A common generalization of Hall's theorem and Vizing's edge-coloring theorem

landon rabern

LBD Data

Miami University Colloquium
November 6, 2014

Hall's theorem

- given finite sets $A_1, A_2, \ldots, A_n$

Hall's theorem

- given finite sets $A_1, A_2, \ldots, A_n$
- a system of distinct representatives (SDR) is a choice of $a_i \in A_i$ for all i where $a_i \neq a_j$ for $i \neq j$

Hall's theorem

- given finite sets $A_1, A_2, \ldots, A_n$
- a system of distinct representatives (SDR) is a choice of $a_i \in A_i$ for all i where $a_i \neq a_j$ for $i \neq j$
- when can we pick an SDR?

Hall's theorem

- given finite sets $A_1, A_2, \ldots, A_n$
- a system of distinct representatives (SDR) is a choice of $a_i \in A_i$ for all i where $a_i \neq a_j$ for $i \neq j$
- when can we pick an SDR?
- if k of the sets together have fewer than k elements, we can't

Hall's theorem

- given finite sets $A_1, A_2, \ldots, A_n$
- a system of distinct representatives (SDR) is a choice of $a_i \in A_i$ for all i where $a_i \neq a_j$ for $i \neq j$
- when can we pick an SDR?
- if k of the sets together have fewer than k elements, we can't
  - $A_1 = \{1, 2\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2\}$

Hall's theorem

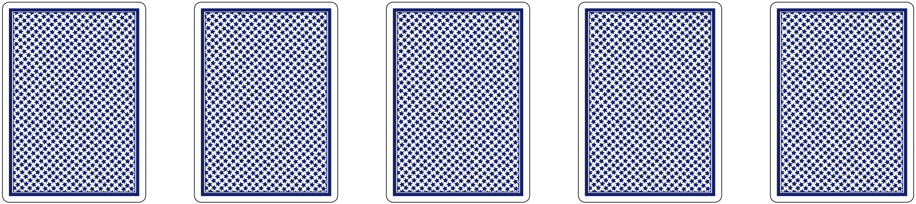
- given finite sets $A_1, A_2, \ldots, A_n$
- a system of distinct representatives (SDR) is a choice of $a_i \in A_i$ for all i where $a_i \neq a_j$ for $i \neq j$
- when can we pick an SDR?
- if k of the sets together have fewer than k elements, we can't
  - $A_1 = \{1, 2\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2\}$
- Hall's theorem: this is the only thing that can go wrong

$$\text{SDR exists} \;\Leftrightarrow \left|\bigcup_{i \in I} A_i\right| \geq |I| \text{ for all } I \subseteq \{1, \ldots, n\}$$

some card games

the simplest variation

- Dealer vs. Player



- Dealer vs. Player
- the deck has just many copies of the high spade cards

some card games

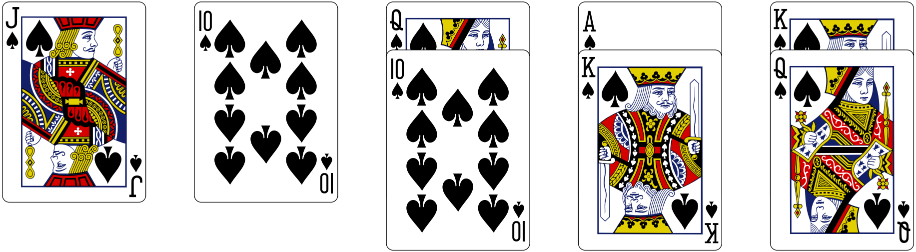
the simplest variation

- Dealer vs. Player
- the deck has just many copies of the high spade cards
- Dealer makes 5 stacks of cards with no duplicates, all cards face-up

- Dealer vs. Player
- the deck has just many copies of the high spade cards
- Dealer makes 5 stacks of cards with no duplicates, all cards face-up
- Player wins if he can pick a Royal Flush, one card from each stack

