Edge-coloring via fixable subgraphs

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1 Introduction

All multigraphs are loopless. Let G be a multigraph and L a list assignment on $V(G)$ and $\text{pot}(L) = \bigcup_{v \in V(G)} L(v)$. An *$L$-pot* is a set X containing $\text{pot}(L)$. An edge-coloring π of G such that $\pi(x) \in L(x) \cap L(y)$ for all $xy \in E(G)$ is called an *L-edge-coloring*.

2 Completing edge-colorings

Our goal is to convert a partial k-edge-coloring of a multigraph M into a (total) k-edge-coloring of M. For a partial k-edge-coloring π of M, let M_π be the subgraph of M induced on the uncolored edges and let L_π be the list assignment on the vertices of M_π given by $L_\pi(v) = [k] - \{\tau \mid \pi(vx) = \tau \text{ for some } vx \in E(M)\}$.

Kempe chains give a powerful technique for converting a partial k-edge-coloring into a total k-edge-coloring. The idea is to repeatedly exchange colors on two-colored paths until M_π has an edge-coloring ζ such that $\zeta(xy) \in L_\zeta(x) \cap L_\zeta(y)$ for all $xy \in E(M_\pi)$. In this sense the original list assignment L_π on M_π is *fixable*. In the next section, we give an abstract definition of this notion that frees us from the embedding in the ambient graph M.

Let G be a multigraph, L a list assignment on $V(G)$ and P an arbitrary L-pot. Throughout this section, G, L and P will refer to these objects.

2.1 Fixable graphs

For different colors $a, b \in P$, let $S_{L,a,b}$ be all the vertices of G that have exactly one of a or b in their list; more precisely, $S_{L,a,b} = \{v \in V(G) \mid |\{a, b\} \cap L(v)| = 1\}$.

Definition 1. G is *(L, P)-fixable* if either

(1) G has an L-edge-coloring; or

(2) there are different $a, b \in P$ such that for every partition $X_1, \dots, X_t$ of $S_{L,a,b}$ into sets of size at most two, there is $J \subseteq [t]$ so that G is (L', P)-fixable where L' is formed from L by swapping a and b in $L(v)$ for every $v \in \bigcup_{i \in J} X_i$.

We write L-fixable as shorthand for $(L, \text{pot}(L))$-fixable. When G is (L, P)-fixable, the choices of a, b and J in each application of (2) determine a tree where all leaves have lists satisfying (1). The *height* of (L, P) is the minimum possible height of such a tree. We write $h_G(L, P)$ for this height and let $h_G(L, P) = \infty$ when G is not (L, P)-fixable.

Lemma 2.1. *If a multigraph M has a partial k-edge-coloring π such that M_π is $(L_\pi, [k])$-fixable, then M is k-edge-colorable.*

Proof. Choose a partial k-edge-coloring π of M such that M_π is $(L_\pi, [k])$-fixable minimizing $h_{M_\pi}(L_\pi, [k])$. If $h_{M_\pi}(L_\pi, [k]) = 0$, then (1) must hold for M_π and L_π; that is, M_π has an edge-coloring ζ such that $\zeta(x) \in L_\pi(x) \cap L_\pi(y)$ for all $xy \in E(M_\pi)$. But that means that $\pi \cup \zeta$ is the desired k-edge-coloring of M.

So, we may assume that $h_{M_\pi}(L_\pi, [k]) > 0$. Let $a, b \in [k]$ be a choice in (2) that leads to a tree of height $h_{M_\pi}(L_\pi, [k])$. Let H be the subgraph of M induced on all edges e with $\pi(e) \in \{a, b\}$. Let S be the vertices in M_π with degree exactly one in H. Consider the component C_x in H for each $x \in S$. We have $|V(C_x) \cap S| \in \{1, 2\}$ and hence the components of H give a partition $X_1, \ldots, X_t$ of S into sets of size at most two. Moreover, exchanging colors a and b on C_x has the effect of swapping a and b in $L_\pi(v)$ for each $v \in V(C_x) \cap S$. Hence we can achieve the needing swapping of colors in the lists in (2) by exchanging colors on the components of H. By (2) there is $J \subseteq [t]$ so that M_π is $(L', [k])$-fixable where L' is formed from L_π by swapping a and b in $L_\pi(v)$ for every $v \in \bigcup_{i \in J} X_i$. Choose such a J that leads to a tree of height $h_{M_\pi}(L_\pi, [k])$. Let π' be the partial k-edge-coloring of M created from π by performing the color exchanges to create L' from L_π. Then $M_{\pi'}$ is $(L_{\pi'}, [k])$-fixable and $h_{M_{\pi'}}(L_{\pi'}, [k]) < h_{M_\pi}(L_\pi, [k])$, contradicting the minimality of $h_{M_\pi}(L_\pi, [k])$. □

