Edge lower bounds via discharging notes

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1 Introduction

For a graph G, let $d(G)$ be the average degree of G. Let $\mathcal{T}_k$ be the Gallai trees with maximum degree at most $k-1$, excepting K_k.

2 Gallai's bound via discharging

Theorem 2.1 (Gallai)**.** *For $k \geq 4$ and $G \neq K_k$ a k-AT-critical graph, we have*

$$d(G) > k - 1 + \frac{k-3}{k^2-3}.$$

Proof. Start with initial charge function $\mathrm{ch}(v) = d_G(v)$. Have each k^+-vertex give charge $\frac{k-1}{k^2-3}$ to each of its $(k-1)$-neighbors. Then let the vertices in each component of the low vertex subgraph share their total charge equally. Let $\mathrm{ch}^*(v)$ be the resulting charge function. We finish the proof by showing that $\mathrm{ch}^*(v) \geq k - 1 + \frac{k-3}{k^2-3}$ for all $v \in V(G)$.

If v is a k^+-vertex, then $ch^*(v) \geq d_G(v) - \frac{k-1}{k^2-3} d_G(v) = \left(1 - \frac{k-1}{k^2-3}\right) d_G(v) \geq \left(1 - \frac{k-1}{k^2-3}\right) k = k - 1 + \frac{k-3}{k^2-3}$ as desired.

Let T be a component of the low vertex subgraph. Then the vertices in T receive total charge

$$\frac{k-1}{k^2-3} \sum_{v \in V(T)} k - 1 - d_G(v) = \frac{k-1}{k^2-3} \left((k-1)|T| - 2\left\|T\right\|\right).$$

So, after distributing this charge out equally, each vertex in T receives charge

$$\frac{1}{|T|} \frac{k-1}{k^2-3} ((k-1)|T| - 2\left\|T\right\|) = \frac{k-1}{k^2-3} \left((k-1) - d(T)\right).$$

By Lemma 2.2, this is at least

$$\frac{k-1}{k^2-3} \left((k-1) - \left(k - 2 + \frac{2}{k-1}\right)\right) = \frac{k-1}{k^2-3} \left(\frac{k-3}{k-1}\right) = \frac{k-3}{k^2-3}.$$

Hence each low vertex ends with charge at least $k - 1 + \frac{k-3}{k^2-3}$ as desired. $\square$

Lemma 2.2 (Gallai)**.** *For $k \geq 4$ and $T \in \mathcal{T}_k$, we have $d(T) < k - 2 + \frac{2}{k-1}$.*

Proof. Suppose not and choose a counterexample T minimizing $|T|$. Then T has at least two blocks. Let B be an endblock of T. If B is K_t for $2 \leq t \leq k-2$, then remove the non-separating vertices of B from T to get T'. By minimality of $|T|$, we have

$$2\,\|T\| - t(t-1) = 2\,\|T'\| < \left(k-2+\frac{2}{k-1}\right)|T'| = \left(k-2+\frac{2}{k-1}\right)|T| - \left(k-2+\frac{2}{k-1}\right)(t-1).$$

Hence we have the contradiction

$$2\,\|T\| < \left(k-2+\frac{2}{k-1}\right)|T| + (t+2-k-\frac{2}{k-1})(t-1) \leq \left(k-2+\frac{2}{k-1}\right)|T|.$$

The case when B is an odd cycle is the same as the above, a longer cycle just makes things better. Finally, if $B = K_{k-1}$, remove all vertices of B from T to get T'. By minimality of $|T|$, we have

$$\begin{aligned}
2\,\|T\| - (k-1)(k-2) - 2 = 2\,\|T'\| & \\
&< \left(k-2+\frac{2}{k-1}\right)|T'| \\
&= \left(k-2+\frac{2}{k-1}\right)|T| - \left(k-2+\frac{2}{k-1}\right)(k-1).
\end{aligned}$$

Hence $2\,\|T\| < \left(k-2+\frac{2}{k-1}\right)|T|$, a contradiction. □

3 An initial improved bound

Lemma 2.2 is best possible as can be seen by the family of graphs with blocks on a path alternating K_{k-1} and K_2. But we have reducible configurations (see the last section for the precise statements) that place restrictions on K_{k-1} blocks. To state these restrictions, we need the following auxiliary bipartite graph.

