Lemma 1. *For any connected graph G and any $z \in V(G)$, there exists a total ordering $\leq$ of $V(G)$ with z minimum, such that $G[x \mid x \leq y]$ is connected for each $y \in V(G)$.*

Proof. Let G be a connected graph and $z \in V(G)$. Let H be a maximal induced subgraph of G which has such an ordering with z minimum. If $H \neq G$, then, since G is connected, some $w \in V(G) - V(H)$ has an edge into H. Thus we may add w to H as the last vertex in the ordering contradicting the maximality of H. Hence $H = G$ and we have our ordering. $\square$

Lemma 2. *Let G be an incomplete 2-connected graph with $\delta(G) \geq 3$. Then G contains an induced P_3, say abc, such that $G - a - c$ is connected.*

Proof. Since G is connected and not complete, it contains induced P_3's. If G is 3-connected, any induced P_3 will do. Otherwise, let $\{b, x\} \subseteq V(G)$ be a cutset. Since $G - b$ is not 2-connected, it has at least two endblocks B_1, B_2. But G is 2-connected, so b must be adjacent to noncutvertices $a \in B_1$ and $c \in B_2$. Thus $G - a - c$ is connected since $d(b) \geq 3$. Whence abc is our desired P_3. $\square$

Theorem 3 (Brooks 1941)**.** *Every graph with $\Delta \geq 3$ satisfies $\chi \leq \max\{\omega, \Delta\}$.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Plainly, G must be regular, 2-connected and not complete. Let abc be the induced P_3 guaranteed by Lemma 2. By Lemma 1, we have an ordering $b, x_1, x_2, \ldots, x_k$ of $V(G-a-c)$ such that $G[b, x_1, \ldots, x_i]$ is connected for each $1 \leq i \leq k$. Thus, greedily coloring with $V(G)$ ordered $a, c, x_k, x_{k-1}, \ldots, x_1, b$ uses only $\Delta(G)$ colors. $\square$