The given definition of L-fixable was originally motivated as a generalization of the Fixer-Breaker game in [3] from complete graphs to arbitrary graphs. The direct generalization of that game gives us less power because it does not take the fact that two-colored paths cannot cross into account. Interpreted in the Fixer-Breaker game, the choice of partition in (2) is forcing Breaker to choose two-colored paths in a way that is consistent with being embedded in *some* graph. For stars the two games have identical winning conditions because the obvious necessary condition is sufficient, but in general the extra power does make more graphs fixable. However, it is more natural to phrase some proofs in terms of the weaker game, so we define that here.

Definition 2. G is *weakly (L, P)-fixable* if either

(1) G has an L-edge-coloring; or

(2) there are different $a, b \in P$ and $v \in S_{L,a,b}$ such that for every $X \subseteq S_{L,a,b}$ with $|X| \leq 2$ and $v \in X$, it holds that G is (L', P)-fixable where L' is formed from L by swapping a and b in $L(v)$ for every $v \in X$.

Lemma 2.2. *If G is weakly (L, P)-fixable, then G is (L, P)-fixable.*

Proof. Clearly if (1) holds we are done. So, assume we have different $a, b \in P$ and $v \in S_{L,a,b}$ as in (2). Given a partition $X_1, \ldots, X_t$ of $S_{L,a,b}$ into sets of size at most two, let i be the index with $v \in X_i$. Then $J = \{i\}$ is the desired subset of $[t]$. □

2.2 A necessary condition

Since the edges incident to a given vertex must all get different colors, we have the following.

Lemma 2.3. *If G is (L, P)-fixable, then $|L(v)| \geq d_G(v)$ for all $v \in V(G)$.*

By considering the maximum size of matchings in each color, we get a more interesting necessary condition. For $C \subseteq \text{pot}(L)$ and $H \subseteq G$, let $H_{L,C}$ be the subgraph of H induced on the vertices v with $L(v) \cap C \neq \emptyset$. When L is clear from context, we may write H_C for $H_{L,C}$. If $C = \{\alpha\}$, we may write H_α for H_C. For $H \subseteq G$, put

$$\psi_L(H) = \sum_{\alpha \in \text{pot}(L)} \left\lfloor \frac{|H_{L,\alpha}|}{2} \right\rfloor .$$

Each term in the sum gives an upper bound on the size of a matching in color α. So $\psi_L(H)$ is an upper bound on the number of edges in a partial L-edge-coloring of H. We say that (H, L) is *abundant* if $\psi_L(H) \geq \|H\|$ and that (G, L) is *superabundant* if for every $H \subseteq G$, the pair (H, L) is abundant.

Lemma 2.4. *If G is (L, P)-fixable, then (G, L) is superabundant.*

Proof. Suppose to the contrary that G is (L, P)-fixable and there is $H \subseteq G$ such that (H, L) is not abundant. We show that for all different $a, b \in P$ there is a partition $X_1, \dots, X_t$ of $S_{a,b}$ into sets of size at most two, such that for all $J \subseteq [t]$, the pair (H, L') is not abundant where L' is formed from L by swapping a and b in $L(v)$ for every $v \in \bigcup_{i \in J} X_i$. Since G can never be edge-colored from a list assignment that is not superabundant, this contradicts the (L, P)-fixability of G.

Pick different $a, b \in P$. Let $S = S_{L,a,b} \cap V(H)$ and let S_a be the $v \in S$ with $a \in L(v)$. Put $S_b = S \setminus S_a$. Swapping a and b will only effect the terms $\left\lfloor \frac{|S_a|}{2} \right\rfloor$ and $\left\lfloor \frac{|S_b|}{2} \right\rfloor$ in $\psi_L(H)$. So, if $\psi_L(H)$ is increased by the swapping, it must be that both $|S_a|$ and $|S_b|$ are odd and after swapping they are both even. Say $S_a = \{a_1, \dots, a_p\}$ and $S_b = \{b_1, \dots, b_q\}$. By symmetry, we may assume $p \leq q$. For $i \in [p]$, let $X_i = \{a_i, b_i\}$. Since both p and q are odd, $q - p$ is even, so we get a partition by, for each $j \in \left[\frac{q-p}{2}\right]$, letting $X_{p+j} = \{b_{p+2j-1}, b_{p+2j}\}$. For any $i \in [p]$, swapping a and b in $L(v)$ for every $v \in X_i$ maintains $|S_a|$ and $|S_b|$. For any $j \in \left[\frac{q-p}{2}\right]$, swapping a and b in $L(v)$ for every $v \in X_{p+j}$ maintains the parity of $|S_a|$ and $|S_b|$. So no choice of J can increase $\psi_L(H)$ and hence (H, L') is never abundant. □

Intuitively, superabundance requires the potential for a large enough matching in each color. If instead we require the existence of a large enough matching in each color, we get a stronger condition that has been studied before. For a multigraph H, let $\nu(H)$ be the number of edges in a maximum matching of H. For a list assignment L on H, put $\eta_L(H) = \sum_{\alpha \in \text{pot}(L)} \nu(H_\alpha)$. Note that we always have $\psi_L(H) \geq \eta_L(H)$.

