A better lower bound on average degree of k-list-critical graphs.

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1 A general bound

Let $q_k(G)$ be the number of non-cut vertices in G that appear in copies of K_{k-1}. Let $\beta_k(G)$ be the independence number of the subgraph of G induced on the vertices of degree $k-1$. When k is defined in context, we just write $q(G)$ and $\beta(G)$.

Sections 2 and 3 prove the following upper bounds on $q(\mathcal{L})$ and $\beta(\mathcal{L})$.

Lemma 1.1. *Let G be a non-complete k-list-critical graph where $k \geq 5$. Let $\mathcal{L}$ be the subgraph of G induced on $(k-1)$-vertices, $\mathcal{H}^-$ the subgraph of G induced on k-vertices, $\mathcal{H}$ the subgraph of G induced on k^+-vertices, $\mathcal{H}^+$ the subgraph of G induced on $(k+1)^+$-vertices and $\mathcal{D}$ the components of $\mathcal{L}$ containing K_{k-1}. Then*

$$q(\mathcal{L}) \leq |\mathcal{D}| + 4\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\|,$$

and if $k \geq 7$, then

$$q(\mathcal{L}) \leq 2|\mathcal{D}| + 3\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\|.$$

Lemma 1.2. *Let G be a k-list-critical graph. Let $\mathcal{L}$ be the subgraph of G induced on $(k-1)$-vertices and $\mathcal{H}$ the subgraph of G induced on k^+-vertices. If $2 \leq \lambda \leq \frac{6(k-1)}{k}$, then*

$$\beta(\mathcal{L}) \leq \frac{2}{\lambda}\|\mathcal{H}\| + \frac{2\|G\| - (k-2)|G| - \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|\mathcal{H}| - 1}{k-1}.$$

Definition 1. A quadruple (p, h, z, f) of functions from $\mathbb{N}$ to $\mathbb{R}$ is *r-Gallai* if for every $k \geq r$ and Gallai tree $T \neq K_k$ with $\Delta(T) \leq k-1$, the following hold:

- if $K_{k-1} \subseteq T$, then $2\|T\| \leq (k-3+p(k))|T| + h(k)q(T) + z(k)\beta(T) + f(k)$; and
- if $K_{k-1} \not\subseteq T$, then $2\|T\| \leq (k-3+p(k))|T| + h(k)q(T) + z(k)\beta(T) + \max\{0, f(k)\}$.

Theorem 1.3. *Let (p, h, z, f) be 7-Gallai. If $k \geq 7$ and $2 \leq z(k) \leq \frac{6(k-1)}{k}$, then for any non-complete k-list-critical graph G,*

$$d(G) \geq \frac{2 - p(k) - \frac{z(k)}{k-1} + \frac{\frac{z(k)}{k-1} - (2h(k) + f(k))c(\mathcal{L})}{|G|}}{k + 1 + 3h(k) - p(k) - \frac{(k-2)z(k)}{2(k-1)}},$$

where $\mathcal{L}$ is the subgraph of G induced on $(k-1)$-vertices.

Proof. Let $\mathcal{L}$ be the subgraph of G induced on $(k-1)$-vertices, $\mathcal{H}^-$ the subgraph of G induced on k-vertices, $\mathcal{H}$ the subgraph of G induced on k^+-vertices, $\mathcal{H}^+$ the subgraph of G induced on $(k+1)^+$-vertices and $\mathcal{D}$ the components of $\mathcal{L}$ containing K_{k-1}. Plainly, the following bounds hold.

$$2\|G\| \geq k|G| - |\mathcal{L}| \tag{1}$$

$$2\|G\| \geq (k+1)|G| - \left|\mathcal{H}^-\right| - 2|\mathcal{L}| \tag{2}$$

$$2\|G\| \geq k\left|\mathcal{H}^-\right| + (k-1)|\mathcal{L}| + \left\|\mathcal{H}^+, \mathcal{L}\right\| \tag{3}$$

$$\|\mathcal{H}, \mathcal{L}\| = (k-1)|\mathcal{L}| - 2\|\mathcal{L}\| \tag{4}$$

Since (p, h, z, f) is 7-Gallai,

$$2\|\mathcal{L}\| \leq (k-3+p(k))|\mathcal{L}| + f(k)|\mathcal{D}| + h(k)q(\mathcal{L}) + z(k)\beta(\mathcal{L}) \tag{5}$$

By Lemma 1.1,

$$q(\mathcal{L}) \leq 2|\mathcal{D}| + 3\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\|,$$

plugging this into (5) gives

$$2\|\mathcal{L}\| \leq (k-3+p(k))|\mathcal{L}| + 3h(k)\left|\mathcal{H}^-\right| + h(k)\left\|\mathcal{H}^+, \mathcal{L}\right\| + z(k)\beta(\mathcal{L}) + S_1, \tag{6}$$

where

$$S_1 := (2h(k) + f(k))|\mathcal{D}|.$$

Now using (1) and (6),

$$\begin{aligned}
2\|G\| &= 2\|\mathcal{H}\| + 2\|\mathcal{H}, \mathcal{L}\| + 2\|\mathcal{L}\| \\
&= 2\|\mathcal{H}\| + 2((k-1)|\mathcal{L}| - 2\|\mathcal{L}\|) + 2\|\mathcal{L}\| \\
&= 2\|\mathcal{H}\| + 2(k-1)|\mathcal{L}| - 2\|\mathcal{L}\| \\
&\geq 2\|\mathcal{H}\| + (k+1-p(k))|\mathcal{L}| - 3h(k)\left|\mathcal{H}^-\right| - h(k)\left\|\mathcal{H}^+, \mathcal{L}\right\| - z(k)\beta(\mathcal{L}) - S_1
\end{aligned} \tag{7}$$

