Landon Rabern

problem

The

Rephrasing

Solving the rephrased

A spectrum of

Further improvements

Improving Brooks' theorem

Landon Rabern

Arizona State University

October 28, 2011

Landon Rabern

Some background

The Ore-degree

the problem

problem
A spectrum

generalizations

Further improvements

Outline

- 1 A prison problem
- 2 Some background
- 3 The Ore-degree
- 4 Rephrasing the problem
- Solving the rephrased problem Kierstead and Kostochka's proof Problem solved Proof outline

Mozhan's lemma
The recoloring algorithm

The recoloring algorithm

6 A spectrum of generalizations
Generalizing maximum degree
The generalized bound
The lower bound on t_{ϵ} What about Δ_{0} ?

Further improvements

Landon Rabern

A prison problem

Some background

The Ore-degre

Rephrasing the problem

Solving the rephrased problem

A spectrum o generalization

Further improvements

A prison problem

A prison problem

You are a warden in a prison with five large cells. You need to put all the inmates into the cells, but to prevent fighting you cannot put a pair of inmates that have fought before into the same cell. Each inmate in the prison has fought with at most six other inmates and none of the inmates who have fought with six others have fought with each other. Under what conditions can you complete your task?

Landon Rabern

A prison problem

Some background

The Ore-degre

the problen

Solving the rephrased problem

A spectrum o generalization

Further improvements

A prison problem

A prison problem

You are a warden in a prison with five large cells. You need to put all the inmates into the cells, but to prevent fighting you cannot put a pair of inmates that have fought before into the same cell. Each inmate in the prison has fought with at most six other inmates and none of the inmates who have fought with six others have fought with each other. Under what conditions can you complete your task?

 plainly, if there is a group of six inmates who have all fought one another, then you cannot complete your task

Landon Rabern

A prison

Some backgroun

The Ore-degree

the probler

Solving the rephrased problem

A spectrum o generalization

Further improvements

A prison problem

A prison problem

You are a warden in a prison with five large cells. You need to put all the inmates into the cells, but to prevent fighting you cannot put a pair of inmates that have fought before into the same cell. Each inmate in the prison has fought with at most six other inmates and none of the inmates who have fought with six others have fought with each other. Under what conditions can you complete your task?

- plainly, if there is a group of six inmates who have all fought one another, then you cannot complete your task
- is this simple necessary condition sufficient?

Landon Rabern

problem

The

Rephrasing the problem

Solving th rephrased problem

A spectrum o generalization

Further improvements

Greedy coloring

• $C := \{c_1, c_2, c_3, \ldots\}$ an infinite set of colors

Landon Rabern

A prison problem

The Control of the Co

Rephrasing the problem

Solving th rephrased problem

A spectrum of generalization

Further

Greedy coloring

- $C := \{c_1, c_2, c_3, \ldots\}$ an infinite set of colors
- G has vertices ordered v_1, v_2, \ldots, v_n

Landon Rabern

Some background

The Ore-degree

Rephrasing the problem

Solving the rephrased problem

A spectrum of generalization

Further improvements

Greedy coloring

- $C := \{c_1, c_2, c_3, \ldots\}$ an infinite set of colors
- G has vertices ordered v_1, v_2, \ldots, v_n
- go through the vertices in order, coloring v_i with the first color not used on a neighbor of v_i

Landon Rabern

A prison problem

The Ore degree

Rephrasing

Solving the rephrased problem

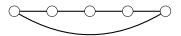
problem
A spectrum

generalizations

Further improvements

Greedy coloring

- $C := \{c_1, c_2, c_3, \ldots\}$ an infinite set of colors
- G has vertices ordered v_1, v_2, \ldots, v_n
- go through the vertices in order, coloring v_i with the first color not used on a neighbor of v_i



Landon Rabern

A prison problem

background The

Rephrasing

Solving the rephrased problem

problem
A spectrum

generalization

Further improvements

Greedy coloring

- $C := \{c_1, c_2, c_3, \ldots\}$ an infinite set of colors
- G has vertices ordered v_1, v_2, \ldots, v_n
- go through the vertices in order, coloring v_i with the first color not used on a neighbor of v_i



Landon Rabern

A prison problem

The Ore-degree

Rephrasing the problem

Solving the rephrased problem

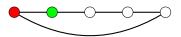
problem

A spectrum

generalization

Greedy coloring

- $C := \{c_1, c_2, c_3, \ldots\}$ an infinite set of colors
- G has vertices ordered v_1, v_2, \ldots, v_n
- go through the vertices in order, coloring v_i with the first color not used on a neighbor of v_i



