LIST BORODIN-KOSTOCHKA FOR LARGE Δ

1. The setup

Theorem 1. *There exists Δ_0 such that every graph G with $\chi_l(G) \geq \Delta(G) \geq \Delta_0$ contains a $K_{\Delta(G)}$.*

Suppose the theorem is false and choose a counterexample G minimizing $|G|$. Put $\Delta := \Delta(G)$ and let L be a bad $(\Delta - 1)$-assignment on G. Then, by minimality of $|G|$, any proper induced subgraph of G is L-colorable. In particular, every vertex has degree either Δ or $\Delta - 1$, we call these *high* and *low* vertices respectively. We need the following concept.

Definition 1. A graph H is called d_1-*choosable* if H is f-choosable where $f(v) := d(v) - 1$.

If G had an induced d_1-choosable subgraph H, then we could L-color $G - H$ by minimality and then complete the L-coloring to all of G. So, G has no d_1-choosable induced subgraphs. We will use results from [1] where many graphs were shown to be d_1-choosable.

The proof strategy is the same as Reed's [2] for chromatic number, except some more care must be taken when lists have small intersection and $K_{\Delta-1}$'s require special attention.

2. The decomposition

For a vertex v, put $G_v := G[N(v)]$.

Definition 2. A clique with at least $\frac{3}{4}\Delta + 1$ vertices is called *big.*

Definition 3. A vertex v is called *sparse* if $\|G_v\| < \frac{2}{5}\Delta^2$.

We'll use the following easy consequence of d_1-choosability lemmas in [1] to show that nonsparse vertices are in a big clique.

Lemma 2. *If B is a graph with $\delta(B) \geq \frac{|B|+1}{2}$ such that $K_1 * B$ is not d_1-choosable, then $\omega(B) \geq |B| - 1$ or $B = E_3 * K_4$.*

Note that the neighborhoods we will be looking at are huge, so the $B = E_3 * K_4$ case will never happen here.

Lemma 3. *Any nonsparse vertex is contained in a big clique.*

Proof. Suppose we have a vertex x which is not contained in a big clique. By applying Lemma 2 repeatedly, we get a sequence $y_1, \ldots, y_{\lfloor \frac{\Delta}{4} \rfloor} \in N(x)$ such that

$$|N(y_i) \cap (N(x) - \{y_1, \ldots, y_{i-1}\})| \leq \frac{1}{2}(\Delta + 1 - i).$$

Hence x is sparse since

$$\|G_x\| \leq \binom{\Delta}{2} - \frac{1}{2}\sum_{i=1}^{\left\lfloor \frac{\Delta}{4} \right\rfloor}(\Delta - i) < \frac{2}{5}\Delta^2.$$

□

We need another d_1-choosability lemma from [1].

Lemma 4. *If B is a graph such that $K_6 * B$ is not d_1-choosable, then $\omega(B) \geq |B| - 1$.*

Let $C_1, \dots, C_t$ be the maximal big cliques in G.

Lemma 5. *If $C_i \cap C_j \neq \emptyset$ and $|C_i| \leq |C_j|$, then $|C_i - C_j| \leq 1$.*

Proof. We have $|C_i \cap C_j| \geq |C_i| + |C_j| - (\Delta + 1) \geq \frac{\Delta}{2} \geq 6$. Now the lemma follows from Lemma 4. □

Lemma 6. *No C_i intersects two others.*

Proof. Say C_i intersects C_j and C_k. By Lemma 5, $|C_i \cap C_j| \geq \frac{3}{4}\Delta$ and similarly $|C_i \cap C_k| \geq \frac{3}{4}\Delta$. Hence $|C_i \cap C_j \cap C_k| \geq \frac{1}{2}\Delta \geq 6$ and Lemma 4 gives a contradiction. □

Putting these lemmas together we can partition the nonsparse vertices of G into $D_1, \dots, D_r$ where each D_i is either a maximal big clique C_i or $C_i \cup \{x_i\}$ where x_i has at least $\frac{3}{4}\Delta$ neighbors in C_i. Put $K_i := N(x_i) \cap C_i$ in this latter case and $K_i := C_i$ in the former. Let S be the set of all sparse vertices.

3. The random procedure

For each vertex v, pick $c \in L(v)$ at random to get a possibly improper coloring ζ of G from L. Put $U := \{x \in V(G) \mid \zeta(x) = \zeta(y) \text{ for some } y \in N(x)\}$. Put $H := G - U$, $F := G[U]$ and let π be ζ restricted to $V(H)$. We refer to $V(H)$ as the *colored* vertices and $V(F)$ as the *uncolored* vertices. Also, let J be the resulting list assignment on F; that is, $J(x) := L(x) - \bigcup_{y \in N(x) \cap V(H)} \pi(y)$ for $x \in V(F)$.