Adding $h(k)$ times (3) to (7) gives

$$2\|G\| \geq \frac{2\|\mathcal{H}\| + (k+1+(k-1)h(k)-p(k))|\mathcal{L}| + (k-3)h(k)|\mathcal{H}^-| - z(k)\beta(\mathcal{L}) - S_1}{1+h(k)} \tag{8}$$

Lemma 1.2 gives

$$\beta(\mathcal{L}) \leq \frac{2}{z(k)}\|\mathcal{H}\| + \frac{2\|G\| - (k-2)|G| - \left(\frac{k}{2} + \frac{k-1}{z(k)}\right)|\mathcal{H}| - 1}{k-1}.$$

Plugging this into (8) yields

$$2\|G\| \geq \frac{(k+1+(k-1)h(k)-p(k))|\mathcal{L}| + (k-3)h(k)|\mathcal{H}^-| + \frac{(k-2)z(k)}{k-1}|G| + \left(\frac{kz(k)}{2(k-1)} + 1\right)|\mathcal{H}| + S_2}{1+h(k)+\frac{z(k)}{k-1}}, \tag{9}$$

where

$$S_2 := \frac{z(k)}{k-1} - S_1.$$

Now using $|\mathcal{H}| = |G| - |\mathcal{L}|$ gives

$$2\|G\| \geq \frac{\left(k+(k-1)h(k)-p(k)-\frac{kz(k)}{2(k-1)}\right)|\mathcal{L}| + (k-3)h(k)|\mathcal{H}^-| + \left(\frac{(3k-4)z(k)}{2(k-1)} + 1\right)|G| + S_2}{1+h(k)+\frac{z(k)}{k-1}}. \tag{10}$$

Now using (2) to get a lower bound on $|\mathcal{H}^-|$ gives

$$2\|G\| \geq \frac{\left(k-(k-5)h(k)-p(k)-\frac{kz(k)}{2(k-1)}\right)|\mathcal{L}| + \left((k+1)(k-3)h(k) + \frac{(3k-4)z(k)}{2(k-1)} + 1\right)|G| + S_2}{1+(k-2)h(k)+\frac{z(k)}{k-1}}. \tag{11}$$

Using (13) to get a lower bound on $|\mathcal{L}|$ and simplifying gives

$$\frac{2\|G\|}{|G|} \geq \frac{k^2+3(k-1)h(k)-kp(k)+1-\frac{k^2-3k+4}{2(k-1)}z(k)+\frac{S_2}{|G|}}{k+1+3h(k)-p(k)-\frac{(k-2)z(k)}{2(k-1)}}. \tag{12}$$

Now factoring out $k-1$ gives the desired bound. $\square$

A nearly identical argument, using the other inequality in Lemma 1.1, proves a bound that holds for $k \geq 5$.

Theorem 1.4. *Let (p, h, z, f) be 5-Gallai. If $k \geq 5$ and $2 \leq z(k) \leq \frac{6(k-1)}{k}$, then for any non-complete k-list-critical graph G,*

$$d(G) \geq \frac{2 - p(k) - \frac{z(k)}{k-1} + \frac{\frac{z(k)}{k-1} - (2h(k)+f(k))c(\mathcal{L})}{|G|}}{k + 1 + 4h(k) - p(k) - \frac{(k-2)z(k)}{2(k-1)}},$$

where $\mathcal{L}$ is the subgraph of G induced on $(k-1)$-vertices.

When $k = 4$, we cannot apply Lemma 1.1, but using $h(k) = 0$ and running through the same argument proves the following bound for $k \geq 4$.

Theorem 1.5. *Let $(p, 0, z, f)$ be 4-Gallai. If $k \geq 4$ and $2 \leq z(k) \leq \frac{6(k-1)}{k}$, then for any non-complete k-list-critical graph G,*

$$d(G) \geq \frac{2 - p(k) - \frac{z(k)}{k-1} + \frac{\frac{z(k)}{k-1} - f(k)c(\mathcal{L})}{|G|}}{k + 1 - p(k) - \frac{(k-2)z(k)}{2(k-1)}},$$

where $\mathcal{L}$ is the subgraph of G induced on $(k-1)$-vertices.

When $z(k) < 2$, using Lemma 1.2 worsens the lower bound, so we may as well use $z(k) = 0$. Running through the argument with $z(k) = 0$ proves Lemmas 1.3, 1.4, 1.5 with $z(k) = 0$. Note how this is the exact same form as the bounds proved by Cranston and R. [1].

2 Introduction

Main Theorem. *For $k \geq 7$, every non-complete k-list-critical graph has average degree at least*

$$k - 1 + \frac{(k-3)^2(2k-3)}{k^4 - 2k^3 - 11k^2 + 28k - 14}.$$

3 The Proof

The final piece we need is a bound on the number of edges in Gallai trees, proved in Section 4.

Lemma 3.1. *For all $k \geq 5$ and Gallai trees $T \neq K_k$ with $\Delta(T) \leq k - 1$,*

$$2\|T\| \leq \left(k - 3 + \frac{3k-7}{k^2-4k+5}\right)|T| - \frac{2(k-1)(k-4)}{k^2-4k+5} + \frac{(k-1)(k-4)}{k^2-4k+5}q(T) + 2\beta(T)$$

Proof of Main Theorem. Let $\mathcal{L}$ be the subgraph of G induced on $(k-1)$-vertices, $\mathcal{H}^-$ the subgraph of G induced on k-vertices, $\mathcal{H}$ the subgraph of G induced on k^+-vertices, $\mathcal{H}^+$ the subgraph of G induced on $(k+1)^+$-vertices and $\mathcal{D}$ the components of $\mathcal{L}$ containing K_{k-1}. Plainly, the following bounds hold.

