A better lower bound on average degree of 4-list-critical graphs

Landon Rabern

February 25, 2016

Abstract

We show that for $k \geq 4$, every incomplete k-list-critical graph has average degree at least $k - 1 + \frac{k-3}{k^2-2k+2}$. This improves the best known bound for $k = 4, 5, 6$. The same bound holds for online k-list-critical graphs.

1 Introduction

A graph G is *k-list-critical* if G is not $(k-1)$-choosable, but every proper subgraph of G is $(k-1)$-choosable. For further definitions and notation, see [5, 2]. Table 1 shows some history of lower bounds on the average degree of k-list-critical graphs.

Main Theorem. *For $k \geq 4$, every incomplete k-list-critical graph has average degree at least $k - 1 + \frac{k-3}{k^2-2k+2}$.*

Main Theorem gives a lower bound of $3 + \frac{1}{10}$ for 4-list-critical graphs. This is the first improvement over Gallai's bound of $3 + \frac{1}{13}$. The same proof shows that Main Theorem holds for online k-list-critical graphs as well. The proof does not work for k-Alon-Tarsi-critical graphs since we use the Kernel Lemma.

2 The Proof

The connected graphs in which each block is a complete graph or an odd cycle are called *Gallai trees*. Gallai [4] proved that in a k-critical graph, the vertices of degree $k-1$ induce a disjoint union of Gallai trees. The same is true for k-list-critical graphs ([1, 3]). For a graph T and $k \in \mathbb{N}$, let $\beta_k(T)$ be the independence number of the subgraph of T induced on the vertices of degree $k - 1$. When k is defined in the context, put $\beta(T) := \beta_k(T)$.

Lemma 1. *If $k \geq 4$ and $T \neq K_k$ is a Gallai tree with maximum degree at most $k - 1$, then*

$$2\left\|T\right\| \leq (k-2)\left|T\right| + 2\beta(T).$$

	k-Critical G				k-List Critical G			
k	Gallai [4] $d(G) \geq$	Kriv [9] $d(G) \geq$	KS [7] $d(G) \geq$	KY [8] $d(G) \geq$	KS [7] $d(G) \geq$	KR [5] $d(G) \geq$	CR [2] $d(G) \geq$	Here $d(G) \geq$
4	3.0769	3.1429	—	3.3333	—	—	—	**3.1000**
5	4.0909	4.1429	—	4.5000	—	4.0984	4.1000	**4.1176**
6	5.0909	5.1304	5.0976	5.6000	—	5.1053	5.1076	**5.1153**
7	6.0870	6.1176	6.0990	6.6667	—	6.1149	**6.1192**	6.1081
8	7.0820	7.1064	7.0980	7.7143	—	7.1128	**7.1167**	7.1000
9	8.0769	8.0968	8.0959	8.7500	8.0838	8.1094	**8.1130**	8.0923
10	9.0722	9.0886	9.0932	9.7778	9.0793	9.1055	**9.1088**	9.0853
15	14.0541	14.0618	14.0785	14.8571	14.0610	14.0864	**14.0884**	14.0609
20	19.0428	19.0474	19.0666	19.8947	19.0490	19.0719	**19.0733**	19.0469

Table 1: History of lower bounds on the average degree $d(G)$ of k-critical and k-list-critical graphs G.

Proof. Suppose the lemma is false and choose a counterexample T minimizing $|T|$. Plainly, T has more than one block. Let A be an endblock of T and let x be the unique cutvertex of T with $x \in V(A)$. Consider $T' := T - (V(A) \setminus \{x\})$. By minimality of $|T|$,

$$2\,\|T\| - 2\,\|A\| \leq (k-2)(|T| + 1 - |A|) + 2\beta(T').$$

Since T is a counterexample, $2\,\|A\| > (k-2)(|A|-1)$. So, if $k > 4$, then $A = K_{k-1}$ and if $k = 4$, then A is an odd cycle. In both cases, $d_T(x) = k-1$. Consider $T^* := T - V(A)$. By minimality of $|T|$,

