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Let $c_k^*(\mathcal{L})$ be the number of components of $\mathcal{L}$ containing a copy of K_{k-1}. Let $q_k(\mathcal{L})$ be the number of non-cut vertices in $\mathcal{L}$ that appear in copies of K_{k-1}. Let $\beta_k(\mathcal{L})$ be the independence number of the subgraph of $\mathcal{L}$ induced on the vertices of degree $k-1$. When k is defined in context, we just write $c^*(\mathcal{L})$, $q(\mathcal{L})$ and $\beta(\mathcal{L})$. Let $\mathcal{H}(G)$ be the subgraph of G induced on vertices of degree greater than $\delta(G)$. Let $\mathcal{L}(G)$ be the subgraph of G induced on vertices of degree $\delta(G)$.

Definition 1. The *maximum independent cover number* of a graph G is the maximum $\mathrm{mic}(G)$ of $\|I, V(G) \setminus I\|$ over all independent sets I of G.

Definition 2. A graph G is *OC-reducible* to H if H is a nonempty induced subgraph of G which is online f_H-choosable where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not OC-reducible to any nonempty induced subgraph, then it is *OC-irreducible*.

Lemma 1. *Every OC-irreducible graph G satisfies*

$$2\|G\| > (\delta(G) - 1)|G| + \mathrm{mic}(G).$$

Lemma 2. *If G is an OC-irreducible graph where $\mathcal{H}(G)$ is edgeless, $\Delta := \Delta(G) = \delta(G) + 1$ and $\mathcal{L} := \mathcal{L}(G)$, then*

$$2\|\mathcal{L}\| > \left(\Delta - 2 - \frac{2}{\Delta - 2}\right)|\mathcal{L}| + \frac{\Delta(\Delta-1)}{\Delta - 2}\beta_\Delta(\mathcal{L}).$$

Proof. Let G be such a graph. Put $\mathcal{H} := \mathcal{H}(G)$ and $\mathcal{L} := \mathcal{L}(G)$. Since $\mathcal{H}$ is edgeless,

$$\begin{aligned}\Delta|\mathcal{H}| &= \|\mathcal{H}, \mathcal{L}\| \\ &= (\Delta - 1)|\mathcal{L}| - 2\|\mathcal{L}\|,\end{aligned} \tag{1}$$

so, by Lemma 1,

$$\begin{aligned}(\Delta-1)|\mathcal{L}| + \Delta|\mathcal{H}| &= 2\|G\| \\ &> (\Delta - 2)|G| + \mathrm{mic}(G) \\ &\geq (\Delta-2)|G| + \Delta|\mathcal{H}| + (\Delta-1)\beta_\Delta(\mathcal{L}) \\ &= (\Delta-2)|\mathcal{L}| + (2\Delta-2)|\mathcal{H}| + (\Delta-1)\beta_\Delta(\mathcal{L}),\end{aligned}$$

so simplifying and using (1) again gives

$$\begin{aligned}|\mathcal{L}| &> (\Delta - 2)\,|\mathcal{H}| + (\Delta-1)\beta_\Delta(\mathcal{L})\\ &= \frac{\Delta-2}{\Delta}\left((\Delta-1)\,|\mathcal{L}| - 2\,\|\mathcal{L}\|\right) + (\Delta-1)\beta_\Delta(\mathcal{L}),\end{aligned}$$

now some mild manipulation yields the desired bound. □

Lemma 3. $\left(\frac{3k-7}{k^2-4k+5}, \frac{(k-1)(k-4)}{k^2-4k+5}, 2, \frac{-2(k-1)(k-4)}{k^2-4k+5}\right)$ *is 5-Gallai.*

Lemma 4. *Let G be a non-complete AT-irreducible graph with $\delta(G) = k-1$ where $k \geq 5$. Let $\mathcal{L}$ be the subgraph of G induced on $(k-1)$-vertices, $\mathcal{H}^-$ the subgraph of G induced on k-vertices and $\mathcal{H}^+$ the subgraph of G induced on $(k+1)^+$-vertices. Then*

$$q(\mathcal{L}) \leq c^*(\mathcal{L}) + 4\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\|,$$

and if $k \geq 7$, then

$$q(\mathcal{L}) \leq 2c^*(\mathcal{L}) + 3\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\|.$$

Lemma 5. *If G is an OC-irreducible graph where $\mathcal{H}(G)$ is edgeless, $\Delta := \Delta(G) = \delta(G)+1 \geq 7$, $\mathcal{L} := \mathcal{L}(G)$ and $\mathcal{H} := \mathcal{H}(G)$, then*

$$2\,\|\mathcal{L}\| \leq \left(\Delta - 3 + \frac{3\Delta-7}{\Delta^2-4\Delta+5}\right)|\mathcal{L}| + \frac{3(\Delta-1)(\Delta-4)}{\Delta^2-4\Delta+5}\,|\mathcal{H}| + 2\beta_\Delta(\mathcal{L}).$$

Proof. Combine the second inequality in Lemma 4 with Lemma 3. □

Lemma 6. *If G is an OC-irreducible graph where $\mathcal{H}(G)$ is edgeless, $\Delta := \Delta(G) = \delta(G)+1 \geq 7$, $\mathcal{L} := \mathcal{L}(G)$ and $\mathcal{H} := \mathcal{H}(G)$, then*

$$\left(\Delta - 10 + \frac{4(\Delta+1)}{\Delta^2-4\Delta+5}\right)|\mathcal{L}| + \left(\Delta^2 - 3\Delta + 4\right)\beta_\Delta(\mathcal{L}) < 0,$$

in particular, $\Delta \leq 9$.

Proof. Combine Lemma 2 with Lemma 5 and the fact that $|\mathcal{H}| < \frac{|L|}{\Delta-2}$. □

1 Further Reducibility Lemmas

Lemma 7. *Let G be a directed graph and $x_1x_2 \in E(G)$ such that $d^-(x_i) = 1$ for all $i \in [2]$. If $y \in V(G) \setminus \{x_1, x_2\}$, then*

$$EE(G + x_1y + x_2y) - EO(G + x_1y + x_2y) = EE(G) - EO(G).$$

2 Using Minimality

Definition 3. A graph A is a *child* of a graph G if there exists $H \lhd G$ and an epimorphism $f\colon H \twoheadrightarrow A$.

Note that the child-of relation is a strict partial order on the set of (finite simple) graphs $\mathcal{G}$. We call this the *child order* on $\mathcal{G}$ and denote it by '$\prec$'.

Lemma 8. *The ordering $\prec$ is well-founded on $\mathcal{G}$. That is, every non-empty subset of $\mathcal{G}$ has a minimal element under $\prec$.*

Proof. Let $\mathcal{T}$ be a non-empty subset of $\mathcal{G}$. Pick $G \in \mathcal{T}$ with the minimum number of vertices. Since any child of G must have fewer vertices, we see that G is minimal in $\mathcal{T}$ with respect to $\prec$. □

Definition 4. Let $\mathcal{T}$ be a collection of graphs. A minimal graph in $\mathcal{T}$ under the child order is called a $\mathcal{T}$*-mule*.

Let Q_k be the collection of graphs G with $\mathrm{AT}(G) = \Delta(G) = k$ such that no two vertices of degree k are adjacent. We'd like to show that $Q_k = \emptyset$ for $k \geq 6$. If not, then a Q_k-mule exists.

Lemma 9. *Let G be a Q_k-mule. If x is a k-vertex, then x has at most one neighbor in any $K_{k-1} \subseteq \mathcal{L}(G)$.*