For a k-AT-critical graph G, let $\mathcal{L}(G)$ be the subgraph of G induced on the $(k-1)$-vertices and $\mathcal{H}(G)$ the subgraph of G induced on the k-vertices. For $T \in \mathcal{T}_k$, let $W^k(T)$ be the set of vertices of T that are contained in some K_{k-1} in T. Let $\mathcal{B}_k(G)$ be the bipartite graph with one part $V(\mathcal{H}(G))$ and the other part the components of $\mathcal{L}(G)$. Put an edge between $y \in V(\mathcal{H}(G))$ and a component T of $\mathcal{L}(G)$ if and only if $N(y) \cap W^k(T) \neq \emptyset$. Then Lemma 4.2 says that $\mathcal{B}_k(G)$ is 2-degenerate.

We can use this fact to refine our discharging argument. Let ϵ and γ be parameters that we will determine where $\epsilon \leq \gamma < 2\epsilon$. Start with initial charge function $\mathrm{ch}(v) = d_G(v)$.

1. Each k^+-vertex gives charge ϵ to each of its $(k-1)$-neighbors not in a K_{k-1},

2. Each $(k+1)^+$-vertex give charge γ to each of its $(k-1)$-neighbors in a K_{k-1},

3. Let $Q = \mathcal{B}_k(G)$. Repeat the following steps until Q is empty.

    (a) For each component T of $\mathcal{L}(G)$ in Q that has degree at most two in Q do the following:

i. For each $v \in V(\mathcal{H}(G)) \cap V(Q)$ such that $\left|N_G(v) \cap W^k(T)\right| = 2$, pick one $x \in N_G(v) \cap W^k(T)$ and send charge γ from v to x,

ii. Remove T from Q.

(b) Pick $v \in V(\mathcal{H}(G)) \cap V(Q)$ with degree at most two in Q. Send charge γ from v to each $x \in N_G(v) \cap W^k(T)$ for each component T of $\mathcal{L}(G)$ where $vT \in E(Q)$.

(c) Remove v from Q.

4. Have the vertices in each component of $\mathcal{L}(G)$ share their total charge equally.

Let $\mathrm{ch}^*(v)$ be the resulting charge function. Here is some intuition for why this might be a useful refinement. In (3b), v sends charge to at most two different T and so, by Lemma 4.1 (or our 'beyond degree choosability' classification), v loses charge at most 3γ. On the other hand, from (3a) each component T of $\mathcal{L}(G)$ receives charge γ for all but at most two non-separating vertices in a K_{k-1} (the at most two is coming from Lemma 4.1 and the fact that we leave T in Q until it has degree at most two and when it does, we send up to two extra γ to T in (3ai) as needed). Note that (3ai) doesn't cause any $v \in V(\mathcal{H}(G))$ to lose more than 3γ, because it only gets enacted when the component T is about to be removed, after that v does not have two neighbors in another component. So, we can get each T almost as much charge as we could hope for without losing too much from the k-vertices. We don't have the same control over $(k+1)^+$-vertices, but it won't matter since they have extra charge to start with and sending γ to every $(k-1)$-neighbor will leave enough charge (we'll use $\gamma < 2\epsilon$ here).

To analyze this discharging procedure we need a bound like Lemma 2.2, but taking into account the number of edges in $\mathcal{B}_k(G)$. We can do this by taking into account the number of non-separating vertices in K_{k-1}'s in T. To this end, for $T \in \mathcal{T}_k$, let $q(T)$ be the number of non-separating vertices in a K_{k-1} in T. We give a family of such bounds. Without more reducible configurations we can't hope to do better than average degree $k-3$ because of K_{k-2} components, that is why the bound below has $(k-3+p(k))|T|$, a slight worsening of average degree $k-3$.