$$2\|G\| \geq k|G| - |\mathcal{L}| \tag{13}$$

$$2\|G\| \geq (k+1)|G| - \left|\mathcal{H}^-\right| - 2|\mathcal{L}| \tag{14}$$

$$2\|G\| \geq k\left|\mathcal{H}^-\right| + (k-1)|\mathcal{L}| + \left\|\mathcal{H}^+, \mathcal{L}\right\| \tag{15}$$

$$\|\mathcal{H}, \mathcal{L}\| = (k-1)|\mathcal{L}| - 2\|\mathcal{L}\| \tag{16}$$

Put

$$p(k) := \frac{3k-7}{k^2-4k+5}$$
$$h(k) := \frac{(k-1)(k-4)}{k^2-4k+5}$$
$$f(k) := 2h(k)$$

By Lemma 3.1,

$$2\left\|\mathcal{L}\right\| \le (k-3+p(k))\left|\mathcal{L}\right| - f(k)c(\mathcal{L}) + h(k)q(\mathcal{L}) + 2\beta(\mathcal{L}) \tag{17}$$

By Lemma 1.1,

$$q(\mathcal{L}) \le 2\left|\mathcal{D}\right| + 3\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\|,$$

plugging this into (17) and using the fact that $c(\mathcal{L}) \ge |\mathcal{D}|$ gives

$$2\left\|\mathcal{L}\right\| \le (k-3+p(k))\left|\mathcal{L}\right| + 3h(k)\left|\mathcal{H}^-\right| + h(k)\left\|\mathcal{H}^+, \mathcal{L}\right\| + 2\beta(\mathcal{L}). \tag{18}$$

Now using (13) and (18),

$$\begin{aligned}
2\left\|G\right\| &= 2\left\|\mathcal{H}\right\| + 2\left\|\mathcal{H},\mathcal{L}\right\| + 2\left\|\mathcal{L}\right\| \\
&= 2\left\|\mathcal{H}\right\| + 2((k-1)\left|\mathcal{L}\right| - 2\left\|\mathcal{L}\right\|) + 2\left\|\mathcal{L}\right\| \\
&= 2\left\|\mathcal{H}\right\| + 2(k-1)\left|\mathcal{L}\right| - 2\left\|\mathcal{L}\right\| \\
&\ge 2\left\|\mathcal{H}\right\| + (k+1-p(k))\left|\mathcal{L}\right| - 3h(k)\left|\mathcal{H}^-\right| - h(k)\left\|\mathcal{H}^+, \mathcal{L}\right\| - 2\beta(\mathcal{L})
\end{aligned} \tag{19}$$

Adding $h(k)$ times (15) to (19) gives

$$(1+h(k))\left(2\left\|G\right\|\right) \ge 2\left\|\mathcal{H}\right\| + (k+1+(k-1)h(k)-p(k))\left|\mathcal{L}\right| + (k-3)h(k)\left|\mathcal{H}^-\right| - 2\beta(\mathcal{L}) \tag{20}$$

Lemma 1.2 gives

$$\beta(\mathcal{L}) \le \left\|\mathcal{H}\right\| + \frac{2\left\|G\right\| - (k-2)\left|G\right| - \left(k-\frac{1}{2}\right)\left|\mathcal{H}\right|}{k-1}.$$

Plugging this into (20) yields

$$\begin{aligned}
\left(1+h(k)+\frac{2}{k-1}\right)\left(2\left\|G\right\|\right) &\ge (k+1+(k-1)h(k)-p(k))\left|\mathcal{L}\right| \\
&\quad + (k-3)h(k)\left|\mathcal{H}^-\right| + \frac{2(k-2)}{k-1}\left|G\right| + \frac{2k-1}{k-1}\left|\mathcal{H}\right|.
\end{aligned} \tag{21}$$

Now using $|\mathcal{H}| = |G| - |\mathcal{L}|$ gives

$$\begin{aligned}
\left(1+h(k)+\frac{2}{k-1}\right)\left(2\left\|G\right\|\right) &\ge \left(k+1+(k-1)h(k)-p(k)-\frac{2k-1}{k-1}\right)\left|\mathcal{L}\right| \\
&\quad + (k-3)h(k)\left|\mathcal{H}^-\right| + \frac{4k-5}{k-1}\left|G\right|.
\end{aligned} \tag{22}$$

Now using (14) to get a lower bound on $|\mathcal{H}^-|$ gives

$$\begin{aligned}
\left(1+(k-2)h(k)+\frac{2}{k-1}\right)\left(2\left\|G\right\|\right) &\ge \left(k+1-(k-5)h(k)-p(k)-\frac{2k-1}{k-1}\right)\left|\mathcal{L}\right| \\
&\quad + \left((k+1)(k-3)h(k)+\frac{4k-5}{k-1}\right)\left|G\right|.
\end{aligned} \tag{23}$$

Finally, using (13) to get a lower bound on $|\mathcal{L}|$ gives

$$\frac{2\left\|G\right\|}{\left|G\right|} \ge \frac{k(k+1)+3(k-1)h(k)-kp(k)+\frac{4k-5}{k-1}-\frac{k(2k-1)}{k-1}}{k+2+3h(k)-p(k)+\frac{2}{k-1}-\frac{2k-1}{k-1}}. \tag{24}$$

Plugging in the values for $p(k)$ and $h(k)$ and simplifying proves the theorem. $\square$

Proof of Main Theorem 5,6. Let $\mathcal{L}$ be the subgraph of G induced on $(k-1)$-vertices, $\mathcal{H}^-$ the subgraph of G induced on k-vertices, $\mathcal{H}$ the subgraph of G induced on k^+-vertices, $\mathcal{H}^+$ the subgraph of G induced on $(k+1)^+$-vertices and $\mathcal{D}$ the components of $\mathcal{L}$ containing K_{k-1}. Plainly, the following bounds hold.

