Short fans and the 5/6 bound for line graphs

Daniel W. Cranston Landon Rabern

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Abstract

In 2011, the second author conjectured that every line graph G satisfies $\chi(G) \le \max\left\{\omega(G), \frac{5\Delta(G)+8}{6}\right\}$. This conjecture is best possible, as shown by replacing each edge in a 5-cycle by k parallel edges, and taking the line graph. In this paper we prove the conjecture. We also develop more general techniques and results that will likely be of independent interest, due to their use in attacking the Goldberg–Seymour conjecture.

1 Overview

By *graph* we mean multigraph without loops. Our notation follows Diestel [1]. In [3], the second author showed that $\chi(G) \le \max\left\{\omega(G), \frac{7\Delta(G)+10}{8}\right\}$ for every line graph G. In the same paper, he conjectured that $\chi(G) \le \max\left\{\omega(G), \frac{5\Delta(G)+8}{6}\right\}$. This conjecture is best possible, as shown by replacing each edge in a 5-cycle by k parallel edges, and taking the line graph. In this paper we prove the latter inequality. Along the way, we develop more general techniques and results that will likely be of independent interest. The main result of this paper is the following theorem.

Theorem 16 ($\frac{5}{6}$-Theorem)**.** *If Q is a line graph, then*

$$\chi(Q) \le \max\left\{\omega(Q), \frac{5\Delta(Q)+8}{6}\right\}.$$

For every graph G, we have $\chi'(G) \ge \left\lceil \frac{\|G\|}{\left\lfloor \frac{|G|}{2} \right\rfloor} \right\rceil$, since each color class has size at most $\left\lfloor \frac{|G|}{2} \right\rfloor$. Likewise, the same bound holds for any subgraph H. Thus $\chi'(G) \ge \max_{H \subseteq G} \left\lceil \frac{\|H\|}{\left\lfloor \frac{|H|}{2} \right\rfloor} \right\rceil$ (where the max is over all subgraphs H with at least two vertices). For convenience, we let $\mathcal{W}(G) := \max_{H \subseteq G} \left\lceil \frac{\|H\|}{\left\lfloor \frac{|H|}{2} \right\rfloor} \right\rceil$. Goldberg [**?**, 2] and Seymour [**?**, **?**] each conjectured that this lower bound holds with equality, whenever $\chi'(G) > \Delta(G) + 1$.

Goldberg–Seymour Conjecture. *Every graph G satisfies*

$$\chi'(G) \leq \max\{\mathcal{W}(G), \Delta(G) + 1\}.$$

The Goldberg–Seymour conjecture is the major open problem in the area of edge-coloring graphs. Most of our work goes toward proving the following intermediate result, in Section 6.

Theorem 13 (Weak $\frac{5}{6}$-Theorem)**.** *If Q the line graph of a graph G, then*

$$\chi(Q) \leq \max\left\{\mathcal{W}(G), \Delta(G) + 1, \frac{5\Delta(Q) + 8}{6}\right\}.$$

Finally, in Section 7 we show that the Weak $\frac{5}{6}$-Theorem does indeed imply the $\frac{5}{6}$-Theorem.

2 Tashkinov Trees

A graph G is *elementary* if $\chi'(G) = \mathcal{W}(G)$. Let $[k]$ denote $\{1, \ldots, k\}$. For a path or cycle Q, let $\ell(Q)$ denote the length of Q. A graph G is *critical* if $\chi'(G - e) < \chi'(G)$ for all $e \in E(G)$. For a graph G and a partial k-edge-coloring φ, for each vertex $v \in V(G)$, let $\varphi(v)$ denote the set of colors used in φ on edges incident to v. Let $\overline{\varphi}(v) = [k] \setminus \varphi(v)$. A color c is *seen* by a vertex v if $c \in \varphi(v)$ and c is *missed* by v if $c \in \overline{\varphi}(v)$. Given a partial k-edge-coloring φ, a set $W \subseteq V(G)$ is *elementary* with respect to φ (henceforth, *w.r.t.* φ) if each color in $[k]$ is missed by at most one vertex of W. More formally, $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$ for all distinct $u, v \in W$. A *defective color* for a set $X \subseteq V(G)$ (w.r.t. φ) is a color used on more than one edge from X to $V(G) \setminus X$. A set X is *strongly closed* w.r.t. φ if X has no defective color. Elementary and strongly closed sets are of particular interest because of the following theorem, proved implicitly by Andersen [**?**] and Goldberg [2]; see also [**?**, Theorem 1.4].

Theorem 1. *Let G be a graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G)$. If G is critical, then G is elementary if and only if there exists $uv \in E(G)$, a k-edge-coloring φ of $G - uv$, and a set X with $u, v \in X$ such that X is both elementary and strongly closed w.r.t. φ.*

A *Tashkinov tree* w.r.t. φ is a sequence $v_0, e_1, v_1, e_2, \ldots, v_{t-1}, e_t, v_t$ such that all v_i are distinct, $e_i = v_j v_i$ and $\varphi(e_i) \in \overline{\varphi}(v_\ell)$ for some j and ℓ with $0 \leq j < i$ and $0 \leq \ell < i$. A *Vizing fan* (or simply *fan*) is a Tashkinov tree that induces a star. Tashkinov trees are of interest because of the following lemma.

Tashkinov's Lemma. *Let G be a graph with $\chi'(G) = k+1$, for some integer $k \geq \Delta(G)+1$ and choose $e \in E(G)$ such that $\chi'(G - e) < \chi'(G)$. Let φ be a k-edge-coloring of $G - e$. If T is a Tashkinov tree w.r.t. φ and e, then $V(T)$ is elementary w.r.t. φ.*

In view of Theorem 1 and Tashkinov's Lemma, to prove that a graph G is elementary, it suffices to find an edge e, a k-edge-coloring φ of $G - e$, and a Tashkinov tree T containing e

such that $V(T)$ is strongly closed. This motivates our next two lemmas. But first, we need a few more definitions.

Let $t(G)$ be the maximum number of vertices in a Tashkinov tree over all $e \in E(G)$ and all k-edge-colorings φ of $G - e$. Let $\mathcal{T}(G)$ be the set of all triples (T, e, φ) such that $e \in E(G)$, φ is a k-edge-coloring of $G - e$ and T is a Tashkinov tree with respect to e and φ with $|T| = t(G)$. Notice that, by definition, we have $\mathcal{T}(G) \neq \emptyset$. For a k-edge-coloring φ of $G - e$, a maximal Tashkinov tree starting with e may not be unique. However, if T_1 and T_2 are both such trees, then it is easy to show that $V(T_1) \subseteq V(T_2)$; by symmetry, also $V(T_2) \subseteq V(T_1)$, so $V(T_1) = V(T_2)$. Let G be a critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G)+1$. Let φ be a k-edge-coloring of $G - e_0$ for some $e_0 \in E(G)$. For $v \in V(G)$ and colors α, β, let $P_v(\alpha, \beta)$ be the maximal connected subgraph of G that contains v and is induced by edges with color α or β. So $P_v(\alpha, \beta)$ is a path or a cycle. For a k-edge-coloring φ of $G - v_0v_1$, we often let $P = P_{v_1}(\alpha, \beta)$ for some $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Clearly P must end at v_0 (or we can swap colors α and β on P and color v_0v_1 with α), so let $v_1, \ldots, v_r, v_0$ denote the vertices of P in order. To *rotate the α, β coloring on $P \cup \{v_0v_1\}$ by one*, we uncolor v_1v_2 and use its color on v_0v_1. To *rotate the α, β coloring on $P \cup \{v_0v_1\}$ by j*, we rotate the α, β coloring by one j times in succession. (When we do not specify j, we allow j to take any value from 1 to r.)

