# A GAME GENERALIZING HALL'S THEOREM

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Abstract. We characterize the initial positions from which the first player has a winning strategy in a certain two-player game. This provides a generalization of Hall's Theorem. Vizing's Theorem on edge-coloring follows from a special case.

## 1. Introduction

A *set system* is a finite family of finite sets. A *transversal* of a set system $\mathcal{S}$ is an injection $f\colon \mathcal{S} \hookrightarrow \bigcup \mathcal{S}$ such that $f(S) \in S$ for each $S \in \mathcal{S}$. Hall's Theorem [4] gives the precise conditions under which a set system has a transversal.

**Theorem 1.1** (Hall [4])**.** *A set system $\mathcal{S}$ has a transversal if and only if $|\bigcup \mathcal{W}| \geq |\mathcal{W}|$ for each $\mathcal{W} \subseteq \mathcal{S}$.*

We generalize this by analyzing winning strategies in a two-player game played on a set system by *Fixer* (henceforth dubbed **F**) and *Breaker*. Fixer wins the game by eventually modifying the set system so that it has a transversal; if Breaker has a strategy to prevent this forever, then we say that Breaker wins. Additionally, when playing on the set system $\mathcal{S}$, we provide a *pot* $P$ with $\bigcup \mathcal{S} \subseteq P$. Fixer moves first and he can do the following.

**Fixer's turn.** Pick $x \in P$ and $S \in \mathcal{S}$ with $x \notin S$ and replace $S$ with $S \cup \{x\} \smallsetminus \{y\}$ for some $y \in S$.

For $k \in \mathbb{N}$, let $[k] = \{1, \ldots, k\}$. For each $t \in [|\mathcal{S}| - 1]$, we have a different rule for Breaker. We denote Breaker by $\mathbf{B}_t$ when he is playing with the following rule.

**Breaker's turn.** If **F** modified $S \in \mathcal{S}$ by inserting $x$ and removing $y$, $\mathbf{B}_t$ can pick up to $t$ sets in $\mathcal{S} \smallsetminus \{S\}$ and modify them by swapping $x$ for $y$ or $y$ for $x$.

To state the main theorem, we need additional notation. For $\mathcal{W} \subseteq \mathcal{S}$ and $x \in P$ define the *degree* in $\mathcal{W}$ of $x$, written $d_{\mathcal{W}}(x)$, by

$$d_{\mathcal{W}}(x) = \left|\{S \in \mathcal{W}\colon\ x \in S\}\right|.$$

Define the *t-value* of $\mathcal{W} \subseteq \mathcal{S}$, written $\nu_t(\mathcal{W})$, by

$$\nu_t(\mathcal{W}) = \sum_{x \in \bigcup \mathcal{W}} \left\lfloor \frac{d_{\mathcal{W}}(x) - 1}{t + 1} \right\rfloor.$$

Intuitively, this measures how much **F** can increase $|\bigcup \mathcal{W}|$ without $\mathbf{B}_t$ undoing the progress. For instance, if $d_{\mathcal{W}}(y) \leq t + 1$ and **F** swaps $x$ in for $y$ at $W$, then $B_t$ can change all instances of $x$ to $y$, since $x$ appears in at most $t$ other sets. In this case $y$ contributes nothing to the $t$-value of $\mathcal{W}$. Our main theorem shows that this intuition is correct.

**Theorem 1.2.** *In a set system $\mathcal{S}$ with $\bigcup \mathcal{S} \subseteq P$ and $|P| \geq |\mathcal{S}|$, $\mathbf{F}$ has a winning strategy against $\mathbf{B}_t$ if and only if $|\bigcup \mathcal{W}| \geq |\mathcal{W}| - \nu_t(\mathcal{W})$ for each $\mathcal{W} \subseteq \mathcal{S}$.*

We can recover Hall's Theorem from the case $t = |\mathcal{S}| - 1$; that is, $\mathbf{B}_t$ can remove all $y$'s in $\mathcal{S}$ rendering $\mathbf{F}$'s move equivalent to swapping the names of $x$ and $y$, that is, rendering it useless. In Section 3 we show that Vizing's Theorem on edge-coloring is a quick corollary of this result. In fact, the strategy employed by $\mathbf{F}$ is based, in part, on the proofs of Vizing's Theorem by Ehrenfeucht, Faber, and Kierstead [2] and by Schrijver [5]. For a graph $G$, let $\chi'(G)$ be the edge-chromatic number of $G$ and let $\Delta(G)$ be the maximum degree of $G$.