Definition 4. A vertex in $v \in V(G)$ is called *safe* if it is colored or $|J(v)| \geq d_F(v) + 1$.

Note that if every vertex is safe, then we can easily complete the L-coloring to all of G. Our goal will be to show that the random procedure will, with positive probability, produce a partial coloring where every sparse vertex is safe and the uncolored nonsparse vertices satisfy conditions that will allow the coloring to be completed. Now we make this precise. Consider the following events:

- S_v, for $v \in S$: the event that v is not safe.
- E_i, for $i \in [r]$ where $|C_i| \leq \Delta - 2$: the event that C_i does not contain two uncolored safe vertices.
- Q_i, for $i \in [r]$ where $|C_i| \leq \Delta - 2$: the event that K_i does not contain two uncolored vertices.
- F_i, for $i \in [r]$ where $|C_i| = \Delta - 1$, every $x \in G - C_i$ has $|N(x) \cap C_i| \leq \sqrt{\Delta}\log(\Delta)$ and there are at most $\log^2(\Delta)$ vertices $x \in G - C_i$ with $|N(x) \cap C_i| > \frac{\sqrt{\Delta}}{\log(\Delta)}$: the event that C_i does not contain two uncolored safe vertices.

- P_i, for $i \in [r]$ where $|C_i| = \Delta - 1$ and either some $x \in G - C_i$ has $|N(x) \cap C_i| > \sqrt{\Delta}\log(\Delta)$ or more than $\log^2(\Delta)$ vertices $x \in G - C_i$ have $|N(x) \cap C_i| > \frac{\sqrt{\Delta}}{\log(\Delta)}$: the event that every $x \in G - C_i$ has at most two "good clumps" in K_i.

It remains to define "good clumps". To do so we need a lemma.

Lemma 7. *Let K be a $\Delta - 1$ clique in G and $x \in G - K$ with $|N(x) \cap K| \geq 4$. Then every vertex in $|N(x) \cap K|$ is high and there is a partition $\{Z_1, \ldots, Z_m\}$ of $N(x) \cap K$ such that for each $i \in [m]$ we have $|Z_i| \leq 5$ and $L(u) = L(v)$ for all $u, v \in Z_i$. Moreover, $|L(v) - L(w)| \leq 1$ for all $v, w \in N(x) \cap K$.*

Proof. Put $A := N(x) \cap K$ and $Q := G[\{x\} \cup K]$. For any L-coloring γ of $G - Q$, let L_γ be the resulting list assignment on Q.

First, suppose there is an L-coloring γ of $G - Q$ such that $L_\gamma(u) \neq L_\gamma(v)$ for some $u, v \in A$. Pick $y \in K - A$. If $L_\gamma(x) \cap L_\gamma(y) \neq \emptyset$, then coloring x and y the same leaves a list assignment on $K - y$ which is completable by Hall's theorem. Hence we must have $L_\gamma(x) \cap L_\gamma(y) = \emptyset$. Thus $|L_\gamma(x) \cup L_\gamma(y)| \geq \Delta$. Put $Pot(T) := \bigcup_{v \in T} L_\gamma(v)$ for $T \subseteq A$. If there is $c \in (L_\gamma(x) \cup L_\gamma(y)) - Pot(A)$, then coloring x and y so that c is used leaves a list assignment on $K - y$ which is completable by Hall's theorem. In particular, we must have $|Pot(A)| \geq \Delta$. Now, if we color x and y arbitrarily we can complete the coloring unless there exists $T \subseteq A$ with $|T| = |A| - 1$ and $|Pot(T)| \leq \Delta - 2$. Thus we can pick a color in $L_\gamma(x) \cup L_\gamma(y)$ which is not in any of T's lists giving a coloring that is again easily completable.

Therefore $L_\gamma(u) = L_\gamma(v)$ for all $u, v \in A$ for every L-coloring γ of $G - Q$. In particular, no vertex of A is low and $|L(v) - L(w)| \leq 1$ for all $v, w \in A$. Suppose there exists $Z \subseteq A$ with $|Z| \geq 6$ such that $L(u) = L(v)$ for all $u, v \in Z$. Then every $v \in Z$ has exactly one neighbor z_v in $G - Q$. Put $N := \{z_v \mid v \in Z\}$. If $|N| = 1$, then G contains $K_6 * E_3$ violating Lemma 4. If some L-coloring γ of $G - Q$ assigned two vertices of N different colors, then L_γ would give different lists for two vertices of A, a contradiction. Hence N is an independent set and adding an edge between two vertices of N in $G - Q$ must create a K_Δ by minimality of $|G|$. By counting degrees this is plainly impossible for $|N| \geq 3$. For $|N| = 2$, both vertices have $\Delta - 2$ neighbors in $G - Q$ and one has at least 3 vertices in Z, impossible.

