Proof of Brooks' theorem. Suppose the theorem is false and choose a counterexample G minimizing $|G|$. Let $\{A_1, A_2\}$ be a separation of G minimizing $k := |A_1 \cap A_2|$. If $A_1 \cap A_2$ is a clique, then by minimality of $|G|$, we have Δ-colorings of each $G[A_i]$ which use k colors on $A_1 \cap A_2$. By permuting color names if necessary we can combine these to get a $\Delta(G)$-coloring of G, a contradiction. In particular, $k \geq 2$ and if $k = 2$, then $A_1 \cap A_2$ is independent.

Suppose $k = 2$ and say $A_1 \cap A_2 = \{u, v\}$. By symmetry we may assume that every Δ-coloring of $G[A_1]$ gives u and v different colors and every Δ-coloring of $G[A_2]$ gives u and v the same color (otherwise we could combine the colorings into a Δ-coloring of G as above). By minimality of k, each of u and v have a neighbor on both sides of the separation. Hence u and v must each have $\Delta - 1$ neighbors in A_1 and 1 neighbor in A_2. But then since $\Delta \geq 3$, any Δ-coloring of $G[A_2 - \{u, v\}]$ can be extended to a Δ-coloring of $G[A_2]$ where u and v receive the same color, a contradiction.

Thence $k \geq 3$. Since G is not a disjoint union of cliques, we have an induced P_3 xyz in G. Since $k \geq 3$, $G - x - z$ is connected and hence we may order $V(G)$ as $x, z, v_1, \ldots, v_n, y$ so that each v_i has a neighbor to the right. But this is a contradiction since greedily coloring in this order uses at most Δ colors as x and z get the same color, each v_i has at most $\Delta - 1$ neighbors to the left and y has two neighbors (x and z) colored the same. $\square$