

# Edge-coloring via fixable subgraphs

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## 1 Introduction

All multigraphs are loopless.

## 2 Completing edge-colorings

Our goal is to convert a partial  $k$ -edge-coloring of a multigraph  $M$  into a (total)  $k$ -edge-coloring of  $M$ . For a partial  $k$ -edge-coloring  $\pi$  of  $M$ , let  $M_\pi$  be the subgraph of  $M$  induced on the uncolored edges and let  $L_\pi$  be the list assignment on the vertices of  $M_\pi$  given by  $L_\pi(v) = [k] - \{\tau \mid \pi(vx) = \tau \text{ for some } vx \in E(M)\}$ .

Kempe chains give a powerful technique for converting a partial  $k$ -edge-coloring into a total  $k$ -edge-coloring. The idea is to repeatedly exchange colors on two-colored paths until  $M_\pi$  has an edge-coloring  $\zeta$  such that  $\zeta(xy) \in L_\zeta(x) \cap L_\zeta(y)$  for all  $xy \in E(M_\pi)$ . In this sense the original list assignment  $L_\pi$  on  $M_\pi$  is *fixable*. In the next section, we give an abstract definition of this notion that frees us from the embedding in the ambient graph  $M$ .

### 2.1 Fixable graphs

Let  $G$  be a multigraph and  $L$  a list assignment on  $V(G)$ . For different colors  $a, b \in \text{Pot}(L)$ , let  $S_{a,b}$  be all the vertices of  $G$  that have exactly one of  $a$  or  $b$  in their list; more precisely,  $S_{a,b} = \{v \in V(G) \mid |\{a, b\} \cap L(v)| = 1\}$ . We say that  $G$  is  *$L$ -fixable* if either

1.  $G$  has an edge-coloring  $\pi$  such that  $\pi(xy) \in L(x) \cap L(y)$  for all  $xy \in E(G)$ ; or
2. there are different  $a, b \in \text{Pot}(L)$  such that for every partition  $P_1, \dots, P_k$  of  $S_{a,b}$  into sets of size at most two, there is  $J \subseteq [k]$  so that  $G$  is  $L'$ -fixable where  $L'$  is formed from  $L$  by swapping  $a$  and  $b$  in  $L(v)$  for every  $v \in \bigcup_{i \in J} P_i$ .

**Lemma 2.1.** *If a multigraph  $M$  has a partial  $k$ -edge-coloring  $\pi$  such that  $M_\pi$  is  $L_\pi$ -fixable, then  $M$  is  $k$ -edge-colorable.*

*Proof.*

□

## 2.2 A necessary condition

Since the edges incident to a given vertex must all get different colors, we have the following.

**Lemma 2.2.** *If  $G$  is  $L$ -fixable, then  $|L(v)| \geq d_G(v)$  for all  $v \in V(G)$ .*

By considering the maximum size of matchings in each color, we get a much more interesting necessary condition. For  $C \subseteq \text{Pot}(L)$  and  $H \subseteq G$ , let  $H_{L,C}$  be the subgraph of  $H$  induced on the vertices  $v$  with  $L(v) \cap C \neq \emptyset$ . When  $L$  is clear from context, we may write  $H_C$  for  $H_{L,C}$ . If  $C = \{\alpha\}$ , we may write  $H_\alpha$  for  $H_C$ . For  $H \subseteq G$ , put

$$\psi_L(H) = \sum_{\alpha \in \text{Pot}(L)} \left\lfloor \frac{|H_{L,\alpha}|}{2} \right\rfloor.$$

Each term in the sum gives an upper bound on the size of a matching in color  $\alpha$ . So  $\psi_L(H)$  is an upper bound on the the number of edges in a partial  $L$ -edge-coloring of  $H$ . We say that  $(H, L)$  is *abundant* if  $\psi_L(H) \geq \|H\|$  and that  $(G, L)$  is *superabundant* if for every  $H \subseteq G$ , the pair  $(H, L)$  is abundant.

**Lemma 2.3.** *If  $G$  is  $L$ -fixable, then  $(G, L)$  is superabundant.*

*Proof.* □

## 2.3 Fixability of stars

When  $G$  is a star, the conjunction of our two necessary conditions is sufficient. This generalizes Vizing fans [2]; in the next section we will define “Kierstead-Tashkinov-Vizing assignments” and show that they are always superabundant. In [1], the second author proved a common generalization of Theorem 2.4 and Hall’s theorem. In particular, Theorem 2.4 holds for multistars as well; the proof for multistars is nearly identical, but notationally cumbersome.