Landon Rabern

problem

The Ore-degree

Rephrasing the problem

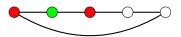
Solving the rephrased problem

A spectrum of generalization

Further improvements

Greedy coloring

- $C := \{c_1, c_2, c_3, \ldots\}$ an infinite set of colors
- G has vertices ordered v_1, v_2, \ldots, v_n
- go through the vertices in order, coloring v_i with the first color not used on a neighbor of v_i



Landon Rabern

A prison problem

The

Rephrasing

Solving the rephrased problem

problem

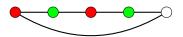
A spectrum

generalizations

Further improvements

Greedy coloring

- $C := \{c_1, c_2, c_3, \ldots\}$ an infinite set of colors
- G has vertices ordered v_1, v_2, \ldots, v_n
- go through the vertices in order, coloring v_i with the first color not used on a neighbor of v_i



Landon Rabern

Some background

The Ore-degree

Rephrasing the problem

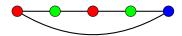
Solving the rephrased problem

A spectrum

Further improvements

Greedy coloring

- $C := \{c_1, c_2, c_3, \ldots\}$ an infinite set of colors
- G has vertices ordered v_1, v_2, \ldots, v_n
- go through the vertices in order, coloring v_i with the first color not used on a neighbor of v_i



Landon Rabern

problem

The Ore-degree

the problem
Solving the

rephrased problem

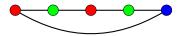
generalization

Further improvements

Greedy coloring

- $C := \{c_1, c_2, c_3, \ldots\}$ an infinite set of colors
- G has vertices ordered v_1, v_2, \ldots, v_n
- go through the vertices in order, coloring v_i with the first color not used on a neighbor of v_i

For example, say $C := \{ red, green, blue, cyan, \ldots \}$ and G is the 5-cycle:



• if G has maximum degree k, then v_i has at most k colored neighbors, so greedy coloring uses at most k+1 colors

Landon Rabern

Some background

The Ore-degree

Rephrasing the problem

Solving the rephrased problem

A spectrum of generalization

Further

Brooks' theorem

 χ(G) := the minimum number of colors needed to color the vertices of G so that adjacent vertices receive different colors

Landon Rabern

Some background

The Ore-degree

the problem

Solving the rephrased problem

generalization

Further improvements

Brooks' theorem

- χ(G) := the minimum number of colors needed to color the vertices of G so that adjacent vertices receive different colors
- $\omega(G)$:= the number of vertices in a largest complete subgraph of G

Landon Rabern

problem

The Ore-degree

Rephrasing the problem

Solving the rephrased problem

a spectrum of generalization

Further improvements

Brooks' theorem

- χ(G) := the minimum number of colors needed to color the vertices of G so that adjacent vertices receive different colors
- ω(G) := the number of vertices in a largest complete subgraph of G
- $\Delta(G) :=$ the maximum degree of G

Landon Rabern

problem

The Ore-degree

Rephrasing

Solving the rephrased problem

A spectrum

Further

Brooks' theorem

- χ(G) := the minimum number of colors needed to color the vertices of G so that adjacent vertices receive different colors
- ω(G) := the number of vertices in a largest complete subgraph of G
- $\Delta(G)$:= the maximum degree of G

Theorem (Brooks 1941)

Every graph with $\Delta \geq 3$ satisfies $\chi \leq \max\{\omega, \Delta\}$.

Landon Rabern

problem

The One down

Rephrasing the problem

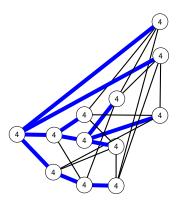
Solving th rephrased problem

A spectrum of generalization

Further improvements

Proof sketch

Any incomplete 2-connected graph with $\Delta \geq 3$ has a spanning tree where the root has two nonadjacent leaves as neighbors.



Landon Rabern

A prison problem

background The

Rephrasing

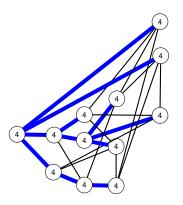
Solving the rephrased problem

A spectrum of generalization

Further improvements

Proof sketch

Any incomplete 2-connected graph with $\Delta \geq 3$ has a spanning tree where the root has two nonadjacent leaves as neighbors.



