A slightly better lower bound on the number of edges in (online) list critical graphs

October 15, 2015

1 Introduction

Let $\mathcal{T}_k$ be the Gallai trees with maximum degree at most $k-1$, excepting K_k. For a graph G, let $W^k(G)$ be the set of vertices of G that are contained in some K_{k-1} in G.

Definition 1. A graph G is *AT-reducible* to H if H is a nonempty induced subgraph of G which is f_H-AT where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not AT-reducible to any nonempty induced subgraph, then it is *AT-irreducible*.

2 Reducible Configurations

This first lemma tells us how a single high vertex can interact with the low vertex subgraph. This is the version Hal and i used, it (and more) follows from the classification in "mostlow".

Lemma 2.1. *Let $k \geq 5$ and let G be a graph with $x \in V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *$G - x$ has t components $H_1, H_2, \dots, H_t$, and all are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k-1$ for all $v \in V(G-x)$; and*
4. *$\left|N(x) \cap W^k(H_i)\right| \geq 1$ for $i \in [t]$; and*
5. *$d_G(x) \geq t+2$.*

Then G is f-AT where $f(x) = d_G(x) - 1$ and $f(v) = d_G(v)$ for all $v \in V(G-x)$.

To deal with more than one high vertex we need the following auxiliary bipartite graph. For a graph G, $\{X, Y\}$ a partition of $V(G)$ and $k \geq 4$, let $\mathcal{B}_k(X, Y)$ be the bipartite graph with one part Y and the other part the components of $G[X]$. Put an edge between $y \in Y$ and a component T of $G[X]$ iff $N(y) \cap W^k(T) \neq \emptyset$. The next lemma tells us that we have a reducible configuration if this bipartite graph has minimum degree at least three.

Lemma 2.2. *Let $k \geq 7$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. $K_k \not\subseteq G$; and
2. the components of $G - Y$ are in $\mathcal{T}_k$; and
3. $d_G(v) \le k - 1$ for all $v \in V(G - Y)$; and
4. with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $\delta(\mathcal{B}) \ge 3$.

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.

We also have the following version with asymmetric degree condition on $\mathcal{B}$. The point here is that this works for $k \ge 5$. As we'll see in the next section, the consequence is that we trade a bit in our size bound for the proof to go through with $k \in \{5, 6\}$.

Lemma 2.3. *Let $k \ge 5$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. $K_k \not\subseteq G$; and
2. the components of $G - Y$ are in $\mathcal{T}_k$; and
3. $d_G(v) \le k - 1$ for all $v \in V(G - Y)$; and
4. with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $d_{\mathcal{B}}(y) \ge 4$ for all $y \in Y$ and $d_{\mathcal{B}}(T) \ge 2$ for all components T of $G - Y$.

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.

3 Improved bounds on σ

Let $T \in \mathcal{T}_k$. Then each block of T is regular. Say type$(B) = b$ if B is $(b-1)$-regular. Let $x \in V(T)$ and let $B_1, \dots, B_\ell$ be the blocks of T containing x where B_i is of type b_i. Then we say that $\text{type}_T(x) = (b_1, \dots, b_\ell)$. For an endblock B of T, let x_B be the cutvertex of T contained in B and put $T_B := T - (V(B) \setminus \{x\})$. For $b \ge 1$, put $t(b) := 2 - \frac{2}{b}$. For $T \in \mathcal{T}_k$ and $x \in V(T)$ put

$$\sigma_T(x) := k - 2 + \frac{2}{k-1} - d_T(x).$$

For $T \in \mathcal{T}_k$ and $x \in V(T)$ with $\text{type}_T(x) = (b_1, \dots, b_\ell)$, put

$$\sigma'_T(x) := \sigma_T(x) - 2 + \sum_{i \in [\ell]} t(b_i).$$

Furthermore, put

$$\sigma(T) := \sum_{x \in V(T)} \sigma_T(x),$$

and

$$\sigma'(T) := \sum_{x \in V(T)} \sigma'_T(x).$$

Lemma 3.1. *Let $T \in \mathcal{T}_k$ where $k \geq 4$. Then,*

(a) If B is a block of T, then $\sigma(B) = 2$ if $B = K_{k-1}$ and $\sigma(B) \geq k - 2 + \frac{2}{k-1}$ otherwise,

(b) If B is an endblock of T, then $\sigma(T) = \sigma(T_B) + \sigma(B) - \left(k - 2 + \frac{2}{k-1}\right)$.