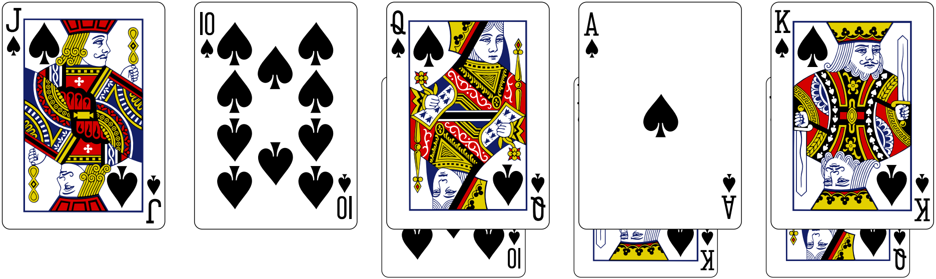
some card games

example, a Player win



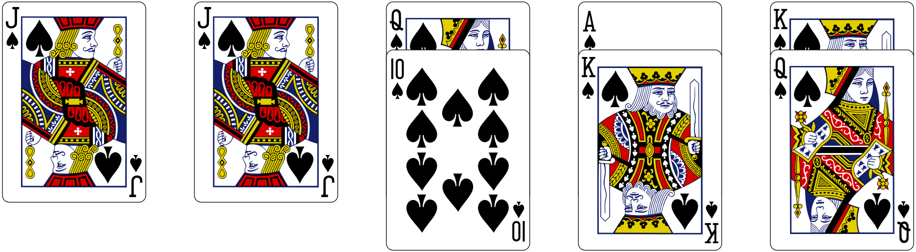
some card games

example, a Player win



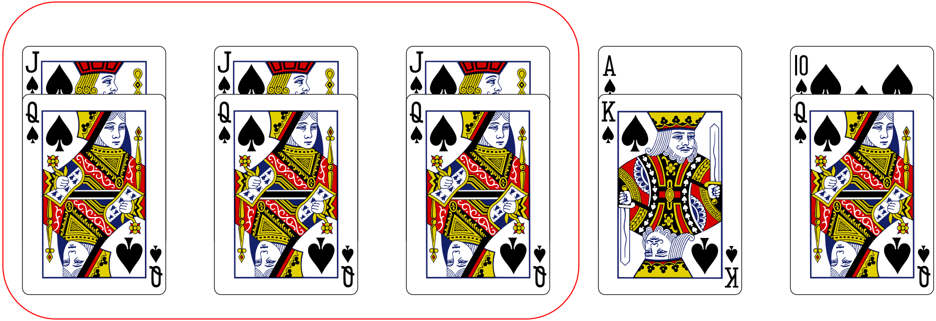
some card games

example, a Dealer win

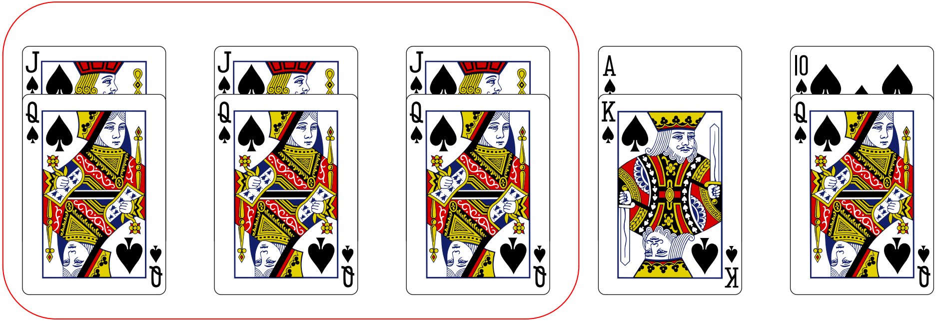


- Player cannot win if there is a set of k stacks that together have fewer than k different cards

- Player cannot win if there is a set of k stacks that together have fewer than k different cards



- Player cannot win if there is a set of k stacks that together have fewer than k different cards
- Hall's theorem says: **Player wins otherwise**



- this isn't a fun game, far too easy for Dealer to win

- this isn't a fun game, far too easy for Dealer to win
- to make a better game, we allow Player to modify some of the stacks

- this isn't a fun game, far too easy for Dealer to win
- to make a better game, we allow Player to modify some of the stacks

Player's Move

Player can pick any card A from the deck and swap it for another card B in one stack (not containing A).

- this isn't a fun game, far too easy for Dealer to win
- to make a better game, we allow Player to modify some of the stacks

Player's Move

Player can pick any card A from the deck and swap it for another card B in one stack (not containing A).

Dealer's Move

Dealer can (i) do nothing or (ii) swap A and B in one other stack.

- this isn't a fun game, far too easy for Dealer to win
- to make a better game, we allow Player to modify some of the stacks

Player's Move

Player can pick any card A from the deck and swap it for another card B in one stack (not containing A).

Dealer's Move

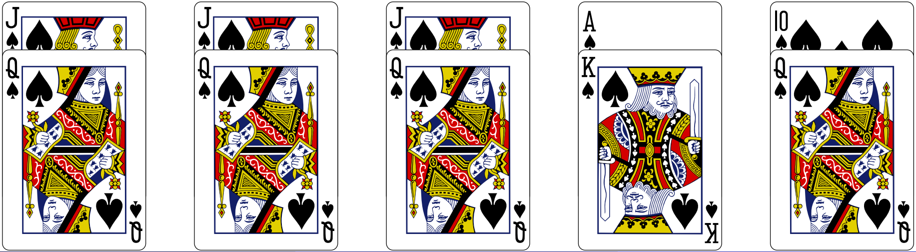
Dealer can (i) do nothing or (ii) swap A and B in one other stack.

Winning

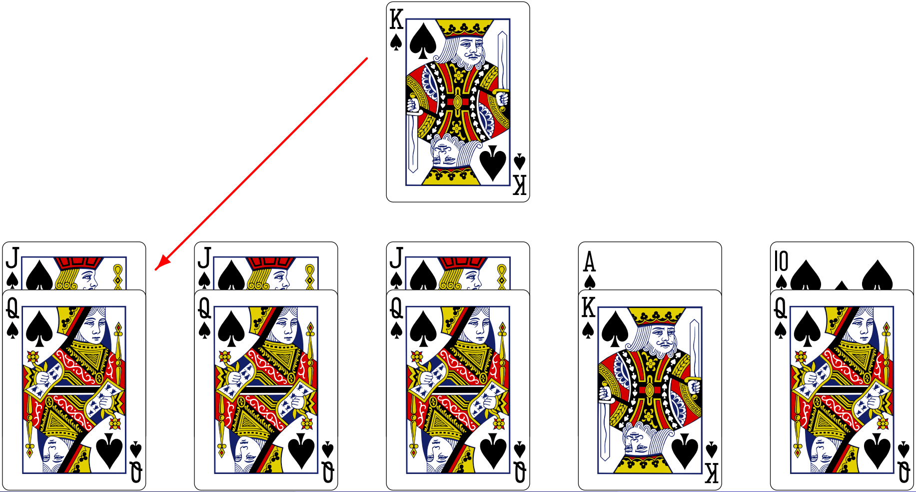
Player wins if he can pick a Royal Flush at the start of one of his turns, otherwise Dealer wins.

some card games

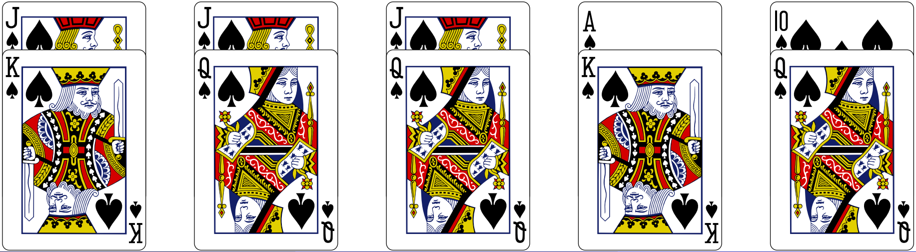
example, a Player win



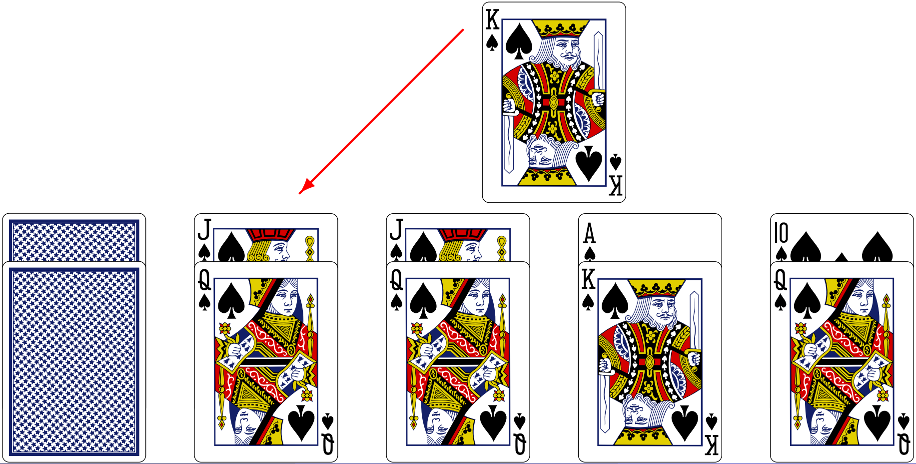
- Player picks a King from the deck and swaps it for a Queen in the first stack