Now taking maximal subsets of A of vertices all having the same list gives the desired partition. □

The Z_i in the partition in Lemma 7 are called *clumps* of x in K. Note that there exists Y such that for any $i \neq j$ we have $L(v) \cap L(z) = Y$ for $v \in Z_i$ and $w \in Z_j$. For $i \in [m]$ and $v \in Z_i$ we let α_i be the unique element of $L(v) - Y$. We call α_i the *special* color for Z_i.

Now let $i \in [r]$ where $|C_i| = \Delta - 1$ and some $x \in G - C_i$ has $|N(x) \cap C_i| \geq 4$. A clump $Z_j \subseteq N(x) \cap C_i$ is *good* if there is uncolored $z_i \in Z_i$ such that α_i is not used on any neighbor of z_i and the unique y in $N(z_i) - C_i - \{x\}$ is colored with a color that is either not in $L(z_i)$ or is used on C_i.

Now suppose we have a partial coloring π where none of the bad events S_v, E_i, Q_i, F_i and P_i occur. We color the D_i corresponding to P_i events first. Suppose $x \in G - C_i$ has 3 good clumps Z_1, Z_2, Z_3 in K_i with corresponding vertices z_1, z_2, z_3. Since $\alpha_1 \notin L(z_2), L(z_3)$, coloring z_1 with α_1 leaves a list assignment we can complete greedily by coloring z_2 and z_3 last. However, we need to be careful to not break the other such D_i in the process. So, we first color the respective z_1 in each such D_i. After all of those have been colored, we greedily color the rest of each D_i. It still needs to be checked that when we color z_1 with α_1 we don't lose the ability to do the same with α_1 on some other D_j. To see this, note that x has at least 3 neighbors in C_i and thus is contain in no other C_j with $|C_j| = \Delta - 1$. Moreover, z_1's only possible other neighbor y outside C_i is already colored by assumption. Now consider the D_i that have two safe uncolored vertices in C_i. If $C_i \neq K_i$, then since Q_i doesn't happen x_i has two uncolored neighbors, color it first. Now color C_i greedily saving the two safe uncolored vertices in C_i for last. Now we can finish the coloring on the sparse vertices greedily. Therefore if we can prevent all the bad events from happening we get our desired contradiction.

It is easy to see that any given event depends on less than $3\Delta^5$ others, so the result will follow by showing that $\Pr(S_v), \Pr(E_i), \Pr(Q_i), \Pr(F_i), \Pr(P_i) \leq \Delta^{-6}$. The following sections prove these bounds.

4. $\Pr(S_v) \leq \Delta^{-6}$

We know $\|G_v\| < \frac{2}{5}\Delta^2$. Put $A := \left\{x \in N(v) \mid |L(x) \cap L(v)| \geq \frac{2}{3}\Delta\right\}$ and $B := N(v) - A$. Note that for $x, y \in A$ we have $|L(x) \cap L(y)| \geq \frac{1}{3}\Delta$ and for $x \in B$ we have $|L(x) - L(v)| \geq \frac{1}{3}\Delta$.

Let A_v be the random variable that counts the number of nonadjacent pairs $x, y \in A$ such that, $\zeta(x) = \zeta(y)$ and $\zeta(z) \neq \zeta(x)$ for all $z \in N(v) - \{x, y\} \cup N(x) \cup N(y)$.

Let B_v be the random variable that counts the number of $x \in B$ such that $\zeta(x) \notin L(v)$ and $\zeta(z) \neq \zeta(x)$ for all $z \in N(v) - \{x\} \cup N(x)$.

Put $Z_v := A_v + B_v$. Then $\mathrm{E}(Z_v) = \mathrm{E}(A_v) + \mathrm{E}(B_v)$. We prove the bound $\mathrm{E}(Z_v) \geq \frac{\Delta}{1000}$ and then use Azuma's inequality to prove that $\Pr(|Z_v - \mathrm{E}(Z_v)| > \frac{\Delta}{1000} - 2) \leq \Delta^{-6}$. The conclusion $\Pr(S_v) \leq \Delta^{-6}$ is then immediate.

We know that G_v has at least $\binom{\Delta-1}{2} - \frac{2}{5}\Delta^2 \geq \frac{\Delta^2}{12}$ nonadjacent pairs. Let b be the number of nonadjacent pairs in G_v that intersect B. Plainly, $G[A]$ contains at least $\frac{\Delta^2}{12} - b$ nonadjacent pairs and $b \leq |B|\,\Delta$.