Landon Rabern

A prison problem

Some background

Ore-degree

Solving the rephrased

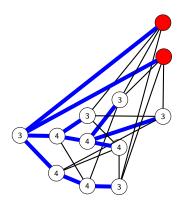
problem

generalization

Further improvements

Proof sketch

Any incomplete 2-connected graph with $\Delta \geq 3$ has a spanning tree where the root has two nonadjacent leaves as neighbors.



Landon Rabern

A prison problem

background

Rephrasing

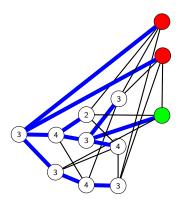
Solving the rephrased problem

A spectrum of generalization

Further improvements

Proof sketch

Any incomplete 2-connected graph with $\Delta \geq 3$ has a spanning tree where the root has two nonadjacent leaves as neighbors.



Landon Rabern

A prison problem

Some background

Rephrasing

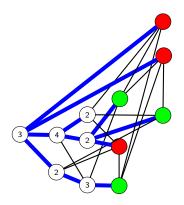
Solving the rephrased problem

A spectrum of generalization

Further improvements

Proof sketch

Any incomplete 2-connected graph with $\Delta \geq 3$ has a spanning tree where the root has two nonadjacent leaves as neighbors.



Landon Rabern

A prison problem

background The

Rephrasing

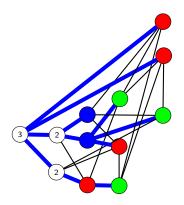
Solving the rephrased problem

A spectrum of generalization

Further

Proof sketch

Any incomplete 2-connected graph with $\Delta \geq 3$ has a spanning tree where the root has two nonadjacent leaves as neighbors.



Landon Rabern

A prison problem

Some background

Rephrasing

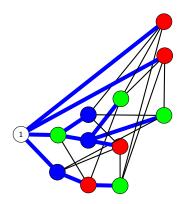
Solving the rephrased problem

A spectrum of generalization

Further improvements

Proof sketch

Any incomplete 2-connected graph with $\Delta \geq 3$ has a spanning tree where the root has two nonadjacent leaves as neighbors.



Landon Rabern

A prison problem

Some background

Ore-degree

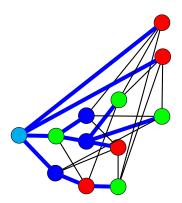
Solving the rephrased

A spectrum of

Further

Proof sketch

Any incomplete 2-connected graph with $\Delta \geq 3$ has a spanning tree where the root has two nonadjacent leaves as neighbors.



Landon Rabern

A prison problem

backgroun

The Ore-degree

the problem

Solving the rephrased problem

A spectrum

Further improvements

The Ore-degree

Definition

The Ore-degree of an edge xy in a graph G is

$$\theta(xy) := d(x) + d(y).$$

The *Ore-degree* of a graph *G* is

$$\theta(G) := \max_{xy \in E(G)} \theta(xy).$$

A prison

ome ackground

The Ore-degree

Rephrasing the problem

Solving the rephrased problem

A spectrum

Further

Definition

The Ore-degree of an edge xy in a graph G is

$$\theta(xy) := d(x) + d(y).$$

The Ore-degree of a graph G is

$$\theta(G) := \max_{xy \in E(G)} \theta(xy).$$

• every graph satisfies $\left|\frac{\theta}{2}\right| \leq \Delta$

Landon Rabern

A prison problem

Some backgroun

The Ore-degree

the problem

Solving the rephrased problem

A spectrum

generalization Further

The Ore-degree

Definition

The Ore-degree of an edge xy in a graph G is

$$\theta(xy) := d(x) + d(y).$$

The Ore-degree of a graph G is

$$\theta(G) := \max_{xy \in E(G)} \theta(xy).$$

- every graph satisfies $\left|\frac{\theta}{2}\right| \leq \Delta$
- greedy coloring (in any order) shows that every graph satisfies $\chi \leq \left|\frac{\theta}{2}\right| + 1$

Landon Rabern

problem

Some backgroup

The Ore-degree

Rephrasing the problem

rephrased problem

A spectrum of generalization

Further improvements

Kierstead and Kostochka's generalization

Theorem (Kierstead and Kostochka 2009)

Every graph with $\theta \geq 12$ satisfies $\chi \leq \max \left\{ \omega, \left\lfloor \frac{\theta}{2} \right\rfloor \right\}$.

Landon Rabern

A prison problem

backgroun

The Ore-degree

the problem

Solving the rephrased problem

generalization

Further improvements

Kierstead and Kostochka's generalization

Theorem (Kierstead and Kostochka 2009)

Every graph with $\theta \ge 12$ satisfies $\chi \le \max \{\omega, \lfloor \frac{\theta}{2} \rfloor \}$.

