A slightly better lower bound on the number of edges in (online) list critical graphs

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1 Introduction

Let $\mathcal{T}_k$ be the Gallai trees with maximum degree at most $k-1$, excepting K_k. For a graph G, let $W^k(G)$ be the set of vertices of G that are contained in some K_{k-1} in G.

Definition 1. A graph G is *AT-reducible* to H if H is a nonempty induced subgraph of G which is f_H-AT where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not AT-reducible to any nonempty induced subgraph, then it is *AT-irreducible*.

2 Reducible Configurations

This first lemma tells us how a single high vertex can interact with the low vertex subgraph. This is the version Hal and i used, it (and more) follows from the classification in "mostlow".

Lemma 2.1. *Let $k \geq 5$ and let G be a graph with $x \in V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *$G - x$ has t components $H_1, H_2, \ldots, H_t$, and all are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k-1$ for all $v \in V(G-x)$; and*
4. *$\left|N(x) \cap W^k(H_i)\right| \geq 1$ for $i \in [t]$; and*
5. *$d_G(x) \geq t+2$.*

Then G is f-AT where $f(x) = d_G(x) - 1$ and $f(v) = d_G(v)$ for all $v \in V(G-x)$.

To deal with more than one high vertex we need the following auxiliary bipartite graph. For a graph G, $\{X, Y\}$ a partition of $V(G)$ and $k \geq 4$, let $\mathcal{B}_k(X, Y)$ be the bipartite graph with one part Y and the other part the components of $G[X]$. Put an edge between $y \in Y$ and a component T of $G[X]$ iff $N(y) \cap W^k(T) \neq \emptyset$. The next lemma tells us that we have a reducible configuration if this bipartite graph has minimum degree at least three.

Lemma 2.2. *Let $k \geq 7$ and let G be a graph with $Y \subseteq V(G)$ such that:*

	k-Critical G				k-ListCritical G		
k	Gallai [1] $d(G) \geq$	Kriv [5] $d(G) \geq$	KS [4] $d(G) \geq$	KY [3] $d(G) \geq$	KS [4] $d(G) \geq$	KR [2] $d(G) \geq$	Here $d(G) \geq$
4	3.0769	3.1429	—	3.3333	—	—	
5	4.0909	4.1429	—	4.5000	—	4.0984	**4.1**
6	5.0909	5.1304	5.0976	5.6000	—	5.1053	**5.1082**
7	6.0870	6.1176	6.0990	6.6667	—	6.1149	**6.1204**
8	7.0820	7.1064	7.0980	7.7143	—	7.1128	**7.1181**
9	8.0769	8.0968	8.0959	8.7500	8.0838	8.1094	**8.1143**
10	9.0722	9.0886	9.0932	9.7778	9.0793	9.1055	**9.1100**
15	14.0541	14.0618	14.0785	14.8571	14.0610	14.0864	**14.0892**
20	19.0428	19.0474	19.0666	19.8947	19.0490	19.0719	**19.0738**

Table 1: History of lower bounds on the average degree $d(G)$ of k-critical and k-list-critical graphs G.

1. *$K_k \not\subseteq G$; and*
2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*
4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $\delta(\mathcal{B}) \geq 3$.*

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.

We also have the following version with asymmetric degree condition on $\mathcal{B}$. The point here is that this works for $k \geq 5$. As we'll see in the next section, the consequence is that we trade a bit in our size bound for the proof to go through with $k \in \{5, 6\}$.

Lemma 2.3. *Let $k \geq 5$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*
4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $d_{\mathcal{B}}(y) \geq 4$ for all $y \in Y$ and $d_{\mathcal{B}}(T) \geq 2$ for all components T of $G - Y$.*

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.