First let's consider $\mathrm{E}(A_v)$. Let $x, y \in A$ be nonadjacent. Since $|L(x) \cap L(y)| \geq \frac{1}{3}\Delta$, the probability that x and y get the same color and this color is not used on any of the rest of $N(v) \cup N(x) \cup N(y)$ is at least $(3\Delta)^{-1}(1 - (\Delta - 1)^{-1})^{3\Delta - 3} \geq (3\Delta)^{-1}3^{-3}$. Thus $\mathrm{E}(A_v) \geq (\frac{\Delta^2}{12} - b)\Delta^{-1}3^{-4} \geq \frac{\Delta}{1000} - \frac{b}{81\Delta}$.

Now consider $\mathrm{E}(B_v)$. Let $x \in B$. Since $|L(x) - L(v)| \geq \frac{1}{3}\Delta$, the probability that x gets a color not in $L(v)$ and this color is not used on any of the rest of $N(v) \cup N(x)$ is at least $\frac{1}{3}(1 - (\Delta - 1)^{-1})^{2\Delta - 2} \geq 3^{-4}$. Hence $\mathrm{E}(B_v) \geq \frac{|B|}{81} \geq \frac{b}{81\Delta}$. Therefore $\mathrm{E}(Z_v) \geq \frac{\Delta}{1000}$.

Now we need Azuma's inequality. The concentration analysis is identical to the coloring case in Reed's proof. We reproduce it here for completeness.

Lemma 8 (Azuma)**.** *Let X be a random variable determined by n trials $T_1, \ldots, T_n$ such that for each i and any two possible sequences of outcomes $t_1, \ldots, t_i$ and $t_1, \ldots, t_{i-1}, t_i'$:*

$$|\mathrm{E}(X \mid T_1 = t_1, \ldots, T_i = t_i) - \mathrm{E}(X \mid T_1 = t_1, \ldots, T_i = t_i')| \leq c_i,$$

then $\Pr(|X - \mathrm{E}(X)| > t) \leq 2e^{\frac{-t^2}{2\sum c_i^2}}$.

Since we colored the vertices of G independently, we can apply Azuma using any ordering. Order $V(G)$ as $w_1, \ldots, w_n$ so that $N(v)$ comes last and let w_s be the last vertex not in $N(v)$. Changing $\zeta(w_i)$ from β to τ only affects the vertices using β or τ and thus changes the conditional expected value by at most 2. For $w_i \notin N(v)$, the probability that changing w_i's color will affect Z_v is at most the probability that one of w_i's two colors is also assigned to one of its neighbors in $N(v)$. Say w_i has d_i neighbors in $N(v)$. Then the most changing w_i can change $\mathrm{E}(Z_v)$ is $c_i := 2\frac{2d_i}{\Delta-1} = \frac{4d_i}{\Delta-1}$. Now $\sum_{i=1}^{s} d_i \leq \Delta^2$ and thus $\sum_{i=1}^{s} c_i \leq 4\Delta + 4\frac{\Delta}{\Delta-1} \leq 4\Delta + 5$. As each $c_i \leq 5$, we have $\sum_{i=1}^{s} c_i^2 \leq 21\Delta$ and hence $\sum_i c_i^2 \leq 25\Delta$. Now using $t := \frac{\Delta}{1000} - 2$ in Azuma gives $\Pr(Z_v < 2) < 2e^{\frac{-(\frac{\Delta}{1000}-2)^2}{50\Delta}} \leq \Delta^{-6}$ for large enough Δ.

5. $\Pr(E_i) \leq \Delta^{-6}$

The following is an immediate consequence of the d_1-choosability lemmas in [1].

Lemma 9. *Let C be a maximal big clique. Each $v \in C$ has at most one neighbor with more than 4 neighbors in C and no such neighbor if v is low.*

Lemma 10. *Let C be a maximal big clique with $|C| \leq \Delta - 2$. There are at least $\frac{3}{28}\Delta$ disjoint P_3's xyz with $y \in C$ and $x, z \notin C$ such that x and z each have at most 4 neighbors in C.*

Proof. Consider a maximal such set of P_3's. Let A be all the central vertices of these P_3's and B all the ends. Then each $v \in B$ has at most 3 neighbors in $C - A$ and by Lemma 9 and maximality, each $v \in C - A$ has at most 2 neighbors in $G - C - B$ and at most 1 if v is low. Thus $6|A| = 3|B| \geq \|C - A, B\| \geq (\Delta - |C| - 1)|C - A| \geq |C| - |A|$. Hence $|A| \geq \frac{3}{28}\Delta$. □

We need to force safe uncolored vertices in C_i. If the lists have small intersections this might not happen with high probability. We handle this case using minimality of $|G|$ instead.

