Edge lower bounds via discharging notes

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1 Introduction

For a graph G, let $d(G)$ be the average degree of G. Let $\mathcal{T}_k$ be the Gallai trees with maximum degree at most $k-1$, excepting K_k.

2 Gallai's bound via discharging

Theorem 2.1 (Gallai)**.** *For $k \geq 4$ and $G \neq K_k$ a k-AT-critical graph, we have*

$$d(G) > k - 1 + \frac{k-3}{k^2-3}.$$

Proof. Start with initial charge function $\mathrm{ch}(v) = d_G(v)$. Have each k^+-vertex give charge $\frac{k-1}{k^2-3}$ to each of its $(k-1)$-neighbors. Then let the vertices in each component of the low vertex subgraph share their total charge equally. Let $\mathrm{ch}^*(v)$ be the resulting charge function. We finish the proof by showing that $\mathrm{ch}^*(v) \geq k - 1 + \frac{k-3}{k^2-3}$ for all $v \in V(G)$.

If v is a k^+-vertex, then $ch^*(v) \geq d_G(v) - \frac{k-1}{k^2-3} d_G(v) = \left(1 - \frac{k-1}{k^2-3}\right) d_G(v) \geq \left(1 - \frac{k-1}{k^2-3}\right) k = k - 1 + \frac{k-3}{k^2-3}$ as desired.

Let T be a component of the low vertex subgraph. Then the vertices in T receive total charge

$$\frac{k-1}{k^2-3} \sum_{v \in V(T)} k - 1 - d_G(v) = \frac{k-1}{k^2-3} \left((k-1)|T| - 2\left\|T\right\|\right).$$

So, after distributing this charge out equally, each vertex in T receives charge

$$\frac{1}{|T|} \frac{k-1}{k^2-3} ((k-1)|T| - 2\left\|T\right\|) = \frac{k-1}{k^2-3} \left((k-1) - d(T)\right).$$

By Lemma 2.2, this is at least

$$\frac{k-1}{k^2-3} \left((k-1) - \left(k - 2 + \frac{2}{k-1}\right)\right) = \frac{k-1}{k^2-3} \left(\frac{k-3}{k-1}\right) = \frac{k-3}{k^2-3}.$$

Hence each low vertex ends with charge at least $k - 1 + \frac{k-3}{k^2-3}$ as desired. $\square$

Lemma 2.2 (Gallai)**.** *For $k \geq 4$ and $T \in \mathcal{T}_k$, we have $d(T) < k - 2 + \frac{2}{k-1}$.*

Proof. Suppose not and choose a counterexample T minimizing $|T|$. Then T has at least two blocks. Let B be an endblock of T. If B is K_t for $2 \leq t \leq k-2$, then remove the non-separating vertices of B from T to get T'. By minimality of $|T|$, we have

$$2\|T\| - t(t-1) = 2\|T'\| < \left(k-2+\frac{2}{k-1}\right)|T'| = \left(k-2+\frac{2}{k-1}\right)|T| - \left(k-2+\frac{2}{k-1}\right)(t-1).$$

Hence we have the contradiction

$$2\|T\| < \left(k-2+\frac{2}{k-1}\right)|T| + (t+2-k-\frac{2}{k-1})(t-1) \leq \left(k-2+\frac{2}{k-1}\right)|T|.$$

The case when B is an odd cycle is the same as the above, a longer cycle just makes things better. Finally, if $B = K_{k-1}$, remove all vertices of B from T to get T'. By minimality of $|T|$, we have

$$\begin{aligned} 2\|T\| - (k-1)(k-2) - 2 = 2\|T'\| & \\ &< \left(k-2+\frac{2}{k-1}\right)|T'| \\ &= \left(k-2+\frac{2}{k-1}\right)|T| - \left(k-2+\frac{2}{k-1}\right)(k-1). \end{aligned}$$

Hence $2\|T\| < \left(k-2+\frac{2}{k-1}\right)|T|$, a contradiction. □

3 An initial improved bound

Lemma 2.2 is best possible as can be seen by the family of graphs with blocks on a path alternating K_{k-1} and K_2. But we have reducible configurations (see the last section for the precise statements) that place restrictions on K_{k-1} blocks. To state these restrictions, we need the following auxiliary bipartite graph.