**Corollary 1.3** (Vizing [7])**.** *If $G$ is a simple graph, then $\chi'(G) \leq \Delta(G) + 1$.*

There is a "multiplicity" version of Hall's Theorem in which the representatives sought for the sets in the family are disjoint subsets of specified sizes. When each set $S$ is asked to have $\eta(S)$ representatives in the "$\eta$-transversal", the desired subsets can be found by making $\eta(S)$ copies of each set $S$ and applying Hall's Theorem. In Sections 4 and 5 we generalize this folklore extension of Hall's Theorem and use the generalization to give a non-standard proof of the following result from which classical edge-coloring results and various "adjacency lemmas" follow (see [6] for the standard proof and how these consequences are derived). Let $xy$ be an edge in a multigraph $G$. We denote the multiplicity of $xy$ by $\mu(xy)$. Additionally, $xy$ is *critical* if $\chi'(G - xy) < \chi'(G)$.

**Corollary 1.4.** *Let $G$ be a multigraph satisfying $\chi'(G) \geq \Delta(G) + 1$. For each critical edge $xy$ in $G$, there exists $X \subseteq N(x)$ with $y \in X$ and $|X| \geq 2$ such that*

$$\sum_{v \in X} (d(v) + \mu(xv) + 1 - \chi'(G)) \geq 2.$$

## 2. The proof

*Proof of Theorem 1.2.* First we prove necessity of the condition. Suppose we have $\mathcal{W} \subseteq \mathcal{S}$ with $|\bigcup \mathcal{W}| < |\mathcal{W}| - \nu_t(\mathcal{W})$. We show that no matter what moves $\mathbf{F}$ makes, $\mathbf{B}_t$ can maintain this invariant. We then always have $|\bigcup \mathcal{W}| < |\mathcal{W}|$ and hence $\mathcal{W}$ can never have a transversal.

Suppose $\mathbf{F}$ modifies $S \in \mathcal{S}$ by inserting $x$ and removing $y$ to get $S'$. If $S \notin \mathcal{W}$, then $\mathbf{B}_t$ does not need to do anything, so we may assume $S \in \mathcal{W}$. Put $\mathcal{W}' = \mathcal{W} \cup \{S'\} \smallsetminus \{S\}$.

If $d_{\mathcal{W}}(x) = 0$, then $|\bigcup \mathcal{W}'| = |\bigcup \mathcal{W}| + 1$. Now $\mathbf{B}_t$ swaps $x$ in for $y$ in $\min\{t, d_{\mathcal{W}'}(y)\}$ sets of $\mathcal{W}'$ to form $\mathcal{W}^*$. If $d_{\mathcal{W}'}(y) \leq t$, then $d_{\mathcal{W}^*}(y) = 0$ and we have $|\bigcup \mathcal{W}^*| = |\bigcup \mathcal{W}|$; hence the invariant is maintained. Otherwise $\nu_t(\mathcal{W}^*) < \nu_t(\mathcal{W})$ because the degree of $y$ has decreased by $t + 1$, and again the invariant is maintained.

Hence we may assume $d_{\mathcal{W}}(x) > 0$. Now $|\bigcup \mathcal{W}'| \leq |\bigcup \mathcal{W}|$. In order to have a chance to destroy the invariant, $\mathbf{F}$ must achieve $\nu_t(\mathcal{W}') > \nu_t(\mathcal{W})$. This requires $d_{\mathcal{W}'}(x) - 1$ to be a multiple of $t+1$ and $d_{\mathcal{W}'}(y)$ to not be a multiple of $t+1$; in particular, $d_{\mathcal{W}'}(y) \neq d_{\mathcal{W}'}(x) - 1$. If $d_{\mathcal{W}'}(y) < d_{\mathcal{W}'}(x) - 1$, then $\mathbf{B}_t$ swaps $y$ in for $x$ in one set in $\mathcal{W}' \smallsetminus \{S'\}$. Doing so maintains the invariant, since now every element has the same degree in the new set system as in $\mathcal{W}$. Otherwise, $d_{\mathcal{W}'}(y) > d_{\mathcal{W}'}(x) - 1$ and $\mathbf{B}_t$ swaps $x$ in for $y$ in $\min\{t, d_{\mathcal{W}'}(y) + 1 - d_{\mathcal{W}'}(x)\}$ sets of $\mathcal{W}'$. This reduces the contribution from $y$ without further increasing the contribution from $x$ and thereby maintains the invariant.

