Short fans and the 5/6 bound for line graphs

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Abstract

In 2011, the second author conjectured that every line graph G satisfies $\chi(G) \leq \max\left\{\omega(G), \frac{5\Delta(G)+8}{6}\right\}$. This conjecture is best possible, as shown by replacing each edge in a 5-cycle by k parallel edges, and taking the line graph. In this paper we prove the conjecture. We also develop more general techniques and results that will likely be of independent interest, due to their use in attacking the Goldberg–Seymour conjecture.

1 Overview

By *graph* we mean multigraph without loops. Our notation follows Diestel [3][1]. In [8], the second author showed that $\chi(G) \leq \max\left\{\omega(G), \frac{7\Delta(G)+10}{8}\right\}$ for every line graph G. In the same paper, he conjectured that $\chi(G) \leq \max\left\{\omega(G), \frac{5\Delta(G)+8}{6}\right\}$. This conjecture is best possible, as shown by replacing each edge in a 5-cycle by k parallel edges, and taking the line graph. In this paper we prove the latter inequality. Along the way, we develop more general techniques and results that will likely be of independent interest. The main result of this paper is the following theorem.

Theorem 20 ($\frac{5}{6}$-Theorem)**.** *If Q is a line graph, then*

$$\chi(Q) \leq \max\left\{\omega(Q), \frac{5\Delta(Q)+8}{6}\right\}.$$

For every graph G, we have $\chi'(G) \geq \left\lceil \frac{\|G\|}{\left\lfloor\frac{|G|}{2}\right\rfloor} \right\rceil$, since in any proper edge-coloring each color class has size at most $\left\lfloor\frac{|G|}{2}\right\rfloor$. Likewise, the same bound holds for any subgraph H. Thus

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[1]In particular, $|G|$ denotes $|V(G)|$ and $\|G\|$ denotes $|E(G)|$.

$\chi'(G) \geq \max_{H \subseteq G} \left\lceil \frac{\|H\|}{\lfloor \frac{|H|}{2} \rfloor} \right\rceil$ (where the max is over all subgraphs H with at least two vertices). For convenience, we let $\mathcal{W}(G) := \max_{H \subseteq G} \left\lceil \frac{\|H\|}{\lfloor \frac{|H|}{2} \rfloor} \right\rceil$. Goldberg [5, 6] and Seymour [10, 11] each conjectured that this lower bound holds with equality, whenever $\chi'(G) > \Delta(G) + 1$.

Goldberg–Seymour Conjecture. *Every graph G satisfies*

$$\chi'(G) \leq \max\{\mathcal{W}(G), \Delta(G) + 1\}.$$

The Goldberg–Seymour Conjecture is the major open problem in the area of edge-coloring multigraphs. Most of our work goes toward proving the following intermediate result, in Section 4. This theorem is a weakened version of both the Goldberg–Seymour Conjecture and our main result, the $\frac{5}{6}$-Theorem.

Theorem 11 (Weak $\frac{5}{6}$-Theorem)**.** *If Q the line graph of a graph G, then*

$$\chi(Q) \leq \max\left\{\mathcal{W}(G), \Delta(G) + 1, \frac{5\Delta(Q) + 8}{6}\right\}.$$

Finally, in Section 5 we show that the Weak $\frac{5}{6}$-Theorem does indeed imply the $\frac{5}{6}$-Theorem. To conclude, in Section 6 we prove strengthenings of Reed's Conjecture for line graphs that follow from the general lemmas we prove earlier in the paper.

2 Tashkinov Trees

A graph G is *elementary* if $\chi'(G) = \mathcal{W}(G)$. Let $[k]$ denote $\{1, \ldots, k\}$. For a path or cycle Q, let $\ell(Q)$ denote the length of Q. A graph G is *critical* if $\chi'(G - e) < \chi'(G)$ for all $e \in E(G)$. For a graph G and a partial k-edge-coloring φ, for each vertex $v \in V(G)$, let $\varphi(v)$ denote the set of colors used in φ on edges incident to v. Let $\overline{\varphi}(v) = [k] \setminus \varphi(v)$. A color c is *seen* by a vertex v if $c \in \varphi(v)$ and c is *missing* by v if $c \in \overline{\varphi}(v)$. Given a partial k-edge-coloring φ, a set $W \subseteq V(G)$ is *elementary* with respect to φ (henceforth, *w.r.t.* φ) if each color in $[k]$ is missing at no more than one vertex of W. More formally, $\overline{\varphi}(u) \cap \overline{\varphi}(v) = \emptyset$ for all distinct $u, v \in W$. A *defective color* for a set $X \subseteq V(G)$ (w.r.t. φ) is a color used on more than one edge from X to $V(G) \setminus X$. A set X is *strongly closed* w.r.t. φ if X has no defective color. Elementary and strongly closed sets are of particular interest because of the following theorem, proved implicitly by Andersen [1] and Goldberg [6]; see also [12, Theorem 1.4].

elementary graph
$\ell(Q)$
critical
seen
missing
elementary set
defective color
strongly closed

Theorem 1. *Let G be a graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G)$. If G is critical, then G is elementary if and only if there exists $uv \in E(G)$, a k-edge-coloring φ of $G - uv$, and a set X with $u, v \in X$ such that X is both elementary and strongly closed w.r.t. φ.*

A *Tashkinov tree* w.r.t. φ is a sequence $v_0, e_1, v_1, e_2, \dots, v_{t-1}, e_t, v_t$ such that all v_i are distinct, $e_i = v_j v_i$ and $\varphi(e_i) \in \overline{\varphi}(v_\ell)$ for some j and ℓ with $0 \le j < i$ and $0 \le \ell < i$. A *Vizing fan* (or simply *fan*) is a Tashkinov tree that induces a star. Tashkinov trees are of interest because of the following lemma.

Tashkinov tree

Vizing fan

Tashkinov's Lemma. *Let G be a graph with $\chi'(G) = k+1$, for some integer $k \ge \Delta(G)+1$ and choose $e \in E(G)$ such that $\chi'(G-e) < \chi'(G)$. Let φ be a k-edge-coloring of $G-e$. If T is a Tashkinov tree w.r.t. φ and e, then $V(T)$ is elementary w.r.t. φ.*

In view of Theorem 1 and Tashkinov's Lemma, to prove that a graph G is elementary, it suffices to find an edge e, a k-edge-coloring φ of $G-e$, and a Tashkinov tree T containing e such that $V(T)$ is strongly closed. This motivates our next two lemmas. But first, we need a few more definitions.

Let $t(G)$ be the maximum number of vertices in a Tashkinov tree over all $e \in E(G)$ and all k-edge-colorings φ of $G-e$. Let $\mathcal{T}(G)$ be the set of all triples (T, e, φ) such that $e \in E(G)$, φ is a k-edge-coloring of $G-e$, and T is a Tashkinov tree with respect to e and φ with $|T| = t(G)$. Notice that, by definition, we have $\mathcal{T}(G) \neq \emptyset$. For a k-edge-coloring φ of $G-e$, a maximal Tashkinov tree starting with e may not be unique. However, if T_1 and T_2 are both such trees, then it is easy to show that $V(T_1) \subseteq V(T_2)$; by symmetry, also $V(T_2) \subseteq V(T_1)$, so $V(T_1) = V(T_2)$. Let G be a critical graph with $\chi'(G) = k+1$ for some integer $k \ge \Delta(G)+1$. Let φ be a k-edge-coloring of $G-e_0$ for some $e_0 \in E(G)$. For $v \in V(G)$ and colors α, β, let $P_v(\alpha, \beta)$ be the maximal connected subgraph of G that contains v and is induced by edges with color α or β. So $P_v(\alpha, \beta)$ is a path or a cycle. For a k-edge-coloring φ of $G - v_0v_1$, we often let $P = P_{v_1}(\alpha, \beta)$ for some $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Clearly P must end at v_0 (or we can swap colors α and β on P and color v_0v_1 with α), so let $v_1, \dots, v_r, v_0$ denote the vertices of P in order. To *rotate the α, β coloring on $P \cup \{v_0v_1\}$ by one*, we uncolor v_1v_2 and use its color on v_0v_1. To *rotate the α, β coloring on $P \cup \{v_0v_1\}$ by j*, we rotate the α, β coloring by one j times in succession. (When we do not specify j, we allow j to take any value from 1 to r.)

$t(G)$

$\mathcal{T}(G)$

$P_v(\alpha, \beta)$

rotate coloring

Lemma 2. *Let G be a non-elementary critical graph with $\chi'(G) = k+1$ for some integer $k \ge \Delta(G)+1$. For every $v_0v_1 \in E(G)$, k-edge-coloring φ of $G - v_0v_1$, $\alpha \in \overline{\varphi}(v_0)$, and $\beta \in \overline{\varphi}(v_1)$, we have $|P_{v_1}(\alpha, \beta)| < t(G)$.*

Proof. Suppose the lemma is false and choose $v_0v_1 \in E(G)$, a k-edge-coloring φ of $G - v_0v_1$, $\alpha \in \overline{\varphi}(v_0)$, and $\beta \in \overline{\varphi}(v_1)$, such that $|P_{v_1}(\alpha, \beta)| \ge t(G)$. Let $P = P_{v_1}(\alpha, \beta)$; see Figure 1. Let (T, v_0v_1, φ) be a Tashkinov tree that begins with edges $v_0v_1, v_1v_2, \dots, v_{r-1}v_r$. Now $V(T) = V(P)$ since $t(G) \ge |T| \ge |P| \ge t(G)$. By hypothesis G is non-elementary, so Theorem 1 implies that $V(T)$ is not strongly closed; thus, T has a defective color δ with respect to φ. Choose $\tau \in \overline{\varphi}(v_2)$. Let $Q = P_{v_2}(\tau, \delta)$. Since T is maximal, δ is not missing at any vertex of T, and since $V(T)$ is elementary, τ is not missing at any vertex of T other than v_2. As a result, Q ends outside $V(T)$. Now Q could leave $V(T)$ and re-enter it repeatedly, but Q ends outside $V(T)$, so there is a last vertex $w \in V(Q) \cap V(T)$; say Q ends at $z \in V(G) \setminus V(T)$. Let $\pi \notin \{\alpha, \beta\}$ be a color missing at w. Since $\tau \in \overline{\varphi}(v_2)$ and $\pi \in \overline{\varphi}(w)$ and $|T| = t(G)$,

