

# SPARSE GRAPHS ADMIT HOMOMORPHISMS INTO ODD CYCLES

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ABSTRACT.

## 1. INTRODUCTION

All graphs under consideration are nonempty finite simple graphs. For graphs  $G$  and  $H$ , we indicate the existence of a homomorphism from  $G$  to  $H$  or lack thereof by  $G \rightarrow H$  and  $G \nrightarrow H$ , respectively. We write  $H \trianglelefteq G$  to indicate that  $H$  is an induced subgraph of  $G$ , when we want the containment to be proper, we write  $H \triangleleft G$ .

## 2. POTENTIAL FUNCTIONS

Kostochka and Yancey [3] used “potential functions” to great effect in proving lower bounds on the number of edges in critical graphs. Here we generalize this idea and prove some basic facts.

**Definition 1.** For positive integers  $a$  and  $b$ , the  $(a, b)$ -*potential function* is the function from graphs to  $\mathbb{Z}$  given by  $\rho_{a,b}(G) := a|G| - b\|G\|$ . Additionally, put

$$\hat{\rho}_{a,b}(G) := \min_{H \trianglelefteq G} \rho_{a,b}(H).$$

The invariant  $\hat{\rho}_{a,b}(G)$  is a measure of the sparseness of  $G$ , the larger  $\hat{\rho}_{a,b}(G)$  is, the sparser  $G$  is. For example, if  $\hat{\rho}_{a,b}(G) \geq 0$ , then  $\text{mad}(G) \leq \frac{2a}{b}$  where  $\text{mad}(G)$  is the maximum average degree of  $G$ .

For any fixed graph  $T$ , we are interested in proving results of the form: any sufficiently sparse graph admits a homomorphism into  $T$ . To do so, it will be useful to get the benefits of having a minimum counterexample without being bound to a fixed inductive context. To achieve this, we use *mules* as introduced in [2, 4].

### 2.1. Mules.

**Definition 2.** If  $G$  and  $H$  are graphs, an *epimorphism* is a graph homomorphism  $f: G \twoheadrightarrow H$  such that  $f(V(G)) = V(H)$ . We indicate this with the arrow  $\twoheadrightarrow$ .

**Definition 3.** Let  $G$  be a graph. A graph  $A$  is called a *child* of  $G$  if  $A \neq G$  and there exists  $H \trianglelefteq G$  and an epimorphism  $f: H \twoheadrightarrow A$ .

Note that the child-of relation is a strict partial order on the set of (finite simple) graphs  $\mathcal{G}$ . We call this the *child order* on  $\mathcal{G}$  and denote it by ‘ $\prec$ ’. By definition, if  $H \triangleleft G$  then  $H \prec G$ .

$$\begin{array}{ccc}
H & \xrightarrow{\iota} & G \\
h \downarrow & & \downarrow h' \\
Q & \xrightarrow{\iota} & G_h
\end{array}$$

FIGURE 1. The commutative diagram for  $G_h$ .

**Lemma 1.** *The ordering  $\prec$  is well-founded on  $\mathcal{G}$ ; that is, every nonempty subset of  $\mathcal{G}$  has a minimal element under  $\prec$ .*

*Proof.* Let  $\mathcal{T}$  be a nonempty subset of  $\mathcal{G}$ . Pick  $G \in \mathcal{T}$  minimizing  $|V(G)|$  and then maximizing  $|E(G)|$ . Since any child of  $G$  must have fewer vertices or more edges (or both), we see that  $G$  is minimal in  $\mathcal{T}$  with respect to  $\prec$ .  $\square$

**Definition 4.** Let  $\mathcal{T}$  be a collection of graphs. A minimal graph in  $\mathcal{T}$  under the child order is called a  $\mathcal{T}$ -mule.

## 2.2. Basic facts.

For a graph  $T$  together with positive integers  $a$ ,  $b$  and  $c$ , let  $\mathcal{C}_{T,a,b,c}$  be the set of graphs  $G$  such that  $G \not\rightarrow T$  and  $\hat{\rho}_{a,b}(G) \geq c$ .

**Lemma 2.** *Let  $G$  be a  $\mathcal{C}_{T,a,b,c}$ -mule. If  $H \triangleleft G$ , then  $H \rightarrow T$ .*

*Proof.* Since  $\hat{\rho}_{a,b}(H) \geq \hat{\rho}_{a,b}(G) \geq c$  and  $H \prec G$ , we must have  $H \rightarrow T$  since  $G$  is a  $\mathcal{C}_{T,a,b,c}$ -mule.  $\square$

**Definition 5.** Let  $H$  be an induced subgraph of a graph  $G$  and  $h: H \rightarrow Q$  an epimorphism onto some graph  $Q$ . Let  $G_h$  be the image of the natural extension of  $h$  to an epimorphism  $h'$  defined on  $G$ ; that is,  $G_h$  and  $h'$  are such that the diagram in Figure 1 commutes (where  $\iota$  indicates the inclusion map).

