A WEIRD PROOF OF BROOKS' THEOREM

Let G be a graph. A partition $P := (V_0, V_1)$ of $V(G)$ will be called *normal* if it achieves the minimum value of $(\Delta(G) - 1) \|V_0\| + \|V_1\|$. Note that if P is a normal partition, then $\Delta(G[V_0]) \leq 1$ and $\Delta(G[V_1]) \leq \Delta(G) - 1$. The *$P$-components* of G are the components of $G[V_i]$ for $i \in [2]$. A P-component is called an *obstruction* if it is a K_2 in $G[V_0]$ or a $K_{\Delta(G)}$ in $G[V_1]$ or an odd cycle in $G[V_1]$ when $\Delta(G) = 3$. A path $x_1 x_2 \cdots x_k$ is called *P-acceptable* if x_1 is contained in an obstruction and for different $i, j \in [k]$, x_i and x_j are in different P-components. For a subgraph H of G and $x \in V(G)$, we put $N_H(x) := N(x) \cap V(H)$.

Lemma 1. *Let G be a graph with $\Delta(G) \geq 3$. If G doesn't contain $K_{\Delta(G)+1}$, then $V(G)$ has an obstruction-free normal partition.*

Proof. Suppose the lemma is false. Among the normal partitions having the minimum number of obstructions, choose $P := (V_0, V_1)$ and a maximal P-acceptable path $x_1 x_2 \cdots x_k$ so as to minimize k.

Let A and B be the P-components containing x_1 and x_k respectively. Put $X := N_A(x_k)$. First, suppose $|X| = 0$. Then moving x_1 to the other part of P creates another normal partition P' having the minimum number of obstructions. But $x_2 x_3 \cdots x_k$ is a maximal P'-acceptable path, violating the minimality of k. Hence $|X| \geq 1$.

Pick $z \in X$. Moving z to the other part of P destroys the obstruction A, so it must create an obstruction containing x_k and hence B. Since obstructions are complete graphs or odd cycles, the only possibility is that $\{z\} \cup V(B)$ induces an obstruction. Put $Y := N_B(z)$. Then, since obstructions are regular, $N_B(x) = Y$ for all $x \in X$ and $|Y| = \delta(B) + 1$. In particular, X is joined to Y in G.

Suppose $|X| \geq 2$. Then, similarly to above, moving z and then x_k to their respective other parts of P shows that $\{x_k\} \cup V(A - z)$ induces an obstruction. Since obstructions are regular, we must have $|N_{A-z}(x_k)| = \Delta(A)$ and hence $|X| \geq \Delta(A) + 1$. Thus $|X \cup Y| = \Delta(A) + \delta(B) + 2 = \Delta(G) + 1$. Suppose X is not a clique and pick nonadjacent $v_1, v_2 \in X$. It is easily seen that moving v_1, v_2 and then x_k to their respective other parts violates normality of P. Hence X is a clique. Suppose Y is not a clique and pick nonadjacent $w_1, w_2 \in Y$. Pick $z' \in X - \{z\}$. Now moving z and then w_1, w_2 and then z' to their respective other parts again violates normality of P. Hence Y is a clique. But X is joined to Y, so $X \cup Y$ induces a $K_{\Delta(G)+1}$ in G, a contradiction.

Hence we must have $|X| = 1$. Suppose $X \neq \{x_1\}$. First, suppose A is K_2. Then moving z to the other part of P creates another normal partition Q having the minimum number of obstructions. In Q, $x_k x_{k-1} \cdots x_1$ is a maximal Q-acceptable path since the Q-components containing x_2 and x_k contain all of x_1's neighbors in that part. Running through the above argument using Q gets us to the same point with A not K_2. Hence we may assume A is not K_2.

Move each of $x_1, x_2, \ldots, x_k$ in turn to their respective other parts of P. Then the obstruction A was destroyed by moving x_1 and for $1 \leq i < k$, the obstruction created by moving x_i

was destroyed by moving x_{i+1}. Thus, after the moves, x_k is contained in an obstruction. By minimality of k, it must be that $\{x_k\} \cup V(A - x_1)$ induces an obstruction and hence $|X| \geq 2$, a contradiction.

Therefore $X = \{x_1\}$. But then moving x_1 to the other part of P creates an obstruction containing both x_2 and x_k. Hence $k = 2$. Since x_1x_2 is maximal, x_2 can have no neighbor in the other part besides x_1. But now moving x_1 and then x_2 to their respective other parts creates a partition violating the normality of P. □

Theorem 2 (Brooks 1941)**.** *If a connected graph G is not complete and not an odd cycle, then $\chi(G) \leq \Delta(G)$.*

Proof. Suppose not and choose a counterexample G minimizing $\Delta(G)$. Plainly, $\Delta(G) \geq 3$. By Lemma 1, $V(G)$ has an obstruction-free normal partition (V_0, V_1). Since $G[V_0]$ has maximum degree at most one and contains no K_2's, we see that V_0 is independent. Since $G[V_1]$ is obstruction-free, applying minimality of $\Delta(G)$ gives $\chi(G[V_1]) \leq \Delta(G[V_1]) < \Delta(G)$. Hence $\chi(G) \leq \Delta(G)$, a contradiction. □