3 Improved bounds on σ

Let $T \in \mathcal{T}_k$. Then each block of T is regular. Say type$(B) = b$ if B is $(b-1)$-regular. Let $x \in V(T)$ and let $B_1, \dots, B_\ell$ be the blocks of T containing x where B_i is of type b_i. Then we say that type$_T(x) = (b_1, \dots, b_\ell)$. For an endblock B of T, let x_B be the cutvertex of T contained in B and put $T_B := T - (V(B) \setminus \{x\})$. For $b \geq 1$, put $t(b) := 2 - \frac{2}{b}$. For $T \in \mathcal{T}_k$ and $x \in V(T)$ put

$$\sigma_T(x) := k - 2 + \frac{2}{k-1} - d_T(x).$$

For $T \in \mathcal{T}_k$ and $x \in V(T)$ with type$_T(x) = (b_1, \dots, b_\ell)$, put

$$\sigma'_T(x) := \sigma_T(x) - 2 + \sum_{i \in [\ell]} t(b_i).$$

Furthermore, put

$$\sigma(T) := \sum_{x \in V(T)} \sigma_T(x),$$

and

$$\sigma'(T) := \sum_{x \in V(T)} \sigma'_T(x).$$

Lemma 3.1. *Let $T \in \mathcal{T}_k$ where $k \geq 4$. Then,*

(a) If B is a block of T, then $\sigma(B) = 2$ if $B = K_{k-1}$ and $\sigma(B) \geq k - 2 + \frac{2}{k-1}$ otherwise,

(b) If B is an endblock of T, then $\sigma(T) = \sigma(T_B) + \sigma(B) - \left(k - 2 + \frac{2}{k-1}\right)$.

Proof. Immediate from the definitions (see Kostochka and Stiebitz [4]). □

Lemma 3.2. *If $T \in \mathcal{T}_k$ and $k \geq 4$, then $\sigma(T) \geq \sigma'(T) + 2$.*

Proof. Suppose the lemma is false and let T be a counterexample with the minimum number of blocks. First, suppose T has one block. Then, T is complete or an odd cycle. If T is complete, then $T = K_b$ with $b \in [k-1]$ and hence $\sigma'(T) = \sigma(T) + (t(b) - 2)b = \sigma(T) - 2$. If instead T is an odd cycle, then $\sigma'(T) = \sigma(T) + (t(3) - 2)|T| = \sigma(T) - \frac{2}{3}|T| \leq \sigma(T) - 2$. Hence T must have at least two blocks.

Let B be an endblock of T. Say type$(B) = b$ and type$_T(x_B) = (b_1, \dots, b_\ell)$ where $b_\ell = b$. Then type$_{T_B}(x_B) = (b_1, \dots, b_{\ell-1})$. Therefore, we have

$$\sigma'_{T_B}(x_B) = k - 2 + \frac{2}{k-1} - d_{T_B}(x_B) - 2 + \sum_{i \in [\ell-1]} t(b_i),$$

and

$$\sigma'_B(x_B) = k - 2 + \frac{2}{k-1} - d_B(x_B) + t(b_\ell) - 2,$$

and

$$\sigma'_T(x_B) = k - 2 + \frac{2}{k-1} - d_T(x_B) - 2 + \sum_{i \in [\ell]} t(b_i).$$

Since $d_T(x_B) = d_{T_B}(x_B) + d_B(x_B)$, we have

$$\sigma'(T) = \sigma'(T_B) + \sigma'(B) + 2 - \left(k - 2 + \frac{2}{k-1}\right).$$

By our minimality condition on T, we have

$$\sigma(T_B) \geq \sigma'(T_B) + 2,$$

and

$$\sigma(B) \geq \sigma'(B) + 2.$$

Putting this all together with Lemma 3.1 gives the contradiction

$$\sigma'(T) \leq \sigma(T_B) + \sigma'(B) - \left(k - 2 + \frac{2}{k-1}\right) = \sigma(T) - 2.$$