**Lemma 3.** *Let  $G$  be a  $\mathcal{C}_{T,a,b,c}$ -mule and  $Q$  an arbitrary graph. If  $H \trianglelefteq G$  with  $H \neq Q$  such that  $H \rightarrow Q$ , then  $\rho_{a,b}(H) > \hat{\rho}_{a,b}(Q)$ .*

*Proof.* Suppose to the contrary that there is  $H \trianglelefteq G$  with  $H \neq Q$  such that  $H \rightarrow Q$  and  $\rho_{a,b}(H) \leq \hat{\rho}_{a,b}(Q)$ . Let  $h$  be an epimorphism from  $H$  onto  $Q$ . Since  $G$  is a  $\mathcal{C}_{T,a,b,c}$ -mule,  $G_h$  cannot be a child of  $G$ . But we have an epimorphism  $h'$  from  $G$  onto  $G_h$  and  $G_h \neq G$  since  $H \neq Q$ , so it must be that  $G_h \notin \mathcal{C}_{T,a,b,c}$ . Since  $G \rightarrow G_h$  and  $G \not\rightarrow T$ , we must have  $G_h \not\rightarrow T$ . Therefore  $\hat{\rho}_{a,b}(G_h) < c$ . Pick  $W \trianglelefteq G_h$  with  $\rho_{a,b}(W) < c$ . Since  $W \not\subseteq G$ , we must have  $V(W) \cap V(Q) \neq \emptyset$ . Hence  $\rho_{a,b}(G[(V(W) - V(Q)) \cup V(H)]) \leq \rho_{a,b}(W) - \hat{\rho}_{a,b}(Q) + \rho_{a,b}(H) \leq \rho_{a,b}(W) - \hat{\rho}_{a,b}(Q) + \hat{\rho}_{a,b}(Q) = \rho_{a,b}(W) < c$ , a contradiction since  $\hat{\rho}_{a,b}(G) \geq c$ .  $\square$

We have the following basic bound on the potential of non-complete subgraphs of  $G$ .

**Corollary 4.** *Let  $G$  be a  $\mathcal{C}_{T,a,b,c}$ -mule. If  $H \trianglelefteq G$  is not complete and  $\chi(H) \leq \frac{2a}{b}$ , then  $\rho_{a,b}(H) > a$ .*

*Proof.* Suppose  $\chi(H) = k \leq \frac{2a}{b}$ . Then there is an epimorphism from  $H$  onto  $K_k$  given by contracting all color classes in a  $k$ -coloring of  $H$ . Since  $H \neq K_k$ , Lemma 3 gives  $\rho_{a,b}(H) > \hat{\rho}_{a,b}(K_k)$ . But  $\hat{\rho}_{a,b}(K_k) = \min_{t \in [k]} at - b \binom{t}{2} = a$  since  $k \leq \frac{2a}{b}$ , so we have the desired bound.  $\square$

There is room for improvement in the proof of Lemma 3; in particular, the bound

$$\rho_{a,b}(G[(V(W) - V(Q)) \cup V(H)]) \leq \rho_{a,b}(W) - \hat{\rho}_{a,b}(Q) + \rho_{a,b}(H)$$

can be improved in many cases. Put  $X := V(W) - V(Q)$  and  $Y := h^{-1}(V(W) \cap V(Q))$ . We did not count any of the edges between  $X$  and  $H - Y$  in our estimate, so in fact we can improve the upper bound to

$$\rho_{a,b}(W) - \hat{\rho}_{a,b}(Q) + \rho_{a,b}(H) - b|E_G(X, H - Y)|.$$

Furthermore, we gain when a vertex in  $X$  has more than  $|V(W) \cap V(Q)|$  neighbors in  $Y$  because multi-edges get replaced by single edges. Taking this into account, improves the upper bound to

$$\rho_{a,b}(W) - \hat{\rho}_{a,b}(Q) + \rho_{a,b}(H) - b|E_G(X, H - Y)| - b \sum_{v \in X} \max\{0, |N_G(v) \cap Y| - |V(W) \cap V(Q)|\}.$$

Finally, we may be able to improve on the use of  $\hat{\rho}_{a,b}(Q)$  when  $|V(W) \cap V(Q)| > 1$ . There is a tension between this final improvement and the previous two. For many  $Q$  in the wild,  $\rho_{a,b}(Q') > \rho_{a,b}(K_1)$  for all  $Q' \trianglelefteq Q$  with  $|Q'| > 1$ . When this occurs, our upper bound is improved by 1 unless  $|V(W) \cap V(Q)| = 1$  and  $|E(X, H - Y)| = 0$  and  $|N(v) \cap Y| \leq 1$  for all  $v \in X$ .

## REFERENCES

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