Edge Lower Bounds for List Critical Graphs, via Discharging

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February 7, 2016

Abstract

A graph G is k-critical if G is not $(k-1)$-colorable, but every proper subgraph of G is $(k-1)$-colorable. A graph G is k-choosable if G has an L-coloring from every list assignment L with $|L(v)| = k$ for all v, and a graph G is *k-list-critical* if G is not $(k-1)$-choosable, but every proper subgraph of G is $(k-1)$-choosable. The problem of bounding (from below) the number of edges in a k-critical graph has been widely studied, starting with work of Gallai and culminating with the seminal results of Kostochka and Yancey, who essentially solved the problem. In this paper, we improve the best lower bound on the number of edges in a k-list-critical graph. Our proof uses the discharging method, which makes it simpler and more modular than previous work in this area.

1 Introduction

A k-coloring of a graph G assigns to each vertex of G a color from $\{1, \ldots, k\}$ such that adjacent vertices get distinct colors. A graph G is *k-colorable* if it has a k-coloring and its chromatic number, $\chi(G)$, is the least integer t such that G is t-colorable. Further, G is *k-critical* when $\chi(G) = k$ and every proper subgraph H of G has $\chi(H) < k$. For a graph G with $\chi(G) = k$, every minimal subgraph H such that $\chi(H) = k$ must be k-critical. As a result, many questions about the chromatic number of a graph can be reduced to corresponding questions about k-critical graphs. One natural question is how few edges an n-vertex k-critical graph G can have? Since $\delta(G) \geq k-1$, clearly $2||G|| \geq (k-1)|G|$. Brooks' theorem shows that if G is a connected graph, other than K_k, then this bound can be slightly improved. Dirac proved that every k-critical graph G satisfies

$$2||G|| \geq (k-1)|G| + k - 3.$$

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Now let

$$g_k(n,c) = k - 1 + \frac{k-3}{(k-c)(k-1)+k-3}n.$$

For $k \geq 4$ and $|G| \geq k+2$, Gallai improved Dirac's bound to $2||G|| \geq g_k(|G|, 0)$. This result was subsequently strengthened by Krivelevich [11] to $2||G|| \geq g_k(|G|, 2)$ and by Kostochka and Stiebitz [10], for $k \geq 6$, to $2||G|| \geq g_k(|G|, (k-5)\alpha_k)$, where $\alpha_k = \frac{1}{2} - \frac{1}{(k-1)(k-2)}$. In a recent breakthrough, Kostochka and Yancey [8] proved that every k-critical graph G satisfies

$$||G|| \geq \left\lceil \frac{(k+1)(k-2)|G| - k(k-3)}{2(k-1)} \right\rceil.$$

This bound is tight for $k = 4$ and $|G| \geq 6$. Also, for each $k \geq 5$, it is tight for infinitely many values of $|G|$.

This result of Kostochka and Yancey has numerous applications to coloring problems. For example, it gives a short proof of Grötzsch's theorem [7], that every triangle-free planar graph is 3-colorable. It also yields short proofs of a series of results on coloring with respect to Ore degree [5, 12, 9]. Thus, it is natural to consider the same question for more general types of coloring, such as list coloring, online list coloring, and Alon–Tarsi number (all of which are defined below). Gallai's bound [3] also holds for list coloring, as well as online list coloring ([10, 13]). In contrast, Krivelevich's proof [11] does not work for list coloring, since it uses a lemma of Stiebitz [16], which says that in a color-critical graph, the subgraph induced by vertices of degree at least k has no more components than the subgraph induced by vertices of degree $k-1$ (but no analogous lemma is known for list coloring). For list coloring Kostochka and Stiebitz [10] gave the first improvement over Gallai's bound. Table 1, at the end of this section, gives the values of these bounds for small k.

Recently, Kierstead and the second author [6] further improved the lower bound and extended it to online list coloring as well as to the Alon-Tarsi number. Their proof combined a global averaging argument from Kostochka and Stiebitz [10] with improved reducibility lemmas. Here we use these same reducibility lemmas, but replace the global averaging argument with a discharging argument. The discharging argument is more intuitive and will be easier to modify in the future for use with new reducibility lemmas. The improvement in our lower bound on the number of edges in a list critical graph comes from an improved upper bound on the average degree of Gallai trees. To state our results we need some definitions.

List coloring was introduced by Vizing [17] and independently by Erdős, Rubin, and Taylor [2]. A *list assignment* L assigns to each vertex v of a graph G a set of allowable colors. An *L-coloring* is a proper coloring φ of G such that $\varphi(v) \in L(v)$ for all v. An *f-assignment* is a list assignment L such that $|L(v)| = f(v)$ for all v; a *k-assignment* is an f-assignment such that $f(v) = k$ for all v; a *d_0-assignment* is an f-assignment such that $f(v) = d(v)$ for all v. A graph G is *k-choosable* (resp. *f-choosable* or *d_0-choosable*) if G is L-colorable whenever L is a k-assignment (resp. f-assignment or d_0-assignment). The *list chromatic number*, $\chi_\ell(G)$, of G is the least integer t such that G is t-choosable.

L-coloring

d_0-choosable

f-choosable

Online list coloring allows for the possibility that the lists are being revealed as the graph is being colored. This notion was introduced independently by Zhu [18] and Schauz [14] (who called it *paintability*). A graph G is *online f-list colorable* if either (i) G is an independent set and $f(v) \geq 1$ for all v or else (ii) for every subset $S \subseteq V(G)$, there exists an independent set $I \subseteq S$ such that $G - I$ is online f'-list colorable, where $f'(v) = f(v) - 1$ for all $v \in S \setminus I$ and $f'(v) = f(v)$ for all $v \in V(G) \setminus S$. The online list chromatic number, $\chi_{\mathrm{OL}}(G)$, of G is the least k such that G is online f-list colorable when $f(v) = k$. If $\chi_{\mathrm{OL}}(G) \le k$, then G is *online k-list colorable*. Note that if G is online k-list colorable, then G is k-choosable. (Given L, we can take S to be, successively, $\{v : i \in L(v)\}$, as i ranges through all elements of $\cup_{v \in V} L(v)$.)