**Theorem 2.4.** *If  $G$  is a star, then  $G$  is  $L$ -fixable if and only if  $(G, L)$  is superabundant and  $|L(v)| \geq d_G(v)$  for all  $v \in V(G)$ .*

*Proof.* □

## 2.4 Kierstead-Tashkinov-Vizing assignments

Many edge-coloring results have been proved using a specific kind of superabundant pair  $(G, L)$  where superabundance can be proved via a special ordering. That is, the orderings given by the definition of Vizing fans, Kierstead paths, and Tashkinov trees. In this section, we show how superabundance easily follows from these orderings.

We say that a list assignment  $L$  on  $G$  is a *Kierstead-Tashkinov-Vizing assignment* (henceforth *KTV-assignment*) if for some  $xy \in E(G)$ , there is a total ordering ‘ $<$ ’ of  $V(G)$  such that

1. there is an edge-coloring  $\pi$  of  $G - xy$  such that  $\pi(uv) \in L(u) \cap L(v)$  for each  $uv \in E(G - xy)$ ;

2.  $x < z$  for all  $z \in V(G - x)$ ;
3.  $G[w \mid w \leq z]$  is connected for all  $z \in V(G)$ ;
4. for each  $wz \in E(G - xy)$ , there is  $u < \max\{w, z\}$  such that  $\pi(wz) \in L(u) - \{\pi(e) \mid e \in E(u)\}$ ;
5. there are different  $s, t \in V(G)$  such that  $L(s) \cap L(t) - \{\pi(e) \mid e \in E(s) \cup E(t)\} \neq \emptyset$ .

**Lemma 2.5.** *If  $L$  is a KTV-assignment on  $G$ , then  $(G, L)$  is superabundant.*

*Proof.* Let  $L$  be a KTV-assignment on  $G$ , and let  $H \subseteq G$ . We will show that  $(H, L)$  is abundant. Clearly it suffices to consider the case when  $H$  is an induced subgraph, so we assume this. Property (1) gives that  $G - xy$  has an edge-coloring  $\pi$ , so  $\psi_L(H) \geq \|H\| - 1$ ; also  $\psi_L(H) \geq \|H\|$  if  $\{x, y\} \not\subseteq V(H)$ . Furthermore  $\psi_L(H) \geq \|H\|$  if  $s$  and  $t$  from property (5) are both in  $V(H)$ , since then  $\psi_L(H)$  gains 1 over the naive lower bound, due to the color in  $L(s) \cap L(t)$ . So  $V(G) - V(H) \neq \emptyset$ .

Now choose  $z \in V(G) - V(H)$  that is smallest under  $<$ . Put  $H' = G[w \mid w \leq z]$ . By the minimality of  $z$ , we have  $H' - z \subseteq H$ . By property (2),  $|H'| \geq 2$ . By property (3),  $H'$  is connected and thus there is  $w \in V(H' - z)$  adjacent to  $z$ . So, we have  $w < z$  and  $wz \in E(G) - E(H)$ . Now  $\pi(wz) \in L(w)$ . By the definition of a KTV-assignment, property (4) implies that there exists  $u$  with  $u < \max\{w, z\} = z$  and  $\pi(wz) \in L(u) - \{\pi(e) \mid e \in E(u)\}$ . Then  $u \in V(H' - z) \subseteq V(H)$  and again we gain 1 over the naive lower bound on  $\psi_L(H)$ , due to the color in  $L(u) \cap L(w)$ . So  $\psi_L(H) \geq \|H\|$ .  $\square$

## 2.5 Stars with one edge subdivided

**Theorem 2.6.** *Let  $G$  be a star with one edge subdivided and root  $r$ . If  $(G, L)$  is superabundant,  $|L(v)| \geq d_G(v)$  for all  $v \in V(G)$  and  $|L(r)| > d_G(r)$ , then  $G$  is  $L$ -fixable.*

*Proof.*  $\square$

## 3 Applications

## 4 Conjectures

**Conjecture 4.1.** *Any multigraph  $G$  is  $L$ -fixable if  $(G, L)$  is superabundant and  $|L(v)| > d_G(v)$  for all  $v \in V(G)$ .*

**Conjecture 4.2.** *Any tree  $G$  is  $L$ -fixable if  $(G, L)$  is superabundant and  $|L(v)| > d_G(v)$  for all  $v \in V(G)$ .*

**Conjecture 4.3.** *Any path  $G$  is  $L$ -fixable if  $(G, L)$  is superabundant and  $|L(v)| > d_G(v)$  for all  $v \in V(G)$ .*

## References

- [1] Rabern, L., *A game generalizing Hall's Theorem*, Discrete Mathematics **320** (2014), no. 6, 87–91.
- [2] V.G. Vizing, *Vertex coloring with given colors*, Metody Diskretn. Anal. **29** (1976), 3–10 (in Russian).