Lemma 3.1. *Let $K \geq 7$ and $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$, $h\colon \mathbb{N} \to \mathbb{R}$ be such that for all $k \geq K$ we have*

1. *$f(k) \geq t(t+2-k-p(k))$ for all $t \in [k-2]$; and*
2. *$f(k) \geq (5-k-p(k))s$ for all $s \geq 5$; and*
3. *$f(k) \geq (k-1)(1-p(k)-h(k))$; and*
4. *$p(k) \geq h(k)+5-k$; and*
5. *$p(k) \geq \frac{2}{k-2}$; and*
6. *$p(k) \geq \frac{1+h(k)}{k-2}$; and*
7. *$(k-1)p(k)+(k-3)h(k) \geq k+1$; and*
8. *$p(k) \geq \frac{3-f(k)}{3(k-2)}$; and*

Then for $k \geq K$ and $T \in \mathcal{T}_k$, we have

$$2\|T\| \leq (k-3+p(k))\,|T| + f(k) + h(k)q(T).$$

Proof. Suppose not and choose a counterexample T minimizing $|T|$. First, suppose T is K_t for $t \in [k-2]$. Then $t(t-1) > (k-3+p(k))t + f(k)$ contradicting (1). If T is C_{2r+1} for $r \geq 2$, then $2(2r+1) > (k-3+p(k))(2r+1) + f(k)$ and hence $f(k) < (5-k-p(k))(2r+1)$ contradicting (2). If T is K_{k-1}, then $(k-1)(k-2) > (k-3+p(k))(k-1) + f(k) + h(k)(k-1)$ contradicting (3).

Hence T has at least two blocks. Let B be an endblock of T and x_B the cutvertex of T contained in B. Let $T' = T - (V(B) \setminus \{x_B\})$. Then, by minimality of $|T|$, we have

$$2\|T'\| \leq (k-3+p(k))\,|T'| + f(k) + h(k)q(T').$$

Hence
$$2\|T\| - 2\|B\| \leq (k-3+p(k))\,(|T| - (|B|-1)) + f(k) + h(k)q(T').$$
Since T is a counterexample, this gives

$$2\|B\| > (k-3+p(k))(|B|-1) + h(k)\left(q(T) - q(T')\right). \qquad (*)$$

Suppose B is K_t for $3 \leq t \leq k-3$ or B is an odd cycle. Then $q(T') = q(T)$, $2\|B\| \leq |B|\,(|B|-1)$ and $2\|B\| = 2\,|B|$ if $|B| > k-3$. Since $p(k) \geq \frac{4}{k-2}$ by (5), this contradicts *.

If B is K_2, then $q(T') \leq q(T)+1$ and * gives $2 > k-3+p(k)-h(k)$ contradicting (4).

To handle the cases when B is K_{k-2} or K_{k-1} we need to remove x_B from T as well. Let $T^* = T - V(B)$. Then, by minimality of $|T|$, we have

$$2\|T^*\| \leq (k-3+p(k))\,|T^*| + f(k) + h(k)q(T^*).$$

Hence

$$2\|T\| - 2\|B\| - 2(d_T(x_B) - d_B(x_B)) \leq (k-3+p(k))\,(|T| - |B|) + f(k) + h(k)q(T^*).$$

Since T is a counterexample and B is complete, this gives

$$2\|B\| > (k-3+p(k))\,|B| - 2(d_T(x_B) + 1 - |B|) + h(k)\left(q(T) - q(T^*)\right),$$

which is
$$2\|B\| > (k-1+p(k))\,|B| - 2d_T(x_B) - 2 + h(k)\left(q(T) - q(T^*)\right). \qquad (**)$$

Suppose B is K_{k-1}. Then $d_T(x_B) = k-1$ and $q(T^*) \leq q(T) - (k-2) + 1 = q(T) - (k-3)$. From **, we have

$$\begin{aligned}(k-1)(k-2) &> (k-1+p(k))(k-1) - 2(k-1) - 2 + h(k)(k-3)\\ &= (k-1)(k-2) + p(k)(k-1) - (k+1) + h(k)(k-3),\end{aligned}$$

contradicting (7).

Finally, suppose every endblock of T is K_{k-2}. Then, for any endblock B, we have $d_T(x_B) = k-1$ or $d_T(x_B) = k-2$. In the former case, $q(T) = q(T^*)$ and in the latter case $q(T^*) \leq q(T) + 1$.