The following generalization of Hall's theorem was proved by Marcotte and Seymour [2] and independently by Cropper, Gyárfás and Lehel [1]. By a *multitree* we mean a tree that possibly has edges of multiplicity greater than one.

Lemma 2.5. *Let T be a multitree and L a list assignment on $V(T)$. If $\eta_L(H) \geq \|H\|$ for all $H \subseteq T$, then T has an L-edge-coloring.*

2.3 Basic properties

For $i \in [2]$, let G_i be a multigraph, L_i a list assignment on $V(G_i)$ and P_i an L_i-pot. Let $L_1 \cup L_2$ be the list assignment L' on $G_1 \cup G_2$ given by $L'(v) = L_1(v) \cup L_2(v)$ for all $v \in V(G_1) \cap V(G_2)$ and $L'(v) = L_i(v)$ for all $v \in V(G_i) \setminus V(G_{3-i})$ and $i \in [2]$.

Lemma 2.6. *Suppose $G_1 \subseteq G_2$ and G_2 is (L_2, P_2)-fixable. If $L_2(v) \subseteq L_1(v)$ for all $v \in V(G_1)$ and $P_2 \subseteq P_1$, then G_1 is (L_1, P_1)-fixable.*

Lemma 2.7. *Suppose G_i is (L_i, P_i)-fixable for $i \in [2]$. If $P_1 \cap P_2 = \emptyset$, then $G_1 \cup G_2$ is $(L_1 \cup L_2, P_1 \cup P_2)$-fixable.*

Lemma 2.8. *Suppose G_1 is (L_1, P_1)-fixable. If G_2 has an L_2-edge-coloring π such that $L_1(v) \cap \{\pi(vx) \mid vx \in E(G_2)\} = \emptyset$ for all $v \in V(G_1)$, then $G_1 \cup G_2$ is $(L_1 \cup L_2, P_1 \cup \text{pot}(L_2))$-fixable.*

Proof. Suppose the lemma is false and choose a counterexample minimizing $\|G_2\|$ and subject to that minimizing $h_{G_1}(L_1, P_1)$. Suppose $\|G_2\| > 1$. Pick $xy \in E(G_2)$. Let $G_2' = G_2 - xy$ and define L_2' by $L_2'(v) = L_2(v)$ for $v \notin \{x, y\}$ and $L_2'(v) = L_2(v) \setminus \{\pi(xy)\}$ for $v \in \{x, y\}$. Applying minimality of $\|G_2\|$ we conclude that $G_1 \cup G_2'$ is $(L_1 \cup L_2', P_1 \cup \text{pot}(L_2'))$-fixable. Now let $G_2'' = G_2[x, y]$ and $L_2''(x) = L_2''(y) = \{\pi(xy)\}$. Since $\|G_2\| > 1$, we can apply minimality of $\|G_2\|$ again to conclude that $G_1 \cup G_2' \cup G_2''$ is $(L_1 \cup L_2' \cup L_2'', P_1 \cup \text{pot}(L_2' \cup L_2''))$-fixable, but $G_1 \cup G_2' \cup G_2'' = G_1 \cup G_2$ and $L_1 \cup L_2' \cup L_2'' = L_1 \cup L_2$, so this is a contradiction.

Hence, we must have $\|G_2\| = 1$. Let xy be G_2's only edge. If $h_{G_1}(L_1, P_1) = 0$, then G_1 has an L_1-edge-coloring and hence $G_1 \cup G_2$ has an $(L_1 \cup L_2)$-edge-coloring, so we must have $h_{G_1}(L_1, P_1) > 0$. By the definition of (L_1, P_1)-fixable, we have $a, b \in P_1$ such that for every partition $X_1, \ldots, X_t$ of $S_{L_1,a,b}$ into sets of size at most two, there is $J \subseteq [t]$ so that $h_{G_1}(L', P_1) < h_{G_1}(L_1, P_1)$ where L' is formed from L_1 by swapping a and b in $L_1(v)$ for every $v \in \bigcup_{i \in J} X_i$.