Proof. Immediate from the definitions (see Kostochka and Stiebitz [1]). □

Lemma 3.2. *If $T \in \mathcal{T}_k$ and $k \geq 4$, then $\sigma(T) \geq \sigma'(T) + 2$.*

Proof. Suppose the lemma is false and let T be a counterexample with the minimum number of blocks. First, suppose T has one block. Then, T is complete or an odd cycle. If T is complete, then $T = K_b$ with $b \in [k-1]$ and hence $\sigma'(T) = \sigma(T) + (t(b) - 2)b = \sigma(T) - 2$. If instead T is an odd cycle, then $\sigma'(T) = \sigma(T) + (t(3) - 2)|T| = \sigma(T) - \frac{2}{3}|T| \leq \sigma(T) - 2$. Hence T must have at least two blocks.

Let B be an endblock of T. Say $\text{type}(B) = b$ and $\text{type}_T(x_B) = (b_1, \ldots, b_\ell)$ where $b_\ell = b$. Then $\text{type}_{T_B}(x_B) = (b_1, \ldots, b_{\ell-1})$. Therefore, we have

$$\sigma'_{T_B}(x_B) = k - 2 + \frac{2}{k-1} - d_{T_B}(x_B) - 2 + \sum_{i \in [\ell-1]} t(b_i),$$

and

$$\sigma'_B(x_B) = k - 2 + \frac{2}{k-1} - d_B(x_B) + t(b_\ell) - 2,$$

and

$$\sigma'_T(x_B) = k - 2 + \frac{2}{k-1} - d_T(x_B) - 2 + \sum_{i \in [\ell]} t(b_i).$$

Since $d_T(x_B) = d_{T_B}(x_B) + d_B(x_B)$, we have

$$\sigma'(T) = \sigma'(T_B) + \sigma'(B) + 2 - \left(k - 2 + \frac{2}{k-1}\right).$$

By our minimality condition on T, we have

$$\sigma(T_B) \geq \sigma'(T_B) + 2,$$

and

$$\sigma(B) \geq \sigma'(B) + 2.$$

Putting this all together with Lemma 3.1 gives the contradiction

$$\sigma'(T) \leq \sigma(T_B) + \sigma'(B) - \left(k - 2 + \frac{2}{k-1}\right) = \sigma(T) - 2.$$

□

Let $T \in \mathcal{T}_k$ and $x \in V(T)$ with $\mathrm{type}_T(x) = (b_1, \ldots, b_\ell)$. We always have $\sum_{i\in[\ell]} b_i \leq k + \ell - 1$. We say that x is *full* if $\sum_{i\in[\ell]} b_i = k + \ell - 1$. When positive integers $k, b_1, \ldots, b_\ell$ are such that $\sum_{i\in[\ell]} b_i < k + \ell - 1$, put

$$\Gamma_{k,(b_1,\ldots,b_\ell)} := 1 - \frac{3 - 2\ell - \frac{2}{k-1} + \sum_{i\in[\ell]} \frac{2}{b_i}}{k + \ell - 1 - \sum_{i\in[\ell]} b_i},$$

when $\sum_{i\in[\ell]} b_i = k + \ell - 1$, put

$$\Gamma_{k,(b_1,\ldots,b_\ell)} := \ell.$$

Lemma 3.3. *If $T \in \mathcal{T}_k$, then $\sigma'_T(x) \geq \Gamma_{k,\mathrm{type}_T(x)}\,(k - 1 - d_T(x))$ for all $x \in V(T)$.*