□

Let $T \in \mathcal{T}_k$ and $x \in V(T)$ with $\text{type}_T(x) = (b_1, \ldots, b_\ell)$. We always have $\sum_{i \in [\ell]} b_i \leq k + \ell - 1$. We say that x is *full* if $\sum_{i \in [\ell]} b_i = k + \ell - 1$. When positive integers $k, b_1, \ldots, b_\ell$ are such that $\sum_{i \in [\ell]} b_i < k + \ell - 1$, put

$$\Gamma_{k,(b_1,\ldots,b_\ell)} := 1 - \frac{3 - 2\ell - \frac{2}{k-1} + \sum_{i \in [\ell]} \frac{2}{b_i}}{k + \ell - 1 - \sum_{i \in [\ell]} b_i},$$

when $\sum_{i \in [\ell]} b_i = k + \ell - 1$, put

$$\Gamma_{k,(b_1,\ldots,b_\ell)} := \ell.$$

Lemma 3.3. *If $T \in \mathcal{T}_k$, then $\sigma'_T(x) \geq \Gamma_{k,\text{type}_T(x)}\,(k - 1 - d_T(x))$ for all $x \in V(T)$.*

Proof. We have

$$\sigma'_T(x) = \sigma_T(x) - 2 + \sum_{i \in [\ell]} t(b_i).$$

So, $\sigma'_T(x) \geq c\,(k - 1 - d_T(x))$ for some c and $x \in V(T)$ with $\text{type}(x) = (b_1, \ldots, b_\ell)$ if and only if

$$(1 - c)\left(k - 1 - \sum_{i \in [\ell]} (b_i - 1)\right) + \sum_{i \in [\ell]} \left(2 - \frac{2}{b_i}\right) \geq 3 - \frac{2}{k-1}.$$

A quick computation shows that $\Gamma_{k,(b_1,\ldots,b_\ell)}$ is the largest such c that works when x is not full. When x is full, the first term is zero, so c is irrelevant. We need

$$\sum_{i \in [\ell]} \left(2 - \frac{2}{b_i}\right) \geq 3 - \frac{2}{k-1}.$$

Since x is full, we have $\sum_{i \in [\ell]} b_i = k + \ell - 1$. In particular, since $K_k \not\subseteq T$, we must have $\ell \geq 2$ and hence $b_i \geq 2$ for all $i \in [\ell]$. So, if $\ell \geq 3$, then we have

$$\sum_{i \in [\ell]} \left(2 - \frac{2}{b_i}\right) \geq \ell \geq 3 - \frac{2}{k-1}.$$

Hence we may assume $\ell = 2$. So, $b_1 + b_2 = k + 1$. Simplifying the inequality we need gives

$$\frac{k+1}{b_1(k+1-b_1)} = \frac{b_1+b_2}{b_1 b_2} \leq \frac{1}{2} + \frac{1}{k-1}.$$

The worst case is when $b_1 = 2$, in which case we have

$$\frac{k+1}{b_1(k+1-b_1)} = \frac{1}{2} + \frac{1}{k-1}.$$

□

So, now our task is to prove lower bounds on $\Gamma_{k,(b_1,\ldots,b_\ell)}$.

Lemma 3.4. *Let $k \geq 6$ and let $b_1, \ldots, b_\ell$ be positive integers. We have*

$$\Gamma_{k,(b_1,\ldots,b_\ell)} \geq \begin{cases} 0 & \text{if } \ell = 1 \text{ and } b_1 = k-1, \\ \frac{1}{2} - \frac{1}{(k-1)(k-2)} & \text{if } \ell = 1 \text{ and } b_1 = k-2, \\ 1 - \frac{3k-5}{(k-1)^2} & \text{if } \ell = 1 \text{ and } b_1 = 1, \\ \frac{2}{3} - \frac{4}{3(k-1)(k-3)} & \text{if } \ell = 1 \text{ and } 2 \leq b_1 \leq k-3, \\ 1 - \frac{1}{k-1} & \text{if } \ell = 2, \\ \ell - 2 + \frac{2}{k-1} & \text{if } \ell \geq 3. \end{cases}$$

Proof. Just compute. The $\ell \geq 3$ case can be improved, but not sure we need it. □

The bound in Lemma 3.3 gives $\sigma'_T(x) \geq 0$ when x is a full vertex. We can often do better.

