A common generalization of Hall's theorem and Vizing's edge-coloring theorem

landon rabern

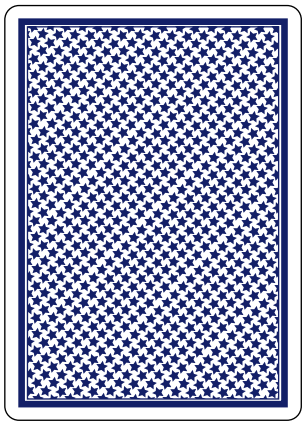
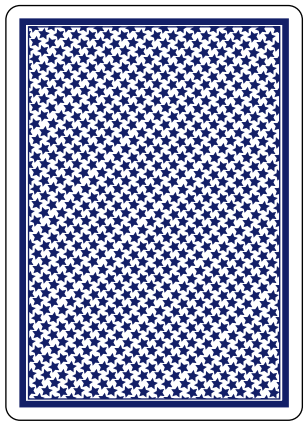
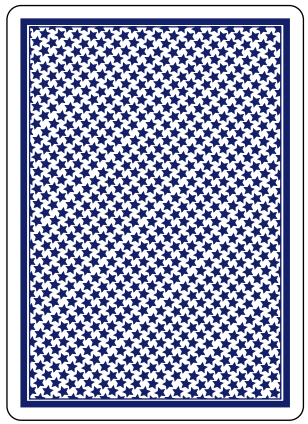
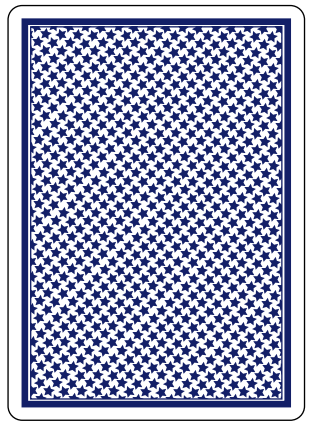
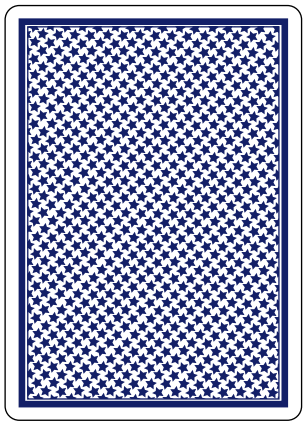
LBD Data

Miami University Colloquium
November 6, 2014

some card games

the simplest variation

- two players, Dealer and Player

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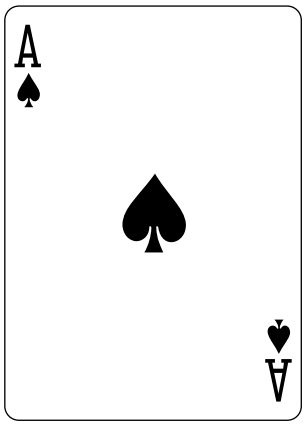

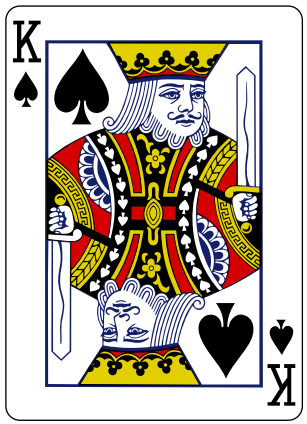

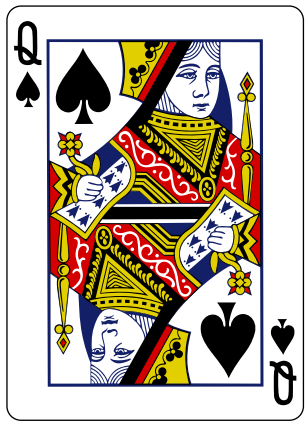

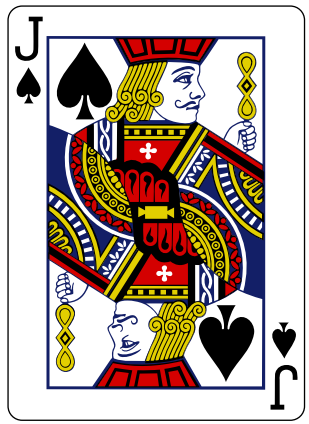

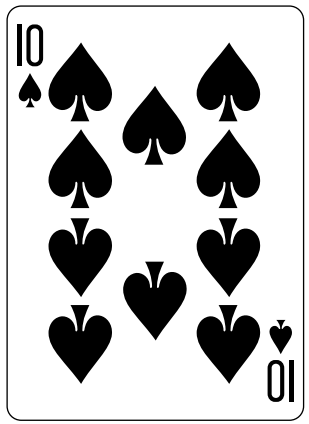

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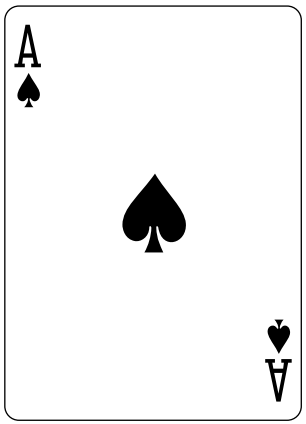

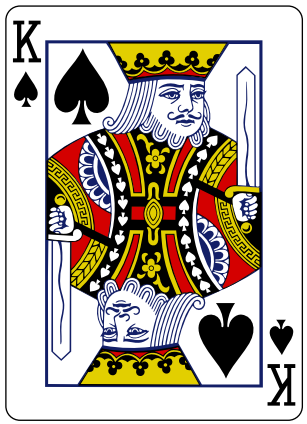

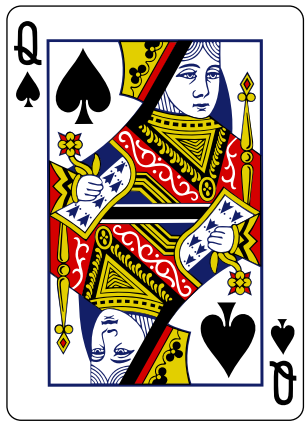

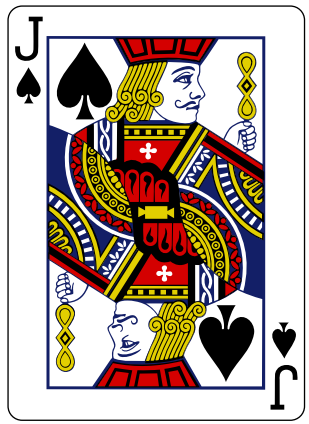

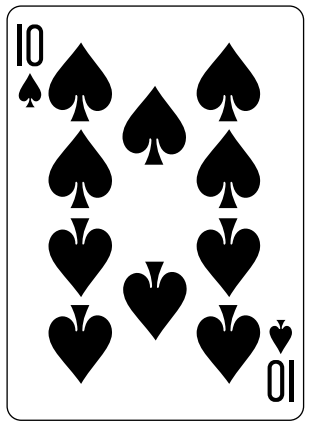

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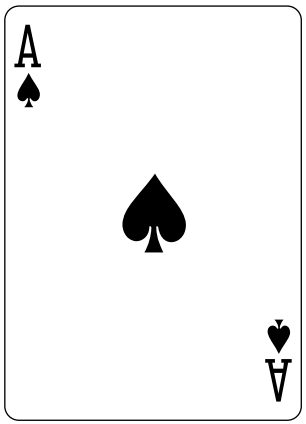
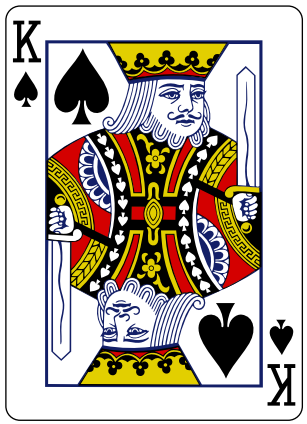
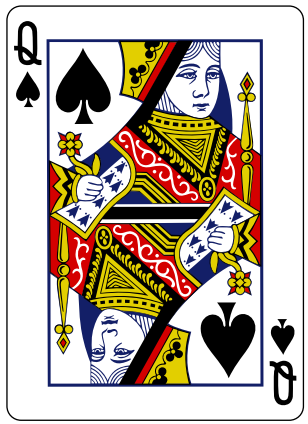
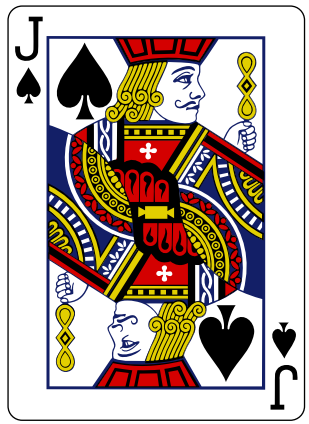
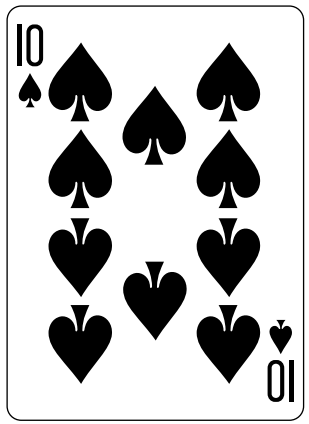