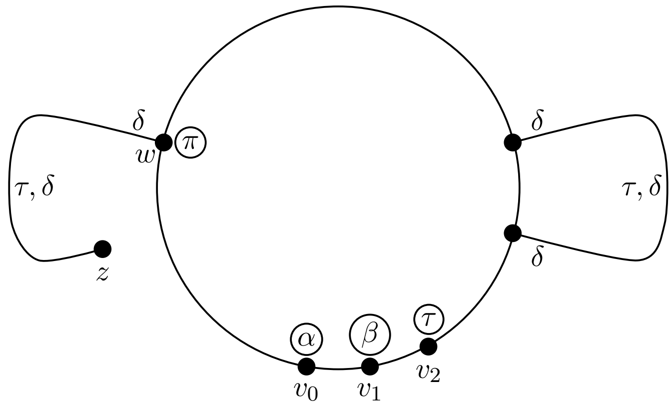

Figure 1: $Q \cup (P_{v_1}(\alpha, \beta) + v_0v_1)$ in the proof of Lemma 2. By recoloring some of wQz, we form a larger Tashkinov tree.

no edge colored τ or π leaves $V(T)$. So we can swap τ and π on every edge in $G - V(T)$ without changing the fact that T is a Tashkinov tree with $|T| = t(G)$. After swapping τ and π, we swap δ and π on the subpath of Q from w to z. Since π is missing at w, the $\delta - \pi$ path starting at z must end at w. Now δ is missing at w, but δ was defective in φ, so some other edge e colored δ still leaves $V(T)$. Adding e gets a larger Tashkinov tree, which is a contradiction. $\square$

3 Short vertices and long vertices

A vertex $v \in V(G)$ is *short* if every Vizing fan rooted at v (taken over all k-colorings of $G - e$, over all edges e incident to v) has at most 3 vertices, including v. Otherwise, v is *long*. Let $\nu(T)$ be the number of long vertices in a Tashkinov tree T.

short vertex
long vertex
$\nu(T)$

Now we can outline our proof of the $\frac{5}{6}$-Conjecture. We will show in Section 5 (and at the end of Section 4) that the $\frac{5}{6}$-Conjecture is implied by the Goldberg–Seymour Conjecture. More precisely, if Q is the line graph of graph G and $\chi(Q) = \chi'(G) \le \max\{\mathcal{W}(G), \Delta(G)+1\}$, then also $\chi(Q) \le \max\{\omega(Q), \frac{5\Delta(Q)+8}{6}\}$. So here it suffices to prove the bound $\chi'(G) \le \max\left\{\mathcal{W}(G), \Delta(G)+1, \frac{5\Delta(Q)+8}{6}\right\}$. We consider cases based on $\nu(T)$, for some Tashkinov tree $T \in \mathcal{T}(G)$.

In the present section, we show that if G has a maximum Tashkinov tree T that contains no long vertices, i.e., $\nu(T) = 0$, then G is elementary. In fact, Lemma 7 implies that the same is true when $\nu(T) = 1$. In the proof of Theorem 11, we show that if G is a minimal counterexample to the $\frac{5}{6}$-Conjecture, then every long vertex v has $d(v) < \frac{3}{4}\Delta(G)$. This implies that $\nu(T) < 4$, since otherwise the number of colors missing at vertices of T is more than $4(k - \frac{3}{4}\Delta(G)) > k$, which contradicts that $V(T)$ is elementary. So it remains to consider the case $\nu(T) \in \{2, 3\}$.

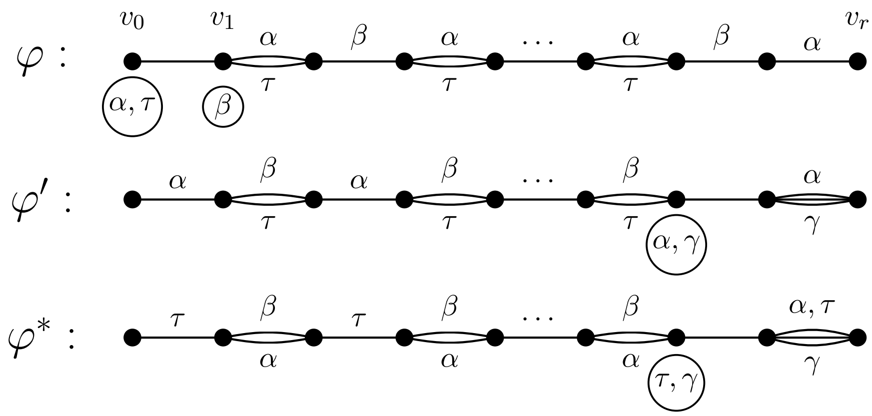

Figure 2: Edge-colorings φ, φ', and φ^* in the proof of the Parallel Edge Lemma.

In Section 4, we introduce *k-thin graphs*; these are essentially graphs for which $\mu(G)$ is not too large. Using a lemma from [8], we show that every minimal counterexample to the $\frac{5}{6}$-Conjecture must be k-thin. We then extend the ideas of the present section to handle the case when $\nu(T) \in \{2, 3\}$. Much like when $\nu(T) \geq 4$, we show that T has too many colors missing at its vertices to be elementary.

Short vertices were introduced in [2], where they were motivated by a version of the following lemma in the context of proving a strengthening of Reed's Conjecture for line graphs.

Parallel Edge Lemma. *Let G be a critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. Let φ be a k-edge-coloring of $G - v_0v_1$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Let $P = v_1v_2 \cdots v_r$ be an α, β path with edges $e_i = v_iv_{i+1}$ for all $i \in [r-1]$. If v_i is short for all odd i, then for each $\tau \in \overline{\varphi}(v_0)$ and for all odd $i \in [r-1]$ there are edges $f_i = v_iv_{i+1}$ such that $\varphi(f_i) = \tau$.*

Proof. Suppose not and choose a counterexample minimizing r. By minimality of r, we have $\varphi(v_{r-1}v_r) = \alpha$ and we have $f_i = v_iv_{i+1}$ for all odd $i \in [r-2]$ such that $\varphi(f_i) = \tau$; see Figure 2. Swap α and β on e_i for all $i \in [r-3]$ and then color v_0v_1 (call this edge e_0) with α and uncolor e_{r-2}. Let φ' be the resulting coloring. Since $k \geq \Delta(G) + 1$, some color other than α is missing at v_{r-2}; let γ be such a color. Now v_{r-1} is short since $r - 1$ is odd (since P starts and ends with α), so there is an edge $e = v_{r-1}v_r$ with $\varphi'(e) = \gamma$. Swap τ and α on e_i for all i with $0 \leq i \leq r - 3$ to get a new coloring φ^*. Now γ and τ are both missing at v_{r-2} in φ^*. Since v_{r-1} is short, the fan with v_{r-2}, v_{r-1}, v_r and e implies that there is an edge $f_{r-1} = v_{r-1}v_r$ with $\varphi^*(f_{r-1}) = \tau$. But we have never recolored f_{r-1}, so $\varphi(f_{r-1}) = \tau$, which is a contradiction. □

Lemma 3. *Let G be a non-elementary critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. Choose $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$ for some $v_0v_1 \in E(G)$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $P = P_{v_1}(\alpha, \beta)$. Now P contains a long vertex. In particular, $\nu(T) \geq 1$.*

Proof. Suppose every vertex of P is short. Applying the Parallel Edge Lemma to P shows that for every $\tau \in \overline{\varphi}(v_0)$, there is an edge in T colored τ incident to every $v \in V(P - v_0)$. The same is also true of every other color missing at some vertex of P; to see this, we rotate the α, β coloring of $P \cup \{v_0v_1\}$ and repeat the same argument. Hence $V(P) = V(T)$, which contradicts Lemma 2. □

Theorem 4. *If G is a critical graph in which every vertex is short, then*

$$\chi'(G) \leq \max\{\mathcal{W}(G), \Delta(G) + 1\}.$$

Proof. Suppose not and let G be a counterexample. Let $k = \chi'(G) - 1$, and note that $k \geq \Delta(G) + 1$. Since $\mathcal{T}(G) \neq \emptyset$, by applying Lemma 3 we conclude that G is elementary. Hence $\chi'(G) = \mathcal{W}(G)$, which is a contradiction. □

For a path Q, recall that $\ell(Q)$ denotes the length of Q. For $x, y \in V(Q)$, let xQy denote the subpath of Q with end vertices x and y, and let $d_Q(x, y) = \ell(xQy)$, i.e., the distance from x to y along Q.

Lemma 5. *Let G be a critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. Let φ be a k-edge-coloring of $G - v_0v_1$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. If $\tau \in \overline{\varphi}(x)$ for some $x \in V(C)$ and there is a τ-colored edge from $y \in V(C)$ to $w \in V(G) \setminus V(C)$, then C has a subpath Q with long endpoints z_1, z_2 such that $x \in V(Q)$, $y \notin V(Q - z_1 - z_2)$ and the distance from x to z_i along Q is odd for each $i \in [2]$. Moreover, for each $i \in [2]$, there are no τ-colored edges between z_i and its neighbors along C.*

Proof. Let G, α, β, τ, x, and y be as in the statement of the lemma. Choose z_1 (resp. z_2) to be the first vertex at an odd distance from x along C in the clockwise (resp. counterclockwise) direction with no incident τ-colored edge parallel to some edge of C. Let Q be the subpath of C with endpoints z_1 and z_2 that contains x. By the choice of z_1 each vertex w between x and z_1 with $d_Q(x, w)$ odd has a τ-colored edge parallel to some edge of C. The presence of these edges implies the same for each w for which $d_Q(x, w)$ is even. By the proof of the Parallel Edge Lemma, z_1 must be long, since otherwise it would have an incident τ-colored edge parallel to some edge of C. The same argument applies to z_2. □

4 Thin graphs

Let G be a critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. For vertices $x \in V(G)$ and $S \subseteq V(G) \setminus \{x\}$, we say that x is *S-short* if every Vizing fan F rooted at x with $(\{x\} \cup S) \subseteq V(F)$, has $|F| \leq 3$ (with respect to any k-edge-coloring of $G - xy$). Otherwise, x is *S-long*. For brevity, when $S = \{y\}$, we may write y-short instead of $\{y\}$-short. It is worth noting that in the Parallel Edge Lemma we can weaken the hypothesis that v_i is short for all odd i to require only that v_i is v_{i-1}-short for all odd i, since this is what we use in the proof.