Lemma 3.5. *Let $T \in \mathcal{T}_k$ for $k \geq 5$. If $x \in V(T)$ is a full vertex of type $(b_1, \ldots, b_\ell)$, then*

$$\sigma'_T(x) \geq \begin{cases} 0 & \text{if } \ell = 2, b_1 = 2 \text{ and } b_2 = k-1, \\ \frac{1}{3} - \frac{2}{(k-1)(k-2)} & \text{if } \ell = 2 \text{ and } b_1, b_2 \leq k-2, \\ \frac{1}{3} + \frac{2}{k-1} & \text{if } \ell \geq 3, \end{cases}$$

Proof. We have $\sigma'_T(x) = 2\ell - 3 + \frac{2}{k-1} - \sum_{i \in [\ell]} \frac{2}{b_i}$. Now, just compute. □

In the proof of the bound in [2], we handled K_{k-1}'s separately and for the other components we used a bound of $\frac{1}{2} - \frac{1}{(k-1)(k-2)}$ as a lower bound on $\Gamma_{k,(b_1,\ldots,b_\ell)}$ (which can be seen from Lemma 3.4). As the bound in Lemma 3.4 shows, if we want to improve on this we need to handle the $\ell = 1$ and $b_1 = k-2$ case where we get $\frac{1}{2} - \frac{1}{(k-1)(k-2)}$ exactly. When $T = K_{k-2}$ is a component, for $x \in V(T)$ we have

$$\begin{aligned} \sigma'_T(x) &= k - 2 + \frac{2}{k-1} - (k-3) + \left(-2 + 2 - \frac{2}{k-2}\right) \\ &= 1 + \frac{2}{k-1} - \frac{2}{k-2} \\ &= \left(\frac{1}{2} - \frac{1}{(k-1)(k-2)}\right)(k - 1 - d_T(x)). \end{aligned}$$

So, to improve on $\frac{1}{2} - \frac{1}{(k-1)(k-2)}$, we need to do get some extra weight from somewhere. We know $\sigma(T) \geq \sigma'(T)+2$ by Lemma 3.2. It turns out that in the proof below, that $+2$ is wasted on all components except those containing a K_{k-1}. Since components in general could be arbitrary size, distributing this extra $+2$ to the vertices of the component will not give us a useful improvement. But in the case that $T = K_{k-2}$, this gets us an extra $\frac{2}{k-2}$ weight for each vertex in T, which allows us to improve $\frac{1}{2} - \frac{1}{(k-1)(k-2)}$ to $\frac{1}{2} + \frac{1}{k-2} - \frac{1}{(k-1)(k-2)} = \frac{1}{2} + \frac{1}{k-1}$ in this case. It remains to find extra weight for K_{k-2} blocks that are not components. Suppose $B = K_{k_2}$ is a block in a component T and let x_B be the cutvertex of T contained in B. We are going to get our extra weight from x_B. First, if x_B is not full, then type$(x_B) = (2, k-2)$ and Lemma 3.4 gives $\Gamma_{k,(b_1,b_2} \geq 1 - \frac{1}{k-1}$. So, x_B contributes $\left(1 - \frac{1}{k-1}\right)(k-1-d_T(x_B)) = \left(1 - \frac{1}{k-1}\right)$ which is an extra $1 - \frac{1}{k-1} - \left(\frac{1}{2} - \frac{1}{(k-1)(k-2)}\right) = \frac{1}{2} - \frac{k-3}{(k-1)(k-2)}$. Distributing this evenly amongst the vertices of the K_{k-2} gets each vertex an extra $\frac{1}{2(k-2)} - \frac{k-3}{(k-1)(k-2)(k-2)}$. Suppose instead that x_B is full. Then Lemma 3.5 shows that that there is unused weight $\frac{1}{3} - \frac{2}{(k-1)(k-2)}$. Distributing this weight to the at most $k-3$ non-full vertices in T gets each vertex an extra $\frac{1}{3(k-3)} - \frac{2}{(k-1)(k-2)(k-3)}$. This latter gain is the smaller of the two, adding this to $\frac{1}{2} - \frac{1}{(k-1)(k-2)}$ gives $\frac{1}{2} + \frac{k-5}{3(k-2)(k-3)}$. All of the bounds in Lemma 3.4 are at most this for $k \geq 5$, except $\frac{1}{2} - \frac{1}{(k-1)(k-2)}$ (which we have already fixed) and $1 - \frac{3k-5}{(k-1)^2}$ which occurs for components with a single vertex. We can easily fix these using the same trick of using the extra $+2$ like we did for K_{k-2} components. In the notation of the next section, this proves the following.