Call an endblock *extreme* if it is the end of a longest path in the block-tree of T. First, suppose there is an extreme endblock B with $d_T(x_B) = k-2$. Then B is adjacent to a K_2 block which is adjacent to a block A. If A is not K_{k-1}, then $q(T) = q(T^*)$ and hence

$$(k-2)(k-3) > (k-1+p(k))(k-2) - 2(k-2) - 2 = (k-2)(k-3) - 2 + (k-2)p(k),$$

contradicting (5). Hence A is K_{k-1}. By the extremality of B, all but at most one K_2 block adjacent to A is adjacent to a K_{k-2} endblock. Let $B_1 = B, B_2, \ldots, B_t$ be these K_{k-2} endblocks. $\hat{T} = T \setminus \left(A \cup \bigcup_{i\in[t]} B_i\right)$. If $|\hat{T}| = 0$, then

$$2\left\|T\right\| = t + (k-1)(k-2) + t(k-2)(k-3),$$

and

$$|T| = k-1+t(k-2),$$

since T is a counterexample and $q(T) = k-1-t$, this gives (after some simplification)

$$t+k-1 > p(k)(k-1+t(k-2)) + f(k) + h(k)(k-1-t),$$

which gives

$$p(k) < \frac{t+k-1-f(k)-h(k)(k-1-t)}{k-1+t(k-2)} = \frac{t(h(k)+1)+(k-1)(1-h(k))-f(k)}{,}$$

applying (3), we get

$$p(k) < \frac{t+k-1-f(k)-h(k)(k-1-t)}{k-1+t(k-2)} = \frac{t(h(k)+1)+(k-1)p(k)}{k-1+t(k-2)},$$

so

$$p(k) < \frac{t(h(k)+1)}{t(k-2)} = \frac{1+h(k)}{k-2},$$

contradicting (6).

So, for every extreme block B of T we have $d_T(x_B) = k-1$. Pick an extreme block B and let C be the odd cycle adjacent to B. We claim that all but at most one vertex of C is in an endblock. Since B is the end of a longest path, C cannot have two noncutvertices that are both not in endblocks, for then we could get a longer path. So, to prove our claim, it will suffice to show that every vertex of C is a cut-vertex. Suppose $v \in V(C)$ is not a cut-vertex. Then $d_T(v) = 2$ and hence by minimality of $|T|$

$$2\left\|T\right\| - 4 \le (k-3+p(k))\left(|T|-1\right) + f(k) + h(k)q(T-v),$$

Since $q(T-v) = q(T)$, the fact that T is a counterexample implies

$$4 > k-3+p(k),$$

a contradiction since $k \ge K \ge 7$ and $p(k) > 0$. So, we have shown that all but one vertex of C is in an endblock. Let $B_1 = B, B_2, \ldots, B_t$ be the endblocks adjacent to C.

First, suppose $V(C) \cup \bigcup_{i \in [t]} V(B_i) = V(T)$. Then

$$2\,\|T\| = t + t(k-2)(k-3)$$

and

$$|T| = t(k-2),$$

so since T is a counterexample and $q(T) = 0$, we have

$$t > p(k)t(k-2) + f(k),$$

which is

$$p(k) < \frac{t - f(k)}{t(k-2)} = \frac{1}{k-2} - \frac{f(k)}{t(k-2)},$$

which contradicts (5) when $f(k) \geq 0$. When $f(k) < 0$, the worst case is when $t = 3$ and hence

$$p(k) < \frac{3 - f(k)}{3(k-2)},$$

which contradicts (8).

Hence $V(C) \cup \bigcup_{i \in [t]} V(B_i) \neq V(T)$. Since $\delta(T) \geq 3$, C must be adjacent to a non-endblock D.

First, suppose D is an odd cycle. By extremality, every block X adjacent to D with $V(X) \cap V(D) = V(C) \cap V(D)$ is an odd cycle. Let $C_1 = C, C_2, \ldots, C_s$ be these blocks. Each C_i is adjacent to $|C_i| - 1$ endblocks just like C. Let $C' = \sum_{i \in [s]} (|C_i| - 1)$. Remove all these blocks and their adjacent endblocks to form T'. Then, by minimality of $|T|$ and since $q(T') = q(T)$, we have

$$(k-2)(k-3)C' + C' + s + 2 > (k - 3 + p(k))(C'(k-2) + 1),$$

which gives

$$C' + s + 2 > k - 3 + p(k)(C'(k-2) + 1),$$

which is

$$p(k) < \frac{C' + s + 5 - k}{C'(k-2) + 1} \leq \frac{C' + s + 5 - k}{(C'+1)(k-2)} \leq \frac{\frac{4}{3}C'}{(C'+1)(k-2)} < \frac{\frac{4}{3}}{k-2},$$

contradicting (5).