Lemma 2. *Let G be a non-elementary critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. For every $v_0v_1 \in E(G)$, k-edge-coloring φ of $G - v_0v_1$, $\alpha \in \overline{\varphi}(v_0)$, and $\beta \in \overline{\varphi}(v_1)$, we have $|P_{v_1}(\alpha, \beta)| < t(G)$.*

Proof. Suppose the lemma is false and choose $v_0v_1 \in E(G)$, a k-edge-coloring φ of $G - v_0v_1$, $\alpha \in \overline{\varphi}(v_0)$, and $\beta \in \overline{\varphi}(v_1)$, such that $|P_{v_1}(\alpha, \beta)| \geq t(G)$. Let $P = P_{v_1}(\alpha, \beta)$. Let (T, v_0v_1, φ) be a Tashkinov tree that begins with edges $v_0v_1, v_1v_2, \ldots, v_{r-1}v_r$. Now $V(T) = V(P)$ since $t(G) \geq |T| \geq |P| \geq t(G)$. By hypothesis G is non-elementary, so Theorem 1 implies that $V(T)$ is not strongly closed; thus, T has a defective color δ with respect to φ. Choose $\tau \in \overline{\varphi}(v_2)$. Let $Q = P_{v_2}(\tau, \delta)$. Since T is maximal, δ is not missing at any vertex of T, and since $V(T)$ is elementary, τ is not missing at any vertex of T other than v_2. As a result, Q ends outside $V(T)$. Now Q could leave $V(T)$ and re-enter it repeatedly, but Q ends outside $V(T)$, so there is a last vertex $w \in V(Q) \cap V(T)$; say Q ends at $z \in V(G) \setminus V(T)$. Let $\pi \notin \{\alpha, \beta\}$ be a color missing at w. Since $\tau \in \overline{\varphi}(v_2)$ and $\pi \in \overline{\varphi}(w)$ and $|T| = t(G)$, no edge colored τ or π leaves $V(T)$. So we can swap τ and π on every edge in $G - V(T)$ without changing the fact that T is a Tashkinov tree with $|T| = t(G)$. After swapping τ and π, we swap δ and π on the subpath of Q from w to z. Since π is missing at w, the $\delta - \pi$ path starting at z must end at w. Now δ is missing at w, but δ was defective in φ, so some other edge e colored δ still leaves $V(T)$. Adding e gets a larger Tashkinov tree, which is a contradiction. $\square$

3 Short vertices

A vertex $v \in V(G)$ is *short* if every Vizing fan rooted at v (taken over all k-colorings of $G - e$, over all edges e incident to v) has at most 3 vertices, including v. Otherwise, v is *long*. Let $\nu(T)$ be the number of long vertices in a Tashkinov tree T.

Now we can outline our proof of the $\frac{5}{6}$-Conjecture. We will show in Section **??** that the $\frac{5}{6}$-Conjecture is implied by the Goldberg–Seymour Conjecture. More precisely, if G is a graph such that $\chi'(G) \le \max\{\mathcal{W}(G), \Delta(G)+1\}$, then also $\chi'(G) \le \frac{5\Delta(G)+8}{6}$. So here it suffices to show that $\chi'(G) \le \max\left\{\mathcal{W}(G), \Delta(G)+1, \frac{5\Delta(G)+8}{6}\right\}$. We consider cases based on $\nu(T)$, for some Tashkinov tree $T \in \mathcal{T}(G)$.

In the present section, we show that if G has a maximum Tashkinov tree T that contains no short vertices, i.e., $\nu(T) = 0$, then G is elementary. In fact, Lemma 7 implies that the same is true when $\nu(T) = 1$. In the proof of Theorem 20, we show that if G is a minimal counterexample to the $\frac{5}{6}$-Conjecture, then every long vertex v has $d(v) < \frac{3}{4}\Delta(G)$. This implies that $\nu(T) < 4$, since otherwise the number of colors missing at vertices of T is more than $4(k - \frac{3}{4}\Delta(G)) > k$, which contradicts that $V(T)$ is elementary. So it remains to consider the case $\nu(T) \in \{2, 3\}$.

In Section 6, we introduce the notion of *k-thin graphs*, which are essentially those for which $\mu(G)$ is not too large. Using a lemma from [3], we show that every minimal counterexample to the $\frac{5}{6}$-Conjecture must be k-thin. We then extend the ideas of the present section to handle the case when $\nu(T) \in \{2, 3\}$. Much like when $\nu(T) \ge 4$, we show that T has too many colors missing at its vertices to be elementary.

Short vertices were introduced in [**?**], where they were motivated by a version of the following lemma in the context of proving a strengthening of Reed's Conjecture for line graphs.

Lemma 3. *Let G be a critical graph with $\chi'(G) = k + 1$ for some integer $k \ge \Delta(G) + 1$. Let φ be a k-edge-coloring of $G - v_0v_1$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Let $P = v_1v_2\cdots v_r$ be an α, β path with edges $e_i = v_iv_{i+1}$ for all $i \in [r-1]$. If v_i is short for all odd i, then for each $\tau \in \overline{\varphi}(v_0)$ there are edges $f_i = v_iv_{i+1}$ for all $i \in [r-1]$ such that $f_i = e_i$ for i even and $\varphi(f_i) = \tau$ for i odd.*

Proof. Suppose not and choose a counterexample minimizing r. By minimality of r, we have $\varphi(v_{r-1}v_r) = \alpha$ and we have $f_i = v_iv_{i+1}$ for all $i \in [r-2]$ such that $f_i = e_i$ for i even and $\varphi(f_i) = \tau$ for i odd. Swap α and β on e_i for all $i \in [r-3]$ and then color v_0v_1 (call this edge e_0) with α and uncolor e_{r-2}. Let φ' be the resulting coloring. Since $k \ge \Delta(G) + 1$, some color other than α is missing at v_{r-2}; let γ be such a color. Now v_{r-1} is short since $r - 1$ is odd (since P starts and ends with α), so there is an edge $e = v_{r-1}v_r$ with $\varphi'(e) = \gamma$. Swap τ and α on e_i for all i with $0 \le i \le r - 3$ to get a new coloring φ^*. Now γ and τ are both missing at v_{r-2} in φ^*. Since v_{r-1} is short, the fan with v_{r-2}, v_{r-1}, v_r and e implies that there is an edge $f_{r-1} = v_{r-1}v_r$ with $\varphi^*(f_{r-1}) = \tau$. But we have never recolored f_{r-1}, so $\varphi(f_{r-1}) = \tau$, which is a contradiction. $\square$

Lemma 4. *Let G be a non-elementary critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. Choose $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$ for some $v_0v_1 \in E(G)$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $P = P_{v_1}(\alpha, \beta)$. Now P contains a long vertex. In particular, $\nu(T) \geq 1$.*

Proof. Suppose every vertex of P is short. Applying Lemma 3 to P shows that for every $\tau \in \overline{\varphi}(v_0)$, there is an edge in T colored τ incident to every $v \in V(P - v_0)$. The same is also true of every $v \in V(P)$; to see this, we rotate the α, β coloring of $P \cup \{v_0v_1\}$ and repeat the same argument. Hence $V(P) = V(T)$, which contradicts Lemma 2. □

Theorem 5. *If G is a critical graph in which every vertex is short, then*

$$\chi'(G) \leq \max\{\mathcal{W}(G), \Delta(G)+1\}.$$

Proof. Suppose not and let G be a counterexample. Let $k = \chi'(G) - 1$, and note that $k \geq \Delta(G)+1$. Since $\mathcal{T}(G) \neq \emptyset$, by applying Lemma 4 we conclude that G is elementary. Hence $\chi'(G) = \mathcal{W}(G)$, which is a contradiction. □

4 An easy bound

In this section, we apply the results of Section 3 to prove an easy bound on $\chi'(G)$. We also show how those results imply Reed's Conjecture, as well as Local and Superlocal strengthenings of Reed's Conjecture, for the class of line graphs.