Lemma 3.6. $\sigma_k(G) \geq 2|\mathcal{D}| + \left(\frac{1}{2} + \frac{k-5}{3(k-2)(k-3)}\right) \sum_{v \in L'} (k - 1 - d_{\mathcal{L}}(v))$.

Below we put this in to get an improved bound in Theorem 4.4.

4 The lower bound

We need the following definitions:

$$\begin{aligned}
\mathcal{L}_k(G) &:= G\left[x \in V(G) \mid d_G(x) < k\right], \\
\mathcal{H}_k(G) &:= G\left[x \in V(G) \mid d_G(x) \geq k\right], \\
\sigma_k(G) &:= \left(k - 2 + \frac{2}{k-1}\right) |\mathcal{L}_k(G)| - 2 \left\|\mathcal{L}_k(G)\right\|, \\
\tau_{k,c}(G) &:= 2 \left\|\mathcal{H}_k(G)\right\| + \left(k - c - \frac{2}{k-1}\right) \sum_{y \in V(\mathcal{H}_k(G))} (d_G(y) - k), \\
g_k(n, c) &:= \left(k - 1 + \frac{k-3}{(k-c)(k-1) + k - 3}\right) n.
\end{aligned}$$

4.1 We only really care about low degree vertices

As proved in [4], a computation gives the following.

Lemma 4.1. *Let G be a graph with $\delta := \delta(G) \geq 3$ and $0 \leq c \leq \delta + 1 - \frac{2}{\delta}$. If $\sigma_{\delta+1}(G) + \tau_{\delta+1,c}(G) \geq c\,|\mathcal{H}_{\delta+1}(G)|$, then $2\,\|G\| \geq g_{\delta+1}(|G|\,, c)$.*

With a lower value of c, we can make it so we only have to care about vertices of degree δ and $\delta + 1$ as follows.

Lemma 4.2. *Let G be a graph with $\delta := \delta(G) \geq 3$ and $0 \leq c \leq \frac{\delta+1}{2} + \frac{1}{\delta}$. Put $H' := \{v \in V(G) : d_G(v) = \delta + 1\}$. If $\sigma_{\delta+1}(G) + \sum_{y \in H'} d_{\mathcal{H}_{\delta+1}}(y) \geq c\,|H'|$, then $2\,\|G\| \geq g_{\delta+1}(|G|\,, c)$.*

Proof. Put $\mathcal{H} := \mathcal{H}_{\delta+1}$ and $k := \delta + 1$. For $y \in V(\mathcal{H})$, put

$$\tau_{k,c}(y) := d_{\mathcal{H}}(y) + \left(k - c + \frac{2}{k-1}\right)(d_G(y) - k).$$

We have

$$\begin{aligned}
\tau_{k,c}(G) &= \sum_{y \in V(\mathcal{H})} \tau_{k,c}(y) \\
&\geq \sum_{y \in H'} d_{\mathcal{H}}(y) + \sum_{y \in V(\mathcal{H}) \setminus H'} \left(d_{\mathcal{H}}(y) + k - c + \frac{2}{k-1}\right) \\
&\geq \sum_{y \in H'} d_{\mathcal{H}}(y) + \left(k - c + \frac{2}{k-1}\right)|\mathcal{H} - H'| \\
&\geq \sum_{y \in H'} d_{\mathcal{H}}(y) + c\,|\mathcal{H} - H'|\,,
\end{aligned}$$

where the last inequality follows since $c \leq \frac{k}{2} + \frac{1}{k-1}$. Now applying Lemma 4.1 proves the lemma. $\square$

4.2 Finishing the proof

We need the following degeneracy lemma.