Now we prove sufficiency. Suppose the condition is not sufficient for $\mathbf{F}$ to have a winning strategy. Among all counterexamples having the fewest sets, choose $\mathcal{S}$ to maximize $|\bigcup\mathcal{S}|$.

First, suppose $|\bigcup\mathcal{S}| \geq |\mathcal{S}|$. Let $C$ be a minimal nonempty subset of $\bigcup\mathcal{S}$ such that $|\mathcal{W}_C| \leq |C|$, where $\mathcal{W}_C = \{S \in \mathcal{S} \mid C \cap S \neq \emptyset\}$ (we can make this choice because $\bigcup\mathcal{S}$ is such a subset). Create a bipartite graph with parts $C$ and $\mathcal{W}_C$ and an edge from $x \in C$ to $S \in \mathcal{W}_C$ if and only if $x \in S$. If $|C| = 1$, then we clearly have a matching of $C$ into $\mathcal{W}_C$. Otherwise, by minimality of $C$, for every set $D$ such that $\emptyset \neq D \subset C$ we have $|\mathcal{W}_D| > |D|$ and hence $|C| = |\mathcal{W}_C|$; now applying Hall's Theorem (for bipartite graphs) gives a matching of $C$ into $\mathcal{W}_C$. This matching gives a transversal $f\colon \mathcal{W}_C \hookrightarrow \bigcup\mathcal{W}_C$ with image $C$. Put $\mathcal{S}' = \mathcal{S} \smallsetminus \mathcal{W}_C$ and $P' = P \smallsetminus C$. The hypotheses of the claim are satisfied by $\mathcal{S}'$ and $P'$. If $\mathbf{F}$ continues to play only using $\mathcal{S}'$ and $\mathcal{P}'$, then $\mathbf{B}_t$ cannot destroy the transversal of $\mathcal{W}_C$ that exists using elements of $C$, even though $\mathbf{B}_t$ may play on all of $\mathcal{S}$, because $\mathbf{F}$ will make no further move involving the elements in that transversal. Now minimality of $|\mathcal{S}|$ gives a contradiction.

Therefore, we may assume $|\bigcup\mathcal{S}| < |\mathcal{S}|$ and hence $\nu_t(\mathcal{S}) \geq 1$. Since $|P| \geq |\mathcal{S}|$, we have $x \in P$ with $d_{\mathcal{S}}(x) = 0$. So, we may choose $y \in P$ with $d_{\mathcal{S}}(y) \geq t+2$. Now $\mathbf{F}$ should swap $x$ in for $y$ in some $S \in \mathcal{S}$ to form $\mathcal{S}'$. Since $d_{\mathcal{S}}(x) = 0$, we have $|\bigcup\mathcal{S}'| > |\bigcup\mathcal{S}|$. We also have $d_{\mathcal{S}'}(y) \geq t+1$. Now $\mathbf{B}_t$ moves and creates $\mathcal{S}^*$. Since $d_{\mathcal{S}^*}(y) \geq d_{\mathcal{S}'}(y) - t > 0$, we have $|\bigcup\mathcal{S}^*| > |\bigcup\mathcal{S}|$. If our modifications changed some $\mathcal{W} \in \mathcal{S}$ so that it now violates the hypotheses, then let $\mathcal{W}^*$ be $\mathcal{W}$ after both player's moves. That is, $|\bigcup\mathcal{W}^*| < |\mathcal{W}^*| - \nu_t(\mathcal{W}^*)$. Since $d_{\mathcal{S}}(x) = 0$, we have $|\bigcup\mathcal{W}^*| \geq |\bigcup\mathcal{W}|$. Thus $\nu_t(\mathcal{W}^*) < \nu_t(\mathcal{W})$ and hence $d_{\mathcal{W}^*}(y) < d_{\mathcal{W}}(y)$. Now $d_{\mathcal{W}^*}(x) > 0$, since any removed $y$ was replaced with $x$. Since $d_{\mathcal{W}}(x) = 0$, we have $|\bigcup\mathcal{W}^*| > |\bigcup\mathcal{W}|$. This leads to $|\mathcal{W}| - \nu_t(\mathcal{W}) \leq |\bigcup\mathcal{W}| < |\bigcup\mathcal{W}^*| < |\mathcal{W}^*| - \nu_t(\mathcal{W}^*)$, and hence $\nu_t(\mathcal{W}^*) \leq \nu_t(\mathcal{W}) - 2$. This is impossible, since $\mathbf{B}_t$ can perform swaps in at most $t$ sets. Since $\mathcal{S}^*$ satisfies the hypotheses of the claim, $|\bigcup\mathcal{S}^*| > |\bigcup\mathcal{S}|$ now implies that $\mathbf{F}$ can win. □

