1. Overview

For a multigraph G, we clearly have $\chi'(G) \geq \lceil |E(G)| / \lfloor |V(G)|/2 \rfloor \rceil$. Likewise, the same bound holds for any subgraph H. Thus, The biggest open problem in edge-coloring is the Goldberg–Seymour conjecture. Over the past two decades, the main tool for attacking this problem has become Tashkinov trees, a vast generalization of Vizing fans and Kierstead paths. The second author proved that if G is a line graph, then $\chi(G) \leq \max\{\omega(G), \frac{7\Delta(G)+10}{8}\}$. In the same paper, he conjectured that $\chi(G) \leq \max\{\omega(G), \frac{5\Delta(G)+8}{6}\}$. This conjecture is best possible, as shown by replacing each edge in a 5-cycle by k parallel edges, and taking the line graph. We call the latter inequality the $\frac{5}{6}$-Conjecture, and in this paper we prove it. Along the way, we develop more general techniques and results that will likely be of independent interest, due to their use in approaching the Goldberg–Seymour conjecture.

A graph G is *elementary* if $\chi'(G) = \mathcal{W}(G)$; such graphs satisfy the Goldberg–Seymour Conjecture. A *defective color* for a Tashkinov tree T is a color used on more than one edge from $V(T)$ to $V(G) - V(T)$; a Tashkinov tree is *strongly closed* if it has no defective color. Andersen [] and Goldberg [] showed that if G is critical, then G is elementary if there exists $e \in E(G)$ and $X \subseteq V(G)$ and a k-edge-coloring φ of $G - e$ such that X contains the endpoints of e and X is elementary and strongly closed w.r.t. φ. Thus, to show that G is elementary, it suffices to show that if G is $(k+1)$-critical, then there exists an edge $e \in E(G)$ and a k-coloring φ of $G - e$ such that some maximal Tashkinov tree containing e is strongly closed. The following definition is useful. A vertex $v \in V(G)$ is *special* if every Vizing fan rooted at v (taken over all k-colorings of $G - e$, over all edges e incident to v) has at most 3 vertices, including v. As a warmup, in Section 2 we prove that if $\chi'(G) \geq \Delta(G)+2$ and every vertex of G is special, then G is elementary, i.e., $\chi'(G) = \mathcal{W}(G)$. Next, we push our methods further, allowing our maximal Tashkinov tree to have at most 3 non-special vertices.

In Section 3, we consider the $\frac{5}{6}$-Conjecture. As a consequence of results in Section 2, if G is a minimal counterexample, then every non-special vertex v has $d_G(v) < \frac{3}{4}\Delta(G)$. Since every maximal Tashkinov tree T is elementary, and every non-special vertex misses more than $\frac{k}{4}$ colors, we conclude that T has at most 3 non-special vertices. Thus our results from Section 2 apply. As a consequence, every minimal countexample to the $\frac{5}{6}$-Conjecture is elementary. To complete the proof of the $\frac{5}{6}$-conjecture, we show for each graph G that if $\chi'(G) = \mathcal{W}(G)$, then $\chi'(G) \leq \max\{\omega(G), \frac{5\Delta(G)+8}{6}\}$.

Graphs can have multiple edges.

2. Tashkinov Trees

Recall that a graph G is *elementary* if $\chi'(G) = \mathcal{W}(G)$, as defined above. We also use the following notation. Let $[k]$ denote $\{1, \ldots, k\}$. A graph G is *critical* if $\chi'(G - e) < \chi'(G)$ for all $e \in E(G)$. For a graph G and a partial k-edge-coloring φ, for each vertex $v \in V(G)$, let $\varphi(v)$ denote the set of colors used in φ on edges incident to v. Let $\overline{\varphi}(v) = [k] \setminus \varphi(v)$. A color c is *seen* by a vertex v if $c \in \varphi(v)$ and c is *missed* by v if $c \in \overline{\varphi}(v)$. Given a partial k-edge-coloring φ, a set $W \subseteq V(G)$ is *elementary* (w.r.t. φ) if each color in $[k]$ is missed by at most one vertex of W. More formally, $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$ for all distinct $u, v \in W$. A *defective color* for a set $X \subseteq V(G)$ (w.r.t. φ) is a color used on more than one edge from

X to $V(G) \setminus X$. A set X is *strongly closed* w.r.t. φ if X has no defective color. Elementary and strongly closed sets are of particular interest because of the following theorem, proved implicitly by Andersen [] and Goldberg [].