S-short

S-long

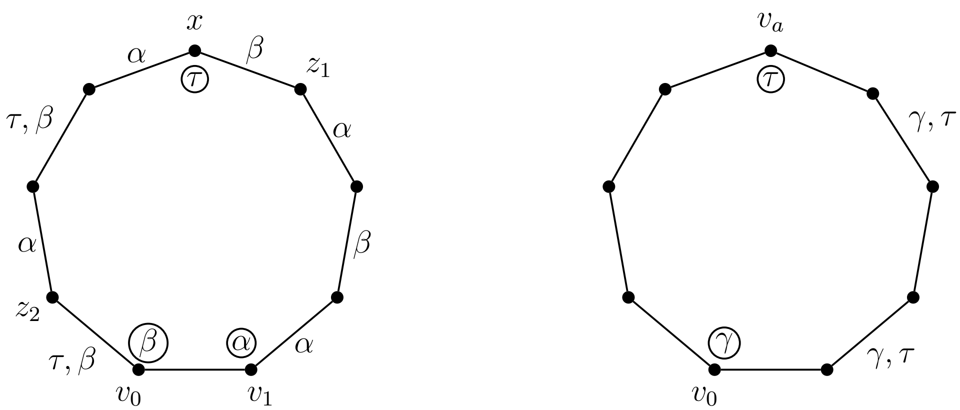

Figure 3: The proofs of Lemmas 6 and 7.

A graph G is *k-thin* if $\mu(G) < 2k - d(x) - d(y)$ for all distinct long $x, y \in V(G)$. In the proof of Theorem 11, we will show that every minimum counterexample to the $\frac{5}{6}$-Conjecture must be k-thin. k-thin

Lemma 6. *Let G be a k-thin, critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. Let φ be a k-edge-coloring of $G - v_0v_1$. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. Let Q be a subpath of C with long end vertices. If all internal vertices of Q are short and $2 \leq \ell(Q) \leq \ell(C) - 2$, then $\ell(Q)$ is even.*

Proof. Suppose to the contrary that we have a subpath Q of C with end vertices long, all internal vertices short, $2 \leq \ell(Q) \leq \ell(C) - 2$, and $\ell(Q)$ odd. Let x and y be the end vertices of Q. Say $C = v_1v_2 \cdots v_rv_0v_1$. By rotating the α, β coloring of C, we may assume that $x = v_0$ and $y = v_a$, where $a \geq 3$ is odd.

We now apply the Parallel Edge Lemma twice, to show that $\mu(v_1v_2) \geq 2k - d(v_0) - d(v_a)$, which contradicts that G is k-thin. More specifically, assume that the edges $v_0v_1, v_1v_2, \ldots$ go clockwise around C. We apply the Parallel Edge Lemma once going clockwise starting from v_0 and once going counterclockwise starting from v_a. The first application implies that every color in $\overline{\varphi}(v_0)$ appears on some edge parallel to v_1v_2; the second implies the same for every color in $\overline{\varphi}(v_a)$. Since $|\overline{\varphi}(v_i)| = k - d(v_i)$ for each $i \in \{0, a\}$ and $\overline{\varphi}(v_0) \cap \overline{\varphi}(v_a) = \emptyset$, the conclusion follows. $\square$

Lemma 7. *Let G be a k-thin, critical graph with $\chi'(G) = k + 1$ for some integer $k \geq \Delta(G) + 1$. Let φ be a k-edge-coloring of $G - v_0v_1$. Suppose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ and let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. If $\ell(C) \geq 5$ and C contains exactly 3 long vertices, then $C = xyAzBx$ where A and B are paths of even length and x, y, z are all long. Moreover, x is y-long and y is x-long.*

Proof. Let G be a graph satisfying the hypotheses, and let x, y, z be the three long vertices. The three subpaths of C with endpoints x, y, and z either (i) all have odd length or (ii) include two paths of even length and one of odd length. If we are in (i), then the longest of these three subpaths violates Lemma 6; so we are in (ii), and also the path of odd length is simply an edge. This proves the first statement. For the second statement, assume to the contrary that x is y-short. By rotating the α, β coloring, we can assume that $y = v_0$ and $x = v_1$. As in the previous lemma, we use the Parallel Edge Lemma (and the comment in the first paragraph of Section 4) to conclude that $\mu(v_1v_2) \geq 2k - d(v_0) - d(z)$. As above, this contradicts that G is k-thin; this contradiction proves the second statement. □

Lemma 8. *Let G be a non-elementary, k-thin, critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G) + 1$. Choose $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$. If $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$, then $P_{v_1}(\alpha, \beta) + v_0v_1$ contains consecutive long vertices.*

Proof. Let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. By Lemma 2, there is $x \in V(C)$ and $\tau \in \overline{\varphi}(x)$ such that there is a τ-colored edge from $y \in V(C)$ to $w \in V(T) \setminus V(C)$. Lemma 5 implies that C has a subpath Q with $x \in V(Q)$ and long endpoints z_1, z_2 such that the distance from x to z_i along Q is odd for each $i \in [2]$. Let Q' be the subpath of C with endpoints z_1 and z_2 that does not contain x. Since C is an odd cycle, $\ell(Q')$ is odd. Let Q^* be a minimum length subpath of Q' with long ends. Now $\ell(Q^*) = 1$ by Lemma 6, as desired. □

Lemma 9. *Let G be a non-elementary, k-thin, critical graph with $\chi'(G) = k+1$ for some integer $k \geq \Delta(G) + 1$. If $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$ and $\nu(T) \leq 3$, then T contains long vertices z_1, z_2, z_3 such that either*

1. *z_1 is $\{z_2, z_3\}$-long and z_2 is z_1-long; or*
2. *z_i is z_j-long and z_j is z_i-long for each $(i, j) \in \{(1, 2), (2, 3)\}$.*

Proof. Choose $\alpha \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$ so that $P_{v_1}(\alpha, \beta)$ contains as many long vertices as possible and, subject to that, $P_{v_1}(\alpha, \beta)$ is as long as possible. Let $C = P_{v_1}(\alpha, \beta) + v_0v_1$. By Lemma 2, there is $x \in V(C)$ and $\tau \in \overline{\varphi}(x)$ such that there is a τ-colored edge from $y \in V(C)$ to $w \in V(T) \setminus V(C)$. By Lemma 8, C has at least 2 long vertices.

First suppose that C contains only 2 long vertices, z_1 and z_2; see the left side of Figure 4. By Lemma 8, z_1 and z_2 are consecutive on C. Lemma 5 implies that C has a subpath Q with endpoints z_1, z_2 such that $x \in V(Q)$ and $y \notin V(Q - z_1 - z_2)$ and for each $i \in [2]$ there are no τ-colored edges between z_i and its neighbors on C. By rotating the α, β coloring of C, we can assume that $x = v_0$ and $\alpha, \tau \in \overline{\varphi}(v_0)$ and $\beta \in \overline{\varphi}(v_1)$. Note that $P_{v_1}(\tau, \beta)$ must end at v_0 (since otherwise we can recolor the Kempe chain and color v_0v_1 with τ). Let $C' = P_{v_1}(\tau, \beta) + v_0v_1$. Note that C' must include v_1Qz_1 and also v_0Qz_2 (the β-colored edges are present by definition and the τ-colored edges are present by the the Parallel Edge Lemma). Thus, $z_1, z_2 \in V(C')$. Since z_1 and z_2 are not consecutive on C' and C' contains no other long vertices by the maximality condition on C, Lemma 8 gives a contradiction.

So instead assume that C contains exactly 3 long vertices. Now we prove that $\ell(C) \geq 5$. Suppose, to the contrary, that $C = v_0v_1v_2$ and each vertex is a long vertex. By Lemma 2, some color that is missing at a vertex of C is used on an edge leaving C. By symmetry, assume that color τ is missing at v_0 and is used on an edge v_1y, where $y \notin V(C)$. Uncolor

v_2v_0 and color v_0v_1 with β. Now the β, τ path starting at v_0 ends at v_2, has length at least 4 and contains v_0, v_1, v_2. So the union of v_0v_2 with this path gives a longer C, a contradiction.

By Lemma 7, $C = z_1z_2Az_3Bz_1$ where A and B are paths of even length. Also, z_1 is z_2-long and z_2 is z_1-long. By Lemma 5, C has a subpath Q with endpoints z_1, z_3 and with $x \in V(Q)$ and $y \notin V(Q - z_1 - z_3)$ such that there are no τ-colored edges between z_i and its neighbors along C for each $i \in \{1, 3\}$ (it could happen that z_3 has a τ-colored edge parallel to an edge of C, so the endpoints of Q are z_1, z_2, but now we get a contradiction as in the previous case, by letting $C' = P_{v_1}(\tau, \beta) + v_0v_1$). By rotating the α, β coloring of C, we may assume that $x = v_0$. Again, let $C' = P_{v_1}(\tau, \beta) + v_0v_1$. We know that C' contains z_1 and z_3 and that z_1 and z_2 are not consecutive on C'. Note also that all long vertices in $V(C')$ must be among z_1, z_2, z_3, since otherwise $\nu(T) \geq 4$, contradicting our hypothesis. So by Lemma 8, either z_1 and z_3 are consecutive on C' or z_2 and z_3 are consecutive on C'.

Suppose that z_2 and z_3 are consecutive on C', and thus connected by a τ-colored edge. Now applying Lemma 7 shows that z_2 is z_3-long and z_3 is z_2-long, so we satisfy (2) in the conclusion of the lemma (by swapping the names of z_1 and z_2).

So instead z_1 and z_3 must be consecutive on C', and thus connected by a τ-colored edge. If $z_1 = v_1$, then we have a fan with an α-colored edge from z_1 to z_2 and a τ-colored edge from z_1 to z_3, so z_1 is $\{z_2, z_3\}$-long.

Now assume that $z_1 \neq v_1$; see the right side of Figure 4. Let z_1' be the predecessor of z_1 on the path from v_0 (through v_1) to z_1. We can shift the coloring so that $z_1'z_1$ is uncolored and z_1z_2 is colored α (as in the proof of the Parallel Edge Lemma). In fact, we can shift either the α, β edges or the τ, β edges. This gives the options that either $\alpha \in \overline{\varphi}(z_1')$ or $\tau \in \overline{\varphi}(z_1')$, whichever we prefer. Suppose we shift the τ, β edges. Now choose $\gamma \in \overline{\varphi}(z_1') - \alpha - \tau$. Consider the γ-colored edge e incident to z_1. If e goes to z_2, then we z_1 is $\{z_2, z_3\}$-long, by colors γ and τ; so we satisfy (1) in the conclusion of the lemma. If instead e goes to z_3, then instead of shifting the τ, β edges we shift the α, β edges; note that this recoloring preserves the fact that γ is missing at z_1'. Now again z_1 is $\{z_2, z_3\}$-long, this time by colors α and γ; so we again satisfy (1) in the conclusion of the lemma.