$$2\,\|G\| \geq k\,|G| - |\mathcal{L}| \tag{25}$$

$$2\,\|G\| \geq (k+1)\,|G| - \left|\mathcal{H}^-\right| - 2\,|\mathcal{L}| \tag{26}$$

$$2\,\|G\| \geq k\left|\mathcal{H}^-\right| + (k-1)\,|\mathcal{L}| + \left\|\mathcal{H}^+, \mathcal{L}\right\| \tag{27}$$

$$\|\mathcal{H}, \mathcal{L}\| = (k-1)\,|\mathcal{L}| - 2\,\|\mathcal{L}\| \tag{28}$$

Put

$$\begin{aligned} p(k) &:= \frac{3k-7}{k^2-4k+5} \\ h(k) &:= \frac{(k-1)(k-4)}{k^2-4k+5} \\ f(k) &:= 2h(k) \end{aligned}$$

By Lemma 3.1,

$$2\,\|\mathcal{L}\| \leq (k-3+p(k))\,|\mathcal{L}| - f(k)c(\mathcal{L}) + h(k)q(\mathcal{L}) + 2\beta(\mathcal{L}) \tag{29}$$

By Lemma 1.1,

$$q(\mathcal{L}) \leq |\mathcal{D}| + 4\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\|,$$

plugging this into (29) and using the fact that $c(\mathcal{L}) \geq |\mathcal{D}|$ gives

$$2\,\|\mathcal{L}\| \leq (k-3+p(k))\,|\mathcal{L}| + 4h(k)\left|\mathcal{H}^-\right| + h(k)\left\|\mathcal{H}^+, \mathcal{L}\right\| + 2\beta(\mathcal{L}). \tag{30}$$

Now using (25) and (30),

$$\begin{aligned} 2\,\|G\| &= 2\,\|\mathcal{H}\| + 2\,\|\mathcal{H}, \mathcal{L}\| + 2\,\|\mathcal{L}\| \\ &= 2\,\|\mathcal{H}\| + 2((k-1)\,|\mathcal{L}| - 2\,\|\mathcal{L}\|) + 2\,\|\mathcal{L}\| \\ &= 2\,\|\mathcal{H}\| + 2(k-1)\,|\mathcal{L}| - 2\,\|\mathcal{L}\| \\ &\geq 2\,\|\mathcal{H}\| + (k+1-p(k))\,|\mathcal{L}| - 4h(k)\left|\mathcal{H}^-\right| - h(k)\left\|\mathcal{H}^+, \mathcal{L}\right\| - 2\beta(\mathcal{L}) \end{aligned} \tag{31}$$

Adding $h(k)$ times (27) to (31) gives

$$(1+h(k))\,(2\,\|G\|) \geq 2\,\|\mathcal{H}\| + (k+1+(k-1)h(k)-p(k))\,|\mathcal{L}| + (k-4)h(k)\left|\mathcal{H}^-\right| - 2\beta(\mathcal{L}) \tag{32}$$

Lemma 1.2 gives

$$\beta(\mathcal{L}) \leq \|\mathcal{H}\| + \frac{2\,\|G\| - (k-2)\,|G| - \left(k-\frac{1}{2}\right)|\mathcal{H}|}{k-1}.$$

Plugging this into (32) yields

$$\begin{aligned} \left(1+h(k)+\frac{2}{k-1}\right)(2\,\|G\|) &\geq (k+1+(k-1)h(k)-p(k))\,|\mathcal{L}| \\ &\quad + (k-4)h(k)\left|\mathcal{H}^-\right| + \frac{2(k-2)}{k-1}\,|G| + \frac{2k-1}{k-1}\,|\mathcal{H}|. \end{aligned} \tag{33}$$

Now using $|\mathcal{H}| = |G| - |\mathcal{L}|$ gives

$$\begin{aligned} \left(1+h(k)+\frac{2}{k-1}\right)(2\,\|G\|) &\geq \left(k+1+(k-1)h(k)-p(k)-\frac{2k-1}{k-1}\right)|\mathcal{L}| \\ &\quad + (k-4)h(k)\left|\mathcal{H}^-\right| + \frac{4k-5}{k-1}\,|G|. \end{aligned} \tag{34}$$

Now using (26) to get a lower bound on $|\mathcal{H}^-|$ gives

$$\begin{aligned}\left(1+(k-3)h(k)+\frac{2}{k-1}\right)(2\|G\|) \geq &\left(k+1-(k-7)h(k)-p(k)-\frac{2k-1}{k-1}\right)|\mathcal{L}| \\ &+\left((k+1)(k-4)h(k)+\frac{4k-5}{k-1}\right)|G|\,.\end{aligned} \tag{35}$$

Finally, using (25) to get a lower bound on $|\mathcal{L}|$ gives

$$\frac{2\|G\|}{|G|} \geq \frac{k(k+1)+4(k-1)h(k)-kp(k)+\frac{4k-5}{k-1}-\frac{k(2k-1)}{k-1}}{k+2+4h(k)-p(k)+\frac{2}{k-1}-\frac{2k-1}{k-1}}. \tag{36}$$

Plugging in the values for $p(k)$ and $h(k)$ and simplifying proves the theorem. □

4 Bounding $q(\mathcal{L})$

This section is devoted to extracting the reusable Lemma 4.1 from the proof of Kierstead and R. [2].

Definition 2. A graph G is *AT-reducible* to H if H is a nonempty induced subgraph of G which is f_H-AT where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not AT-reducible to any nonempty induced subgraph, then it is *AT-irreducible*.