A characterization of connected graphs that are not d_0-choosable was first given by Vizing [17], and later by Erdős, Rubin, and Taylor [2]. Such graphs are called Gallai trees; they are precisely the connected graphs in which each block is a complete graph or an odd cycle. Hladkỳ, Král, and Schauz [4] later characterized the connected graphs that are not online d_0-choosable; again, these are precisely the Gallai trees.

Given a graph G and a list assignment L, a natural way to construct an L-coloring of G is to color G greedily in some order. This approach will always succeed when $|L(v)| \ge d(v) + 1$ for all v. Using digraphs, we can state a weaker sufficient condition. For a vertex ordering σ, form an acyclic digraph D by directing each edge $v_i v_j$ as $v_i \to v_j$ if v_j precedes v_i in σ. Now it suffices to have $|L(v)| \ge d_D^+(v) + 1$. Alon and Tarsi strengthened this result significantly, by allowing certain directed cycles in D. To state their result, we need a few definitions. A digraph D is *eulerian* if $d_D^+(v) = d_D^-(v)$ for all v. A digraph D is *even* if $||D||$ is even, and otherwise D is *odd*. For a digraph D, let $EE(D)$ and $EO(D)$ denote the number of even (resp. odd) spanning eulerian subdigraphs of D. A graph G is *f-Alon–Tarsi* (*f-AT*, for short) if G has an orientation D such that $EE(D) \ne EO(D)$ and $f(v) \ge d_D^+(v) + 1$ for all v. The Alon–Tarsi number, $AT(G)$, of G is the least k such that G is f-AT when $f(v) = k$ for all v. Analogous to the definition for coloring and list-coloring, a graph G is *k-AT-critical* if $\mathrm{AT}(G) = k$ and $\mathrm{AT}(H) < k$ for every proper subgraph H of G. From the definitions, it is easy to check that always $\chi(G) \le \chi_\ell(G) \le \chi_{\mathrm{OL}}(G) \le AT(G) \le \Delta(G) + 1$. Alon and Tarsi [1] proved the following.

f-AT

k-AT-critical

Lemma 1.1. *If a graph G is f-AT for $f\colon V(G) \to \mathbb{N}$, then G is f-choosable.*

Note that always $EE(D) \ge 1$, since the edgeless spanning subgraph is even. If D is acyclic, then $EO(D) = 0$, since every subgraph with edges has a vertex v with $d^+(v) \ge 1 > 0 = d^-(v)$. Thus, Lemma 1.1 generalizes the results we can prove by greedy coloring. Schauz [15] gave a new, constructive proof of this lemma (the original was non-constructive), which allowed him to extend the result to online f-choosability.

Lemma 1.2. *If a graph G is f-AT for $f\colon V(G) \to \mathbb{N}$, then G is online f-choosable.*

In this paper, we prove lower bounds on the number of edges in a k-AT-critical graph. Corollaries 4.2 and 4.4 summarize our main results. Here $d(G)$ denotes the average degree of G.

Corollary 4.2. *If G is a k-AT-critical graph, with $k \ge 7$, and $G \ne K_k$, then*

$$d(G) \ge k - 1 + \frac{(k-3)(2k-5)}{k^3 + k^2 - 15k + 15}.$$

Corollary 4.4. *If G is a k-AT-critical graph, with $k \in \{5, 6\}$, and $G \ne K_k$, then*

$$d(G) \ge k - 1 + \frac{(k-3)(2k-5)}{k^3 + 2k^2 - 18k + 15}.$$

	k-Critical G				k-List Critical G		
k	Gallai [3] $d(G) \geq$	Kriv [11] $d(G) \geq$	KS [10] $d(G) \geq$	KY [8] $d(G) \geq$	KS [10] $d(G) \geq$	KR [6] $d(G) \geq$	Here $d(G) \geq$
4	3.0769	3.1429	—	3.3333	—	—	—
5	4.0909	4.1429	—	4.5000	—	4.0984	**4.1000**
6	5.0909	5.1304	5.0976	5.6000	—	5.1053	**5.1076**
7	6.0870	6.1176	6.0990	6.6667	—	6.1149	**6.1192**
8	7.0820	7.1064	7.0980	7.7143	—	7.1128	**7.1167**
9	8.0769	8.0968	8.0959	8.7500	8.0838	8.1094	**8.1130**
10	9.0722	9.0886	9.0932	9.7778	9.0793	9.1055	**9.1088**
15	14.0541	14.0618	14.0785	14.8571	14.0610	14.0864	**14.0884**
20	19.0428	19.0474	19.0666	19.8947	19.0490	19.0719	**19.0733**

Table 1: History of lower bounds on the average degree $d(G)$ of k-critical and k-list-critical graphs G. (Reproduced from [6] and updated.)

2 Gallai's bound via discharging

One of the earliest results bounding the number of edges in a critical graph is the following theorem, due to Gallai. The key lemma he proved, Lemma 2.2, gives an upper bound on the number of edges in a Gallai tree. The rest of his proof is an easy counting argument. As a warmup, and to illustrate the approach that we take in Section 4, we rephrase this counting in terms of discharging. A *Gallai tree* is a connected graph in which each block is a clique or an odd cycle (these are precisely the d_0-choosable graphs, mentioned in the introduction). Let $\mathcal{T}_k$ denote the set of all Gallai trees of degree at most $k-1$, excluding K_k. A *k-vertex* (resp. k^+-vertex or k^--vertex) is a vertex of degree k (resp. at least k or at most k). A *k-neighbor* of a vertex v is an adjacent k-vertex. For convenience, we write $d(G)$ to denote the average degree of G.

k-vertex
k-neighbor
$d(G)$

Theorem 2.1 (Gallai). *For $k \geq 4$ and G a k-AT-critical graph, with $G \neq K_k$, we have*

$$d(G) > k - 1 + \frac{k-3}{k^2-3}.$$

Proof. We use the discharging method. Each vertex v has initial charge $d_G(v)$. First, each k^+-vertex gives charge $\frac{k-1}{k^2-3}$ to each of its $(k-1)$-neighbors. Now the vertices in each component of the subgraph induced by $(k-1)$-vertices share their total charge equally. Let $\mathrm{ch}^*(v)$ denote the resulting charge on v. We finish the proof by showing that $\mathrm{ch}^*(v) \geq k - 1 + \frac{k-3}{k^2-3}$ for all $v \in V(G)$.