Finally, assume that the γ-colored edge incident to z_1 goes to some vertex other than z_2 and z_3. Now let $C'' = P_{z_1}(\gamma, \beta) + z_1z_1'$. Since $V(C'') \subseteq V(T)$, Lemmas 8 and 7 imply that z_2 and z_3 are adjacent on C'' and furthermore z_2 is z_3-long and z_3 is z_2-long; thus, we satisfy (2) in the conclusion of the lemma. □

We need the following result from [8], which we use to handle the case when G is not k-thin.

Theorem 10 ([8])**.** *If Q is the line graph of a graph G and Q is vertex critical, then*

$$\chi(Q) \leq \max\left\{\omega(Q), \Delta(Q) + 1 - \frac{\mu(G) - 1}{2}\right\}.$$

Now we prove the main result of this section, the Weak $\frac{5}{6}$-Theorem. It encapsulates most of what we will need from the first four sections when we prove our main result, the $\frac{5}{6}$-Theorem, in Section 5.

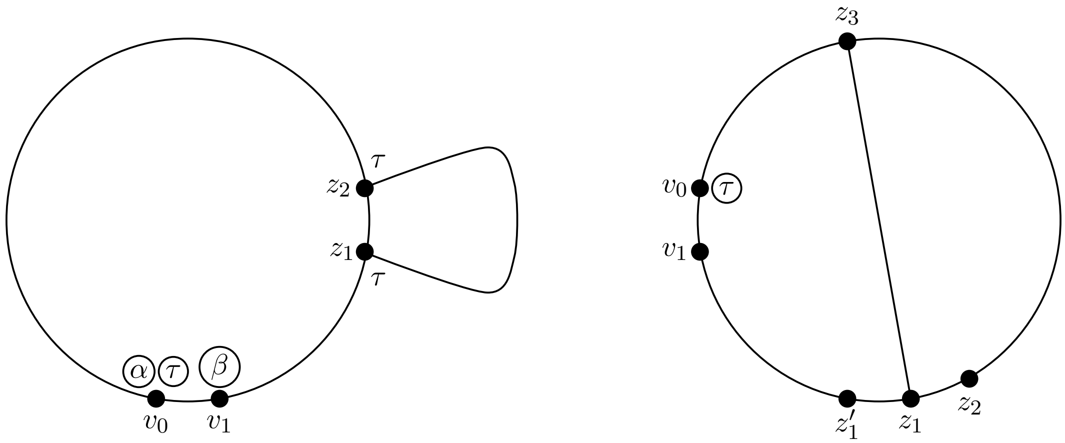

Figure 4: Two parts of the proof of Lemma 9.

Theorem 11 (Weak $\frac{5}{6}$-Theorem)**.** *If Q is the line graph of G, then*

$$\chi(Q) \leq \max\left\{\mathcal{W}(G), \Delta(G)+1, \frac{5\Delta(Q)+8}{6}\right\}.$$

Proof. Suppose the theorem is false and choose a counterexample minimizing $|Q|$. Let $k = \max\left\{\mathcal{W}(G), \Delta(G)+1, \left\lfloor\frac{5\Delta(Q)+8}{6}\right\rfloor\right\}$. Say $Q = L(G)$ for a graph G. The minimality of Q implies that G is critical and $\chi(Q) = k+1$, for some $k \geq \Delta(G)+1$.

The heart of the proof is Claim 1, which roughly says that if x is long, then $d(x) < \frac{3}{4}\Delta(G)$. Moreover, we can improve this bound further if x is the root of a long fan F such that either (i) F has length more than 3 or (ii) some of the other vertices in F have degree less than $\Delta(G)$. The claims thereafter are all essentially applications of Claim 1.

Claim 1. *Let F be a fan rooted at x with respect to a k-edge-coloring of $G - xy$. If $S \subseteq V(F) - x$ and $|S| \geq 3$, then*

$$d(x) \leq \frac{1}{5|S|-11}\left(2|S| - 12 + \sum_{v\in S} d(v)\right).$$

In particular, if $|S| = 3$, then $d(x) \leq \frac{1}{4}\left(-6 + \sum_{v\in S} d(v)\right)$.
Proof: Since F is elementary, we have

$$2 + k - d(x) + \sum_{v\in S} k - d(v) \leq k,$$

so

$$2 + |S|\,k \leq d(x) + \sum_{v\in S} d(v).$$

Using $k \geq \frac{5}{6}(\Delta(Q)+1) - \frac{1}{3} \geq \frac{5}{6}(d(x)+d(v)-\mu(xv)) - \frac{1}{3}$ for each $v \in S$, we get

$$2 + \sum_{v \in S} \left(\frac{5}{6}(d(x)+d(v)-\mu(xv)) - \frac{1}{3} \right) \leq d(x) + \sum_{v \in S} d(v),$$

so multiplying by 6 and rearranging terms gives

$$12 + (5\,|S| - 6)\, d(x) - 2\,|S| \leq \sum_{v \in S} 5\mu(xv) + \sum_{v \in S} d(v).$$

Now $\sum_{v \in S} \mu(xv) \leq d(x)$, so this implies

$$12 + (5\,|S| - 11)\, d(x) - 2\,|S| \leq \sum_{v \in S} d(v).$$

Solving for $d(x)$ gives

$$d(x) \leq \frac{1}{5\,|S| - 11} \left(2\,|S| - 12 + \sum_{v \in S} d(v) \right),$$

and when $|S| = 3$, we get $d(x) \leq \frac{1}{4}\left(-6 + \sum_{v \in S} d(v)\right)$. $\square$

Claim 2. *If $x \in V(G)$ is long, then $d(x) \leq \frac{3}{4}\Delta(G) - 1$.*
Proof: This is immediate from Claim 1, since $d(v) \leq \Delta(G)$ for all $v \in S$. $\square$

Claim 3. *If $x_1x_2 \in E(G)$ such that x_1 is x_2-long and x_2 is x_1-long, then*

$$d(x_i) \leq \frac{2}{3}\Delta(G) - 2 \text{ for all } i \in [2].$$

Proof: By Claim 1, for each $i \in [2]$,

$$d(x_i) \leq \frac{1}{4}\left(-6 + \sum_{v \in S} d(v)\right) \leq \frac{1}{4}\left(-6 + d(x_{3-i}) + 2\Delta(G)\right),$$

Substituting the bound on $d(x_{3-i})$ into that on $d(x_i)$ and simplifying gives for each $i \in [2]$,

$$d(x_i) \leq -2 + \frac{2}{3}\Delta(G).$$

$\square$

Claim 4. *If $x_1x_2, x_1x_3 \in E(G)$ such that x_1 is $\{x_2, x_3\}$-long, x_2 is x_1-long and x_3 is long, then*

$$d(x_1) \leq -\frac{8}{5} + \frac{3}{5}\Delta(G),$$

$$d(x_2) \leq -\frac{7}{5} + \frac{13}{20}\Delta(G).$$

Proof: By Claim 1, we have

$$d(x_1) \leq \frac{1}{4}\left(-6 + \sum_{v \in S} d(v)\right) \leq \frac{1}{4}\left(-6 + d(x_2) + d(x_3) + \Delta(G)\right),$$

$$d(x_2) \leq \frac{1}{4}\left(-6 + \sum_{v \in S} d(v)\right) \leq \frac{1}{4}\left(-6 + d(x_1) + 2\Delta(G)\right).$$

By the same calculation as in Claim 3, these together imply

$$d(x_1) \leq -2 + \frac{2}{5}\Delta(G) + \frac{4}{15}d(x_3).$$

Since x_3 is long, using Claim 2, we get

$$d(x_1) \leq -\frac{34}{15} + \frac{3}{5}\Delta(G),$$

and hence

$$d(x_2) \leq -\frac{61}{15} + \frac{13}{20}\Delta(G).$$

□

Claim 5. *The theorem is true.*
Proof: Let $(T, v_0v_1, \varphi) \in \mathcal{T}(G)$. By Lemma 9, one of the following holds:

1. G is elementary; or
2. G is not k-thin; or
3. $\nu(T) = 3$ and $V(T)$ contains vertices x_1, x_2, x_3 such that x_1 is x_2-long, x_2 is x_1-long, x_2 is x_3-long, and x_3 is x_2-long; or
4. $\nu(T) = 3$ and $V(T)$ contains vertices x_1, x_2, x_3 such that x_1 is $\{x_2, x_3\}$-long, x_2 is x_1-long, and x_3 is long; or
5. $V(T)$ contains four long vertices x_1, x_2, x_3, x_4.

If (1) holds, then $\chi(Q) = \mathcal{W}(G)$, which contradicts our choice of Q as a counterexample. If (2) holds, then Claim 2 implies that $\mu(G) \geq 2k - \frac{3}{2}\Delta(G) + 2$. Now Theorem 10 gives

$$\begin{aligned} k + 1 &\leq \Delta(Q) + 1 - \frac{2k - \frac{3}{2}\Delta(G) + 2}{2} \\ &= \Delta(Q) + 1 - k + \frac{3}{4}\Delta(G) - 1, \end{aligned}$$

so

$$2(k+1) \le \Delta(Q) + 1 + \frac{3}{4}\Delta(G).$$

Substituting $\Delta(G) \le k$ and solving for k gives

$$k \le \frac{4}{5}\Delta(Q) - \frac{4}{5} < \frac{5}{6}\Delta(Q) + \frac{1}{2} \le k,$$

which is a contradiction.

Suppose (3) holds. Now

$$2 + \sum_{i \in [3]} k - d(x_i) \le k,$$

so Claim 3 implies

$$3\left(\frac{2}{3}\Delta(G) - 2\right) \ge 2k + 2,$$

which is a contradiction, since $\Delta(G) \le k$.

Suppose (4) holds. Now

$$2 + \sum_{i \in [3]} k - d(x_i) \le k,$$

so Claims 2 and 4 give

$$\left(\frac{3}{5} + \frac{13}{20} + \frac{3}{4}\right)\Delta(G) - \left(\frac{34}{15} + \frac{16}{15} + 1\right) \ge 2k + 2,$$

which is

$$2\Delta(G) - \frac{13}{3} \ge 2k + 2,$$

again a contradiction, since $\Delta(G) \le k$.

