MORE NOTES

1. Lemma 4.6

Lemma 1.1. *If $K_6 * B$ is not d_1-choosable, then $\omega(B) \geq |B| - 1$.*

Lemma 1.2. *If B is a graph with $\delta(B) \geq \frac{|B|}{2} + 1$ such that $K_1 * B$ is not d_1-choosable, then $\omega(B) \geq |B| - 1$.*

Proof. Suppose the lemma is false and let L be a bad d_1-assignment on B with $|Pot(L)|$ minimal. First note that if B does not contain disjoint nonadjacent pairs x_1, y_1 and x_2, y_2, then $\omega(B) \geq |B| - 1$ by Lemma 1.1.

By the Small Pot Lemma, $|Pot(L)| \leq |B|$. Since $|L(x_i)| + |L(y_i)| \geq |B| + 2$, we have then $|L(x_i) \cap L(y_i)| \geq 2$ for each i. Color x_1 and y_1 with $c_1 \in L(x_1) \cap L(y_1)$ and color x_2 and y_2 with $c_2 \in L(x_2) \cap L(y_2) - c_1$. By the minimum degree condition on B, each component of $B - \{x_1, y_1, x_2, y_2\}$ has a vertex joined to $\{x_1, y_1\}$ or $\{x_2, y_2\}$. Hence we can complete the coloring to all of B and then to the K_1. $\square$

Now let G be our minimum counterexample.

Lemma 1.3. *If $\frac{\Delta(G)+7}{2} \leq t \leq \Delta(G) - 1$, then $\bigcup \mathcal{C}_t$ can be partitioned into sets $D_1, \ldots, D_r$ such that for each $i \in [r]$ at least one of the following holds:*

- *$D_i \in \mathcal{C}_t$,*
- *$D_i = C_i \cup \{x_i\}$ where $C_i \in \mathcal{C}_t$ and $|N(x_i) \cap C_i| \geq t - 1$.*

Moreover, each $v \in V(G) - D_i$ has at most $t - 2$ neighbors in C_i.

Proof. Suppose $|C_i| \leq |C_j|$ and $C_i \cap C_j \neq \emptyset$. Then $|C_i \cap C_j| \geq |C_i| + |C_j| - (\Delta + 1) \geq 6$. It follows from Lemma 1.1 that $|C_i - C_j| \leq 1$.

Now suppose C_i intersects C_j and C_k. By the above, $|C_i \cap C_j| \geq \frac{\Delta(G)+5}{2}$ and similarly $|C_i \cap C_k| \geq \frac{\Delta(G)+5}{2}$. Hence $|C_i \cap C_j \cap C_k| \geq \Delta(G) + 5 - (\Delta(G) - 1) = 6$. By Lemma 1.1 we see that $\omega(G[C_i \cup C_j \cup C_k]) \geq |C_i \cup C_j \cup C_k| - 1$ which is impossible since each of C_i, C_j, C_k are maximal.

The existence of the required partition is immediate. $\square$

This can quickly be turned into a decomposition for d-dense graphs. Call $v \in V(G)$ d-sparse if it has more than $d\Delta$ non-edges in its neighborhood.

Lemma 1.4. *Let $0 \leq d \leq \frac{\Delta}{10} - \frac{3}{2}$. We can partition $V(G)$ into $S, D_1, \ldots, D_r$ so that*

1. *each vertex in S is d-sparse,*
2. *each D_i contains a vertex w_i such that $D_i - w_i$ is a clique of size at least $\Delta - 3d + 1$,*
3. *no vertex outside of D_i has more than $\frac{3\Delta}{4}$ neighbors in D_i and w_i has at least $\frac{3\Delta}{4}$ neighbors in D_i.*

Proof. Put $t := \frac{3}{4}\Delta + 1$, $B := \bigcup \mathcal{C}_t$ and $S := V(G) - B$. Apply Lemma 1.3 to get $D_1, \ldots, D_r$ partitioning B. We claim that some subset of $\{D_1, \ldots, D_r\}$ works. For item (i), we need to check that each $v \in S$ is d-sparse. We know (Lemma 9.2.2 in the other write-up) that each $v \in S$ has more than $\binom{\Delta-1}{2} - \frac{2}{5}\Delta^2 \geq (\frac{\Delta}{10} - \frac{3}{2})\Delta$ non-edges in its neighborhood, so v is d-sparse.

Item (iii) follows by the definition of the D_i. Now item (ii). If for any i, all vertices of D_i are d-sparse, then just move all of D_i into S. So now we may assume that each D_i contains a non-sparse vertex v_i. Clearly, the largest clique in G containing v_i is contained in D_i. Hence it will be enough to show that v_i is in a $\Delta - 3d + 1$ clique. We can do this with the same computation in the proof of Lemma 9.2.2 before. Let x be some v_i. Suppose x is in no $\Delta - 3d + 1$ clique, then using Lemma 1.2, we get a sequence $y_1, \ldots, y_{3d} \in N(x)$ such that

$$|N(y_i) \cap (N(x) - \{y_1, \ldots, y_{i-1}\})| \leq \frac{1}{2}(\Delta + 1 - i).$$

Hence x is d-sparse since it has at least

$$\frac{1}{2}\sum_{i=1}^{3d}(\Delta - i) > d\Delta.$$

non-edges in its neighborhood. □

2. Lemma 5.3

Lemma 2.1. *Each $v \in C_i$ has at most one neighbor outside of C_i with more than 5 neighbors in C_i and no such neighbor if v is low.*

Proof. Suppose otherwise that we have $v \in C_i$ with two neighbors $w_1, w_2 \in V(G) - C_i$ each with 6 or more neighbors in C_i. Put $Q := G[\{w_1, w_2\} \cup C_i - v]$, then v is joined to Q and hence $K_1 * Q \trianglelefteq G$. We show that $K_1 * Q$ must be d_1-choosable. Let L be a bad d_1-assignment on $K_1 * Q$ minimizing $|Pot(L)|$.

First, suppose there are different $z_1, z_2 \in C_i$ such that $\{w_1, z_1\}$ and $\{w_2, z_2\}$ are independent. By the Small Pot Lemma, $|Pot(L)| \leq |Q|$. We have $|L(w_j)| + |L(z_j)| \geq 5 + |Q| - 3 = |Q| + 2$. Hence $|L(w_j) \cap L(z_j)| \geq 2$. Pick $x \in N(w_1) \cap \{C_i - v - z_2\}$. Then after coloring each pair $\{w_1, z_1\}$ and $\{w_2, z_2\}$ with a different color, we can finish the coloring because we saved a color for x and two colors for v.

By maximality of C_i, neither w_1 nor w_2 can be adjacent to all of C_i hence it must be the case that there is $y \in C_i$ such that w_1 and w_2 are joined to $C_i - y$. If w_1 and w_2 aren't adjacent, then G contains $K_6 * E_3$ contradicting Corollary 1.1. Hence C_i intersects the larger clique $\{w_1, w_2\} \cup C_i - \{y\}$, this is impossible by the definition of C_i.

When v is low, an argument similar to the above shows that there can be no z_1 in C_i so that $\{w_1, z_1\}$ is independent, and hence $C_i \cup \{w_1\}$ is a clique contradicting maximality of C_i. □