Theorem 1. *Let G be a graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G)$. If G is critical, then G is elementary if and only if there exists $uv \in E(G)$, a k-edge-coloring φ of $G - uv$, and a set X with $u, v \in X$ such that X is both elementary and strongly closed w.r.t. φ.*

A *Tashkinov tree* w.r.t. φ is a sequence $v_0, e_1, v_1, e_2, \ldots, v_{t-1}, e_t, v_t$ such that all v_i are distinct, $e_i = v_j v_i$ and $\varphi(e_i) \in \overline{\varphi}(v_\ell)$ for some j and ℓ with $0 \leq j < i$ and $0 \leq \ell < i$. Tashkinov trees are of interest because of the following lemma.

Tashkinov's Lemma. *Let G be a graph with $\chi'(G) = k+1$, for some integer $k \geq \Delta(G)+1$ and choose $e \in E(G)$ such that $\chi'(G - e) < \chi'(G)$. Let φ be a k-edge-coloring of $G - e$. If T is a Tashkinov tree w.r.t. φ and e, then $V(T)$ is elementary w.r.t. φ.*

In view of Theorem 1 and Tashkinov's Lemma, to prove that a graph G is elementary, it suffices to find an edge e, a k-edge-coloring φ of $G - e$, and a Tashkinov tree T containing e such that $V(T)$ is strongly closed. This motivates our next two lemmas. But first, we need a few more definitions.

Let $t(G)$ be the maximum number of vertices in a Tashkinov tree over all $e \in E(G)$ and all k-edge-colorings φ of $G - e$. Let $\mathcal{T}(G)$ be the set of all triples (T, e, φ) such that $e \in E(G)$, φ is a k-edge-coloring of $G - e$ and T is a Tashkinov tree with respect to e and φ with $|T| = t(G)$. Notice that, by definition, we have $\mathcal{T}(G) \neq \emptyset$. For a k-edge-coloring φ of $G - e$, a maximal Tashkinov tree starting with e may not be unique. However, if T_1 and T_2 are both such trees, then it is easy to show that $V(T_1) \subseteq V(T_2)$; by symmetry, also $V(T_2) \subseteq V(T_1)$, so $V(T_1) = V(T_2)$. Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$. Let φ be a k-edge-coloring of $G - e_0$ for some $e_0 \in E(G)$. For $v \in V(G)$ and colors α, β, let $P_v(\alpha, \beta)$ be the maximal connected subgraph of G that contains v and is induced by edges with color α or β. So $P_v(\alpha, \beta)$ is a path or a cycle.

Lemma 2. *Let G be a non-elementary critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$. For every $v_0 v_1 \in E(G)$, k-edge-coloring φ of $G - v_0 v_1$ and for all $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ we have $|P_{v_1}(\alpha, \beta)| < t(G)$.*

Proof. Suppose the lemma is false and choose $v_0 v_1 \in E(G)$, a k-edge-coloring φ of $G - v_0 v_1$, $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ such that $|P_{v_1}(\alpha, \beta)| \geq t(G)$. Put $P = P_{v_1}(\alpha, \beta)$. Clearly P must end at v_0 (or we can swap colors α and β on P and color $v_0 v_1$), so let $v_1, \ldots, v_r, v_0$ denote the vertices of P in order. Let $(T, v_0 v_1, \varphi)$ be a Tashkinov tree that begins with edges $v_0 v_1, v_1 v_2, \ldots, v_{r-1} v_r$. Then $V(T) = V(P)$ since $t(G) \geq |T| \geq |P| \geq t(G)$. Since G is non-elementary, Theorem 1 implies that $V(T)$ is not strongly closed, so T has a defective color δ with respect to φ. Choose $\tau \in \overline{\varphi}(v_2)$. Let $Q = P_{v_2}(\tau, \delta)$. Since T is maximal, δ is not missing at any vertex of T; since $V(T)$ is elementary, τ is not missing at any vertex of T other than v_2. As a result, Q ends outside $V(T)$. Now Q could leave $V(T)$ and re-enter it repeatedly, but Q ends outside $V(T)$, so there is a last vertex $w \in V(Q) \cap V(T)$; say Q ends at $z \in V(G) \setminus V(T)$. Let $\pi \notin \{\alpha, \beta\}$ be a color missing at w. Since $|T| = t(G)$, no edge colored τ or π leaves $V(T)$. So, we can swap τ and π on every edge in $G - V(T)$ without changing the fact that T is a Tashkinov tree with $|T| = t(G)$. Now swap δ and π on the subpath of Q from w to z; since π is missing at w, the $\delta - \pi$ path does end at w. Now δ is missing at w, but δ was defective in φ, so some other edge e colored δ still leaves $V(T)$, adding e gets a larger Tashkinov tree, a contradiction. □

3. Special vertices

Recall that a vertex $v \in V(G)$ is *special* if every Vizing fan rooted at v (taken over all k-colorings of $G - e$, over all edges e incident to v) has at most 3 vertices, including v. Let $\nu(T)$ be the number of non-special vertices in T.