Lemma 4.1. *Let G be a non-complete AT-irreducible graph with $\delta(G) = k-1$ where $k \geq 5$. Let $\mathcal{L}$ be the subgraph of G induced on $(k-1)$-vertices, $\mathcal{H}^-$ the subgraph of G induced on k-vertices, $\mathcal{H}$ the subgraph of G induced on k^+-vertices, $\mathcal{H}^+$ the subgraph of G induced on $(k+1)^+$-vertices and $\mathcal{D}$ the components of $\mathcal{L}$ containing K_{k-1}. Then*

$$q(\mathcal{L}) \leq |\mathcal{D}| + 4\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\|,$$

and if $k \geq 7$, then

$$q(\mathcal{L}) \leq 2|\mathcal{D}| + 3\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\|.$$

Observation. The hypotheses of Lemma 4.1 are satisfied by non-complete k-critical, k-list-critical, online k-list-critical and k-AT-critical graphs.

The proof of Lemma 4.1 requires the following four lemmas from [2].

Lemma 4.2. *Let G be a graph and $f\colon V(G) \to \mathbb{N}$. If $\|G\| > \sum_{v \in V(G)} f(v)$, then G has an induced subgraph H such that $d_H(v) > f(v)$ for each $v \in V(H)$.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Then $|G| \geq 3$ and we have $x \in V(G)$ with $d_G(x) \leq f(x)$. But now $\|G - x\| > \sum_{v \in V(G-x)} f(v)$, contradicting minimality of $|G|$. □

Let $\mathcal{T}_k$ be the Gallai trees with maximum degree at most $k-1$, excepting K_k. For a graph G, let $W^k(G)$ be the set of vertices of G that are contained in some K_{k-1} in G.

Lemma 4.3. *Let $k \geq 5$ and let G be a graph with $x \in V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *$G - x$ has t components $H_1, H_2, \ldots, H_t$, and all are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k-1$ for all $v \in V(G-x)$; and*
4. *$\left|N(x) \cap W^k(H_i)\right| \geq 1$ for $i \in [t]$; and*
5. *$d_G(x) \geq t+2$.*

Then G is f-AT where $f(x) = d_G(x) - 1$ and $f(v) = d_G(v)$ for all $v \in V(G-x)$.

For a graph G, $\{X, Y\}$ a partition of $V(G)$ and $k \geq 4$, let $\mathcal{B}_k(X, Y)$ be the bipartite graph with one part Y and the other part the components of $G[X]$. Put an edge between $y \in Y$ and a component T of $G[X]$ iff $N(y) \cap W^k(T) \neq \emptyset$. The next lemma tells us that we have a reducible configuration if this bipartite graph has minimum degree at least three.

Lemma 4.4. *Let $k \geq 7$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*
4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $\delta(\mathcal{B}) \geq 3$.*

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.

We also have the following version with asymmetric degree condition on $\mathcal{B}$. The point here is that this works for $k \geq 5$. The consequence is that we trade a bit in our bound for the proof to go through with $k \in \{5, 6\}$.

Lemma 4.5. *Let $k \geq 5$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*
4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $d_{\mathcal{B}}(y) \geq 4$ for all $y \in Y$ and $d_{\mathcal{B}}(T) \geq 2$ for all components T of $G - Y$.*

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.

Proof of Lemma 4.1. Put $W := W^k(\mathcal{L})$ and $L' := V(\mathcal{L}) \setminus W$. Define an auxiliary bipartite graph F with parts A and B where:

1. $B = V(\mathcal{H}^-)$ and A is the disjoint union of the following sets A_1, A_2 and A_3,
2. $A_1 = \mathcal{D}$ and each $T \in \mathcal{D}$ is adjacent to all $y \in B$ where $N(y) \cap W^k(T) \neq \emptyset$,
3. For each $v \in L'$, let $A_2(v)$ be a set of $|N(v) \cap B|$ vertices connected to $N(v) \cap B$ by a matching in F. Let A_2 be the disjoint union of the $A_2(v)$ for $v \in L'$,
4. For each $y \in B$, let $A_3(y)$ be a set of $d_{\mathcal{H}}(y)$ vertices which are all joined to y in F. Let A_3 be the disjoint union of the $A_3(y)$ for $y \in B$.

Define $f \colon V(F) \to \mathbb{N}$ by $f(v) = 1$ for all $v \in A_1 \cup A_2 \cup A_3$ and $f(v) = 3$ for all $v \in B$. First, suppose $\|F\| > \sum_{v \in V(F)} f(v)$. Then by Lemma 4.2, F has an induced subgraph Q such that $d_Q(v) > f(v)$ for each $v \in V(Q)$. In particular, $V(Q) \subseteq B \cup A_1$ and $d_Q(v) \geq 4$ for $v \in B \cap V(Q)$ and $d_Q(v) \geq 2$ for $v \in A_1 \cap V(Q)$. Put $Y := B \cap V(Q)$ and let X be $\bigcup_{T \in V(Q) \cap A_1} V(T)$. Now $Z := G[X \cup Y]$ satisfies the hypotheses of Lemma 4.5, so Z has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for $v \in X$. Since $Y \subseteq B$ and $X \subseteq V(\mathcal{L})$, we have $f(v) = k - 1 + d_{G'}(v) - d_G(v)$ for all $v \in V(G')$. Hence, G is AT-reducible to G', a contradiction. Therefore $\|F\| \leq \sum_{v \in V(F)} f(v) = 3|B| + |\mathcal{D}| + |A_2| + |A_3|$. By Lemma 4.3, for each $y \in B$ we have $d_F(y) \geq k - 1$. Hence $\|F\| \geq (k-1)|B|$. This gives $(k-4)|B| \leq |\mathcal{D}| + |A_2| + |A_3|$. Now the first inequality in the lemma follows since $B = V(\mathcal{H}^-)$, $|A_3| = \sum_{v \in V(\mathcal{H}^-)} d_{\mathcal{H}}(v)$ and

$$\begin{aligned}|A_2| &= -q(\mathcal{L}) + \|\mathcal{H}, \mathcal{L}\| \\ &= -q(\mathcal{L}) + k\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\| - \sum_{v \in V(\mathcal{H}^-)} d_{\mathcal{H}}(v).\end{aligned}$$