Let G be a graph. The *claw-degree* of $x \in V(G)$ is

$$d_{\text{claw}}(x) := \max_{\substack{S \subseteq N(x) \\ |S|=3}} \frac{1}{4}\left(d(x) + \sum_{v \in S} d(v)\right),$$

where $d_{\text{claw}}(x) := 0$ when $|N(x)| \leq 2$. The *claw-degree* of G is

$$d_{\text{claw}}(G) := \max_{x \in V(G)} d_{\text{claw}}(x).$$

Theorem 6. *If G is a graph, then*

$$\chi'(G) \leq \max\left\{\mathcal{W}(G), \Delta(G)+1, \left\lceil \frac{4}{3} d_{claw}(G) \right\rceil\right\}.$$

Proof. Suppose not and choose a counterexample G with the fewest edges; note that G is critical. Let $k = \chi'(G) - 1$, so $k \geq \left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil$. By Theorem 5, G has a long vertex x. Choose $xy_1 \in E(G)$ and a k-edge-coloring φ of $G - xy_1$ such that φ has a fan F of length 3 rooted at x with leaves y_1, y_2, y_3. Since $V(F)$ is elementary,

$$2 + k - d(x) + \sum_{i \in [3]} k - d(y_i) \leq k,$$

and hence

$$d_{\text{claw}}\,(x) \geq \frac{1}{4}\left(d(x) + \sum_{i\in[3]} d(y_i)\right) \geq \frac{3k+2}{4}.$$

This gives the contradiction

$$\left\lceil \frac{4}{3} d_{\text{claw}}\,(G) \right\rceil \leq k \leq \frac{4}{3} d_{\text{claw}}\,(G) - \frac{2}{3}.$$

□

Reed [4] conjectured that $\chi(G) \leq \left\lceil \frac{\omega(G)+\Delta+1}{2} \right\rceil$ for every graph G. This is the average of a trivial lower bound $\omega(G)$ and a trivial upper bound $\Delta(G)+1$. King [**?**] conjectured the stronger bound $\chi(G) \leq \max_{v\in V(G)} \left\lceil \frac{\omega(v)+d(v)+1}{2} \right\rceil$, where $\omega(v)$ is the size of the largest clique containing v, which is now known to hold for many classes of graphs, including line graphs [**?**]. Here we show that for line graphs this bound is an easy consequence of our more general lemmas from Section 3.

Theorem 7. *If G is a critical graph that is not a thickened cycle, then*

$$\chi'(G) \leq \max\left\{\Delta(G)+1, \left\lceil \frac{4}{3} d_{claw}\,(G) \right\rceil \right\}.$$

Proof. Suppose not and let G be a counterexample. By Theorem 6, G is elementary. Since G is critical and elementary, $|G|$ is odd and

$$k = \frac{2(\|G\|-1)}{|G|-1}. \tag{1}$$

Let $x \in V(G)$ with $|N(x)| \geq 3$. Put $M := |N(x)|$,

$$P := \sum_{v\in N(x)} d_G(v),$$

$$S_2 := 2 + \sum_{v\in V(G)\setminus N(x)} \Delta(G) - d_G(v),$$

$$S_3 := k - (\Delta(G)+1).$$

Then

$$2(\|G\|-1) = \Delta(G)(|G|-M) - S_2 + P. \tag{2}$$

Since

$$\frac{2(\|G\|-1)}{|G|-1} = k = \Delta(G)+1+S_3,$$

using (6), we get

$$P = (|G|-1)(\Delta(G)+1+S_3) - \Delta(G)(|G|-M) + S_2,$$

which is

$$P = \Delta(G)(M-1) + |G| - 1 + S_2 + S_3(|G|-1). \tag{3}$$

Let $N(x) = \{v_0, v_1, \ldots, v_{M-1}\}$ and consider

$$R := \sum_{i=0}^{M-1} \frac{1}{3}\left(d_G(x) + d_G(v_i) + d_G(v_{i+1}) + d_G(v_{i+2})\right),$$

where indices are taken modulo M. Since $k \geq \frac{4}{3} d_{\text{claw}}(G)$, there is $S_4 \geq 0$ such that

$$Mk - S_4 = R = \frac{M}{3} d_G(x) + P,$$

so

$$Mk = \frac{M}{3} d_G(x) + \Delta(G)(M-1) + |G| - 1 + S_2 + S_3(|G|-1) + S_4,$$

which gives, for some $S_5 \geq 0$,

$$3 + S_5 = M = \frac{S_2 + (S_3+1)(|G|-1) + S_4 - \Delta(G)}{S_3 + 1 - \frac{1}{3} d_G(x)},$$

hence

$$(3+S_5)(S_3+1) - \left(1 + \frac{S_5}{3}\right) d_G(x) = S_2 + (S_3+1)(|G|-1) + S_4 - \Delta(G),$$

so

$$\left(1 + \frac{S_5}{3}\right) d_G(x) = \Delta(G) - (S_2 - 2) + (4 + S_5 - |G|)\, S_3 + (2 + S_5 - |G|) - S_4.$$

Suppose $4 + S_5 - |G| \leq 0$. Then

$$\left(1 + \frac{S_5}{3}\right) d_G(x) \leq \Delta(G) - (S_2 - 2) - 2.$$

Now $S_2 \geq 2 + \Delta(G) - d_G(x)$, so we have

$$\left(1 + \frac{S_5}{3}\right) d_G(x) \leq d_G(x) - 2,$$

a contradiction since $S_5 \geq 0$. So, we must have $4 + S_5 - |G| > 0$, that is,

$$|G| \leq S_5 + 3 = |N(x)| \leq |G| - 1,$$

a contradiction. □

For a graph Q and $r \in \mathbb{N} \cup \{\infty\}$, put

$$\mathcal{C}_r(Q) := \{X \subseteq V(Q) : X \text{ is a maximal clique with } |X| < r \text{ or } X \text{ is a clique with } |X| = r\}.$$

Put

$$\gamma_r(Q) := \max_{X \in \mathcal{C}_r(Q)} \frac{1}{|X|} \sum_{v \in X} \frac{d(v) + \omega(v) + 1}{2}.$$

In terms of γ_r, the local version of Reed's conjecture for line graphs proved by Chudnovsky et al. and the superlocal version proved by King and Edwards are the following two theorems.

Theorem 8 (Chudnovsky et al.)**.** *If Q is a line graph, then $\chi(Q) \leq \lceil \gamma_1(Q) \rceil$.*

Theorem 9 (King and Edwards)**.** *If Q is a line graph, then $\chi(Q) \leq \lceil \gamma_2(Q) \rceil$.*

Note that if $a, b \in \mathbb{N} \cup \{\infty\}$ with $a \leq b$, then $\gamma_a(Q) \geq \gamma_b(Q)$, so Theorem 9 implies Theorem 8. We prove the next bound in the sequence.

Theorem 10. *If Q is a line graph, then $\chi(Q) \leq \lceil \gamma_3(Q) \rceil$.*

King and Edwards conjectured the fractional version of the following bound without the round-up for all graphs.

Conjecture 11. *If Q is a line graph, then $\chi(Q) \leq \lceil \gamma_\infty(Q) \rceil$.*

It is not obvious that γ_r is monotone, so let's prove that first.

Lemma 12. *If A and B are graphs with $A \subseteq B$ and $r \in \mathbb{N} \cup \{\infty\}$, then $\gamma_r(A) \leq \gamma_r(B)$.*

Proof. This is true for all graphs on at most 8 vertices and line graphs of graphs on at most 8 vertices. Don't have a proof yet. Is this obvious? □

Lemma 13. *Let $Q = L(G)$ where G is a critical graph. If G is not a thickened cycle, then $\chi(Q) \leq \lceil \gamma_3(Q) \rceil$.*

Proof. Suppose G is not a thickened cycle. For $uv \in E(G)$, put

$$f(uv) := \max\{d_G(u) + \frac{1}{2}(d_G(v) - \mu(uv)), d_G(v) + \frac{1}{2}(d_G(u) - \mu(uv)\}.$$

For $uv \in E(G)$, we have

$$\begin{aligned} f(uv) &= \frac{d_G(u) + d_G(v) - \mu(uv) + \max\{d_G(u), d_G(v)\}}{2} \\ &\leq \frac{d_Q(uv) + \omega(uv) + 1}{2}. \end{aligned}$$