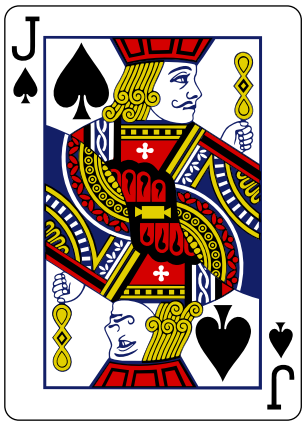


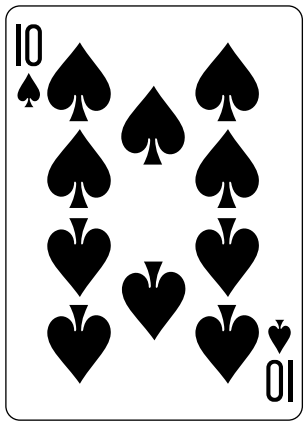


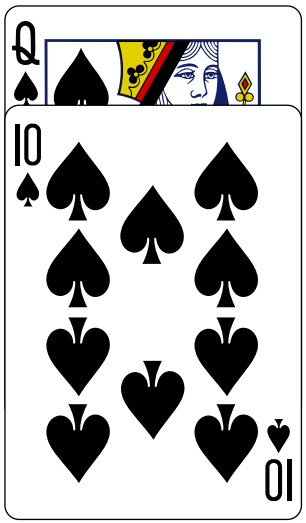


some card games

example, a Player win







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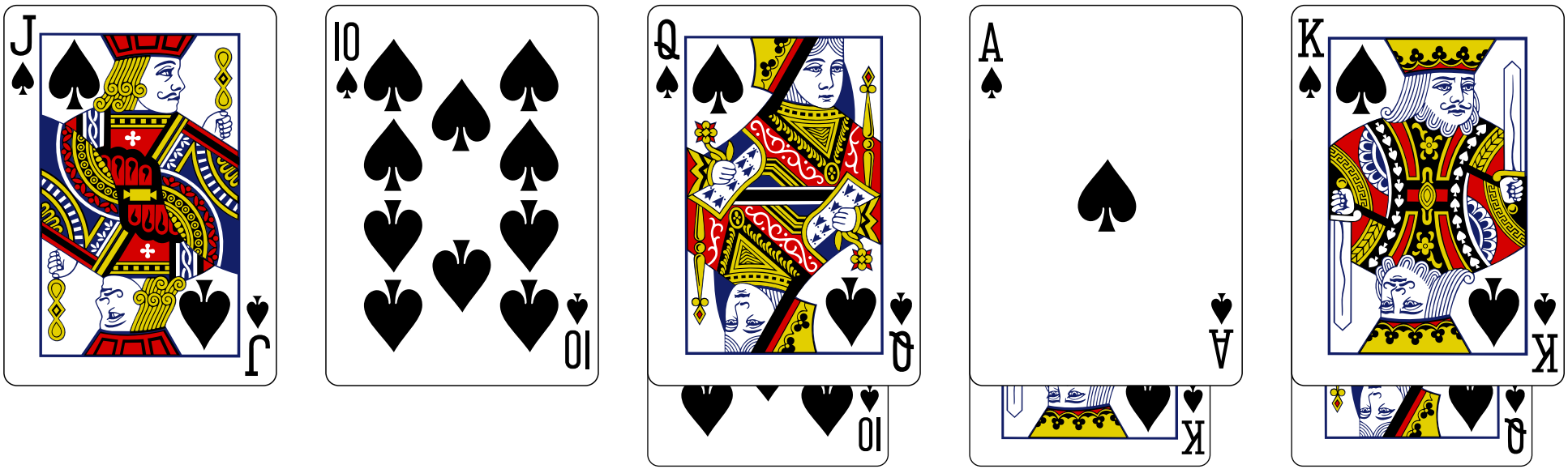
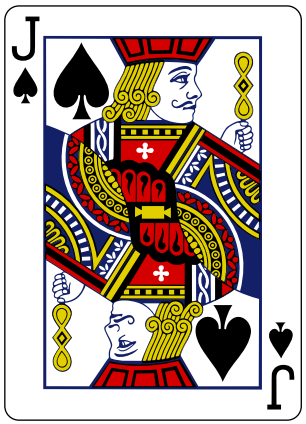
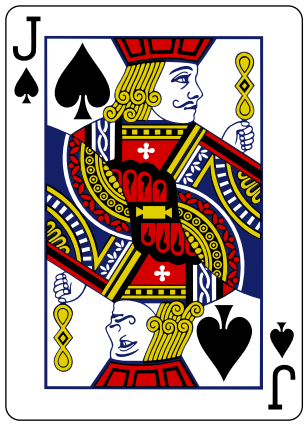
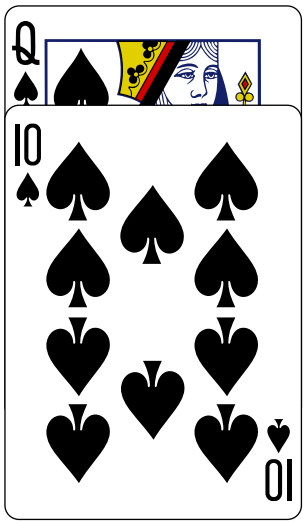


some card games

example, a Dealer win

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winning condition

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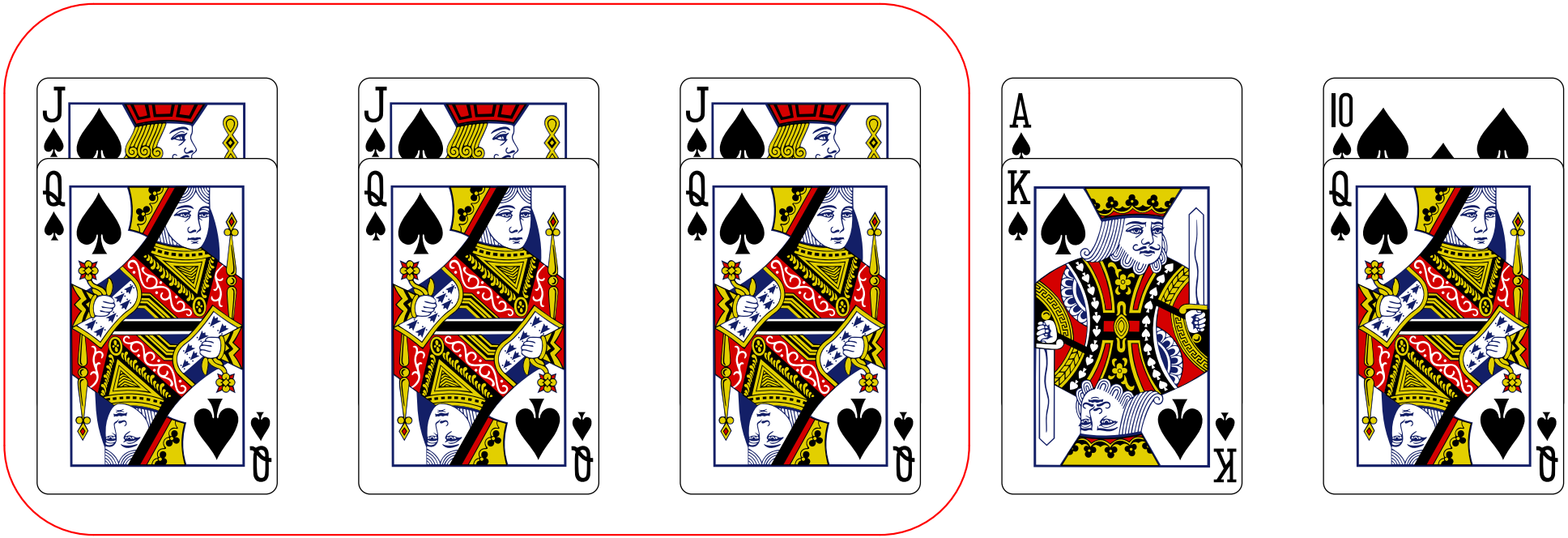

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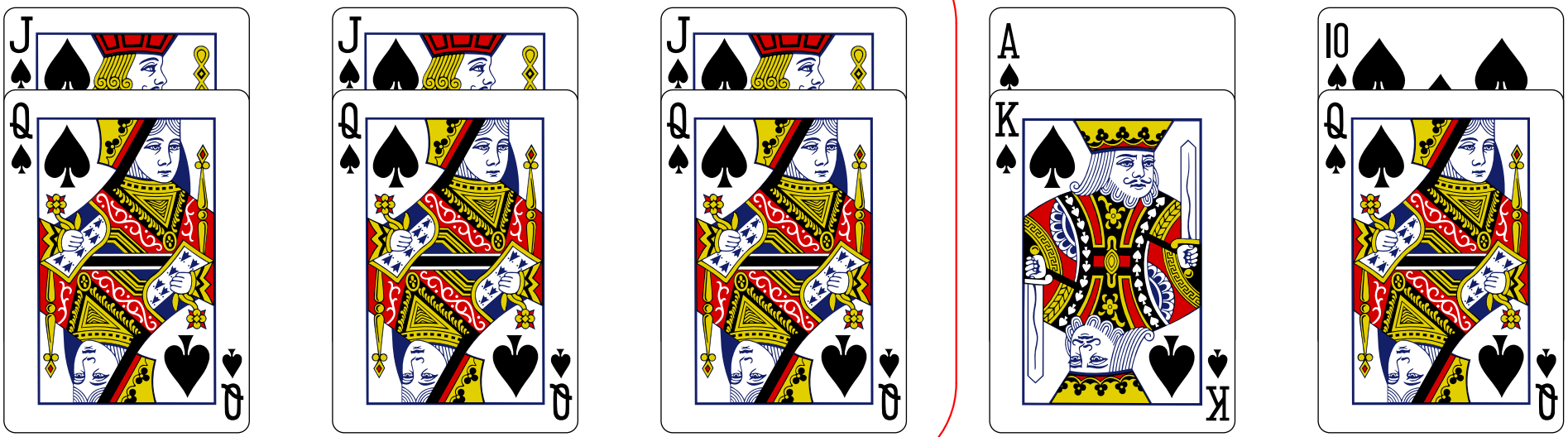
winning condition

- Player cannot win if there is a set of k stacks that together have fewer than k different cards
- Hall's theorem says: **Player wins otherwise**



some card games

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Dealer can either do nothing or swap A and B in at most one other stack.

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Dealer's Move

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Winning

Player wins if he can pick a Royal Flush at the start of one of his turns, otherwise Dealer wins.

some card games

example, a Player win

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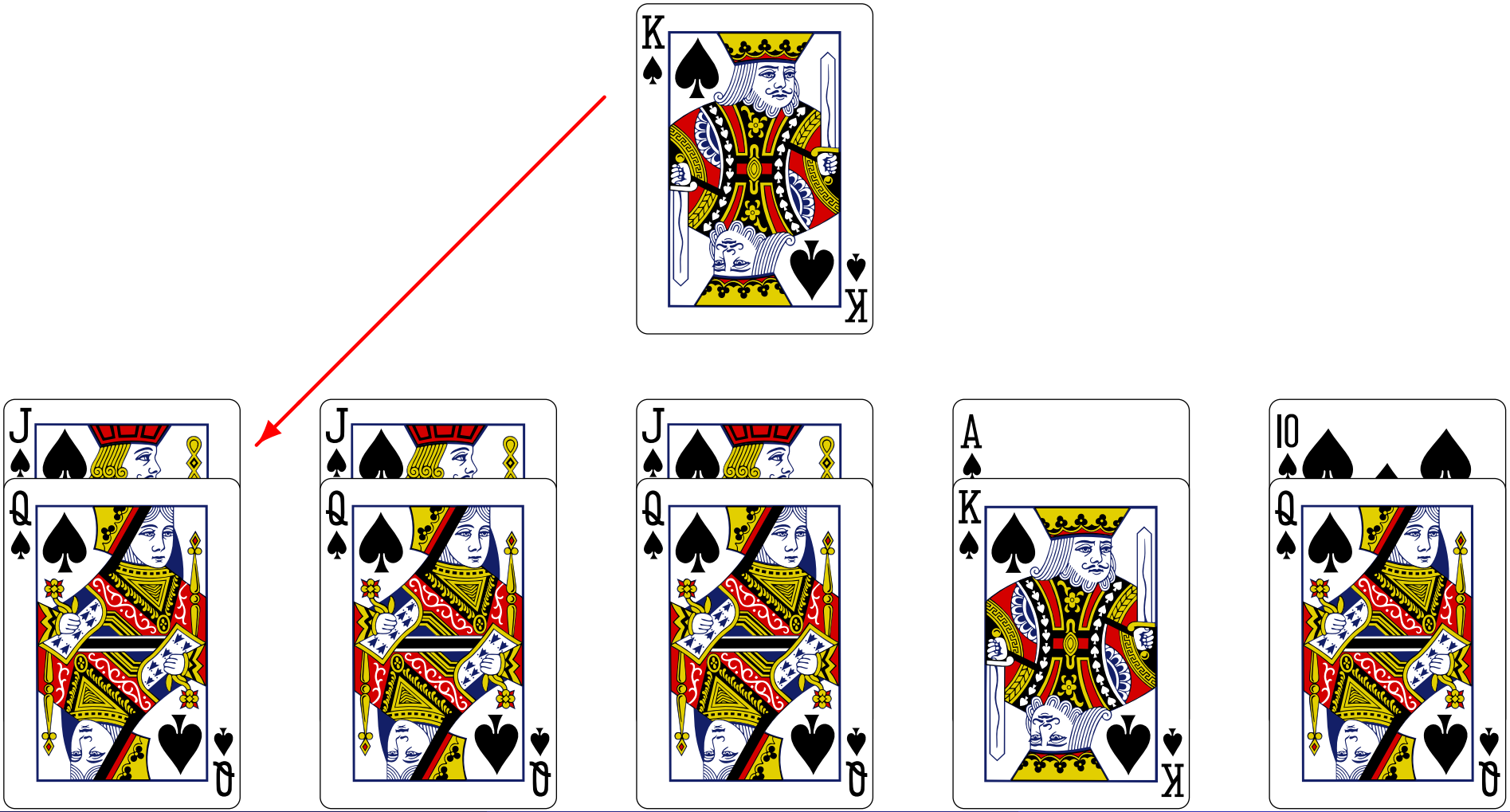
example, a Player win

