma 5.1,  $G$  contains an induced subgraph  $H$  such that  $(H, h_x)$  is AT, a contradiction. If there is an edge between  $B_1$  and  $B_2$ , then by Claim 1,  $G$  is the rightmost graph in Figure 1, a contradiction.

**Claim 8.** *Every noncutvertex of  $G - x$  is adjacent to  $x$ .*

Suppose  $G - x$  has a noncutvertex  $v$  with  $v \nleftrightarrow x$ . Then  $G - v$  is 2-connected and  $h_x$ -minimal, so by minimality of  $|G|$ , we conclude that  $d_{G-v}(x) \leq 2$ ,  $G - v$  is complete, or  $(G - v, h_x) \in \mathcal{D}$ . The first three clearly cannot occur, so we have  $(G - v, h_x) \in \mathcal{D}$ .

**Subclaim 8a.**  *$G - v$  has an induced path  $v_1v_2v_3v_4$  such that  $d_G(v_2) = d_G(v_3) = 2$ .*

Otherwise,  $G - v$  is one of the graphs in Figure 1. But  $G - v$  cannot be the leftmost, middle, or rightmost graph in Figure 1 because then  $G$  would contain the graph in Figure 3, Figure 7, and Figure 5 as an induced subgraph, respectively.

**Subclaim 8b.** *The block  $B$  containing  $v$  is  $K_3$ .*

By Claim 4,  $B$  is complete. Some neighbor  $w$  of  $v$  must have gone from degree 3 to degree 2. Since  $v$  is only adjacent to vertices in  $B$ , the only way for this to happen is if  $B = K_3$ .

**Subclaim 8c.**  *$G - v$  is the result of applying the operation in Corollary 5.3 to a graph  $F$  in Figure 1 one time.* None of the graphs in Figure 1 have an induced path  $v_1v_2v_3v_4$  such that  $d_G(v_2) = d_G(v_3) = 2$ .

**Subclaim 8d.**  *$F$  is not the rightmost graph in Figure 1.* For this one, removing any edge leaves an AT graph, so Corollary 5.3 cannot be applied.

**Subclaim 8e.**  *$F$  is not the middle graph in Figure 1.* For this one, removing any edge in the triangle leaves an AT graph, so Corollary 5.3 cannot be applied to those edges. But then  $G$  is one of the graphs in Figure 8, impossible.

**Subclaim 8f.** *Claim 8 is true.* By the previous subclaims,  $F$  must be the leftmost graph in Figure 1. For this one, removing any of the edges not incident to the vertex labeled 1 leaves an AT graph, so Corollary 5.3 cannot be applied to those edges. But then  $G$  contains an induced even subdivision of Figure 4 or the leftmost graph in Figure 6, impossible.

**Claim 9.** *Every internal block of  $G - x$  consists entirely of cutvertices.* Suppose otherwise that we have an internal block  $B$  of  $G - x$  containing a noncutvertex  $v$ . By Claim 8,  $x \leftrightarrow v$ . Note that by Lemma 2.2 and Figure 4, we get that Figure 5 is AT with either or both of the bottom left and bottom right edge subdivided once. But  $G$  contains at least one of these with edges subdivided twice some number of times as an induced subgraph, a contradiction.

**Claim 10.**  *$x$ 's neighbors are precisely the noncutvertices in the endblocks of  $G - x$ .* By Claim 5,  $x$  is adjacent to all these vertices. By Claim 9 and Claim 2,  $x$  is not adjacent to any other vertex.

**Claim 11.**  *$G - x$  has at least three endblocks.* If not, then by Claim 6,  $G - x$  has exactly two endblocks  $B_1$  and  $B_2$ . Since  $d_G(x) \geq 3$ , Claim 7, Claim 9 and Claim 10 show that  $G - x$  is a triangle  $w_1w_2w_3$  with a path  $w_1y_1y_2 \dots y_t$  emanating from  $w_1$ . Since  $(G, h_x)$  is not AT, Lemma 2.2 and Figure 4 show that  $t$  must be even. But then  $(G, h_x) \in \mathcal{D}$  since  $G$  is formed from the leftmost graph in Figure 1 by applying Corollary 5.3 some number of times, a contradiction.

**Claim 12.** *Every endblock of  $G - x$  is  $K_2$ .* Suppose  $G - x$  has a  $K_3$  endblock  $B$ .

**Subclaim 12a.** *The component of  $G - N(x_B)$  containing  $x$  is not degree-AT.* Since  $G$  is  $h_x$ -minimal, this follows by Lemma 5.1.