Lemma 4.3. *Let G be a graph and $f \colon V(G) \to \mathbb{N}$. If $\|G\| > \sum_{v \in V(G)} f(v)$, then G has an induced subgraph H such that $d_H(v) > f(v)$ for each $v \in V(H)$.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Then $|G| \geq 3$ and we have $x \in V(G)$ with $d_G(x) \leq f(x)$. But now $\|G - x\| > \sum_{v \in V(G-x)} f(v)$, contradicting minimality of $|G|$. $\square$

Theorem 4.4. *If G is an AT-irreducible graph with $\delta(G) \geq 4$ and $\omega(G) \leq \delta(G)$, then $2\,\|G\| \geq g_{\delta(G)+1}(|G|\,, c)$ where $c := (\delta(G) - 2)\left(\frac{1}{2} + \frac{\delta(G)-4}{3(\delta(G)-1)(\delta(G)-2)}\right)$ when $\delta(G) \geq 6$ and $c := (\delta(G) - 3)\left(\frac{1}{2} + \frac{\delta(G)-4}{3(\delta(G)-1)(\delta(G)-2)}\right)$ when $\delta(G) \in \{4, 5\}$.*

Proof. Put $k := \delta(G) + 1$, $\mathcal{L} := \mathcal{L}_k(G)$ and $\mathcal{H} := \mathcal{H}_k(G)$. Put $W := W^k(\mathcal{L})$, $L' := V(\mathcal{L}) \setminus W$ and $H' := \{v \in V(\mathcal{H}) : d_G(v) = k\}$. By Lemma 4.2, it will be sufficient to prove that

$$S := \sigma_k(G) + \sum_{y \in H'} d_{\mathcal{H}}(y) \geq c\,|H'|\,.$$

Let $\mathcal{D}$ be the components of $\mathcal{L}$ containing K_{k-1}. By Lemma 3.6, we have

$$\sigma_k(G) \geq 2|\mathcal{D}| + \left(\frac{1}{2} + \frac{k-5}{3(k-2)(k-3)}\right) \sum_{v \in L'} (k-1-d_{\mathcal{L}}(v)).$$

Now we define an auxiliary bipartite graph F with parts A and B where:

1. $B = H'$ and A is the disjoint union of the following sets A_1, A_2 and A_3,
2. $A_1 = \mathcal{D}$ and each $T \in \mathcal{D}$ is adjacent to all $y \in H'$ where $N(y) \cap W^k(T) \neq \emptyset$,
3. For each $v \in L'$, let $A_2(v)$ be a set of $|N(v) \cap H'|$ vertices connected to $N(v) \cap H'$ by a matching in F. Let A_2 be the disjoint union of the $A_2(v)$ for $v \in L'$,
4. For each $y \in H'$, let $A_3(y)$ be a set of $d_{\mathcal{H}}(y)$ vertices which are all joined to y in F. Let A_3 be the disjoint union of the $A_3(y)$ for $y \in H'$.

Case 1. $\delta \geq 6$.

Define $f\colon V(F) \to \mathbb{N}$ by $f(v) = 1$ for all $v \in A_2 \cup A_3$ and $f(v) = 2$ for all $v \in B \cup A_1$. First, suppose $\|F\| > \sum_{v \in V(F)} f(v)$. Then by Lemma 4.3, F has an induced subgraph Q such that $d_Q(v) > f(v)$ for each $v \in V(Q)$. In particular, $V(Q) \subseteq B \cup A_1$ and $\delta(Q) \geq 3$. Put $Y := B \cap V(Q)$ and let X be $\bigcup_{T \in V(Q) \cap A_1} V(T)$. Now $H := G[X \cup Y]$ satisfies the hypotheses of Lemma 2.2, so H has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for $v \in X$. Since $Y \subseteq H'$ and $X \subseteq \mathcal{L}$, we have $f(v) = \delta(G) + d_{G'}(v) - d_G(v)$ for all $v \in V(G')$. Hence, G is AT-reducible to G', a contradiction.