- Player picks a King from the deck and swaps it for a Queen in the first stack



some card games

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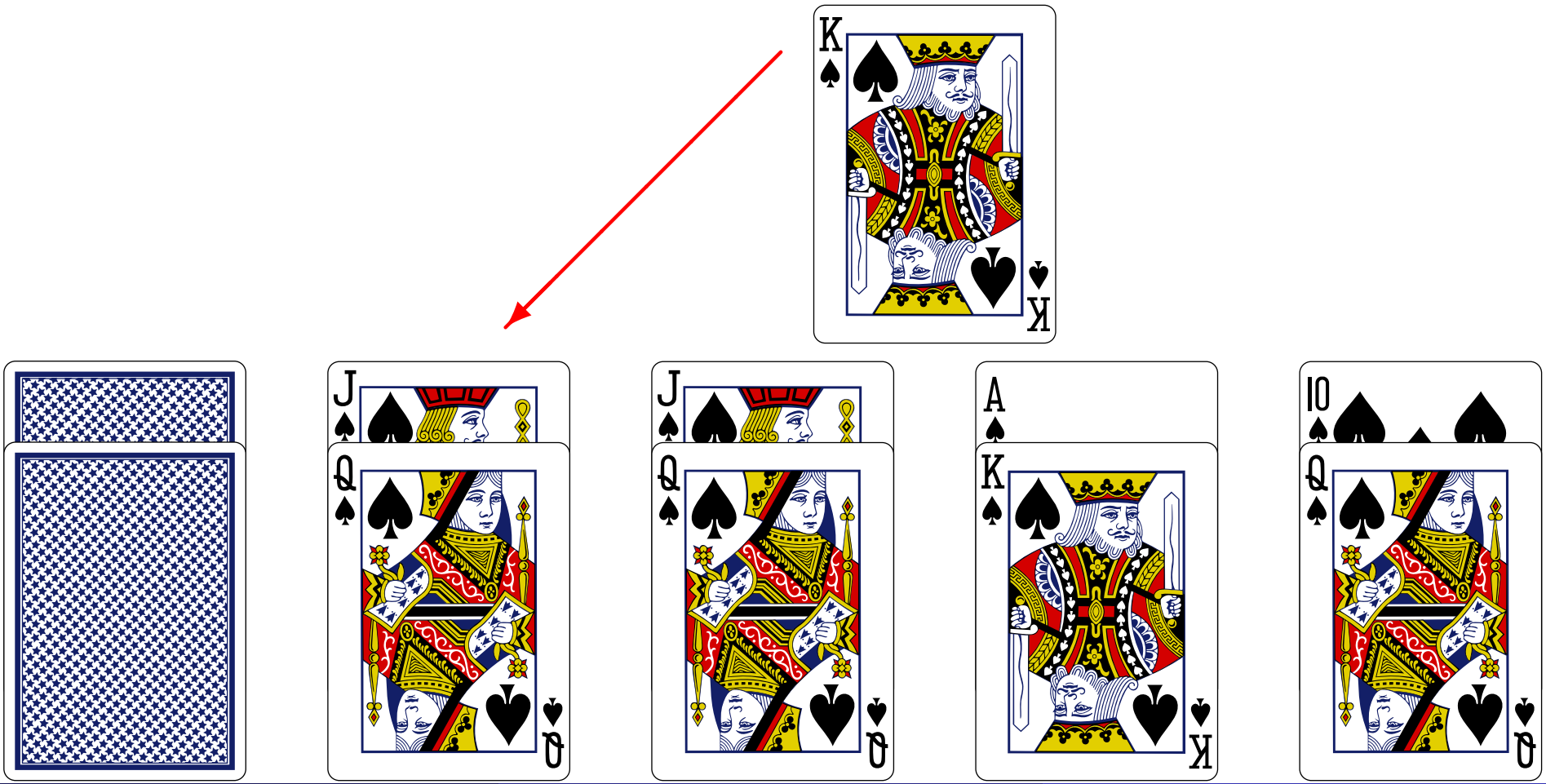

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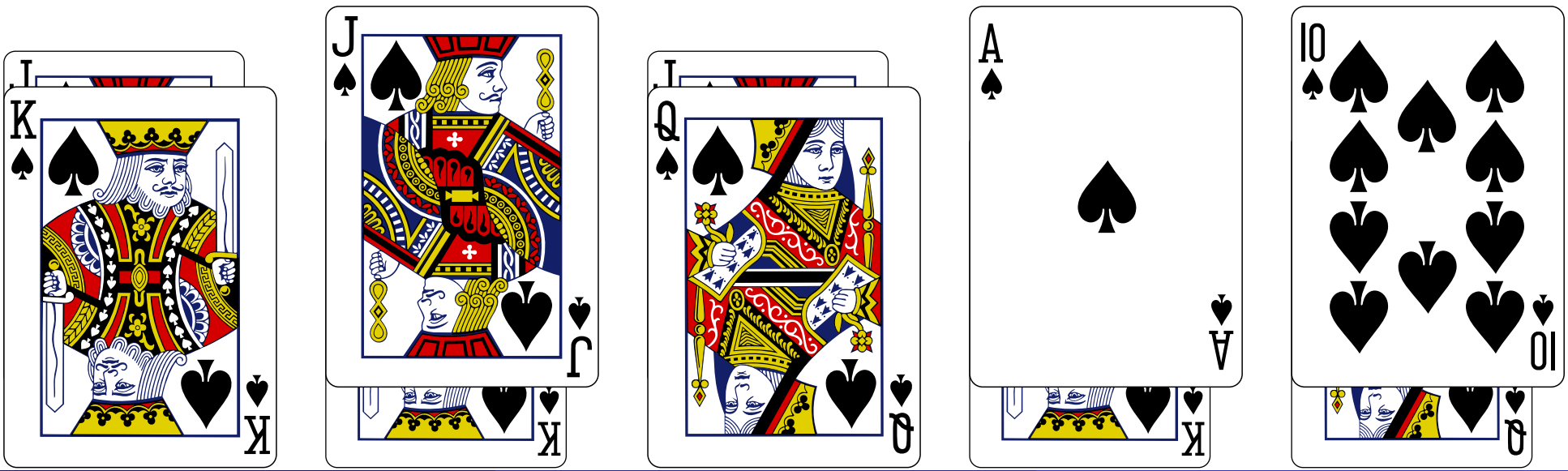
- Player wins no matter what Dealer does



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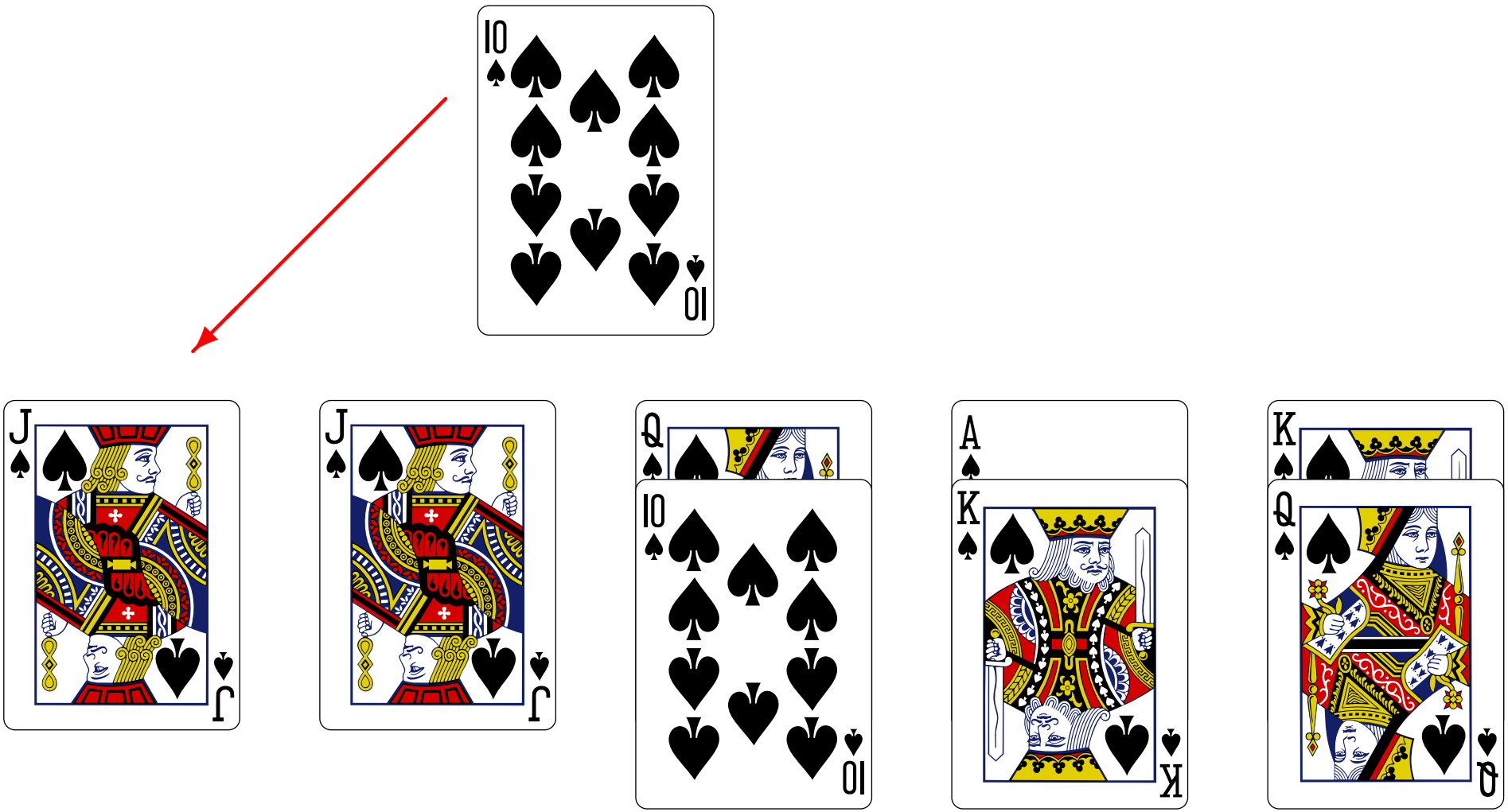