If v is a k^+-vertex, then $ch^*(v) \geq d_G(v) - \frac{k-1}{k^2-3} d_G(v) = \left(1 - \frac{k-1}{k^2-3}\right) d_G(v) \geq \left(1 - \frac{k-1}{k^2-3}\right) k = k - 1 + \frac{k-3}{k^2-3}$ as desired.

Instead, let T be a component of the subgraph induced by $(k-1)$-vertices. Now the vertices in T receive total charge

$$\frac{k-1}{k^2-3} \sum_{v \in V(T)} (k - 1 - d_T(v)) = \frac{k-1}{k^2-3} \left((k-1)|T| - 2\,\|T\|\right).$$

So, after distributing this charge equally, each vertex in T receives charge

$$\frac{1}{|T|}\frac{k-1}{k^2-3}((k-1)|T|-2\,\|T\|)=\frac{k-1}{k^2-3}\left((k-1)-d(T)\right).$$

By Lemma 2.2, which we prove next, this is greater than

$$\frac{k-1}{k^2-3}\left((k-1)-\left(k-2+\frac{2}{k-1}\right)\right)=\frac{k-1}{k^2-3}\left(\frac{k-3}{k-1}\right)=\frac{k-3}{k^2-3}.$$

Hence, each $(k-1)$-vertex ends with charge greater than $k-1+\frac{k-3}{k^2-3}$, as desired. $\square$

Lemma 2.2 (Gallai)**.** *For $k \geq 4$ and $T \in \mathcal{T}_k$, we have $d(T) < k-2+\frac{2}{k-1}$.*

Proof. Suppose the lemma is false and choose a counterexample T minimizing $|T|$. Now T has at least two blocks. Let B be an endblock of T. If B is K_t for some $t \in \{2,\ldots,k-2\}$, then remove the non-cut vertices of B from T to get T'. By the minimality of $|T|$, we have

$$2\,\|T\|-t(t-1)=2\,\|T'\|<\left(k-2+\frac{2}{k-1}\right)|T'|=\left(k-2+\frac{2}{k-1}\right)\left(|T|-(t-1)\right).$$

Hence, we have the contradiction

$$2\,\|T\|<\left(k-2+\frac{2}{k-1}\right)|T|+(t+2-k-\frac{2}{k-1})(t-1)\leq\left(k-2+\frac{2}{k-1}\right)|T|.$$

The case when B is an odd cycle is similar to that above, when $t=3$; a longer cycle just makes the inequality stronger. Finally, if $B=K_{k-1}$, remove all vertices of B from T to get T'. By the minimality of $|T|$, we have

$$\begin{aligned}2\,\|T\|-(k-1)(k-2)-2&=2\,\|T'\|\\&<\left(k-2+\frac{2}{k-1}\right)|T'|\\&=\left(k-2+\frac{2}{k-1}\right)|T|-\left(k-2+\frac{2}{k-1}\right)(k-1).\end{aligned}$$

Hence, $2\,\|T\|<\left(k-2+\frac{2}{k-1}\right)|T|$, a contradiction. $\square$

3 A refined bound on $||T||$

Lemma 2.2 is essentially best possible, as shown by a path of copies of K_{k-1}, with each successive pair of copies linked by a copy of K_2. When the path T has m copies of K_{k-1}, we get $2||T||=m(k-1)(k-2)+2(m-1)=(k-2+\frac{2}{k-2})|T|-2$. And a small modification to the proof above yields $2||T||\leq(k-2+\frac{2}{k-2})|T|-2$. Fortunately, this is not the end of the story.

We see two potential places that we could improve the bound in Theorem 2.1. For each graph G, we could show that either (i) the bound in Lemma 2.2 is loose or (ii) many of the k^+-vertices finish with extra charge, because they have incident edges leading to other k^+-vertices (rather than only $(k-1)$-vertices, as allowed in the proof of Theorem 2.1). A good way to quantify this slackness in the proof is with the parameter $q(T)$, which denotes the number of non-cut vertices in T that appear in copies of K_{k-1}. When $q(T)$ is small relative to $|T|$, we can save as in (i) above. And when it is large, we can save as in (ii). In the direction of (i), we now prove a bound on $||T||$ akin to that in Lemma 2.2, but which is stronger when $q(T) \leq |T|\frac{k-3}{k-1}$. In Section 4 we do the discharging; at that point we handle case (2), using a reducibility lemma proved in [6]. $q(T)$

Without more reducible configurations we can't hope to prove $d(T) < k-3$, since each component T could be a copy of K_{k-2}. This is why our next bound on $2||T||$ has the form $(k-3+p(k))|T|$; since we will always have $p(k) > 0$, this is slightly worse than average degree $k-3$. To get the best edge bound we will take $p(k) = \frac{3k-5}{k^2-4k+5}$, but we prefer to prove the more general formulation, which shows that previous work of Gallai [3] and Kostochka and Steibitz [10] fits the same pattern. This general version will also be more convenient for the discharging. It is helpful to handle separately the cases $K_{k-1} \not\subseteq T$ and $K_{k-1} \subseteq T$. The former is simpler, since it implies $q(T) = 0$, so we start there.

Lemma 3.1. *Let $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$. For all $k \geq 5$ and $T \in \mathcal{T}_k$ with $K_{k-1} \not\subseteq T$, we have*

$$2\,\|T\| \leq (k-3+p(k))\,|T| + f(k)$$

whenever p and f satisfy all of the following conditions:

1. *$p(k) \geq \frac{-f(k)}{k-2}$; and*
2. *$p(k) \geq \frac{-f(k)}{5} + 5 - k$; and*
3. *$0 \geq f(k) \geq -k+2$; and*
4. *$p(k) \geq \frac{3}{k-2}$.*

Proof. A general outline for the proof is that it mirrors that of Lemma 2.2, and we add as hypotheses all of the conditions that we need along the way.

Suppose the lemma is false and choose a counterexample T minimizing $|T|$. If T is K_t for some $t \in \{2, k-2\}$, then $t(t-1) > (k-3+p(k))t + f(k)$. After substituting $p(k) \geq \frac{-f(k)}{k-2}$ from (1), this simplifies to $-t(k-2) > f(k)$, which contradicts (3). If T is C_{2r+1} for $r \geq 2$, then $2(2r+1) > (k-3+p(k))(2r+1) + f(k)$ and hence $(5-k-p(k))(2r+1) > f(k)$. Since $f(k) \leq 0$, this contradicts (2). (Note that we only use conditions (1), (2), and (3) when T has a single block; these are the base cases when the proof is phrased using induction.)