So (5) must hold. But now

$$2 + \sum_{i \in [4]} k - d(x_i) \le k,$$

so using Claim 2 gives

$$4\left(\frac{3}{4}\Delta(G) - 1\right) \ge 3k + 2,$$

a contradiction since $\Delta(G) \le k$. □

This finishes the final case of Claim 5, which proves the theorem. □

In the previous theorem, we showed that $\chi(Q) \le \max\left\{\mathcal{W}(Q), \Delta(G) + 1, \frac{5\Delta(Q)+8}{6}\right\}$. Now we show that if the maximum is attained by the second argument (and Q is vertex critical), then Q satisfies the $\frac{5}{6}$-Conjecture. We use the following lemma, which is implicit in [8]; see the proof of Lemma 9 therein.

Lemma 12. *If Q is the line graph of a graph G and Q is vertex critical, then*

$$\chi(Q) \leq \max\left\{\Delta(G), \Delta(Q) + 1 + 2\mu(G) - \Delta(G)\right\}.$$

Corollary 13. *If Q is the line graph of a critical graph G and $\chi(Q) \leq \Delta(G) + 1$, then*

$$\chi(Q) \leq \max\left\{\omega(Q), \frac{5\Delta(Q) + 8}{6}\right\}.$$

Proof. Let $k + 1 = \chi(Q) \leq \Delta(G) + 1$. Suppose $\chi(Q) > \omega(Q)$. Now Lemma 12 gives

$$k + 1 = \chi(Q) \leq \Delta(Q) + 1 + 2\mu(G) - k,$$

so solving for $\mu(G)$ gives

$$\mu(G) \geq k - \frac{\Delta(Q)}{2}.$$

Applying Theorem 10 gives

$$k + 1 = \chi(Q) \leq \Delta(Q) + 1 - \frac{k - \frac{\Delta(Q)}{2} - 1}{2},$$

and solving for $k + 1$ yields

$$\chi(Q) = k + 1 \leq \frac{5}{6}\Delta(Q) + \frac{4}{3} = \frac{5\Delta(Q) + 8}{6}.$$ □

Since $\omega(Q) \leq \max\{\Delta(G), \mathcal{W}(G)\}$, Theorem 11 and Corollary 13 together imply the following.

Corollary 14. *If Q is the line graph of a graph G, then*

$$\chi(Q) \leq \max\left\{\Delta(G), \mathcal{W}(G), \frac{5\Delta(Q) + 8}{6}\right\}.$$

Proof. Let Q be the line graph of a graph G. We assume that G is critical. If not, then choose $\widehat{G} \subseteq G$ such that $\widehat{G}$ is critical and $\chi'(\widehat{G}) = \chi'(G)$. Let $\widehat{Q} := L(\widehat{G})$. Now $\chi(Q) = \chi(\widehat{Q}) \leq \max\{\Delta(\widehat{Q}), \mathcal{W}(\widehat{Q}), \frac{5\Delta(\widehat{Q})+8}{6}\} \leq \max\{\Delta(Q), \mathcal{W}(Q), \frac{5\Delta(Q)+8}{6}\}$, as desired.

If $\chi(Q) > \Delta(G) + 1$, then Theorem 11 implies that $\chi(Q) \leq \max\{\mathcal{W}(G), \frac{5\Delta(Q)+8}{6}\}$. Otherwise, $\chi(Q) \leq \Delta(G) + 1$; since G is critical, Corollary 13 implies $\chi(Q) \leq \{\omega(Q), \frac{5\Delta(Q)+8}{6}\}$. Since $\omega(G) \leq \max\{\Delta(G), \mathcal{W}(G)\}$, the result follows. □

5 The $\frac{5}{6}$-Conjecture

In this section, we prove our main result, Theorem 20, that $\chi(Q) \le \max\{\omega(Q), \frac{5\Delta(Q)+8}{6}\}$, when Q is the line graph of a graph G. Throughout, we may assume that Q is a minimal counterexample, so Q is vertex critical. Observe that if $\chi(Q) \le \Delta(G)+1$, then the result follows immediately from Corollary 13. Thus, we may assume that $\chi(Q) > \Delta(G)+1$. Now, in view of Corollary 14, it suffices to show that $\chi(Q) = \chi'(G) \le \mathcal{W}(G)$ implies $\chi(Q) \le \max\{\omega(Q), \frac{5\Delta(Q)+8}{6}\}$.

Our approach is to show that if Q is a minimal counterexample and Q is the line graph of G, then $|N(x)| = 2$ for nearly every vertex $x \in V(G)$. This implies that the simple graph underlying G is very close to a cycle. By Corollary 14, we can assume that G is elementary, i.e., $\chi(Q) = \chi'(G) = W(G)$. Thus $\chi(Q) = \left\lceil \frac{2|E(G)|}{|V(G)|-1} \right\rceil \ge \Delta(G)+2$. So, to show that $|N(x)| = 2$ for nearly every $x \in V(G)$, it suffices to show that when $|N(x)| \ge 3$, we have $d(x) \le \frac{3}{5}\Delta(G)$. This is an immediate consequence of Lemma 18, which is the key technical result that we prove in this section.

It is helpful to note that in the present section the only results we use from previous sections are Corollary 13 and Corollary 14 (as well as Theorem 10, which is proved in [8]). In Lemmas 16–18, we prove bounds on $|N(x)|$ for each $x \in V(G)$. Ultimately, these lemmas yield Corollary 19, which says that $|N(x)| \ge 3$ for at most one vertex $x \in V(G)$. This corollary plays a key role in the proof of our main result, Theorem 20.

For reference, we record the following proposition, which is Proposition 1.3 in [12].

Proposition 15. *Let G be a graph that is critical and elementary, with $\chi'(G) = k+1$ for some integer $k \ge \Delta(G)$. Now $\|G\|$ is odd and $k = \frac{2(\|G\|-1)}{|G|-1}$.*

Lemma 16. *Let G be a critical, elementary graph with $\chi'(G) = k+1$ where $k \ge \Delta(G)+1$. Put $Q := L(G)$. If $k = \epsilon\,(\Delta(Q)+1)+\beta$, then for all $x \in V(G)$,*

$$|N(x)| = \frac{\epsilon\left(|G| - \Delta(G) - d_G(x) - 1 + S_1 + S_2 + S_3\left(|G|-1\right)\right)}{(1-\epsilon)\Delta(G) - \epsilon d_G(x) + 1 - \beta + S_3},$$

where

$$S_1 := \sum_{v \in N(x)} \Delta(Q) - d_Q(xv),$$

$$S_2 := 2 + \sum_{v \in V(G)\setminus N(x)} \Delta(G) - d_G(v),$$

$$S_3 := k - (\Delta(G)+1).$$

Proof. The details of the proof are somewhat tedious, however, the idea is simple. We write down a system of equations and solve for $|N(x)|$. The main insight needed is knowing which quantities to consider. We use ϵ, β, S_1, S_2, S_3, and $\sum_{v \in N(x)} d_G(v)$.

Since G is critical and elementary, Proposition 15 implies that $|G|$ is odd and

$$k = \frac{2(\|G\| - 1)}{|G| - 1}. \tag{1}$$

Choose $x \in V(G)$ and put $M := |N(x)|$ and

$$P := \sum_{v \in N(x)} d_G(v).$$

Grouping edges by whether or not they are incident to any neighbor of x gives

$$2(\|G\| - 1) = \Delta(G)(|G| - M) - S_2 + P. \tag{2}$$

By (1), and by the definition of S_3,

$$\frac{2(\|G\| - 1)}{|G| - 1} = k = \Delta(G) + 1 + S_3.$$

Now clearing the denominator and using (2) to substitute, we get

$$P = (|G| - 1)(\Delta(G) + 1 + S_3) - \Delta(G)(|G| - M) + S_2.$$

After regrouping terms, this is

$$P = \Delta(G)(M - 1) + |G| - 1 + S_2 + S_3(|G| - 1). \tag{3}$$

Using $k = \epsilon\,(\Delta(Q) + 1) + \beta$, and the definition of S_1, we get

$$kM = \beta M + \epsilon S_1 + \epsilon \sum_{v \in N(x)} d_G(x) + d_G(v) - \mu(xv).$$

Since $\sum_{v \in N(x)} \mu(xv) = d_G(x)$, we have

$$kM = \beta M + \epsilon S_1 + \epsilon d_G(x)(M - 1) + \epsilon P. \tag{4}$$

Substituting (3) into (4) and solving for M gives

$$M = \frac{\epsilon\,(|G| - \Delta(G) - d_G(x) - 1 + S_1 + S_2 + S_3\,(|G| - 1))}{(1 - \epsilon)\Delta(G) - \epsilon d_G(x) + 1 - \beta + S_3},$$

as desired. □

Using $\epsilon = \frac{5}{6}$ in Lemma 16, we get the following.

Lemma 17. *Let G be a critical, elementary graph with $\chi'(G) = k+1$ where $k \geq \Delta(G)+1$. Put $Q := L(G)$. If $k = \frac{5}{6}(\Delta(Q)+1) + \beta$, then for all $x \in V(G)$,*

$$|N(x)| = \frac{5\left(|G| - \Delta(G) - d_G(x) - 1 + S_1 + S_2 + S_3\left(|G|-1\right)\right)}{\Delta(G) - 5d_G(x) + 6(1 - \beta + S_3)},$$

where

$$S_1 := \sum_{v \in N(x)} \Delta(Q) - d_Q(xv),$$

$$S_2 := 2 + \sum_{v \in V(G) \setminus N(x)} \Delta(G) - d_G(v),$$

$$S_3 := k - (\Delta(G) + 1).$$

Lemma 18. *Let G be a critical, elementary graph with $\chi'(G) = k+1$, where $k \geq \Delta(G)+1$. Put $Q := L(G)$. If $k = \frac{5}{6}(\Delta(Q)+1) + \beta$, where $\beta \geq -\frac{1}{3}$, then for all $x \in V(G)$ with $|N(x)| \geq 3$,*

$$d_G(x) \leq \frac{3}{5}\Delta(G) - \frac{1}{|N(x)|-2} \sum_{v \in V(G) \setminus N[x]} \Delta(G) - d_G(v).$$

If additionally, $|N(x)| \leq \frac{5}{8}|G|$, then

$$d_G(x) \leq \frac{|N(x)|}{5\left(|N(x)|-2\right)}\Delta(G) - \frac{1}{|N(x)|-2} \sum_{v \in V(G) \setminus N[x]} \Delta(G) - d_G(v).$$

Proof. We may assume that $|G| \geq 5$, since $|G|$ is odd by Proposition 15, and if $|G| = 3$, then $|N(x)| \leq 2$ for all $x \in V(G)$, so there is nothing to prove.