Proof. We have

$$\sigma'_T(x) = \sigma_T(x) - 2 + \sum_{i\in[\ell]} t(b_i).$$

So, $\sigma'_T(x) \geq c\,(k - 1 - d_T(x))$ for some c and $x \in V(T)$ with $\mathrm{type}(x) = (b_1, \ldots, b_\ell)$ if and only if

$$(1 - c)\left(k - 1 - \sum_{i\in[\ell]}(b_i - 1)\right) + \sum_{i\in[\ell]}\left(2 - \frac{2}{b_i}\right) \geq 3 - \frac{2}{k-1}.$$

A quick computation shows that $\Gamma_{k,(b_1,\ldots,b_\ell)}$ is the largest such c that works when x is not full. When x is full, the first term is zero, so c is irrelevant. We need

$$\sum_{i\in[\ell]}\left(2 - \frac{2}{b_i}\right) \geq 3 - \frac{2}{k-1}.$$

Since x is full, we have $\sum_{i\in[\ell]} b_i = k + \ell - 1$. In particular, since $K_k \not\subseteq T$, we must have $\ell \geq 2$ and hence $b_i \geq 2$ for all $i \in [\ell]$. So, if $\ell \geq 3$, then we have

$$\sum_{i\in[\ell]}\left(2 - \frac{2}{b_i}\right) \geq \ell \geq 3 - \frac{2}{k-1}.$$

Hence we may assume $\ell = 2$. So, $b_1 + b_2 = k + 1$. Simplifying the inequality we need gives

$$\frac{k+1}{b_1(k + 1 - b_1)} = \frac{b_1 + b_2}{b_1 b_2} \leq \frac{1}{2} + \frac{1}{k-1}.$$

The worst case is when $b_1 = 2$, in which case we have

$$\frac{k+1}{b_1(k + 1 - b_1)} = \frac{1}{2} + \frac{1}{k-1}.$$

$\square$

Note that $\Gamma_{k,\mathrm{type}_T(x)}\,(k-1-d_T(x)) = 0$ whenever $x \in W^k(T)$. To make that explicit, put

$$\sigma_T^*(x) := \begin{cases} \Gamma_{k,\mathrm{type}_T(x)}\,(k-1-d_T(x)) & \text{if } x \in V(T) \setminus W^k(T), \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, put

$$\sigma^*(T) := \sum_{x \in V(T)} \sigma_T^*(x).$$

With those definitions, the following is immediate from Lemma 3.2 and Lemma 3.3.

Lemma 3.4. *If $T \in \mathcal{T}_k$ and $k \geq 4$, then $\sigma(T) \geq \sigma^*(T) + 2$.*

So, now our task is to prove lower bounds on $\Gamma_{k,(b_1,\ldots,b_\ell)}$.

Lemma 3.5. *Let $k \geq 6$ and let $b_1, \ldots, b_\ell$ be positive integers. We have*

$$\Gamma_{k,(b_1,\ldots,b_\ell)} \geq \begin{cases} 0 & \text{if } \ell = 1 \text{ and } b_1 = k-1, \\ \frac{1}{2} - \frac{1}{(k-1)(k-2)} & \text{if } \ell = 1 \text{ and } b_1 = k-2, \\ 1 - \frac{3k-5}{(k-1)^2} & \text{if } \ell = 1 \text{ and } b_1 = 1, \\ \frac{2}{3} - \frac{4}{3(k-1)(k-3)} & \text{if } \ell = 1 \text{ and } 2 \leq b_1 \leq k-3, \\ 1 - \frac{1}{k-1} & \text{if } \ell = 2, \\ \ell - 2 + \frac{2}{k-1} & \text{if } \ell \geq 3. \end{cases}$$

Proof. Just compute, for now i checked things with wolfram alpha. The $\ell \geq 3$ case can be improved, but not sure we need it. $\square$

The bound in Lemma 3.3 gives $\sigma_T'(x) \geq 0$ when x is a full vertex. We can often do better.