## 3. Vizing's Theorem for simple graphs

Vizing's Theorem for simple graphs follows from a very special case of Theorem 1.2.

**Corollary 3.1.** *If $\mathcal{S} = \{S_1, \ldots, S_k\}$ with $|S_k| \geq 1$, $|S_i| \geq 2$ for all $i \in [k-1]$, and $|P| \geq k$, then $\mathbf{F}$ has a winning strategy against $\mathbf{B}_1$.*

*Proof.* For $\mathcal{W} \subseteq \mathcal{S}$, we have

$$\nu_1(\mathcal{W}) \geq \sum_{x \in \bigcup\mathcal{W}} \frac{d_{\mathcal{W}}(x) - 2}{2} = \frac{1}{2}\sum_{S \in \mathcal{W}} |S| - \left|\bigcup\mathcal{W}\right| \geq \frac{1}{2}(2\,|\mathcal{W}| - 1) - \left|\bigcup\mathcal{W}\right|.$$

Hence $\nu_1(\mathcal{W}) \geq |\mathcal{W}| - |\bigcup\mathcal{W}|$, as desired. □

*Proof of Vizing's Theorem for simple graphs (Corollary 1.3).* Suppose not and let $G$ be a counterexample with fewest vertices. Let $v_1, \ldots, v_k$ be the neighbors of a vertex $v$ of maximum degree in $G$. By minimality of $|G|$, we have a $(k+1)$-edge-coloring of $G - v$. Among the $k+1$ colors used in this coloring, let $S_i$ be the set of those not appearing on edges incident to $v_i$. Each $v_i$ has degree at most $k-1$ in $G - v$ and hence $|S_i| \geq 2$. Also, if $a \in S_i$ and $b \notin S_i$ we may exchange colors on a maximum length path $M$ starting at $v_i$ and alternating between colors $b$ and $a$. After the exchange, $S_i$ has lost $a$ and gained $b$. Consider $S_j$ for some $j \in [k] - \{i\}$. If $v_j$ is not in $M$ or is an internal vertex in $M$, then $S_j$ is maintained. If $v_j$ is the endpoint of $M$ then $S_j$ has changed by either swapping $a$ in for $b$ or by swapping $b$ in for $a$. Therefore, performing the exchange translates into an $\mathbf{F}$ move followed by a $\mathbf{B}_1$ move on the set system $\{S_1, \ldots, S_k\}$. Since $\mathbf{F}$ can always make any of his legal moves this way, we may apply Corollary 3.1 to get a transversal of the $S_i$. Now we can extend the $(k+1)$-edge-coloring to all of $G$ by using the corresponding element of the transversal on $vv_i$ for each $i \in [k]$. $\square$

*Remark.* In the proof above, the legal moves of the second player may be more restricted than those of $\mathbf{B}_1$. For example, if $G$ is bipartite, then $M$ must have even length. Hence if $M$ ends at some $v_j$, then it ends with an edge colored $a$; that is, the second player is only allowed to swap $a$ in for $b$. Fixer can win against this opponent when all $S_i$ are nonempty by just being greedy; this proves Kőnig's Theorem on edge-coloring. Are there any other interesting weak opponents that we can take advantage of?

