The Borodin-Kostochka Conjecture for choosability and graphs with large maximum degree

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Abstract

Brooks' Theorem states that for a graph G with maximum degree $\Delta(G)$ at least 3, the chromatic number is at most $\Delta(G)$ when the clique number of G is at most $\Delta(G)$. Vizing proved that the list chromatic number is also at most $\Delta(G)$ under the same conditions. Borodin and Kostochka conjectured that a graph G with maximum degree at least 9 must be $(\Delta(G)-1)$-colorable when G has clique number at most $\Delta(G)-1$; this was proven for graphs with maximum degree at least 10^{14} by Reed. In this paper, we prove the analogous result for the list chromatic number; namely, we prove that a graph G with $\Delta(G) \geq 10^{20}$ is $(\Delta(G)-1)$-choosable when G has clique number at most $\Delta(G)-1$.

1 Introduction

Let K_n be the complete graph on n vertices and let E_n be the empty graph on n vertices. For a graph G, let $\Delta(G)$, $\omega(G)$, $\chi(G)$, and $\chi_l(G)$ denote the maximum degree, clique number, chromatic number, and list chromatic number of G, respectively. It is a trivial fact that a graph can be properly colored with $\Delta(G)+1$ colors. Interestingly enough, $\Delta(G)+1$ happens to be the least upper bound on $\omega(G)$. In 1941, Brooks [2] proved the following classical result that connects $\Delta(G)$, $\omega(G)$, and $\chi(G)$.

Theorem 1.1. *[2] For a graph G with $\Delta(G) \geq 3$, if $\omega(G) \leq \Delta(G)$, then $\chi(G) \leq \Delta(G)$.*

The condition on the maximum degree is tight, as the conclusion does not follow for odd cycles. Actually, in 1976, Vizing [8] showed that the analogous result holds for the list chromatic number under the same conditions.

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Theorem 1.2. *[2] For a graph G with $\Delta(G) \geq 3$, if $\omega(G) \leq \Delta(G)$, then $\chi_l(G) \leq \Delta(G)$.*

Shortly after, in 1977, Borodin and Kostochka [1] conjectured a similar type of result when the upper bound on the clique number is one less. The condition on the maximum degree is tight, as there exist graphs with maximum degree less then 9 where the conclusion is not true. We state the contrapositive.

Conjecture 1.3. *[1] Every graph G satisfying $\chi(G) \geq \Delta(G) \geq 9$ contains a $K_{\Delta(G)}$.*

There are various partial results regarding this conjecture. Kostochka [4] proved the following result, which guarantees a clique of size almost the maximum degree. A relaxation on the lower bound on the maximum degree conditions allows a theorem by Mozhan [6], which ensures a bigger, but still less than the maximum degree, clique. By drastically increasing the lower bound on the maximum degree, Reed [7] finally shows the existence of a clique of size equal to the maximum degree using probabilistic arguments.

Theorem 1.4. *[4] Every graph G satisfying $\chi(G) \geq \Delta(G)$ contains a $K_{\Delta(G)-28}$.*

Theorem 1.5. *[6] Every graph G satisfying $\chi(G) \geq \Delta(G) \geq 31$ contains a $K_{\Delta(G)-3}$.*

Theorem 1.6. *[7] Every graph G satisfying $\chi(G) \geq \Delta(G) \geq 10^{14}$ contains a $K_{\Delta(G)}$.*

There is even a paper by Cranston and Rabern [3] that takes a deep look into minimum counterexamples to Conjecture 1.3. We will use a couple results from their paper for a structural lemma in this paper. Recall the following generalization of choosability.

Definition 1.7. For an integer r, a graph G is d_r*-choosable* if G is f-choosable where $f(v) = d(v) - r$.

In this paper, we address Conjecture 1.3 for the list chromatic number. We prove that the conjecture is true even for the list chromatic number when the maximum degree is sufficiently large. The main result in this paper is the following.

Theorem 1.8. *For a graph G with $\Delta(G) \geq 10^{20}$, if $\omega(G) \leq \Delta(G)-1$, then $\chi_l(G) \leq \Delta(G)-1$.*

Throughout this paper, unless specified otherwise, G will be a minimum counterexample in terms of the number of vertices for Theorem 1.8 with maximum degree Δ. Note that every vertex of G must have degree either Δ or $\Delta - 1$, and G cannot have a d_1-choosable graph as an induced subgraph, even though every proper subgraph of G is $(\Delta - 1)$-choosable. Let $L(v)$ be the list of colors assigned to a vertex $v \in V(G)$.

We will show that G is actually $(\Delta - 1)$-choosable, proving such a counterexample G cannot exist. Section 2 will first introduce some useful lemmas. The proof will come in two steps. Section 3 is the first step, which is to construct a decomposition of G that will facilitate the second step. Section 4 is the second step, which is to show that G is actually $(\Delta - 1)$-choosable via a probabilistic argument involving the Lovász Local Lemma and Azuma's Inequality.

2 Lemmas

The lemmas in this section will reveal some aspects of the list assignment to the vertices of cliques of G. In particular, the list assignment of vertices in maximum cliques are analyzed.

Lemma 2.1. *[3] If B is a graph with $\omega(B) \leq |B| - 2$, then $K_6 \vee B$ is d_1-choosable.*

The above lemma from [3] shows that G cannot have $K_6 \vee B$ as an induced subgraph if $\omega(B) \leq |B| - 2$. Recall that $G_1 \vee G_2$ denotes the *join* of G_1 and G_2.

Definition 2.2. A vertex v is *safe* if $d(v) - (\Delta - 2)$ instances of any of the following happen:

1. there is a color not in $L(v)$ coloring a vertex of $N(v)$;
2. there is a color assigned to two vertices of $N(v)$.

Note that a safe vertex always has a color in its list that can be used on it since at most $\Delta - 2$ colors can appear in its neighborhood. Now we show that the lists of all but at most one vertex in a clique of G has many colors in common. This lemma will be used for a more detailed analysis in Lemma 2.4.

Lemma 2.3. *If C is a clique of G, then there exists $C' \subset C$ such that $|C'| = |C| - 1$ and $|L(x) \cap L(y)| \geq |C| - 3$ for all $x, y \in C$.*

Proof. Let $L'(v)$ be the remaining colors available for v after a $(\Delta - 1)$-list coloring of $G - C$, which exists by the minimality of G. For $v \in C$, since v has at most $\Delta - (|C| - 1)$ neighbors outside C, it follows that $|L'(v)| \geq |C| - 2$.

Since G is a minimum counterexample, a system of distinct representatives for $L'(v)$ where $v \in C$ does not exist. Thus, by Hall's theorem, there exists a subset F of C such that the union of the $L'(v)$ for v in F has size less than $|F|$. Since each L' list has size at least $|C| - 2$, $|F|$ has size at least $|C| - 1$. If $|F|$ has size $|C| - 1$ then every vertex in F has the same L' list and we are done. Otherwise, F is C, and the union of the lists for the vertices in C has at most $|C| - 1$ elements. We can assume there are two vertices x and y with distinct lists as otherwise we are done. Now the union of these L' lists has size exactly $|C| - 1$ and every vertex in C has at most one color missing from each of the L' lists of x and y. $\square$

Lemma 2.4. *For a $(\Delta - 1)$-clique C of G and a vertex $w \notin C$ such that $|N(w) \cap C| \geq 5$, the following holds:*

(i) there exists a set S of $\Delta - 2$ colors that are in $L(v)$ for every $v \in N(w) \cap C$;

(ii) each vertex in $N(w) \cap C$ has degree Δ;

(iii) each vertex $y \notin C \cup \{w\}$ has at most 4 neighbors in $N(w) \cap C$;

(iv) for each $v \in N(w) \cap C$, the color in $L(v) - S$ appears in the lists of at most 5 vertices in $N(w) \cap C$.