Choose $x \in V(G)$ with $|N(x)| \geq 3$. Let $S_4 = |N(x)| - 2$, and note that $S_4 \geq 1$. Applying Lemma 17 and simplifying using $S_1 \geq 0$ and $\beta \geq -\frac{1}{3}$ gives

$$(5 + 5S_4)d_G(x) \leq (7 + S_4)\Delta(G) - 5|G| + 21 + S_3(-5|G| + 17 + 6S_4) + 8S_4 - 5S_2. \quad (5)$$

Put

$$t := \sum_{v \in V(G) \setminus N[x]} \Delta(G) - d_G(v).$$

Now $S_2 = t + 2 + \Delta(G) - d_G(x)$. Using this in (5), we get

$$5S_4 d_G(x) \leq (2 + S_4)\Delta(G) - 5|G| + 11 + S_3(-5|G| + 17 + 6S_4) + 8S_4 - 5t. \quad (6)$$

When $|N(x)| \leq \frac{5}{8}|G|$, i.e., $S_4 \leq \frac{5}{8}|G| - 2$, we have

$$-5|G| + 11 + S_3(-5|G| + 17 + 6S_4) + 8S_4 \leq 0,$$

so dividing (6) through by $5S_4$ gives the desired bound.

So instead assume $S_4 > \frac{5}{8}|G| - 2$. Rearranging (6) gives

$$5S_4 d_G(x) \leq 3S_4\Delta(G) - (2S_4 - 2)\Delta(G) - 5|G| + 11 + S_3(-5|G| + 17 + 6S_4) + 8S_4 - 5t \quad (7)$$

Now $S_4 = |N(x)| - 2 = |G| - 3 + S_5$, where

$$S_5 := S_4 + 3 - |G| \leq 0.$$

Thus

$$-5|G| + 15 + 5S_4 + S_3(-5|G| + 15 + 5S_4) - 5S_5(1 + S_3) = 0.$$

Subtracting this equality from (7) gives

$$5S_4 d_G(x) \leq 3S_4\Delta(G) - (2S_4 - 2)\Delta(G) - 4 + 2S_3 + (S_3 + 3)S_4 + 5S_5(1 + S_3) - 5t \qquad (8)$$

If

$$-(2S_4 - 2)\Delta(G) - 4 + 2S_3 + (S_3 + 3)S_4 + 5S_5(1 + S_3) \leq 0,$$

then we have the desired bound

$$d_G(x) \leq \frac{3}{5}\Delta(G) - \frac{1}{|N(x)| - 2}\sum_{v \in V(G)\setminus N[x]} \Delta(G) - d_G(v).$$

So assume instead that

$$-4 + 2S_3 + (S_3 + 3)S_4 - (2S_4 - 2)\Delta(G) + 5S_5(1 + S_3) > 0,$$

which we rewrite as

$$(2 + S_4)S_3 + 3S_4 > (2S_4 - 2)\Delta(G) + 4 - 5S_5(S_3 + 1). \qquad (9)$$

By Shannon's theorem $k + 1 \leq \frac{3}{2}\Delta(G)$, so $S_3 \leq \frac{\Delta(G)}{2} - 2$. After plugging in for S_3 on the left side and solving for S_4, we get

$$S_5 + |G| - 3 = S_4 < \frac{6\Delta(G) - 16 + 10S_5(S_3 + 1)}{3\Delta(G) - 2} = 2 + \frac{10S_5(S_3 + 1) - 12}{3\Delta(G) - 2},$$

so

$$|G| < 5 + \frac{(10S_3 - 3\Delta(G) + 12)S_5 - 12}{3\Delta(G) - 2}.$$

Since $S_5 \leq 0$, this implies $|G| \leq 3$, unless $10S_3 - 3\Delta(G) + 12 < 0$. So $S_3 < \frac{3}{10}\Delta(G) - \frac{6}{5}$. Since $S_5 \leq 0$, (9) implies

$$(2 + S_4)S_3 + 3S_4 > (2S_4 - 2)\Delta(G) + 4$$

Substituting $S_3 < \frac{3}{10}\Delta(G) - \frac{6}{5}$ gives

$$|G| - 3 \leq S_4 < \frac{26\Delta(G) - 64}{17\Delta(G) - 18} < 2,$$

which contradicts that $|G| \geq 5$. $\square$

Corollary 19. *Let G be a critical, elementary graph with $\chi'(G) = k+1$, where $k \geq \Delta(G)+1$. Put $Q := L(G)$. If $k = \frac{5}{6}(\Delta(Q)+1)+\beta$, where $\beta \geq -\frac{1}{3}$, then there is at most one $x \in V(G)$ with $|N(x)| \geq 3$.*

Proof. Since G is critical and elementary, Proposition 15 implies that $|G|$ is odd and

$$\frac{2(\|G\|-1)}{|G|-1} = k \geq \Delta(G)+1,$$

so

$$2\|G\| \geq \Delta(G)\,|G| + |G| - \Delta(G) + 1.$$

In particular,

$$\sum_{v \in V(G)} \Delta(G) - d_G(v) \leq \Delta(G) - 1 - |G|. \tag{10}$$

By Lemma 18, every $x \in V(G)$ with $|N(x)| \geq 3$ has $d_G(x) \leq \frac{3}{5}\Delta(G)$, so there are at most two such x since $\frac{2}{5}+\frac{2}{5}+\frac{2}{5} > 1$. Suppose there are x_1, x_2 with $|N(x_1)| \geq |N(x_2)| \geq 3$.

Choose $z \in V(G)$ with $d_G(z) = \Delta(G)$. By Lemma 18, $|N(z)| = 2$, so $\mu(G) \geq \frac{1}{2}\Delta(G)$. By Theorem 10, $\mu(G) < \frac{1}{3}(\Delta(Q)+1)$. We conclude

$$\Delta(G) < \frac{2}{3}(\Delta(Q)+1). \tag{11}$$

First, suppose x_1 and x_2 are adjacent. Since Q is vertex-critical,

$$\begin{aligned}
k &\leq d_Q(x_1x_2)\\
&= d_G(x_1) + d_G(x_2) - \mu(x_1x_2) - 1\\
&\leq \frac{6}{5}\Delta(G) - \mu(x_1x_2) - 1,
\end{aligned}$$

So, $\Delta(G) > \frac{5}{6}k$. By (11),

$$\frac{5}{6}k < \Delta(G) < \frac{2}{3}(\Delta(Q)+1),$$

and hence $k < \frac{4}{5}(\Delta(Q)+1)$, a contradiction.

So instead assume x_1 and x_2 are non-adjacent. Suppose also that $|N(x_2)| \leq \frac{5}{8}|G|$. If $|N(x_2)| \geq 4$, then $d_G(x_2) \leq \frac{2}{5}\Delta(G)$. Now (10) and Lemma 18 give

$$\begin{aligned}
\Delta(G) - 1 - |G| &\geq \sum_{v \in V(G)} \Delta(G) - d_G(v)\\
&\geq (\Delta(G) - d_G(x_1)) + (\Delta(G) - d_G(x_2))\\
&\geq \left(\frac{2}{5} + \frac{3}{5}\right)\Delta(G) = \Delta(G),
\end{aligned}$$

a contradiction.

So assume that $|N(x_2)| = 3$. Since x_1 and x_2 are non-adjacent, Lemma 18 gives

$$d_G(x_2) \le \frac{3}{5}\Delta(G) - (\Delta(G) - d_G(x_1)) \le \frac{1}{5}\Delta(G).$$

Similar to above, we get a contradiction since $\Delta(G) > \left(\frac{4}{5} + \frac{2}{5}\right)\Delta(G) > \Delta(G)$.

So assume that $|N(x_i)| > \frac{5}{8}|G|$ for each $i \in [2]$. In particular, there is $y \in N(x_1) \cap N(x_2)$. Since $|N(y)| = 2$, by symmetry we may assume $\mu(x_1 y) \ge \frac{1}{2} d_G(y)$. Hence, using

$$\begin{aligned}
k &\le d_Q(x_1 y) \\
&= d_G(x_1) + d_G(y) - \mu(x_1 y) < d_G(x_1) + \frac{1}{2} d_G(y) \\
&\le \frac{3}{5}\Delta(G) + \frac{1}{2}\Delta(G) = \frac{11}{10}\Delta(G) \\
&< \frac{11}{15}(\Delta(Q) + 1),
\end{aligned}$$

where the final inequality holds by (11). This contradiction completes the proof. □

Now we prove the Main Theorem.

Theorem 20 ($\frac{5}{6}$-Theorem)**.** *If Q is a line graph, then*

$$\chi(Q) \le \max\left\{\omega(Q), \frac{5\Delta(Q) + 8}{6}\right\}.$$

Proof. It is convenient to note that, since $\chi(Q)$ is an integer, $\chi(Q) \le \frac{5\Delta(Q)+8}{6}$ if and only if $\chi(Q) \le \left\lceil \frac{5\Delta(Q)+3}{6} \right\rceil$. For the present proof, it is simpler to work with the latter formulation.

Suppose the theorem is false and choose a counterexample Q minimizing $|Q|$. Now $Q = L(G)$ for a critical graph G. Say $\chi(Q) = \chi'(G) = k + 1$. So $k = \max\left\{\omega(Q), \left\lceil \frac{5\Delta(Q)+3}{6} \right\rceil\right\}$ by the minimality of $|Q|$. By Corollary 14,

$$k + 1 \le \max\left\{\mathcal{W}(G), \left\lceil \frac{5\Delta(Q) + 3}{6} \right\rceil\right\},$$

so

$$\mathcal{W}(G) = \left\lceil \frac{5\Delta(Q) + 3}{6} \right\rceil + 1 = \chi(Q).$$

Therefore G is elementary and $k = \frac{5}{6}(\Delta(Q) + 1) + \beta$ for some $\beta \ge -\frac{1}{3}$. By Corollary 13, $k \ge \Delta(G) + 1$. Let H be the underlying simple graph of G. We may apply Corollary 19 to conclude that there is at most one $x \in V(G)$ with $d_H(x) \ge 3$. Since G is critical, $\delta(H) \ge 2$ and H has no cut vertices. Hence H is a cycle.