- Player picks a King from the deck and swaps it for a Queen in the first stack



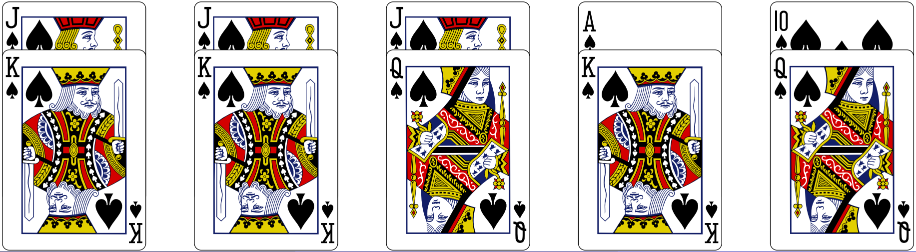
- Player picks a King from the deck and swaps it for a Queen in the first stack
- Dealer can swap a King and Queen in one of the other stacks



some card games

example, a Player win

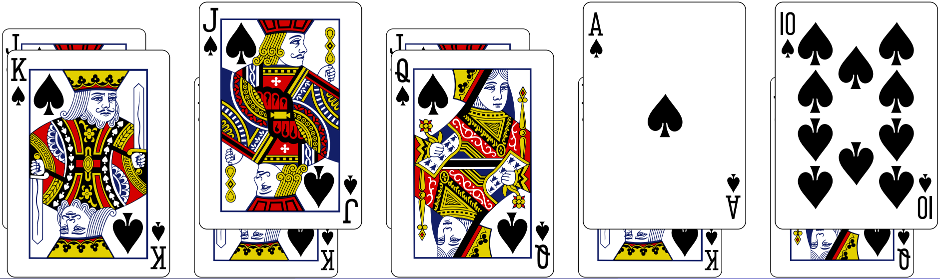
- Player picks a King from the deck and swaps it for a Queen in the first stack
- Dealer can swap a King and Queen in one of the other stacks



some card games

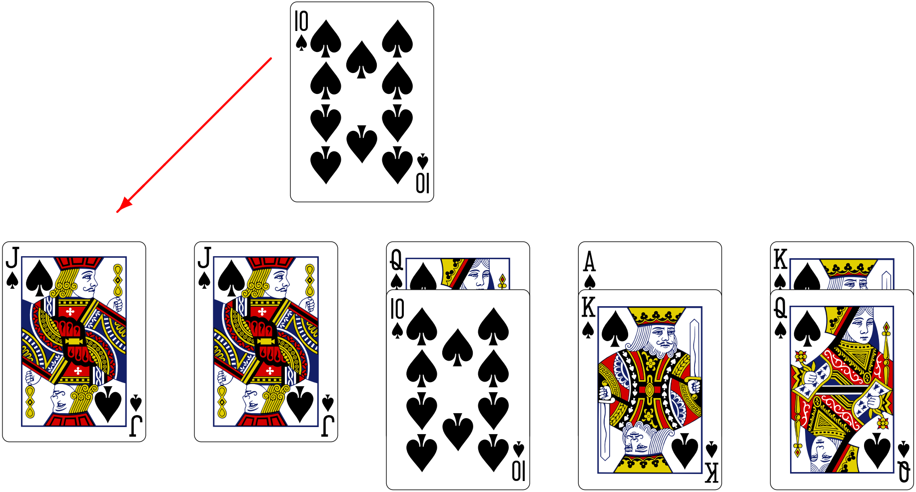
example, a Player win

- Player picks a King from the deck and swaps it for a Queen in the first stack
- Dealer can swap a King and Queen in one of the other stacks
- Player wins no matter what Dealer does



