**embodied graph theory**

# Contents

# preface

this comes prior to the face.

# graphs

This book is about groups of people. Not about individual people, but about the structure of relationships in the group. Each person wears a belt with many loops. There are ropes with caribiners on both ends for attaching two people by their belt-loops. We say that two people are *joined* if they are connected by a rope. A *graph* is a group of *people* together with some joinings. If $G$ is a graph, then $P(G)$ is its group of people and $R(G)$ its collection of ropes. We write $|G|$ for the number of people in $P(G)$ and $\|G\|$ for the number of ropes in $R(G)$. The group of people to which $p$ is joined is her *neighborhood*, written $N(v)$. For the size of $v$'s neighborhood $|N(v)|$, we write $d(v)$ and call this the *degree* of $v$.

joined

$P(G)$, $R(G)$

$|G|$, $\|G\|$

neighborhood

$N(v)$

$d(v)$, degree

# grouping people

The entire book concerns one simple task: we want to group the people of a given graph so that joined people are in different groups. With sufficiently many groups and no preferences about what the groupings should look like, this is easy, we just put each person in her own group. Things get interesting when we ask how few different groups we can use. We are definitely going to need at least zero groups and that will only do for the graph with no people at all. Given one group, we can handle all graphs with no joins. With two groups, we can do any path and any cycle with an even number of people. But, we can't handle a triangle or any other cycle with an odd number of people. In fact, odd cycles are really the only thing that will prevent us from using just two groups. A graph $H$ is a *subgraph* of a graph $G$, written $H \subseteq G$ if $P(H) \subseteq P(G)$ and $R(H) \subseteq R(G)$. When $H \subseteq G$, we say that $G$ *contains* $H$. If $p \in P(G)$, then $G - p$ is the graph we get by removing $p$ from the group along with all the ropes attached to her.

subgraph, $\subseteq$

contains
$G - v$

It will be convenient to have all the members of a group wear the same color shirt. So, we might have the red shirted group and the blue shirted group, etc. A graph is $k$-colorable if we can group its people into (at most) $k$ groups such that joined people are in different groups. A 0-colorable graph is *empty*, a 1-colorable graph is *ropeless* and a 2-colorable graph is *bipartite*.

$k$-colorable

empty
ropeless
bipartite

Theorem 1. *A graph is* 2*-colorable just in case it contains no odd cycle.*

Proof. A graph containing an odd cycle clearly can't be 2-colored. For the other implication, suppose there is a graph that is not 2-colorable and doesn't contain an odd cycle. Then we may pick such a graph $G$ with $|G|$ as small as possible. Surely, $|G| > 0$, so we may pick $v \in P(G)$. If $x, y \in N(v)$, then $x$ is not joined to $y$ since then $xyz$ would be an odd cycle. So we can construct a graph $H$ from $G$ by removing $v$ and identifying all of $N(v)$ to a new person $x_v$. Any odd cycle in $H$ would contain $x_v$ and hence give rise to an odd cycle in $G$ passing through $v$. So $H$ contains no odd cycle. Since $|H| < |G|$, applying the theorem to $H$ gives a 2-coloring of $H$, say into the red and blue groups where $x_v$ is in the red group. But this gives a 2-coloring of $G$ by putting all people in $N(v)$ in the red group and putting $v$ in the blue group, a contradiction. □

Well, this is embarrassing, coloring appears to be easy. Fortunately, things get more interesting when we move up to three colors.

Theorem 2. *3-coloring is hard supposing other things we think are hard are actually hard.*

Proof. We need a concise proof of this without having to introduce too much background. Please submit a pull request on GitHub. □

**basic estimates**

chromatic number $\chi(G)$

Even though finding the minimum number of colors needed to color a graph is hard in general (supposing it is), we can still look for lower and upper bounds on this value. The *chromatic number* $\chi(G)$ of a graph $G$ is the smallest $k$ for which $G$ is $k$-colorable. The simplest thing we can do is give each person a different colored shirt.