## 4. The multiplicity version

To deal with edge-coloring of multigraphs, we need to generalize our game slightly. Instead of looking for a transversal, we will look for a system of disjoint sets of representatives. For $\eta \colon \mathcal{S} \to \mathbb{N}$, an *$\eta$-transversal* of $\mathcal{S}$ is a function $f \colon \mathcal{S} \to \mathcal{P}\left(\bigcup \mathcal{S}\right)$ such that $f(S) \subseteq S$, with $|f(S)| = \eta(S)$ for $S \in \mathcal{S}$ and $f(A) \cap f(B) = \emptyset$ when $A \neq B$. By making $\eta(S)$ copies of each $S \in \mathcal{S}$ and applying Hall's Theorem, we get the following folklore result.

**Theorem 4.1.** *A set system $\mathcal{S}$ has an $\eta$-transversal if and only if $\left|\bigcup \mathcal{W}\right| \geq \sum_{W \in \mathcal{W}} \eta(W)$ for each $\mathcal{W} \subseteq \mathcal{S}$.*

Call the game where $\mathbf{F}$ wins by creating an $\eta$-transversal *the $\eta$-game*. We can use the same idea of making $\eta(S)$ copies of each $S \in \mathcal{S}$ to get a multiplicity version of Theorem 1.2. For simplicity, we pull the implementation of this idea out into Lemma 4.3. In the proof of Theorem 1.2, we implicitly proved Lemma 4.3 in the special case where $\eta(x) = 1$ for all $x \in X$. This special case is well-known; for instance, it was used by Borodin, Kostochka and Woodall [1] in strengthening Galvin's Theorem about list edge-coloring of bipartite graphs [3]. To help develop the reader's intuition, we first state the special case.

**Lemma 4.2.** *Let $G$ be a bipartite graph with nonempty parts $X$ and $Y$. If $|X| \leq |Y|$ and $Y$ has no isolated vertices, then $G$ contains a nonempty matching $M$ whose vertex set is $S \cup N(S)$ for some $S \subseteq X$.*

**Lemma 4.3.** *Let $G$ be a bipartite graph with nonempty parts $X$ and $Y$. If $\eta \colon X \to \mathbb{N}$ and $|N_G(X)| \geq \sum_{x \in X} \eta(x)$, then $G$ has a nonempty subgraph $G'$ with parts $X' \subseteq X$ and $Y' \subseteq Y$ such that $d_{G'}(x) = \eta(x)$ for $x \in X'$, $d_{G'}(y) = 1$ for $y \in Y'$, and $N_G(Y') = X'$.*

*Proof.* First, we prove the special case where $\eta(x) = 1$ for all $x \in X$. Since $|N_G(Y)| \leq |X| \leq |N_G(X)|$, we may let $Y' \subseteq Y$ be a minimal nonempty subset of $N_G(X)$ such that $|N_G(Y')| \leq |Y'|$. Let $X' = N_G(Y')$. If $|Y'| = 1$, then $|X'| = 1$ and we clearly have a matching of $Y'$ into $X'$. Otherwise, by minimality of $Y'$, for every set $D$ such that $\emptyset \neq D \subset Y'$, we have

$|N_G(D)| > |D|$ and hence $|X'| = |Y'|$; now applying Hall's Theorem (for bipartite graphs) gives a matching of $Y'$ into $X'$ as desired.

For the general case, create a bipartite graph $H$ with parts $X^*$ and $Y$ from $G$ by replacing each $x \in X$ with $\eta(x)$ identical copies of $x$. By assumption, $|N_H(X^*)| = |N_G(X)| \geq |X^*|$. Now apply the special case to $H$ to get a matching $M$ of $Y' \subseteq Y$ into $N_H(Y')$. Since all copies of $x \in X$ have the same neighborhood in $H$, a copy of $x$ is in $N_H(Y')$ if and only if all copies of $x$ are. For $x \in N_G(Y')$, we thus have all $\eta(x)$ copies of $x$ matched into $Y'$ by $M$. Let $X' = N_G(Y')$ and let $G'$ be the subgraph of $G$ with parts $X'$ and $Y'$ such that $xy \in E(G')$ if and only if some copy of $x$ is matched to $y$ by $M$. We have shown that $G'$ has the desired properties. $\square$