Kierstead and Kostochka conjectured that the 12 could be reduced to 10. That this would be best possible can be seen from the following example which has $\theta=9$, $\omega=4$ and $\chi=5$.

> Landon Rabern

A prison problem

background

The Ore-degree

the problem

Solving the rephrased problem

A spectrum o generalization

Further improvements

Kierstead and Kostochka's generalization

Theorem (Kierstead and Kostochka 2009)

Every graph with $\theta \ge 12$ satisfies $\chi \le \max \{\omega, \lfloor \frac{\theta}{2} \rfloor \}$.

Kierstead and Kostochka conjectured that the 12 could be reduced to 10. That this would be best possible can be seen from the following example which has $\theta=9$, $\omega=4$ and $\chi=5$.

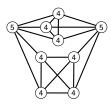


Figure: O_5 , a counterexample with $\theta = 9$.

Landon Rabern

A prison problem

backgroun

Ore-degree

Rephrasing the problem

Solving th rephrased problem

A spectrum egeneralizatio

Further improvements

Rephrasing the problem

Definition

A graph G is called *vertex critical* if $\chi(G - v) < \chi(G)$ for each $v \in V(G)$.

Landon Rabern

A prison problem

Some backgroun

The Ore-degree

Rephrasing the problem

Solving the rephrased problem

A spectrum of generalizations

Further improvements

Rephrasing the problem

Definition

A graph G is called *vertex critical* if $\chi(G - v) < \chi(G)$ for each $v \in V(G)$.

Definition

Let G be a vertex critical graph. The *low vertex subgraph* $\mathcal{L}(G)$ is the graph induced on the vertices of degree $\chi(G)-1$. The *high vertex subgraph* $\mathcal{H}(G)$ is the graph induced on the vertices of degree at least $\chi(G)$.

Landon Rabern

A prison problem

Some background

The Ore-degree

Rephrasing the problem

Solving the rephrased problem

A spectrum of generalizations

Further improvements

Rephrasing the problem

Definition

A graph G is called *vertex critical* if $\chi(G - v) < \chi(G)$ for each $v \in V(G)$.

Definition

Let G be a vertex critical graph. The *low vertex subgraph* $\mathcal{L}(G)$ is the graph induced on the vertices of degree $\chi(G)-1$. The *high vertex subgraph* $\mathcal{H}(G)$ is the graph induced on the vertices of degree at least $\chi(G)$.

Problem

Prove that $K_{\Delta(G)+1}$ is the only vertex critical graph G with $\chi(G) \geq \Delta(G) \geq 6$ such that $\mathcal{H}(G)$ is edgeless.

Landon Rabern

problem

backgroun

Rephrasing

Solving the rephrased

problem Kierstead and

algorithm

Kostochka's proof Problem solved Proof outline Mozhan's lemma The recoloring

A spectrum of

Further improvements

Kierstead and Kostochka's proof

 the proof is high-tech and clean, it uses both of the following

Landon Rabern

problem

The

Rephrasing

Solving the rephrased problem

Kierstead and Kostochka's

proof
Problem solved
Proof outline
Mozhan's lemma
The recoloring

A spectrum of

Further improvement

- the proof is high-tech and clean, it uses both of the following
- Alon and Tarsi's algebraic list coloring theorem

Landon Rabern

problem

The

Rephrasing

Solving the rephrased problem

problem Kierstead and Kostochka's

proof
Problem solved
Proof outline
Mozhan's lemma
The recoloring
algorithm

A spectrum of

Further improvement

- the proof is high-tech and clean, it uses both of the following
- Alon and Tarsi's algebraic list coloring theorem
- a result of Stiebitz from 1982 proving a conjecture of Gallai stating that $\mathcal{H}(G)$ has at most as many components as $\mathcal{L}(G)$

Landon Rabern

problem

The

Rephrasing

rephrased problem

Kierstead and Kostochka's proof Problem solve Proof outline

The recoloring algorithm

A spectrum of

generalization

Further improvement

- the proof is high-tech and clean, it uses both of the following
- Alon and Tarsi's algebraic list coloring theorem
- a result of Stiebitz from 1982 proving a conjecture of Gallai stating that $\mathcal{H}(G)$ has at most as many components as $\mathcal{L}(G)$
- using these it is basically just a counting argument

