Hitting maximum cliques

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Outline

Introduction

Finding a stable set hitting every maximum clique in a graph can be very useful for coloring problems.

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Observation

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Observation

A graph is perfect iff every induced subgraph has a stable set hitting every maximum clique.

Kostochka [8] gave the following sufficient condition.

Lemma (Kostochka 1980)

A graph satisfying $\omega \geq \Delta + \frac{3}{2} - \sqrt{\Delta}$ *has a stable set hitting every maximum clique.*

In [10], we improved this as follows.

Lemma (Rabern 2009)

There exists a positive constant $c < 1$ such that every graph satisfying $\omega > c(\Delta + 1)$ has a stable set hitting every maximum clique.

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- expanding it to a maximal stable set and removing it leaves a graph with $\omega \leq 3$, $\chi = 4$ and $\Delta \leq 3$, contradicting minimality of Δ

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- expanding it to a maximal stable set and removing it leaves a graph with $\omega \leq 3$, $\chi = 4$ and $\Delta \leq 3$, contradicting minimality of Δ
- thus a countexample to Brooks' theorem minimizing Δ must have $\Delta = 3$

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- Reed conjectures that every graph satisfies $\chi \leq \left\lceil \frac{\omega+\Delta+1}{2} \right\rceil$
- when proving this conjecture for a hereditary class of graphs, a minimum counterexample must have $\omega \leq \frac{3}{4}(\Delta + 1)$
- this was used in [10] to simplify the proof of Reed's conjecture for line graphs given in [7]

An outline of the proof

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- use lemmas of Hajnal and Kostochka to show that each component of the maximum clique graph has many universal vertices
- consider the subgraph induced on these universal vertices and partition it by component
- find an independent transversal
- this is our desired stable set hitting all maximum cliques

Hajnal's clique collection lemma

Lemma (Hajnal 1965)

For a collection $\mathcal{Q}$ of maximum cliques in a graph G we have

$$\left|\bigcup \mathcal{Q}\right| + \left|\bigcap \mathcal{Q}\right| \geq 2\omega(G).$$

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- consider $W := (Q_1 \cap \bigcup_{i=2}^{r} Q_i) \cup \bigcap_{i=2}^{r} Q_i$
- W is a clique, so the following machinations give a contradiction

Machinations

$$
\begin{aligned}
\omega(G) &\geq |W| \\
&= |(Q_1 \cap \bigcup_{i=2}^r Q_i) \cup \bigcap_{i=2}^r Q_i| \\
&= |Q_1 \cap \bigcup_{i=2}^r Q_i| + |\bigcap_{i=2}^r Q_i| - |\bigcap_{i=1}^r Q_i \cap \bigcup_{i=2}^r Q_i| \\
&= |Q_1| + |\bigcup_{i=2}^r Q_i| - |\bigcup_{i=1}^r Q_i| + |\bigcap_{i=2}^r Q_i| - |\bigcap_{i=1}^r Q_i| \\
&= \omega(G) + |\bigcup_{i=2}^r Q_i| + |\bigcap_{i=2}^r Q_i| - |\bigcup_{i=1}^r Q_i| - |\bigcap_{i=1}^r Q_i| \\
&\geq \omega(G) + 2\omega(G) - (|\bigcup_{i=1}^r Q_i| + |\bigcap_{i=1}^r Q_i|) \\
&> \omega(G).
\end{aligned}
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The clique graph

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Clique graph

For a collection of cliques $\mathcal{Q}$ in a graph, let $X_{\mathcal{Q}}$ be the intersection graph of $\mathcal{Q}$; that is, the vertex set of $X_{\mathcal{Q}}$ is $\mathcal{Q}$ and there is an edge between $Q_1 \neq Q_2 \in \mathcal{Q}$ iff Q_1 and Q_2 intersect.