Let D be an induced subgraph such that $T \setminus D$ is connected. (We will choose D to be a connected subgraph contained in at most three blocks of T.) Let $T' = T \setminus D$. By the minimality of $|T|$, we have

$$2\,\|T'\| \leq (k-3+p(k))\,|T'| + f(k).$$

Since T is a counterexample, subtracting this inequality from the inequality for $2||T||$ gives

$$2\,\|T\| - 2\,\|T'\| > (k-3+p(k))|D|. \tag{*}$$

Suppose T has an endblock B that is K_t for some $t \in \{3, \ldots, k-3\}$; let x_B be a cut vertex of B and let $D = B - x_B$. Now (*) gives $2\|T\| - 2\|T'\| = |B|(|B|-1) > (k-3+p(k))(|B|-1)$, which is a contradiction, since $|B| \le k-3$ and $p(k) > 0$. Suppose instead that T has an endblock B that is an odd cycle. Again, let $D = B - x_B$. Now we get $2|B| > (k-3+p(k))(|B|-1)$. This simplifies to $|B| < 1 + \frac{2}{k-5+p(k)}$, which is a contradiction, since the denominator is always at least 1 (using (4) when $k = 5$). Finally suppose that T has an endblock B that is K_2. Now (*) gives $2 > k-3+p(k)$, which is again a contradiction, since $k \ge 5$ and $p(k) > 0$.

To handle the case when B is K_{k-2} we need to remove x_B from T as well, so we simply let $D = B$. Since $B = K_{k-2}$, we have either $d_T(x_B) = k-2$ or $d_T(x_B) = k-1$. When $d_T(x_B) = k-2$, we have

$$(k-2)(k-3) + 2 > (k-3+p(k))(k-2),$$

contradicting (4).

The only remaining case is when B is K_{k-2} and $d_T(x_B) = k-1$. Each case above applied when B was any endblock of T, so we may assume that every endblock of T is a copy of K_{k-2} that shares a vertex with an odd cycle. Choose an endblock B that is the end of a longest path in the block-tree of T. Let C be the odd cycle sharing a vertex x_B with B. Consider a neighbor y of x_B on C that either (i) lies only in C or (ii) lies also in an endblock A that is a copy of K_{k-2} (such a neighbor exists because B is at the end of a longest path in the block-tree). In (i), let $D = B \cup \{y\} + yx_B$; in (ii), let $D = B \cup A + yx_B$.

In (i), equation (*) gives

$$(k-2)(k-3) + 2(3) > (k-3+p(k))(k-1).$$

This simplifies to $6 > k-3+(k-1)p(k)$, and eventually, by (4), to $6 > k + \frac{3}{k-2}$, which yields a contradiction.

In (ii), equation (*) gives

$$2(k-2)(k-3) + 2(3) > 2(k-3+p(k))(k-2),$$

which simplifies to

$$3 > (k-2)p(k),$$

again contradicting (4). $\square$

Lemma 3.1 gives the tightest bound on $||T||$ when $p(k) = \frac{3}{k-2}$ and $f(k) = -3$. However, for the discharging in Section 4, it will be convenient to apply Lemma 3.1 with a larger $p(k)$, to match the best value of $p(k)$ that works in the analogous lemma for $K_{k-1} \subseteq T$. We now prove such a lemma. Its statement is similar to the previous one, but with an extra term in the bound, as well as slightly different hypotheses.

Lemma 3.2. *Let $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$, $h\colon \mathbb{N} \to \mathbb{R}$. For all $k \ge 5$ and $T \in \mathcal{T}_k$ with $K_{k-1} \subseteq T$, we have*

$$2\|T\| \le (k-3+p(k))|T| + f(k) + h(k)q(T)$$

whenever p, f, and h satisfy all of the following conditions:

1. $f(k) \geq (k-1)(1-p(k)-h(k))$; *and*

2. $p(k) \geq \frac{3}{k-2}$; *and*

3. $p(k) \geq h(k) + 5 - k$; *and*

4. $p(k) \geq \frac{2+h(k)}{k-2}$; *and*

5. $(k-1)p(k) + (k-3)h(k) \geq k+1$.

Proof. The proof is similar to that of Lemma 3.1. The main difference is that now our only base case is $T = K_{k-1}$. For this reason, we replace hypotheses (1), (2), and (3) of Lemma 3.1, which we used only for the base cases of that proof, with our new hypothesis (1), which we use for the current base case. When some endblock B is an odd cycle or K_t, with $t \in \{3, \ldots, k-3\}$, the induction step is identical to that in Lemma 3.1, since deleting D does not change $q(T)$. It is easy to check that, as needed, $K_{k-1} \subseteq T \setminus D$. Thus, we need only to consider the induction step when T has an endblock B that is K_2, K_{k-2}, or K_{k-1}. As we will see, these three cases require hypotheses (3), (4), and (5), respectively.

Let T be a counterexample minimizing $|T|$. Let D be an induced subgraph such that $T \setminus D$ is connected, and let $T' = T \setminus D$. The same argument as in Lemma 3.1 now gives

$$2||T|| - 2||T'|| > (k-3+p(k))\,|D| + h(k)\,(q(T) - q(T')) \,. \qquad (^{**})$$

If B is K_2, then $q(T') \leq q(T) + 1$ and (**) gives $2 > k-3+p(k)-h(k)$, contradicting (3). So every endblock of B is K_{k-2} or K_{k-1}. To handle these cases, we will need to remove x_B from T as well. Suppose some endblock B is K_{k-1} and $K_{k-1} \subseteq T \setminus B$. Let $D = B$. Now $q(T') \leq q(T) - (k-2) + 1$. So (**) gives

$$(k-1)(k-2) + 2 > (k-3+p(k))(k-1) + h(k)(k-3).$$

This simplifies to $k+1 > (k-1)p(k) + (k-3)h(k)$, which contradicts (5). Thus, at most one endblock of T is K_{k-1}. Since the cases above apply when B is any endblock, each other endblock must be K_{k-2}. Let B be such an endblock, and x_B its cut vertex. So $d_T(x_B) = k-2$ or $d_T(x_B) = k-1$. In the former case, $q(T') \leq q(T) + 1$, and in the latter, $q(T) = q(T')$. If $d_T(x_B) = k-2$, then (**) gives

$$(k-2)(k-3) + 2 > (k-3+p(k))(k-2) - h(k),$$

which simplifies to $\frac{2+h(k)}{k-2} > p(k)$, and contradicts (4).