Lemma 11. *There exists $C_i' \subset C_i$ with $|C_i'| = |C_i| - 1$ such that for $x, y \in C_i'$ we have $|L(x) \cap L(y)| \geq \frac{2}{3}\Delta$.*

Proof. Suppose not and consider an L-coloring of $G - C_i$. Let L' be the resulting list assignment on C_i. Then $|L'(v)| \geq |C_i| - 2$ for all $v \in C_i$. By assumption, for each $v \in C_i$ we have $x, y \in C_i - \{v\}$ with $|L(x) \cap L(y)| < \frac{2}{3}\Delta$. But then $|L'(x) \cup L'(y)| \geq 2(\Delta - 1 - (\Delta + 1 - |C_i|)) - \frac{2}{3}\Delta \geq |C_i|$. Hence we can complete the L-coloring to C_i by Hall's theorem, a contradiction. □

We will find the desired uncolored safe vertices in C_i'. By Lemma 10, there are at least $\frac{\Delta}{10}$ paths acb where $c \in C_i'$ and $a, b \notin C_i'$ such that a and b each have at most 4 neighbors in C. Let T_i be the union of all the vertices in these paths. For some such fixed path we want to bound the probability that c is uncolored and safe and the colors used on a and b are used on none of the rest of T_i. To do so, we distinguish three cases.

Case 1. $|L(a) \cap L(c)| < \frac{2}{3}\Delta$ *and* $|L(b) \cap L(c)| < \frac{2}{3}\Delta$

For $\alpha \in L(a) - L(c)$, $\beta \in L(b) - L(c)$, $z \in C_i' - T_i$ and $\gamma \in L(c) \cap L(z)$ where α, β, γ are all different, let $A_{\alpha,\beta,\gamma,z}$ be the event that all of the following hold:

(1) α is assigned to a and none of the rest of $T_i \cup N(a)$,
(2) β is assigned to b and none of the rest of $T_i \cup N(b)$,
(3) γ is assigned to c and z and none of the rest of T_i.

Then $\Pr(A_{\alpha,\beta,\gamma,z}) \geq (\Delta-1)^{-1}(1-(\Delta-1)^{-1})^{|T_i \cup N(a)|}(\Delta-1)^{-1}(1-(\Delta-1)^{-1})^{|T_i \cup N(b)|}(\Delta-1)^{-2}(1-(\Delta-1)^{-1})^{|T_i|} \geq (\Delta-1)^{-4}3^{-5}$.

Plainly, the $A_{\alpha,\beta,\gamma,z}$ are disjoint for different sets of indices. Since $|L(a)-L(c)| \geq \frac{\Delta}{3}$, we have $\frac{\Delta}{3}$ choices for α. Similarly we then have $\frac{\Delta}{3}-1$ choices for β. For z we have at least $\frac{3}{4}\Delta - \frac{1}{10}\Delta \geq \frac{\Delta}{3}$ choices. Since $|L(z) \cap L(c)| \geq \frac{2}{3}\Delta$, we then have at least $\frac{2}{3}\Delta - 2$ choices for γ for each z. In total we have at least $\Delta^4 3^{-4}$ choices and thus the probability that $A_{\alpha,\beta,\gamma,z}$ holds for some choice of indices is at least 3^{-9}.

Case 2. $|L(a) \cap L(c)| < \frac{2}{3}\Delta$ *and* $|L(b) \cap L(c)| \geq \frac{2}{3}\Delta$

For $y \in C'_i - T_i - N(b)$, $z \in C'_i - T_i$, $\alpha \in L(a) - L(c)$, $\beta \in L(b) \cap L(y)$ and $\gamma \in L(c) \cap L(z)$ where α, β, γ are all different, let $A_{\alpha,\beta,\gamma,y,z}$ be the event that all of the following hold:

(1) α is assigned to a and none of the rest of $T_i \cup N(a)$,
(2) β is assigned to b and y and none of the rest of $T_i \cup N(b) \cup N(y)$,
(3) γ is assigned to c and z and none of the rest of T_i.

Then $\Pr(A_{\alpha,\beta,\gamma,y,z}) \geq (\Delta-1)^{-1}(1-(\Delta-1)^{-1})^{|T_i \cup N(a)|}(\Delta-1)^{-2}(1-(\Delta-1)^{-1})^{|T_i \cup N(b) \cup N(y)|}(\Delta-1)^{-2}(1-(\Delta-1)^{-1})^{|T_i|} \geq (\Delta-1)^{-5}3^{-6}$.