Proof. of (*i*). Let $L'(v)$ be the remaining colors available for v after a $(\Delta-1)$-list coloring of $G-C-w$, which exists by the minimality of G. We want to show $L'(x) = L'(y)$ for all $x, y \in N(w) \cap C$. Assume for the sake of contradiction that there exists a color α such that $N(w) \cap C$ can be partitioned into two nonempty sets A and B where a vertex $v \in N(w) \cap C$ is in A if and only if $\alpha \in L'(v)$. Choose vertices $x \in C - N(w)$, $a \in A$, and $b \in B$. We may assume $|B| \geq 2$ since if $|B| = 1$, then choose a color in $L'(b) - L'(a)$ instead of α to form the partition. Note that $|L'(w)| \geq \Delta - 1 - (\Delta - 5) = 4$, $|L'(x)| \geq \Delta - 3$, and $|L'(b)| \geq \Delta - 2$.

If there exists a color $c \in L'(w) \cap L'(x)$, then color both w and x with c. If $c = \alpha$, then we can complete the coloring by coloring the vertices in B last since every vertex in B is safe. If $c \neq \alpha$, then color a with α, and now we can complete the coloring by coloring the vertices in B last since every vertex in B is safe.

If $L'(w)$ and $L'(x)$ are disjoint, then since $|L'(w) \cup L'(x) - L'(b)| \geq |L'(w)| + |L'(x)| - |L'(b)| \geq \Delta + 1 - (\Delta - 2) = 3$, either $L'(w) - L'(b)$ or $L'(x) - L'(b)$ has at least two colors. Without loss of generality, assume $|L'(w) - L'(b)| \geq 2$, and therefore there must be a color $c \in L'(w) - L'(b)$ that is different from α. We can complete the coloring by first coloring w with c and a with α, and then coloring the vertices in B last since they are safe. $\square$

Proof. of (*ii*). Let $L'(v)$ be the remaining colors available for v after a $(\Delta-1)$-list coloring of $G-C-w$, which exists by the minimality of G. Assume for the sake of contradiction that there exists a vertex $v \in N(w) \cap C$ with degree $\Delta - 1$. Choose a vertex $x \in C - N(w)$. Note that $|L'(w)| \geq \Delta - 1 - (\Delta - 5) = 4$, $|L'(x)| \geq \Delta - 3$, and $|L(v)| = |L'(v)| = \Delta - 1$.

If there exists a color $c \in L'(w) \cap L'(x)$, then color both w and x with c. We can complete the coloring by coloring v last since v is safe.

If $L'(w)$ and $L'(x)$ are disjoint, then since $|L'(w) \cup L'(x) - L'(v)| \geq |L'(w)| + |L'(x)| - |L'(v)| \geq \Delta + 1 - (\Delta - 1) = 2$, either $L'(w) - L'(v)$ or $L'(x) - L'(v)$ has one color c that is not in $L(v)$. Again, we can complete the coloring by first coloring w or x with c, and coloring v last since v is safe. $\square$

Proof. of (*iii*). Suppose not. Let $y \notin C \cup \{w\}$ have at least 5 neighbors in $N(w) \cap C$. Let $x \in C - N(w)$ and let $z \in C - x$ be a vertex not adjacent to y. Such a z must exist since if not, then y and w must be adjacent to all the vertices in C except x. Now, x, w, y form an independent set since adding any edge would create a Δ-clique. This implies that there exists an induced $E_3 \vee K_6$, which is d_1-choosable, which is a contradiction.

Let $L'(v)$ be the remaining colors available for v after a $(\Delta-1)$-list coloring of $G-C-w-y$, which exists by the minimality of G. Note that $|L'(y)| \geq 4$ and $|L'(w)| \geq 4$ since y and w each have at least 5 neighbors in C. Also, $|L'(x)| \geq \Delta - 3$ and $|L'(z)| \geq \Delta - 3$ since x and z have at most 2 neighbors in $G-C-w-y$. Let $v \in N(y) \cap C$ so that $|L'(v)| = |L(v)| = \Delta - 1$.

Without loss of generality assume $L'(w) \cap L'(x) = \emptyset$, which implies $|L'(w) \cup L'(x) - L'(v)| \geq \Delta + 3 - (\Delta - 1) = 2$. If $L'(y) \cap L'(z) = \emptyset$, then by the same logic $|L'(y) \cup L'(z) - L'(v)| \geq 2$. Now, color y or z with a color $c' \in L'(y) \cup L'(z) - L'(v)$, and color w or x with a color c that is neither in $L'(v)$ nor c'. Such color c exists since $|L'(w) \cup L'(x) - L'(v)| \geq 2$. If there exists a color c' in $L'(y) \cap L'(z)$, then color y and z with c', and color w or x with a color c that is neither in $L'(v)$ nor c. Such color c exists since $|L'(w) \cup L'(x) - L'(v)| \geq 2$. Either way, we can color every vertex in C, leaving v and one more vertex u in $N(y) \cap N(w) \cap C$ to color last. Since u has the same list as v, we can color u since v is uncolored and there exists a color in $N(u)$ that is not in the list of u. Now we can finish the coloring since v is safe.

Now assume $L'(w) \cap L'(x) \neq \emptyset$ and $L'(y) \cap L'(z) \neq \emptyset$. If $|(L'(w) \cap L'(x)) \cup (L'(y) \cap L'(z))| \geq 2$, then we can find two different colors c and c' where we can color w, x with c and y, z with c'. We finish the coloring by coloring every vertex in $C - N(y) \cap N(w)$ first, and coloring the vertices in $N(y) \cap N(w) \cap C$ last since every vertex in $N(y) \cap N(w) \cap C$ is safe.

The only remaining case is when $L'(w) \cap L'(x) = L'(y) \cap L'(z) = \{c\}$ for some color c. In this case, $|(L'(w) - L(x)) \cup (L'(x) - L'(w)) - L'(v)| \geq 1$, which implies that there is a color c' in either $L'(w)$ or $L'(x)$ that is not in the list of v. So now we can color y and z with c and color either w or x with c' first, and then color vertices in $C - N(y)$. Since vertices in $N(y) \cap N(w) \cap C$ must have the same list by a previous result, we can finish the coloring since every vertex in $N(y) \cap N(w) \cap C$ is safe. $\square$

Proof. of (iv). Suppose not. Assume a color in $L(v) - S$ appears in a set P of at least 6 vertices. Consider the set Q of neighbors of vertices in P that are not in $C \cup \{w\}$. Note that the neighbors of vertices in Q that are in P partition P since each vertex in P has exactly one neighbor not in $C \cup \{w\}$ by a previous result. By another previous result, since a vertex outside of $C \cup \{w\}$ has at most 5 neighbors in $N(w) \cap C$, there must be at least 2 vertices in Q. Also, Q must be an independent set. Otherwise, if there is an edge with two endpoints in Q, then the endpoints of this edge will receive different colors in a $(\Delta - 1)$-list coloring of $G - C - w$, which exists by the minimality of G. Now, the vertices in P could not have had the same list of colors, which is a contradiction to a previous result.

Adding any edge with endpoints in Q must create a Δ-clique in $G - C - w$. If $v \in Q$ has $d_P(v) \geq 3$, this is impossible since v cannot have $\Delta - 2$ neighbors outside P. This implies that $|Q| \geq 3$ and $d_P(v) \leq 2$ for all $v \in Q$. Note that three vertices $x, y, z \in Q$ must have at least $\Delta - 3$ common neighbors not in $C \cup \{w\}$. These common neighbors and x, y, z induce a copy of $E_3 \vee K_6$, which is d_1-choosable, which is a contradiction. $\square$

Definition 2.5. Let C be a $(\Delta - 1)$-clique of G and let a vertex $w \notin C$ be such that $|N(w) \cap C| \geq 6$. The *core* of $N(w) \cap C$ is the set of $\Delta - 2$ colors S that are in $L(v)$ for every $v \in N(w) \cap C$. For a vertex $v \in N(w) \cap C$, the *special color* of v is the color in $L(v) - S$, and the *external neighbor* of v is the one vertex that is adjacent to v that is not in $C \cup \{w\}$.

3 The Decomposition

In this section, we will construct a decomposition of G that will be very helpful in the next section. Here are a few definitions and a lemma from page 158 of [5].

Definition 3.1. A vertex v of G is *d-sparse* if the subgraph induced by its neighborhood contains fewer than $\binom{\Delta}{2} - d\Delta$ edges. Otherwise, v is *d-dense*.

Definition 3.2. Given a graph H, let the *pot* of H, denoted $Pot(L)$, be $\bigcup_{v \in V(H)} L(v)$.