Suppose $k \geq 7$. Define $f\colon V(F) \to \mathbb{N}$ by $f(v) = 1$ for all $v \in A_2 \cup A_3$ and $f(v) = 2$ for all $v \in B \cup A_1$. First, suppose $\|F\| > \sum_{v \in V(F)} f(v)$. Then by Lemma 4.2, F has an induced subgraph Q such that $d_Q(v) > f(v)$ for each $v \in V(Q)$. In particular, $V(Q) \subseteq B \cup A_1$ and $\delta(Q) \geq 3$. Put $Y := B \cap V(Q)$ and let X be $\bigcup_{T \in V(Q) \cap A_1} V(T)$. Now $Z := G[X \cup Y]$ satisfies the hypotheses of Lemma 4.4, so Z has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for $v \in X$. Since $Y \subseteq B$ and $X \subseteq V(\mathcal{L})$, we have $f(v) = k - 1 + d_{G'}(v) - d_G(v)$ for all $v \in V(G')$. Hence, G is AT-reducible to G', a contradiction.

Therefore $\|F\| \leq \sum_{v \in V(F)} f(v) = 2(|B| + |\mathcal{D}|) + |A_2| + |A_3|$. By Lemma 4.3, for each $y \in B$ we have $d_F(y) \geq k - 1$. Hence $\|F\| \geq (k-1)\,|B|$. This gives $(k-3)\,|B| \leq 2\,|\mathcal{D}| + |A_2| + |A_3|$. Now the second inequality in the lemma follows as before. $\square$

5 Bounding $\beta(\mathcal{L})$

This section is devoted to extracting the reusable Lemma 5.1 from the proof of R. [**?**].

Definition 3. A graph G is *OC-reducible* to H if H is a nonempty induced subgraph of G which is online f_H-choosable where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not OC-reducible to any nonempty induced subgraph, then it is *OC-irreducible*.

Lemma 5.1. *Let G be an OC-irreducible graph with $\delta(G) = k-1$. Let $\mathcal{L}$ be the subgraph of G induced on $(k-1)$-vertices and $\mathcal{H}$ the subgraph of G induced on k^+-vertices. If $2 \leq \lambda \leq \frac{6(k-1)}{k}$, then*

$$\beta(\mathcal{L}) \leq \frac{2}{\lambda}\,\|\mathcal{H}\| + \frac{2\,\|G\| - (k-2)\,|G| - \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|\mathcal{H}| - 1}{k-1}.$$

Observation. The hypotheses of Lemma 5.1 are satisfied by k-critical, k-list-critical and online k-list-critical graphs.

The proof of Lemma 5.1 requires the following lemma from Kierstead and R. [3] that generalizes a kernel technique of Kostochka and Yancey [4].

Definition. The *maximum independent cover number* of a graph G is the maximum $\operatorname{mic}(G)$ of $\|I, V(G) \setminus I\|$ over all independent sets I of G.

Kernel Magic. *Every OC-irreducible graph G with $\delta(G) = k-1$ satisfies*

$$2\,\|G\| \geq (k-2)\,|G| + \operatorname{mic}(G) + 1.$$

Theorem 5.2. *[Lowenstein, et al.] If G is a connected graph then*

$$\alpha(G) \geq \frac{2}{3}\,|G| - \frac{1}{4}\,\|G\| - \frac{1}{3}.$$

Corollary 5.3. *If G is a connected graph then*

$$\alpha(G) \geq \frac{2}{3}\,|G| - \frac{1}{3}\,\|G\|.$$

Proof. By Theorem 5.2,

$$\alpha(G) \geq \frac{2}{3}\,|G| - \frac{1}{3}\,\|G\| + \frac{1}{12}\,\|G\| - \frac{1}{3},$$

so, the corollary holds if $\frac{1}{12}\,\|G\| \geq \frac{1}{3}$. If not, then $\|G\| < 4$, so G is K_1, K_2, P_3 or K_3 which all satisfy the desired bound. $\square$

Proof of Lemma 5.1. Fix λ with $2 \leq \lambda \leq \frac{6(k-1)}{k}$. Let M be the maximum of $\|I, V(G) \setminus I\|$ over all independent sets I of G with $I \subseteq \mathcal{H}$. Since the vertices in $\mathcal{L}$ with $k-1$ neighbors in $\mathcal{L}$ have no neighbors in $\mathcal{H}$,

$$\operatorname{mic}(G) \geq M + (k-1)\beta(\mathcal{L}). \tag{37}$$

Claim 1. *If C is a component of $G[\mathcal{H}]$, then*

$$M \geq \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|C| - \left(\frac{2(k-1)}{\lambda}\right)\|C\|.$$

First, suppose $\|C\| < |C|$. Then $\|C\| = |C| - 1$ and C is a tree. If $|C| \geq 2$, then

$$\begin{aligned} M &\geq k\alpha(C) \\ &\geq k\frac{|C|}{2} \\ &\geq \left(\frac{k}{2} - \frac{k-1}{\lambda}\right)|C| + \frac{2(k-1)}{\lambda} \\ &= \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|C| - \left(\frac{2(k-1)}{\lambda}\right)(|C|-1) \\ &= \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|C| - \left(\frac{2(k-1)}{\lambda}\right)\|C\|. \end{aligned}$$

If instead, $|C| = 1$, then $M \geq k \geq \left(\frac{k}{2} + \frac{k-1}{\lambda}\right) = \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|C| - \left(\frac{2(k-1)}{\lambda}\right)\|C\|$ since $\lambda \geq 2$.