Again the $A_{\alpha,\beta,\gamma,y,z}$ are disjoint for different sets of indices. For y we have at least $|C'_i| - |T_i \cap C'_i| - |N(b) \cap C_i| \geq \frac{3}{4}\Delta - 1 - \frac{\Delta}{10} - 4 \geq \frac{\Delta}{9}$ choices. For each y we have at least $\frac{2}{3}\Delta$ choices for β. The rest are similar to above and in total we have at least $\Delta^5 3^{-6}$ choices and thus the probability that $A_{\alpha,\beta,\gamma,y,z}$ holds for some choice of indices is at least 3^{-12}.

Case 3. $|L(a) \cap L(c)| \geq \frac{2}{3}\Delta$ *and* $|L(b) \cap L(c)| \geq \frac{2}{3}\Delta$

For $x \in C'_i - T_i - N(a)$, $y \in C'_i - T_i - N(b)$, $z \in C'_i - T_i$, $\alpha \in L(a) \cap L(c)$, $\beta \in L(b) \cap L(y)$ and $\gamma \in L(c) \cap L(z)$ where α, β, γ are all different, let $A_{\alpha,\beta,\gamma,x,y,z}$ be the event that all of the following hold:

(1) α is assigned to a and y and none of the rest of $T_i \cup N(a) \cup N(x)$,
(2) β is assigned to b and y and none of the rest of $T_i \cup N(b) \cup N(y)$,
(3) γ is assigned to c and z and none of the rest of T_i.

Then $\Pr(A_{\alpha,\beta,\gamma,x,y,z}) \geq (\Delta-1)^{-2}(1-(\Delta-1)^{-1})^{|T_i \cup N(a) \cup N(x)|}(\Delta-1)^{-2}(1-(\Delta-1)^{-1})^{|T_i \cup N(b) \cup N(y)|}(\Delta-1)^{-2}(1-(\Delta-1)^{-1})^{|T_i|} \geq (\Delta-1)^{-6}3^{-7}$.

Again the $A_{\alpha,\beta,\gamma,x,y,z}$ are disjoint for different sets of indices. In total we get at least $\Delta^6 3^{-8}$ choices and thus the probability that $A_{\alpha,\beta,\gamma,x,y,z}$ holds for some choice of indices is at least 3^{-15}.

Now we have at least $\frac{\Delta}{10}$ such triples. So if M_i counts the number of uncolored safe vertices in C_i we have $\mathrm{E}(M_i) \geq 10^{-9}\Delta$. The concentration details are identical to Reed's proof and we conclude $Pr(M_i < 2) < \Delta^{-6}$.

6. $\Pr(Q_i) \leq \Delta^{-6}$

If $\zeta(x) = \zeta(y)$ for different $x, y \in K_i$, then x and y will be uncolored and Q_i cannot hold. Thus it is enough to show that all vertices of K_i getting different colors is unlikely. Just like Lemma 11, we can find $K_i' \subset K_i$ with $|K_i'| = |K_i| - 1$ such that for $x, y \in K_i'$ we have $|L(x) \cap L(y)| \geq \frac{2}{3}\Delta$.

Let $x, y \in K_i'$. The probability that x and y get the same color and this color is used on none of the rest of $N(x) \cup N(y)$ is at least $\frac{2}{3\Delta}(1 - (\Delta - 1)^{-1})^{2\Delta - 2} \geq \frac{2}{3^3\Delta}$. Since there are at least $\frac{1}{2}(\frac{2}{3}\Delta)^2$ such pairs, the expected number of pairs getting the same color is at least $3^{-4}\Delta$. An application of Azuma's inequality very similar to the sparse case now proves $\Pr(Q_i) \leq \Delta^{-6}$.

7. $\Pr(F_i) \leq \Delta^{-6}$

In this case we must have $C_i = K_i$ since no vertex outside C_i has $\frac{3}{4}\Delta$ neighbors in C_i. Since low vertices don't make things harder, we will assume there are no low vertices in C_i. In particular, for a low vertex, we don't need a triple as in the follow lemma, but just one good neighbor outside because we only need to save one color on a low vertex's neighborhood to make it safe.