So, D is not an odd cycle. Then D is not adjacent to an endblock. By extremality of B, all but at most one block adjacent to D is adjacent to an endblock. Let $C_1 = C, C_2, \ldots, C_s$ be the blocks adjacent to D that are also adjacent to an endblock. Then, since C_i is also the penultimate vertex on a longest path in the block-tree, each C_i is an odd cycle and all but one vertex of C is in an endblock as above. Let $C' = \sum_{i \in [s]} (|C_i| - 1)$. Note $1 \leq s \leq \frac{C'}{2}$. Let Q be the vertices of T that are in a block adjacent to some C_i.

Suppose $Q = V(T)$ and $D = K_r$ for $2 \leq p \leq k-2$. Then, since T is a counterexample and $q(T) = 0$, we have

$$(k-2)(k-3)C' + C' + s + (r)(r-1) > (k - 3 + p(k))(C'(k-2) + r) + f(k),$$

which gives,

$$C' + s + (r)(r - k + 2) > p(k)(C'(k-2) + r) + f(k),$$

so,

$$p(k) < \frac{C' + s + r(r - k + 2) - f(k)}{C'(k-2) + r},$$

which achieves its maximum when $r = k - 2$, so we have

$$p(k) < \frac{C' + s - f(k)}{(C'+1)(k-2)} \leq \frac{\frac{3C'}{2} - f(k)}{(C'+1)(k-2)} \leq \frac{3 - f(k)}{3(k-2)},$$

contradicting (8).

Otherwise, there is a block F adjacent to D but not adjacent to an endblock. Let y be the vertex that D and F share. We apply minimality of T to $T' = T - (Q \setminus \{y\})$. Then, since T is a counterexample and $q(T') = q(T)$, we have

$$|D|\left(|D| - 1\right) + C' + s + C'(k-2)(k-3) > (k - 3 + p(k))(|D| - 1 + C'(k-2)),$$

simplifying, we get

$$(|D| - (k-3))(|D| - 1) + C' + s > p(k)(C'(k-2) + |D| - 1),$$

which is

$$p(k) < \frac{(|D| - (k-3))(|D| - 1) + C' + s}{C'(k-2) + |D| - 1}.$$

Suppose $|D| \leq k - 3$. The worst case is when $|D| = k - 3$, using $s \leq \frac{C'}{2}$ we get

$$p(k) < \frac{\frac{3}{2}C'}{C'(k-2) + k - 4} = \frac{\frac{3}{2}C'}{(C'+1)\left(k - 4 + 2\frac{C'}{C'+1}\right)} \leq \frac{3}{2(k - 4 + \frac{4}{3})} \leq \frac{2}{k-2},$$

where in the penultimate inequality we used $C' \geq 2$. This contradicts (5).

Hence, we must have $D = K_{k-2}$. Suppose $F = K_2$ and let $T^* = T - Q$. Then $q(T^*) \leq q(T) + 1$ and applying minimality of $|T|$, we get

$$1 + (k-2)(k-3) + C' + s + C'(k-2)(k-3) > (k - 3 + p(k))(|D| + C'(k-2)) - h(k),$$

simplifying, we get

$$1 + C' + s > p(k)(k - 2 + C'(k-2)) - h(k),$$

so,

$$p(k) < \frac{1 + C' + s + h(k)}{(C'+1)(k-2)} = \frac{1}{k-2} + \frac{s + h(k)}{(C'+1)(k-2)} \leq \frac{\frac{C'}{2} + h(k)}{(C'+1)(k-2)}.$$

Since $C' \geq 2$, this gives

$$p(k) < \frac{1 + h(k)}{3(k-2)},$$

which contradicts (6).

So, F must be an odd cycle. All but at most one block adjacent to F is either an endblock, adjacent to an endblock or at distance two from an endblock. Let Q be such a block adjacent to F. If Q is an endblock, then $Q = K_{k-2}$. If Q is at distance two from an endblock, then the intervening block must be an odd cycle by extremality, but then by then $Q = K_{k-2}$ for otherwise we can run one of the above cases on that subtree instead. If Q is adjacent to an endblock, then Q cannot be K_2 since then we can remove Q and its endblock without increasing $q(T)$. So, in that case, Q is an odd cycle. Let $V(D) \cap V(F) = \{z\}$ and let $w \in V(F) \setminus \{z\}$ be adjacent to z and in a block Q that is at distance at most two from an endblock. So, Q is either an endblock K_{k-2}, a K_{k-2} of the form of D, or an odd cycle. Let S be the graph formed from T by removing D and Q and their subtrees. Let $C_1 = C, C_2, \ldots, C_s$ be the blocks adjacent to D that are also adjacent to an endblock. Let $C' = \sum_{i\in[s]} (|C_i| - 1)$. We have $q(S) = q(T)$.