**Theorem 4.4.** *In a set system $\mathcal{S}$ with $\bigcup \mathcal{S} \subseteq P$ and $|P| \geq \sum_{S \in \mathcal{S}} \eta(S)$, $\mathbf{F}$ has a winning strategy against $\mathbf{B}_t$ in the $\eta$-game if and only if $|\bigcup \mathcal{W}| \geq \sum_{W \in \mathcal{W}} \eta(W) - \nu_t(\mathcal{W})$ for each $\mathcal{W} \subseteq \mathcal{S}$.*

*Proof.* First we prove necessity of the condition. We note that the proof of necessity is identical to that in Theorem 1.2 aside from changing the invariant we are maintaining. Consider $\mathcal{W} \subseteq \mathcal{S}$ with $|\bigcup \mathcal{W}| < \sum_{W \in \mathcal{W}} \eta(W) - \nu_t(\mathcal{W})$. We show that no matter what moves $\mathbf{F}$ makes, we can maintain this invariant. We then always have $|\bigcup \mathcal{W}| < \sum_{W \in \mathcal{W}} \eta(W)$, and hence $\mathcal{W}$ can never have an $\eta$-transversal.

Suppose $\mathbf{F}$ modifies $S \in \mathcal{S}$ by inserting $x$ and removing $y$ to get $S'$. If $S \notin \mathcal{W}$, then $\mathbf{B}_t$ does not need to do anything, so we may assume $S \in \mathcal{W}$. Put $\mathcal{W}' = \mathcal{W} \cup \{S'\} \setminus \{S\}$.

If $d_{\mathcal{W}}(x) = 0$, then $|\bigcup \mathcal{W}'| = |\bigcup \mathcal{W}| + 1$. Now $\mathbf{B}_t$ swaps $x$ in for $y$ in $\min\{t, d_{\mathcal{W}'}(y)\}$ sets of $\mathcal{W}'$ to form $\mathcal{W}^*$. If $d_{\mathcal{W}'}(y) \leq t$, then $d_{\mathcal{W}^*}(y) = 0$ and we have $|\bigcup \mathcal{W}^*| = |\bigcup \mathcal{W}|$; hence the invariant is maintained. Otherwise, $\nu_t(\mathcal{W}^*) < \nu_t(\mathcal{W})$ because the degree of $y$ has decreased by $t+1$, and again the invariant is maintained.

Hence we may assume $d_{\mathcal{W}}(x) > 0$. Now $|\bigcup \mathcal{W}'| \leq |\bigcup \mathcal{W}|$. In order to have a chance to destroy the invariant, $\mathbf{F}$ must achieve $\nu_t(\mathcal{W}') > \nu_t(\mathcal{W})$. This requires $d_{\mathcal{W}'}(x) - 1$ to be a multiple of $t+1$ and $d_{\mathcal{W}'}(y)$ to not be a multiple of $t+1$; in particular, $d_{\mathcal{W}'}(y) \neq d_{\mathcal{W}'}(x) - 1$. If $d_{\mathcal{W}'}(y) < d_{\mathcal{W}'}(x) - 1$, then $\mathbf{B}_t$ swaps $y$ in for $x$ in one set in $\mathcal{W}' \setminus \{S'\}$. Doing so maintains the invariant, since now every element has the same degree in the new set system as in $\mathcal{W}$. Otherwise, $d_{\mathcal{W}'}(y) > d_{\mathcal{W}'}(x) - 1$ and $\mathbf{B}_t$ swaps $x$ in for $y$ in $\min\{t, d_{\mathcal{W}'}(y) + 1 - d_{\mathcal{W}'}(x)\}$ sets of $\mathcal{W}'$. This reduces the contribution from $y$ without further increasing the contribution from $x$ and thereby maintains the invariant.

Now we prove sufficiency. Suppose the condition is not sufficient for $\mathbf{F}$ to have a winning strategy. Among all counterexamples having the fewest sets, choose $\mathcal{S}$ to maximize $|\bigcup \mathcal{S}|$.