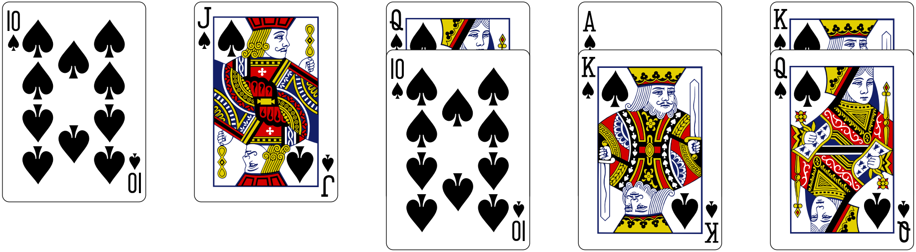
some card games

example, a Dealer win



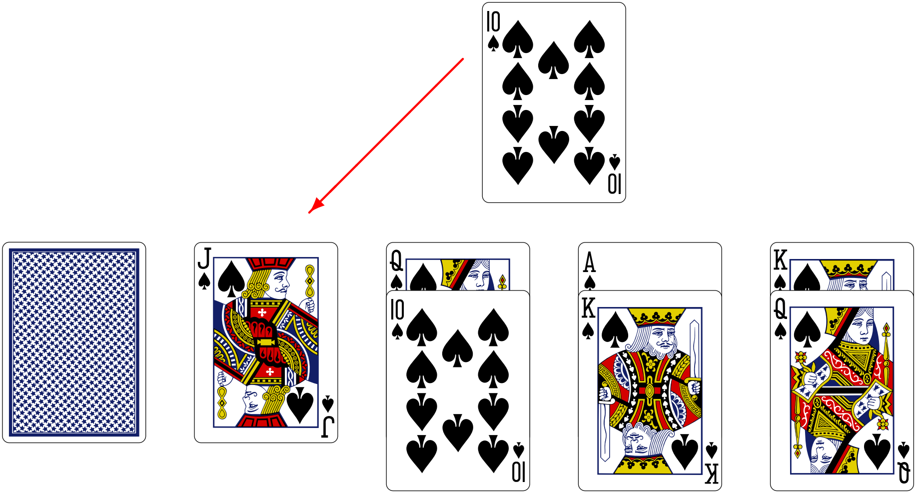
some card games

example, a Dealer win



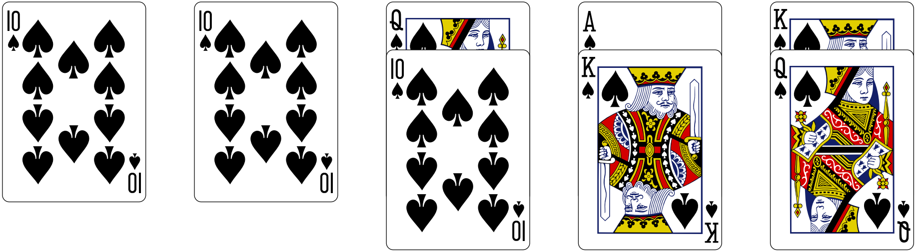
some card games

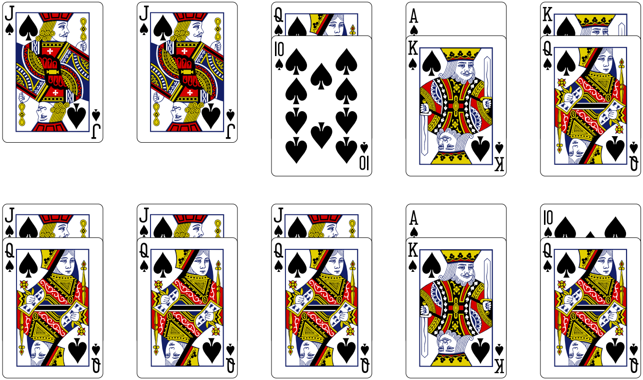
example, a Dealer win



some card games

example, a Dealer win

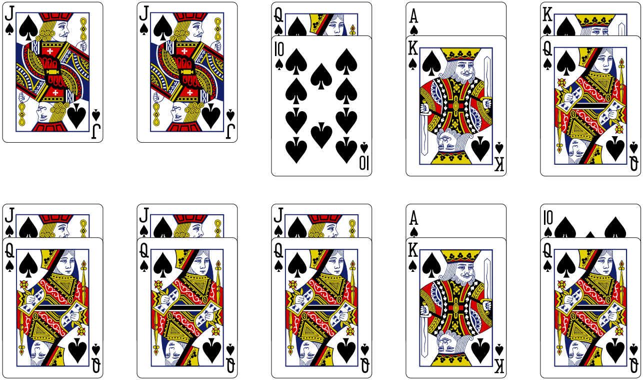




some card games

what was the difference?

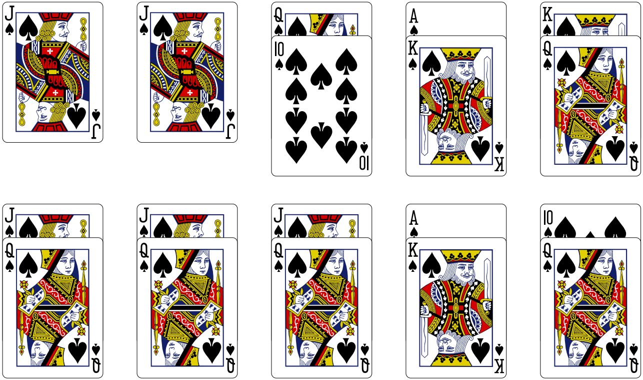
- in the top game, Dealer can prevent Player from increasing the number of different cards in the first two stacks



some card games

what was the difference?

- in the top game, Dealer can prevent Player from increasing the number of different cards in the first two stacks
- in the bottom game, Dealer cannot prevent prevent Player from increasing the number of different cards in the first three stacks



some card games

necessary condition

- if the same card appears on three stacks, Player can force the addition of a new card to these stacks

- if the same card appears on three stacks, Player can force the addition of a new card to these stacks
- it is not hard to show that this is essentially all Player can do

- if the same card appears on three stacks, Player can force the addition of a new card to these stacks
- it is not hard to show that this is essentially all Player can do
- this suggests a necessary condition

- if the same card appears on three stacks, Player can force the addition of a new card to these stacks
- it is not hard to show that this is essentially all Player can do
- this suggests a necessary condition

Degree

The *degree* of a card C in a set of stacks S is the number of times C appears in S. We write $d_S(C)$ for this quantity.

- if the same card appears on three stacks, Player can force the addition of a new card to these stacks
- it is not hard to show that this is essentially all Player can do
- this suggests a necessary condition

Degree

The *degree* of a card C in a set of stacks S is the number of times C appears in S. We write $d_S(C)$ for this quantity.

Necessary Condition

If Player can win, then for every set of stacks S we must have

$$\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil \geq |S|.$$

Degree

The *degree* of a card C in a set of stacks S is the number of times C appears in S. We write $d_S(C)$ for this quantity.

Necessary Condition

If Player can win, then for every set of stacks S we must have

$$\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil \geq |S|.$$

- in Hall's theorem, each C is 'worth' 1 in $\sum_{C \in \bigcup S} 1 = \left|\bigcup S\right| \geq |S|$

Degree

The *degree* of a card C in a set of stacks S is the number of times C appears in S. We write $d_S(C)$ for this quantity.

Necessary Condition

If Player can win, then for every set of stacks S we must have

$$\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil \geq |S|.$$

- in Hall's theorem, each C is 'worth' 1 in $\sum_{C \in \bigcup S} 1 = \left|\bigcup S\right| \geq |S|$
- Player can turn $2t+1$ of the same card into $t+1$ different cards, so C is 'worth' $\left\lceil \frac{d_S(C)}{2} \right\rceil$

- given a set of stacks S with $\displaystyle\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil < |S|$

- given a set of stacks S with $\displaystyle\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil < |S|$
- Dealer's strategy: maintain this invariant

- given a set of stacks S with $\displaystyle\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil < |S|$
- Dealer's strategy: maintain this invariant
  - this is good enough since then $\left|\bigcup S\right| \leq \displaystyle\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil < |S|$ always

- given a set of stacks S with $\displaystyle\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil < |S|$
- Dealer's strategy: maintain this invariant
  - this is good enough since then $\left|\bigcup S\right| \le \displaystyle\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil < |S|$ always
  - if Player swaps A in for B, increasing $\left\lceil \frac{d_S(A)}{2} \right\rceil + \left\lceil \frac{d_S(B)}{2} \right\rceil$, then $d_S(A)$ and $d_S(B)$ both changed from even to odd

- given a set of stacks S with $\displaystyle\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil < |S|$
- Dealer's strategy: maintain this invariant
  - this is good enough since then $|\bigcup S| \leq \displaystyle\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil < |S|$ always
  - if Player swaps A in for B, increasing $\left\lceil \frac{d_S(A)}{2} \right\rceil + \left\lceil \frac{d_S(B)}{2} \right\rceil$, then $d_S(A)$ and $d_S(B)$ both changed from even to odd
  - so, Dealer can swap A for B somewhere else, decreasing $\left\lceil \frac{d_S(A)}{2} \right\rceil + \left\lceil \frac{d_S(B)}{2} \right\rceil$