THEOREM 3. *If $G$ is a graph, then $\chi(G) \leq |G|$.*

complete

maximum degree $\Delta(G)$

The only graphs that attain the upper bound in Theorem 3 are the *complete* graphs; those in which any two people are joined. We can usually do much better by just arbitrarily putting colored shirts on people, reusing colors when we can. The *maximum degree* $\Delta(G)$ of a graph $G$ is the largest degree of any person in $G$; that is

$$\Delta(G) := \max_{v \in V(G)} d(v).$$

We are going to need smooth language to talk about putting various colored shirts on people. Let's say that "to color a person red" means to put a red shirt on them, etc.

THEOREM 4. *If $G$ is a graph, then $\chi(G) \leq \Delta(G) + 1$.*

PROOF. Suppose there is a graph $G$ that is not $(\Delta(G)+1)$-colorable. Then we may pick such a graph $G$ with as few people as possible. Surely $G$ has at least one person, so we may pick $v \in V(G)$. Then $|G - v| < |G|$ and $\Delta(G - v) \leq \Delta(G)$, so applying the theorem to $G - v$ gives a $(\Delta(G-v)+1)$-coloring of $G - v$. But $v$ has at most $\Delta(G)$ neighbors, so there is some color, say red, not used on $N(v)$, coloring $v$ red gives a $(\Delta(G)+1)$-coloring of $G$, a contradiction. □

clique

$\omega(G)$

Both complete graphs and odd cycles attain the upper bound in Theorem 4. Theorem 1 says we can do better for graphs that don't contain odd cycles. We can also do better for graphs that don't contain large complete subgraphs. A group of people $S$ in a graph $G$ is a *clique* if the people in $S$ are pairwise joined. The *clique number* of a graph $G$, written $\omega(G)$, is the number of people in a largest clique in $G$.

THEOREM 5. *If $G$ is a graph, then $\chi(G) \geq \omega(G)$.*

independent

$\alpha(G)$

A group of people $S$ in a graph $G$ is *independent* if the people in $S$ are pairwise non-joined. The *independence number* of a graph $G$, written $\alpha(G)$, is the number of people in a largest independent set in $G$.

THEOREM 6. *If $G$ is a graph with $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$, then $\chi(G) \leq \Delta(G)$.*

PROOF. Suppose there is a graph $G$ with $\Delta(G) \geq 3$ and $\omega(G) \leq \Delta(G)$ that is not $\Delta(G)$-colorable. Then we may pick such a graph $G$ with as few people as possible. Let $S$ be a maximal independent set in $G$. Since $S$ is maximal, every person in $G - S$ has a neighbor in $S$, so $\Delta(G) > \Delta(G - S)$. If red is an unused color in a $\chi(G-S)$-coloring of $G - S$, then by coloring all people in $S$ red we get a $(\chi(G-S)+1)$-coloring of $G$. So, $\Delta(G) + 1 \leq \chi(G) \leq \chi(G-S)+1$. We conclude $\chi(G-S) > \Delta(G-S)$ and thus $\Delta(G) = \chi(G-S) = \Delta(G-S)+1$ by Theorem 4. Since $|G - S| < |G|$, applying the theorem to $G - S$ shows that $\Delta(G-S) < 3$ or

$\Delta(G-S) < \omega(G-S)$. So, either $\chi(G-S) = \Delta(G) = 3$ or $\omega(G-S) \geq \Delta(G)$. In the former case, let $X$ be the people group of an odd cycle in $G-S$ guaranteed by Theorem 1. In the latter case, let $X$ be a $\Delta(G)$-clique in $G-S$.