Small Pot Lemma 3.3. *[3] Let H be a graph and $f : V(H) \to \mathbb{N}$ with $f(v) < |H|$ for all $v \in V(H)$. If H is not f-choosable, then H has a bad f-assignment L such that $|Pot(L)| < |H|$.*

Lemma 3.4. *For any graph B with $\delta(B) \geq \frac{|B|}{2} + 1$ and $\omega(B) \leq |B| - 2$, the graph $K_1 \vee B$ is d_1-choosable.*

Proof. By the Small Pot Lemma 3.3 it suffices to prove that all d_1-assignments L on $K_1 \vee B$ with $|Pot(L)| \leq |B|$ are good. Let L be such a list assignment.

First, suppose B contains disjoint nonadjacent pairs $\{x_1, y_1\}$ and $\{x_2, y_2\}$. Since $|L(x_i)| + |L(y_i)| \geq |B|+2$, we have $|L(x_i) \cap L(y_i)| \geq 2$ for each i. Color x_1 and y_1 with $c_1 \in L(x_1) \cap L(y_1)$ and color x_2 and y_2 with $c_2 \in L(x_2) \cap L(y_2) - c_1$. By the minimum degree condition on B, each component of $B - \{x_1, y_1, x_2, y_2\}$ has a vertex joined to $\{x_1, y_1\}$ or $\{x_2, y_2\}$. Hence we can complete the coloring to all of B and then to the K_1. Thus L is good.

So, we may assume there are no disjoint nonadjacent pairs. Now let K be a maximum clique in B. Then we know $|K| \leq |B| - 2$ so we can pick $x, y \in B - K$. The only possibility is that there is $z \in K$ such that both x and y are joined to $K - z$. Since K is maximum x is not adjacent to y and hence B is a $K_{|B|-3} \vee E_3$. By Lemma 2.1, $|B| \leq 4$. Since $d_B(y) = |B| - 3$, this violates our minimum degree condition on B. □

Lemma 3.5. *We can partition $V(G)$ into $S, D_1, \ldots, D_l$ so that*

(i) each vertex of S is sparse;

(ii) each D_i contains a vertex w_i such that $D_i - w_i$ is a clique of size at least $\Delta - 8\Delta^\alpha + 1$;

(iii) no vertex outside of D_i has more than $\frac{3\Delta}{4}$ neighbors in D_i and w_i has at least $\frac{3\Delta}{4}$ neighbors in D_i.

Proof. Let $C_1, \ldots, C_s$ be the maximal cliques in G with at least $\frac{3\Delta}{4} + 1$ vertices. Suppose $|C_i| \leq |C_j|$ and $C_i \cap C_j \neq \emptyset$. Then $|C_i \cap C_j| \geq |C_i| + |C_j| - (\Delta + 1) \geq 6$. It follows from Lemma 2.1 that $|C_i - C_j| \leq 1$. Now suppose C_i intersects C_j and C_k. By the above, $|C_i \cap C_j| \geq \frac{3\Delta}{4}$. Hence $|C_i \cap C_j \cap C_k| \geq \frac{\Delta}{2} \geq 6$. By Lemma 2.1 we see that $\omega(G[C_i \cup C_j \cup C_k]) \geq |C_i \cup C_j \cup C_k| - 1$ which is impossible since each of C_i, C_j, C_k are maximal. Hence $\bigcup_{i \in [s]} C_i$ can be partitioned into sets $F_1, \ldots, F_r$ so that each F_j is either one of the C_i or one of the C_i and an extra vertex w_i with at least $\frac{3\Delta}{4}$ neighbors in C_i.

Put $d = \Delta^\alpha$ and let $D_1, \ldots, D_l$ be all the F_j such that some vertex in F_j is d-dense and let S be $V(G) - \bigcup_{i \in [l]} D_i$. Then (iii) follows by construction. It remains to check (i) and (ii).

We show that if $v \in V(G)$ is d-dense, then it is in a $(\Delta - 8d + 2)$-clique. Since we know that any $v \in S$ is either in no $(\frac{3\Delta}{4} + 1)$-clique (and hence in no $(\Delta - 8d + 2)$-clique) or is d-sparse, (i) follows. Also, since each F_j contains a d-dense vertex, (ii) follows as well.

So, suppose $v \in V(G)$ is d-dense but in no $(\Delta - 8d + 2)$-clique. Then applying Lemma 3.4 repeatedly, we get a sequence $y_1, \ldots, y_{8d} \in N(v)$ such that

$$|N(y_i) \cap (N(x) - \{y_1, \ldots, y_{i-1}\})| \leq \frac{1}{2}(\Delta + 1 - i).$$

Hence the number of non-edges in v's neighborhood is at least

$$\frac{1}{2}\sum_{i=1}^{8d}(\Delta - i) > d\Delta.$$

$\square$

Definition 3.6. For convenience, let $K_i = \begin{cases} C_i & \text{if } D_i = C_i \\ C_i \cap N(w_i) & \text{if } D_i = C_i \cup \{w_i\} \end{cases}$

Definition 3.7. Also for convenience, partition the set of C_i into the following three sets (if some C_i can be either (ii) or (iii), then just choose an arbitrary one):

(i) $\mathcal{P}_1$: the set of C_i such that $|C_i| \leq \Delta - 2$;

(ii) $\mathcal{P}_2$: the set of C_i such that $|C_i| = \Delta - 1$ and every vertex outside C_i has at most Δ^c neighbors in C_i;

(iii) $\mathcal{P}_3$: the set of C_i such that $|C_i| = \Delta - 1$ and some vertex outside C_i has more than Δ^d neighbors inside C_i.

TODO: NOTE: Of course, for this to be a partition, it must be the case that $d \leq c$. For $\Delta_0 = 10^{20}$, the values $c = d = 0.29$ work. The reason I have c and d separate is so that I can do calculations more efficiently when something changes in the proof.

Now we prove a structure lemma that will be crucial in the following sections. We use a lemma from [3] to prove the lemma needed.

Lemma 3.8. *[3] Let H be a d_0-choosable graph such that $F := K_1 \vee H$ is not d_1-choosable and let L be a bad d_1-assignment on F minimizing $|Pot(L)|$. If some nonadjacent pair in H has intersecting lists, then $|Pot(L)| \leq |H| - 1$.*

Lemma 3.9. *Each $v \in C_i$ of G has at most one neighbor outside of C_i with more than 4 neighbors in C_i, and no such neighbor if v has degree $\Delta - 1$.*

Proof. Suppose there exists $v \in C_i$ with two neighbors $w_1, w_2 \in V(G) - C_i$, each with 5 or more neighbors in C_i. Put $Q := G[\{w_1, w_2\} \cup C_i - v]$, so that v is joined to Q and hence $K_1 \vee Q$ is an induced subgraph of G. We will show that $K_1 \vee Q$ must be d_1-choosable. Note that Q is d_0-choosable since it contains a K_4 without one edge. Let L be a bad d_1-assignment on $K_1 \vee Q$ minimizing $|Pot(L)|$.

First, suppose there are different $z_1, z_2 \in C_i$ such that $\{w_1, z_1\}$ and $\{w_2, z_2\}$ are independent. By the Small Pot Lemma 3.3, $|Pot(L)| \leq |Q|$. Thus $|L(w_1)| + |L(z_1)| \geq 4 + |Q| - 3 > |Pot(L)|$ and therefore w_1 and w_2 have intersecting lists. Applying Lemma 3.8 shows that $|Pot(L)| \leq |Q| - 1$.

Now $|L(w_j)| + |L(z_j)| \geq 4 + |Q| - 3 \geq |Pot(L)| + 2$. Hence $|L(w_j) \cap L(z_j)| \geq 2$. Pick $x \in N(w_1) \cap \{C_i - v - z_2\}$. Then after coloring each pair $\{w_1, z_1\}$ and $\{w_2, z_2\}$ with a different color, we can finish the coloring because we saved a color for x and two colors for v.

By maximality of C_i, neither w_1 nor w_2 can be adjacent to all of C_i hence it must be the case that there is $y \in C_i$ such that w_1 and w_2 are joined to $C_i - y$. If w_1 and w_2 aren't adjacent, then G contains $K_6 \vee E_3$ contradicting Lemma 2.1. Hence C_i intersects the larger clique $\{w_1, w_2\} \cup C_i - \{y\}$, this is impossible by the definition of C_i.