Lemma 3.6. *Let $T \in \mathcal{T}_k$ for $k \geq 5$. If $x \in V(T)$ is a full vertex of type $(b_1, \ldots, b_\ell)$, then*

$$\sigma_T'(x) \geq \begin{cases} 0 & \text{if } \ell = 2, b_1 = 2 \text{ and } b_2 = k-1, \\ \frac{2}{k-1} + \frac{1}{3} - \frac{2}{k-2} & \text{if } \ell = 2 \text{ and } b_1, b_2 \leq k-2, \\ \frac{2}{k-1} + \frac{1}{3} & \text{if } \ell \geq 3, \end{cases}$$

Proof. We have $\sigma_T'(x) = 2\ell - 3 + \frac{2}{k-1} - \sum_{i \in [\ell]} \frac{2}{b_i}$. $\square$

4 The lower bound

We need the following definitions:

$$\begin{aligned}
\mathcal{L}_k(G) &:= G\left[x \in V(G) \mid d_G(x) < k\right], \\
\mathcal{H}_k(G) &:= G\left[x \in V(G) \mid d_G(x) \geq k\right], \\
\sigma_k(G) &:= \left(k - 2 + \frac{2}{k-1}\right)\left|\mathcal{L}_k(G)\right| - 2\left\|\mathcal{L}_k(G)\right\|, \\
\tau_{k,c}(G) &:= 2\left\|\mathcal{H}_k(G)\right\| + \left(k - c - \frac{2}{k-1}\right)\sum_{y \in V(\mathcal{H}_k(G))}\left(d_G(y) - k\right), \\
g_k(n,c) &:= \left(k - 1 + \frac{k-3}{(k-c)(k-1)+k-3}\right)n.
\end{aligned}$$

4.1 We only really care about low degree vertices

As proved in [1], a computation gives the following.

Lemma 4.1. *Let G be a graph with $\delta := \delta(G) \geq 3$ and $0 \leq c \leq \delta + 1 - \frac{2}{\delta}$. If $\sigma_{\delta+1}(G) + \tau_{\delta+1,c}(G) \geq c\left|\mathcal{H}_{\delta+1}(G)\right|$, then $2\left\|G\right\| \geq g_{\delta+1}(\left|G\right|, c)$.*

With a lower value of c, we can make it so we only have to care about vertices of degree δ and $\delta+1$ as follows.

Lemma 4.2. *Let G be a graph with $\delta := \delta(G) \geq 3$ and $0 \leq c \leq \frac{\delta+1}{2} + \frac{1}{\delta}$. Put $H' := \{v \in V(G) : d_G(v) = \delta + 1\}$. If $\sigma_{\delta+1}(G) + \sum_{y \in H'} d_{\mathcal{H}_{\delta+1}}(y) \geq c\left|H'\right|$, then $2\left\|G\right\| \geq g_{\delta+1}(\left|G\right|, c)$.*

Proof. Put $\mathcal{H} := \mathcal{H}_{\delta+1}$ and $k := \delta + 1$. For $y \in V(\mathcal{H})$, put

$$\tau_{k,c}(y) := d_{\mathcal{H}}(y) + \left(k - c + \frac{2}{k-1}\right)(d_G(y) - k).$$

We have

$$\begin{aligned}
\tau_{k,c}(G) &= \sum_{y \in V(\mathcal{H})} \tau_{k,c}(y) \\
&\geq \sum_{y \in H'} d_{\mathcal{H}}(y) + \sum_{y \in V(\mathcal{H}) \setminus H'}\left(d_{\mathcal{H}}(y) + k - c + \frac{2}{k-1}\right) \\
&\geq \sum_{y \in H'} d_{\mathcal{H}}(y) + \left(k - c + \frac{2}{k-1}\right)\left|\mathcal{H} - H'\right| \\
&\geq \sum_{y \in H'} d_{\mathcal{H}}(y) + c\left|\mathcal{H} - H'\right|,
\end{aligned}$$

where the last inequality follows since $c \leq \frac{k}{2} + \frac{1}{k-1}$. Now applying Lemma 4.1 proves the lemma. $\square$

4.2 Finishing the proof

We need the following degeneracy lemma.