Since G is critical, we have $|N(v)| \geq 2$ for all $v \in V(G)$. Since G is not a thickened cycle, we may choose $x \in V(G)$ and $S \subseteq N(x)$ with $|S| = 3$ such that x and S achieve maximality in the definition of $d_{claw}(G)$. Say $S = \{v_1, v_2, v_3\}$. Then

$$\begin{aligned}
\left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil &= \left\lceil \left(\frac{4}{3}\right)\left(\frac{1}{4}\right)\left(d_G(x) + \sum_{i\in[3]} d_G(v_i)\right)\right\rceil \\
&\leq \left\lceil \frac{1}{3}\sum_{i\in[3]} d_G(v_i) + \frac{1}{2}(d_G(x) - \mu(xv_i))\right\rceil \\
&\leq \left\lceil \frac{1}{3}\sum_{i\in[3]} f(xv_i)\right\rceil \\
&\leq \left\lceil \frac{1}{3}\sum_{i\in[3]} \frac{d_Q(xv_i) + \omega(xv_i) + 1}{2}\right\rceil \\
&\leq \lceil \gamma_3(Q) \rceil .
\end{aligned}$$

By Theorem 7, we have

$$\chi(Q) \leq \max\left\{\Delta(G) + 1, \lceil \gamma_3(Q) \rceil\right\}. \tag{4}$$

Let $M \subseteq V(Q)$ be a maximum clique in Q. Then $|M| \geq 3$, so we can choose $X \subseteq M$ with $|X| = 3$ maximizing

$$\frac{1}{3}\sum_{v\in X} \frac{d(v) + \omega(v) + 1}{2}.$$

We have

$$\begin{aligned}
\gamma_3(Q) &\geq \frac{1}{3}\sum_{v\in X} \frac{d(v) + \omega(v) + 1}{2} \\
&\geq \frac{1}{|M|}\sum_{v\in M} \frac{d(v) + \omega(v) + 1}{2} \\
&\geq \omega(Q) + \sum_{v\in M} \frac{d(v) + 1 - \omega(v)}{2}.
\end{aligned}$$

If $V(Q) = M$, then $\lceil \gamma_3(Q) \rceil = \omega(Q) = \Delta(Q) = \chi(Q)$, as desired. Otherwise, some $v \in M$ has $d(v) \geq \omega(Q)$ and hence $\lceil \gamma_3(Q) \rceil \geq \omega(Q) + 1 \geq \Delta(Q) + 1$. Using (4), this gives $\chi(Q) \leq \lceil \gamma_3(Q) \rceil$. □

Proof of Theorem 11. Suppose Theorem 11 is false, and choose a counterexample Q minimizing $|Q|$. Say $Q = L(G)$. Minimality of $|Q|$ and Lemma 12 imply that G is critical. So Lemma 13 gives a contradiction unless G is a thickened cycle. We should be able to finish, for example if King and Edwards conjecture is true (that $\gamma_\infty(Q)$ is an upper bound on fractional chromatic number), then since G is a circular-interval graph, it has the round-up property and so $\chi(Q) \leq \lceil \gamma_\infty(Q) \rceil \leq \gamma_3(Q)$. But it should be simpler than that, we can compute the chromatic number directly. $\square$

5 Properties of long vertices

For a path Q, recall that $\ell(Q)$ denotes the length of Q. For $x, y \in V(Q)$, let xQy denote the subpath of Q with endvertices x and y, and let $d_Q(x,y) = \ell(xQy)$, i.e., the distance from x to y along Q.

Lemma 14. *Let G be a critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. Let φ be a k-edge-coloring of $G - v_0v_1$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $C = P_{v_1}(\alpha,\beta) + v_0v_1$. If $\tau \in \overline{\varphi}(x)$ for some $x \in V(C)$ and there is a τ-colored edge from $y \in V(C)$ to $w \in V(G) \setminus V(C)$, then C has a subpath Q with long endpoints z_1, z_2 such that $x \in V(Q)$, $y \notin V(Q - z_1 - z_2)$ and the distance from x to z_i along Q is odd for each $i \in [2]$. Moreover, for each $i \in [2]$, there are no τ-colored edges between z_i and its neighbors along C.*

Proof. Let G, α, β, τ, x, and y be as in the statement of the lemma. Choose z_1 (resp. z_2) to be the first vertex at an odd distance from x along C in the clockwise (resp. counterclockwise) direction with no incident τ-colored edge parallel to some edge of C. Let Q be the subpath of C with endpoints z_1 and z_2 that contains x. By the choice of z_1 each vertex w between x and z_1 with $d_Q(xw)$ odd has a τ-colored edge parallel to some edge of C. The presence of these edges implies the same for each w for which $d_Q(xw)$ is even. By the proof of the Parallel Edge Lemma, z_1 must be long, since otherwise it would have an incident τ-colored edge parallel to some edge of C. The same argument applies to z_2. $\square$

6 Thin graphs

Let G be a critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. For vertices $x \in V(G)$ and $S \subseteq V(G) \setminus \{x\}$, we say that x is *S-short* if every Vizing fan F rooted at x with $S \subseteq V(F)$, has $|F| \leq 3$ (with respect to any k-edge-coloring of $G - xy$). Otherwise, x is *S-long*. For brevity, when $S = \{y\}$, we may write y-short instead of $\{y\}$-short. It is worth noting that in Lemma 3 we can weaken the hypothesis that v_i is short for all odd i to require only that v_i is v_{i-1}-short for all odd i, since this is what we use in the proof.

A graph G is *k-thin* if $\mu(G) < 2k - d(x) - d(y)$ for all long $x, y \in V(G)$. In the proof of Theorem 20, we will show that every counterexample to the $\frac{5}{6}$-Conjecture must be k-thin.

Lemma 15. *Let G be a k-thin, critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G)+1$. Let φ be a k-edge-coloring of $G - v_0v_1$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $C = P_{v_1}(\alpha,\beta) + v_0v_1$. Let Q be a subpath of C with long end vertices. If all internal vertices of Q are short and $2 \leq \ell(Q) \leq \ell(C) - 2$, then $\ell(Q)$ is even.*

Proof. Suppose to the contrary that we have a subpath Q of C with end vertices long, all internal vertices short, $2 \leq \ell(Q) \leq \ell(C) - 2$, and $\ell(Q)$ odd. Let x and y be the end vertices of Q. Say $C = v_1v_2 \cdots v_rv_0v_1$. By rotating the α, β coloring of C, we may assume that $x = v_0$ and $y = v_a$, where $a \geq 3$ is odd.

We now apply Lemma 3 twice, to show that $\mu(v_1v_2) \geq 2k - d(v_0) - d(v_a)$, which contradicts that G is k-thin. More specifically, assume that the edges $v_0v_1, v_1v_2, \ldots$ go clockwise around C. We apply Lemma 3 once going clockwise starting from v_0 and once going counterclockwise starting from v_a. The first application implies that every color in $\overline{\varphi}(v_0)$ appears on some edge parallel to v_1v_2; the second implies the same for every color in $\overline{\varphi}(v_a)$. Since $|\overline{\varphi}(v_i)| = k - d(v_i)$ for each $i \in \{0, a\}$ and $\overline{\varphi}(v_0) \cap \overline{\varphi}(v_a) = \emptyset$, the conclusion follows. □

Lemma 16. *Let G be a k-thin, critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. Let φ be a k-edge-coloring of $G - v_0v_1$. Suppose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. If C contains exactly 3 long vertices, then $C = xyAzBx$ where A and B are paths of even length and x, y, z are all long. Moreover, x is y-long and y is x-long.*

Proof. Let G be a graph satisfying the hypotheses, and let x, y, z be the three long vertices. The three subpaths of C with endpoints x, y, and z either (i) all have odd length or (ii) include two paths of even length and one of odd length. First assume that $\ell(C) \geq 5$. If we are in (i), then the longest of these three subpaths violates Lemma 15; so we are in (ii), and also the path of odd length is simply an edge. This proves the first statement. For the second statement, assume to the contrary that x is y-short. By rotating the α, β coloring, we can assume that $y = v_0$ and $x = v_1$. As in the previous lemma, we use Lemma 3 (and the comment in the first paragraph of Section 6) to conclude that $\mu(v_1v_2) \geq 2k - d(v_0) - d(z)$. As above, this contradicts that G is k-thin; this contradiction proves the second statement. □