Lemma 3. *Let G be a critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$. Let φ be a k-edge-coloring of $G - v_0v_1$. Suppose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Let $P = v_1v_2 \cdots v_r$ be an $\alpha - \beta$ path with edges $e_i = v_iv_{i+1}$ for $1 \leq i \leq r - 1$. If v_i is special for all odd i, then for any $\tau \in \overline{\varphi}(v_0)$ there are edges $f_i = v_iv_{i+1}$ for $1 \leq i \leq r - 1$ such that $f_i = e_i$ for i even and $\varphi(f_i) = \tau$ for i odd.*

Proof. Suppose not and choose a counterexample minimizing r. By minimality of r, we have $\varphi(v_{r-1}v_r) = \alpha$ and we have $f_i = v_iv_{i+1}$ for $1 \leq i \leq r - 2$ such that $f_i = e_i$ for i even and $\varphi(f_i) = \tau$ for i odd. Swap α and β on e_i for $1 \leq i \leq r-3$ and then color v_0v_1 (call this edge e_0) with α and uncolor e_{r-2}. Let φ' be the resulting coloring. Since $k \geq \Delta(G) + 1$, some color other than α is missing at v_{r-2}; let γ be such a color. Now v_{r-1} is special since $r - 1$ is odd (since P starts and ends with α), so there is an edge $e = v_{r-1}v_r$ with $\varphi'(e) = \gamma$. Swap τ and α on e_i for $0 \leq i \leq r - 3$ to get a new coloring φ^*. Now γ and τ are both missing at v_{r-2} in φ^*. Since v_{r-1} is special, the fan with v_{r-2}, v_{r-1}, v_r and e implies that there is an edge $f_{r-1} = v_{r-1}v_r$ with $\varphi^*(f_{r-1}) = \tau$. But we have never recolored f_{r-1}, so $\varphi(f_{r-1}) = \tau$, a contradiction. □

Lemma 4. *Let G be a non-elementary critical graph with $\chi'(G) = k + 1$ for an integer $k \geq \Delta(G) + 1$. Let $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$ for some $v_0v_1 \in E(G)$. Let $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and put $P = P_{v_1}(\alpha, \beta)$. The P contains a non-special vertex. In particular, $\nu(T) \geq 1$.*

Proof. Suppose every vertex of P is special. Applying Lemma 3 to P shows that every $\tau \in \overline{\varphi}(v_0)$, there is a τ-edge in T incident to every $v \in V(P - v_0)$. By symmetry, the same is true of every $v \in V(P)$. Hence $V(P) = V(T)$ contradicting Lemma 2. □

Theorem 5. *If G is a critical graph in which every vertex is special, then*

$$\chi'(G) \leq \max\left\{\left\lceil \chi'_f(G) \right\rceil, \Delta(G) + 1\right\}.$$

Proof. Suppose G is a critical graph in which every vertex is special and put $k = \chi'(G) - 1$. Then $k \geq \Delta(G) + 1$. Since $\mathcal{T}(G) \neq \emptyset$, applying Lemma 4, we conclude that G is elementary. Hence $\chi'(G) = \left\lceil \chi'_f(G) \right\rceil$, a contradiction. □

4. The easy bound

Let G be a graph. The *claw-degree* of $x \in V(G)$ is

$$d_{\text{claw}}(x) := \max_{\substack{S \subseteq N(x) \\ |S|=3}} \frac{1}{4}\left(d(x) + \sum_{v \in S} d(v)\right).$$

The *claw-degree* of G is

$$d_{\text{claw}}(G) := \max_{x \in V(G)} d_{\text{claw}}(x).$$

Theorem 6. *If G is a graph, then*

$$\chi'(G) \leq \max\left\{\left\lceil \chi'_f(G) \right\rceil, \Delta(G)+1, \left\lceil \frac{4}{3} d_{claw}(G) \right\rceil\right\}.$$