For a k-AT-critical graph G, let $\mathcal{L}(G)$ be the subgraph of G induced on the $(k-1)$-vertices and $\mathcal{H}(G)$ the subgraph of G induced on the k-vertices. For $T \in \mathcal{T}_k$, let $W^k(T)$ be the set of vertices of T that are contained in some K_{k-1} in T. Let $\mathcal{B}_k(G)$ be the bipartite graph with one part $V(\mathcal{H}(G))$ and the other part the components of $\mathcal{L}(G)$. Put an edge between $y \in V(\mathcal{H}(G))$ and a component T of $\mathcal{L}(G)$ if and only if $N(y) \cap W^k(T) \neq \emptyset$. Then Lemma 4.2 says that $\mathcal{B}_k(G)$ is 2-degenerate.

We can use this fact to refine our discharging argument. Let ϵ and γ be parameters that we will determine where $\epsilon \leq \gamma < 2\epsilon$. Start with initial charge function $\text{ch}(v) = d_G(v)$.

1. Each k^+-vertex gives charge ϵ to each of its $(k-1)$-neighbors not in a K_{k-1},

2. Each $(k+1)^+$-vertex give charge γ to each of its $(k-1)$-neighbors in a K_{k-1},

3. Let $Q = \mathcal{B}_k(G)$. Repeat the following steps until Q is empty.

    (a) Remove all components T of $\mathcal{L}(G)$ in Q that have degree at most two in Q.

(b) Pick $v \in V(\mathcal{H}(G)) \cap V(Q)$. Send charge γ from v to each $x \in N_G(v) \cap W^k(T)$ for each component T of $\mathcal{L}(G)$ where $vT \in E(Q)$.

(c) Remove v from Q.

4. Have the vertices in each component of $\mathcal{L}(G)$ share their total charge equally.

Let $\text{ch}^*(v)$ be the resulting charge function. Here is some intuition for why this might be a useful refinement. In (3b), v sends charge to at most two different T and so, by Lemma 4.1 (or our 'beyond degree choosability' classification), v loses charge at most 3γ. On the other hand, from (3a) each component T of $\mathcal{L}(G)$ receives charge γ for all but at most four non-separating vertices in a K_{k-1} (the at most four again coming from Lemma 4.1 and the fact that we leave T in Q until it has degree at most two). So, we can get each T almost as much charge as we could hope for without losing too much from the k-vertices. We don't have the same control over $(k+1)^+$-vertices, but it won't matter since they have extra charge to start with and sending γ to every $(k-1)$-neighbor will leave enough charge (we'll use $\gamma < 2\epsilon$ here).

To analyze this discharging procedure we need a bound like Lemma 2.2, but taking into account the number of edges in $\mathcal{B}_k(G)$. We can do this by taking into account the number of non-separating vertices in K_{k-1}'s in T. To this end, for $T \in \mathcal{T}_k$, let $q(T)$ be the number of non-separating vertices in a K_{k-1} in T. We give a family of such bounds. Without more reducible configurations we can't hope to do better than average degree $k-3$ because of K_{k-2} components, that is why the bound below has $(k-3+p(k))|T|$, a slight worsening of average degree $k-3$.

Lemma 3.1. *Let $K \in \mathbb{N}$ and $p\colon \mathbb{N} \to \mathbb{R}$, $f\colon \mathbb{N} \to \mathbb{R}$, $h\colon \mathbb{N} \to \mathbb{R}$ be such that for all $k \geq K \geq 4$ we have*

1. *$f(k) \geq t(t+2-k-p(k))$ for all $t \in [k-2]$; and*
2. *$f(k) \geq (5-k-p(k))s$ for all $s \geq 5$; and*
3. *$f(k) \geq (k-1)(1-p(k)-h(k))$; and*
4. *$p(k) \geq h(k)+5-k$; and*
5. *$p(k) \geq \frac{4}{k-2}$; and*
6. *$p(k) \geq \frac{2+h(k)}{k-2}$; and*
7. *$(k-1)p(k)+(k-3)h(k) \geq k+1$.*

Then for $k \geq K$ and $T \in \mathcal{T}_k$, we have

$$2\,\|T\| \leq (k-3+p(k))\,|T| + f(k) + h(k)q(T).$$

Proof. Suppose not and choose a counterexample T minimizing $|T|$. First, suppose T is K_t for $t \in [k-2]$. Then $t(t-1) > (k-3+p(k))t + f(k)$ contradicting (1). If T is C_{2r+1} for $r \geq 2$, then $2(2r+1) > (k-3+p(k))(2r+1) + f(k)$ and hence $f(k) < (5-k-p(k))(2r+1)$

contradicting (2). If T is K_{k-1}, then $(k-1)(k-2) > (k-3+p(k))(k-1)+f(k)+h(k)(k-1)$ contradicting (3).

