tarpit notes

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1 The game

Let $\mathcal{Q}_n$ be the words of length n in the alphabet $\{A, B, C\}$. For $w \in Q_n$, let w_i be the i-th letter of w; that is, $w = w_1 w_2 \cdots w_n$. For $w \in \mathcal{Q}_n$ and $x \in \{A, B, C\}$, let $\mathcal{I}_x(w) = \{i \mid w_i = x\}$. A *2-partition* of a set S is a partition of S into sets of size two and at most one set of size one.

A game is specified by a tuple $(\mathcal{W}, n)$ where $\mathcal{W} \subseteq \mathcal{Q}_n$ is the set of *won* positions. The $(\mathcal{W}, n)$-*game* is played as follows. At the start of the game, Player 1 picks an initial $w \in \mathcal{Q}_n$. After that, the players alternate turns (with Player 1 going first) until $w \in \mathcal{W}$ at the start of Player 1's turn. Player 2 wins if the game ends and Player 1 wins otherwise.

Player 1 Move. For each $x \in \{A, B, C\}$, choose a 2-partition $\mathcal{P}_x$ of $\mathcal{I}_x(w)$.

Player 2 Move. Pick $x \in \{A, B, C\}$ as well as $p \in \mathcal{P}_x$ and for each $i \in p$, change w_i to the letter in $\{A, B, C\} \setminus \{x, w_i\}$.

Question. For what pairs $(\mathcal{W}, n)$ does Player 2 have a winning strategy?

2 Tarpits

We say that $T \subseteq \mathcal{Q}_n$ is a *tarpit* if for each $w \in T$, Player 1 can move such that all of Player 2's possible modifications of w still lie in T. Tarpits are great for Player 1, if T is a tarpit in $\mathcal{Q}_n$ disjoint from $\mathcal{W}$, then Player 1 can just pick the initial w to be in T to guarantee a win. In fact, if we know the minimal tarpits in $\mathcal{Q}_n$ then we know exactly which pairs $(\mathcal{W}, n)$ Player 2 wins on. For $n \in \mathbb{N}$, let $\mathcal{T}_n$ be the set of minimal tarpits in $\mathcal{Q}_n$.

Lemma 2.1. *Player 2 wins the* $(\mathcal{W}, n)$*-game if and only if* $\mathcal{W} \cap T \neq \emptyset$ *for every* $T \in \mathcal{T}_n$.

Proof. If there is $T \in \mathcal{T}_n$ for which $\mathcal{W} \cap T = \emptyset$, then Player 1 wins by definition. For the other direction, suppose $\mathcal{W} \cap T \neq \emptyset$ for every $T \in \mathcal{T}_n$. Then Player 2 should play by the strategy: choose $x \in \{A, B, C\}$ and $p \in \mathcal{P}_x$ such that, after modification, w is the word seen least recently (with words that have never been seen being top choice, breaking ties arbitrarily). Suppose there is a game where Player 2 plays by this strategy, but Player 1 wins. Say the sequence of words Player 1 encounters at the start of his turns is $w^1, w^2, w^3, \ldots$. Since $|Q_n|$ is finite, there is k such that each word appearing in $w^k, w^{k+1}, w^{k+2}, \ldots$ appears infinitely many times. Let $S = \left\{w^k, w^{k+1}, w^{k+2}, \ldots\right\}$. Going out far enough in the sequence, the words in S have been seen more recently than any other word in Q_n. So, by Player 2's strategy, he would escape S if he could. Since each word in S is encountered infinitely many times, Player 2 has the opportunity to escape S from any word in S. That means that S must be a tarpit. But, $S \cap \mathcal{W} = \emptyset$ since Player 2 does not win and hence S contains a minimal tarpit disjoint from $\mathcal{W}$, contradicting our assumption. $\square$

The strategy for Player 2 in the proof of Lemma 2.1 has the interesting property that it is a winning strategy whenever a winning strategy exists.

3 Transversal hypergraphs

A *transversal* of $\mathcal{T}_n$ is a subset S of $\mathcal{Q}_n$ such that $S \cap T \neq \emptyset$ for all $T \in \mathcal{T}_n$. Let $\mathcal{H}_n$ be the hypergraph with vertex set $\mathcal{Q}_n$ and edge set the minimal transversals of $\mathcal{T}_n$. As an immediate consequence of Lemma 2.1, we have the following.

Lemma 3.1. *Player 2 wins the $(\mathcal{W}, n)$-game if and only if $\mathcal{W}$ contains an edge of $\mathcal{H}_n$.*

Conjecture 3.2. *If $n \in \mathbb{N}$, then $\mathcal{H}_n$ is k-uniform for some k.*