$$2\,\|T\| - 2\,\|A\| - 2 \leq (k-2)(|T| - |A|) + 2\beta(T^*).$$

Since T is a counterexample, $2\,\|A\| + 2 > (k-2)\,|A| + 2(\beta(T) - \beta(T^*))$. In T^*, all of x's neighbors have degree at most $k-2$. But $d_T(x) = k-1$, so some vertex in $\{x\} \cup N(x)$ is in a maximum independent set of degree $k-1$ vertices in T. Hence $\beta(T^*) \leq \beta(T) - 1$, which gives

$$2\,\|A\| > (k-2)\,|A|\,,$$

a contradiction since $k \geq 4$. □

Definition 1. The *maximum independent cover number* of a graph G is the maximum $\mathrm{mic}(G)$ of $\|I, V(G) \setminus I\|$ over all independent sets I of G.

Theorem 2 (Kierstead and R. [6])**.** *Every k-list-critical graph G satisfies*

$$2\,\|G\| \geq (k-2)\,|G| + \mathrm{mic}(G) + 1.$$

Main Theorem. *For $k \geq 4$, every incomplete k-list-critical graph has average degree at least $k - 1 + \frac{k-3}{k^2-2k+2}$.*

Proof. Let $G \neq K_k$ be a k-list-critical graph. Let $\mathcal{L} \subseteq V(G)$ be the vertices with degree $k-1$ and let $\mathcal{H} = V(G) \setminus \mathcal{L}$. Put $\|\mathcal{L}\| := \|G[\mathcal{L}]\|$ and $\|\mathcal{H}\| := \|G[\mathcal{H}]\|$. By Lemma 1,

$$2\,\|\mathcal{L}\| \leq (k-2)|\mathcal{L}| + 2\beta(\mathcal{L})$$

Hence,

$$\begin{aligned}
2\,\|G\| &= 2\,\|\mathcal{H}\| + 2\,\|\mathcal{H}, \mathcal{L}\| + 2\,\|\mathcal{L}\| \\
&= 2\,\|\mathcal{H}\| + 2((k-1)\,|\mathcal{L}| - 2\,\|\mathcal{L}\|) + 2\,\|\mathcal{L}\| \\
&= 2\,\|\mathcal{H}\| + 2(k-1)\,|\mathcal{L}| - 2\,\|\mathcal{L}\| \\
&\geq 2\,\|\mathcal{H}\| + k\,|\mathcal{L}| - 2\beta(\mathcal{L}),
\end{aligned}$$

which is

$$\beta(\mathcal{L}) \geq \|\mathcal{H}\| + \frac{k}{2}\,|\mathcal{L}| - \|G\|\,. \tag{1}$$

Let M be the maximum of $\|I, V(G) \setminus I\|$ over all independent sets I of G with $I \subseteq \mathcal{H}$. Then

$$\text{mic}(G) \geq M + (k-1)\beta(\mathcal{L}).$$

Applying Lemma 2 and using (1) gives

$$\begin{aligned}
2\,\|G\| &\geq (k-2)\,|G| + M + (k-1)\beta(\mathcal{L}) + 1 \\
&\geq (k-2)\,|G| + M + (k-1)\left(\|\mathcal{H}\| + \frac{k}{2}\,|\mathcal{L}| - \|G\|\right) + 1 \\
&= (k-2)\,|G| + M + (k-1)\,\|\mathcal{H}\| + \frac{k(k-1)}{2}\,|\mathcal{L}| - (k-1)\,\|G\| + 1.
\end{aligned}$$

Hence

$$(k+1)\,\|G\| \geq (k-2)\,|G| + M + (k-1)\,\|\mathcal{H}\| + \frac{k(k-1)}{2}\,|\mathcal{L}| + 1 \tag{2}$$