So, we may assume $\|C\| \geq |C|$. Applying Corollary 5.3, we conclude

$$\begin{aligned} M &\geq k\alpha(C) \\ &\geq \frac{2k}{3}|C| - \frac{k}{3}\|C\| \\ &= \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|C| - \left(\frac{2(k-1)}{\lambda}\right)\|C\| + \left(\frac{k}{6} - \frac{k-1}{\lambda}\right)|C| - \left(\frac{k}{3} - \frac{2(k-1)}{\lambda}\right)\|C\| \\ &= \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|C| - \left(\frac{2(k-1)}{\lambda}\right)\|C\| + \left(\frac{k-1}{\lambda} - \frac{k}{6}\right)|C| \\ &\geq \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|C| - \left(\frac{2(k-1)}{\lambda}\right)\|C\|, \end{aligned}$$

where in the final inequality we used $\lambda \leq \frac{6(k-1)}{k}$.

Claim 2. *Lemma 5.1 is true.* Combining (37) and Claim 1 gives

$$\mathrm{mic}(G) \geq \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|\mathcal{H}| - \left(\frac{2(k-1)}{\lambda}\right)\|\mathcal{H}\| + (k-1)\beta(\mathcal{L}). \tag{38}$$

Applying Kernel Magic using (38) and solving for $\beta(\mathcal{L})$ proves the claim. □

6 Gallai tree bounds

Lemma 6.1. *Let $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$, $z\colon \mathbb{N} \to \mathbb{R}$. For all $k \geq 6$ and $T \in \mathcal{T}_k$ with $K_{k-1} \not\subseteq T$, we have*

$$2\|T\| \leq (k - 3 + p(k))|T| + f(k) + z(k)\beta(T)$$

whenever p, f and z satisfy all of the following conditions:

1. $p(k) \geq \frac{-f(k)}{k-2}$; *and*

2. $p(k) \geq \frac{-f(k)}{5} + 5 - k$; *and*

3. $0 \geq f(k) \geq -k + 2$; *and*

4. $p(k) \geq \frac{3-z(k)}{k-2}$.

Proof. A general outline for the proof is that it mirrors that of Lemma **??**, and we add as hypotheses all of the conditions that we need along the way.

Suppose the lemma is false and choose a counterexample T minimizing $|T|$. If T is K_t for some $t \in \{2, k-2\}$, then $t(t-1) > (k-3+p(k))t + f(k)$. After substituting $p(k) \geq \frac{-f(k)}{k-2}$ from (1), this simplifies to $-t(k-2) > f(k)$, which contradicts (3). If T is C_{2r+1} for $r \geq 2$, then $2(2r+1) > (k-3+p(k))(2r+1)+f(k)$ and hence $(5-k-p(k))(2r+1) > f(k)$. Since $f(k) \leq 0$, this contradicts (2). (Note that we only use conditions (1), (2), and (3) when T has a single block; these are the base cases when the proof is phrased using induction.)

Let D be an induced subgraph such that $T \setminus D$ is connected. (We will choose D to be a connected subgraph contained in at most three blocks of T.) Let $T' = T \setminus D$. By the minimality of $|T|$, we have

$$2\,\|T'\| \leq (k-3+p(k))\,|T'| + f(k) + z(k)\beta(T').$$

Since T is a counterexample, subtracting this inequality from the inequality for $2||T||$ gives

$$2\,\|T\| - 2\,\|T'\| > (k-3+p(k))|D| + z(k)(\beta(T) - \beta(T')). \tag{*}$$

Suppose T has an endblock B that is K_t for some $t \in \{3, \ldots, k-3\}$; let x_B be a cut vertex of B and let $D = B - x_B$. Now (*) gives $2\,\|T\| - 2\,\|T'\| = |B|\,(|B|-1) > (k-3+p(k))(|B|-1)$, which is a contradiction, since $|B| \leq k-3$ and $p(k) > 0$. Suppose instead that T has an endblock B that is an odd cycle. Again, let $D = B - x_B$. Now we get $2|B| > (k-3+p(k))(|B|-1)$. This simplifies to $|B| < 1 + \frac{2}{k-5+p(k)}$, which is a contradiction, since the denominator is always at least 1 (using (4) when $k=5$). Finally suppose that T has an endblock B that is K_2. Now (*) gives $2 > k-3+p(k)$, which is again a contradiction, since $k \geq 5$ and $p(k) > 0$.

To handle the case when B is K_{k-2} we need to remove x_B from T as well, so we simply let $D = B$. Since $B = K_{k-2}$, we have either $d_T(x_B) = k-2$ or $d_T(x_B) = k-1$. When $d_T(x_B) = k-2$, we have

$$(k-2)(k-3) + 2 > (k-3+p(k))(k-2),$$

contradicting (4).

The only remaining case is when B is K_{k-2} and $d_T(x_B) = k-1$. Each case above applied when B was any endblock of T, so we may assume that every endblock of T is a copy of K_{k-2} that shares a vertex with an odd cycle. Choose an endblock B that is the end of a longest path in the block-tree of T. Let C be the odd cycle sharing a vertex x_B with B. Consider a neighbor y of x_B on C that either (i) lies only in C or (ii) lies also in an endblock A that is a copy of K_{k-2} (such a neighbor exists because B is at the end of a longest path in the block-tree). In (i), let $D = B \cup \{y\} + yx_B$; in (ii), let $D = B \cup A + yx_B$.

