Lemma 1. *Let G be a graph and $\pi\colon V(G) \to [k]$ a proper k-coloring of G. Suppose that π has no G-independent transversal, but for every $e \in E(G)$, π has a $(G-e)$-independent transversal. Then for every $xy \in E(G)$ there is $J \subseteq [k]$ with $\pi(x), \pi(y) \in J$ and an induced matching M of $G\left[\pi^{-1}(J)\right]$ with $xy \in M$ such that*

1. *$\bigcup M$ totally dominates $G\left[\pi^{-1}(J)\right]$,*
2. *the multigraph with vertex set J and an edge between $a, b \in J$ for each $uv \in M$ with $\pi(u) = a$ and $\pi(v) = b$ is a (simple) tree. In particular $|M| = |J| - 1$.*

Proof. Suppose the lemma is false and choose a counterexample G with $\pi\colon V(G) \to [k]$ so as to minimize k. Let $xy \in E(G)$. By assumption π has a $(G - xy)$-independent transversal T. Note that we must have $x, y \in T$ lest T be a G-independent transversal of π.

By symmetry we may assume that $\pi(x) = k-1$ and $\pi(y) = k$. Put $X := \pi^{-1}(k-1)$, $Y := \pi^{-1}(k)$ and $H := G - N(\{x, y\}) - E(X, Y)$. Define $\zeta\colon V(H) \to [k-1]$ by $\zeta(v) := \min\{\pi(v), k-1\}$. Note that since $x, y \in T$, we have $|\zeta^{-1}(i)| \geqslant 1$ for each $i \in [k-2]$. Put $Z := \zeta^{-1}(k-1)$. Then $Z \neq \varnothing$ for otherwise $M := \{xy\}$ totally dominates $G[X \cup Y]$ giving a contradiction.

Suppose ζ has an H-independent transversal S. Then we have $z \in S \cap Z$ and by symmetry we may assume $z \in X$. But then $S \cup \{y\}$ is a G-independent transversal of π, a contradiction.

Let $H' \subseteq H$ be a minimal spanning subgraph such that ζ has no H'-independent transversal. Now $d(z) \geqslant 1$ for each $z \in Z$ for otherwise $T - \{x, y\} \cup \{z\}$ would be an H'-independent transversal of ζ. Pick $zw \in E(H')$. By minimality of k, we have $J \subseteq [k-1]$ with $\zeta(z), \zeta(w) \in J$ and an induced matching M of $H'\left[\zeta^{-1}(J)\right]$ with $zw \in M$ such that

1. $\bigcup M$ totally dominates $H'\left[\zeta^{-1}(J)\right]$,
2. the multigraph with vertex set J and an edge between $a, b \in J$ for each $uv \in M$ with $\zeta(u) = a$ and $\zeta(v) = b$ is a (simple) tree.

Put $M' := M \cup \{xy\}$ and $J' := J \cup \{k\}$. Since H' is a spanning subgraph of H, $\bigcup M$ totally dominates $H\left[\zeta^{-1}(J)\right]$ and hence $\bigcup M'$ totally dominates $G\left[\pi^{-1}(J')\right]$. Moreover, the multigraph in (2) for M' and J' is formed by splitting the vertex $k-1 \in J$ in two vertices and adding an edge between them and hence it is still a tree. This final contradiction proves the lemma. $\square$