- given a set of stacks S with $\displaystyle\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil < |S|$
- Dealer's strategy: maintain this invariant
  - this is good enough since then $|\bigcup S| \leq \displaystyle\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil < |S|$ always
  - if Player swaps A in for B, increasing $\left\lceil \frac{d_S(A)}{2} \right\rceil + \left\lceil \frac{d_S(B)}{2} \right\rceil$, then $d_S(A)$ and $d_S(B)$ both changed from even to odd
  - so, Dealer can swap A for B somewhere else, decreasing $\left\lceil \frac{d_S(A)}{2} \right\rceil + \left\lceil \frac{d_S(B)}{2} \right\rceil$
  - Dealer has maintained $\displaystyle\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil < |S|$

- **this necessary condition is also suffcient**

- **this necessary condition is also suffcient**

Winning Condition

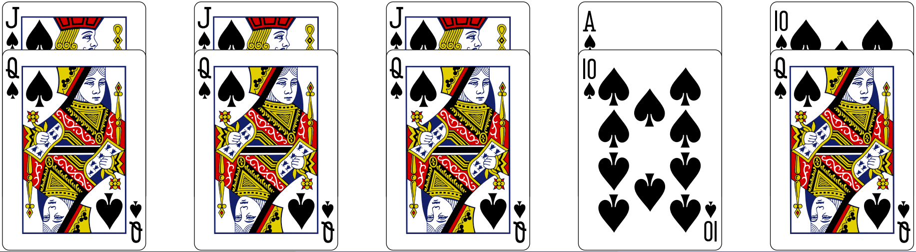
Player can win if and only if for every set of stacks S we have

$$\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil \geq |S|.$$

some card games

proof idea

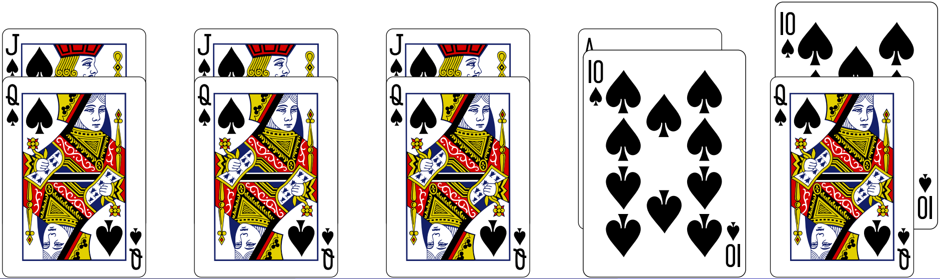
1. Player looks for a set of card types that give a system of distinct representatives of all the stacks containing them



some card games

proof idea

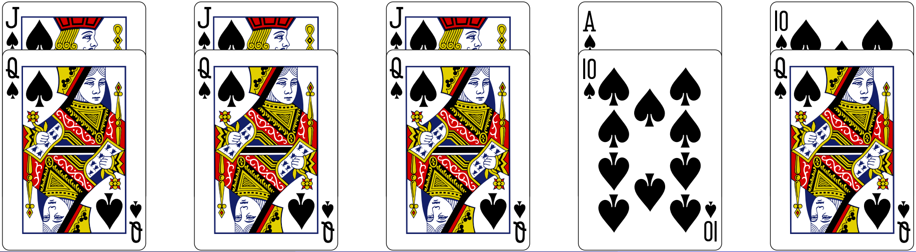
1. Player looks for a set of card types that give a system of distinct representatives of all the stacks containing them



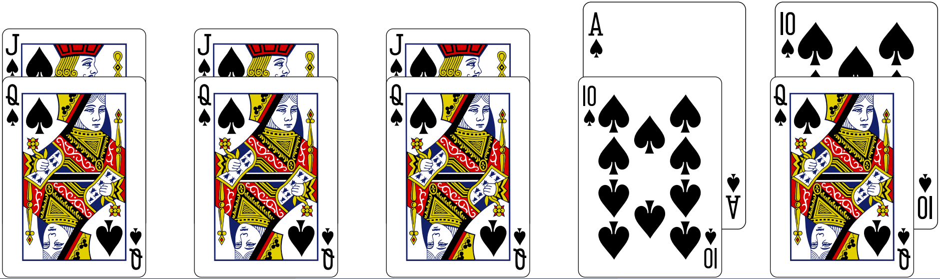
some card games

proof idea

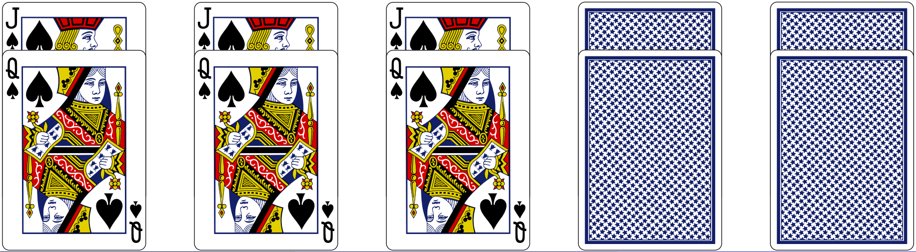
1. Player looks for a set of card types that give a system of distinct representatives of all the stacks containing them



1. Player looks for a set of card types that give a system of distinct representatives of all the stacks containing them



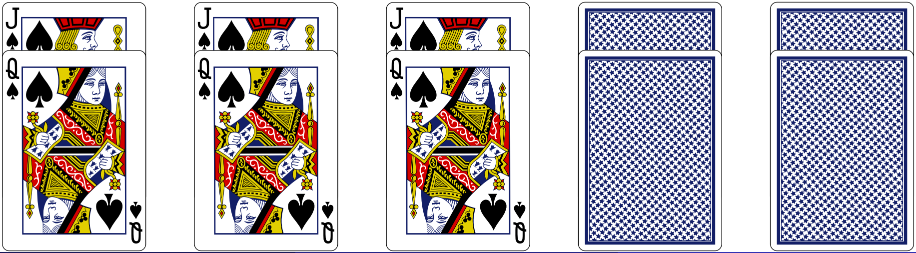
1. Player looks for a set of card types that give a system of distinct representatives of all the stacks containing them
2. Player calls those stacks done and never plays with those card types again