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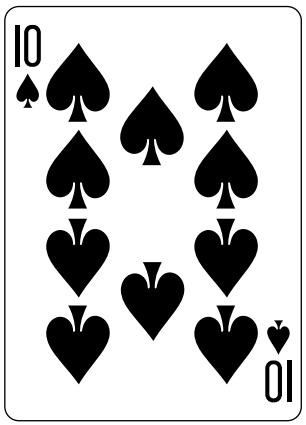

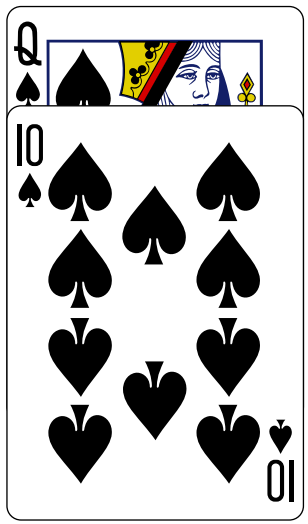

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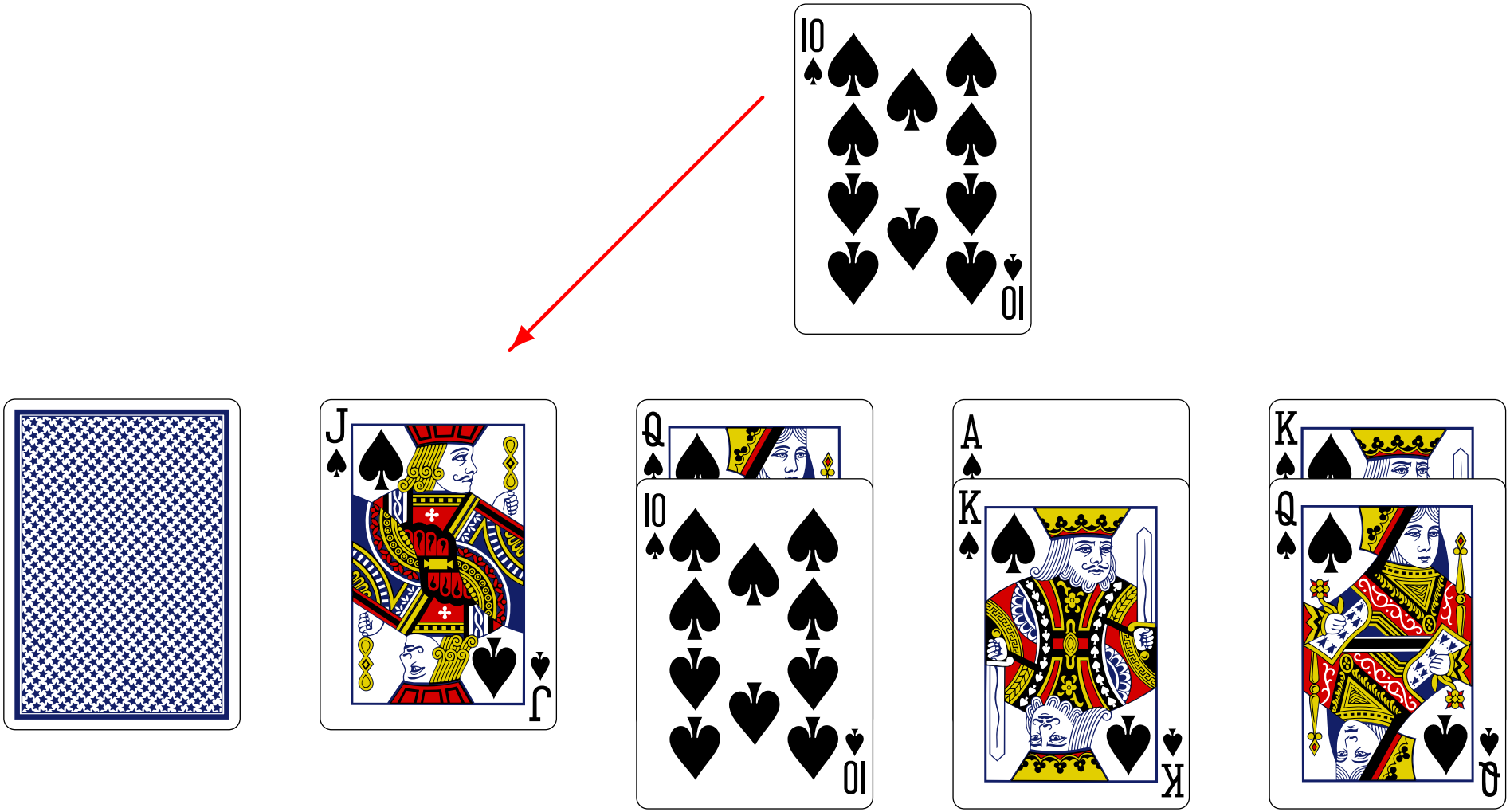

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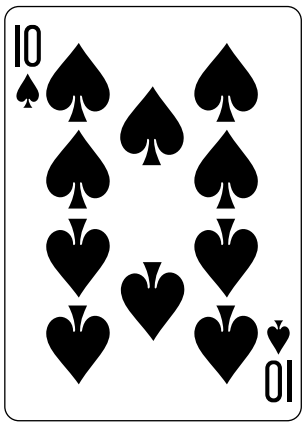
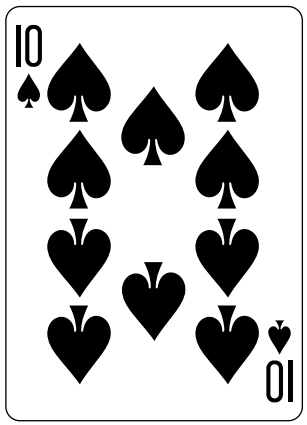
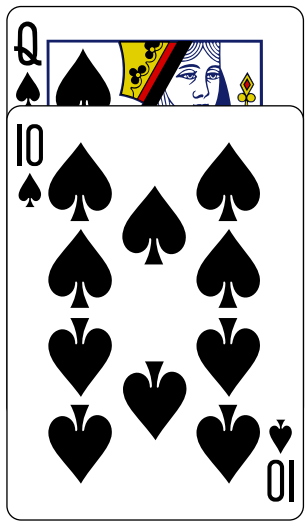
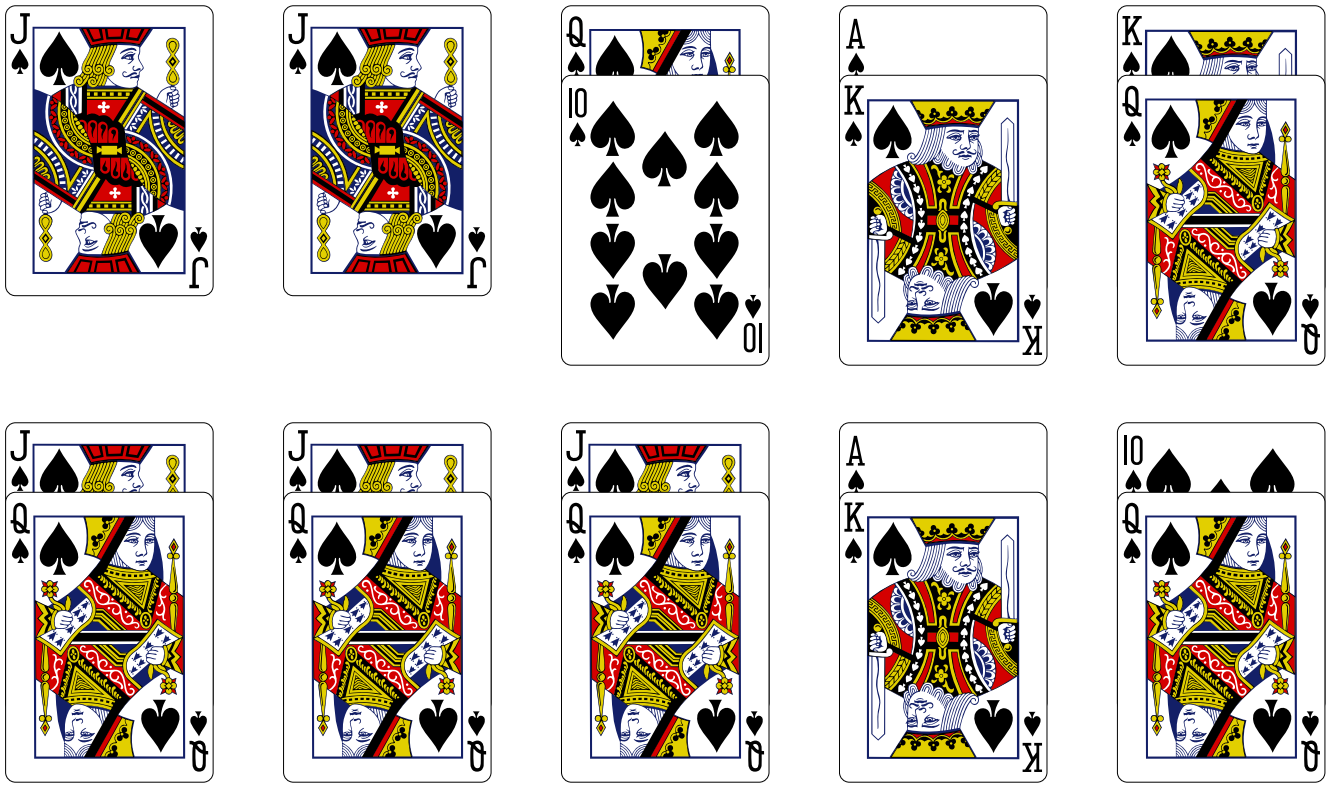