Proof. Suppose not and choose a counterexample G minimizing $\|G\|$; note that G critical. Let $k = \chi'(G) - 1$, so $k \geq \left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil$. By Theorem 5, G has a non-special vertex x. Choose $xy_1 \in E(G)$ and a k-edge-coloring φ of $G - xy_1$ such that φ has a fan F of length 3 rooted at x with leaves y_1, y_2, y_3. Since $V(F)$ is elementary,

$$2 + k - d(x) + \sum_{i \in [3]} k - d(y_i) \leq k,$$

and hence

$$d_{\text{claw}}(x) \geq \frac{1}{4}\left(d(x) + \sum_{i \in [3]} d(y_i)\right) \geq \frac{3k+2}{4}.$$

This gives the contradiction

$$\left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil \leq k \leq \frac{4}{3} d_{\text{claw}}(G) - \frac{2}{3}.$$

□

TODO: ADD REED, LOCAL REED AND SUPERLOCAL REED CONSEQUENCES.

5. Properties of non-special vertices

For a path Q and $x, y \in V(Q)$, let $d_Q(x,y) = \ell(Q)$.

Lemma 7. *Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$. Let φ be a k-edge-coloring of $G - v_0v_1$. Suppose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. If $\tau \in \overline{\varphi}(x)$ for some $x \in V(C)$ and there is a τ-colored edge from $y \in V(C)$ to $w \in V(G) \setminus V(C)$, then C has a subpath Q with $x \in V(Q)$, $y \notin V(Q)$ and non-special endpoints z_1, z_2 such that $d_Q(x, z_i)$ is odd for $i \in [2]$. Moreover, there are no τ-colored edges between z_i and $N_C(z_i)$ for $i \in [2]$.*

Proof. TODO: FILL THIS SPACE WITH PROOF. □

6. Thin graphs

Let G be a critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$. For vertices $x, y \in V(G)$, we say that x is *y-special* if every Vizing fan rooted at x, with respect to any k-edge-coloring of $G - xy$, has at most 3 vertices. We say that G is *k-thin* if $\mu(G) < 2k - d(x) - d(y)$ for all non-special $x, y \in V(G)$.

Lemma 8. *Let G be a k-thin, critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$. Let φ be a k-edge-coloring of $G - v_0v_1$. Suppose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. If Q is a subpath of C with non-special end vertices and all special internal vertices such that $2 \leq \ell(Q) \leq \ell(C) - 2$, then $\ell(Q)$ is even.*

Proof. Suppose to the contrary that we have a subpath Q of C with non-special end vertices and all special internal vertices, such that $\ell(Q) \leq \ell(C) - 2$ and $\ell(Q)$ is odd. Let x and y be the end vertices of Q. Say $C = v_1v_2 \cdots v_rv_0v_1$. By rotating the $\alpha - \beta$ coloring of C, we may assume that $x = v_1$ and $y = v_a$ where $a \geq 4$ is even.

Apply Lemma 3 twice, to show that $\mu(v_2v_3) \geq 2k - d(v_1) - d(v_a)$, violating k-thinness. □

Lemma 9. *Let G be a k-thin, critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$. Let φ be a k-edge-coloring of $G - v_0v_1$. Suppose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. If C contains exactly 3 non-special vertices, then $C = xyAzBx$ where A and B are paths of even length and x, y, z are all non-special. Moreover, x is y-non-special and y is x-non-special.*

Proof. Immediate from Lemma 8 and Lemma 3. □

Lemma 10. *Let G be a non-elementary, k-thin, critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$. Let $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$. Suppose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Then there are consecutive non-special vertices on $P_{v_1}(\alpha, \beta) + v_0v_1$.*

Proof. Put $C = P_{v_1}(\alpha, \beta) + v_0v_1$. By Lemma 2, there is $x \in V(C)$ and $\tau \in \overline{\varphi}(x)$ such that there is a τ-colored edge from $y \in V(C)$ to $w \in V(T) \setminus V(C)$. By Lemma 7, C has a subpath Q with $x \in V(Q)$, $y \notin V(Q)$ and non-special endpoints z_1, z_2 such that $d_Q(x, z_i)$ is odd for $i \in [2]$. Let Q' be the subpath of Q with endpoints z_1 and z_2 that contains y. Since C is an odd cycle, $\ell(Q')$ is odd. Let Q^* be a minimum length subpath of Q' with non-special ends. Then $\ell(Q^*) = 1$ by Lemma 8, as desired. □

We say that x is *$\{y, z\}$-non-special* if there is a fan rooted at x that contains y and z. TODO: MERGE THIS DEFINITION WITH SPECIAL.