Kostochka's lemma

Lemma (Kostochka 1980)

Let G be a graph satisfying $\omega > \frac{2}{3}(\Delta + 1)$. If $\mathcal{Q}$ is a collection of maximum cliques in G such that $X_{\mathcal{Q}}$ is connected, then $\cap\mathcal{Q} \neq \emptyset$.

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- then $X_{\mathcal{Z}}$ is connected and hence by minimality of $|\mathcal{Q}|$, $\cap\mathcal{Z} \neq \emptyset$
- in particular $|\cup\mathcal{Z}| \leq \Delta(G) + 1$
- thus $|\cup Q| \leq |A - B| + |\cup\mathcal{Z}| \leq 2(\Delta(G) + 1) - \omega(G) < 2\omega(G)$ contradicting Hajnal

Independent transversals

Definition

An *independent transversal* of a partition $\{V_1, \ldots, V_r\}$ of a the vertex set of a graph G is a stable set $\{v_1, \ldots, v_r\} \subseteq V(G)$ such that $v_i \in V_i$ for each i.

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Alon [2] proved a simple sufficient condition probabilistically.

Lemma (Alon 1988)

A partition $\{V_1, \ldots, V_r\}$ of the vertex set of a graph G has an independent transversal if $|V_i| \geq 2e\Delta(G)$ for each i.

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- hence an independent transversal exists

Putting it all together

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- hence $k \geq \frac{11}{13}(\Delta(G) + 1) \geq \frac{4e}{13}(\Delta(G) + 1) \geq 2e\Delta(H)$
- by Alon's lemma, we have an independent transversal through the K_i and this is the desired stable set hitting every maximum clique

Improving the constant

In [10] we proved that $c = \frac{3}{4}$ works in the same way as above using the following lemma of Haxell [5].

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- Haxell's proof is elementary and uses some somewhat delicate induction
- there are also proofs based on topological connectivity of the independent set complex

Improving the constant

Building on observations by Aharoni, Berger and Ziv [1] about the proof of Haxell's lemma, King [6] proved the following lopsided version of Haxell's lemma. Using this, he proved that $c = \frac{2}{3}$ works.

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Lemma (King 2009)

A partition $\{V_1, \ldots, V_r\}$ of the vertex set of a graph G has an independent transversal if there exists a positive integer k such that for each i we have $\min\{k, |V_i| - k\} \geq \max_{v \in V_i} d(v)$.

Tightness of King's lemma

- let G_r be line graph of a 5-cycle where each edge has multiplicity r; that is, $G_r := L(r \cdot C_5)$

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R. Aharoni, E. Berger, and R. Ziv, *Independent systems of representatives in weighted graphs*, Combinatorica **27** (2007), no. 3, 253–267.

N. Alon, *The linear arboricity of graphs*, Israel Journal of Mathematics **62** (1988), no. 3, 311–325.

O.V. Borodin and A.V. Kostochka, *On an upper bound of a graph's chromatic number, depending on the graph's degree and density*, Journal of Combinatorial Theory, Series B **23** (1977), no. 2-3, 247–250.

A. Hajnal, *A theorem on k-saturated graphs*, Canadian Journal of Mathematics (1965), 720.

P.E. Haxell, *A note on vertex list colouring*, Combinatorics, Probability and Computing **10** (2001), no. 4, 345–347.

A.D. King, *Hitting all maximum cliques with a stable set using lopsided independent transversals*, CoRR **abs/0911.1741** (2009).

A.D. King, B.A. Reed, and A. Vetta, *An upper bound for the chromatic number of line graphs*, European Journal of Combinatorics **28** (2007), no. 8, 2182–2187.

A.V. Kostochka, *Degree, density, and chromatic number*, Metody Diskret. Anal. **35** (1980), 45–70.

L. Rabern, *On hitting all maximum cliques with an independent set*, ArXiv e-prints (2009).

______, *On hitting all maximum cliques with an independent set*, Journal of Graph Theory **66** (2011), no. 1, 32–37.