some card games

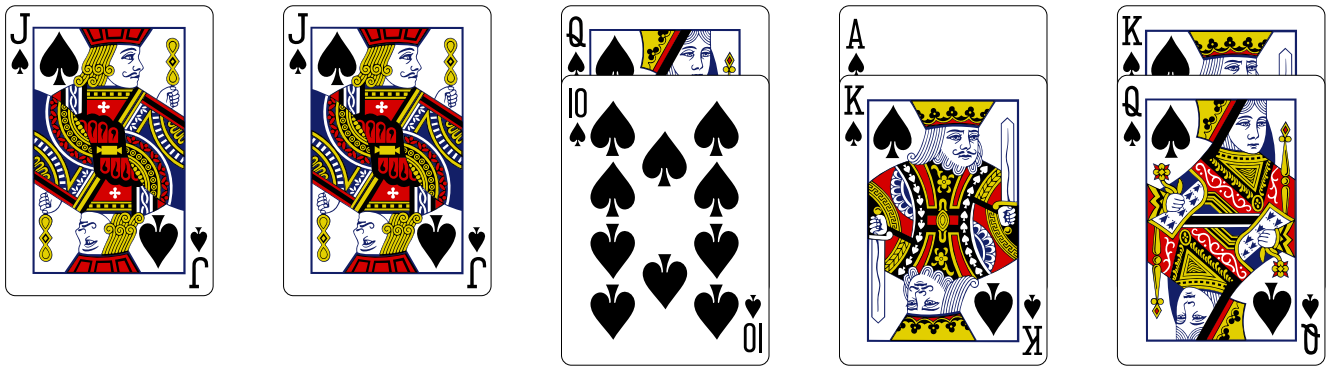
what was the difference?



some card games

what was the difference?

- in the top game, Dealer can prevent Player from increasing the number of different cards in the first two stacks



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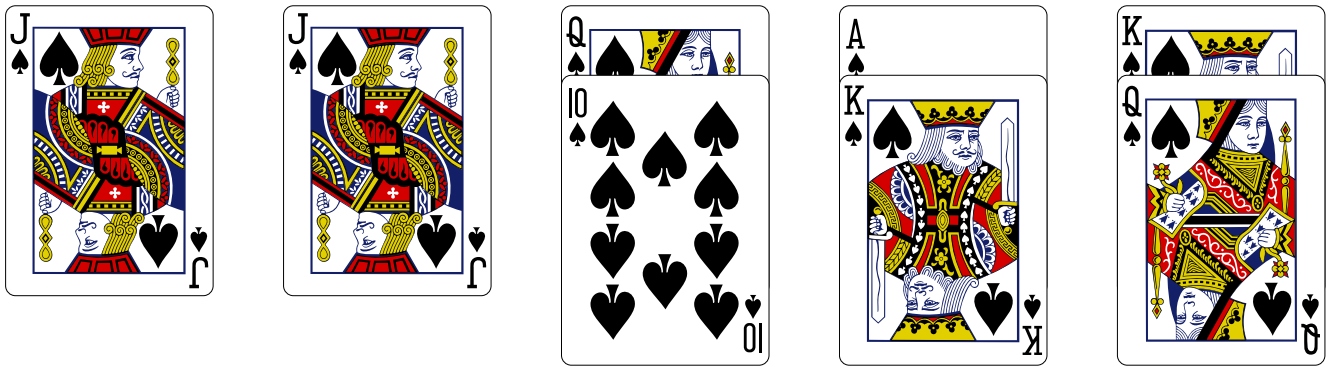
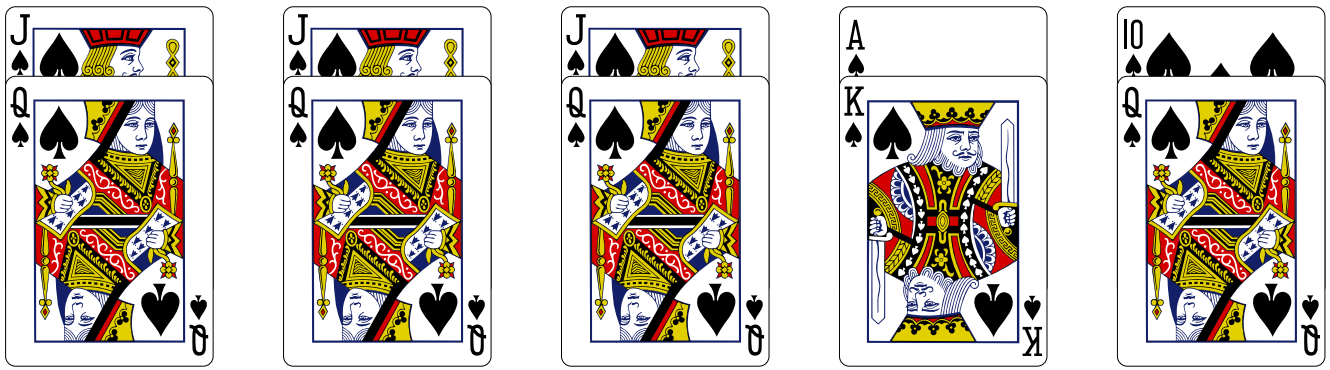
what was the difference?

- in the top game, Dealer can prevent Player from increasing the number of different cards in the first two stacks
- in the bottom game, Dealer cannot prevent prevent Player from increasing the number of different cards in the first three stacks





some card games

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The *degree* of a card C in a set of stacks S is the number of times C appears in S. We write $d_S(C)$ for this quantity.

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Necessary Condition

If Player has a winning strategy, then for every set of stacks S we must have

$$\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{2} \right\rceil \geq |S|.$$

- **this necessary condition is also suffcient**

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Winning Condition

Player has a winning strategy if and only if for every set of stacks S we have

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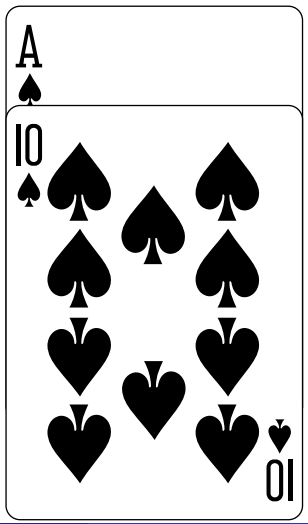
1. Player looks for a set of card types that give a system of distinct representatives of all the stacks containing them



some card games

proof idea

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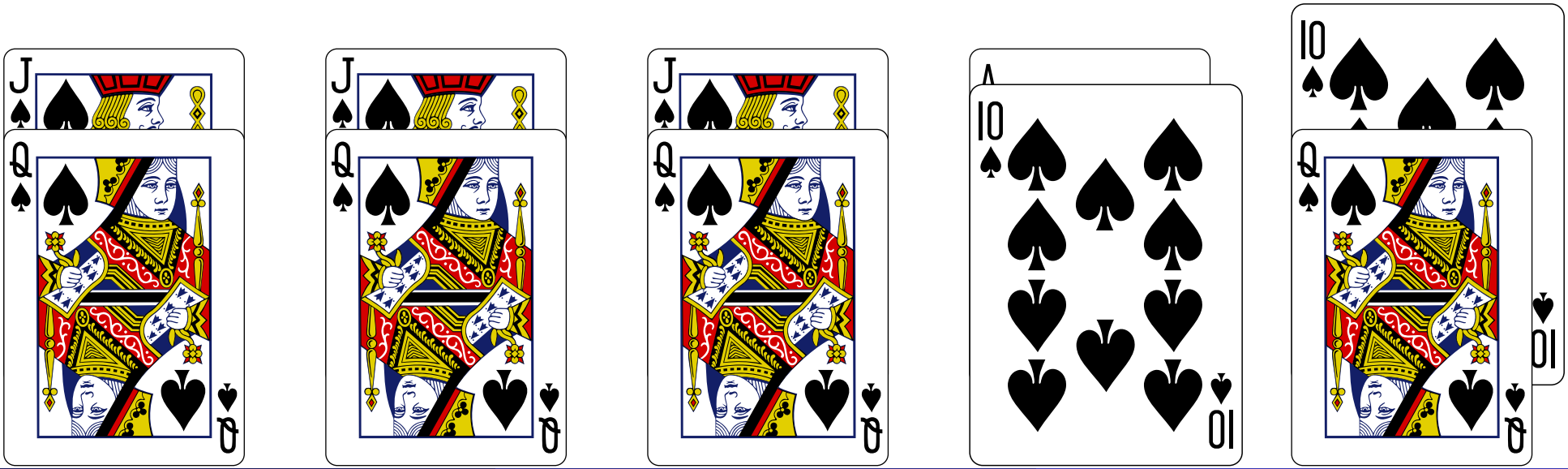

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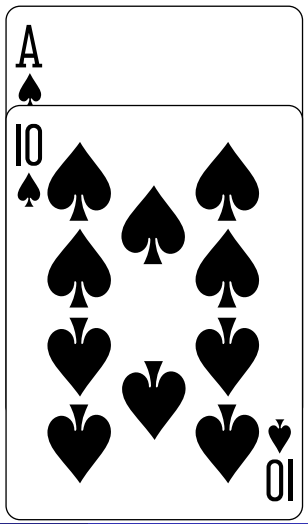