Lemma 11. *Let G be a non-elementary, k-thin, critical graph with $\chi'(G) = k+1$ for an integer $k \geq \Delta(G)+1$. Let $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$ with $\nu(T) \leq 3$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ so that $C = P_{v_1}(\alpha, \beta) + v_0v_1$ contains as many non-special vertices as possible. Then C contains non-special vertices z_1, z_2, z_3 such that either*

- *z_1 is $\{z_2, z_3\}$-special and z_2 is z_1-special; or*
- *z_i is z_{3-i}-non-special and z_{i+1} is z_{4-i}-non-special for $i \in [2]$.*

Proof. By Lemma 2, there is $x \in V(C)$ and $\tau \in \overline{\varphi}(x)$ such that there is a τ-colored edge from $y \in V(C)$ to $w \in V(T) \setminus V(C)$.

First, suppose C contains only 2 non-special vertices, z_1 and z_2. By Lemma 10, z_1 and z_2 are consecutive on C. By Lemma 7, C has a subpath Q with $x \in V(Q)$ and $y \notin V(Q)$ with endpoints z_1, z_2 and there are no τ-colored edges between z_i and $N_C(z_i)$ for $i \in [2]$.

By rotating the $\alpha - \beta$ coloring of C, we may assume that $x = v_0$. Consider $C' = P_{v_1}(\tau, \beta) + v_0v_1$. Since z_1 and z_2 are not consecutive on C' and C' contains no other non-special vertices by the maximality condition on C, Lemma 10 gives a contradiction.

So, C contains exactly 3 non-special vertices, z_1, z_2 and z_3. By Lemma 9, $C = z_1z_2Az_3Bz_1$ where A and B are paths of even length. Also, z_1 is z_2-special and z_2 is z_1-special.

By Lemma 7, C has a subpath Q with $x \in V(Q)$ and $y \notin V(Q)$ with endpoints z_1, z_3 and there are no τ-colored edges between z_i and $N_C(z_i)$ for $i \in \{1, 3\}$ (it could happen that z_3 is bypassed and the endpoints are z_1, z_2, but then we get a contradiction as in the previous case). By rotating the $\alpha - \beta$ coloring of C, we may assume that $x = v_0$. Consider $C' = P_{v_1}(\tau, \beta) + v_0v_1$. We know that C' contains z_1 and z_3 and that z_1 and z_2 are not consecutive on C'. By Lemma 10, either z_1 and z_3 are consecutive on C' or z_2 and z_3 are consecutive on Q'.

Suppose z_2 and z_3 are consecutive on C' and connected by a τ edge. Then applying Lemma 8 shows that z_2 is z_3-special and z_3 is z_2-special, so we win.

So it must be that z_1 and z_3 are consecutive on C' and connected by a τ edge. If $z_1 = v_1$, we have a fan with an α-edge from z_1 to z_2 and a τ edge from z_1 to z_3, so z_1 is $\{z_2, z_3\}$-special. When $z_1 \neq v_1$, we get the same thing, but have to shift the coloring over to z_1 like in the proof of Lemma 3 using another missing color γ at z_1. $\square$

Theorem 12 (from strengthening Brooks paper). *If Q is the line graph of a graph G and Q is vertex critical, then*

$$\chi(Q) \leq \max\left\{\omega(Q), \Delta(Q) + 1 - \frac{\mu(G) - 1}{2}\right\}.$$

Theorem 13. *If Q is a line graph, then*

$$\chi(Q) \leq \max\left\{\lceil \chi_f(Q) \rceil, \left\lceil \frac{5\Delta(Q) + 3}{6} \right\rceil\right\}.$$

Proof. Suppose the theorem is false and choose a counterexample minimizing $|Q|$. Let $k = \max\left\{\lceil \chi_f(Q) \rceil, \left\lceil \frac{5\Delta(Q)+3}{6} \right\rceil\right\}$. Say $Q = L(G)$ for a graph G. Minimality of Q implies that G is k-edge-critical.