Hence, all but at most one endblock of T is a copy of K_{k-2} with a cut vertex that is also in an odd cycle. Let B be such an endblock at the end of a longest path in the block-tree of T, and let C be the odd cycle sharing a vertex x_B with B. Consider a neighbor y of x_B on C that either (i) lies only in block C or (ii) lies also in an endblock A that is a copy of K_{k-2} (such a neighbor exists because B is at the end of a longest path in the block-tree). In (i), let $D = B \cup \{y\} + yx_B$; in (ii), let $D = B \cup A + yx_B$. Let $T' = T \setminus V(D)$. In each case, we have $q(T') = q(T)$, so the analysis is identical to that in the proof of Lemma 3.1. $\square$

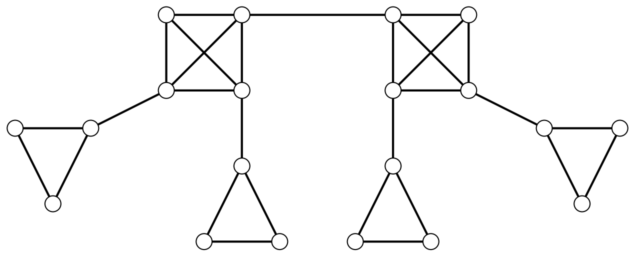

Figure 1: The construction when $k = 5$ and $m = 2$.

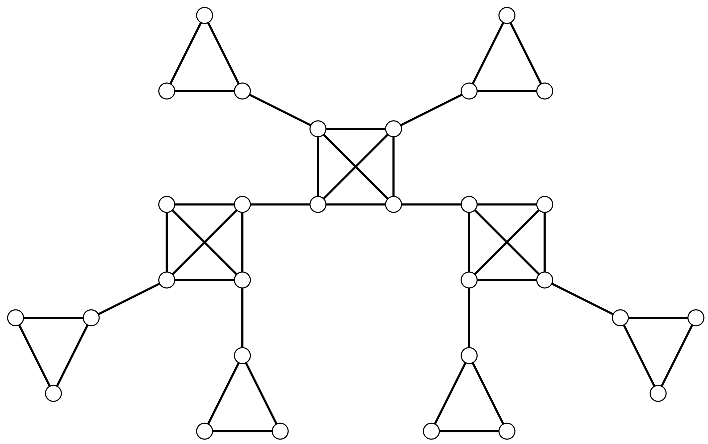

Figure 2: The construction when $k = 5$ and $m = 3$.

Now let's see some examples of using Lemma 3.1 and Lemma 3.2. What happens if we take $h(k) = 0$ in Lemma 3.2? By hypothesis (5), we need $(k-1)p(k) \geq k+1$ and hence $p(k) \geq 1 + \frac{2}{k-1}$. Taking $p(k) = 1 + \frac{2}{k-1}$, (1) requires $f(k) \geq -2$. Using $h(k) = 0$, $p(k) = 1 + \frac{2}{k-1}$, and $f(k) = -2$, all of the conditions are satisfied in both of Lemmas 3.1 and 3.2, so we conclude $2\,\|T\| \leq \left(k - 2 + \frac{2}{k-1}\right)|T| - 2$ for every $T \in \mathcal{T}_k$ when $k \geq 5$. This is the previously mentioned slight refinement of Gallai's Lemma 2.2.

Instead, let's make $p(k)$ as small as Lemma 3.2 allows. By (4), $h(k) \leq (k-2)p(k) - 2$. Plugging this into (5) and solving, we get $p(k) \geq \frac{3k-5}{k^2-4k+5}$. Now $\frac{3k-5}{k^2-4k+5} \geq \frac{3}{k-2}$ for $k \geq 5$, so $p(k) = \frac{3k-5}{k^2-4k+5}$ satisfies (2). With $h(k) = \frac{k(k-3)}{k^2-4k+5}$, we also satisfy (3), (4), and (5). Now with $f(k) = -\frac{2(k-1)(2k-5)}{k^2-4k+5}$, condition (1) is satisfied, so by Lemma 3.2 we have the following.

Corollary 3.3. *For $k \geq 5$ and $T \in \mathcal{T}_k$ with $K_{k-1} \subseteq T$, we have*

$$2\,\|T\| \leq \left(k - 3 + \frac{3k-5}{k^2-4k+5}\right)|T| - \frac{2(k-1)(2k-5)}{k^2-4k+5} + \frac{k(k-3)}{k^2-4k+5}q(T).$$

If we put a similar bound of Kostochka and Stiebitz [10] into this form, we get the following.

Lemma 3.4 (Kostochka–Stiebitz)**.** *For $k \geq 7$ and $T \in \mathcal{T}_k$, we have*

$$2\,\|T\| \leq \left(k - 3 + \frac{4(k-1)}{k^2-3k+4}\right)|T| - \frac{4(k^2-3k+2)}{k^2-3k+4} + \frac{k^2-3k}{k^2-3k+4}q(T).$$

Note that $\frac{3k-5}{(k-5)(k-1)} < \frac{4(k-1)}{k^2-3k+4}$ for $k \geq 7$.

In Section 4, we will see that the bound we get on $d(G)$ is primarily a function of the $p(k)$ with which we apply Lemma 3.2: the smaller $p(k)$ is, the better bound we get on $d(G)$. So it useful to note that the choice of $p(k)$ in Corollary 3.3 is best possible. We now give a construction to prove this. Let X be a K_{k-1} with $k-3$ pendant edges. At the end of each pendant edge, put a K_{k-2}. Make a path of copies of X by adding one edge between the K_{k-1} in each copy of X (in the only way possible to keep the degrees at most $k-1$). Let T be the path made like this from m copies of X. Now $q(T) = 2$ (from the copies of K_{k-1} at the ends of the path), so if T satisfies the bound in Lemma 3.2, then we must have

$$\begin{aligned}&m((k-1)(k-2) + (k-3)(k-2)(k-3) + 2(k-3)) + 2(m-1)\\ \leq&(k-3+p(k))m(k-1+(k-2)(k-3)) + f(k) + 2h(k),\end{aligned}$$

which reduces to

$$m(k-1) + 2m(k-3) + 2(m-1) \leq m(k-1+(k-2)(k-3))p(k) + f(k) + 2h(k).$$

Now solving for $p(k)$ gives

$$p(k) \geq \frac{m(k-1) + 2m(k-3) + 2(m-1) - f(k) - 2h(k)}{m((k-1) + (k-2)(k-3))},$$

which simplifies to

$$p(k) \geq \frac{3k-5}{k^2-4k+5} - \frac{2+f(k)+2h(k)}{m(k^2-4k+5)}.$$

Since we can make m arbitrarily large, this implies $p(k) \geq \frac{3k-5}{k^2-4k+5}$, as desired.