Lemma 17. *Let G be a non-elementary, k-thin, critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. Choose $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$. If $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$, then $P_{v_1}(\alpha, \beta) + v_0v_1$ contains consecutive long vertices.*

Proof. Let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. By Lemma 2, there is $x \in V(C)$ and $\tau \in \overline{\varphi}(x)$ such that there is a τ-colored edge from $y \in V(C)$ to $w \in V(T) \setminus V(C)$. Lemma 14 implies that C has a subpath Q with $x \in V(Q)$ and long endpoints z_1, z_2 such that the distance from x to z_i along Q is odd for each $i \in [2]$. Let Q' be the subpath of C with endpoints z_1 and z_2 that does not contain x. Since C is an odd cycle, $\ell(Q')$ is odd. Let Q^* be a minimum length subpath of Q' with long ends. Now $\ell(Q^*) = 1$ by Lemma 15, as desired. □

Lemma 18. *Let G be a non-elementary, k-thin, critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. If $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$ and $\nu(T) \leq 3$, then T contains long vertices z_1, z_2, z_3 such that either*

1. *z_1 is $\{z_2, z_3\}$-long and z_2 is z_1-long; or*
2. *z_i is z_j-long and z_j is z_i-long for each $(i, j) \in \{(1, 2), (2, 3)\}$.*

Proof. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ so that $P_{v_1}(\alpha, \beta)$ contains as many long vertices as possible; let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. By Lemma 2, there is $x \in V(C)$ and $\tau \in \overline{\varphi}(x)$ such that there is a τ-colored edge from $y \in V(C)$ to $w \in V(T) \setminus V(C)$. By Lemma 17, C has at least two long vertices.

First suppose that C contains only 2 long vertices, z_1 and z_2. By Lemma 17, z_1 and z_2 are consecutive on C. Lemma 14 implies that C has a subpath Q with endpoints z_1, z_2 such that $x \in V(Q)$ and $y \notin V(Q - z_1 - z_2)$ and for each $i \in [2]$ there are no τ-colored edges between z_i and its neighbors on C. By rotating the α, β coloring of C, we can assume that $x = v_0$ and $\alpha, \tau \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Note that $P_{v_1}(\tau, \beta)$ must end at v_0 (since otherwise we can recolor the Kempe chain and color v_0v_1 with τ). Let $C' = P_{v_1}(\tau, \beta) + v_0v_1$. Note that C' must include v_1Qz_1 and also v_0Qz_2 (the β-colored edges are present by definition and the τ-colored edges are present by the Parallel Edge Lemma). Thus, $z_1, z_2 \in V(C')$. Since z_1 and z_2 are not consecutive on C' and C' contains no other long vertices by the maximality condition on C, Lemma 17 gives a contradiction.

So instead C contains exactly 3 long vertices, z_1, z_2, and z_3. By Lemma 16, $C = z_1z_2Az_3Bz_1$ where A and B are paths of even length. Also, z_1 is z_2-long and z_2 is z_1-long.

By Lemma 14, C has a subpath Q with endpoints z_1, z_3 and with $x \in V(Q)$ and $y \notin V(Q - z_1 - z_3)$ such that there are no τ-colored edges between z_i and its neighbors along C for each $i \in \{1, 3\}$ (it could happen that z_3 has a τ-colored edge parallel to an edge of C, so the endpoints of Q are z_1, z_2, but now we get a contradiction as in the previous case, by letting $C' = P_{v_1}(\tau, \beta) + v_0v_1$). By rotating the α, β coloring of C, we may assume that $x = v_0$. Again, let $C' = P_{v_1}(\tau, \beta) + v_0v_1$. We know that C' contains z_1 and z_3 and that z_1 and z_2 are not consecutive on C'. Note also that all long vertices in $V(C')$ must be among z_1, z_2, z_3, since otherwise $\nu(T) \geq 4$, contradicting our hypothesis. So by Lemma 17, either z_1 and z_3 are consecutive on C' or z_2 and z_3 are consecutive on C'.

Suppose that z_2 and z_3 are consecutive on C', and thus connected by a τ-colored edge. Now applying Lemma 16 shows that z_2 is z_3-long and z_3 is z_2-long, so we satisfy (2) in the conclusion of the lemma (by swapping the names of z_1 and z_2).

So instead z_1 and z_3 must be consecutive on C', and thus connected by a τ-colored edge. If $z_1 = v_1$, then we have a fan with an α-colored edge from z_1 to z_2 and a τ-colored edge from z_1 to z_3, so z_1 is $\{z_2, z_3\}$-long.

Now assume that $z_1 \neq v_1$. Let z_1' be the predecessor of z_1 on the path from v_0 (through v_1) to z_1. We can shift the coloring so that $z_1'z_1$ is uncolored and z_1z_2 is colored α (as in the proof of the Parallel Edge Lemma). In fact, we can shift either the α, β edges or the τ, β edges. This gives the options that either $\alpha \in \overline{\varphi}(z_1')$ or $\tau \in \overline{\varphi}(z_1')$, whichever we prefer. Suppose we shift the τ, β edges. Now choose $\gamma \in \overline{\varphi}(z_1') - \alpha - \tau$. Consider the γ-colored edge e incident to z_1. If e goes to z_2, then we z_1 is $\{z_2, z_3\}$-long, by colors γ and τ; so we satisfy (1) in the conclusion of the lemma. If instead e goes to z_3, then instead of shifting the τ, β edges we shift the α, β edges; note that this recoloring preserves the fact that γ is missing at z_1'. Now again z_1 is $\{z_2, z_3\}$-long, this time by colors α and γ; so we again satisfy (1) in the conclusion of the lemma.

Finally, assume that the γ-colored edge incident to z_1 goes to some vertex other than z_2

and z_3. Now let $C'' = P_{z_1}(\gamma, \beta) + z_1 z_1'$. Since $V(C'') \subseteq V(T)$, Lemmas 17 and 16 imply that z_2 and z_3 are adjacent on C'' and furthermore z_2 is z_3-long and z_3 is z_2-long; thus, we satisfy (2) in the conclusion of the lemma. □

We need the following result from [3], which we use to handle the case when G is not k-thin.

Theorem 19 ([3])**.** *If Q is the line graph of a graph G and Q is vertex critical, then*

$$\chi(Q) \leq \max\left\{\omega(Q), \Delta(Q) + 1 - \frac{\mu(G) - 1}{2}\right\}.$$

Now we prove the main result of this section.

Theorem 20. *If Q is the line graph of G, then*

$$\chi(Q) \leq \max\left\{\mathcal{W}(Q), \Delta(G) + 1, \frac{5\Delta(Q) + 8}{6}\right\}.$$

Proof. Suppose the theorem is false and choose a counterexample minimizing $|Q|$. Let $k = \max\left\{\mathcal{W}(Q), \Delta(G) + 1, \frac{5\Delta(Q)+8}{6}\right\}$. Say $Q = L(G)$ for a graph G. The minimality of Q implies that G is critical and $\chi(Q) = k + 1$, for some $k \geq \Delta(G) + 1$.

The heart of the proof is Claim 1, which roughly says that if x is long, then $d(x) < \frac{3}{4}\Delta(G)$. Moreover, we can improve this bound further if x is the root of a long fan F such that either (i) F has length more than 3 or (ii) some of the other vertices in F have degree less than $\Delta(G)$. The claims thereafter are all essentially applications of Claim 1.