Let $\mathcal{C}$ be the components of $G[\mathcal{H}]$. Then $\alpha(C) \geq \frac{|C|}{\chi(C)}$ for all $C \in \mathcal{C}$. Whence

$$M + (k-1)\,\|\mathcal{H}\| \geq \sum_{C \in \mathcal{C}} k\frac{|C|}{\chi(C)} + (k-1)\,\|C\|\,. \tag{3}$$

If $\mathcal{L} = \emptyset$, then G has average degree at least $k \geq k - 1 + \frac{k-3}{k^2-2k+2}$. So, assume $\mathcal{L} \neq \emptyset$. Then $G[\mathcal{H}]$ is $(k-1)$-colorable by k-list-criticality of G. In particular, $\chi(C) \leq k-1$ for every $C \in \mathcal{C}$. We claim that for every $C \in \mathcal{C}$,

$$k\frac{|C|}{\chi(C)} + (k-1)\,\|C\| \geq \left(k - \frac{1}{2}\right)|C|\,. \tag{4}$$

If $C \in \mathcal{C}$ is not a tree, then $\|C\| \geq |C|$ and hence $k\frac{|C|}{\chi(C)} + (k-1)\,\|C\| \geq k\frac{|C|}{k-1} + (k-1)\,|C| \geq (k-\frac{1}{2})\,|C|$. If C is a tree, then $\chi(C) \leq 2$ and hence $k\frac{|C|}{\chi(C)} + (k-1)\,\|C\| \geq k\frac{|C|}{2} + (k -$

1)$(|C|-1) \geq (k-\frac{1}{2})\,|C|$ unless $|C|=1$. This proves (4) since the bound is trivially satisfied when $|C|=1$.

Now combining (2), (3) and (4) with the basic bound

$$|\mathcal{L}| \geq k\,|G| - 2\,\|G\|\,,$$

gives

$$\begin{aligned}(k+1)\,\|G\| &\geq (k-2)\,|G| + \left(k-\frac{1}{2}\right)|\mathcal{H}| + \frac{k(k-1)}{2}\,|\mathcal{L}| + 1\\ &= \left(2k-\frac{5}{2}\right)|G| + \frac{k^2-3k+1}{2}\,|\mathcal{L}| + 1\\ &\geq \left(2k-\frac{5}{2}\right)|G| + \frac{k^2-3k+1}{2}\,(k\,|G| - 2\,\|G\|) + 1.\end{aligned}$$

After some algebra, this becomes

$$2\,\|G\| \geq \left(k-1+\frac{k-3}{k^2-2k+2}\right)|G| + \frac{2}{k^2-2k+2}.$$

That proves the theorem. □

Problem. The right side of equation (4) in the above proof can be improved to $k\,|C|$ unless C is a K_2 where both vertices have degree k in G. If these K_2's could be handled, the average degree bound would improve to $k-1+\frac{k-3}{(k-1)^2}$. Handle the K_2's.

3 An Improvement?

Lemma 3. *Let $p\colon \mathbb{N} \to \mathbb{R}$ with $\frac{2}{k-2} \leq p(k) \leq 1$ for all $k \in \mathbb{N}$. If $k \geq 4$ and $T \neq K_k$ is a Gallai tree with maximum degree at most $k-1$, then*

$$2\,\|T\| \leq (k-3+p(k))\,|T| + (k-1)(1-p(k)) + (2+(k-1)(1-p(k)))\beta(T).$$

Note that this bound will pick up an extra $(k-1)(1-p(k))$ per component in a Gallai forest.

Theorem 4.