Claim 0. *Let F be a fan rooted at x with respect to a k-edge-coloring of $G - xy$. If $|F| = 4$, then*

$$d(x) \leq \frac{1}{4}\left(-4 + \sum_{v \in V(F - x)} d(v)\right).$$

Proof: Since F is elementary, we have

$$2 + k - d(x) + \sum_{v \in V(F - x)} k - d(v) \leq k,$$

so
$$2 + (|F| - 1)k \le d(x) + \sum_{v \in V(F-x)} d(v).$$
Using $6k \ge 5(\Delta(Q) + 1) - 2 \ge 5(d(x) + d(v) - \mu(xv)) - 2$ for each $v \in V(F - x)$, we get
$$2 + \sum_{v \in V(F-x)} \left(\frac{5}{6}(d(x) + d(v) - \mu(xv)) - \frac{1}{3}\right) \le d(x) + \sum_{v \in V(F-x)} d(v),$$
so
$$12 + (5|F| - 11)\, d(x) - 2|F| \le \sum_{v \in V(F-x)} 5\mu(xv) + \sum_{v \in V(F-x)} d(v).$$
Now $\sum_{v \in V(F-x)} \mu(xv) \le d(x)$, so this implies
$$12 + (5|F| - 16)\, d(x) - 2|F| \le \sum_{v \in V(F-x)} d(v).$$
Using $|F| = 4$ gives
$$d(x) \le \frac{1}{4}\left(-4 + \sum_{v \in V(F-x)} d(v)\right).$$
□

Claim 1. *If $x \in V(G)$ with $d(x) > \frac{3}{4}\Delta(G) - 1$, then x is special.*
Proof: This is immediate from Claim 0, since $d(v) \le \Delta(G)$ for all $v \in V(F - x)$. □

Claim 2. *If $x_1x_2 \in E(G)$ with*
$$d(x_i) > \frac{2}{3}\Delta(G) - \frac{4}{3},$$
for at least one $i \in [2]$, then x_1 is x_2-special or x_2 is x_1-special.
Proof: Suppose x_1 is not x_2-special and x_2 is not x_1-special. By Claim 0, for each $i \in [2]$,
$$d(x_i) \le \frac{1}{4}\left(-4 + \sum_{v \in V(F-x)} d(v)\right) \le \frac{1}{4}\left(-4 + d(x_{3-i}) + 2\Delta(G)\right),$$
Substituting the bound on $d(x_{3-i})$ into that on $d(x_i)$ and simplifying gives for each $i \in [2]$,
$$d(x_i) \le -\frac{4}{3} + \frac{2}{3}\Delta(G).$$
□

Claim 3. *The theorem is true.*
Proof: Let $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$. By Lemma 11, one of the following holds:
1. (1) G is elementary; or
2. (2) G is not thin; or
3. (3) $\nu(T) = 3$ and $E(T)$ contains non-special $x_1, x_2, x_3 \in V(T)$ such that x_1 is x_2-non-special, x_2 is x_1-non-special, x_2 is x_3-non-special and x_3 is x_2-non-special; or

(4) $\nu(T) = 3$ and $E(T)$ contains non-special $x_1, x_2, x_3 \in V(T)$ such that x_1 is $\{x_2, x_3\}$-non-special and x_2 is x_1-non-special; or

(5) $V(T)$ contains four non-special vertices x_1, x_2, x_3, x_4.

If (1) holds, then $k + 1 = \lceil \chi_f(Q) \rceil \le k$, a contradiction.

If (2) holds, then by Claim 1 we have $\mu(G) \ge 2k - 2\frac{3(1-\epsilon)}{2\epsilon-1}\Delta(G)$. Hence Theorem 12 gives

$$k + 1 \le \Delta(Q) + 1 - k + \frac{3(1-\epsilon)}{2\epsilon - 1}\Delta(G) + \frac{1}{2},$$

so

$$2(k+1) \le \Delta(Q) + \frac{5}{2} + \frac{3(1-\epsilon)}{2\epsilon - 1}\Delta(G).$$

Since $k \ge \Delta(G) + 1$, this gives

$$k + 1 < \frac{\Delta(Q) + \frac{5}{2}}{2 - \frac{3(1-\epsilon)}{2\epsilon-1}},$$

which is a contradiction when $\epsilon > \frac{4}{5}$.

Suppose (3) holds. So

$$2 + \sum_{i \in [3]} k - d(x_i) \le k,$$

using Claim 2, this gives

$$3\left(\frac{2(1-\epsilon)}{3\epsilon - 2}\right)\Delta(G) \ge 2k + 2,$$

which is a contradiction when $\epsilon \ge \frac{5}{6}$.