First, suppose $|\bigcup \mathcal{S}| \geq \sum_{S \in \mathcal{S}} \eta(S)$. Let $G$ be the bipartite graph with parts $\mathcal{S}$ and $\bigcup \mathcal{S}$ and an edge from $S \in \mathcal{S}$ to $y \in \bigcup \mathcal{S}$ if and only if $y \in S$. Apply Lemma 4.3 to get a nonempty subgraph $G'$ of $G$ with parts $X' \subseteq \mathcal{S}$ and $Y' \subseteq \bigcup \mathcal{S}$ such that $d_{G'}(x) = \eta(x)$ for $x \in X'$, $d_{G'}(y) = 1$ for $y \in Y'$ and $N_G(Y') = X'$. Letting $Q = \bigcup \mathcal{S}$, the function $f \colon X' \to \mathcal{P}(Q)$ given by $f(S) = N_{G'}(S)$ is an $\eta$-transversal of $X'$ with $\bigcup_{x \in X'} f(x) = Y'$. Put $\mathcal{S}' = \mathcal{S} \setminus V(G')$ and $P' = P \setminus V(G')$. The hypotheses of the claim are satisfied by $\mathcal{S}'$ and $P'$. If $\mathbf{F}$ continues to play only using $\mathcal{S}'$ and $\mathcal{P}'$, then $\mathbf{B}_t$ cannot destroy the transversal of $X'$ that exists using elements of $Y'$, even though $\mathbf{B}_t$ may play on all of $\mathcal{S}$, because $\mathbf{F}$ will make no further move involving the elements in that $\eta$-transversal. Now minimality of $|\mathcal{S}|$ gives a contradiction.

Therefore we must have $|\bigcup \mathcal{S}| < \sum_{S\in\mathcal{S}} \eta(S)$ and hence $\nu_t(\mathcal{S}) \geq 1$. Since $|P| \geq \sum_{S\in\mathcal{S}} \eta(S)$, we have $x \in P$ with $d_{\mathcal{S}}(x) = 0$. So, we may choose $y \in P$ with $d_{\mathcal{S}}(y) \geq t+2$. Now $\mathbf{F}$ should swap $x$ in for $y$ in some $S \in \mathcal{S}$ to form $\mathcal{S}'$. Since $d_{\mathcal{S}}(x) = 0$, we have $|\bigcup \mathcal{S}'| > |\bigcup \mathcal{S}|$. We also have $d_{\mathcal{S}'}(y) \geq t+1$. Now $\mathbf{B}_t$ moves and creates $\mathcal{S}^*$. Since $d_{\mathcal{S}^*}(y) \geq d_{\mathcal{S}'}(y) - t > 0$, we have $|\bigcup \mathcal{S}^*| > |\bigcup \mathcal{S}|$. If our modifications changed some $\mathcal{W} \in \mathcal{S}$ so it now violates the hypotheses, then let $\mathcal{W}^*$ be $\mathcal{W}$ after both player's moves. That is, $|\bigcup \mathcal{W}^*| < \sum_{W\in\mathcal{W}^*} \eta(W) - \nu_t(\mathcal{W}^*)$.

Since $d_{\mathcal{S}}(x) = 0$, we have $|\bigcup \mathcal{W}^*| \geq |\bigcup \mathcal{W}|$. Thus $\nu_t(\mathcal{W}^*) < \nu_t(\mathcal{W})$ and hence $d_{\mathcal{W}^*}(y) < d_{\mathcal{W}}(y)$. Now $d_{\mathcal{W}^*}(x) > 0$, since any removed $y$ was replaced with $x$. Since $d_{\mathcal{W}}(x) = 0$, we have $|\bigcup \mathcal{W}^*| > |\bigcup \mathcal{W}|$.

This leads to $\sum_{W\in\mathcal{W}} \eta(W) - \nu_t(\mathcal{W}) \leq |\bigcup \mathcal{W}| < |\bigcup \mathcal{W}^*| < \sum_{W\in\mathcal{W}^*} \eta(W) - \nu_t(\mathcal{W}^*)$ and hence $\nu_t(\mathcal{W}^*) \leq \nu_t(\mathcal{W}) - 2$. This is impossible since $\mathbf{B}_t$ can perform swaps in at most $t$ sets. Since $\mathcal{S}^*$ satisfies the hypotheses of the claim, $|\bigcup \mathcal{S}^*| > |\bigcup \mathcal{S}|$ now implies that $\mathbf{F}$ can win. $\square$

## 5. The fan equation

In [7], Vizing proved Corollary 1.3 with an argument based on "fans". Taking this type of argument further proves Corollary 1.4—the so-called "fan equation" (see [6]). We show that Corollary 1.4 follows easily from Theorem 4.4.