Landon Rabern

problem

The

Rephrasing

Solving the

problem
Kierstead and

Kostochka's proof Problem solved Proof outline

The recoloring algorithm

A spectrum of generalizations

Further improvement

- the proof is high-tech and clean, it uses both of the following
- Alon and Tarsi's algebraic list coloring theorem
- a result of Stiebitz from 1982 proving a conjecture of Gallai stating that $\mathcal{H}(G)$ has at most as many components as $\mathcal{L}(G)$
- using these it is basically just a counting argument
- unfortunately, it only works for $\Delta \geq 7$

Landon Rabern

To get down to $\Delta=6$, go low-tech and get dirty.

problem

The

Rephrasing

Solving the rephrased problem

Kierstead and Kostochka's proof

Problem solved

Mozhan's lemma The recoloring algorithm

A spectrum of generalizations

Further improvements

Landon Rabern

A prison problem

The

Rephrasing the problem

Solving the rephrased problem

problem
Kierstead ar

proof Problem solved

Mozhan's lemma The recoloring

A spectrum of

Further improvement

To get down to $\Delta = 6$, go low-tech and get dirty.

Theorem (Rabern 2010)

 $K_{\Delta(G)+1}$ is the only vertex critical graph G with

$$\chi(G) \ge \Delta(G) \ge 6$$
 and $\omega(\mathcal{H}(G)) \le \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2$.

Landon Rabern

A prison problem

The

Rephrasing the problem

Solving the rephrased problem

Kierstead an

Problem solved

Mozhan's lemma The recoloring

A spectrum or

Further improvement

To get down to $\Delta = 6$, go low-tech and get dirty.

Theorem (Rabern 2010)

 $K_{\Delta(G)+1}$ is the only vertex critical graph G with $\chi(G) \geq \Delta(G) \geq 6$ and $\omega(\mathcal{H}(G)) \leq \left|\frac{\Delta(G)}{2}\right| - 2$.

• setting $\omega(\mathcal{H}(G))=1$ proves Kierstead and Kostochka's conjecture

Landon Rabern

To get down to $\Delta=6$, go low-tech and get dirty.

problem

Some background

Rephrasing

Rephrasing the problem

rephrased problem

Kierstead an Kostochka's

Problem solved
Proof outline
Mozhan's lemma
The recoloring

A spectrum of

Further improvement

Theorem (Rabern 2010)

 $K_{\Delta(G)+1}$ is the only vertex critical graph G with $\chi(G) \geq \Delta(G) \geq 6$ and $\omega(\mathcal{H}(G)) \leq \left\lfloor \frac{\Delta(G)}{2} \right\rfloor - 2$.

- setting $\omega(\mathcal{H}(G))=1$ proves Kierstead and Kostochka's conjecture
- equivalently, as long as there is no group of six inmates who have all fought one another, you (the warden) can complete your inmate-cell-assignment task

Landon Rabern

A prison problem

backgroun

The Ore-degree

Rephrasing the problem

Solving th rephrased problem

Kierstead and Kostochka's proof Problem solve

Proof outline Mozhan's lemma The recoloring algorithm

A spectrum of

Further improvements

Proof outline

ullet start with a minimal counterexample G

Landon Rabern

A prison problem

Some backgroun

The Ore-degree

Rephrasing the problem

Solving the rephrased

Kierstead and Kostochka's proof Problem solve

Mozhan's lemn The recoloring algorithm

A spectrum of generalizations

Further improvement

Proof outline

- start with a minimal counterexample G
- for any induced subgraph H, $\Delta-1$ coloring G-H leaves a list assignment L on H where $|L(v)| \geq \deg(v)-1$

Landon Rabern

A prison problem

The

Rephrasing the problem

problem

Kierstead ar
Kostochka's

Problem solved
Proof outline
Mozhan's lemm
The recoloring

A spectrum o

Further

Proof outline

- start with a minimal counterexample G
- for any induced subgraph H, $\Delta-1$ coloring G-H leaves a list assignment L on H where $|L(v)| \geq \deg(v)-1$

Goal

Construct a subgraph H for which such a list assignment can always be completed.