First, suppose $Q = K_{k-2}$. Then, by minimality of $|T|$, we get

$$(k-2)(k-3)C' + C' + s + 2(k-3)(k-2) + 1 > (k-3+p(k))(C'+2)(k-2),$$

hence

$$C' + s + 1 > p(k)(C'+2)(k-2),$$

which gives

$$p(k) < \frac{C'+s+1}{(C'+2)(k-2)} = \frac{1}{k-2} + \frac{s-1}{(C'+2)(k-2)} \le \frac{1}{k-2} + \frac{\frac{C'}{2}-1}{(C'+2)(k-2)} < \frac{\frac{3}{2}}{k-2},$$

contradicting (5).

Instead, suppose Q is a K_{k-2} of the form of D. Let $H_1, H_2, \ldots, H_r$ be the blocks adjacent to Q that are also adjacent to an endblock. Let $H' = \sum_{i\in[r]} (|H_i| - 1)$. Then, by minimality of $|T|$, we get

$$(k-2)(k-3)(C'+D')+C'+D'+s+r+2(k-3)(k-2)+1 > (k-3+p(k))(C'+D'+2)(k-2),$$

hence

$$C' + D' + s + r + 1 > p(k)(C'+D'+2)(k-2),$$

which is

$$p(k) < \frac{C'+D'+s+r+1}{(C'+D'+2)(k-2)} = \frac{1}{k-2} - \frac{s+r-1}{(C'+D'+2)(k-2)} < \frac{2}{k-2},$$

contradicting (5).

So, Q is an odd cycle and all of the blocks adjacent to Q (besides possibly F) are K_{k-2} endblocks (this is because, by extremality, the only other thing they could be is an odd cycle, but then the same proof that showed D is K_{k-2} gives a contradiction). Since $\delta(T) \ge 3$, all but at most one vertex in Q is adjacent to an endblock. Now we are in a case similar to when D was an odd cycle before, just with length one less. Remove Q and its subtree to get S'. Let $A_1, A_2, \ldots, A_u$ be the endblocks adjacent to Q and put $A' = \sum_{i\in[u]} (|A_i| - 1)$. Then $q(S') = q(T)$ and hence by minimality we have,

$$(k-2)(k-3)A' + A' + u + (k-2)(k-3) + 2 > (k-3+p(k))((k-2)(A'+1)),$$

so,

$$A' + u + 2 > p(k)((k-2)(A'+1)),$$

which is

$$p(k) < \frac{A' + u + 2}{(k-2)(A'+1)} \leq= \frac{1}{k-2} + \frac{u+1}{(k-2)(A'+1)} \leq \frac{1}{k-2} + \frac{\frac{A'}{2}+1}{(k-2)(A'+1)} < \frac{2}{k-2},$$

contradicting (5).

□

Now some examples of using Lemma 3.1. What happens if we take $h(k) = 0$? Then, by (7), we need $(k-1)p(k) \geq k+1$ and hence $p(k) \geq 1 + \frac{2}{k-1}$. Taking $p(k) = 1 + \frac{2}{k-1}$, (3) requires $f(k) \geq -2$. Using $f(k) = -2$, all of the other conditions are satisfied and we conclude $2\|T\| \leq \left(k - 2 + \frac{2}{k-1}\right)|T| - 2$ for every $T \in \mathcal{T}_k$ when $k \geq 4$. This is a slight refinement of Gallai's Lemma 2.2.

Instead, let's make $p(k)$ as small as Lemma 3.1 will let us. By (6), $h(k) \leq (k-2)p(k) - 2$, plugging this in to (7) and solving we get $p(k) \geq \frac{3k-5}{k^2-4k+5}$. Now $\frac{3k-5}{k^2-4k+5} \geq \frac{3}{k-2}$ for $k \geq 7$, so $p(k) = \frac{3k-5}{k^2-4k+5}$ satisfies (5). With $h(k) = \frac{k(k-3)}{k^2-4k+5}$, (6) and (7) are also satisfied. Now with $f(k) = -\frac{2(k-1)(2k-5)}{k^2-4k+5}$, all the conditions of Lemma 3.1 are satisfied and hence we have the following.