4 Discharging

4.1 Overview and Discharging Rules

Now we use the discharging method, together with the edge bound lemmas of the previous section, to give an improved bound on $d(G)$ for every k-critical graph G. It is helpful to view our proof here as a refinement and strengthening of the proof of Gallai's bound, in Section 2. For $T \in \mathcal{T}_k$, let $W^k(T)$ be the set of vertices of T that are contained in some K_{k-1} in T. For a k-AT-critical graph G, let $\mathcal{L}(G)$ denote the subgraph of G induced on the $(k-1)$-vertices and $\mathcal{H}(G)$ the subgraph of G induced on the k-vertices.

$W^k(T)$ $\mathcal{L}(G)$ $\mathcal{H}(G)$

Note that in the proof of Gallai's bound, all $(k+1)^+$-vertices finish with extra charge; $(k+1)$-vertices have extra charge almost 1 and vertices of higher degree have even more. Our idea to improve the bound on $d(G)$ is to have the k-vertices give slightly less charge, ϵ, to their $(k-1)$-neighbors. Now all k^+-vertices finish with extra charge. But components of $\mathcal{L}(G)$ have less charge, so we need to give them more charge from $(k+1)^+$-neighbors. How much charge will each component T of $\mathcal{L}(G)$ receive? This depends on $||T||$. If $||T||$ is small, then T has many external neighbors, so T will receive lots of charge. If $||T||$ is large, then Lemma 3.2 implies that $q(T)$ is also large. So our plan is to send charge γ to T via each edge incident to a vertex in $W^k(T)$, i.e., one counted by $q(T)$. (For comparison with Gallai's bound, we will have $\epsilon < \frac{k-1}{k^2-3} < \gamma$.) If such an incident edge ends at a $(k+1)^+$-vertex v, then v will still finish with sufficient charge. Our concern, of course, is that a k-vertex will give charge γ to too many vertices in $W^k(T)$. We would like to prove that each k-vertex has only a few neighbors in $W^k(T)$. Unfortunately, we believe this is false. However, something similar is true. We can assign each k-vertex to "sponsor" some adjacent vertices in $W^k(T)$, so that each k-vertex sponsors at most 3 such neighbors, and in each component T of $\mathcal{L}(G)$ at most two vertices in $W^k(T)$ go unsponsored. This is an immediate consequence of Lemma 5.2, which says that the auxiliary bipartite graph $\mathcal{B}_k(G)$, defined in the next paragraph, is 2-degenerate. Now we give the details.

Let $\mathcal{B}_k(G)$ be the bipartite graph with one part $V(\mathcal{H}(G))$ and the other part the components of $\mathcal{L}(G)$. Put an edge between $y \in V(\mathcal{H}(G))$ and a component T of $\mathcal{L}(G)$ if and only if $N(y) \cap W^k(T) \neq \emptyset$. Now Lemma 5.2 says that $\mathcal{B}_k(G)$ is 2-degenerate. Let ϵ and γ be parameters, to be chosen. Our initial charge function is $\mathrm{ch}(v) = d_G(v)$. We redistribute charge according to the following rules, applied successively. $\mathcal{B}_k(G)$ ϵ, γ

1. Each k^+-vertex gives charge ϵ to each of its $(k-1)$-neighbors not in a K_{k-1}.
2. Each $(k+1)^+$-vertex give charge γ to each of its $(k-1)$-neighbors in a K_{k-1}.
3. Let $Q = \mathcal{B}_k(G)$. Repeat the following steps until Q is empty.
   (a) For each component T of $\mathcal{L}(G)$ in Q with degree at most two in Q do the following:
      i. For each $v \in V(\mathcal{H}(G)) \cap V(Q)$ such that $\left|N_G(v) \cap W^k(T)\right| = 2$, pick one $x \in N_G(v) \cap W^k(T)$ and send charge γ from v to x,
      ii. Remove T from Q.
   (b) For each vertex v of $\mathcal{H}(G)$ in Q with degree at most two in Q do the following:
      i. Send charge γ from v to each $x \in N_G(v) \cap W^k(T)$ for each component T of $\mathcal{L}(G)$ where $vT \in E(Q)$.
      ii. Remove v from Q.
4. Have the vertices in each component of $\mathcal{L}(G)$ share their total charge equally.

First, note that Step 3 will eventually result in Q being empty. This is because $\mathcal{B}_k(G)$ is 2-degenerate, as shown in Lemma 5.2. Next, consider a k-vertex v. In (3bi) v gives away γ to each neighbor in at most two components of $\mathcal{L}(G)$. So it is important that v have few neighbors in these components. Fortunately, this is true. By Lemma 5.1, v has at most 2 neighbors in any component of $\mathcal{L}(G)$. Further, v has at most one component in which it has 2 neighbors. Thus, in (3ai) and (3bi), v gives away a total of at most 3γ. Finally, consider a component T. In (3bi), T receives charge γ via every edge incident in $\mathcal{B}_k(G)$, except possibly two (that are still present when v is deleted in (3aii)). Again, by Lemma 5.2, no such v has three neighbors in T. Further, combining this with Steps (2) and (3ai), T receives γ along all but at most two incident edges leading to k-vertices. Thus, T receives charge at least $\gamma(q(T)-2)$ in Steps (2) and (3).

4.2 Analyzing the Discharging and the Main Result

Here we analyze the charge received by each component T of $\mathcal{L}(G)$. We choose ϵ and γ to maximize the minimum, over all vertices, of the final charge. The following theorem is the main result of this paper.