Therefore $\|F\| \leq \sum_{v \in V(F)} f(v) = 2(|H'| + |\mathcal{D}|) + |A_2| + |A_3|$. By Lemma 2.1, for each $y \in B$ we have $d_F(y) \geq k-1$. Hence $\|F\| \geq (k-1)\,|H'|$. This gives $(k-3)\,|H'| \leq 2\,|\mathcal{D}| + |A_2| + |A_3|$. By our above estimate we have $S \geq 2|\mathcal{D}| + \left(\frac{1}{2} + \frac{k-5}{3(k-2)(k-3)}\right) \sum_{v \in L'} (k-1-d_{\mathcal{L}}(v)) + \sum_{y \in H'} d_{\mathcal{H}}(y) = 2\,|\mathcal{D}| + \left(\frac{1}{2} + \frac{k-5}{3(k-2)(k-3)}\right) |A_2| + |A_3| \geq \left(\frac{1}{2} + \frac{k-5}{3(k-2)(k-3)}\right) (2\,|\mathcal{D}| + |A_2| + |A_3|)$. Hence $S \geq \left(\frac{1}{2} + \frac{k-5}{3(k-2)(k-3)}\right) (k-3)\,|H'|$. Thus our desired bound holds by Lemma 4.2.

Case 2. $\delta \in \{4, 5\}$.

Define $f\colon V(F) \to \mathbb{N}$ by $f(v) = 1$ for all $v \in A_1 \cup A_2 \cup A_3$ and $f(v) = 3$ for all $v \in B$. First, suppose $\|F\| > \sum_{v \in V(F)} f(v)$. Then by Lemma 4.3, F has an induced subgraph Q such that $d_Q(v) > f(v)$ for each $v \in V(Q)$. In particular, $V(Q) \subseteq B \cup A_1$ and $d_Q(v) \geq 4$ for $v \in B \cap V(Q)$ and $d_Q(v) \geq 2$ for $v \in A_1 \cap V(Q)$. Put $Y := B \cap V(Q)$ and let X be $\bigcup_{T \in V(Q) \cap A_1} V(T)$. Now $H := G[X \cup Y]$ satisfies the hypotheses of Lemma 2.3, so H has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for $v \in X$. Since $Y \subseteq H'$ and $X \subseteq \mathcal{L}$, we have $f(v) = \delta(G) + d_{G'}(v) - d_G(v)$ for all $v \in V(G')$. Hence, G is AT-reducible to G', a contradiction.

Therefore $\|F\| \leq \sum_{v \in V(F)} f(v) = 3\,|H'| + |\mathcal{D}| + |A_2| + |A_3|$. By Lemma 2.1, for each $y \in B$ we have $d_F(y) \geq k-1$. Hence $\|F\| \geq (k-1)\,|H'|$. This gives $(k-4)\,|H'| \leq |\mathcal{D}| + |A_2| + |A_3|$. By our above estimate we have $S \geq 2|\mathcal{D}| + \left(\frac{1}{2} + \frac{k-5}{3(k-2)(k-3)}\right) \sum_{v \in L'} (k-1-d_{\mathcal{L}}(v)) + \sum_{y \in H'} d_{\mathcal{H}}(y) = 2\,|\mathcal{D}| + \left(\frac{1}{2} + \frac{k-5}{3(k-2)(k-3)}\right) |A_2| + |A_3| \geq \left(\frac{1}{2} + \frac{k-5}{3(k-2)(k-3)}\right) (|\mathcal{D}| + |A_2| + |A_3|)$. Hence $S \geq \left(\frac{1}{2} + \frac{k-5}{3(k-2)(k-3)}\right) (k-4)\,|H'|$. Thus our desired bound holds by Lemma 4.2. $\square$

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