Choose t such that $|V(H)| = 2t+1$. Let $x_1, \ldots, x_{2t+1}$ denote the multiplicities of the edges in G, and let $X = \sum_{i=1}^{2t+1} x_i$. Since G is elementary, we have $\chi'(G) = \lceil \frac{X}{t} \rceil$. Let $Q = L(G)$ and let v_i be a vertex of Q corresponding to an edge of G counted by x_i. Now $d_Q(v_i) = x_{i-1} + x_i + x_{i+1} - 1$. It suffices to show that there exists $j \in [2t+1]$ such that $\frac{X}{t} \le \frac{5d_Q(v_j)+3}{6}$. We will prove the stronger statement that $\frac{X}{t} \le \frac{5\bar{d}+3}{6}$, where $\overline{d} = \frac{1}{2t+1}\sum_{i=1}^{2t+1} d_Q(v_i)$. Since $\frac{5\bar{d}+3}{6} = \frac{5X}{2(2t+1)} - \frac{1}{3}$, it suffices to have $\frac{5X}{2(2t+1)} - \frac{1}{3} \ge \frac{X}{t}$. Simplifying (for $t \ge 3$) gives $X \ge \frac{1}{3}(4t + 10 + \frac{20}{t-2})$. Since $X \ge 2t+1$, this always holds when $t \ge 6$. When $t = 5$, it suffices to have $X \ge 13$. When $t = 4$, it suffices to have $X \ge 12$, and when $t = 3$, it suffices to have $X \ge 14$. Suppose $t = 5$ and $X \le 12$. Now $\chi'(G) \le \lceil \frac{12}{5} \rceil = 3 \le \left\lceil \frac{5(2)+3}{6} \right\rceil$. Suppose instead that $t = 4$ and $X \le 11$. Now $\chi'(G) \le \lceil \frac{11}{4} \rceil = 3 \le \left\lceil \frac{5(2)+3}{6} \right\rceil$. Suppose instead that $t = 3$ and $X \le 13$. If $X \le 9$, then again $\chi'(G) \le \lceil \frac{9}{3} \rceil = 3 \le \left\lceil \frac{5(2)+3}{6} \right\rceil$. So assume that $X \ge 10$, which implies that $\Delta(Q) \ge 4$. First suppose that $X \le 12$. Now $\chi'(G) \le \lceil \frac{12}{3} \rceil = 4 \le \left\lceil \frac{5(4)+3}{6} \right\rceil$. So instead assume that $X = 13$, which implies that $\Delta(Q) \ge 5$. Now $\chi'(G) \le \lceil \frac{13}{3} \rceil = 5 \le \left\lceil \frac{5(5)+3}{6} \right\rceil$, as desired. Finally, we consider $t = 2$. Now $\frac{5\bar{d}+3}{6} = \frac{X}{2} - \frac{1}{3}$ and $\frac{X}{t} = \frac{X}{2}$, so always $\frac{5\bar{d}+3}{6} < \frac{X}{t}$. However, $\left\lceil \frac{5\bar{d}+3}{6} \right\rceil = \lceil \frac{X}{2} - \frac{1}{3} \rceil = \lceil \frac{X}{2} \rceil$, for all integers X, which completes the proof. □

We suspect that our Main Theorem can be extended to the larger class of quasi-line graphs (those for which the neighborhood of each vertex is covered by two cliques).

Conjecture 21. *If Q is a quasi-line graph, then*

$$\chi(Q) \le \max\left\{\omega(Q), \frac{5\Delta(Q)+8}{6}\right\}.$$

6 Strengthenings of Reed's Conjecture

In this section, we show how the results of Section 3 imply Reed's Conjecture, as well as Local and Superlocal strengthenings of Reed's Conjecture, for the class of line graphs.

Let G be a graph. The *claw-degree* of $x \in V(G)$ is claw-degree

$$d_{\text{claw}}(x) := \max_{\substack{S \subseteq N(x) \\ |S|=3}} \frac{1}{4}\left(d(x) + \sum_{v \in S} d(v)\right),$$

where $d_{\text{claw}}(x) := 0$ when $|N(x)| \le 2$. The *claw-degree* of G is

$$d_{\text{claw}}(G) := \max_{x \in V(G)} d_{\text{claw}}(x).$$

Theorem 22. *If G is a graph, then*

$$\chi'(G) \leq \max\left\{\mathcal{W}(G), \Delta(G)+1, \left\lceil \frac{4}{3} d_{claw}(G) \right\rceil\right\}.$$

Proof. Suppose not and choose a counterexample G with the fewest edges; note that G is critical. Let $k = \chi'(G) - 1$, so $k \geq \left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil$. By Theorem 4, G has a long vertex x. Choose $xy_1 \in E(G)$ and a k-edge-coloring φ of $G - xy_1$ such that φ has a fan F of length 3 rooted at x with leaves y_1, y_2, y_3. Since no two vertices of F miss a common color,

$$2 + k - d(x) + \sum_{i \in [3]} k - d(y_i) \leq k,$$

and hence

$$\frac{3k+2}{4} \leq \frac{1}{4}\left(d(x) + \sum_{i \in [3]} d(y_i)\right) \leq d_{\text{claw}}(x).$$

This gives the contradiction

$$\left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil \leq k \leq \frac{4}{3} d_{\text{claw}}(G) - \frac{2}{3}.$$

□

Reed [9] conjectured that $\chi(G) \leq \left\lceil \frac{\omega(G)+\Delta+1}{2} \right\rceil$ for every graph G. This is the average of a trivial lower bound $\omega(G)$ and a trivial upper bound $\Delta(G)+1$. King [7] conjectured the stronger bound $\chi(G) \leq \max_{v \in V(G)} \left\lceil \frac{\omega(v)+d(v)+1}{2} \right\rceil$, where $\omega(v)$ is the size of the largest clique containing v; this bound is now known to hold for many classes of graphs, including line graphs [2]. Here we show that for line graphs this bound is an easy consequence of our more general lemmas from Section 3. A *thickened cycle* is a multigraph that has a cycle as its underlying simple graph.

Theorem 23. *If G is a critical graph that is not a thickened cycle, then*

$$\chi'(G) \leq \max\left\{\Delta(G)+1, \left\lceil \frac{4}{3} d_{claw}(G) \right\rceil\right\}.$$

Proof. The proof is in some ways similar to those of Lemmas 16 and 18. It consists largely of straightforward, albeit tedious, algebraic manipulations.

Suppose the theorem is false and let G be a counterexample. By Theorem 22, G is elementary. Since G is critical and elementary, Proposition 15 implies that $|G|$ is odd and

$$k = \frac{2(\|G\| - 1)}{|G| - 1}. \tag{12}$$

Let $x \in V(G)$ with $|N(x)| \geq 3$. Put $M := |N(x)|$,

$$P := \sum_{v \in N(x)} d_G(v),$$

$$S_2 := 2 + \sum_{v \in V(G) \setminus N(x)} \Delta(G) - d_G(v),$$

$$S_3 := k - (\Delta(G) + 1).$$

Now

$$2(\|G\| - 1) = \Delta(G)(|G| - M) - S_2 + P. \tag{13}$$

Since

$$\frac{2(\|G\| - 1)}{|G| - 1} = k = \Delta(G) + 1 + S_3,$$

using (13), we get

$$P = (|G| - 1)(\Delta(G) + 1 + S_3) - \Delta(G)(|G| - M) + S_2,$$

which we rewrite as

$$P = \Delta(G)(M - 1) + |G| - 1 + S_2 + S_3(|G| - 1). \tag{14}$$

Let $N(x) = \{v_0, v_1, \ldots, v_{M-1}\}$ and put

$$R := \sum_{i=0}^{M-1} \frac{1}{3} \left(d_G(x) + d_G(v_i) + d_G(v_{i+1}) + d_G(v_{i+2}) \right),$$

where indices are taken modulo M. Since $k \geq \frac{4}{3} d_{\text{claw}}(G)$, there is $S_4 \geq 0$ such that

$$Mk - S_4 = R = \frac{M}{3} d_G(x) + P.$$

Now substituting (14) we get

$$Mk = \frac{M}{3} d_G(x) + \Delta(G)(M - 1) + |G| - 1 + S_2 + S_3(|G| - 1) + S_4.$$

Since $M \geq 3$ by our choice of x, solving for M gives (for some $S_5 \geq 0$)

$$3 + S_5 = M = \frac{S_2 + (S_3 + 1)(|G| - 1) + S_4 - \Delta(G)}{S_3 + 1 - \frac{1}{3} d_G(x)}.$$

Hence

$$(3 + S_5)(S_3 + 1) - \left(1 + \frac{S_5}{3}\right) d_G(x) = S_2 + (S_3 + 1)(|G| - 1) + S_4 - \Delta(G).$$

Rearranging terms gives

$$\left(1 + \frac{S_5}{3}\right) d_G(x) = \Delta(G) - (S_2 - 2) + (4 + S_5 - |G|) \, S_3 + (2 + S_5 - |G|) - S_4.$$

Suppose $4 + S_5 - |G| \leq 0$. Now

$$\left(1 + \frac{S_5}{3}\right) d_G(x) \leq \Delta(G) - (S_2 - 2) - 2.$$

By definition, $S_2 \geq 2 + \Delta(G) - d_G(x)$, so we have

$$\left(1 + \frac{S_5}{3}\right) d_G(x) \leq d_G(x) - 2,$$

a contradiction since $S_5 \geq 0$. So, we must have $4 + S_5 - |G| > 0$, that is,

$$|G| \leq S_5 + 3 = |N(x)| \leq |G| - 1,$$

a contradiction. $\square$

For a graph Q and $r \in \mathbb{N} \cup \{\infty\}$, put

$$\mathcal{C}_r(Q) := \{X \subseteq V(Q) : X \text{ is a maximal clique with } |X| < r \text{ or } X \text{ is a clique with } |X| = r\}.$$

Put

$$\gamma_r(Q) := \max_{X \in \mathcal{C}_r(Q)} \frac{1}{|X|} \sum_{v \in X} \frac{d(v) + \omega(v) + 1}{2}.$$

In terms of γ_r, the local version of Reed's conjecture for line graphs, proved by Chudnovsky et al. [2], and the superlocal version, proved by Edwards and King [4], are the following two theorems.