Suppose (4) holds. Doing the bounding like in the proof of Claim 2 gets a contradiction when $\epsilon \ge \frac{5}{6}$.

So (5) must hold. But then

$$2 + \sum_{i \in [4]} k - d(x_i) \le k,$$

using Claim 1 gives

$$\frac{12(1-\epsilon)}{2\epsilon - 1}\Delta(G) \ge 3k + 2,$$

which is a contradiction when $\epsilon \ge \frac{5}{6}$. $\square$

$\square$

7. The $\frac{5}{6}$-Conjecture

Lemma 14. *If H is a connected multigraph and $G = L(H)$, then $\mathcal{W}(H) \le \max\{\omega(G), \frac{5}{6}(\Delta(G)+1) + \frac{3}{6}\}$.*

Proof. Let $d = d_H(x)$, $\Delta = \Delta(H)$, and $h = |H|$. Also, let $p = \sum_{v \in N(x)} d_H(v)$ and let $t = \Delta h - 2||H||$. Note that $0 < t \le \Delta$. Also $p \ge Md - t$. Now summing over $N_H(x)$ gives

$$|N(x)|(\Delta h - t)/(h-1) > 5/6((|N(x)|-1)d + |N(x)|\Delta - t) + |N(x)|/2$$

Solving for $|N(x)|$ gives

$$|N(x)| < (5d + 5t)/(3 + 5d + 5\Delta - 6(\Delta h - t)/(h-1)).$$

Since the numerator and denominator are linear in t, the right side is maximized at one end of the interval $1 \le t \le D$. Letting $t = D$, gives $|N(x)| < (5d + 5\Delta)/(3 + 5d - \Delta)$, like you had originally. Letting $t = 1$, gives $|N(x)| < (5d+5)/(3+5d+5\Delta-6(\Delta h-1)/(h-1))$, which requires a little more analysis, akin to what you wrote in your most recent email.

Does that look right to you?

I did the analysis a little differently, but I got to the same conclusion: Substituting $d \ge 4D/5$ gives that if $M \ge 3$, then we must have $h \le 4$, which implies $h \le 3$, which contradicts $M \ge 3$.

So, I think I believe it. I also agree there must be an easier way. One thing that seems a little magical is that when $5/6 - M/(h-1) \ge 0$ all of the h's go away.

$w(H)$ really has a ceiling in its definition, not sure how much that changes things. without, it is the fractional chromatic index.

i think we get some gain as well from the $\Delta(H) + 2$ in place of $\Delta(H)$ we get as i wrote in the previous emails. Maybe this helps with the ceiling.

We can use $|H|$ odd to get a bit better on the ceiling in what you wrote since the top is even (divide both by two before doing ceiling approximation).

Thinking about your comment that we can assume H is critical, we can, but not how i was setting it up. Probably you are already thinking something like this:

Assume Goldberg. Take minimum counterexample to 5/6 conjecture, say $G = L(H)$. The H is critical. From the argument like in strengthening of Brooks, we get $\chi(G) \ge \Delta(H) + 2$. By Goldberg this implies

$$\chi(G) = \max_{Q \subseteq H \text{ s.t. } |Q| \ge 3 \text{ and odd}} \left\lceil \frac{2||Q||}{|Q|-1} \right\rceil$$

If the max is achieved at a proper subgraph of H, then there is an edge we can remove without decreasing the max, but this decreases the chromatic number by criticality and the max is a lower bound, so impossible. Therefore, $|H|$ is odd and