some card games

proof idea

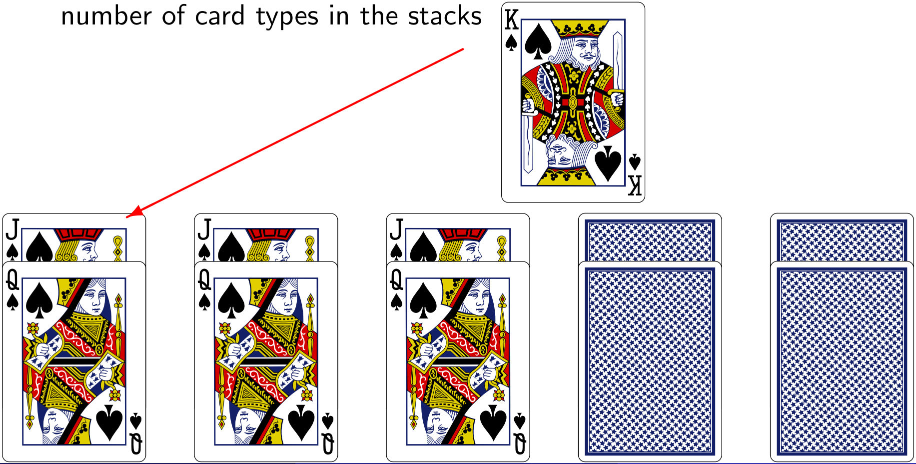
3. if no such set of card types exists, then Hall's theorem shows that there is at least one card appearing on none of the remaining stacks



some card games

proof idea

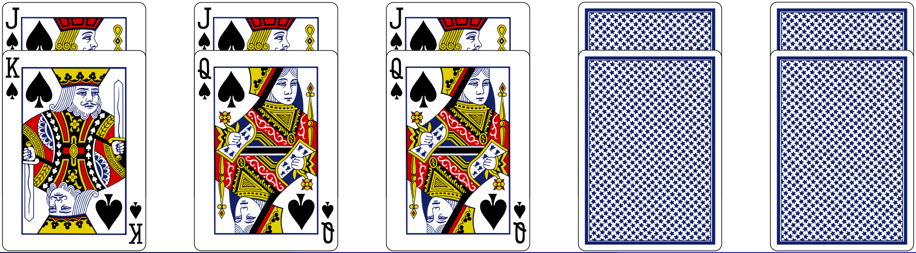
3. if no such set of card types exists, then Hall's theorem shows that there is at least one card appearing on none of the remaining stacks
4. but then some card appears at least thrice, so Player can increase the number of card types in the stacks



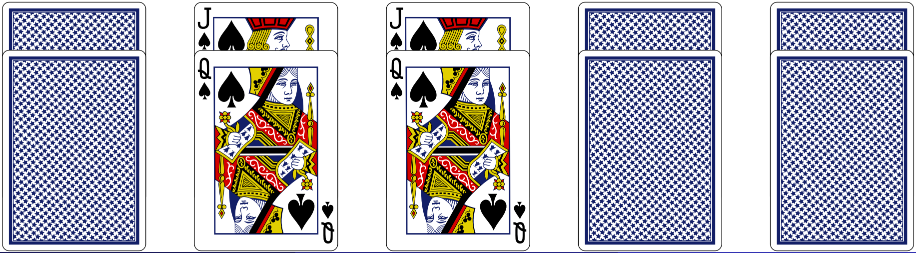
some card games

proof idea

3. if no such set of card types exists, then Hall's theorem shows that there is at least one card appearing on none of the remaining stacks
4. but then some card appears at least thrice, so Player can increase the number of card types in the stacks
5. goto step 1



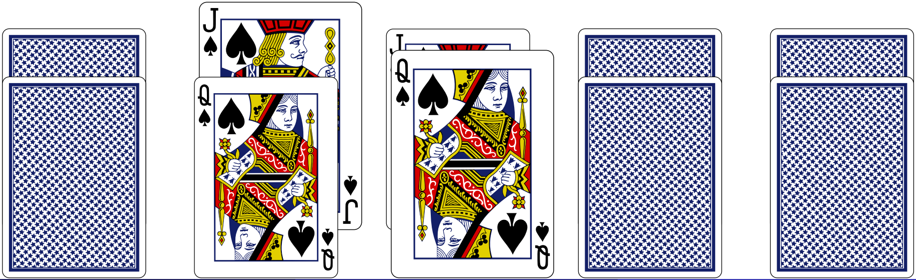
3. if no such set of card types exists, then Hall's theorem shows that there is at least one card appearing on none of the remaining stacks
4. but then some card appears at least thrice, so Player can increase the number of card types in the stacks
5. goto step 1



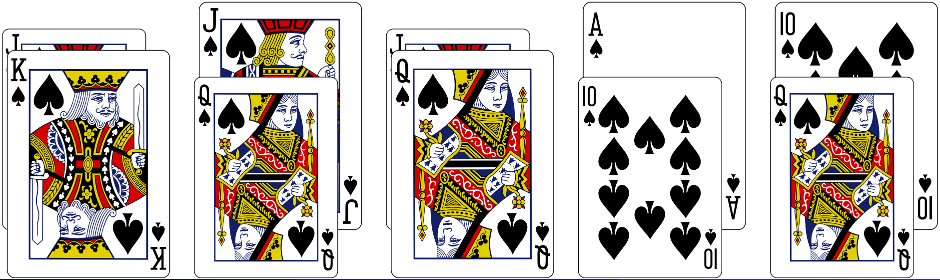
some card games

proof idea

3. if no such set of card types exists, then Hall's theorem shows that there is at least one card appearing on none of the remaining stacks
4. but then some card appears at least thrice, so Player can increase the number of card types in the stacks
5. goto step 1



3. if no such set of card types exists, then Hall's theorem shows that there is at least one card appearing on none of the remaining stacks
4. but then some card appears at least thrice, so Player can increase the number of card types in the stacks
5. goto step 1



A generalization of Hall's theorem

making it harder for Player

- allow Dealer to make more swaps in response to Player's move

A generalization of Hall's theorem

making it harder for Player

- allow Dealer to make more swaps in response to Player's move
- for each $t \geq 1$, the *t*-game allows Dealer to make up to t swaps

A generalization of Hall's theorem

making it harder for Player

- allow Dealer to make more swaps in response to Player's move
- for each $t \geq 1$, the *t*-game allows Dealer to make up to t swaps

Winning Condition

Player can win in the t-game if and only if for every set of stacks S we have

$$\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{t+1} \right\rceil \geq |S|.$$

A generalization of Hall's theorem

making it harder for Player

- allow Dealer to make more swaps in response to Player's move
- for each $t \geq 1$, the t-game allows Dealer to make up to t swaps

Winning Condition

Player can win in the t-game if and only if for every set of stacks S we have

$$\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{t+1} \right\rceil \geq |S|.$$

- Hall's theorem is the winning condition in the $(k-1)$-game when there are k total stacks:

A generalization of Hall's theorem

making it harder for Player

- allow Dealer to make more swaps in response to Player's move
- for each $t \geq 1$, the t-game allows Dealer to make up to t swaps