Claim 1. *Let F be a fan rooted at x with respect to a k-edge-coloring of $G - xy$. If $S \subseteq V(F) - x$ and $|S| \geq 3$, then*

$$d(x) \leq \frac{1}{5|S| - 11}\left(2|S| - 12 + \sum_{v \in S} d(v)\right).$$

In particular, if $|S| = 3$, then $d(x) \leq \frac{1}{4}\left(-6 + \sum_{v \in S} d(v)\right)$.
<u>Proof:</u> Since F is elementary, we have

$$2 + k - d(x) + \sum_{v \in S} k - d(v) \leq k,$$

so

$$2 + |S|k \leq d(x) + \sum_{v \in S} d(v).$$

Using $k \geq \frac{5}{6}(\Delta(Q)+1) - \frac{1}{3} \geq \frac{5}{6}(d(x)+d(v)-\mu(xv)) - \frac{1}{3}$ for each $v \in S$, we get

$$2 + \sum_{v \in S} \left(\frac{5}{6}(d(x)+d(v)-\mu(xv)) - \frac{1}{3} \right) \leq d(x) + \sum_{v \in S} d(v),$$

so multiplying by 6 and rearranging terms gives

$$12 + (5|S|-6)\, d(x) - 2|S| \leq \sum_{v \in S} 5\mu(xv) + \sum_{v \in S} d(v).$$

Now $\sum_{v \in S} \mu(xv) \leq d(x)$, so this implies

$$12 + (5|S|-11)\, d(x) - 2|S| \leq \sum_{v \in S} d(v).$$

Solving for $d(x)$ gives

$$d(x) \leq \frac{1}{5|S|-11} \left(2|S| - 12 + \sum_{v \in S} d(v) \right),$$

and when $|S| = 3$, we get $d(x) \leq \frac{1}{4}\left(-6 + \sum_{v \in S} d(v)\right)$. □

Claim 2. *If $x \in V(G)$ is long, then $d(x) \leq \frac{3}{4}\Delta(G) - 1$.*
Proof: This is immediate from Claim 1, since $d(v) \leq \Delta(G)$ for all $v \in S$. □

Claim 3. *If $x_1x_2 \in E(G)$ such that x_1 is x_2-long and x_2 is x_1-long, then*

$$d(x_i) \leq \frac{2}{3}\Delta(G) - 2 \textit{ for all } i \in [2].$$

Proof: By Claim 1, for each $i \in [2]$,

$$d(x_i) \leq \frac{1}{4}\left(-6 + \sum_{v \in S} d(v)\right) \leq \frac{1}{4}\left(-6 + d(x_{3-i}) + 2\Delta(G)\right),$$

Substituting the bound on $d(x_{3-i})$ into that on $d(x_i)$ and simplifying gives for each $i \in [2]$,

$$d(x_i) \leq -2 + \frac{2}{3}\Delta(G).$$

□

Claim 4. *If $x_1x_2, x_1x_3 \in E(G)$ such that x_1 is $\{x_2, x_3\}$-long, x_2 is x_1-long and x_3 is long, then*

$$d(x_1) \leq -\frac{8}{5} + \frac{3}{5}\Delta(G),$$

$$d(x_2) \le -\frac{7}{5} + \frac{13}{20}\Delta(G).$$

Proof: By Claim 1, we have

$$d(x_1) \le \frac{1}{4}\left(-6 + \sum_{v\in S} d(v)\right) \le \frac{1}{4}\left(-6 + d(x_2) + d(x_3) + \Delta(G)\right),$$

$$d(x_2) \le \frac{1}{4}\left(-6 + \sum_{v\in S} d(v)\right) \le \frac{1}{4}\left(-6 + d(x_1) + 2\Delta(G)\right).$$

By the same calculation as in Claim 3, these together imply

$$d(x_1) \le -2 + \frac{2}{5}\Delta(G) + \frac{4}{15}d(x_3).$$

Since x_3 is long, using Claim 2, we get

$$d(x_1) \le -\frac{34}{15} + \frac{3}{5}\Delta(G),$$

and hence

$$d(x_2) \le -\frac{61}{15} + \frac{13}{20}\Delta(G).$$

$\square$

Claim 5. *The theorem is true.*
Proof: Let $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$. By Lemma 18, one of the following holds:

1. G is elementary; or
2. G is not k-thin; or
3. $\nu(T) = 3$ and $V(T)$ contains vertices x_1, x_2, x_3 such that x_1 is x_2-long, x_2 is x_1-long, x_2 is x_3-long, and x_3 is x_2-long; or
4. $\nu(T) = 3$ and $V(T)$ contains vertices x_1, x_2, x_3 such that x_1 is $\{x_2, x_3\}$-long, x_2 is x_1-long, and x_3 is long; or
5. $V(T)$ contains four long vertices x_1, x_2, x_3, x_4.

If (1) holds, then $\chi(Q) = \lceil \chi_f(Q) \rceil$, which contradicts our choice of Q as a counterexample. If (2) holds, then Claim 2 implies that $\mu(G) \ge 2k - \frac{3}{2}\Delta(G) + 2$. Now Theorem 19 gives

$$\begin{aligned} k + 1 &\le \Delta(Q) + 1 - \frac{2k - \frac{3}{2}\Delta(G) + 2}{2} \\ &= \Delta(Q) + 1 - k + \frac{3}{4}\Delta(G) - 1, \end{aligned}$$

so

$$2(k+1) \leq \Delta(Q) + 1 + \frac{3}{4}\Delta(G).$$

Substituting $\Delta(G) \leq k$ and solving for k gives

$$k \leq \frac{4}{5}\Delta(Q) - \frac{4}{5} < \frac{5}{6}\Delta(Q) + \frac{1}{2} \leq k,$$

which is a contradiction.

Suppose (3) holds. Now

$$2 + \sum_{i \in [3]} k - d(x_i) \leq k,$$

so Claim 3 implies

$$3\left(\frac{2}{3}\Delta(G) - 2\right) \geq 2k + 2,$$

which is a contradiction, since $\Delta(G) \leq k$.

Suppose (4) holds. Now

$$2 + \sum_{i \in [3]} k - d(x_i) \leq k,$$

so Claims 2 and 4 give

$$\left(\frac{3}{5} + \frac{13}{20} + \frac{3}{4}\right)\Delta(G) - \left(\frac{34}{15} + \frac{16}{15} + 1\right) \geq 2k + 2,$$

which is

$$2\Delta(G) - \frac{13}{3} \geq 2k + 2,$$

again a contradiction, since $\Delta(G) \leq k$.

So (5) must hold. But now

$$2 + \sum_{i \in [4]} k - d(x_i) \leq k,$$

so using Claim 2 gives

$$4\left(\frac{3}{4}\Delta(G) - 1\right) \geq 3k + 2,$$

a contradiction since $\Delta(G) \leq k$. □

This finishes the final case of Claim 5, which proves the theorem. □

In the previous theorem, we showed that $\chi(Q) \leq \max\left\{\mathcal{W}(Q), \Delta(G) + 1, \frac{5\Delta(Q)+8}{6}\right\}$. Now we show that if the maximum is attained by the second argument, then G satisfies the $\frac{5}{6}$-Conjecture. We use the following lemma, which is implicit in [3]; see the proof of Lemma 9 therein.

Lemma 21. *If Q is the line graph of a graph G and Q is vertex critical, then*

$$\chi(Q) \leq \max\left\{\Delta(G), \Delta(Q) + 1 + 2\mu(G) - \Delta(G)\right\}.$$

Corollary 22. *If Q is the line graph of a critical graph G and $\chi(Q) \leq \Delta(G) + 1$, then*

$$\chi(Q) \leq \max\left\{\omega(Q), \frac{5\Delta(Q) + 8}{6}\right\}.$$

Proof. Let $k + 1 = \chi(Q) \leq \Delta(G) + 1$. Suppose $\chi(Q) > \omega(Q)$. Then Lemma 21 gives

$$k + 1 = \chi(Q) \leq \Delta(Q) + 1 + 2\mu(G) - k,$$

so solving for $\mu(G)$ gives

$$\mu(G) \geq k - \frac{\Delta(Q)}{2}.$$

Applying Theorem 19 gives

$$k + 1 = \chi(Q) \leq \Delta(Q) + 1 - \frac{k - \frac{\Delta(Q)}{2} - 1}{2},$$

and solving for $k + 1$ yields

$$\chi(Q) = k + 1 \leq \frac{5}{6}\Delta(Q) + \frac{4}{3} = \frac{5\Delta(Q) + 8}{6}.$$

□

Since $\omega(Q) \leq \max\{\Delta(G), \mathcal{W}(G)\}$, Theorem 20 and Corollary 22 together imply the following.