$$\chi(G) = \left\lceil \frac{2||H||}{|H|-1} \right\rceil$$

so,

$$\left\lceil \frac{2||H||}{|H|-1} \right\rceil \geq \Delta(H) + 2$$

$$2||H||/(|H|-1) \geq \Delta(H) + 1$$

using

$$\Delta(H)|H| \geq 2||H||,$$

using $\Delta(H)|H| \geq 2||H||$, I get

$$\Delta(H) \geq |H| - 1,$$

□

I think we should be able to prove that the conjecture follows from Goldberg–Seymour. That lemma you proved is pretty useful. We can assume that H is critical, which implies that $|N(x)| \geq 2$ for all x in H. Now let J be the simple graph underlying H. We know that $\delta(J) \geq 2$. Let $B = \{x \in H s.t. d_J(x) \geq 3\}$. That lemma implies that $|B| \leq 4$. Further, if $|B| = 4$, then each vertex of B has degree 3 in J. If $|B| = 3$, then two vertices of B have degree 3 in J and one has degree 4 in J. Otherwise $|B| \leq 2$. Now if J has a vertex x of degree at least 5, and $|B| = 2$, then the other vertex in B has degree 3 in J. Now x must be a cut-vertex (since J is formed by identifying one vertex in multiple disjoint cycles, exactly one of which has a chord). But a cut-vertex in J is also a cut-vertex in H, which is a contradiction. Thus, we only need consider the cases when $|B| = 3$ and $|B| = 4$, which have degree sequences $3, 3, 3, 3, 2, \ldots 2.$ and $4, 3, 3, 2, \ldots, 2$. $|B| = 4$ is a subdivided K_4 or a subdivision of a 4-cycle where one matching has multiplicity 2. $|B| = 3$ is a subdivision of a triangulated 5-cycle. I haven't worked out those cases, but I don't think they should be too hard.

Lemma 15. *Suppose $G = L(H)$ and G is a minimal counterexample to the $\frac{5}{6}$-Conjecture. Let $k = \frac{5}{6}(\Delta(G)+1)$. If T is a Tashkinov tree w.r.t. a k-edge-coloring φ of $H - e$, then*

$$\sum_{v\in V(T)} d_H(v)(5d_T(v)-6) \leq -12 + 5\sum_{e\in E(T)} \mu_H(e)$$

Proof. Since T is elementary, the sets of colors missing at vertices of T are disjoint, so $2 + \sum_{v\in V(T)}(k - d_H(v)) \leq k$. Rewriting this gives $k(|V(T)|-1) \leq -2 + \sum_{v\in V(T)} d_H(v)$. For each edge $xy \in E(T)$, we have $k = \frac{5}{6}(\Delta(G)+1) \geq \frac{5}{6}(d_H(x) + d_H(y) - \mu_H(xy))$. Summing over all $|T|-1$ edges gives

$$\begin{aligned}
-2 + \sum_{v\in V(T)} d_H(v) &\geq k(|V(T)|-1) \\
&\geq \frac{5}{6}(\Delta(G)+1)(|T|-1) \\
&\geq \frac{5}{6}\sum_{uv\in E(T)}(d_H(u) + d_H(v) - \mu_H(uv)) \\
&= \frac{5}{6}\sum_{v\in V(T)} d_H(v)d_T(v) - \frac{5}{6}\sum_{uv\in E(T)}\mu_H(uv)
\end{aligned}$$

To prove the lemma, we take the first and last expressions in the inequality chain, multiply by 6, then rearrange terms. □

Corollary 16. *If $G = L(H)$ and G is a minimal counterexample to the $\frac{5}{6}$-Conjecture, then each $x \in V(H)$ is special if $d_H(x) > \frac{3}{4}\Delta(H) - 3$.*

Proof. Suppose that x is a non-special vertex. Choose e incident to x and a k-edge-coloring φ of $G - e$ such that there exists a Vizing fan T rooted at x with $|T| \geq 4$. Since every edge in F is incident to x, we have $\sum_{e\in E(T)} \mu_H(e) \leq d_H(x)$. From Lemma 15, we have

$$\begin{aligned}
-12 + 5d_H(x) &\geq -12 + 5\sum_{e\in E(T)}\mu_H(e) \\
&\geq \sum_{v\in T}(5d_T(v)-6)d_H(v) \\
&\geq (5d_T(x)-6)d_H(x) + \sum_{v\in T-x}(5d_T(v)-6)d_H(v) \\
&= (5(|T|-1)-6)d_H(x) - \sum_{v\in V(T-x)} d_H(v),
\end{aligned}$$

where the final equality holds because each vertex $v \in T - x$ is a leaf. Now rearranging terms gives

$$\begin{aligned} -12 + \sum_{v \in V(T-x)} d_H(v) &\geq (5(|T|-1) - 11)d_H(x) \\ -12 + (|T|-1)\Delta(H) &\geq (5(|T|-16)d_H(x) \\ d_H(x) &\leq \frac{-12 + (|T|-1)\Delta(H)}{5|T|-16} \\ d_H(x) &\leq \frac{-12 + 3\Delta(H)}{4} = \frac{3}{4}\Delta(H) - 3, \end{aligned}$$

where the final inequality holds because $|T| \geq 4$ and the right side decreases as a function of $|T|$. $\square$