Winning Condition

Player can win in the t-game if and only if for every set of stacks S we have

$$\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{t+1} \right\rceil \geq |S|.$$

- Hall's theorem is the winning condition in the $(k-1)$-game when there are k total stacks:
  - $1 \leq d_S(C) \leq k$, so $\left\lceil \frac{d_S(C)}{t+1} \right\rceil = 1$

A generalization of Hall's theorem

making it harder for Player

- allow Dealer to make more swaps in response to Player's move
- for each $t \geq 1$, the t-game allows Dealer to make up to t swaps

Winning Condition

Player can win in the t-game if and only if for every set of stacks S we have

$$\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{t+1} \right\rceil \geq |S|.$$

- Hall's theorem is the winning condition in the $(k-1)$-game when there are k total stacks:
  - $1 \leq d_S(C) \leq k$, so $\left\lceil \frac{d_S(C)}{t+1} \right\rceil = 1$
  - so, the sum equals $|\bigcup S|$

A generalization of Hall's theorem

making it harder for Player

- allow Dealer to make more swaps in response to Player's move
- for each $t \geq 1$, the *t-game* allows Dealer to make up to t swaps

Winning Condition

Player can win in the t-game if and only if for every set of stacks S we have

$$\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{t+1} \right\rceil \geq |S|.$$

- Hall's theorem is the winning condition in the $(k-1)$-game when there are k total stacks:
    - $1 \leq d_S(C) \leq k$, so $\left\lceil \frac{d_S(C)}{t+1} \right\rceil = 1$
    - so, the sum equals $|\bigcup S|$
    - Player's moves are useless

- assign colors to the edges of a graph so that incident edges get different colors

edge coloring

setup

- assign colors to the edges of a graph so that incident edges get different colors
- how few colors can we use?

edge coloring

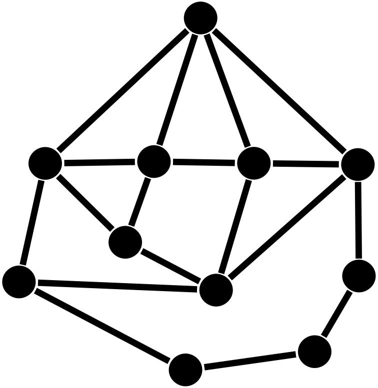
setup

- assign colors to the edges of a graph so that incident edges get different colors
- how few colors can we use?

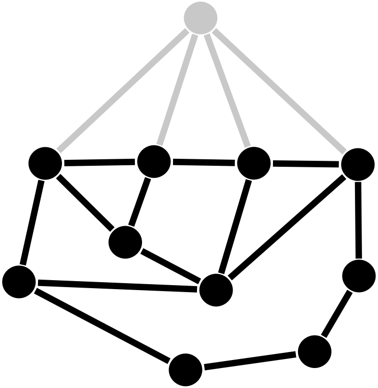
Vizing's theorem

Any simple graph can be edge-colored using at most one more color than its maximum degree.

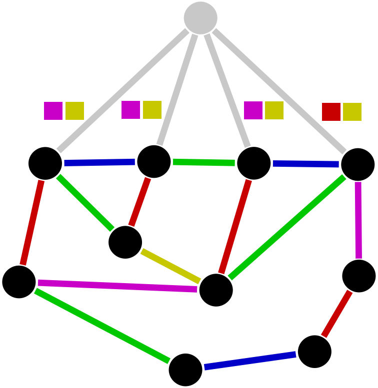
- proceed by induction on the number of vertices



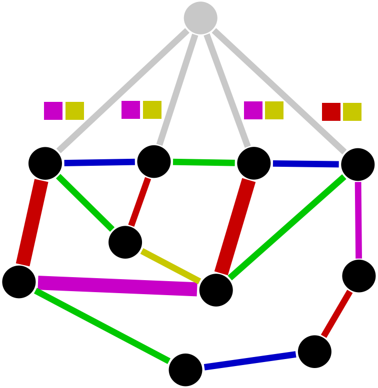
- proceed by induction on the number of vertices
- remove a vertex and edge-color the rest with one more color than its maximum degree



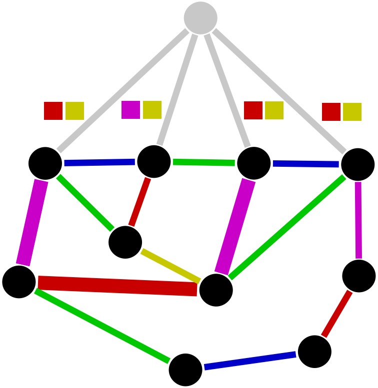
- proceed by induction on the number of vertices
- remove a vertex and edge-color the rest with one more color than its maximum degree



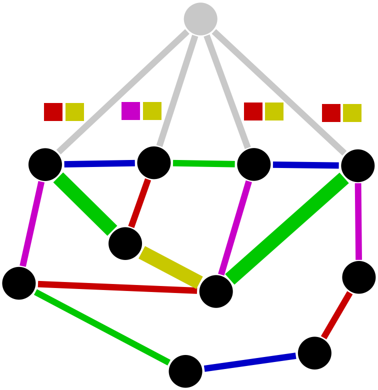
- proceed by induction on the number of vertices
- remove a vertex and edge-color the rest with one more color than its maximum degree



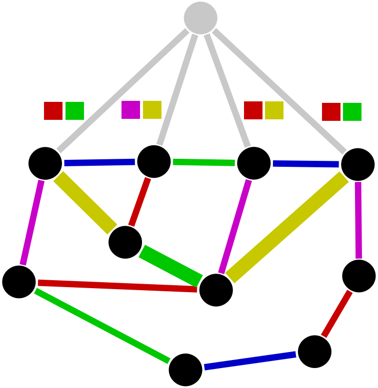
- proceed by induction on the number of vertices
- remove a vertex and edge-color the rest with one more color than its maximum degree



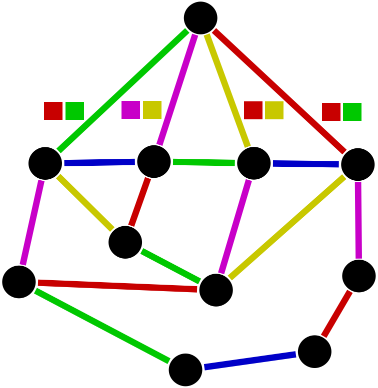
- proceed by induction on the number of vertices
- remove a vertex and edge-color the rest with one more color than its maximum degree



- proceed by induction on the number of vertices
- remove a vertex and edge-color the rest with one more color than its maximum degree