**Subclaim 12b.** *The component of  $G - N(x_B)$  containing  $x$  is triangle-free.* If not, then by Claim 11,  $G - N(x_B)$  contains an induced subgraph containing  $x$  that is a cycle  $y_1 \dots y_t y_1$  plus the edge  $y_1 y_3$  with  $t \geq 4$ . If  $t$  is even, then  $G - N(x_B)$  contains an induced even cycle with at most one chord which is degree-AT, otherwise  $G - N(x_B)$  contains the induced even cycle  $y_1 y_3 \dots y_t y_1$  which is also degree-AT; this contradicts Subclaim 12a.

**Subclaim 12c.** *The other block  $D$  containing  $x_B$  is  $K_2$ .* If not, then  $G$  must have either an induced even subdivision of the leftmost graph in Figure 6 or an induced even subdivision of Figure 4 (both path length parities are covered).

**Subclaim 12d.** *Claim 12 is true.* By Subclaim 12b and 12c,  $G - x$  has exactly one  $K_3$  internal component, call it  $Q$ , and the rest of the internal components are  $K_2$ . Moreover,  $Q$  intersects  $D$  and in particular, a shortest path from  $Q$  to  $x$  passing through  $B$  has length three. By Claim 1 and Claim 11,  $G - x$  has exactly three endblocks  $B_1 = B, B_2$  and  $B_3$ . For  $i \in [3]$ , let  $\ell_i$  be the length of the shortest path from  $Q$  to  $x$  passing through  $B_i$ . We know  $\ell_1 = 3$ . If both  $\ell_2$  and  $\ell_3$  are even, then  $G$  would have an induced even subdivision of Figure 5, a contradiction. So, at least one of  $\ell_2, \ell_3$  are odd and hence  $G$  contains an induced even subdivision of Figure 4, a contradiction.

**Claim 13.** *All internal blocks of  $G - x$  are  $K_2$  or  $K_3$ .* If not, then by Claim 4,  $G - x$  has a  $K_t$  block with  $t \geq 4$ . But this is impossible by Lemma 5.7 since  $G$  is  $h_x$ -minimal.

**Claim 14.**  *$G - x$  has at least four endblocks.* If not, then by Claim 11,  $G - x$  has exactly three endblocks  $B_1, B_2$  and  $B_3$ , all of them  $K_2$  by Claim 12. By Claim 1, Claim 9 and Claim 13,  $G = T_{a_1, a_2, a_3}$  for some  $a_1, a_2, a_3 \in \mathbb{N}$ . By Lemma 5.6, either all or none of  $a_1, a_2, a_3$  are even. If all, then  $(G, h_x) \in \mathcal{D}$  since  $G$  is formed from the leftmost graph in Figure 1 by applying Corollary 5.3 some number of times, a contradiction. If none, then  $(G, h_x) \in \mathcal{D}$  since  $G$  is formed from the middle graph in Figure 1 by applying Corollary 5.3 some number of times, a contradiction.

**Claim 15.** *The lemma is true.* If not, then by Claim 1, Claim 9, Claim 13 and Claim 14,  $G$  contains an induced subgraph that violates  $h_x$ -minimality by Lemma 5.8.  $\square$

Putting Lemma 5.1 and Lemma 5.9 together we get the following classification of  $h_x$ -minimal graphs.

**Theorem 5.10.** *Let  $G$  be  $h_x$ -minimal for  $x \in V(G)$  and let  $\mathcal{B}$  be the set of blocks of  $G$  containing  $x$ . Then  $(G, h_x)$  is AT if and only if*

1.  $\mathcal{B}$  contains at least two degree-AT graphs; or
2.  $G$  is 2-connected,  $G$  is not complete,  $(G, h_x) \notin \mathcal{D}$  and  $d_G(x) \geq 3$ .

## 6 Choosability and Paintability

The same classification holds. Just need to show none of the bad ones are choosable. Also, show that when we move up to two vertices  $x, y$  with  $h(x) = h(y) = 1$ , AT, choosability and paintability all separate.

## 7 Applications

When this is in a critical graph, we get  $h_x$ -minimality for free. If  $x$  has  $t \geq 5$  neighbors in one component of the low vertex subgraph, then those neighbors are all in one  $K_t$  block. If  $t = 4$ , then the neighbors are either a  $K_4$  block or we create the Moser Spindle.

## References

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