Since $S$ is maximal and $\omega(G) \leq \Delta(G)$, there are $x_1, x_2 \in X$ and $y_1, y_2 \in S$ such that $x_1$ is joined to $y_1$ and $x_2$ is joined to $y_2$. Construct a graph $H$ from $G-X$ by adding the rope $y_1y_2$. Since $|H| < |G|$, applying the theorem to $H$ shows that $\omega(H) > \Delta(G)$ or $\chi(H) \leq \Delta(G)$. Suppose $\chi(H) \leq \Delta(G)$. Then there is a $\Delta(G)$-coloring of $G-X$ where $y_1$ and $y_2$ receive different colors, say red and blue respectively. Pick the first person $z$ in a shortest path $P$ from $x_1$ to $x_2$ in $X$ that has a blue colored neighbor in $V(H)$. Each person in $X$ has $\Delta(G)-1$ neighbors in $X$ and hence at most one neighbor in $V(H)$. So, $z \neq x_1$ since $x_1$ already has a red colored neighbor in $V(H)$. Let $w$ be be the person preceding $z$ on $P$. Then $w$ has no blue colored neighbor. Since $X$ is the person group of a cycle or a complete graph, there is a path $Q$ from $w$ to $z$ passing through every person of $X$. Color $w$ blue and then proceed along $Q$, coloring one person at a time. Since each person we encounter before we get to $z$ has at most $\Delta(G)-1$ colored neighbors, we always have an available color to use. But, $z$ is joined to both $w$ and another blue colored person in $V(H)$, so there is an available color for $z$ as well. This gives a $\Delta(G)$-coloring of $G$, a contradiction.

So, $\omega(H) > \Delta(G)$. In particular, $y_1$ and $y_2$ each have exactly one neighbor in $X$ and $\Delta(G)-1$ neighbors in the same $\Delta(G)-1$ clique $A$ in $G-X$. Since $S$ is maximal and $|X| \geq 3$, there must be joined $x_3 \in X \setminus \{x_1, x_2\}$ and $y_3 \in S \setminus \{y_1, y_2\}$. Applying the same argument with $x_3, y_3$ in place of $x_2, y_2$ shows that $y_1$ and $y_3$ each have exactly one neighbor in $X$ and $\Delta(G)-1$ neighbors in the same $\Delta(G)-1$ clique $B$ in $G-X$. Now $|A \cap B| = |A| + |B| - |A \cup B| \geq 2(\Delta(G)-1) - d(y_1) \geq \Delta(G) - 2 > 0$. But there can't be a person in $A \cap B$ since she would be joined to $y_1, y_2, y_3$ as well as $\Delta(G)-2$ people in $A$ and thus have degree greater than $\Delta(G)$, a contradiction.

# edge coloring

It is also useful to consider coloring the edges of a graph so that incident edges receive different colors. This appears to be at odds with our previous claim that this book was only about coloring vertices of graph; fortunately, edge coloring is just a special case of vertex coloring. If $G$ is a graph, the *line graph* of $G$, written $L(G)$ is the graph with vertex set $E(G)$ where two edges of $G$ are adjacent in $L(G)$ if they are incident in $G$. Coloring the edges of $G$ is equivalent to coloring the vertices of $L(G)$.

For graphs with maximum degree zero (that is, no edges at all), we can get by with zero colors. With just one color we can edge color any graph with maximum degree at most one. We will definitely always need at least $\Delta(G)$ colors to edge color a graph $G$. Could we be so fortunate that the pattern continues and we can edge color any graph $G$ with only $\Delta(G)$-colors? Not quite, but we can do so for bipartite (2-colorable) graphs. A graph is $k$-edge-colorable if we can color its edges with (at most) $k$ colors such that incident edges receive different colors. A color $c$ us *used* at a vertex $v$ of $G$ if an edge incident to $v$ in $G$ is colored with $c$. Otherwise, $c$ is *available* at $v$. A *path* in $G$ is a sequence of pairwise distinct vertices $x_1x_2\cdots x_k$ such that $x_i$ is adjacent to $x_{i+1}$ for $i \in [k-1]$.

used
available
path

THEOREM 7. *If $G$ is a bipartite graph, then $G$ is $\Delta(G)$-edge-colorable.*