Lemma 12. *There are at least $\frac{1}{4}\sqrt{\Delta}\log\Delta$ disjoint P_3's xyz with $y \in C_i$ and $x, z \notin C_i$ such that x and z each have at most $\frac{\sqrt{\Delta}}{\log(\Delta)}$ neighbors in C_i.*

Proof. Since there are at most $\log^2(\Delta)$ vertices outside C_i which have more than $\frac{\sqrt{\Delta}}{\log(\Delta)}$ neighbors in C_i and all of these vertices have at most $\sqrt{\Delta}\log(\Delta)$ neighbors in C_i, removing all their neighbors from C_i we are left with a set A of of vertices all of whose neighbors outside C_i have at most $\frac{\sqrt{\Delta}}{\log(\Delta)}$ neighbors in C_i. Now $|A| \geq \Delta - 1 - \log^2(\Delta)\sqrt{\Delta}\log(\Delta) \geq \frac{\Delta}{2}$. Now pick P_3's xyz with $y \in A$ in turn removing the neighbors of x and z each time. We get at least $\frac{|A|}{2\frac{\sqrt{\Delta}}{\log(\Delta)}} \geq \frac{1}{4}\sqrt{\Delta}\log\Delta$ disjoint P_3's.

Now the proof of the expected value is the same as the proof of $\Pr(E_i) \leq \Delta^{-6}$, except that we have fewer P_3's to multiply by at the end. So, if M_i counts the number of uncolored safe vertices in C_i, we have $\mathrm{E}(M_i) \geq 3^{-15}(\frac{1}{4})\sqrt{\Delta}\log\Delta \geq 10^{-9}\sqrt{\Delta}\log\Delta$.

Now, the application of Azuma is the same as in the $\Pr(E_i) \leq \Delta^{-6}$ case, except we use $t := 10^{-9}\sqrt{\Delta}\log\Delta - 2$, which gives gives $\Pr(M_i < 2) < 2e^{\frac{-(10^{-9}\sqrt{\Delta}\log\Delta - 2)^2}{\Delta}} \leq \Delta^{-6}$ for large enough Δ. □

8. $\Pr(P_i) \leq \Delta^{-6}$

Case 1. *Some $x \in G - C_i$ has $|N(x) \cap C_i| > \sqrt{\Delta}\log(\Delta)$.*

If $C_i \neq K_i$, then take x to be x_i. Let $Z_1, \ldots, Z_m$ be the clumps of x in K_i and for $j \in [m]$, let α_j be the color the Z_j clump has that the others do not. By Lemma 11, we may as well assume that $|L(x) \cap L(y)| \geq \frac{2}{3}\Delta$ for all $x, y \in K_i$ (the cost is one vertex which changes nothing).

Pick $z_j \in Z_j$ arbitrarily. By Lemma 9, any neighbor of z_j in $G - C_i - x$ (of which there is at most one) has at most 4 neighbors in C_i. Thus, by symmetry, for each $j \in \left[\frac{m}{4}\right]$ we can pick $y_j \in G - C_i - x$ such that $y_j z_j \in E(G)$ and the y_j are all different. Put $A := N(x) \cap K_i$. Then $\frac{m}{4} \geq (\frac{1}{4})(\frac{1}{5})|A| \geq \frac{1}{20}\sqrt{\Delta}\log(\Delta)$.

Now, for fixed $j \in \left[\frac{m}{4}\right]$, we bound the probability that z_j is uncolored, α_j is not used on any neighbor of z_j and y_j is colored with a color that is either not in $L(z_j)$ or is used on C_i. Let T_i be the union of all the y_j's and z_j's. We distinguish two cases.

Subcase 1a. $|L(y_j) \cap L(z_j)| < \frac{2}{3}\Delta$

For $\beta \in L(y_j) - L(z_j)$, $w \in C_i - T_i$ and $\gamma \in L(z_j) \cap L(w)$ where $\beta, \gamma \neq \alpha_j$ and $\beta \neq \gamma$, let $F_{\beta,\gamma,w}$ be the event that all of the following hold:

(1) β is assigned to y_j and none of the rest of $T_i \cup N(y_j)$,
(2) γ is assigned to z_j and w and none of the rest of T_i,
(3) α_j is assigned to no neighbor of z_j.

The probability of (3) is at least $\left(\frac{\Delta-2}{\Delta-1}\right)^{\Delta} = (1-(1-\Delta)^{-1})^{\Delta} \geq \frac{1}{3}$. Hence $\Pr(F_{\beta,\gamma,w}) \geq \frac{1}{3}(\Delta-1)^{-1}(1-(\Delta-1)^{-1})^{|T_i \cup N(y_j)|}(\Delta-1)^{-2}(1-(\Delta-1)^{-1})^{|T_i|} \geq (\Delta-1)^{-3}3^{-4}$.

Now we have at least $\frac{\Delta}{3}$ choices for β, $\frac{\Delta}{2}$ choices for w and $\frac{2}{3}\Delta$ choices for γ for each w. Thus the probability that $F_{\beta,\gamma,w}$ holds for some choice of indices is at least 3^{-6}.