When v is low, an argument similar to the above shows that there can be no z_1 in C_i with $\{w_1, z_1\}$ independent, and hence $C_i \cup \{w_1\}$ is a clique contradicting maximality of C_i. $\square$

4 The Coloring

Now we will show that there exists a $(\Delta - 1)$-list coloring. We will conduct the naive coloring procedure, which is defined below, to a subgraph of G, namely, $G - \bigcup_{C_i \in \mathcal{P}_3} C_i$. Using the Lovász Local Lemma, we will show that with positive probability, the naive coloring procedure will produce a coloring in which none of the bad events happen. The bad events are defined in a way that if none of them happen, then we can finish off the list coloring in a greedy fashion.

Definition 4.1. The *naive coloring procedure* is the following:

(i) For each vertex, choose a color in its list uniformly at random and use it on the vertex;

(ii) Uncolor any vertex that receives the same color as one of its neighbors.

For $C_i \in \mathcal{P}_3$, let w_i be a vertex with the maximum number of neighbors in C_i and let K_i' be the neighbors of w_i that are in C_i.

Definition 4.2. The *bad events* are the following events:

(i) For $C_i \in \mathcal{P}_1$, let $\mathcal{E}_{1,i}$ be the event that K_i does not contain two uncolored safe vertices;

(ii) For $C_i \in \mathcal{P}_2$, let $\mathcal{E}_{2,i}$ be the event that K_i does not contain two uncolored safe vertices;

(iii) For $C_i \in \mathcal{P}_3$, let $\mathcal{E}_{3,i}$ be the event that K_i' does not contain two uncolored safe vertices;

(iv) For sparse vertex v, let $\mathcal{S}_v$ be the event that v is not safe.

We will apply the local lemma to prove that with positive probability, none of the bad events above happen. To do so, we need to calculate the probability of each (bad) event and the dependencies. Note that each event is mutually independent to all but at most Δ^5 events. The task of proving that the probability of each (bad) event is at most Δ^{-6} will be done in the following subsections.

We will apply the naive coloring procedure to $G - \bigcup_{C_i \in \mathcal{P}_3} C_i$. Assuming none of the bad events happen, we will obtain a proper list coloring of G in the following way: First color all the vertices in the dense sets that are not the two uncolored safe vertices. This is possible since each vertex we are coloring in this phase is adjacent to the two uncolored (safe) vertices. Then, for each dense set, color the remaining uncolored vertices, which must be safe by the first phase. Now, we can finish the proper list coloring of G since the only vertices that are possibly uncolored are the sparse vertices, which are all safe.

4.1 $Pr(\mathcal{E}_{1,i}) \leq \Delta^{-6}$

Let C_i' be a subset of C_i with one less vertex where every two vertices in C_i' have at least $|C_i| - 3$ colors in common in their lists; such a C_i' exists by Lemma 2.3. Let $\mathcal{T}_i$ be vertices in a set of maximum P_3 where the center vertex is in C_i' and the other two vertices are not.

Claim 4.3. *There are at least $\frac{3}{28}\Delta$ such P_3.*

Proof. Consider a maximal set of P_3. Let A be the central vertices and let B be the endpoints of these P_3. Then, each $v \in B$ has at most 3 neighbors in $C - A$ and by the previous lemma and maximality, each $v \in C - A$ has at most 2 neighbors $G - C - B$. Thus, $6|A| = 3|B| \geq ||C - A, B|| \geq |C| - |A|$. Hence, $|A| \geq \frac{3}{28}\Delta$. $\square$

Consider a set T_i of $\frac{3}{28}\Delta$ such P_3. For some fixed P_3, we want to bound the probability that the center vertex c is uncolored and safe, and the colors used on the two end vertices, a and b, are used on none of the rest of T_i. To do so, we distinguish three cases.

Case 1. When $|L(a) \cap L(c)| < \frac{2}{3}\Delta$ and $|L(b) \cap L(c)| < \frac{2}{3}\Delta$.

For $\alpha \in L(a) - L(c)$, $\beta \in L(b) - L(c)$, $z \in C_i' - T_i$, and $\gamma \in L(c) \cap L(z)$, where α, β, γ are all different, let $A_{\alpha,\beta,\gamma,z}$ be the event that all of the following hold:

(i) vertex a gets α and none of the rest of $N(a) \cup T_i$;

(ii) vertex b gets β and none of the rest of $N(b) \cup T_i$;

(iii) vertices c and z get γ and none of the rest of T_i.

Then, $Pr(A_{\alpha,\beta,\gamma,z}) \geq (\Delta - 1)^{-4}\left(1 - \frac{1}{\Delta-1}\right)^{|T_i \cup N(a)|}\left(1 - \frac{1}{\Delta-1}\right)^{|T_i \cup N(b)|}\left(1 - \frac{1}{\Delta-1}\right)^{|T_i|} \geq (\Delta - 1)^{-4}3^{-(2+3\frac{999}{1000}\Delta^{1-c})} \geq \Delta^{-4}3^{-2.1}$ for sufficiently large c and Δ. See calculation 5.2.

The $A_{\alpha,\beta,\gamma,z}$ are disjoint for different sets of indices. Since $|L(a) - L(c)| \geq \frac{\Delta}{3}$, we have $\frac{\Delta}{3}$ choices for α. Similarly, we have $\frac{\Delta}{3} - 1$ choices for β. There are $|C_i'| - |T_i| \geq \Delta - \frac{3}{28}\Delta - o(\Delta)$, which is about $\frac{25}{28}\Delta$ choices for z. Since $|L(z) \cap L(c)| \geq \frac{2}{3}\Delta$, there are $|C_i'| - 4$, which is about Δ, choices for γ.

Thus, there are $\frac{25}{28}3^{-2}\Delta^4$ choices for indices, and thus the probability that $A_{\alpha,\beta,\gamma,z}$ holds for some choice of indices is at least $\frac{25}{28}3^{-4.1}$, which is about 0.00987.

Case 2. When $|L(a) \cap L(c)| < \frac{2}{3}\Delta$ and $|L(b) \cap L(c)| \geq \frac{2}{3}\Delta$.

For $\alpha \in L(a) - L(c)$, $y \in C_i' - T_i - N(b)$, $\beta \in L(b) \cap L(y)$, $z \in C_i' - T_i$, and $\gamma \in L(c) \cap L(z)$, where α, β, γ are all different, let $A_{\alpha,\beta,\gamma,y,z}$ be the event that all of the following hold:

(i) vertex a gets α and none of the rest of $N(a) \cup T_i$;

(ii) vertices b and y get β and none of the rest of $N(b) \cup N(y) \cup T_i$;

(iii) vertices c and z get γ and none of the rest of T_i.

Then, $Pr(A_{\alpha,\beta,\gamma,y,z}) \geq (\Delta-1)^{-1}\left(1-\frac{1}{\Delta-1}\right)^{|T_i \cup N(a)|}(\Delta-1)^{-2}\left(1-\frac{1}{\Delta-1}\right)^{|T_i\cup N(b)\cup N(y)|}(\Delta-1)^{-2}\left(1-\frac{1}{\Delta-1}\right)^{|T_i|} \geq (\Delta-1)^{-5}3^{-(3+3\frac{999}{1000}\Delta^{1-c})} \geq \Delta^{-5}3^{-3.1}$ for sufficiently large c and Δ. See calculation 5.2.

The $A_{\alpha,\beta,\gamma,y,z}$ are disjoint for different sets of indices. Since $|L(a)-L(c)| \geq \frac{\Delta}{3}$, we have $\frac{\Delta}{3}$ choices for α. For y, we have at least $|C_i'|-|T_i\cap C_i'|-|N(b)\cap C_i'| \geq \Delta - \frac{3}{28}\Delta - o(\Delta)$, which is about $\frac{25}{28}\Delta$ choices. For each y, we have about $\frac{2}{3}\Delta$ choices since $|L(y)\cap L(b)| \geq |C_i|-4$. There are $|C_i'|-|T_i| \geq \Delta - \frac{3}{28}\Delta - o(\Delta)$, which is about $\frac{25}{28}\Delta$ choices for z. Since $|L(z)\cap L(c)| \geq \frac{2}{3}\Delta$, there are $|C_i'|-4$, which is about Δ, choices for γ.