Theorem 4.1. *Let $k \geq 7$ and $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$, $h\colon \mathbb{N} \to \mathbb{R}$. If G is a k-AT-critical graph, and $G \neq K_k$, then*

$$d(G) \geq k - 1 + \frac{2 - p(k)}{k + 2 + 3h(k) - p(k)},$$

whenever p, f, and h satisfy all of the following conditions:

1. $f(k) \geq (k-1)(1 - p(k) - h(k))$*; and*
2. $p(k) \geq \frac{3}{k-2}$*; and*
3. $p(k) \geq h(k) + 5 - k$*; and*
4. $p(k) \geq \frac{2+h(k)}{k-2}$*; and*
5. $(k-1)p(k) + (k-3)h(k) \geq k + 1$*; and*
6. $2(h(k)+1) + f(k) \leq 0$*; and*
7. $p(k) + (k-5)h(k) \leq k + 1$.

Before we prove Theorem 4.1, we show that two previous results on this problem follow immediately from this theorem. Note that conditions (1)–(5) are the hypotheses of Lemma 3.2. As a first test, let $p(k) = 1 - \frac{2}{k-1}$, $f(k) = -2$ and $h(k) = 0$. Now the hypotheses of Theorem 4.1 are satisfied when $k \geq 7$, and we get Gallai's bound: $d(G) \geq k - 1 + \frac{k-3}{k^2-3}$. Next, let's use the Kostochka–Stiebitz bound, that is, $p(k) = \frac{4(k-1)}{k^2-3k+4}$, $f(k) = -\frac{4(k^2-3k+2)}{k^2-3k+4}$ and $h(k) = \frac{k^2-3k}{k^2-3k+4}$. Again, the hypotheses of Theorem 4.1 are satisfied when $k \geq 7$ and we get

$$d(G) \geq k - 1 + \frac{2(k-2)(k-3)}{(k-1)(k^2+3k-12)}.$$

This is exactly the bound in the paper of Kierstead and the second author [6].

Finally, to get our sharpest bound on $d(G)$, we use the bound in Corollary 3.3, that is, $p(k) = \frac{3k-5}{k^2-4k+5}$, $f(k) = -\frac{2(k-1)(2k-5)}{k^2-4k+5}$, and $h(k) = \frac{k(k-3)}{k^2-4k+5}$. The hypotheses of Theorem 4.1 are satisfied when $k \geq 7$ and we get $d(G) \geq k - 1 + \frac{(k-3)(2k-5)}{k^3+k^2-15k+15}$. This is better than the bound in [6] for $k \geq 7$. We record this as our main corollary.

Corollary 4.2. *If G is a k-AT-critical graph, with $k \geq 7$, and $G \neq K_k$, then*

$$d(G) \geq k - 1 + \frac{(k-3)(2k-5)}{k^3 + k^2 - 15k + 15}.$$

Now we prove Theorem 4.1

Proof of Theorem 4.1. Our discharging procedure in the previous section gives charge ϵ to a component T for every incident edge not ending in a K_{k-1}. The number of such edges is exactly

$$-q(T) + \sum_{v \in V(T)} (k - 1 - d_T(v)) = (k-1)\,|T| - 2\,\|T\| - q(T),$$

so we let $A(T)$ denote this quantity. When $K_{k-1} \subseteq T$, since (1)–(5) hold, Lemma 3.2 gives

$$2\,\|T\| \leq (k - 3 + p(k))\,|T| + f(k) + h(k)q(T).$$

So, when $K_{k-1} \subseteq T$ we get

$$\begin{aligned} A(T) &\geq (k-1)\,|T| - q(T) - ((k - 3 + p(k))\,|T| + f(k) + h(k)q(T)) \\ &= (2 - p(k))\,|T| - f(k) - (h(k)+1)q(T). \end{aligned}$$

Hence, in total T receives charge at least

$$\begin{aligned} \epsilon A(T) + \gamma(q(T) - 2) &\geq \epsilon(2 - p(k))|T| - \epsilon f(k) - \epsilon(h(k)+1)q(T) + \gamma q(T) - 2\gamma \\ &= \epsilon(2 - p(k))|T| + q(T)(\gamma - \epsilon(h(k)+1)) - (2\gamma + \epsilon f(k)) \end{aligned}$$

Our goal is to make $\epsilon(2-p(k))$ as large as possible, while ensuring that the final two terms are nonnegative. To make the second term 0, we let $\gamma = \epsilon(h(k)+1)$. Now the final term becomes $-\epsilon(2(h(k)+1) + f(k))$. For simplicity, we have added, as (6), that $2(h(k)+1) + f(k) \leq 0$. (Since we typically take $h(k) > 0$, as in Corollary 4.2, it is precisely this requirement that necessitates the use of $f(k)$ in Lemma 3.2.) Thus, T receives charge at least

$$\epsilon\,(2 - p(k))\,|T|\,,$$

so each of its vertices gets at least $\epsilon(2 - p(k))$. We also need each k-vertex to end with enough charge, and each of these loses at most $3\gamma + (k-3)\epsilon$. So we take

$$1 - (3\gamma + (k-3)\epsilon) = \epsilon\,(2 - p(k))\,,$$

which gives

$$\begin{aligned} \epsilon &= \frac{1}{k + 2 + 3h(k) - p(k)}, \\ \gamma &= \frac{h(k)+1}{k + 2 + 3h(k) - p(k)}. \end{aligned}$$

Thus, after discharging, each k-vertex finishes with charge at least $k - 1 + \epsilon(2 - p(k))$. The same bound holds for each $(k-1)$-vertex in a component T with a K_{k-1}.

When $K_{k-1} \not\subseteq T$, we have $q(T) = 0$. Applying Lemma 3.1 with $f(k) = 0$ and $p(k)$ as in the present theorem, we get

$$2\,\|T\| \leq (k - 3 + p(k))\,|T|\,,$$

and hence

$$A(T) \geq (2 - p(k))\,|T|\,.$$

So T receives sufficient charge.

