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For a graph G , let $\beta_k(G)$ be the independence number of the subgraph of G induced on the vertices of degree $k - 1$. When k is defined in context, we just write $\beta(G)$. Let $\mathcal{H}(G)$ be the subgraph of G induced on vertices of degree greater than $\delta(G)$. Let $\mathcal{L}(G)$ be the subgraph of G induced on vertices of degree $\delta(G)$.

Definition 1. The *maximum independent cover number* of a graph G is the maximum $\text{mic}(G)$ of $\|I, V(G) \setminus I\|$ over all independent sets I of G .

Definition 2. A graph G is *OC-reducible* to H if H is a nonempty induced subgraph of G which is online f_H -choosable where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not OC-reducible to any nonempty induced subgraph, then it is *OC-irreducible*.

Lemma 1. Every OC-irreducible graph G satisfies

$$2 \|G\| > (\delta(G) - 1) |G| + \text{mic}(G).$$

Lemma 2. If G is an OC-irreducible graph where $\mathcal{H}(G)$ is edgeless, $\Delta := \Delta(G) = \delta(G) + 1$ and $\mathcal{L} := \mathcal{L}(G)$, then

$$2 \|\mathcal{L}\| > \left(\Delta - 2 - \frac{2}{\Delta - 2} \right) |\mathcal{L}| + \frac{\Delta(\Delta - 1)}{\Delta - 2} \beta(\mathcal{L}) + \frac{\Delta}{\Delta - 2}.$$

Proof. Let G be such a graph. Put $\mathcal{H} := \mathcal{H}(G)$ and $\mathcal{L} := \mathcal{L}(G)$. Since \mathcal{H} is edgeless,

$$\begin{aligned} \Delta |\mathcal{H}| &= \|\mathcal{H}, \mathcal{L}\| \\ &= (\Delta - 1) |\mathcal{L}| - 2 \|\mathcal{L}\|, \end{aligned} \tag{1}$$

so, by Lemma 1,

$$\begin{aligned} (\Delta - 1) |\mathcal{L}| + \Delta |\mathcal{H}| &= 2 \|G\| \\ &> (\Delta - 2) |G| + \text{mic}(G) \\ &\geq (\Delta - 2) |G| + \Delta |\mathcal{H}| + (\Delta - 1) \beta(\mathcal{L}) \\ &= (\Delta - 2) |\mathcal{L}| + (2\Delta - 2) |\mathcal{H}| + (\Delta - 1) \beta(\mathcal{L}), \end{aligned}$$

so simplifying and using (1) again gives

$$\begin{aligned} |\mathcal{L}| &> (\Delta - 2) |\mathcal{H}| + (\Delta - 1) \beta(\mathcal{L}) + 1 \\ &= \frac{\Delta - 2}{\Delta} ((\Delta - 1) |\mathcal{L}| - 2 \|\mathcal{L}\|) + (\Delta - 1) \beta(\mathcal{L}), \end{aligned}$$

now some mild manipulation yields the desired bound. \square

Definition 3. A quadruple (p, h, z, f) of functions from \mathbb{N} to \mathbb{R} is *r-Gallai* if for every $k \geq r$ and Gallai tree $T \neq K_k$ with $\Delta(T) \leq k - 1$, the following hold:

- if $K_{k-1} \subseteq T$, then $2 \|T\| \leq (k - 3 + p(k)) |T| + h(k)q(T) + z(k)\beta(T) + f(k)$; and
- if $K_{k-1} \not\subseteq T$, then $2 \|T\| \leq (k - 3 + p(k)) |T| + z(k)\beta(T)$.

Lemma 3. If $z: \mathbb{N} \rightarrow \mathbb{R}$ is such that $z(k) = 0$ or $2 \leq z(k) \leq \frac{k(k-3)}{k-2}$ for all $k \in \mathbb{N}$, then (p, h, z, f) is 5-Gallai, where

$$h(k) := \frac{k(k-3) - (k-2)z(k)}{k^2 - 4k + 5},$$

$$p(k) := \frac{2 + h(k)}{k - 2},$$

$$f(k) := (k - 1)(1 - h(k) - p(k)).$$

Corollary 4. $\left(\frac{2}{k-2}, 0, \frac{k(k-3)}{k-2}, \frac{(k-1)(k-4)}{k-2}\right)$ is 7-Gallai.

Theorem 5. There does not exist an OC-irreducible graph G where $\Delta(G) = \delta(G) + 1 \geq 7$ and $\mathcal{H}(G)$ is edgeless.

Proof. If one did, then combining Lemma 2 and Corollary 4 would give the contradiction

$$(\Delta - 2) |\mathcal{H}| \leq |\mathcal{L}| < \Delta + \frac{4}{\Delta - 6} - \frac{2\Delta}{\Delta - 6} \beta(\mathcal{L}).$$

\square