Subcase 1b. $|L(y_j) \cap L(z_j)| \geq \frac{2}{3}\Delta$ For $y \in C_i - T_i - N(y_j)$, $\beta \in L(y_j) \cap L(y)$, $w \in C_i - T_i$ and $\gamma \in L(z_j) \cap L(w)$ where $\beta, \gamma \neq \alpha_j$ and $\beta \neq \gamma$, let $F_{\beta,\gamma,y,w}$ be the event that all of the following hold:

(1) β is assigned to y_j and y and none of the rest of $T_i \cup N(y_j) \cup N(y)$,
(2) γ is assigned to z_j and w and none of the rest of T_i,
(3) α_j is assigned to no neighbor of z_j.

We have $\Pr(F_{\beta,\gamma,y,w}) \geq \frac{1}{3}(\Delta-1)^{-2}(1-(\Delta-1)^{-1})^{|T_i \cup N(y_j) \cup N(y)|}(\Delta-1)^{-2}(1-(\Delta-1)^{-1})^{|T_i|} \geq (\Delta-1)^{-4}3^{-5}$.

Now for y we have at least $|K_i| - |C_i \cap T_i| - |N(y_j) \cap C_i| \geq \frac{\Delta}{2}$ choices and for each y we have at least $\frac{2}{3}\Delta$ choices for β. Thus the probability that $F_{\beta,\gamma,y,w}$ holds for some choice of indices is at least 3^{-7}.

Therefore the expected number of good clumps is at least $3^{-7}(\frac{1}{20})\sqrt{\Delta}\log(\Delta) \geq 10^{-5}\sqrt{\Delta}\log(\Delta)$. Changing any color will affect the conditional expectations by at most 2 and a similar computation for Azuma shows that $\Pr(F_i) \leq \Delta^{-6}$. The key here is that $(\sqrt{\Delta}\log(\Delta))^2$ grows faster that Δ.

Case 2. *More than* $\log^2(\Delta)$ *vertices* $x \in G - C_i$ *have* $|N(x) \cap C_i| > \frac{\sqrt{\Delta}}{\log(\Delta)}$.

We must have $C_i = K_i$. Let $x_1, \ldots, x_k$ be $k := \lceil \log^2(\Delta) \rceil$ different vertices in $G - C_i$ which have $|N(x_j) \cap C_i| > \frac{\sqrt{\Delta}}{\log(\Delta)}$ for each $j \in [k]$.

The computation for the expected number of good clumps for each x_j is the same as Case 1 and so we expect at least $10^{-5}\frac{\sqrt{\Delta}}{\log(\Delta)}$ good clumps for each x_j. Thus in total we expect $10^{-5}\sqrt{\Delta}\log(\Delta)$ good clumps over the $\log^2(\Delta)$ sets. Let X count this total number of good clumps. We show that $\Pr(X < 3\log^2(\Delta)) \leq \Delta^{-6}$ and hence at least one x_j has at least 3 good clumps with high enough probability.

If we applied Azuma with the information we have now we'd be in trouble because many of the x_j's could use the same special color and hence changing a vertex to that color would change the conditional expectation by a lot. We need one further structural lemma that guarantees at most 4 of the x_j's use any given special color.

Lemma 13. *Let K be a $\Delta - 1$ clique in G and $x_1, x_2, x_3, x_4, x_5 \in G - K$ with $|N(x_j) \cap K| \geq 5$ such that the $N(x_j) \cap K$ are pairwise disjoint. Then no color is special for all the x_j.*

Proof. Suppose otherwise that some color α is special for all the x_j. Put $A_j := N(x_j) \cap K$. Just like in the proof of Lemma 7, any L-coloring of $G - (K \cup \{x_1, \ldots, x_5\})$ must not leave α available on any of the vertices in A_j for any $j \in [5]$. Pick $z_j \in A_j$ for each j and let y_j be the neighbor of z_j in $G - (K \cup \{x_1, \ldots, x_5\})$. Put $N := \{y_1, \ldots, y_5\}$. By Lemma 9, $|N| \geq 2$. Now just like in Lemma 7, by using minimality of $|G|$ we see that adding any edge between vertices in N must create a K_Δ and then counting degrees gives a contradiction. □

Now when we change a color we change the conditional expectation by at most 8. A similar computation to before bounds $\sum_j c_j^2 \leq 500\Delta$. Applying Azuma with $t = 10^{-5}\sqrt{\Delta}\log(\Delta) - 3\log^2(\Delta)$ gives $\Pr(X < 3\log^2(\Delta)) < 2e^{\frac{-(10^{-5}\sqrt{\Delta}\log(\Delta) - 3\log^2(\Delta))^2}{500\Delta}} \leq \Delta^{-6}$ for large Δ.

References

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