Landon Rabern

A prison problem

backgroun

The Ore-degree

Solving th

rephrased problem

Kostochka's proof Problem solved Proof outline Mozhan's lemma The recoloring algorithm

A spectrum or

Further improvement

Proof outline

- start with a minimal counterexample G
- for any induced subgraph H, $\Delta-1$ coloring G-H leaves a list assignment L on H where $|L(v)| \geq \deg(v)-1$

Goal

Construct a subgraph H for which such a list assignment can always be completed.

 we need H to have large degrees to get large lists, so H will be "dense"

Landon Rabern

problem

The

Ore-degree Rephrasin

Solving the

Kierstead and Kostochka's proof Problem solved **Proof outline** Mozhan's lemm

The recoloring algorithm

A spectrum of generalizations

Further improvement

Proof outline

- start with a minimal counterexample G
- for any induced subgraph H, $\Delta-1$ coloring G-H leaves a list assignment L on H where $|L(v)| \geq \deg(v)-1$

Goal

Construct a subgraph H for which such a list assignment can always be completed.

- we need H to have large degrees to get large lists, so H will be "dense"
- first, use minimality of G to exclude some troublesome H's

Landon Rabern

A prison problem

The backgroun

Ore-degree Rephrasir

Solving th

Kierstead and Kostochka's proof Problem solved Proof outline Mozhan's lemm The recoloring

A spectrum of generalization

Further improvement

Proof outline

- start with a minimal counterexample G
- for any induced subgraph H, $\Delta-1$ coloring G-H leaves a list assignment L on H where $|L(v)| \geq \deg(v)-1$

Goal

Construct a subgraph H for which such a list assignment can always be completed.

- we need H to have large degrees to get large lists, so H will be "dense"
- first, use minimality of G to exclude some troublesome H's
- run the following recoloring algorithm to construct H

Landon Rabern

problem

The

Rephrasing the problem

Solving the rephrased problem

Kierstead and Kostochka's proof

Problem solved Proof outline

Mozhan's lemma The recoloring algorithm

A spectrum or

Further improvement

Partitioned colorings

Definition

Let G be a vertex critical graph. Let $a \ge 1$ and r_1, \ldots, r_a be such that $1 + \sum_i r_i = \chi(G)$. By a (r_1, \ldots, r_a) -partitioned coloring of G we mean a proper coloring of G of the form

$$\{\{x\}, L_{11}, L_{12}, \dots, L_{1r_1}, L_{21}, L_{22}, \dots, L_{2r_2}, \dots, L_{a1}, L_{a2}, \dots, L_{ar_a}\}.$$

Here $\{x\}$ is a singleton color class and each L_{ij} is a color class.

Rephrasing the problem

Solving the rephrased problem

problem Kierstead an Kostochka's

Problem solve Proof outline

Mozhan's lemma The recoloring

A spectrum o

Further improvement

Lemma (Mozhan 1983)

Let G be a vertex critical graph. Let $a \ge 1$ and r_1, \ldots, r_a be such that $1 + \sum_i r_i = \chi(G)$. Of all (r_1, \ldots, r_a) -partitioned colorings of G pick one minimizing

$$\sum_{i=1}^{a} \left\| G \left[\bigcup_{j=1}^{r_i} L_{ij} \right] \right\|.$$

Remember that $\{x\}$ is a singleton color class in the coloring. Put $U_i := \bigcup_{j=1}^{r_i} L_{ij}$ and let $Z_i(x)$ be the component of x in $G[\{x\} \cup U_i]$. If $d_{Z_i(x)}(x) = r_i$, then $Z_i(x)$ is complete if $r_i \geq 3$ and $Z_i(x)$ is an odd cycle if $r_i = 2$.

Landon Rabern

A prison problem

backgroun The

Rephrasing

Solving th

rephrased problem

Kierstead an Kostochka's proof

Problem solved Proof outline

Mozhan's lemn The recoloring algorithm

A spectrum of

Further improvement

The recoloring algorithm

• take a $(\lfloor \frac{\Delta-1}{2} \rfloor, \lceil \frac{\Delta-1}{2} \rceil)$ -partitioned coloring minimizing the above function

Landon Rabern

problem

The

Rephrasing

Solving the rephrased

rephrased problem Kierstead ar

proof Problem solved

Proof outline Mozhan's lemn The recoloring

algorithm
A spectrum of

generalization

Further improvement

- take a $\left(\left\lfloor\frac{\Delta-1}{2}\right\rfloor, \left\lceil\frac{\Delta-1}{2}\right\rceil\right)$ -partitioned coloring minimizing the above function
- prove that we may assume that x is a low vertex

Landon Rabern

problem

The

Rephrasing

Solving the rephrased problem

Kostochka's proof Problem solved Proof outline

Mozhan's lemn The recoloring algorithm

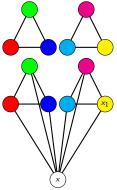
A spectrum or

Further improvement

- take a $\left(\left\lfloor\frac{\Delta-1}{2}\right\rfloor, \left\lceil\frac{\Delta-1}{2}\right\rceil\right)$ -partitioned coloring minimizing the above function
- prove that we may assume that x is a low vertex
- by Mozhan's lemma, the neighborhood of x in each part induces a clique or an odd cycle