Corollary 23. *If Q is the line graph of a graph G, then*

$$\chi(Q) \leq \max\left\{\Delta(G), \mathcal{W}(G), \frac{5\Delta(Q) + 8}{6}\right\}.$$

7 The $\frac{5}{6}$-Conjecture

Lemma 24. *Let G be a critical, elementary graph with $\chi'(G) = k + 1$ where $k \geq \Delta(G) + 1$. Put $Q := L(G)$. If $k = \epsilon\left(\Delta(Q) + 1\right) + \beta$, then for all $x \in V(G)$,*

$$|N(x)| = \frac{\epsilon\left(|G| - \Delta(G) - d_G(x) - 1 + S_1 + S_2 + S_3\left(|G| - 1\right)\right)}{(1 - \epsilon)\Delta(G) - \epsilon d_G(x) + 1 - \beta + S_3},$$

where

$$S_1 := \sum_{v \in N(x)} \Delta(Q) - d_Q(xv),$$

$$S_2 := 2 + \sum_{v \in V(G) \setminus N(x)} \Delta(G) - d_G(v),$$

$$S_3 := k - (\Delta(G) + 1).$$

Proof. Since G is critical and elementary, $|G|$ is odd and

$$k = \frac{2(\|G\| - 1)}{|G| - 1}. \tag{5}$$

Let $x \in V(G)$, put $M := |N(x)|$ and

$$P := \sum_{v \in N(x)} d_G(v).$$

Then

$$2(\|G\| - 1) = \Delta(G)(|G| - M) - S_2 + P. \tag{6}$$

Since

$$\frac{2(\|G\| - 1)}{|G| - 1} = k = \Delta(G) + 1 + S_3,$$

using (6), we get

$$P = (|G| - 1)(\Delta(G) + 1 + S_3) - \Delta(G)(|G| - M) + S_2,$$

which is

$$P = \Delta(G)(M - 1) + |G| - 1 + S_2 + S_3(|G| - 1). \tag{7}$$

Also, using $k = \epsilon\,(\Delta(Q) + 1) + \beta$, we get

$$kM = \beta M + \epsilon S_1 + \epsilon \sum_{v \in N(x)} d_G(x) + d_G(v) - \mu(xv),$$

Since $\sum_{v \in N(x)} \mu(xv) = d_G(x)$,we have

$$kM = \beta M + \epsilon S_1 + \epsilon d_G(x)(M - 1) + \epsilon P. \tag{8}$$

Plugging (7) into (8) and solving for M gives

$$M = \frac{\epsilon\,(|G| - \Delta(G) - d_G(x) - 1 + S_1 + S_2 + S_3\,(|G| - 1))}{(1 - \epsilon)\Delta(G) - \epsilon d_G(x) + 1 - \beta + S_3},$$

as desired. $\square$

Using $\epsilon = \frac{5}{6}$, we get the following.

Lemma 25. *Let G be a critical, elementary graph with $\chi'(G) = k+1$ where $k \geq \Delta(G)+1$. Put $Q := L(G)$. If $k = \frac{5}{6}(\Delta(Q)+1) + \beta$, then for all $x \in V(G)$,*

$$|N(x)| = \frac{5\left(|G| - \Delta(G) - d_G(x) - 1 + S_1 + S_2 + S_3\left(|G|-1\right)\right)}{\Delta(G) - 5d_G(x) + 6(1-\beta+S_3)},$$

where

$$S_1 := \sum_{v \in N(x)} \Delta(Q) - d_Q(xv),$$

$$S_2 := 2 + \sum_{v \in V(G) \setminus N(x)} \Delta(G) - d_G(v),$$

$$S_3 := k - (\Delta(G)+1).$$

Lemma 26. *Let G be a critical, elementary graph with $\chi'(G) = k+1$ where $k \geq \Delta(G)+1$. Put $Q := L(G)$. If $k = \frac{5}{6}(\Delta(Q)+1) + \beta$ where $\beta \geq -\frac{1}{3}$, then for all $x \in V(G)$ with $|N(x)| \geq 3$,*

$$d_G(x) \leq \frac{3}{5}\Delta(G) - \frac{1}{|N(x)|-2} \sum_{v \in V(G)\setminus N[x]} \Delta(G) - d_G(v).$$

If additionally, $|N(x)| \leq \frac{5}{8}|G|$, then

$$d_G(x) \leq \frac{|N(x)|}{5\left(|N(x)|-2\right)}\Delta(G) - \frac{1}{|N(x)|-2} \sum_{v \in V(G)\setminus N[x]} \Delta(G) - d_G(v).$$

Proof. Say $|N(x)| = 2 + S_4$ for some $S_4 \geq 1$. Applying Lemma 25 and simplifying using $S_1 \geq 0$ and $\beta \geq -\frac{1}{3}$ gives

$$(5+5S_4)d_G(x) \leq (7+S_4)\Delta(G) - 5|G| + 21 + S_3(-5|G| + 17 + 6S_4) + 8S_4 - 5S_2. \tag{9}$$

Put

$$t := \sum_{v \in V(G)\setminus N[x]} \Delta(G) - d_G(v).$$

Then $S_2 = t + 2 + \Delta(G) - d_G(x)$. Using this in (9), we get

$$5S_4 d_G(x) \leq (2+S_4)\Delta(G) - 5|G| + 11 + S_3(-5|G| + 17 + 6S_4) + 8S_4 - 5t. \tag{10}$$

The desired bound follows when $S_4 \leq \frac{5}{8}|G| - 2$, since then

$$-5|G| + 11 + S_3(-5|G| + 17 + 6S_4) + 8S_4 \leq 0.$$

So, suppose $S_4 > \frac{5}{8}|G| - 2$. Rearranging (10), we get

$$5S_4 d_G(x) \leq 3S_4\Delta(G) - 5|G| + 11 + S_3(-5|G| + 17 + 6S_4) + 8S_4 - (2S_4 - 2)\Delta(G) - 5t \tag{11}$$

Now $S_4 = |N(x)| - 2 = |G| - 3 + S_5$, where

$$S_5 := S_4 + 3 - |G| \leq 0.$$

Thus

$$-5|G| + 15 + 5S_4 + S_3(-5|G| + 15 + 5S_4) - 5S_5(1 + S_3) = 0,$$

so

$$5S_4 d_G(x) \leq 3S_4\Delta(G) - 4 + 2S_3 + (S_3 + 3)S_4 - (2S_4 - 2)\Delta(G) + 5S_5(1 + S_3) - 5t \tag{12}$$

If

$$-4 + 2S_3 + (S_3 + 3)S_4 - (2S_4 - 2)\Delta(G) + 5S_5(1 + S_3) \leq 0,$$

then we have the contradiction

$$d_G(x) \leq \frac{3}{5}\Delta(G) - \frac{1}{|N(x)| - 2} \sum_{v \in V(G) \setminus N[x]} \Delta(G) - d_G(v).$$

So, we have

$$-4 + 2S_3 + (S_3 + 3)S_4 - (2S_4 - 2)\Delta(G) + 5S_5(1 + S_3) > 0,$$

which is

$$(2 + S_4)S_3 + 3S_4 > (2S_4 - 2)\Delta(G) + 4 - 5S_5(S_3 + 1). \tag{13}$$

By Shannon's theorem $k + 1 \leq \frac{3}{2}\Delta(G)$, so $S_3 \leq \frac{\Delta(G)}{2} - 2$. After plugging in for S_3 on the left side and solving for S_4, we get

$$S_5 + |G| - 3 = S_4 < \frac{6\Delta(G) - 16 + 10S_5(S_3 + 1)}{3\Delta(G) - 2} = 2 + \frac{5S_5(S_3 + 1) - 12}{3\Delta(G) - 2},$$

so

$$|G| < 5 + \frac{(10S_3 - 3\Delta(G) + 12)S_5 - 12}{3\Delta(G) - 2}.$$

Since $S_5 \leq 0$, this implies $|G| \leq 3$, unless $10S_3 - 3\Delta(G) + 12 < 0$. So $S_3 < \frac{3}{10}\Delta(G) - \frac{6}{5}$. Since $S_5 \leq 0$, (13) implies

$$(2 + S_4)S_3 + 3S_4 > (2S_4 - 2)\Delta(G) + 4$$

Plugging in $S_3 < \frac{3}{10}\Delta(G) - \frac{6}{5}$ gives

$$|G| - 3 \leq S_4 < \frac{26\Delta(G) - 64}{17\Delta(G) - 18} < 2,$$

a contradiction. $\square$

Corollary 27. *Let G be a critical, elementary graph with $\chi'(G) = k+1$ where $k \geq \Delta(G)+1$. Put $Q := L(G)$. If $k = \frac{5}{6}(\Delta(Q)+1)+\beta$ where $\beta \geq -\frac{1}{3}$, then there is at most one $x \in V(G)$ with $|N(x)| \geq 3$.*