In (i), equation (*) gives

$$(k-2)(k-3) + 2(3) > (k-3+p(k))(k-1) + z(k).$$

This simplifies to $6 > k-3+(k-1)p(k)+z(k)$. Since $p(k) \geq 0$ by (1) and (3), this implies $z(k) < 9-k$. Also, by (4) we get $6 > k + \frac{3-z(k)}{k-2} > k + \frac{k-6}{k-2}$, which yields a contradiction since $k \geq 6$.

In (ii), equation (*) gives

$$2(k-2)(k-3) + 2(3) > 2(k-3+p(k))(k-2) + z(k),$$

which simplifies to

$$3 - z(k) > (k-2)p(k),$$

contradicting (4). $\square$

Lemma 6.2. *Let $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$, $h\colon \mathbb{N} \to \mathbb{R}$, $z\colon \mathbb{N} \to \mathbb{R}$. For all $k \geq 6$ and $T \in \mathcal{T}_k$ with $K_{k-1} \subseteq T$, we have*

$$2\,\|T\| \leq (k-3+p(k))\,|T| + f(k) + h(k)q(T) + z(k)\beta(T)$$

whenever p, f, h and z satisfy all of the following conditions:

1. *$f(k) \geq (k-1)(1-p(k)-h(k))$; and*

2. $p(k) \geq \frac{3-z(k)}{k-2}$*; and*

3. $p(k) \geq h(k) + 5 - k$*; and*

4. $p(k) \geq \frac{2+h(k)}{k-2}$*; and*

5. $(k-1)p(k) + (k-3)h(k) + z(k) \geq k+1$.

The proof is similar to that of Lemma 6.1. The main difference is that now our only base case is $T = K_{k-1}$. For this reason, we replace hypotheses (1), (2), and (3) of Lemma 6.1, which we used only for the base cases of that proof, with our new hypothesis (1), which we use for the current base case. When some endblock B is an odd cycle or K_t, with $t \in \{3, \ldots, k-3\}$, the induction step is identical to that in Lemma 6.1, since deleting D does not change $q(T)$. It is easy to check that, as needed, $K_{k-1} \subseteq T \setminus D$. Thus, we need only to consider the induction step when T has an endblock B that is K_2, K_{k-2}, or K_{k-1}. As we will see, these three cases require hypotheses (3), (4), and (5), respectively.

Let T be a counterexample minimizing $|T|$. Let D be an induced subgraph such that $T \setminus D$ is connected, and let $T' = T \setminus D$. The same argument as in Lemma 6.1 now gives

$$2||T|| - 2||T'|| > (k-3+p(k))\,|D| + h(k)\,(q(T) - q(T')) + z(k)\,(\beta(T) - \beta(T')) . \tag{**}$$

If B is K_2, then $q(T') \leq q(T) + 1$ and (**) gives $2 > k - 3 + p(k) - h(k)$, contradicting (3). So every endblock of B is K_{k-2} or K_{k-1}. To handle these cases, we will need to remove x_B from T as well. Suppose some endblock B is K_{k-1} and $K_{k-1} \subseteq T \setminus B$. Let $D = B$. Now $q(T') \leq q(T) - (k-2) + 1$. So (**) gives

$$(k-1)(k-2) + 2 > (k-3+p(k))(k-1) + h(k)(k-3) + z(k).$$

This simplifies to $k+1 > (k-1)p(k) + (k-3)h(k) + z(k)$, which contradicts (5). Thus, at most one endblock of T is K_{k-1}. Since the cases above apply when B is any endblock, each other endblock must be K_{k-2}. Let B be such an endblock, and x_B its cut vertex. So $d_T(x_B) = k-2$ or $d_T(x_B) = k-1$. In the former case, $q(T') \leq q(T) + 1$, and in the latter, $q(T) = q(T')$. If $d_T(x_B) = k-2$, then (**) gives

$$(k-2)(k-3) + 2 > (k-3+p(k))(k-2) - h(k),$$

which simplifies to $\frac{2+h(k)}{k-2} > p(k)$, and contradicts (4).

Hence, all but at most one endblock of T is a copy of K_{k-2} with a cut vertex that is also in an odd cycle. Let B be such an endblock at the end of a longest path in the block-tree of T, and let C be the odd cycle sharing a vertex x_B with B. Consider a neighbor y of x_B on C that either (i) lies only in block C or (ii) lies also in an endblock A that is a copy of K_{k-2} (such a neighbor exists because B is at the end of a longest path in the block-tree). In (i), let $D = B \cup \{y\} + yx_B$; in (ii), let $D = B \cup A + yx_B$. Let $T' = T \setminus V(D)$. In each case, we have $q(T') = q(T)$, so the analysis is identical to that in the proof of Lemma 6.1.

Corollary 6.3. *Let* $p \colon \mathbb{N} \to [0,1]$*. For all* $k \geq 6$ *and* $T \in \mathcal{T}_k$ *with* $K_{k-1} \subseteq T$*, we have*

$$2\,\|T\| \leq (k-3+p(k))\,|T| + f(k) + h(k)q(T) + z(k)\beta(T),$$

where

$$\begin{aligned} f(k) &= (k-1)\,(3-(k-1)p(k))\,, \\ h(k) &= (k-2)p(k) - 2, \\ z(k) &= 3k - 5 - (k^2 - 4k + 5)p(k). \end{aligned}$$

References

[1] D. Cranston and L. Rabern, *Edge lower bounds for list critical graphs, via discharging*, arXiv:1602.02589 (2016).

[2] H.A. Kierstead and L. Rabern, *Improved lower bounds on the number of edges in list critical and online list critical graphs*, arXiv:1406.7355 (2014).

[3] ———, *Extracting list colorings from large independent sets*, arXiv:1512.08130 (2015).

[4] A.V. Kostochka and M. Yancey, *Ore's conjecture on color-critical graphs is almost true*, J. Combin. Theory Ser. B **109** (2014), 73–101. MR 3269903