**Corollary 1.4.** *Let $G$ be a multigraph satisfying $\chi'(G) \geq \Delta(G) + 1$. For each critical edge $xy$ in $G$, there exists $X \subseteq N(x)$ with $y \in X$ and $|X| \geq 2$ such that*

$$\sum_{v\in X} \left(d(v) + \mu(xv) + 1 - \chi'(G)\right) \geq 2.$$

*Proof.* Put $k = \chi'(G) - 1$. Consider a $k$-edge-coloring $\pi$ of $G - xy$. For $v \in N(x)$, let $M_v \subseteq [k]$ be the set of colors not incident to $v$ under $\pi$, and let $D_v$ be the set of colors on the edges from $x$ to $v$. The sets $D_v$ for $v \in N(x)$ are pairwise disjoint, $|D_v| = \mu(xv)$ for $v \in N(x) - \{y\}$, and $|D_y| = \mu(xy) - 1$. For $v \in N(x)$, put $S_v = M_v \cup D_v$. We obtain $|S_v| = |D_v| + |M_v| = k + \mu(xv) - d(v)$.

Now we translate the problem into our game. Put $\eta(S_v) = \mu(xv)$. If $v \in N(x)$ and $a \in S_v$ and $b \notin S_v$, then we may exchange colors on a longest path in $G - x$ starting at $v$ and alternating between colors $b$ and $a$. This translates into an $\mathbf{F}$ move followed by a $\mathbf{B}_1$ move in the $\eta$-game with sets $\mathcal{S}_{N(x)}$ where $\mathcal{S}_X = \{S_v \mid v \in X\}$ for $X \subseteq N(x)$. If $\mathbf{F}$ has a winning strategy in this $\eta$-game against $\mathbf{B}_1$, then we can extend the $k$-edge-coloring to all of $G$, giving a contradiction.

Therefore, by Theorem 4.4, we must have $X \subseteq N(x)$ with

$$\left|\bigcup_{v\in X} S_v\right| < \sum_{v\in X} \eta(S_v) - \nu_1(\mathcal{S}_X) = \sum_{v\in X} \mu(xv) - \nu_1(\mathcal{S}_X).$$

Since the sets $D_v$ for $v \in N(x)$ are pairwise disjoint, we have $\left|\bigcup_{v\in X} S_v\right| \geq -1 + \sum_{v\in X} \mu(xv)$, with equality only if $y \in X$. Hence $y \in X$ and $\nu_1(\mathcal{S}_X) = 0$. Since $\chi' \geq \Delta + 1$, we have $|S_y| = k + \mu(xy) - d(y) \geq \mu(xy)$ and hence $|X| \geq 2$. Since $\nu_1(\mathcal{S}_X) = 0$, each color is in at most two elements of $\mathcal{S}_X$. Therefore $\sum_{v\in X} (k + \mu(xv) - d(v)) = \sum_{v\in X} |S_v| \leq 2\left|\bigcup_{v\in X} S_v\right| \leq -2 + 2\sum_{v\in X} \mu(xv)$. The claim follows. $\square$

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## References


1. O. V. Borodin, A. V. Kostochka, and D. R. Woodall, *List edge and list total colourings of multigraphs*, J. Combin. Theory Ser. B **71** (1997), no. 2, 184–204.
2. A. Ehrenfeucht, V. Faber, and H. A. Kierstead, *A new method of proving theorems on chromatic index*, Discrete Mathematics **52** (1984), no. 2–3, 159–164.
3. F. Galvin, *The list chromatic index of a bipartite multigraph*, Journal of Combinatorial Theory, Series B **63** (1995), no. 1, 153–158.
4. P. Hall, *On representatives of subsets*, J. Lond. Math. Soc. **10** (1935), 26–30.
5. A. Schrijver, *Combinatorial Optimization: Polyhedra and Efficiency*, vol. A, Springer Verlag, 2003.
6. M. Stiebitz, D. Scheide, B. Toft, and L.M. Favrholdt, *Graph Edge Coloring: Vizing's Theorem and Goldberg's Conjecture*, vol. 75, Wiley, 2012.
7. V.G. Vizing, *On an estimate of the chromatic class of a p-graph*, Diskret. Analiz **3** (1964), no. 7, 25–30.