We will prove that $G_1 \cup G_2$ is weakly $(L_1 \cup L_2, P_1 \cup \text{pot}(L_2))$-fixable, then the lemma will follow by Lemma 2.2.

Claim 1. *$\pi(xy) \in \{a, b\}$, by symmetry we may assume $b = \pi(xy)$.*

Suppose not. Pick $v \in S_{L_1,a,b}$ arbitrarily.

□

2.4 Fixability of stars

When G is a star, the conjunction of our two necessary conditions is sufficient. This generalizes Vizing fans [4]; in the next section we will define "Kierstead-Tashkinov-Vizing assignments" and show that they are always superabundant. In [3], the second author proved a common generalization of Theorem 2.9 and Hall's theorem. In particular, Theorem 2.9 holds for multistars as well; the proof for multistars is nearly identical, but notationally cumbersome.

Theorem 2.9. *If G is a star, then G is L-fixable if and only if (G, L) is superabundant and $|L(v)| \geq d_G(v)$ for all $v \in V(G)$.*

Proof. □

2.5 Kierstead-Tashkinov-Vizing assignments

Many edge-coloring results have been proved using a specific kind of superabundant pair (G, L) where superabundance can be proved via a special ordering. That is, the orderings given by the definition of Vizing fans, Kierstead paths, and Tashkinov trees. In this section, we show how superabundance easily follows from these orderings.

We say that a list assignment L on G is a *Kierstead-Tashkinov-Vizing* assignment (henceforth *KTV-assignment*) if for some $xy \in E(G)$, there is a total ordering '$<$' of $V(G)$ such that

1. there is and edge-coloring π of $G - xy$ such that $\pi(uv) \in L(u) \cap L(v)$ for each $uv \in E(G - xy)$;
2. $x < z$ for all $z \in V(G - x)$;
3. $G[w \mid w \leq z]$ is connected for all $z \in V(G)$;
4. for each $wz \in E(G - xy)$, there is $u < \max\{w, z\}$ such that $\pi(wz) \in L(u) - \{\pi(e) \mid e \in E(u)\}$;
5. there are different $s, t \in V(G)$ such that $L(s) \cap L(t) - \{\pi(e) \mid e \in E(s) \cup E(t)\} \neq \emptyset$.

Lemma 2.10. *If L is a KTV-assignment on G, then (G, L) is superabundant.*

Proof. Let L be a KTV-assignment on G, and let $H \subseteq G$. We will show that (H, L) is abundant. Clearly it suffices to consider the case when H is an induced subgraph, so we assume this. Property (1) gives that $G - xy$ has an edge-coloring π, so $\psi_L(H) \geq \|H\| - 1$; also $\psi_L(H) \geq \|H\|$ if $\{x, y\} \not\subseteq V(H)$. Furthermore $\psi_L(H) \geq \|H\|$ if s and t from property (5) are both in $V(H)$, since then $\psi_L(H)$ gains 1 over the naive lower bound, due to the color in $L(s) \cap L(t)$. So $V(G) - V(H) \neq \emptyset$.

Now choose $z \in V(G) - V(H)$ that is smallest under $<$. Put $H' = G[w \mid w \leq z]$. By the minimality of z, we have $H' - z \subseteq H$. By property (2), $|H'| \geq 2$. By property (3), H' is connected and thus there is $w \in V(H' - z)$ adjacent to z. So, we have $w < z$ and $wz \in E(G) - E(H)$. Now $\pi(wz) \in L(w)$. By the definition of a KTV-assignment, property (4) implies that there exists u with $u < \max\{w, z\} = z$ and $\pi(wz) \in L(u) - \{\pi(e) | e \in E(u)\}$. Then $u \in V(H' - z) \subseteq V(H)$ and again we gain 1 over the naive lower bound on $\psi_L(H)$, due to the color in $L(u) \cap L(w)$. So $\psi_L(H) \geq \|H\|$. □

2.6 Stars with one edge subdivided

Theorem 2.11. *Suppose G is a star with one edge subdivided and root r. If (G, L) is superabundant, $|L(v)| \geq d_G(v)$ for all $v \in V(G)$ and $|L(r)| > d_G(r)$, then G is L-fixable.*

Proof. □

3 Applications

4 Conjectures

Conjecture 4.1. *Any multigraph G is L-fixable if (G, L) is superabundant and $|L(v)| > d_G(v)$ for all $v \in V(G)$.*

Conjecture 4.2. *Any tree G is L-fixable if (G, L) is superabundant and $|L(v)| > d_G(v)$ for all $v \in V(G)$.*

Conjecture 4.3. *Any path G is L-fixable if (G, L) is superabundant and $|L(v)| > d_G(v)$ for all $v \in V(G)$.*

References

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