An improved Ore-type version of Brooks' theorem for list coloring.

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1 Scratch

Definition. The *maximum independent cover number* of a graph G is the maximum $\mathrm{mic}(G)$ of $\|I, V(G) \setminus I\|$ over all independent sets I of G.

Kernel Magic (Kierstead and R. [4])**.** *Every k-list-critical graph G satisfies*

$$2\|G\| \geq (k-2)|G| + \mathrm{mic}(G) + 1.$$

The connected graphs in which each block is a complete graph or an odd cycle are called *Gallai trees*. Gallai [3] proved that in a k-critical graph, the vertices of degree $k-1$ induce a disjoint union of Gallai trees. The same is true for k-list-critical graphs [1, 2]. For a graph T and $k \in \mathbb{N}$, let $\beta_k(T)$ be the independence number of the subgraph of T induced on the vertices of degree $k-1$ in T. When k is defined in the context, put $\beta(T) := \beta_k(T)$. Let $c(T)$ be the number of components of T.

Lemma 1. *If $k \geq 4$ and $T \neq K_k$ is a Gallai tree with maximum degree at most $k-1$, then for any $p(k)$ with $\frac{2}{k-2} \leq p(k) \leq 1$,*

$$2||T|| \leq (k-3+p(k))|T| + (k-1)(1-p(k)) + (2+(k-1)(1-p(k)))\beta(T).$$

Let G be a k-list-critical graph with $\Delta(G) = k$ such that $\mathcal{H}$ is edgeless. Then

$$k|\mathcal{H}| = \|\mathcal{H}, \mathcal{L}\| = (k-1)|\mathcal{L}| - 2\|\mathcal{L}\|,$$

so

$$2\|\mathcal{L}\| = (k-1)|\mathcal{L}| - k|\mathcal{H}|. \tag{1}$$

Combined with Lemma 1, this gives

$$(2-p(k))|\mathcal{L}| \leq k|\mathcal{H}| + (k-1)(1-p(k))c(\mathcal{L}) + (2+(k-1)(1-p(k)))\beta(\mathcal{L}) \tag{2}$$

Also,

$$2|\mathcal{H}| + |\mathcal{L}| = |G| + |\mathcal{H}| > \mathrm{mic}(G) \geq k|\mathcal{H}| + (k-1)\beta(\mathcal{L}), \tag{3}$$

so with (2), this gives

$$(1 - p(k))\,|\mathcal{L}| < 2\,|\mathcal{H}| + (k-1)(1-p(k))c(\mathcal{L}) + (2-(k-1)p(k))\beta(\mathcal{L}), \tag{4}$$

with $p(k) = 1$, this is

$$(k-3)\beta(\mathcal{L}) < 2\,|\mathcal{H}|\,. \tag{5}$$

Using (4) with $p(k) = \frac{2}{k-2}$ gives

$$\frac{k-4}{k-2}\,|\mathcal{L}| < 2\,|\mathcal{H}| + \frac{(k-1)(k-4)}{k-2}c(\mathcal{L}) - \frac{2}{k-2}\beta(\mathcal{L}), \tag{6}$$

so

$$|\mathcal{L}| < \frac{2(k-2)}{k-4}\,|\mathcal{H}| + (k-1)c(\mathcal{L}) - \frac{2}{k-4}\beta(\mathcal{L}), \tag{7}$$

References

[1] O.V. Borodin, *Criterion of chromaticity of a degree prescription*, Abstracts of IV All-Union Conf. on Th. Cybernetics, 1977, pp. 127–128. 1

[2] P. Erdős, A.L. Rubin, and H. Taylor, *Choosability in graphs*, Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium, vol. 26, 1979, pp. 125–157. 1

[3] T. Gallai, *Kritische Graphen I.*, Publ. Math. Inst. Hungar. Acad. Sci **8** (1963), 165–192 (in German). 1

[4] H.A. Kierstead and L. Rabern, *Extracting list colorings from large independent sets*, arXiv:1512.08130 (2015). 1