Lemma 4.3. *Let G be a graph and $f \colon V(G) \to \mathbb{N}$. If $\|G\| > \sum_{v \in V(G)} f(v)$, then G has an induced subgraph H such that $d_H(v) > f(v)$ for each $v \in V(H)$.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Then $|G| \geq 3$ and we have $x \in V(G)$ with $d_G(x) \leq f(x)$. But now $\|G - x\| > \sum_{v \in V(G-x)} f(v)$, contradicting minimality of $|G|$. □

Theorem 4.4. *If G is an AT-irreducible graph with $\delta(G) \geq 4$ and $\omega(G) \leq \delta(G)$, then $2\|G\| \geq g_{\delta(G)+1}(|G|, c)$ where $c := (\delta(G) - 2)\alpha_{\delta(G)+1}$ when $\delta(G) \geq 6$ and $c := (\delta(G) - 3)\alpha_{\delta(G)+1}$ when $\delta(G) \in \{4, 5\}$.*

Proof. Put $k := \delta(G) + 1$, $\mathcal{L} := \mathcal{L}_k(G)$ and $\mathcal{H} := \mathcal{H}_k(G)$. Put $W := W^k(\mathcal{L})$, $L' := V(\mathcal{L}) \setminus W$ and $H' := \{v \in V(\mathcal{H}) : d_G(v) = k\}$. By Lemma 4.2, it will be sufficient to prove that

$$S := \sigma_k(G) + \sum_{y \in H'} d_{\mathcal{H}}(y) \geq c\,|H'|\,.$$

Let $\mathcal{D}$ be the components of $\mathcal{L}$ containing K_{k-1} and $\mathcal{C}$ the components of $\mathcal{L}$ not containing K_{k-1}. Then $\mathcal{D} \cup \mathcal{C} \subseteq \mathcal{T}_k$ for otherwise some $T \in \mathcal{D} \cup \mathcal{C}$ is d_0-AT and hence f_T-AT and G is AT-reducible. By Lemma **??**, we have $\sigma_k(T) \geq 2 + q_k(T)$ for if $T \in \mathcal{D}$ and $\sigma_k(T) \geq 2 - \alpha_k + q_k(T)$ if $T \in \mathcal{C}$. Hence, we have $\sigma_k(G) = \sum_{T \in \mathcal{D}} \sigma_k(T) + \sum_{T \in \mathcal{C}} \sigma_k(T) \geq 2\,|\mathcal{D}| + (2 - \alpha_k)\,|\mathcal{C}| + \alpha_k \sum_{v \in L'} (k - 1 - d_{\mathcal{L}}(v))$. That is,

$$\sigma_k(G) \geq 2\,|\mathcal{D}| + (2 - \alpha_k)\,|\mathcal{C}| + \alpha_k \sum_{v \in L'} (k - 1 - d_{\mathcal{L}}(v))\,.$$

Now we define an auxiliary bipartite graph F with parts A and B where:

1. $B = H'$ and A is the disjoint union of the following sets A_1, A_2 and A_3,
2. $A_1 = \mathcal{D}$ and each $T \in \mathcal{D}$ is adjacent to all $y \in H'$ where $N(y) \cap W^k(T) \neq \emptyset$,
3. For each $v \in L'$, let $A_2(v)$ be a set of $|N(v) \cap H'|$ vertices connected to $N(v) \cap H'$ by a matching in F. Let A_2 be the disjoint union of the $A_2(v)$ for $v \in L'$,
4. For each $y \in H'$, let $A_3(y)$ be a set of $d_{\mathcal{H}}(y)$ vertices which are all joined to y in F. Let A_3 be the disjoint union of the $A_3(y)$ for $y \in H'$.

Case 1. $\delta \geq 6$.