Corollary 3.2. *For $k \geq 7$ and $T \in \mathcal{T}_k$, we have*

$$2\|T\| \leq \left(k - 3 + \frac{3k-5}{k^2-4k+5}\right)|T| - \frac{2(k-1)(2k-5)}{k^2-4k+5} + \frac{k(k-3)}{k^2-4k+5}q(T).$$

If we put the Kostochka-Stiebitz bound on $\sigma(T)$ into this form we get the following.

Lemma 3.3 (Kostochka-Stiebitz)**.** *For $k \geq 7$ and $T \in \mathcal{T}_k$, we have*

$$2\|T\| \leq \left(k - 3 + \frac{4(k-1)}{k^2-3k+4}\right)|T| - \frac{4(k^2-3k+2)}{k^2-3k+4} + \frac{k^2-3k}{k^2-3k+4}q(T).$$

Note that $\frac{3k-5}{(k-5)(k-1)} < \frac{4(k-1)}{k^2-3k+4}$ for $k \geq 13$.

3.1 Analyzing the discharging

Our discharging procedure gives charge ϵ to a component T for every incident edge not ending in a K_{k-1}. The number of such edges is exactly

$$A(T) := -q(T) + \sum_{v \in V(T)} k - 1 - d_T(v) = (k-1)|T| - 2\|T\| - q(T).$$

Suppose we have a bound of the form

$$2\|T\| \leq (k - 3 + p(k))|T| + f(k) + h(k)q(T).$$

So, we get

$$A(T) \geq (2 - p(k))\,|T| - f(k) - (h(k)+1)q(T).$$

We will use $\gamma = (h(k)+1)\epsilon$ in order to make the $q(T)$ term cancel. That happens because T receives charge on all but at most two of its non-separating vertices in a K_{k-1}; that is, in discharging steps 2 and 3, T receives charge at least $\gamma \max\{0, q(G) - 2\}$. Hence in total T receives charge at least

$$\epsilon A(T) + \gamma(q(G) - 2) = \epsilon\,(2 - p(k))\,|T| - \epsilon\,(f(k) + 2(h(k)+1))\,.$$

To simplify things, let's impose the requirement $f(k) + 2(h(k)+1) \leq 0$. Then T receives charge at least

$$\epsilon\,(2 - p(k))\,|T|\,.$$

We want the k-vertices to end with enough charge, the worst case is when

$$1 - (3\gamma + (k-3)\epsilon) = \epsilon\,(2 - p(k))\,,$$

and thus

$$\epsilon = \frac{1}{k + 2 + 3h(k) - p(k)},$$

$$\gamma = \frac{h(k)+1}{k + 2 + 3h(k) - p(k)}.$$

It remains to check that the $(k+1)^+$-vertices don't give away too much charge. Let v be a $(k+1)^+$-vertex, then v ends with charge at least

$$d(v) - \gamma d(v) = (1-\gamma)d(v) \geq (1-\gamma)(k+1) = (k+1)\frac{k + 1 + 2h(k) - p(k)}{k + 2 + 3h(k) - p(k)},$$

so we need

$$(k+1)\frac{k + 1 + 2h(k) - p(k)}{k + 2 + 3h(k) - p(k)} \geq k - 1 + \frac{2 - p(k)}{k + 2 + 3h(k) - p(k)},$$

simplifying, we get that we need

$$p(k) + (k-5)h(k) \leq k + 1.$$

Let's just add this as another requirement, it will be easily satisfied by the functions we want to use. We have proved the following.

Theorem 3.4. *Let $K \geq 7$ and $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$, $h\colon \mathbb{N} \to \mathbb{R}$ be functions satisfying*

- $f(k) + 2(h(k)+1) \leq 0$*; and*
- $p(k) + (k-5)h(k) \leq k + 1.$

If for all $k \geq K$ and $T \in \mathcal{T}_k$ we have

$$2\|T\| \leq (k - 3 + p(k))\,|T| + f(k) + h(k)q(T),$$

then for $k \geq K$ and $G \neq K_k$ a k-AT-critical graph, we have

$$d(G) \geq k - 1 + \frac{2 - p(k)}{k + 2 + 3h(k) - p(k)}.$$

As a first test, suppose $p(k) = 1 - \frac{2}{k-1}$, $f(k) = -2$ and $h(k) = 0$. Then the hypotheses of Theorem 3.4 are satisfied with $K = 7$ and we get Gallai's bound $d(G) \geq k - 1 + \frac{k-3}{k^2-3}$.