- proceed by induction on the number of vertices
- remove a vertex and edge-color the rest with one more color than its maximum degree



- **exchanging colors on a two-colored path is just a Player move followed by a Dealer move**

- **exchanging colors on a two-colored path is just a Player move followed by a Dealer move**
- we can make any of Player's legal moves this way, so if the winning conditions are satisfied, Vizing's theorem is true

- **exchanging colors on a two-colored path is just a Player move followed by a Dealer move**
- we can make any of Player's legal moves this way, so if the winning conditions are satisfied, Vizing's theorem is true
- each stack has at least two colors, so counting the 'cards' in two ways we get for each set of stacks S,

$$\sum_{C \in \bigcup S} d_S(C) \geq 2|S|$$

- **exchanging colors on a two-colored path is just a Player move followed by a Dealer move**
- we can make any of Player's legal moves this way, so if the winning conditions are satisfied, Vizing's theorem is true
- each stack has at least two colors, so counting the 'cards' in two ways we get for each set of stacks S,

$$\sum_{C \in \bigcup S} d_S(C) \geq 2|S|$$

- so, we have the desired winning condition

$$\sum_{C \in \bigcup S} \frac{d_S(C)}{2} \geq |S|$$

summary

- we introduced a simple card game

summary

- we introduced a simple card game
  - Player can pick any card A from the deck and swap it for another card B in one stack (not containing A)

summary

- we introduced a simple card game
  - Player can pick any card A from the deck and swap it for another card B in one stack (not containing A)
  - Dealer can (i) do nothing or (ii) swap A and B in one other stack

summary

- we introduced a simple card game
  - Player can pick any card A from the deck and swap it for another card B in one stack (not containing A)
  - Dealer can (i) do nothing or (ii) swap A and B in one other stack
  - Player wins if he can pick a Royal Flush at the start of one of his turns, otherwise Dealer wins

summary

- we introduced a simple card game
  - Player can pick any card A from the deck and swap it for another card B in one stack (not containing A)
  - Dealer can (i) do nothing or (ii) swap A and B in one other stack
  - Player wins if he can pick a Royal Flush at the start of one of his turns, otherwise Dealer wins
- Player can win exactly when a Hall-like condition is satisfied

summary

- we introduced a simple card game
  - Player can pick any card A from the deck and swap it for another card B in one stack (not containing A)
  - Dealer can (i) do nothing or (ii) swap A and B in one other stack
  - Player wins if he can pick a Royal Flush at the start of one of his turns, otherwise Dealer wins
- Player can win exactly when a Hall-like condition is satisfied
- Vizing's edge-coloring theorem is an easy corollary

- we introduced a simple card game
  - Player can pick any card A from the deck and swap it for another card B in one stack (not containing A)
  - Dealer can (i) do nothing or (ii) swap A and B in one other stack
  - Player wins if he can pick a Royal Flush at the start of one of his turns, otherwise Dealer wins
- Player can win exactly when a Hall-like condition is satisfied
- Vizing's edge-coloring theorem is an easy corollary
- taking it further

summary

- we introduced a simple card game
  - Player can pick any card A from the deck and swap it for another card B in one stack (not containing A)
  - Dealer can (i) do nothing or (ii) swap A and B in one other stack
  - Player wins if he can pick a Royal Flush at the start of one of his turns, otherwise Dealer wins
- Player can win exactly when a Hall-like condition is satisfied
- Vizing's edge-coloring theorem is an easy corollary
- taking it further
  - most other classical edge-coloring results follow easily

summary

- we introduced a simple card game
  - Player can pick any card A from the deck and swap it for another card B in one stack (not containing A)
  - Dealer can (i) do nothing or (ii) swap A and B in one other stack
  - Player wins if he can pick a Royal Flush at the start of one of his turns, otherwise Dealer wins
- Player can win exactly when a Hall-like condition is satisfied
- Vizing's edge-coloring theorem is an easy corollary
- taking it further
  - most other classical edge-coloring results follow easily
  - generalizes easily to multigraphs

summary

- we introduced a simple card game
  - Player can pick any card A from the deck and swap it for another card B in one stack (not containing A)
  - Dealer can (i) do nothing or (ii) swap A and B in one other stack
  - Player wins if he can pick a Royal Flush at the start of one of his turns, otherwise Dealer wins
- Player can win exactly when a Hall-like condition is satisfied
- Vizing's edge-coloring theorem is an easy corollary
- taking it further
  - most other classical edge-coloring results follow easily
  - generalizes easily to multigraphs
  - a more general game unifies much of edge-coloring theory

the more general game

- Fixer vs. Breaker

the more general game

- Fixer vs. Breaker
- played on a multigraph G

the more general game

- Fixer vs. Breaker
- played on a multigraph G
- assign a list of colors $L(v)$ to each vertex

the more general game

- Fixer vs. Breaker
- played on a multigraph G
- assign a list of colors $L(v)$ to each vertex
- let the pot be $\bigcup_{v \in V(G)} L(v)$

the more general game

- Fixer vs. Breaker
- played on a multigraph G
- assign a list of colors $L(v)$ to each vertex
- let the pot be $\bigcup_{v \in V(G)} L(v)$
- Fixer wins if at the start of his turn he can construct an edge-coloring π of G where $\pi(xy) \in L(x) \cap L(y)$ for each $xy \in E(G)$

the more general game

- Fixer vs. Breaker
- played on a multigraph G
- assign a list of colors $L(v)$ to each vertex
- let the pot be $\bigcup_{v \in V(G)} L(v)$
- Fixer wins if at the start of his turn he can construct an edge-coloring π of G where $\pi(xy) \in L(x) \cap L(y)$ for each $xy \in E(G)$

Fixer's turn

Pick α in the pot and $v \in V(G)$ with $\alpha \notin L(v)$ and set $L(v) := L(v) \cup \{\alpha\} - \beta$ for some $\beta \in L(v)$.

the more general game

- Fixer vs. Breaker
- played on a multigraph G
- assign a list of colors $L(v)$ to each vertex
- let the pot be $\bigcup_{v \in V(G)} L(v)$
- Fixer wins if at the start of his turn he can construct an edge-coloring π of G where $\pi(xy) \in L(x) \cap L(y)$ for each $xy \in E(G)$

Fixer's turn

Pick α in the pot and $v \in V(G)$ with $\alpha \notin L(v)$ and set $L(v) := L(v) \cup \{\alpha\} - \beta$ for some $\beta \in L(v)$.

Breaker's turn

If Fixer modified $L(v)$ by inserting α and removing β, then Breaker can either do nothing or pick $w \in V(G - v)$ and modify its list by swapping α for β or β for α.