Hence T has at least two blocks. Let B be an endblock of T and x_B the cutvertex of T contained in B. Let $T' = T - (V(B) \setminus \{x_B\})$. Then, by minimality of $|T|$, we have

$$2\,\|T'\| \leq (k-3+p(k))\,|T'| + f(k) + h(k)q(T').$$

Hence

$$2\,\|T\| - 2\,\|B\| \leq (k-3+p(k))\,(|T| - (|B|-1)) + f(k) + h(k)q(T').$$

Since T is a counterexample, this gives

$$2\,\|B\| > (k-3+p(k))(|B|-1) + h(k)\,(q(T) - q(T'))\,. \tag{*}$$

Suppose B is K_t for $3 \leq t \leq k-3$ or B is an odd cycle. Then $q(T') = q(T)$, $2\,\|B\| \leq |B|\,(|B|-1)$ and $2\,\|B\| = 2\,|B|$ if $|B| > k-3$. Since $p(k) \geq \frac{4}{k-2}$ by (5), this contradicts *.

If B is K_2, then $q(T') \leq q(T)+1$ and * gives $2 > k-3+p(k)-h(k)$ contradicting (4).

To handle the cases when B is K_{k-2} or K_{k-1} we need to remove x_B from T as well. Let $T^* = T - V(B)$. Then, by minimality of $|T|$, we have

$$2\,\|T^*\| \leq (k-3+p(k))\,|T^*| + f(k) + h(k)q(T^*).$$

Hence

$$2\,\|T\| - 2\,\|B\| - 2(d_T(x_B) - d_B(x_B)) \leq (k-3+p(k))\,(|T| - |B|) + f(k) + h(k)q(T^*).$$

Since T is a counterexample and B is complete, this gives

$$2\,\|B\| > (k-3+p(k))\,|B| - 2(d_T(x_B) + 1 - |B|) + h(k)\,(q(T) - q(T^*))\,,$$

which is

$$2\,\|B\| > (k-1+p(k))\,|B| - 2d_T(x_B) - 2 + h(k)\,(q(T) - q(T^*))\,. \tag{**}$$

Suppose B is K_{k-2}. Then $d_T(x_B) = k-1$ or $d_T(x_B) = k-2$. In the former case, $q(T) = q(T^*)$ and in the latter $q(T^*) \leq q(T)+1$. If $d_T(x_B) = k-1$, we have

$$(k-2)(k-3) > (k-1+p(k))(k-2) - 2(k-1) - 2 = (k-2)(k-3) - 4 + (k-2)p(k),$$

contradicting (5). If instead $d_T(x_B) = k-2$, we have

$$(k-2)(k-3) > (k-1+p(k))(k-2) - 2(k-2) - 2 - h(k) = (k-2)(k-3) - 2 + (k-2)p(k) - h(k),$$

contradicting (6).

Finally, suppose B is K_{k-1}. Then $d_T(x_B) = k-1$ and $q(T^*) \leq q(T) - (k-2) + 1 = q(T) - (k-3)$. From **, we have

$$\begin{aligned}(k-1)(k-2) &> (k-1+p(k))(k-1) - 2(k-1) - 2 + h(k)(k-3) \\ &= (k-1)(k-2) + p(k)(k-1) - (k+1) + h(k)(k-3),\end{aligned}$$

contradicting (7). $\square$

Remark. Some of the conditions can be weakened with more work. For example, the Kostochka-Stiebitz bound Hal and i used satisfies a weaker version of (5) and in that other document it is weakened a bit more. We still need to determine what the best possible conditions are. i think having the whole family could be useful because we can tailor the bound we use for a component T based on $q(T)$ (put less weight on $q(T)$ if it is big).

Now some examples of using Lemma 3.1. What happens if we take $h(k) = 0$? Then, by (7), we need $(k-1)p(k) \geq k+1$ and hence $p(k) \geq 1 + \frac{2}{k-1}$. Taking $p(k) = 1 + \frac{2}{k-1}$, (3) requires $f(k) \geq -2$. Using $f(k) = -2$, all of the other conditions are satisfied and we conclude $2\|T\| \leq \left(k - 2 + \frac{2}{k-1}\right)|T| - 2$ for every $T \in \mathcal{T}_k$ when $k \geq 4$. This is a slight refinement of Gallai's Lemma 2.2.