Now, let's try the Kostochka-Stiebitz bound, that is, $p(k) = \frac{4(k-1)}{k^2-3k+4}$, $f(k) = -\frac{4(k^2-3k+2)}{k^2-3k+4}$ and $h(k) = \frac{k^2-3k}{k^2-3k+4}$. Again, the hypotheses of Theorem 3.4 are satisfied with $K = 7$ and we get

$$d(G) \geq k - 1 + \frac{2(k-2)(k-3)}{(k-1)(k^2+3k-12)}.$$

This is exactly equal to the bound in the paper with Hal!

Now, let's try our bound in Lemma 3.2, that is, $p(k) = \frac{3k-5}{k^2-4k+5}$, $f(k) = -\frac{2(k-1)(2k-5)}{k^2-4k+5}$ and $h(k) = \frac{k(k-3)}{k^2-4k+5}$. The hypotheses of Theorem 3.4 are satisfied with $K = 7$ and we get

$$d(G) \geq k - 1 + \frac{(k-3)(2k-5)}{k^3+k^2-15k+15}.$$

This is better than the bound with Hal for $k \geq 7$.

Possible improvements:

1. Use a better bound on average degree of Gallai trees. i would like to find the best possible family in the form here. How does this bound compare to the hand waiving one in the other document?

2. In the discharging, the k-vertices lost 3γ even though they had degree two in Q because of the possibility of two edges into one component. Can we get this to 2γ somehow, like maybe we can order our picking so that no vertex is picked before the component where it has two edges has been removed.

3. Related to the previous item, improved reducible configurations, a less restrictive condition in Lemma 4.2 taking into account the two edges to a component issue.

4 Reducible Configurations

Definition 1. A graph G is *AT-reducible* to H if H is a nonempty induced subgraph of G which is f_H-AT where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not AT-reducible to any nonempty induced subgraph, then it is *AT-irreducible.*

This first lemma tells us how a single high vertex can interact with the low vertex subgraph. This is the version Hal and i used, it (and more) follows from the classification in "mostlow".

Lemma 4.1. *Let $k \geq 5$ and let G be a graph with $x \in V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*

2. *$G - x$ has t components $H_1, H_2, \ldots, H_t$, and all are in $\mathcal{T}_k$; and*

3. *$d_G(v) \leq k - 1$ for all $v \in V(G - x)$; and*

4. *$\left|N(x) \cap W^k(H_i)\right| \geq 1$ for $i \in [t]$; and*

5. *$d_G(x) \geq t + 2$.*

Then G is f-AT where $f(x) = d_G(x) - 1$ and $f(v) = d_G(v)$ for all $v \in V(G - x)$.

To deal with more than one high vertex we need the following auxiliary bipartite graph. For a graph G, $\{X, Y\}$ a partition of $V(G)$ and $k \geq 4$, let $\mathcal{B}_k(X, Y)$ be the bipartite graph with one part Y and the other part the components of $G[X]$. Put an edge between $y \in Y$ and a component T of $G[X]$ if and only if $N(y) \cap W^k(T) \neq \emptyset$. The next lemma tells us that we have a reducible configuration if this bipartite graph has minimum degree at least three.

Lemma 4.2. *Let $k \geq 7$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*

2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*

3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*

4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $\delta(\mathcal{B}) \geq 3$.*

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.

We also have the following version with asymmetric degree condition on $\mathcal{B}$. The point here is that this works for $k \geq 5$. As we'll see in the next section, the consequence is that we trade a bit in our size bound for the proof to go through with $k \in \{5, 6\}$.

Lemma 4.3. *Let $k \geq 5$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*

2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*

3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*

4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $d_{\mathcal{B}}(y) \geq 4$ for all $y \in Y$ and $d_{\mathcal{B}}(T) \geq 2$ for all components T of $G - Y$.*

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.