Proof. Let $G \neq K_k$ be a k-list-critical graph. Let $\mathcal{L} \subseteq V(G)$ be the vertices with degree $k-1$ and let $\mathcal{H} = V(G) \setminus \mathcal{L}$. Put $\|\mathcal{L}\| := \|G[\mathcal{L}]\|$ and $\|\mathcal{H}\| := \|G[\mathcal{H}]\|$. By Lemma 1,

$$2\,\|\mathcal{L}\| \leq (k-3+p(k))\,|\mathcal{L}| + (k-1)(1-p(k))c(\mathcal{L}) + (2+(k-1)(1-p(k)))\beta(\mathcal{L}).$$

Hence,

$$\begin{aligned}2\,\|G\| &= 2\,\|\mathcal{H}\| + 2\,\|\mathcal{H},\mathcal{L}\| + 2\,\|\mathcal{L}\|\\ &= 2\,\|\mathcal{H}\| + 2((k-1)\,|\mathcal{L}| - 2\,\|\mathcal{L}\|) + 2\,\|\mathcal{L}\|\\ &= 2\,\|\mathcal{H}\| + 2(k-1)\,|\mathcal{L}| - 2\,\|\mathcal{L}\|\\ &\geq 2\,\|\mathcal{H}\| + (k+1-p(k))\,|\mathcal{L}| - (k-1)(1-p(k))c(\mathcal{L}) - (2+(k-1)(1-p(k)))\beta(\mathcal{L}),\end{aligned}$$

which is

$$\beta(\mathcal{L}) \geq \frac{2\|\mathcal{H}\| + (k+1-p(k))|\mathcal{L}| - (k-1)(1-p(k))c(\mathcal{L}) - 2\|G\|}{2+(k-1)(1-p(k))} \tag{5}$$

Let M be the maximum of $\|I, V(G)\setminus I\|$ over all independent sets I of G with $I \subseteq \mathcal{H}$. Then

$$\mathrm{mic}(G) \geq M + (k-1)\beta(\mathcal{L}).$$

To lower bound M, consider the components $\mathcal{C}$ of $G[\mathcal{H}]$. Now $\alpha(C) \geq \frac{|C|}{\chi(C)}$ for all $C \in \mathcal{C}$, so

$$M + (k-1)\|\mathcal{H}\| \geq \sum_{C\in\mathcal{C}} k\frac{|C|}{\chi(C)} + (k-1)\|C\|. \tag{6}$$

If $\mathcal{L} = \emptyset$, then G has average degree at least k which easily satisfies the theorem. So, assume $\mathcal{L} \neq \emptyset$. Then $G[\mathcal{H}]$ is $(k-1)$-colorable by k-list-criticality of G. In particular, $\chi(C) \leq k-1$ for every $C \in \mathcal{C}$. For every $C \in \mathcal{C}$,

$$k\frac{|C|}{\chi(C)} + (k-1)\|C\| \geq \left(k - \frac{1}{2}\right)|C|. \tag{7}$$

To see this, first suppose $C \in \mathcal{C}$ is not a tree. Then $\|C\| \geq |C|$ and hence $k\frac{|C|}{\chi(C)} + (k-1)\|C\| \geq k\frac{|C|}{k-1} + (k-1)|C| \geq (k-\frac{1}{2})|C|$. If C is a tree, then $\chi(C) \leq 2$ and hence $k\frac{|C|}{\chi(C)} + (k-1)\|C\| \geq k\frac{|C|}{2} + (k-1)(|C|-1) \geq (k-\frac{1}{2})|C|$ unless $|C| = 1$. This proves (7) since the bound is trivially satisfied when $|C| = 1$. Therefore,

$$M + (k-1)\|H\| \geq \left(k - \frac{1}{2}\right)|H|. \tag{8}$$

Applying Lemma 2 and using (5) gives

$$\begin{aligned} 2\|G\| &\geq (k-2)|G| + M + (k-1)\beta(\mathcal{L}) + 1 \\ &\geq (k-2)|G| + M + (k-1)\frac{2\|\mathcal{H}\| + (k+1-p(k))|\mathcal{L}| - (k-1)(1-p(k))c(\mathcal{L}) - 2\|G\|}{2+(k-1)(1-p(k))} + 1 \end{aligned}$$

