hypergraph kernel magic notes

January 10, 2017

1 Hypergraph orientations

Let $H = (V, E)$ be a hypergraph. An *orientation* of H is a function q that assigns to each $e \in E$, a subset of e, that is $q(e) \subseteq e$. Given an orientation q of H, the *out-degree* of $v \in V$ is $d_q^+(v) := |\{e \in E : v \in q(e)\}|$. We say that an orientation q of H is *kernel-perfect* if for all induced subhypergraphs $H' = (V', E')$ of H, there is $S \subseteq V'$ such that S is independent and for each $v \in V' \setminus S$, there is $e \in E'$ with $e \cap S \neq \emptyset$ and $v \in q(e)$. Such an S is a *kernel*.

Lemma 1.1. *Let $H = (V, E)$ be a hypergraph and $f : V \to \mathbb{N}$. If H has a kernel-perfect orientation q such that $f(v) > d_q^+(v)$ for all $v \in V$, then H is f-paintable.*

Proof. Suppose not and choose a counterexample $H = (V, E)$ with f so as to minimize $|V|$. Let q be a kernel-perfect orientation of H such that $f(v) > d_q^+(v)$ for all $v \in V$. Since H is not f-paintable, Lister has a winning move, say he chooses $A \subseteq V$ as the vertices that have blue available. Painter should pick a kernel $S \subseteq A$ and color all vertices in S blue. Define a function f' on $H - S$ by $f'(v) = f(v)$ for $v \in V \setminus A$ and $f'(v) = f(v) - 1$ for all $v \in S \setminus A$. Since S is a kernel, the out-degree of each vertex in $S \setminus A$ went down by at least one. Now Painter can win on $H - S$ with f' by minimality of $|V|$, contradicting our choice of A. □

Lemma 1.2. *Let $H = (V, E)$ be a hypergraph and $S \subseteq V$ an independent set. If q is an orientation of H such that $q(e) \geq 1$ for all $e \in E$ and $q(e) \geq 2$ for all $e \subseteq E \setminus S$, then q is kernel-perfect.*

Proof. Suppose not and choose a counterexample $H = (V, E)$ with q so as to minimize $|V|$. Then every proper induced subhypergraph of H has a kernel by minimality of $|V|$. So, it must be that H has no kernel. In particular, S is not a kernel of H with q. So, there is $v \in V \setminus S$ such that $v \notin q(e)$ for every $e \in E$ with $v \in e$ and $e \cap S \neq \emptyset$. Since $q(e) \geq 1$ for all $e \in E$ and $q(e) \geq 2$ for all $e \subseteq E \setminus S$, for each $e \in E$ with $v \in e$, we can choose $x_e \in q(e) \setminus \{v\}$. Let $H' = H - (\{v\} \cup \{x_e : e \in E \text{ with } v \in e\})$. Then H' has a kernel A' by minimality of $|V|$. We claim that $A := A' \cup \{v\}$ is a kernel in H. If A was not independent, then there would be $e \in E$ with $v \in E$ and $e \subseteq A$. But then $x_e \in A'$ which is impossible since x_e is not in H'. So, A in independent. So, A is a kernel since $v \in A$ and $x_e \in q(e)$ for all e containing v. This contradiction completes the proof. □

Theorem 1.3. *Let $H = (V, E)$ be a hypergraph and $f : V \to \mathbb{N}$. If $M \subseteq V$ is independent and*

then H is f-paintable.