Thus, there are $\frac{25^2}{28^2}\Delta^5 3^{-2}2$ choices for indices, and thus the probability that $A_{\alpha,\beta,\gamma,y,z}$ holds for some choice of indices is at least $\frac{25^2}{28^2}3^{-5.1}2$, which is about 0.00587.

Case 3. When $|L(a)\cap L(c)| \geq \frac{2}{3}\Delta$ and $|L(b)\cap L(c)| \geq \frac{2}{3}\Delta$.

For $x \in C_i' - T_i - N(b)$, $\alpha \in L(a)\cap L(c)$, $y \in C_i' - T_i - N(b)$, $\beta \in L(b)\cap L(y)$, $z \in C_i' - T_i$, and $\gamma \in L(c)\cap L(z)$, where α, β, γ are all different, let $A_{\alpha,\beta,\gamma,x,y,z}$ be the event that all of the following hold:

- (i) vertices a and x get α and none of the rest of $N(a)\cup N(x)\cup T_i$;
- (ii) vertices b and y get β and none of the rest of $N(b)\cup N(y)\cup T_i$;
- (iii) vertices c and z get γ and none of the rest of T_i.

Then, $Pr(A_{\alpha,\beta,\gamma,x,y,z}) \geq (\Delta-1)^{-6}\left(1-\frac{1}{\Delta-1}\right)^{|T_i\cup N(a)\cup N(x)|}\left(1-\frac{1}{\Delta-1}\right)^{|T_i\cup N(b)\cup N(y)|}\left(1-\frac{1}{\Delta-1}\right)^{|T_i|}$ $\geq (\Delta-1)^{-6}3^{-(4+3\frac{999}{1000}\Delta^{1-c})} \geq \Delta^{-6}3^{-4.1}$ for sufficiently large c and Δ. See calculation 5.2.

The $A_{\alpha,\beta,\gamma,x,y,z}$ are disjoint for different sets of indices. For y, we have at least $|C_i'|-|T_i\cap C_i'|-|N(b)\cap C_i'| \geq \Delta - \frac{3}{28}\Delta - o(\Delta)$, which is about $\frac{25}{28}\Delta$ choices. For each y, we have about $\frac{2}{3}\Delta$ choices since $|L(y)\cap L(b)| \geq |C_i|-4$. Similarly, for x we have about $\frac{25}{28}\Delta$ choices and for each x, there are about $\frac{2}{3}\Delta$ choices. There are $|C_i'|-|T_i| \geq \Delta - \frac{3}{28}\Delta - o(\Delta)$, which is about Δ choices for z. Since $|L(z)\cap L(c)| \geq \frac{2}{3}\Delta$, there are $|C_i'|-4$, which is about $\frac{25}{28}\Delta$, choices for γ.

Thus, there are $\frac{25^3}{28^3}\Delta^6 3^{-2}2^2$ choices for indices, and thus the probability that $A_{\alpha,\beta,\gamma,x,y,z}$ holds for some choice of indices is at least $\frac{25^3}{28^3}3^{-6.1}2^2$, which is about 0.00349.

Since we have $\frac{3}{28}\Delta$ triples, the expected number of uncolored safe vertices X_i is at least $3.4\cdot 10^{-3}\cdot\frac{3}{28}\Delta$.

Now we use Azuma's Inequality to show that the probability that X_i deviates from the expected value is at most Δ^{-6}. Let the conditional expected value of X_i change by at most c_v when changing the color of v.

If $v \in T_i \cup C_i'$, then $c_v \leq 2$ since changing the color on v affects X_i by at most 2 for any given assignment of colors to the remaining vertices. Thus, the sum of the c_v^2 is at most $4|T_i\cup C_i'| \leq 4(\Delta - o(\Delta) + \frac{3}{28}\Delta)$.

If $v \in V(G)-T_i-C_i'$, then changing the color of v from α to β will only affect X_i if some neighbor of v that is in $T_i\cup C_i'$ receives either α or β. This occurs with probability at most $\frac{2d_v}{\Delta-1}$, where d_v is the number of neighbors of v that are in $T_i\cup C_i'$. Therefore, by changing the color of v, the conditional expectation of X_i changes by at most $c_v = \frac{4d_v}{\Delta-1}$. Since the d_v sum is at most $\left(\Delta - o(\Delta) + \frac{6}{28}\Delta\right)^2 \leq 2\Delta^2$, for sufficiently large Δ, the sum of these c_v is at most $\frac{4}{\Delta}2\Delta^2 = 8\Delta$. As each c_v is at most 4, we see that the sum of c_v^2 is at most 32Δ.

Hence, the sum of all the c_v is at most 40Δ for sufficiently large Δ. Applying Azuma's Inequality yields $Pr(\mathcal{E}_{1,i}) \leq \Delta^{-6}$ for sufficiently large Δ. See Calculation 5.4

4.2 $Pr(\mathcal{E}_{2,i}) \leq \Delta^{-6}$

This subsection is similar to the previous subsection, except a linear (in terms of Δ) number of P_3 is not guaranteed. Let C_i' be a subset of C_i with one less vertex where every two vertices in C_i' have at least $|C_i| - 3$ colors in common in their lists; such a C_i' exists by Lemma 2.3. Let $\mathcal{T}_i$ be vertices in a set of maximum P_3 where the center vertex is in C_i' and the other two vertices are not in C_i'. Since at most one of the two endpoints can have more than 4 neighbors in C_i by Lemma 3.9, it follows that $|\mathcal{T}_i| \geq \frac{\Delta-1}{\Delta^c+4}$. By Calculation 5.1, the number of P_3 is at least $\frac{999}{1000}\Delta^{1-c}$. Consider a set T_i of $\frac{999}{1000}\Delta^{1-c}$ P_3 that are in $\mathcal{T}_i$.

For some such fixed path, we want to bound the probability that the center vertex c is uncolored and safe, and the colors used on the two end vertices, a and b, are used on none of the rest of T_i. To do so, we distinguish three cases.

Case 1. When $|L(a) \cap L(c)| < \frac{2}{3}\Delta$ and $|L(b) \cap L(c)| < \frac{2}{3}\Delta$.

For $\alpha \in L(a) - L(c)$, $\beta \in L(b) - L(c)$, $z \in C_i' - T_i$, and $\gamma \in L(c) \cap L(z)$, where α, β, γ are all different, let $A_{\alpha,\beta,\gamma,z}$ be the event that all of the following hold:

- (i) vertex a gets α and none of the rest of $N(a) \cup T_i$;
- (ii) vertex b gets β and none of the rest of $N(b) \cup T_i$;
- (iii) vertices c and z get γ and none of the rest of T_i.

Then, $Pr(A_{\alpha,\beta,\gamma,z}) \geq (\Delta-1)^{-4}\left(1 - \frac{1}{\Delta-1}\right)^{|T_i \cup N(a)|}\left(1 - \frac{1}{\Delta-1}\right)^{|T_i \cup N(b)|}\left(1 - \frac{1}{\Delta-1}\right)^{|T_i|} \geq (\Delta - 1)^{-4}3^{-(2+3\frac{999}{1000}\Delta^{1-c})} \geq \Delta^{-4}3^{-2.1}$ for sufficiently large c and Δ. See calculation 5.2.

The $A_{\alpha,\beta,\gamma,z}$ are disjoint for different sets of indices. Since $|L(a) - L(c)| \geq \frac{\Delta}{3}$, we have $\frac{\Delta}{3}$ choices for α. Similarly, we have $\frac{\Delta}{3} - 1$ choices for β. There are $|C_i'| - |T_i| \geq \Delta - 2 - \frac{999}{1000}\Delta^{1-c}$, which is about Δ choices for z. Since $|L(z) \cap L(c)| \geq \frac{2}{3}\Delta$, there are $|C_i'| - 4$, which is about Δ, choices for γ.

Thus, there are $\Delta^4 3^{-2}$ choices for indices, and thus the probability that $A_{\alpha,\beta,\gamma,z}$ holds for some choice of indices is at least $3^{-4.1}$, which is about 0.01106.