Landon Rabern

The recoloring algorithm



The recoloring algorithm

swap x with a low vertex x_1 in the right part

Landon Rabern

A prison problem

backgroun

Ore-degre

Rephrasing the problem

Solving th rephrased problem

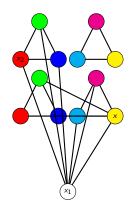
Kierstead an Kostochka's proof

Problem solved Proof outline

The recoloring algorithm

A spectrum o

Further



- swap x with a low vertex x_1 in the right part
- swap x_1 with a low vertex x_2 in the left part

Landon Rabern

A prison problem

The

Rephrasing

Solving the rephrased

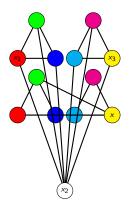
Kierstead and Kostochka's proof Problem solv

Problem solved Proof outline

The recoloring algorithm

A spectrum o

Further improvement



- swap x with a low vertex x_1 in the right part
- swap x_1 with a low vertex x_2 in the left part
- continue swapping back and forth until you wrap around

Landon Rabern

problem

backgroun

Rephrasing

Solving the rephrased

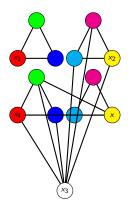
problem Kierstead an Kostochka's proof

Problem solved Proof outline

The recoloring algorithm

A spectrum o

Further improvement



- swap x with a low vertex x_1 in the right part
- swap x_1 with a low vertex x_2 in the left part
- continue swapping back and forth until you wrap around

Landon Rabern

problem

background

Rephrasing

Solving the rephrased

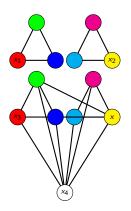
Kierstead and Kostochka's proof Problem solve

Proof outline Mozhan's lemn The recoloring

algorithm
A spectrum o

generalization

Further improvement



- swap x with a low vertex x_1 in the right part
- swap x_1 with a low vertex x_2 in the left part
- continue swapping back and forth until you wrap around

Landon Rabern

A prison problem

Some backgroun

Ore-degre

Rephrasing the problem

Solving the rephrased problem

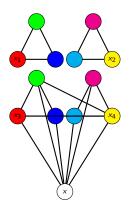
Kierstead and Kostochka's proof Problem solve

Mozhan's lemn

algorithm
A spectrum o

generalization

Further improvement



- swap x with a low vertex x_1 in the right part
- swap x_1 with a low vertex x_2 in the left part
- continue swapping back and forth until you wrap around

Landon Rabern

A prison problem

backgroun

Ore-degre

the problem

Solving th rephrased problem

Kierstead an Kostochka's proof

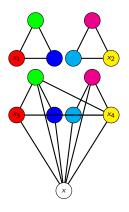
Problem solved

The recoloring algorithm

A spectrum of generalizations

Further improvement

The recoloring algorithm



 use the fact that you wrapped around to show that there are many edges between the two cliques

Landon Rabern

A prison problem

Some backgroun

Ore-degre

Rephrasing the problem

Solving the rephrased

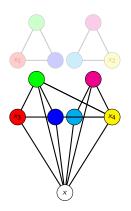
Kierstead and Kostochka's proof

Problem solved Proof outline

The recoloring algorithm

A spectrum of

Further improvement



- use the fact that you wrapped around to show that there are many edges between the two cliques
- we have now constructed the desired large "dense" subgraph

Landon Rabern

A prison problem

The

Rephrasing

Solving the rephrased

A spectrum

generalizations Generalizing

maximum degree
The generalized bound
The lower bound on t_{ϵ}

Further

Generalizing maximum degree

Definition

For $0 \le \epsilon \le 1$, define $\Delta_{\epsilon}(G)$ as

$$\left[\max_{xy\in E(G)}(1-\epsilon)\min\{d(x),d(y)\}+\epsilon\max\{d(x),d(y)\}\right].$$

 Δ_0 ?

Landon Rabern

problem

The

Rephrasing

Solving the rephrased

rephrased problem

A spectrum of generalizations

Generalizing maximum degree The generalized bound

The lower bou on t_{ϵ} What about Δ_{0} ?