Hence

$$X \geq (k-2)|G| + M + \frac{2(k-1)}{2+(k-1)(1-p(k))}\|\mathcal{H}\| + \frac{(k-1)(k+1-p(k))}{2+(k-1)(1-p(k))}|\mathcal{L}| - J. \tag{9}$$

where

$$X = 2\|G\|\left(1 + \frac{k-1}{2+(k-1)(1-p(k))}\right),$$

and

$$J = \frac{(k-1)^2(1-p(k))c(\mathcal{L})}{2+(k-1)(1-p(k))} - 1.$$

Now, by (8),

$$M+\frac{2(k-1)}{2+(k-1)(1-p(k))}\left\|\mathcal{H}\right\| \geq \frac{2\left(k-\frac{1}{2}\right)+k(1-p(k))}{2+(k-1)(1-p(k))}\left|\mathcal{H}\right|$$

With (9) this gives

$$X \geq (k-2)\left|G\right| + \frac{2k-1+k(1-p(k))}{2+(k-1)(1-p(k))}\left|\mathcal{H}\right| + \frac{(k-1)(k+1-p(k))}{2+(k-1)(1-p(k))}\left|\mathcal{L}\right| - J.$$

Using $|G| = |\mathcal{H}| + |\mathcal{L}|$, this becomes

$$X \geq \left(k-2+\frac{2k-1+k(1-p(k))}{2+(k-1)(1-p(k))}\right)\left|G\right| + \frac{k^2-2k-(k-p(k))}{2+(k-1)(1-p(k))}\left|\mathcal{L}\right| - J.$$

Now using the basic bound

$$\left|\mathcal{L}\right| \geq k\left|G\right| - 2\left\|G\right\|,$$

gives

$$X \geq \left(k-2+\frac{2k-1+k(1-p(k))}{2+(k-1)(1-p(k))}\right)\left|G\right| + \frac{k^2-2k-(k-p(k))}{2+(k-1)(1-p(k))}\left(k\left|G\right| - 2\left\|G\right\|\right) - J,$$

so

$$2\left\|G\right\|\left(1+\frac{k^2-2k+p(k)-1}{2+(k-1)(1-p(k))}\right) \geq \left(k-2+\frac{3k-kp(k)-1+k(k^2-2k-(k-p(k)))}{2+(k-1)(1-p(k))}\right)\left|G\right| - J,$$

After some algebra, this becomes

$$2\left\|G\right\| \geq \left(k-1+\frac{k-3}{k^2-k(p(k)+1)+2p(k)}\right)\left|G\right| - \frac{J}{ZZ}.$$

That proves the theorem. $\square$

References

[1] O.V. Borodin, *Criterion of chromaticity of a degree prescription*, Abstracts of IV All-Union Conf. on Th. Cybernetics, 1977, pp. 127–128. 1

[2] D. Cranston and L. Rabern, *Edge lower bounds for list critical graphs, via discharging*, arXiv:1602.02589 (2016). 1, 2

[3] P. Erdős, A.L. Rubin, and H. Taylor, *Choosability in graphs*, Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium, vol. 26, 1979, pp. 125–157. 1

[4] T. Gallai, *Kritische Graphen I.*, Publ. Math. Inst. Hungar. Acad. Sci **8** (1963), 165–192 (in German). 1, 2

[5] H.A. Kierstead and L. Rabern, *Improved lower bounds on the number of edges in list critical and online list critical graphs*, arXiv:1406.7355 (2014). 1, 2

[6] ———, *Extracting list colorings from large independent sets*, arXiv:1512.08130 (2015). 2

[7] A.V. Kostochka and M. Stiebitz, *A new lower bound on the number of edges in colour-critical graphs and hypergraphs*, Journal of Combinatorial Theory, Series B **87** (2003), no. 2, 374–402. 2

[8] A.V. Kostochka and M. Yancey, *Ore's conjecture on color-critical graphs is almost true*, J. Combin. Theory Ser. B **109** (2014), 73–101. MR 3269903 2

[9] M. Krivelevich, *On the minimal number of edges in color-critical graphs*, Combinatorica **17** (1997), no. 3, 401–426. 2