notes on coloring cayley graphs

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1 Basics

Definition 1. For a group G and $A \subseteq G$, the *cayley graph* of G with respect to A is the directed graph with vertex set G and an edge from x to xa for each $x \in G$ and $a \in A$. Write $\mathcal{C}(G, A)$ for this digraph.

We are concerned with coloring undirected graphs without loops, so we want A to not contain the identity element of G and $\frac{1}{A} = A$, where

$$\frac{1}{A} = \left\{a^{-1} \mid a \in A\right\}.$$

Given this, $\mathcal{C}(G, A)$ has all edges directed both ways. Let G_A be the undirected graph with the structure of $\mathcal{C}(G, A)$. We call such G_A a *standard cayley graph.*

Remark. a and b are adjacent in a standard cayley graph G_A just in case $ab^{-1} \in A$.

Conjecture 1.1. *Let G be an abelian group and G_A a standard cayley graph. If $\Delta(G) \geq 9$ and $\omega(G) < \Delta(G)$, then $\chi(G) < \Delta(G)$.*

i am trying to make the $\Delta = 8$ example as a cayley graph of $C_5 \times C_3$, with the standard generators, its missing some edges though so need to throw more into A.

Lemma 1.2. *If a and b are adjacent in a standard cayley graph G_A, then for any independent set X in G_A*

$$\frac{1}{X}a \bigcap \frac{1}{X}b = \emptyset.$$

Proof. Suppose there is $c \in \frac{1}{X}a \bigcap \frac{1}{X}b$. Then $c = x^{-1}a$ and $c = y^{-1}b$ for some $x, y \in X$. So $yx^{-1} = ba^{-1} \in A$, so x and y are adjacent, but they can't be since both in independent set X. □