If instead we make $p(k)$ as small as Lemma 3.1 will let us, which is $p(k) = \frac{4}{k-2}$, we get that $f(k) = -4$ and $h(k) = \frac{k^2-5k+2}{(k-2)(k-3)}$ satisfy the conditions. That is,

Corollary 3.2. *For $k \geq 7$ and $T \in \mathcal{T}_k$, we have*

$$2\|T\| \leq (k - 3 + \frac{4}{k-2})|T| - 4 + \frac{k^2-5k+2}{(k-2)(k-3)}q(T).$$

If we put the Kostochka-Stiebitz bound on $\sigma(T)$ into this form we get the following.

Lemma 3.3 (Kostochka-Stiebitz)**.** *For $k \geq 7$ and $T \in \mathcal{T}_k$, we have*

$$2\|T\| \leq (k - 3 + \frac{4(k-1)}{k^2-3k+4})|T| - \frac{4(k^2-3k+2)}{k^2-3k+4} + \frac{k^2-3k}{k^2-3k+4}q(T).$$

Note that $\frac{4(k-1)}{k^2-3k+4} < \frac{4}{k-2}$, so this does not satisfy condition (5) in Lemma 3.1, it is more like $\frac{3.891}{k-2}$.

3.1 Analyzing the discharging

Our discharging procedure gives charge ϵ to a component T for every incident edge not ending in a K_{k-1}. The number of such edges is exactly

$$A(T) := -q(T) + \sum_{v \in V(T)} k - 1 - d_T(v) = (k-1)|T| - 2\|T\| - q(T).$$

Suppose we have a bound of the form

$$2\|T\| \leq (k - 3 + p(k))|T| + f(k) + h(k)q(T).$$

So, we get

$$A(T) \geq (2 - p(k))|T| - f(k) - (h(k) + 1)q(T).$$

We will use $\gamma = (h(k)+1)\epsilon$ in order to make the $q(T)$ term cancel. That happens because T receives charge on all but at most four of its non-separating vertices in a K_{k-1}; that is, in discharging steps 2 and 3, T receives charge at least $\gamma \max\{0, q(G) - 4\}$. Hence in total T receives charge at least

$$\epsilon A(T) + \gamma \max\{0, q(G) - 4\} = \epsilon (2 - p(k))|T| - \epsilon (f(k) + \min\{4, q(G)\}(h(k) + 1)).$$

For example, using the bound in Corollary 3.2, we get that T receives charge at least

$$\epsilon\left(2-\frac{4}{k-2}\right)|T|+\epsilon\max\{0,4-q(G)\}-\epsilon\min\{4,q(G)\}\frac{k^2-5k+2}{(k-2)(k-3)}.$$

Since $q(T)=0$ when $|T|<k-1$, we conclude that the discharging procedure gives each low vertex charge at least

$$\epsilon\left(2-\frac{4}{k-2}-\frac{k^2-5k+2}{(k-1)(k-2)(k-3)}\right).$$

We want the k-vertices to end with enough charge, the worst case is when

$$1-(3\gamma+(k-3)\epsilon)=\epsilon\left(2-\frac{4}{k-2}-\frac{k^2-5k+2}{(k-1)(k-2)(k-3)}\right),$$

and thus

$$\epsilon=\frac{1}{k-1+\frac{6(k-1)(k-4)}{(k-2)(k-3)}-\frac{4}{k-2}-\frac{k^2-5k+2}{(k-1)(k-2)(k-3)}},$$

which simplifies to

$$\epsilon=\frac{(k-3)(k-2)(k-1)}{k^4-k^3-24k^2+58k-32}.$$

For γ, we get

$$\gamma=\frac{2(k-4)(k-1)^2}{k^4-k^3-24k^2+58k-32}.$$

It remains to check that the $(k+1)^+$-vertices don't give away too much charge. Let v be a $(k+1)^+$-vertex, then v ends with charge at least

$$d(v)-\gamma d(v)=(1-\gamma)d(v)\geq(1-\gamma)(k+1),$$

you can show this is big enough for all k (i checked with wolfram). So, we get the following.