PROOF. Suppose there is a graph $G$ that is not $\Delta(G)$-edge-colorable. Then we may pick such a graph $G$ with $\|G\|$ as small as possible. Now $\|G\| > 0$, since we can surely color a graph with zero edges using zero colors. Let $xy$ be an edge in $G$. Since $\|G - xy\| < \|G\|$, applying the theorem to $G - xy$ gives an edge coloring of $G - xy$ using at most $\Delta(G)$ colors. Now each of $x$ and $y$ are incident to at most $\Delta(G) - 1$ edges in $G - xy$ and $G$ has no $\Delta(G)$-edge-coloring, so there is a color red available at $x$ and a different color blue available at $y$. There is a unique maximal length path $P$ starting at $x$ with edges alternately colored blue and red. Let $z$ be the last vertex of $P$. Since $P$ alternates between two colors, $P$ has even length. In particular, $z \neq y$ since $G$ does not contain an odd cycle by Theorem 1. But then we get a $\Delta(G)$-edge-coloring of $G$ by swapping the colors red and blue along $P$ and coloring $xy$ blue, a contradiction. □

It may come as a surpise that even though we might need more than $\Delta(G)$ colors to edge color a graph $G$, we will only ever need at most one extra color. For bipartite graphs we were able to repair an almost correct coloring by swapping colors along a path because we had control over where this path ended. In the general case we don't have the same control over a path between two vertices, but we can exert some measure of control over paths leaving and entering a larger structure. The larger structure we use here is the whole neighborhood of a vertex.

THEOREM 8. *If $G$ is a graph, then $G$ is $(\Delta(G)+1)$-edge-colorable.*

PROOF. Suppose there is a graph $G$ that is not $(\Delta(G)+1)$-edge-colorable. Then we may pick such a graph $G$ with $\|G\|$ as small as possible. Now $\|G\| > 0$, since we can surely color a graph with zero edges using at most one color. Let $xy$ be an edge in $G$. Since $\|G - xy\| < \|G\|$, applying the theorem to $G - xy$ gives an edge coloring $\pi$ of $G - xy$ using at most $\Delta(G)+1$ colors. Let's name the neighbors of $x$, say $y_0, y_1, \ldots, y_k$ where $y_0 = y$ and $k < \Delta(G)$. By symmetry, we may assume that $1, \ldots, a$ are available at $x$ and $a+1, \ldots, b$ are available at $y_0$ where $a \geq 2$ and $b - a \geq 2$. Also by symmetry, we may assume that $\pi(xy_i) = a + i$ for $i \in [k]$. Now each of $y_i$ for $i \in [k]$ has at least one available color $c_i$. If $c_i \in \{1, \ldots, a\}$ for some $i \in [b-a]$, then we get a $(\Delta(G)+1)$-coloring of $G$ by coloring $xy_i$ with $c_i$ and $xy_0$ with $\pi(xy_i)$, a contradiction.

□

**hardness**

We now know that every graph $G$ can be edge colored with either $\Delta(G)$ or $\Delta(G)+1$ colors. So, edge coloring is basically trivial, right? Furtunately, no it isn't, the collection of graphs requiring $\Delta(G)+1$ colors is very rich. Another way to say this, is that it is a hard problem to decide whether or not edge coloring a given graph $G$ requires $\Delta(G)+1$ colors.

THEOREM 9. *Deciding whether or not edge coloring a given graph $G$ requires $\Delta(G)+1$ colors is hard supposing other things we think are hard are actually hard.*

PROOF. We need a concise proof of this without having to introduce too much background. Please submit a pull request on GitHub. □

# vertex coloring, again

## list coloring

## online list coloring

## kernel tools

**degree again.**

**maximum independent covers.**

## polynomial tools

**combinatorial nullstellensatz.**

**coefficient formulae.**

**a combinatorial interpretation.**

## edge coloring, again

**fans as a greedy strategy**

**acceptable paths**

**acceptable trees**

**edge list coloring**

**more kernel method.**

**2-edge-coloring.**

**improved degree theorem.**

**a hint of quasiline and claw-free graphs.**

# shuffle tool

**destroying non-complete**

**shuffling with more rules**

**looking at the entire recoloring digraph**

## independent transversalls

**randomly**

**Haxell's tool**

**triangles within triangles are neat.**

**big maximum cliques intersect in simple ways**

**going lopsided**

# vertex transitive graphs

## strong coloring

## medium clique implies big clique

**potential potential tool**