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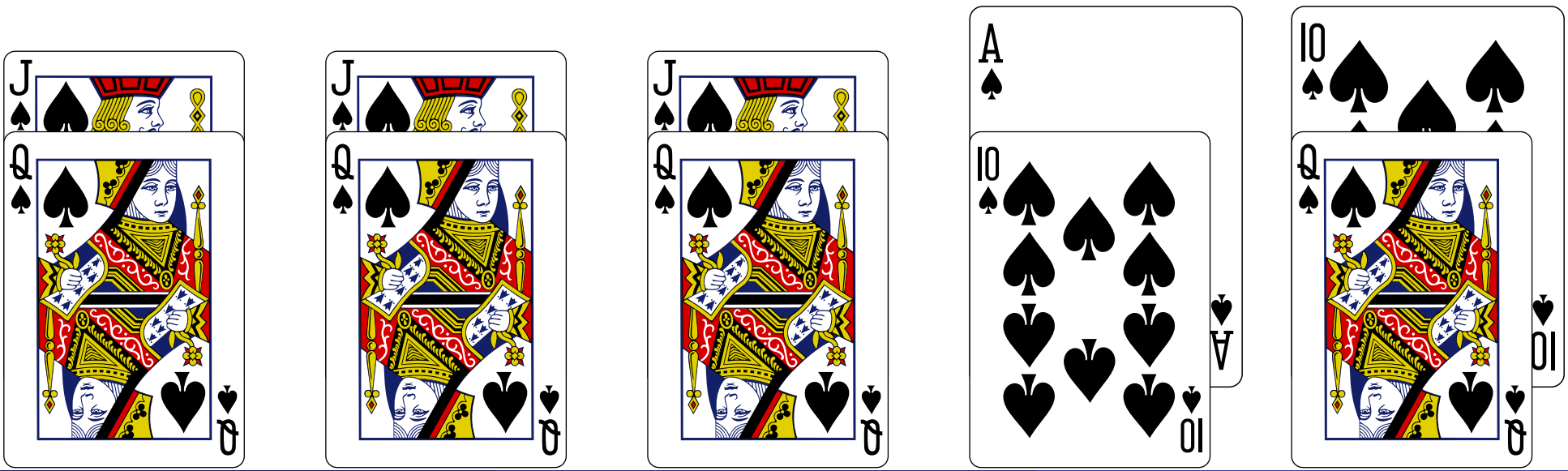
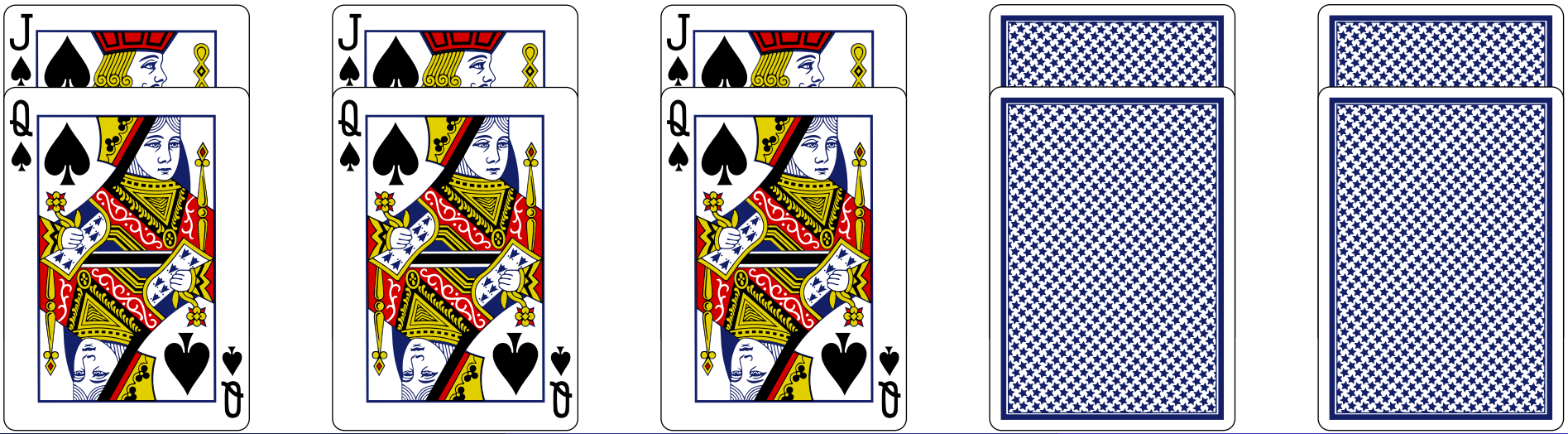


some card games

proof idea

1. Player looks for a set of card types that give a system of distinct representatives of all the stacks containing them
2. Player calls those stacks done and never plays with those card types again



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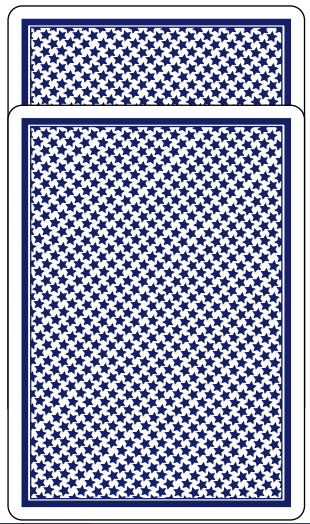
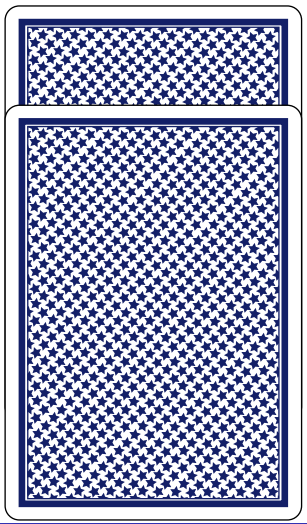

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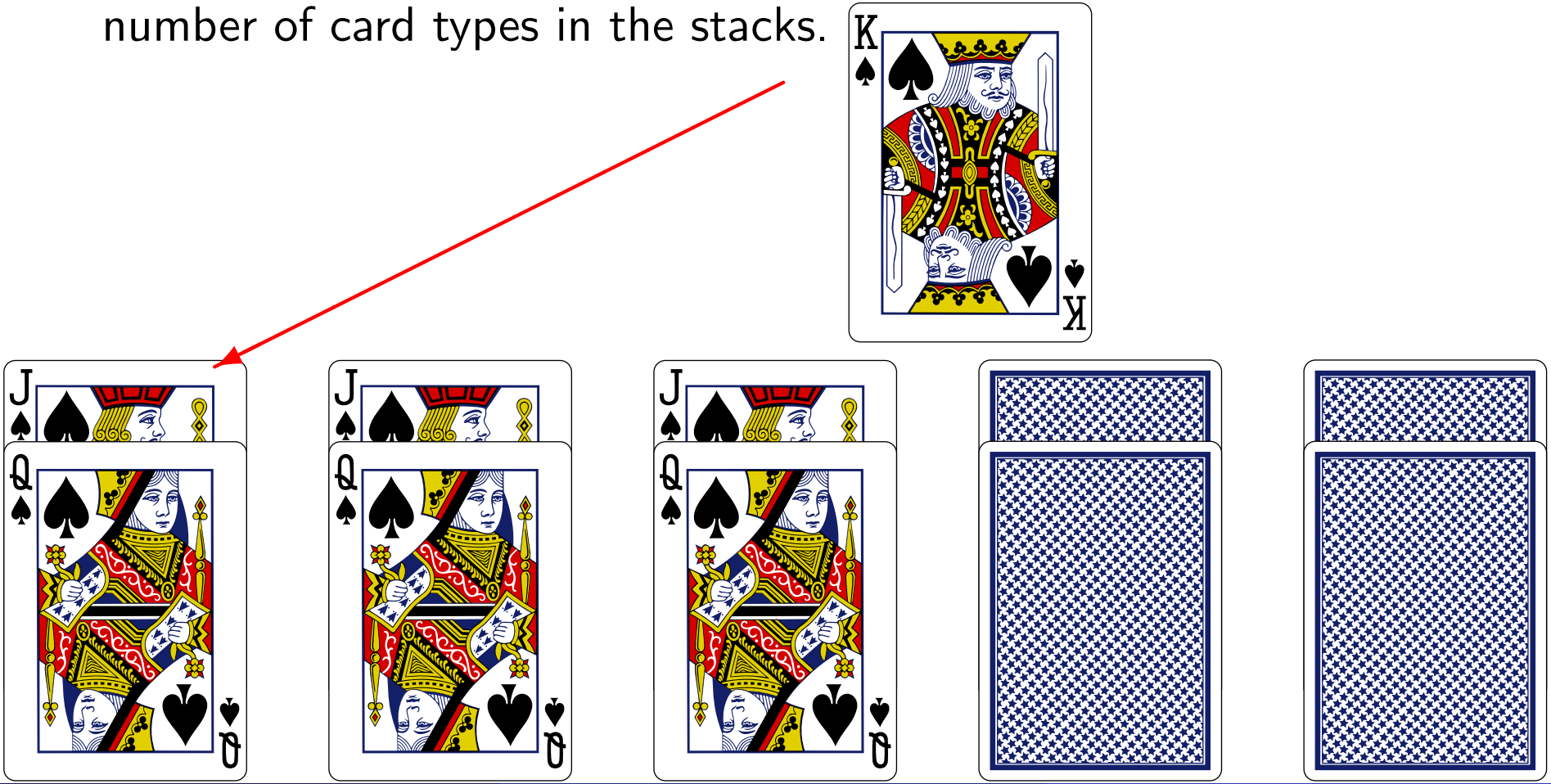
4. but then some card appears at least thrice, so Player can increase the number of card types in the stacks.



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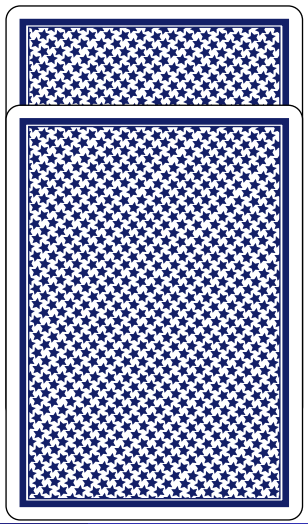
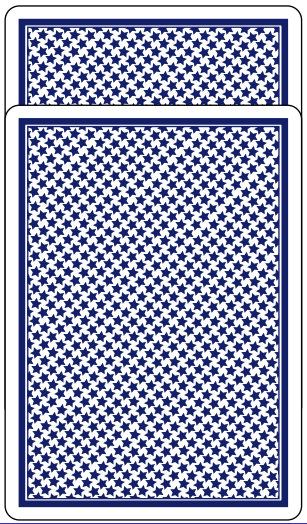

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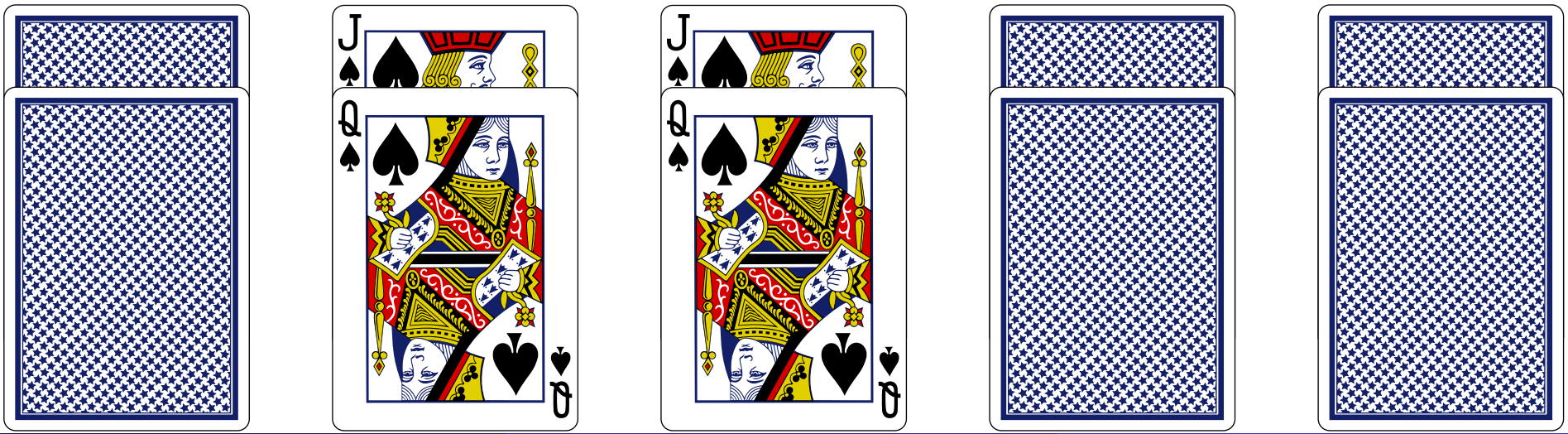