Case 2. When $|L(a) \cap L(c)| < \frac{2}{3}\Delta$ and $|L(b) \cap L(c)| \geq \frac{2}{3}\Delta$.

For $\alpha \in L(a) - L(c)$, $y \in C_i' - T_i - N(b)$, $\beta \in L(b) \cap L(y)$, $z \in C_i' - T_i$, and $\gamma \in L(c) \cap L(z)$, where α, β, γ are all different, let $A_{\alpha,\beta,\gamma,y,z}$ be the event that all of the following hold:

- (i) vertex a gets α and none of the rest of $N(a) \cup T_i$;
- (ii) vertices b and y get β and none of the rest of $N(b) \cup N(y) \cup T_i$;

(iii) vertices c and z get γ and none of the rest of T_i.

Then, $Pr(A_{\alpha,\beta,\gamma,y,z}) \geq (\Delta-1)^{-1}\left(1-\frac{1}{\Delta-1}\right)^{|T_i\cup N(a)|}(\Delta-1)^{-2}\left(1-\frac{1}{\Delta-1}\right)^{|T_i\cup N(b)\cup N(y)|}(\Delta-1)^{-2}\left(1-\frac{1}{\Delta-1}\right)^{|T_i|} \geq (\Delta-1)^{-5}3^{-(3+3\frac{999}{1000}\Delta^{1-c})} \geq \Delta^{-5}3^{-3.1}$ for sufficiently large c and Δ. See calculation 5.2.

The $A_{\alpha,\beta,\gamma,y,z}$ are disjoint for different sets of indices. Since $|L(a)-L(c)| \geq \frac{\Delta}{3}$, we have $\frac{\Delta}{3}$ choices for α. For y, we have at least $|C_i'|-|T_i\cap C_i'|-|N(b)\cap C_i'| \geq \Delta-2-\frac{999}{1000}\Delta^{1-c}-4$, which is about Δ choices. For each y, we have about $\frac{2}{3}\Delta$ choices since $|L(y)\cap L(b)| \geq |C_i|-4$. There are $|C_i'|-|T_i| \geq \Delta-2-\frac{999}{1000}\Delta^{1-c}$, which is about Δ choices for z. Since $|L(z)\cap L(c)| \geq \frac{2}{3}\Delta$, there are $|C_i'|-4$, which is about Δ, choices for γ.

Thus, there are $\Delta^5 3^{-2} 2$ choices for indices, and thus the probability that $A_{\alpha,\beta,\gamma,y,z}$ holds for some choice of indices is at least $3^{-5.1}2$, which is about 0.00737.

Case 3. When $|L(a)\cap L(c)| \geq \frac{2}{3}\Delta$ and $|L(b)\cap L(c)| \geq \frac{2}{3}\Delta$.

For $x \in C_i'-T_i-N(b)$, $\alpha \in L(a)\cap L(c)$, $y \in C_i'-T_i-N(b)$, $\beta \in L(b)\cap L(y)$, $z \in C_i'-T_i$, and $\gamma \in L(c)\cap L(z)$, where α,β,γ are all different, let $A_{\alpha,\beta,\gamma,x,y,z}$ be the event that all of the following hold:

(i) vertices a and x get α and none of the rest of $N(a)\cup N(x)\cup T_i$;

(ii) vertices b and y get β and none of the rest of $N(b)\cup N(y)\cup T_i$;

(iii) vertices c and z get γ and none of the rest of T_i.

Then, $Pr(A_{\alpha,\beta,\gamma,x,y,z}) \geq (\Delta-1)^{-6}\left(1-\frac{1}{\Delta-1}\right)^{|T_i\cup N(a)\cup N(x)|}\left(1-\frac{1}{\Delta-1}\right)^{|T_i\cup N(b)\cup N(y)|}\left(1-\frac{1}{\Delta-1}\right)^{|T_i|} \geq (\Delta-1)^{-6}3^{-(4+3\frac{999}{1000}\Delta^{1-c})} \geq \Delta^{-6}3^{-4.1}$ for sufficiently large c and Δ. See calculation 5.2.

The $A_{\alpha,\beta,\gamma,x,y,z}$ are disjoint for different sets of indices. For y, we have at least $|C_i'|-|T_i\cap C_i'|-|N(b)\cap C_i'| \geq \Delta-2-\frac{999}{1000}\Delta^{1-c}-4$, which is about Δ choices. For each y, we have about $\frac{2}{3}\Delta$ choices since $|L(y)\cap L(b)| \geq |C_i|-4$. Similarly, for x we have about Δ choices and for each x, there are about $\frac{2}{3}\Delta$ choices. There are $|C_i'|-|T_i| \geq \Delta-2-\frac{999}{1000}\Delta^{1-c}$, which is about Δ choices for z. Since $|L(z)\cap L(c)| \geq \frac{2}{3}\Delta$, there are $|C_i'|-4$, which is about Δ, choices for γ.

Thus, there are $\Delta^6 3^{-2} 2^2$ choices for indices, and thus the probability that $A_{\alpha,\beta,\gamma,x,y,z}$ holds for some choice of indices is at least $3^{-6.1}2^2$, which is about 0.00491.

Since we have $\frac{999}{1000}\Delta^{1-c}$ triples, the expected number of uncolored safe vertices X_i is at least $4.9\cdot 10^{-3}\cdot\frac{999}{1000}\Delta^{1-c}$.

Now we use Azuma's Inequality to show that the probability that X_i deviates from the expected value is at most Δ^{-6}. Let the conditional expected value of X_i change by at most c_v when changing the color of v.

If $v \in T_i\cup C_i'$, then $c_v \leq 2$ since changing the color on v affects X_i by at most 2 for any given assignment of colors to the remaining vertices. Thus, the sum of the c_v^2 is at most $4|T_i\cup C_i'| \leq 4(\Delta-1+\frac{1998}{1000}\Delta^{1-c})$.

If $v \in V(G) - T_i - C_i'$, then changing the color of v from α to β will only affect X_i if some neighbor of v that is in $T_i \cup C_i'$ receives either α or β. This occurs with probability at most $\frac{2d_v}{\Delta-1}$, where d_v is the number of neighbors of v that are in $T_i \cup C_i'$. Therefore, by changing the color of v, the conditional expectation of X_i changes by at most $c_v = \frac{4d_v}{\Delta-1}$. Since the d_v sum is at most $2(\Delta - 1 - \frac{1998}{1000}\Delta^{1-c}) + (\Delta-1)\frac{999}{1000}\Delta^{1-c} \leq \frac{\Delta^{2-c}}{5000}$, for c at most $\frac{2}{3}$ and sufficiently large Δ (see calculation 5.3), the sum of these c_v is at most $\frac{4}{\Delta}\frac{\Delta^{2-c}}{5000}$. As each c_v is at most 4, we see that the sum of c_v^2 is at most $\frac{16\Delta^{1-c}}{5000}$.

Hence, the sum of all the c_v is at most $4\Delta - 4 + 8\Delta^{1-c} \leq 4.1\Delta$ for sufficiently large Δ and c. Applying Azuma's Inequality yields $Pr(\mathcal{E}_{2,i}) \leq \Delta^{-6}$ for sufficiently large Δ. See Calculation 5.5

4.3 $Pr(\mathcal{E}_{3,i}) \leq \Delta^{-6}$

Recall that a C_i that corresponds to this case has a vertex w_i outside of C_i that has at least Δ^d neighbors inside C_i and $K_i' = N(w_i) \cap C_i$.

Now uncolor each w_i that is colored. If there exist two vertices in $x, y \in K_i'$ with different special colors such that the external neighbors of x, y are both in $\bigcup_{C_i \in \mathcal{P}_3}(C_i - K_i')$, then color x, y with their special colors; this is possible since none of the neighbors of x, y are colored yet. Note that such K_i' contain at least two safe uncolored vertices, namely, the vertices that do not contain the special colors of x, y in their lists.