It remains to check that the $(k+1)^+$-vertices don't give away too much charge. Let v be a $(k+1)^+$-vertex. Now v ends with charge at least

$$d(v) - \gamma d(v) = (1-\gamma)d(v) \geq (1-\gamma)(k+1) = (k+1)\frac{k+1+2h(k)-p(k)}{k+2+3h(k)-p(k)},$$

so we need to satisfy the inequality

$$(k+1)\frac{k+1+2h(k)-p(k)}{k+2+3h(k)-p(k)} \geq k-1+\frac{2-p(k)}{k+2+3h(k)-p(k)}.$$

This inequality reduces to

$$p(k) + (k-5)h(k) \leq k+1.$$

For simplicity, we have added this as (7), since it is easily satisfied by the p, f, and h we want to use. □

The reason that we require $k \geq 7$ in Theorem 4.1 (and Corollary 4.2) is that the proof uses Lemma 5.2. However, for $k \in \{5,6\}$, Lemma 5.3 can play an analogous role. For $k \geq 7$, Lemma 5.2 implies that if G has no reducible configuration, then $B_k(G)$ is 2-degenerate. For $k \in \{5,6\}$, Lemma 5.3 implies that we can reduce $\mathcal{B}_k(G)$ to the empty graph by repeatedly deleting either a tree component vertex v with $d_{\mathcal{B}_k(G)}(v) \leq 1$ or else a vertex w in $V(\mathcal{B}_k(G)) \cap V(\mathcal{H}(G))$ with $d_{\mathcal{B}_k(G)}(v) \leq 3$. Thus, in the discharging, the tree corresponding to v receives charge at least $\gamma(q(T)-1)$ on edges ending at vertices in $W^k(T)$. Similarly, each k-vertex gives away charge at most $4\gamma + (k-4)\epsilon$. Now, to find the optimal value of ϵ, as in the proof of Theorem 4.1, we solve $(1-(4\gamma+\epsilon(k-4)) = (2-p(k))\epsilon$. This gives $\epsilon = \frac{1}{k+2+4h(k)-p(k)}$ and, again, $\gamma = \epsilon(h(k)+1)$. In place of hypothesis (6), we have the slightly weaker requirement $h(k)+1+f(k) \leq 0$. The result is the following theorem and corollary, for $k \in \{5,6\}$.

Theorem 4.3. *Let $k \in \{5,6\}$ and $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$, $h\colon \mathbb{N} \to \mathbb{R}$. If G is a k-AT-critical graph, and $G \neq K_k$, then*

$$d(G) \geq k-1+\frac{2-p(k)}{k+2+4h(k)-p(k)},$$

whenever p, f, and h satisfy all of the following conditions:

1. *$f(k) \geq (k-1)(1-p(k)-h(k))$; and*

2. *$p(k) \geq \frac{3}{k-2}$; and*

3. *$p(k) \geq h(k)+5-k$; and*

4. *$p(k) \geq \frac{2+h(k)}{k-2}$; and*

5. *$(k-1)p(k)+(k-3)h(k) \geq k+1$; and*

6. *$h(k)+1+f(k) \leq 0$; and*

7. $p(k) + (k-5)h(k) \le k+1$.

To get the best bound on $d(G)$, as in Theorem 4.1, we use $p(k) = \frac{3k-5}{k^2-4k+5}$, $f(k) = -\frac{2(k-1)(2k-5)}{k^2-4k+5}$, and $h(k) = \frac{k(k-3)}{k^2-4k+5}$.

Corollary 4.4. *If G is a k-AT-critical graph, with $k \in \{5,6\}$, and $G \neq K_k$, then*

$$d(G) \ge k - 1 + \frac{(k-3)(2k-5)}{k^3+2k^2-18k+15}.$$

5 Reducible Configurations

In this section, we collect the three main reducibility lemmas used in the proofs of Theorems 4.1 and 4.3. They were proved in [6]. Each lemma describes a class of reducible configurations, and so restricts the structure of AT-critical-graphs. The first says that no k-vertex has three or more neighbors in the same component T of $\mathcal{L}(G)$. Further, for each k-vertex v, at most one component T of $\mathcal{L}(G)$ has two neighbors of v.

Lemma 5.1. *Let $k \ge 5$ and let G be a graph with $x \in V(G)$. Now G is f-AT, where $f(x) = d_G(x) - 1$ and $f(v) = d_G(v)$ for all $v \in V(G-x)$, whenever all of the following hold:*

1. *$K_k \not\subseteq G$; and*
2. *$G - x$ has t components $H_1, H_2, \dots, H_t$, and all are in $\mathcal{T}_k$; and*
3. *$d_G(v) \le k-1$ for all $v \in V(G-x)$; and*
4. *$\left|N(x) \cap W^k(H_i)\right| \ge 1$ for $i \in [t]$; and*
5. *$d_G(x) \ge t+2$.*

To describe reducible configurations with more than one k-vertex we need the following auxiliary bipartite graph, which is a generalization of what we defined in Section 4.1. For a graph G, $\{X,Y\}$ a partition of $V(G)$ and $k \ge 4$, let $\mathcal{B}_k(X,Y)$ be the bipartite graph with one part Y and the other part the components of $G[X]$. Put an edge between $y \in Y$ and a component T of $G[X]$ if and only if $N(y) \cap W^k(T) \neq \emptyset$. The next lemma tells us that we have a reducible configuration if this bipartite graph has minimum degree at least three. In other words, if we have no reducible configuration, then $\mathcal{B}_k(X,Y)$ is 2-degenerate.

Lemma 5.2. *Let $k \ge 7$ and let G be a graph with $Y \subseteq V(G)$. Now G has an induced subgraph G' that is f-AT, where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G'-Y)$, whenever all of the following hold:*

1. *$K_k \not\subseteq G$; and*
2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*
3. *$d_G(v) \le k-1$ for all $v \in V(G-Y)$; and*
4. *with $\mathcal{B} := \mathcal{B}_k(V(G-Y), Y)$ we have $\delta(\mathcal{B}) \ge 3$.*

We also have the following version with asymmetric degree condition on $\mathcal{B}$. The point here is that this works for $k \geq 5$. As we saw in Theorem 4.3 and Corollary 4.4, the consequence is that we trade a bit in our size bound for the proof to go through with $k \in \{5, 6\}$.

Lemma 5.3. *Let $k \geq 5$ and let G be a graph with $Y \subseteq V(G)$. Now G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$ whenever all of the following hold:*

1. *$K_k \not\subseteq G$; and*
2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*
4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $d_{\mathcal{B}}(y) \geq 4$ for all $y \in Y$ and $d_{\mathcal{B}}(T) \geq 2$ for all components T of $G - Y$.*

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