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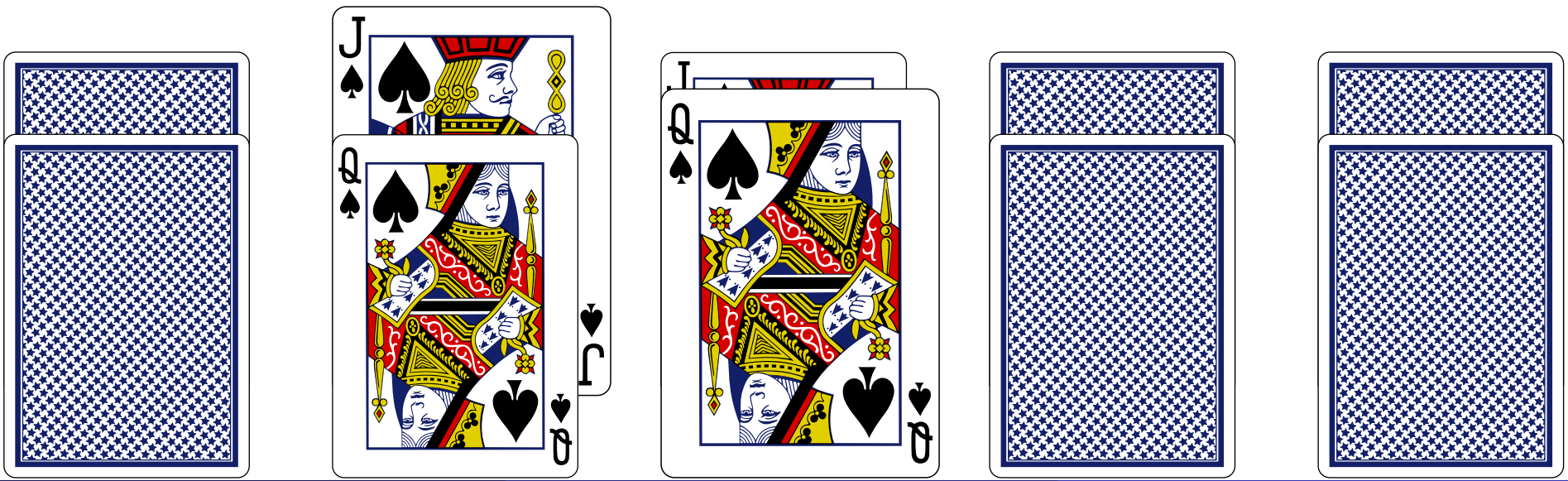

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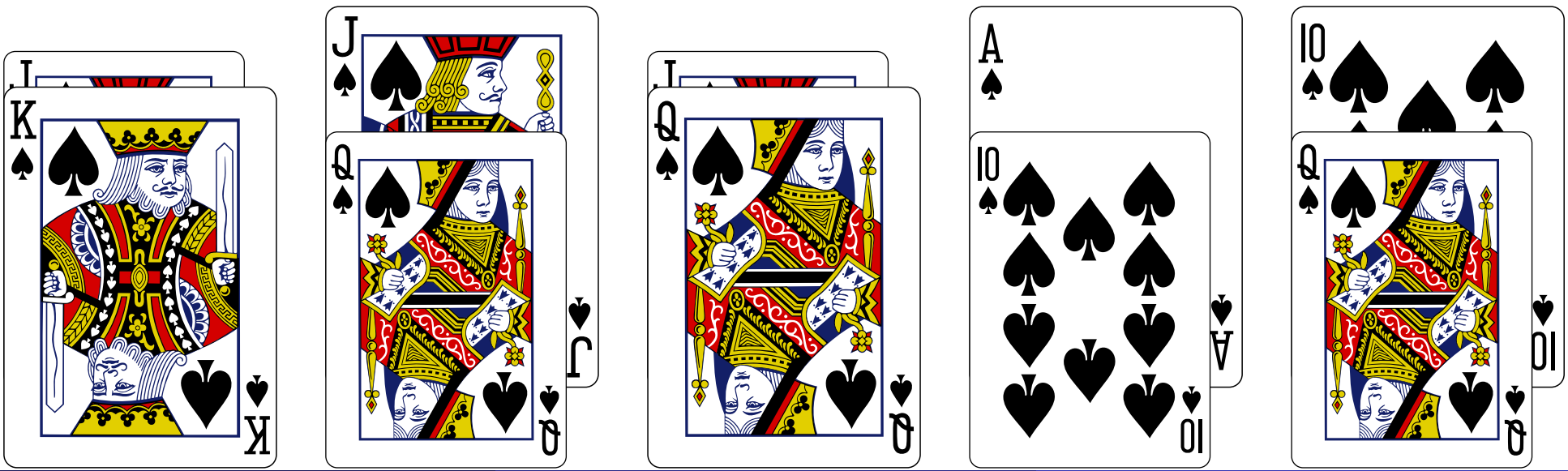

A generalization of Hall's theorem

making it harder for Player

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Player has a winning strategy in the t-game if and only if for every set of stacks S we have

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- Hall's theorem is the winning condition in the $(k-1)$-game when there are k total stacks

edge coloring

setup

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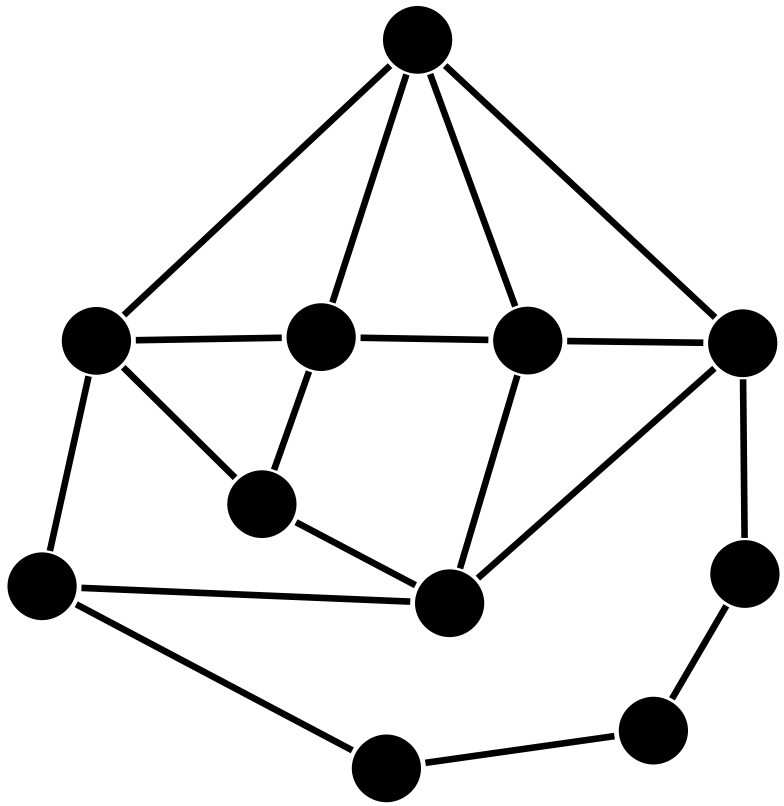
Vizing's theorem

Any simple graph can be edge-colored using at most one more color than its maximum degree.

edge coloring

proof of Vizing's theorem

- take a minimum counterexample to Vizing's theorem, say it has maximum degree Δ



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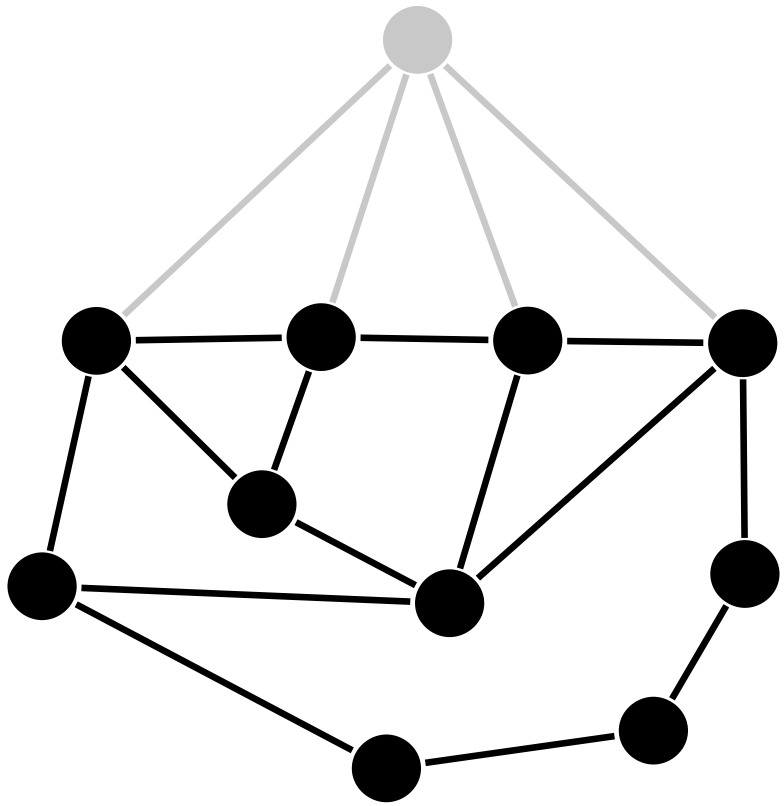

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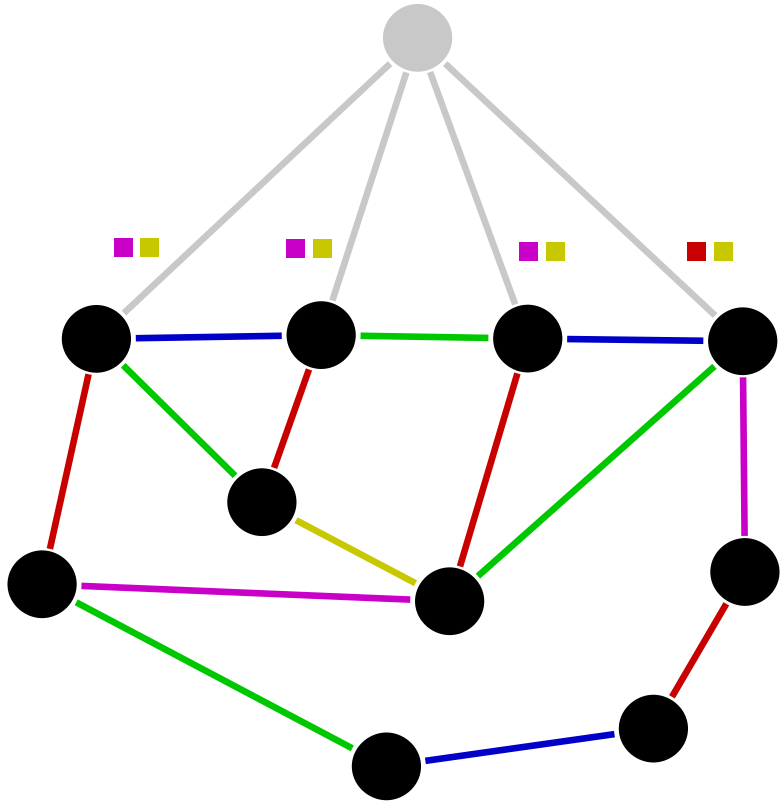