Proof. Since G is critical and elementary, $|G|$ is odd and

$$\frac{2(\|G\|-1)}{|G|-1} = k \geq \Delta(G)+1,$$

so

$$2\|G\| \geq \Delta(G)\,|G| + |G| - \Delta(G) + 1.$$

In particular,

$$\sum_{v \in V(G)} \Delta(G) - d_G(v) \leq \Delta(G) - 1 - |G|\,.$$

By Lemma 26, every $x \in V(G)$ with $|N(x)| \geq 3$ has $d_G(x) \leq \frac{3}{5}\Delta(G)$, so there are at most two such x since $\frac{2}{5}+\frac{2}{5}+\frac{2}{5} > 1$. Suppose there are x_1, x_2 with $|N(x_1)| \geq |N(x_2)| \geq 3$.

Let $z \in V(G)$ with $d_G(z) = \Delta(G)$. Then, by Lemma 26, $|N(z)| = 2$, so $\mu(G) \geq \frac{1}{2}\Delta(G)$. By Theorem 19, $\mu(G) < \frac{1}{3}(\Delta(Q)+1)$. We conclude

$$\Delta(G) < \frac{2}{3}(\Delta(Q)+1). \tag{14}$$

First, suppose $x_1 \leftrightarrow x_2$. Since Q is vertex-critical,

$$\begin{aligned}
k &\leq d_Q(x_1x_2)\\
&= d_G(x_1) + d_G(x_2) - \mu(x_1x_2) - 1\\
&\leq \frac{6}{5}\Delta(G) - \mu(x_1x_2) - 1,
\end{aligned}$$

So, $\Delta(G) > \frac{5}{6}k$. By (14),

$$\frac{5}{6}k < \Delta(G) < \frac{2}{3}(\Delta(Q)+1),$$

and hence $k < \frac{4}{5}(\Delta(Q)+1)$, a contradiction.

So, $x_1 \not\leftrightarrow x_2$. Now suppose $|N(x_2)| \leq \frac{5}{8}|G|$. Since $x_1 \not\leftrightarrow x_2$, Lemma 26 gives

$$d_G(x_2) \leq \frac{3}{5}\Delta(G) - (\Delta(G) - d_G(x_1)) \leq \frac{1}{5}\Delta(G),$$

a contradiction since $\frac{4}{5}+\frac{2}{5} > 1$.

So, $|N(x_i)| > \frac{5}{8}|G|$ for $i \in [2]$. In particular, there is $y \in N(x_1) \cap N(x_2)$. Since $|N(y)| = 2$, by symmetry we may assume $\mu(x_1y) \geq \frac{1}{2}d_G(y)$. Hence, using (14),

$$\begin{aligned}
k &\leq d_Q(x_1y)\\
&= d_G(x_1) + d_G(y) - \mu(x_1y) < d_G(x_1) + \frac{1}{2}d_G(y)\\
&\leq \frac{11}{10}\Delta(G)\\
&< \frac{11}{15}(\Delta(Q)+1),
\end{aligned}$$

a contradiction. $\square$

Theorem 28. *If Q a line graph, then*

$$\chi(Q) \leq \max\left\{\omega(Q), \left\lceil\frac{5\Delta(Q)+3}{6}\right\rceil\right\}.$$

Proof. Suppose the theorem is false and choose a counterexample Q minimizing $|Q|$. Then $Q = L(G)$ for a critical graph G. Say $\chi(Q) = \chi'(G) = k+1$. Then $k = \max\left\{\omega(Q), \left\lceil\frac{5\Delta(Q)+3}{6}\right\rceil\right\}$ by minimality of $|Q|$. By Corollary 23,

$$k+1 \leq \max\left\{\lceil\chi_f(Q)\rceil, \left\lceil\frac{5\Delta(Q)+3}{6}\right\rceil\right\},$$

so

$$\lceil\chi_f(Q)\rceil = \left\lceil\frac{5\Delta(Q)+3}{6}\right\rceil + 1 = \chi(Q).$$

Therefore G is elementary and $k = \frac{5}{6}(\Delta(Q)+1) + \beta$ for some $\beta \geq -\frac{1}{3}$. By Corollary 22, $k \geq \Delta(G)+1$. Let H be the underlying simple graph of G. We may apply Corollary 27 to conclude that there is at most one $x \in V(G)$ with $d_H(x) \geq 3$. Since G is critical, $\delta(H) \geq 2$ and H has no cut vertices. Hence H is a cycle.

Choose t such that $|V(H)| = 2t+1$. Let $x_1, \ldots, x_{2t+1}$ denote the multiplicities of the edges in G, and let $X = \sum_{i=1}^{2t+1} x_i$. Since G is elementary, we have $\chi'(G) = \left\lceil\frac{X}{t}\right\rceil$. Let $Q = L(G)$ and let v_i be a vertex of Q corresponding to an edge of G counted by x_i. Now $d_Q(v_i) = x_{i-1} + x_i + x_{i+1} - 1$. It suffices to show that there exists $j \in [2t+1]$ such that $\frac{X}{t} \leq \frac{5d_Q(v_j)+3}{6}$. We will prove the stronger statement that $\frac{X}{t} \leq \frac{5\overline{d}+3}{6}$, where $\overline{d} = \frac{1}{2t+1}\sum_{i=1}^{2t+1} d_Q(v_i)$. Since $\frac{5\overline{d}+3}{6} = \frac{5X}{2(2t+1)} - \frac{1}{3}$, it suffices to have $\frac{5X}{2(2t+1)} - \frac{1}{3} \geq \frac{X}{t}$. Simplifying (for $t \geq 3$) gives $X \geq \frac{1}{3}(4t + 10 + \frac{20}{t-2})$. Since $X \geq 2t+1$, this always holds when $t \geq 6$. When $t = 5$, it suffices to have $X \geq 13$. When $t = 4$, it suffices to have $X \geq 12$, and when $t = 3$, it suffices to have $X \geq 14$. Suppose $t = 5$ and $X \leq 12$. Now $\chi'(G) \leq \left\lceil\frac{12}{5}\right\rceil = 3 \leq \left\lceil\frac{5(2)+3}{6}\right\rceil$. Suppose instead that $t = 4$ and $X \leq 11$. Now $\chi'(G) \leq \left\lceil\frac{11}{4}\right\rceil = 3 \leq \left\lceil\frac{5(2)+3}{6}\right\rceil$. Finally, suppose that $t = 3$ and $X \leq 13$. If $X \leq 9$, then again Now $\chi'(G) \leq \left\lceil\frac{9}{3}\right\rceil = 3 \leq \left\lceil\frac{5(2)+3}{6}\right\rceil$. So assume that $X \geq 10$, which implies that $\Delta(Q) \geq 4$. First suppose that $X \leq 12$. Now $\chi'(G) \leq \left\lceil\frac{12}{3}\right\rceil = 4 \leq \left\lceil\frac{5(4)+3}{6}\right\rceil$. So instead assume that $X = 13$, which implies that $\Delta(Q) \geq 5$. Now $\chi'(G) \leq \left\lceil\frac{13}{3}\right\rceil = 5 \leq \left\lceil\frac{5(5)+3}{6}\right\rceil$, as desired. $\square$

Conjecture 29. *If Q a quasi-line graph, then*

$$\chi(Q) \leq \max\left\{\omega(Q), \left\lceil\frac{5\Delta(Q)+3}{6}\right\rceil\right\}.$$

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