Theorem 3.4. *For $k\geq 7$ and $G\neq K_k$ a k-AT-critical graph, we have*

$$d(G)\geq k-1+\frac{2k^3-17k^2+43k-26}{k^4-k^3-24k^2+58k-32}.$$

This is better than Gallai's bound and the bound Kostochka-Stiebitz proved, but not as good as the bound Hal and i proved. It is closer to the bound with Hal than it is to the Kostochka-Stiebitz bound. But there is room for improvement all over. Here are some things i see right off:

1. Use a better bound on average degree of Gallai trees than Corollary 3.2, such as the Kostochka-Stiebitz bound or the better one in the other document. i would like to find the best possible family in the form here.

2. Near the end of the proof where we used that $q(T)=0$ when $|T|<k-1$, i think we lost a lot more than we really needed to.

3. In the discharging, the k-vertices lost 3γ even though they had degree two in Q because of the possibility of two edges into one component. Can we get this to 2γ somehow, like maybe we can order our picking so that no vertex is picked before the component where it has two edges has been removed.

4. Related to the previous item, improved reducible configurations, a less restrictive condition in Lemma 4.2 taking into account the two edges to a component issue.

5. Do the averaging like in the Hal paper, or a hybrid. How exactly is this averaging related to the discharging procedure? Is the averaging there better than the discharging here?

4 Reducible Configurations

Definition 1. A graph G is *AT-reducible* to H if H is a nonempty induced subgraph of G which is f_H-AT where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not AT-reducible to any nonempty induced subgraph, then it is *AT-irreducible*.

This first lemma tells us how a single high vertex can interact with the low vertex subgraph. This is the version Hal and i used, it (and more) follows from the classification in "mostlow".

Lemma 4.1. *Let $k \geq 5$ and let G be a graph with $x \in V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*

2. *$G - x$ has t components $H_1, H_2, \ldots, H_t$, and all are in $\mathcal{T}_k$; and*

3. *$d_G(v) \leq k - 1$ for all $v \in V(G - x)$; and*

4. *$\left|N(x) \cap W^k(H_i)\right| \geq 1$ for $i \in [t]$; and*

5. *$d_G(x) \geq t + 2$.*

Then G is f-AT where $f(x) = d_G(x) - 1$ and $f(v) = d_G(v)$ for all $v \in V(G - x)$.

To deal with more than one high vertex we need the following auxiliary bipartite graph. For a graph G, $\{X, Y\}$ a partition of $V(G)$ and $k \geq 4$, let $\mathcal{B}_k(X, Y)$ be the bipartite graph with one part Y and the other part the components of $G[X]$. Put an edge between $y \in Y$ and a component T of $G[X]$ if and only if $N(y) \cap W^k(T) \neq \emptyset$. The next lemma tells us that we have a reducible configuration if this bipartite graph has minimum degree at least three.

Lemma 4.2. *Let $k \geq 7$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*

2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*

3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*

4. *with* $\mathcal{B} := \mathcal{B}_k(V(G-Y), Y)$ *we have* $\delta(\mathcal{B}) \geq 3$.

Then G *has an induced subgraph* G' *that is* f*-AT where* $f(y) = d_{G'}(y) - 1$ *for* $y \in Y$ *and* $f(v) = d_{G'}(v)$ *for all* $v \in V(G' - Y)$.

We also have the following version with asymmetric degree condition on $\mathcal{B}$. The point here is that this works for $k \geq 5$. As we'll see in the next section, the consequence is that we trade a bit in our size bound for the proof to go through with $k \in \{5, 6\}$.

Lemma 4.3. *Let* $k \geq 5$ *and let* G *be a graph with* $Y \subseteq V(G)$ *such that:*

1. $K_k \not\subseteq G$*; and*
2. *the components of* $G - Y$ *are in* $\mathcal{T}_k$*; and*
3. $d_G(v) \leq k - 1$ *for all* $v \in V(G - Y)$*; and*
4. *with* $\mathcal{B} := \mathcal{B}_k(V(G-Y), Y)$ *we have* $d_{\mathcal{B}}(y) \geq 4$ *for all* $y \in Y$ *and* $d_{\mathcal{B}}(T) \geq 2$ *for all components* T *of* $G - Y$.

Then G *has an induced subgraph* G' *that is* f*-AT where* $f(y) = d_{G'}(y) - 1$ *for* $y \in Y$ *and* $f(v) = d_{G'}(v)$ *for all* $v \in V(G' - Y)$.