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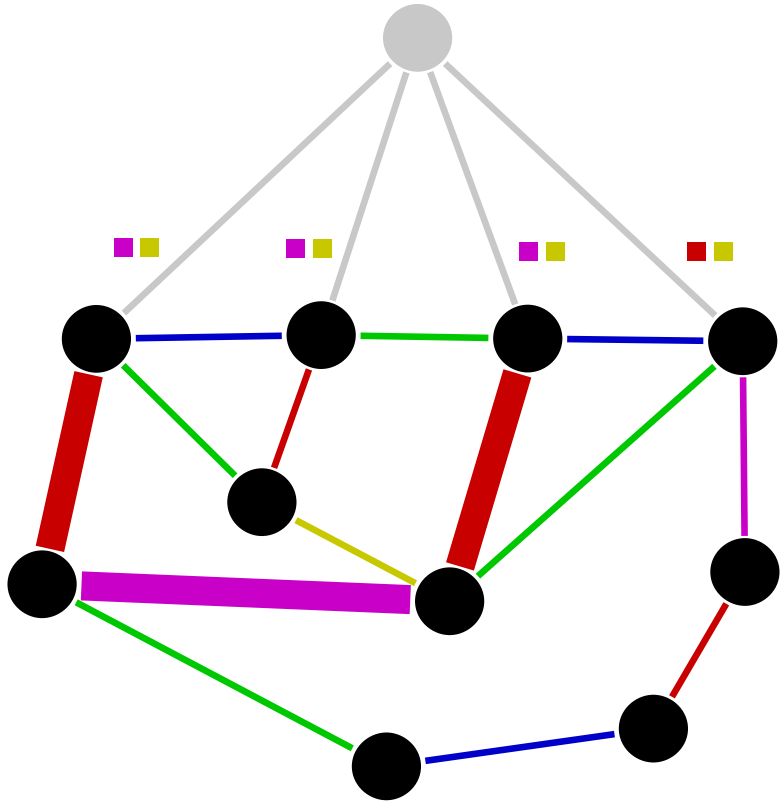

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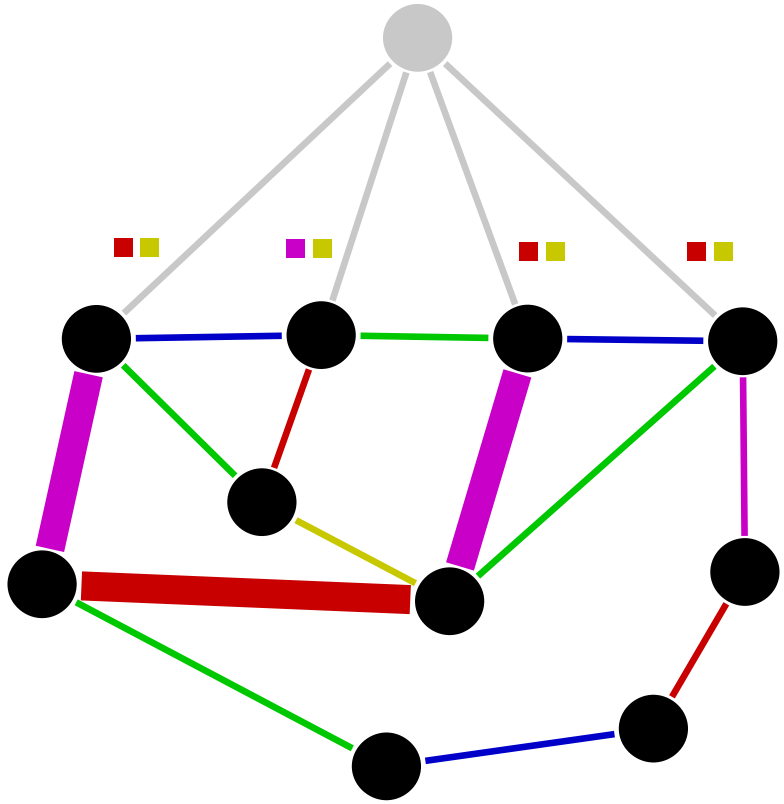

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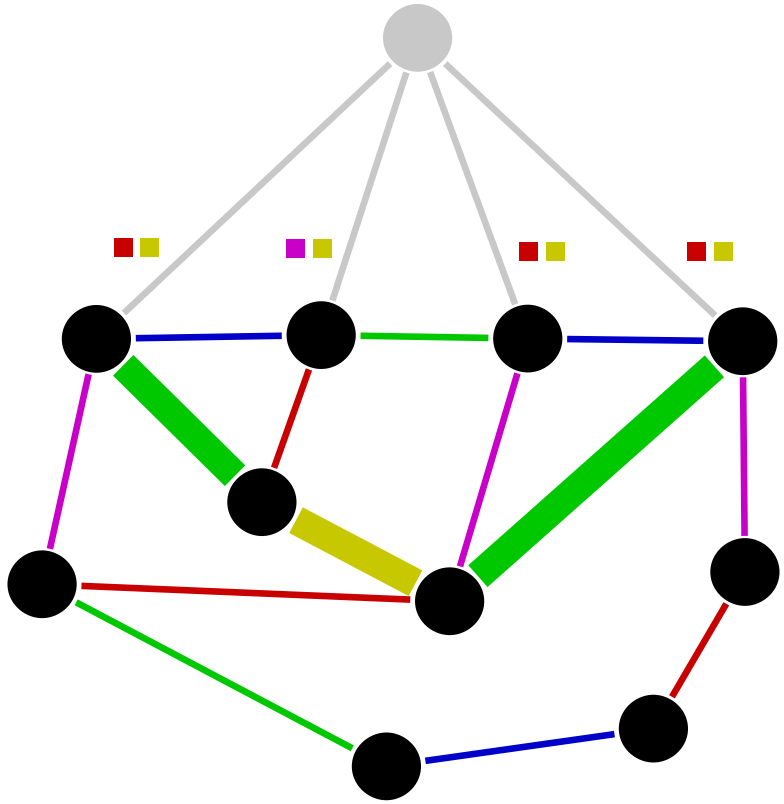

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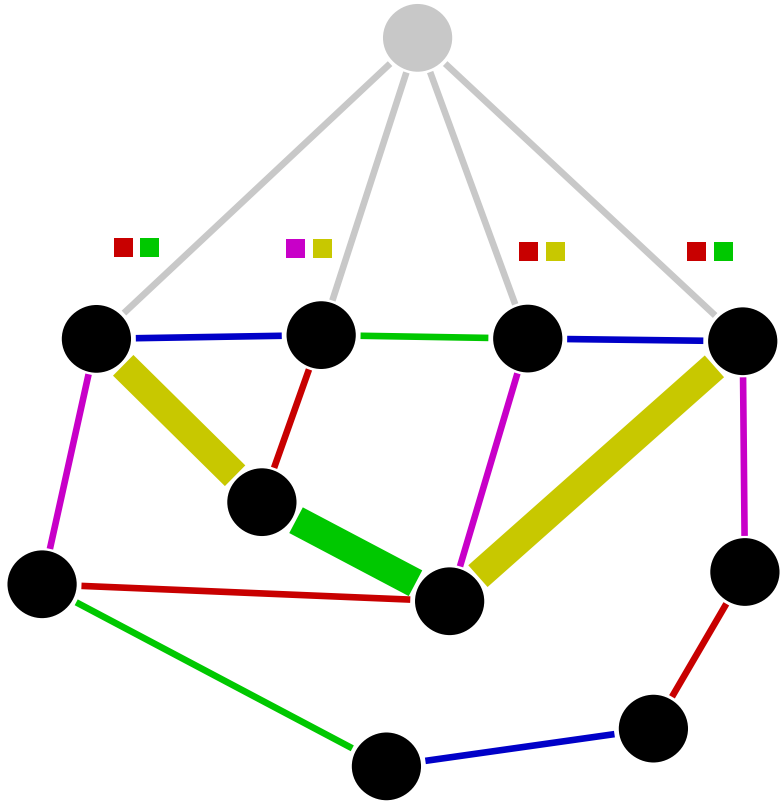

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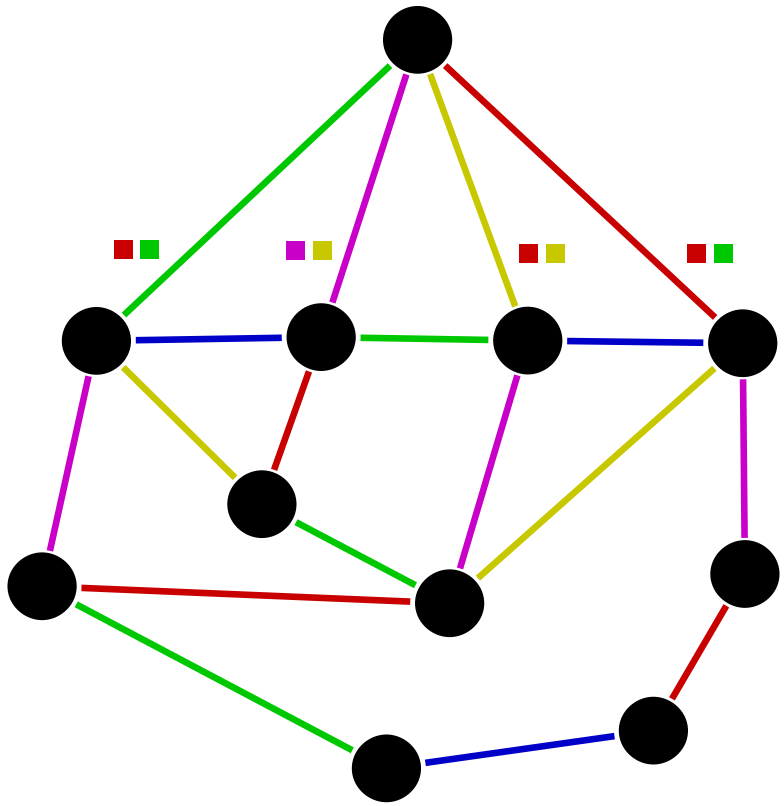

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- so, we have the desired winning condition

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