Let $\mathcal{K}$ be the set of remaining K_i'. Let T_i be a maximum set of vertices in $K_i' \in \mathcal{K}$ such that every vertex in T_i has a different special color, and each vertex in T_i has its external neighbor in $G - \left(\bigcup_{C_i \in \mathcal{P}_3} C_i\right)$. Note that the external neighbors of T_i must all be distinct. Partition $\mathcal{K}$ into two sets $\mathcal{K}_1$ and $\mathcal{K}_2$ so that for $K_i' \in \mathcal{K}$, the set K_i' is in $\mathcal{K}_1$ if and only if $|T_i| \geq \frac{\Delta^d}{5} - 40$.

Claim 4.4. *There exists $Z \subset V(G)$ where $G[Z]$ is a 1-factor and each edge of $G[Z]$ is within the same K_i' in $\mathcal{K}_2$. (There exists a set $Z \subset \bigcup_{K_i' \in \mathcal{K}_2} K_i'$ where exactly two vertices are chosen from each $K_i' \in \mathcal{K}_2$ and no edge from different K_i' has both endpoints chosen.) (There exists a set $Z \subset \bigcup_{K_i' \in \mathcal{K}_2} K_i'$ where $G[Z]$ is a 1-factor and each edge of $G[Z]$ is within the same K_i'.)*

Proof. For each $K_i' \in \mathcal{K}_2$, vertices of at most one special color have their external neighbors in $\bigcup_{C_i \in \mathcal{P}_3}(C_i - K_i')$. Since there are at least $\frac{\Delta^d}{5}$ special colors, there exists at least 40 vertices with different special colors for each $K_i' \in \mathcal{K}_2$ that have their external neighbors in $\bigcup_{K_i' \in \mathcal{K}_2} K_i'$; let R_i be 40 of these vertices for each $K_i' \in \mathcal{K}_2$. Choose two vertices from each R_i uniformly at random. We will apply the local lemma. Let E_e be the (bad) event that both endpoints of an edge e with endpoints in different K_i' is chosen. Thus, $Pr(E_e) \leq \left(\frac{1}{20}\right)^2$. E_{e_1} is mutually independent from E_{e_2} unless e_1 and e_2 have at least one endpoint in the same K_i'. Thus, E_e is mutually independent to all but at most 80 other events. Since $e\left(\frac{1}{20}\right)^2 80 < 1$, we are done. $\square$

For each vertex in Z, color it with its special color; this is possible since no vertex in Z has a colored neighbor. Color these two vertices with their special colors for each $K_i' \in \mathcal{K}_1$.

Now we will finish this section by finally showing that $Pr(\mathcal{E}_{i,3}) \le \Delta^{-6}$.

Claim 4.5. *With high probability, for each $K'_i \in \mathcal{K}_1$, there are at least two vertices in K'_i where the special color of each vertex is available, and their external neighbors are colored.*

Proof. We will actually show that these two vertices are in T_i. Let $\mathcal{U}_i$ be the external neighbors of $\mathcal{T}_i$, which is a maximum set of vertices in T_i that satisfy the following conditions:

(i) the external neighbor of $x \in \mathcal{T}_i$ retains a color that is not the special color of x;

(ii) every external neighbor of a vertex in $\mathcal{T}_i$ has a distinct color.

We will first show that the expectation of $|\mathcal{T}_i|$ is high, and then we will show that $|\mathcal{T}_i|$ is concentrated around its expectation.

The probability that an external neighbor y of $x \in \mathcal{T}_i$ will not receive the special color of x is at least $\frac{\Delta-2}{\Delta-1}$, and the probability that the at most $\Delta - 1$ neighbors of y do not receive the color y received is $\left(1 - \frac{1}{\Delta-1}\right)^{\Delta-1}$, which is about $\frac{1}{e}$. Since $|T_i| \ge \frac{\Delta^d}{5} - 40$, it follows that $E[|\mathcal{T}_i|] \ge \left(\frac{\Delta^d}{5} - 40\right) \cdot \frac{1}{e}$, which is about $\frac{\Delta^d}{5e}$.

Now we use Azuma's Inequality to show that the probability that $|\mathcal{T}_i|$ deviates from the expected value is at most Δ^{-6}. Let the conditional expected value of $|\mathcal{T}_i|$ change by at most c_v when changing the color of v.

If $v \in \mathcal{U}_i - C_i$, then $c_v \le 2$ since changing the color on v affects $|\mathcal{T}_i|$ by at most 2 for any given assignment of colors to the remaining vertices. Thus, the sum of the c_v^2 is at most $4|\mathcal{T}_i|$, which is about $\frac{4\Delta^d}{5}$.

If $v \in V(G) - \mathcal{U}_i - C_i$, then changing the color of v from α to β will only affect $|\mathcal{T}_i|$ if some neighbor of v that is in $\mathcal{U}_i$ receives either α or β. This occurs with probability at most $\frac{2d_v}{\Delta-1}$, where d_v is the number of neighbors of v that are in $\mathcal{U}_i$. Therefore, by changing the color of v, the conditional expectation of $|\mathcal{T}_i|$ changes by at most $c_v = \frac{4d_v}{\Delta-1}$. Since the d_v sum is at most $\left(\frac{\Delta^d}{5} - 40\right)(\Delta - 1)$, the sum of these c_v is at most $\frac{4\Delta^d}{5}$. As each c_v is at most 4, we see that the sum of c_v^2 is at most $\frac{16\Delta^d}{5}$.

Hence, the sum of all the c_v is at most $4\Delta^d$. Applying Azuma's Inequality yields $Pr(\mathcal{E}_{3,i}) \le \Delta^{-6}$ for sufficiently large Δ. See Calculation 5.6 □

Color these two vertices with their special colors for each $K'_i \in \mathcal{K}_1$.

This guarantees that every K'_i that corresponds to this subsection has at least two vertices that are colored with their special colors. Now, the vertices within each K'_i that do not have the colored vertices in their lists are the safe vertices we are looking for.

4.4 $Pr(\mathcal{S}_v) \le \Delta^{-6}$

Recall that v has at least $\Delta^{1+\alpha}$ nonadjacent pairs of vertices in its neighborhood. Let $A = \{x \in N(v) : |L(x) \cap L(v)| \ge \frac{2}{3}\Delta\}$ and $B = N(v) - A$. Note that for $x, y \in A$, we have $|L(x) \cap L(y)| \ge \frac{\Delta}{3}$ and for $x \in B$ we have $|L(x) - L(v)| \ge \frac{\Delta}{3}$. Let b be the number of nonadjacent pairs in $N(v)$ that intersect B, so that $G[A]$ contains at least $\Delta^{1+\alpha} - b$ nonadjacent pairs and $b \leq |B|\Delta$. Let A_v be the random variable that counts the number of colors that appear at least twice in $N(v)$. Let B_v be the random variable that counts the number of colors that appear in $N(v)$ that are not in the list of $L(v)$. Let $Z_v = A_v + B_v$ so that $E[Z_v] = E[A_v] + E[B_v]$. We will prove that $E[Z_v]$ is high, and then use Azuma's Inequality to prove that with high probability, Z_v is concentrated around its mean.

For A_v, let $x, y \in A$ be nonadjacent. We will actually calculate the number of colors that appear exactly twice in $N(v)$. Since $|L(x) \cap L(y)| \geq \frac{\Delta}{3}$, the probability that x, y get the same color and retain it and this color is not used on the rest of $N(v)$ is at least $\frac{\Delta}{3}(\Delta-1)^{-2}(1-(\Delta-1))^{|N(v)\cup N(x)\cup N(y)|} \geq \Delta^{-1}3^{-4}$. Thus, $E[A_v] \geq (\Delta^{1+\alpha} - b)\Delta^{-1}3^{-4}$.

For B_v, let $x \in B$. Since $|L(x) - L(v)| \geq \frac{\Delta}{3}$, the probability that x gets a color not in $L(v)$ and retains it and is not used on the rest of $N(v)$ is at least $\frac{\Delta}{3}(\Delta-1)^{-1}(1-(\Delta-1)^{-1})^{|N(v)\cup N(x)|} \geq 3^{-3}$. Thus $E[B_v] \geq \frac{|B|}{27} \geq \frac{b}{81\Delta}$. Hence, $E[Z_v] \geq \Delta^{\alpha}3^{-4} \geq \frac{\Delta^{\alpha}}{100}$.

Now we use Azuma's Inequality to show that the probability that Z_v deviates from the expected value is at most Δ^{-6}. Let the conditional expected value of Z_w change by at most c_w when changing the color of w.