Define $f \colon V(F) \to \mathbb{N}$ by $f(v) = 1$ for all $v \in A_2 \cup A_3$ and $f(v) = 2$ for all $v \in B \cup A_1$. First, suppose $\|F\| > \sum_{v \in V(F)} f(v)$. Then by Lemma 4.3, F has an induced subgraph Q such that $d_Q(v) > f(v)$ for each $v \in V(Q)$. In particular, $V(Q) \subseteq B \cup A_1$ and $\delta(Q) \geq 3$. Put $Y := B \cap V(Q)$ and let X be $\bigcup_{T \in V(Q) \cap A_1} V(T)$. Now $H := G[X \cup Y]$ satisfies the hypotheses of Lemma 2.2, so H has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$

and $f(v) = d_{G'}(v)$ for $v \in X$. Since $Y \subseteq H'$ and $X \subseteq \mathcal{L}$, we have $f(v) = \delta(G)+d_{G'}(v)-d_G(v)$ for all $v \in V(G')$. Hence, G is AT-reducible to G', a contradiction.

Therefore $\|F\| \le \sum_{v\in V(F)} f(v) = 2(|H'|+|\mathcal{D}|)+|A_2|+|A_3|$. By Lemma 2.1, for each $y \in B$ we have $d_F(y) \ge k-1$. Hence $\|F\| \ge (k-1)\,|H'|$. This gives $(k-3)\,|H'| \le 2\,|\mathcal{D}|+|A_2|+|A_3|$. By our above estimate we have $S \ge 2\,|\mathcal{D}| + \alpha_k \sum_{v\in L'} (k-1-d_{\mathcal{L}}(v)) + \sum_{y\in H'} d_{\mathcal{H}}(y) = 2\,|\mathcal{D}| + \alpha_k\,|A_2| + |A_3| \ge \alpha_k(2\,|\mathcal{D}| + |A_2| + |A_3|)$. Hence $S \ge \alpha_k(k-3)\,|H'|$. Thus our desired bound holds by Lemma 4.2.

Case 2. $\delta \in \{4, 5\}$.

Define $f\colon V(F) \to \mathbb{N}$ by $f(v) = 1$ for all $v \in A_1 \cup A_2 \cup A_3$ and $f(v) = 3$ for all $v \in B$. First, suppose $\|F\| > \sum_{v\in V(F)} f(v)$. Then by Lemma 4.3, F has an induced subgraph Q such that $d_Q(v) > f(v)$ for each $v \in V(Q)$. In particular, $V(Q) \subseteq B \cup A_1$ and $d_Q(v) \ge 4$ for $v \in B \cap V(Q)$ and $d_Q(v) \ge 2$ for $v \in A_1 \cap V(Q)$. Put $Y := B \cap V(Q)$ and let X be $\bigcup_{T\in V(Q)\cap A_1} V(T)$. Now $H := G[X \cup Y]$ satisfies the hypotheses of Lemma 2.3, so H has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for $v \in X$. Since $Y \subseteq H'$ and $X \subseteq \mathcal{L}$, we have $f(v) = \delta(G) + d_{G'}(v) - d_G(v)$ for all $v \in V(G')$. Hence, G is AT-reducible to G', a contradiction.

Therefore $\|F\| \le \sum_{v\in V(F)} f(v) = 3\,|H'|+|\mathcal{D}|+|A_2|+|A_3|$. By Lemma 2.1, for each $y \in B$ we have $d_F(y) \ge k-1$. Hence $\|F\| \ge (k-1)\,|H'|$. This gives $(k-4)\,|H'| \le |\mathcal{D}|+|A_2|+|A_3|$. By our above estimate we have $S \ge 2\,|\mathcal{D}| + \alpha_k \sum_{v\in L'} (k-1-d_{\mathcal{L}}(v)) + \sum_{y\in H'} d_{\mathcal{H}}(y) = 2\,|\mathcal{D}| + \alpha_k\,|A_2| + |A_3| \ge \alpha_k(|\mathcal{D}| + |A_2| + |A_3|)$. Hence $S \ge \alpha_k(k-4)\,|H'|$. Thus our desired bound holds by Lemma 4.2. □

References

[1] A.V. Kostochka and M. Stiebitz, *A new lower bound on the number of edges in colour-critical graphs and hypergraphs*, Journal of Combinatorial Theory, Series B **87** (2003), no. 2, 374–402.