Theorem 24 (Chudnovsky et al.)**.** *If Q is a line graph, then $\chi(Q) \leq \lceil \gamma_1(Q) \rceil$.*

Theorem 25 (Edwards and King)**.** *If Q is a line graph, then $\chi(Q) \leq \lceil \gamma_2(Q) \rceil$.*

Note that if $a, b \in \mathbb{N} \cup \{\infty\}$ with $a \leq b$, then $\gamma_a(Q) \geq \gamma_b(Q)$, so Theorem 25 implies Theorem 24. Edwards and King [4] conjectured that $\gamma_\infty(Q)$ is an upper bound on the fractional chromatic number for all graphs Q.

Conjecture 26 (Edwards and King)**.** *If Q is any graph, then $\chi_f(Q) \leq \gamma_\infty(Q)$.*

A graph parameter f is *monotone* if $f(G) \geq f(H)$ whenever H is an induced subgraph of G. Unfortunately, γ_∞ is not monotone and hence not very induction-friendly.

Lemma 27. *γ_1 and γ_2 are monotone, but γ_r is not monotone for all $r \geq 3$.*

Proof. The first statement is clear. For the second statement, let Q be the graph in Figure 5. For $r \geq 3$, we have $\gamma_r(Q - x) = \frac{11}{2} > \frac{16}{3} = \gamma_r(Q)$. $\square$

As a possible approach to Conjecture 26, we define the following variation on $\gamma_\infty(G)$, which should be more induction-friendly. For $r \in \mathbb{N} \cup \{\infty\}$, put

$$\tilde{\gamma}_r(Q) := \max_{H \subseteq Q} \gamma_r(H).$$

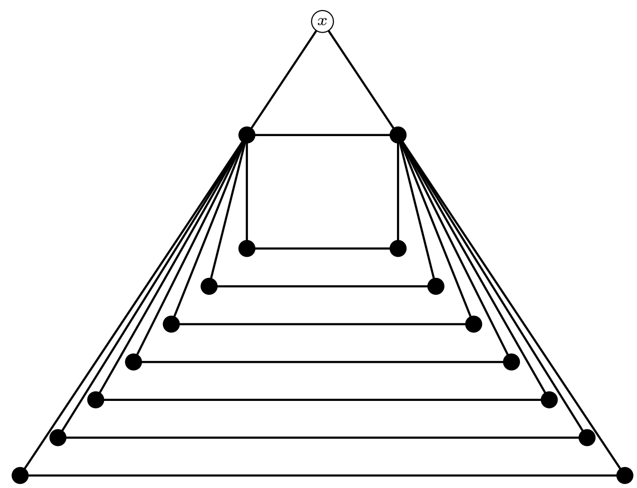

Figure 5: A graph Q where $\gamma_r(Q - x) > \gamma_r(Q)$ for all $r \geq 3$.

Conjecture 28. *If Q is any graph, then $\chi_f(Q) \leq \tilde{\gamma}_\infty(Q)$.*

Conjecture 29. *If Q is a line graph, then $\chi(Q) \leq \lceil \tilde{\gamma}_\infty(Q) \rceil$.*

Theorem 30. *Conjecture 29 follows from Conjecture 28 and the Goldberg-Seymour Conjecture.*

Proof. Assume that Conjecture 28 and the Goldberg–Seymour Conjecture are true and Conjecture 29 is false. Let Q be a minimal counterexample, and say $Q = L(G)$. Since Conjecutre 28 is true, $\chi_f(Q) \leq \tilde{\gamma}_\infty(Q)$. Since the Goldberg–Seymour Conjecture is true, $\chi(Q) \leq \max\{\Delta(G)+1, \lceil \chi_f(Q) \rceil\} \leq \max\{\Delta(G)+1, \lceil \tilde{\gamma}_\infty(Q) \rceil\}$. Since Q is a counterexample $\lceil \tilde{\gamma}_\infty(Q) \rceil < \chi(Q) \leq \Delta(G) + 1 \leq \omega(Q) + 1$. This implies that $\lceil \tilde{\gamma}_\infty(Q) \rceil \leq \omega(Q)$. However, also $\tilde{\gamma}_\infty(Q) \geq \omega(Q)$, by taking X in the definition of $\tilde{\gamma}_\infty$ to be a maximum clique. Further, since Q is connected by minimality, the inequality is strict if $X \subsetneq Q$. So Q is a clique. But this yields the contradiction $\omega(Q) = \lceil \tilde{\gamma}_\infty(Q) \rceil < \chi(Q)$. □

We prove the next bound in the sequence begun by Theorems 24 and 25.

Theorem 31. *If Q is a line graph, then $\chi(Q) \leq \lceil \tilde{\gamma}_3(Q) \rceil$.*

Before proving Theorem 31, we prove the following lemma, which aids in the proof of Theorem 31.

Lemma 32. *Let $Q = L(G)$ where G is a critical graph. If G is not a thickened cycle, then $\chi(Q) \leq \lceil \gamma_3(Q) \rceil$.*

Proof. Suppose G is not a thickened cycle. For $uv \in E(G)$, put

$$f(uv) := \max\{d_G(u) + \frac{1}{2}(d_G(v) - \mu(uv)), d_G(v) + \frac{1}{2}(d_G(u) - \mu(uv)\}.$$

For $uv \in E(G)$, we have

$$\begin{aligned} f(uv) &= \frac{d_G(u) + d_G(v) - \mu(uv) + \max\{d_G(u), d_G(v)\}}{2} \\ &\leq \frac{d_Q(uv) + \omega(uv) + 1}{2}. \end{aligned}$$

Since G is critical, we have $|N(v)| \geq 2$ for all $v \in V(G)$. Since G is not a thickened cycle, we may choose $x \in V(G)$ and $S \subseteq N(x)$ with $|S| = 3$ such that x and S achieve maximality in the definition of $d_{claw}(G)$. Say $S = \{v_1, v_2, v_3\}$. Then

$$\begin{aligned} \left\lceil \frac{4}{3} d_{\text{claw}}(G) \right\rceil &= \left\lceil \left(\frac{4}{3}\right)\left(\frac{1}{4}\right)\left(d_G(x) + \sum_{i \in [3]} d_G(v_i)\right) \right\rceil \\ &\leq \left\lceil \frac{1}{3} \sum_{i \in [3]} d_G(v_i) + \frac{1}{2}(d_G(x) - \mu(xv_i)) \right\rceil \\ &\leq \left\lceil \frac{1}{3} \sum_{i \in [3]} f(xv_i) \right\rceil \\ &\leq \left\lceil \frac{1}{3} \sum_{i \in [3]} \frac{d_Q(xv_i) + \omega(xv_i) + 1}{2} \right\rceil \\ &\leq \lceil \gamma_3(Q) \rceil. \end{aligned}$$

By Theorem 23, we have

$$\chi(Q) \leq \max\{\Delta(G) + 1, \lceil \gamma_3(Q) \rceil\}. \tag{15}$$

Let $M \subseteq V(Q)$ be a maximum clique in Q. Since G is not a thickened cycle, $|M| \geq 3$, so we can choose $X \subseteq M$ with $|X| = 3$ maximizing

$$\frac{1}{3} \sum_{v \in X} \frac{d(v) + \omega(v) + 1}{2}.$$

We have

$$\begin{aligned} \gamma_3(Q) &\geq \frac{1}{3} \sum_{v \in X} \frac{d(v) + \omega(v) + 1}{2} \\ &\geq \frac{1}{|M|} \sum_{v \in M} \frac{d(v) + \omega(v) + 1}{2} \\ &\geq \omega(Q) + \sum_{v \in M} \frac{d(v) + 1 - \omega(v)}{2}. \end{aligned}$$

If $V(Q) = M$, then $\lceil \gamma_3(Q) \rceil = \omega(Q) = \Delta(Q) = \chi(Q)$, as desired. Otherwise, some $v \in M$ has $d(v) \geq \omega(Q)$ and hence $\lceil \gamma_3(Q) \rceil \geq \omega(Q) + 1 \geq \Delta(Q) + 1$. Using (15), this gives $\chi(Q) \leq \lceil \gamma_3(Q) \rceil$. □

Proof of Theorem 31. Suppose Theorem 31 is false, and choose a counterexample Q minimizing $|Q|$. Say $Q = L(G)$. The minimality of $|Q|$ and monotonicity of $\tilde{\gamma}_3$ imply that G is critical. So Lemma 32 gives a contradiction unless G is a thickened cycle.

Suppose that G is a thickened cycle. Since G is elementary, $\|G\|$ is odd, so say $\|G\| = 2t+1$. Denote the vertices of G by $\{v_1, \ldots, v_{2t+1}\}$, and let $x_i = \mu(v_i v_{i+1})$ for each $i \in [2t+1]$. Let $X = \sum_{i=1}^{2t+1} x_i$. First suppose there exists j such that $x_j = 1$. So there exists $e \in E(G)$ such that $G - e$ is bipartite. Thus, $\chi'(G - e) = \Delta(G - e) \leq \Delta(G)$, so $\chi'(G) \leq \Delta(G) + 1$. Since the theorem is trivially true for cycles we can assume $\Delta(G) \geq 3$. We take Y to be any clique of size 3 in Q corresponding to edges incident to a maximum degree vertex of G. Now Y witnesses that $\tilde{\gamma}_3(G) > \Delta(G)$, so the theorem holds.

Now assume instead that $x_i \geq 2$ for all $i \in [2t+1]$. For each i, let u_i^1 and u_i^2 be vertices of Q corresponding to edges counted by x_i. Note that $\frac{d(u_i^j)+\omega(u_i^j)}{2} \geq \frac{x_{i-1}+2x_i+2x_{i+1}}{2}$, for each $i \in [2t+1]$ and each $j \in [2]$. Now averaging over $\{u_i^1, u_{i+1}^1, u_{i+1}^2\}$ gives $\tilde{\gamma}_3(G) \geq \frac{1}{6}\left(x_{i-1} + 4x_i + 6x_{i+1} + 4x_{i+2}\right)$. When we average over all $i \in [2t+1]$, we get $\tilde{\gamma}_3(G) \geq \frac{X5}{4t+2}$. Since G is elementary, we know $\chi'(G) = \left\lceil \frac{X}{t} \right\rceil$. Now the theorem holds since $\frac{1}{t} \leq \frac{5}{4t+2}$, whenever $t \geq 2$. □

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