Changing the color of w from α to β will only affect Z_v if some neighbor of w that is in $N(v)$ receives either α or β. This occurs with probability at most $\frac{2d_w}{\Delta-1}$, where d_w is the number of neighbors of w that are in $N(v)$. Therefore, by changing the color of w, the conditional expectation of Z_v changes by at most $c_w = \frac{4d_w}{\Delta-1}$. Since the d_w sum is at most Δ^2, the sum of these c_w is at most 5Δ. As each c_w is at most 5, we see that the sum of c_w^2 is at most 25Δ.

Hence, the sum of all the c_w is at most 25Δ. Applying Azuma's Inequality yields $Pr(\mathcal{S}_v) \leq \Delta^{-6}$ for sufficiently large Δ. See Calculation 5.7.

5 Calculations

Typed up the calculations so they do not get lost in my notes.

Calculation 5.1. For number of P_3 for $\mathcal{E}_{2,i}$, when proving

$$\frac{\Delta - 1}{\Delta^c + 4} \geq \frac{999}{1000}\Delta^{1-c}$$

Proof.

$$\Leftrightarrow \Delta - 1 \geq \frac{999}{1000}\Delta + 4\frac{999}{1000}\Delta^{1-c} \Leftrightarrow \frac{\Delta - 1000}{3996} \geq \Delta^{1-c} \Leftrightarrow \frac{\ln\left(\frac{\Delta-1000}{3996}\right)}{\ln \Delta} \geq 1 - c$$

$$\Leftrightarrow c \geq 1 - \frac{\ln\left(\frac{\Delta-1000}{3996}\right)}{\ln \Delta} = 0.180008\cdots \text{for } \Delta = 10^{20}$$

□

Calculation 5.2.

$$3\frac{999}{1000}\Delta^{1-c} \leq 0.1\Delta$$

Proof.

$$\Leftrightarrow 30\frac{999}{1000} \leq \Delta^{c} \Rightarrow c \geq \frac{\ln(30\frac{999}{1000})}{\ln\Delta} = 0.0745\cdots \text{for } \Delta = 10^{20}$$

□

Calculation 5.3.

$$2(\Delta - 1 - \frac{1998}{1000}\Delta^{1-c}) + (\Delta - 1)\frac{999}{1000}\Delta^{1-c} \leq \frac{\Delta^{2-c}}{5000}$$

Proof. This is true for c at most $\frac{2}{3}$. □

Calculation 5.4. For $\mathcal{E}_{1,i}$,

$$2\exp\left(\frac{-\left(3.4\cdot 10^{-3}\cdot\frac{3}{28}\Delta - 2\right)^2}{40\Delta}\right) \leq \Delta^{-6}$$

Proof.

$$2\exp\left(\frac{-\left(3.4\cdot 10^{-3}\cdot\frac{3}{28}\Delta - 2\right)^2}{40\Delta}\right) \leq \Delta^{-6}$$

$$\Leftrightarrow (40\ln 2)\Delta + 6\cdot 40\Delta\ln\Delta \leq \left(3.4\cdot 10^{-3}\cdot\frac{3}{28}\right)^2\Delta^2 + 4 - 4\left(3.4\cdot 10^{-3}\cdot\frac{3}{28}\right)\Delta$$

$$\Leftarrow 250\Delta\ln\Delta \leq \left(3.4\cdot 10^{-3}\cdot\frac{3}{28}\right)^2\Delta^2 \Leftarrow 250\ln\Delta \leq 10^{-6}\Delta \Leftrightarrow 250\cdot 10^{6}\ln\Delta \leq \Delta$$

This is true for $\Delta \geq 10^{10}$. □

Calculation 5.5. For $\mathcal{E}_{2,i}$,

$$2\exp\left(\frac{-\left(4.9\cdot 10^{-3}\cdot\frac{999}{1000}\Delta^{1-c} - 2\right)^2}{8.2\Delta}\right) \leq \Delta^{-6}$$

Proof.

$$2\exp\left(\frac{-\left(4.9\cdot 10^{-3}\cdot\frac{999}{1000}\Delta^{1-c} - 2\right)^2}{8.2\Delta}\right) \leq \Delta^{-6}$$

$$\Leftrightarrow (8.2\ln 2)\Delta + 6\cdot 8.2\Delta\ln\Delta \leq \left(4.9\cdot 10^{-3}\cdot\frac{999}{1000}\right)^2\Delta^{2-2c} + 4 - 4\left(4.9\cdot 10^{-3}\cdot\frac{999}{1000}\right)\Delta^{1-c}$$

$$\Leftarrow 50\Delta\ln\Delta \leq \left(4.9\cdot 10^{-3}\cdot\frac{999}{1000}\right)^2\Delta^{2-2c} \Leftarrow c \leq \frac{\ln\left(\frac{\left(4.9\cdot 10^{-3}\cdot\frac{999}{1000}\right)^2\Delta}{50\ln\Delta}\right)}{2\ln\Delta}$$

For $c \leq$	0.3004	0.2905	0.2795	0.2672	0.2535	0.2380
true for $\Delta \geq$	10^{20}	10^{19}	10^{18}	10^{17}	10^{16}	10^{15}

$\square$

Calculation 5.6. For $\mathcal{E}_{3,i}$, Claim 4.5,

$$2\exp\left(\frac{-\left(\frac{\Delta^d}{5e}-2\right)^2}{8\Delta^d}\right) \leq \Delta^{-6}$$

Proof.

$$2\exp\left(\frac{-\left(\frac{\Delta^d}{5e}-2\right)^2}{8\Delta^d}\right) \leq \Delta^{-6} \Leftrightarrow (8\ln 2)\Delta^d + 6\cdot 8\Delta^d \ln\Delta \leq \left(\frac{\Delta^d}{5e}-2\right)^2 = \frac{\Delta^{2d}}{25e^2} - \frac{4\Delta^d}{5e} + 4$$

$$\Leftarrow (8\ln 2)\Delta^d + 6\cdot 8\Delta^d\ln\Delta + \frac{4\Delta^d}{5e} \leq \frac{\Delta^{2d}}{25e^2} \Leftarrow 49\Delta^d\ln\Delta \leq \frac{\Delta^{2d}}{25e^2} \Leftrightarrow d \geq \frac{\ln(49\cdot 25\cdot e^2\cdot\ln\Delta)}{\ln\Delta}$$

For $d \geq$	0.2810	0.2947	0.3097	0.3264	0.3664	0.4495
true for $\Delta \geq$	10^{20}	10^{19}	10^{18}	10^{17}	10^{15}	10^{12}

$\square$

Calculation 5.7. For $\mathcal{S}_v$,

$$2\exp\left(\frac{-\left(\frac{\Delta^\alpha}{100}-2\right)^2}{50\Delta}\right) \leq \Delta^{-6}$$

Proof.

$$2\exp\left(\frac{-\left(\frac{\Delta^\alpha}{100}-2\right)^2}{50\Delta}\right) \leq \Delta^{-6} \Leftrightarrow (50\ln 2)\Delta + 6\cdot 50\Delta\ln\Delta \leq \left(\frac{\Delta^\alpha}{100}-2\right)^2 = \frac{\Delta^{2\alpha}}{10000} - \frac{4\Delta^\alpha}{100} + 4$$

$$\Leftarrow \frac{\Delta^\alpha}{25} + 120\Delta + 300\Delta\ln\Delta \leq \frac{\Delta^{2\alpha}}{10000}$$

For $\Delta \geq 10^{10}$, $\frac{\Delta^\alpha}{25} \leq \frac{0.1\Delta^{2\alpha}}{10000} \Leftrightarrow \frac{10^5}{25} \leq \Delta^\alpha$ and $120\Delta \leq 10\Delta\ln\Delta \Leftrightarrow e^{12} \leq \Delta$.

$$310\Delta\ln\Delta \leq \frac{9\Delta^{2\alpha}}{100000} \Leftrightarrow \alpha \geq \frac{\frac{\ln\left(310\cdot 10^5 øver 9\ln\Delta\right)}{\ln\Delta}+1}{2} = 0.89496 \Leftarrow \alpha \geq \frac{9}{10}$$

$\square$

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