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For a graph G, let $\beta_k(G)$ be the independence number of the subgraph of G induced on the vertices of degree $k-1$. When k is defined in context, we just write $\beta(G)$. Let $\mathcal{H}(G)$ be the subgraph of G induced on vertices of degree greater than $\delta(G)$. Let $\mathcal{L}(G)$ be the subgraph of G induced on vertices of degree $\delta(G)$.

Definition 1. The *maximum independent cover number* of a graph G is the maximum $\operatorname{mic}(G)$ of $\|I, V(G) \setminus I\|$ over all independent sets I of G.

Definition 2. A graph G is *OC-reducible* to H if H is a nonempty induced subgraph of G which is online f_H-choosable where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not OC-reducible to any nonempty induced subgraph, then it is *OC-irreducible*.

Lemma 1. *Every OC-irreducible graph G satisfies*

$$2\|G\| > (\delta(G) - 1)|G| + \operatorname{mic}(G).$$

Lemma 2. *If G is an OC-irreducible graph where $\mathcal{H}(G)$ is edgeless, $\Delta := \Delta(G) = \delta(G) + 1$ and $\mathcal{L} := \mathcal{L}(G)$, then*

$$2\|\mathcal{L}\| > \left(\Delta - 2 - \frac{2}{\Delta - 2}\right)|\mathcal{L}| + \frac{\Delta(\Delta - 1)}{\Delta - 2}\beta_\Delta(\mathcal{L}).$$

Proof. Let G be such a graph. Put $\mathcal{H} := \mathcal{H}(G)$ and $\mathcal{L} := \mathcal{L}(G)$. Since $\mathcal{H}$ is edgeless,

$$\begin{aligned} \Delta|\mathcal{H}| &= \|\mathcal{H}, \mathcal{L}\| \\ &= (\Delta - 1)|\mathcal{L}| - 2\|\mathcal{L}\|, \end{aligned} \tag{1}$$

so, by Lemma 1,

$$\begin{aligned} (\Delta - 1)|\mathcal{L}| + \Delta|\mathcal{H}| &= 2\|G\| \\ &> (\Delta - 2)|G| + \operatorname{mic}(G) \\ &\geq (\Delta - 2)|G| + \Delta|\mathcal{H}| + (\Delta - 1)\beta_\Delta(\mathcal{L}) \\ &= (\Delta - 2)|\mathcal{L}| + (2\Delta - 2)|\mathcal{H}| + (\Delta - 1)\beta_\Delta(\mathcal{L}), \end{aligned}$$

so simplifying and using (1) again gives

$$\begin{aligned}|\mathcal{L}| &> (\Delta-2)\,|\mathcal{H}| + (\Delta-1)\beta_\Delta(\mathcal{L})\\ &= \frac{\Delta-2}{\Delta}\left((\Delta-1)\,|\mathcal{L}| - 2\,\|\mathcal{L}\|\right) + (\Delta-1)\beta_\Delta(\mathcal{L}),\end{aligned}$$

now some mild manipulation yields the desired bound. □

Definition 3. A quadruple (p, h, z, f) of functions from $\mathbb{N}$ to $\mathbb{R}$ is r-*Gallai* if for every $k \geq r$ and Gallai tree $T \neq K_k$ with $\Delta(T) \leq k-1$, the following hold:

- if $K_{k-1} \subseteq T$, then $2\,\|T\| \leq (k-3+p(k))\,|T| + h(k)q(T) + z(k)\beta(T) + f(k)$; and
- if $K_{k-1} \not\subseteq T$, then $2\,\|T\| \leq (k-3+p(k))\,|T| + z(k)\beta(T)$.

Lemma 3. *If $z\colon \mathbb{N} \to \mathbb{R}$ is such that $z(k) = 0$ or $2 \leq z(k) \leq \frac{k(k-3)}{k-2}$ for all $k \in \mathbb{N}$, then (p, h, z, f) is 5-Gallai, where*

$$h(k) := \frac{k(k-3)-(k-2)z(k)}{k^2-4k+5},$$

$$p(k) := \frac{2+h(k)}{k-2},$$

$$f(k) := (k-1)(1-h(k)-p(k)).$$

Corollary 4. $\left(\frac{2}{k-2}, 0, \frac{k(k-3)}{k-2}, \frac{(k-1)(k-4)}{k-2}\right)$ *is 5-Gallai.*

Theorem 5.

Proof. Combining Lemma 2 and Corollary 4 gives

$$\begin{aligned}&\left(\Delta-2-\frac{2}{\Delta-2}\right)|\mathcal{L}| + \frac{\Delta(\Delta-1)}{\Delta-2}\beta_\Delta(\mathcal{L}) <\\ &\left(\Delta-3+\frac{2}{\Delta-2}\right)|\mathcal{L}| + \frac{\Delta(\Delta-3)}{\Delta-2}\beta_\Delta(\mathcal{L}) + \frac{(\Delta-1)(\Delta-4)}{\Delta-2}c_0(\mathcal{L}),\end{aligned}$$

so

$$(\Delta-6)\,|\mathcal{L}| + 2\Delta\beta_\Delta(\mathcal{L}) < (\Delta-1)(\Delta-4)c_0(\mathcal{L}).$$

□