Further improvements

Generalizing maximum degree

Definition

For $0 \leq \epsilon \leq 1$, define $\Delta_{\epsilon}(G)$ as

$$\left[\max_{xy\in E(G)}(1-\epsilon)\min\{d(x),d(y)\}+\epsilon\max\{d(x),d(y)\}\right].$$

Note that $\Delta_1 = \Delta$, $\Delta_{\frac{1}{2}} = \lfloor \frac{\theta}{2} \rfloor$.

Landon Rabern

problem

backgroun

Rephrasing

Solving th rephrased

A spectrum o generalization

Generalizing

The generalized bound

The lower bound on t_{ϵ} What about Δ_{0} ?

Further improvement

The generalized bound

Theorem (Rabern 2010)

Landon Rabern

problem

The Ore degree

Rephrasing the problem

Solving the rephrased problem

A spectrum o

Generalizations

The generalized bound

The lower bou on t_{ϵ} What about Δ_{0} ?

Further improvement

The generalized bound

Theorem (Rabern 2010)

For every $0 < \epsilon \le 1$, there exists t_{ϵ} such that every graph with $\Delta_{\epsilon} \ge t_{\epsilon}$ satisfies $\chi \le \max\{\omega, \Delta_{\epsilon}\}$.

• the proof uses a recoloring algorithm similar to the above

Landon Rabern

problem

The

Rephrasing

Solving the rephrased problem

A spectrum

Generalizing maximum deg

The generalized bound

The lower bou on t_{ϵ} What about $\Delta \alpha$?

Further improvement

The generalized bound

Theorem (Rabern 2010)

- the proof uses a recoloring algorithm similar to the above
- it would be interesting to determine, for each ϵ , the smallest t_{ϵ} that works

Landon Rabern

problem

backgrour The

Rephrasing

Solving the rephrased problem

problem

generalization Generalizing

The generalized bound

The lower bou on t_{ϵ} What about Δ_0 ?

Further improvement

The generalized bound

Theorem (Rabern 2010)

- the proof uses a recoloring algorithm similar to the above
- it would be interesting to determine, for each ε, the smallest t_ε that works
- that $t_1 = 3$ is smallest is Brooks' theorem

Landon Rabern

problem

The Ore-degre

Rephrasing

Solving the rephrased problem

A spectrum

Generalization
Generalizing

The generalized bound

The lower bound on t_{ϵ} What about

Further improvement

The generalized bound

Theorem (Rabern 2010)

- the proof uses a recoloring algorithm similar to the above
- it would be interesting to determine, for each ϵ , the smallest t_{ϵ} that works
- that $t_1 = 3$ is smallest is Brooks' theorem
- the graph O_5 shows that $t_\epsilon=6$ is smallest for $rac{1}{2} \leq \epsilon < 1$

Landon Rabern

problem

The Ore degree

Rephrasing

Solving the rephrased problem

A spectrum of generalization

maximum degre The generalized bound

The lower bou on t_{ϵ} What about

Further improvement

The generalized bound

Theorem (Rabern 2010)

- the proof uses a recoloring algorithm similar to the above
- it would be interesting to determine, for each ϵ , the smallest t_{ϵ} that works
- that $t_1 = 3$ is smallest is Brooks' theorem
- the graph O_5 shows that $t_\epsilon=6$ is smallest for $\frac{1}{2} \leq \epsilon < 1$
- best known general bounds, $\frac{2}{\epsilon} + 1 \le t_{\epsilon} \le \frac{4}{\epsilon} + 2$

Landon Rabern

A prison

Some backgroun

The Ore-degree

Rephrasing the problem

Solving the rephrased problem

A spectrum o

Generalizing maximum degree The generalized

The lower bound on t_{ϵ} What about Δ_0 ?

Further improvements

The lower bound on t_{ϵ}

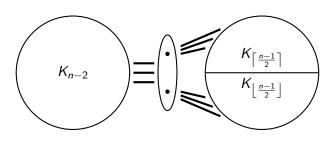


Figure: The graph O_n .

Landon Rabern

A prison problem

background

Rephrasing

Solving the rephrased

A spectrum of generalization

Generalizing maximum degre The generalized

The lower bound on t_{ϵ} What about Δ_0 ?

Further improvement

The lower bound on t_{ϵ}

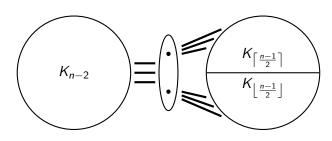


Figure: The graph O_n .

•
$$\chi(O_n)=n>\omega(O_n)$$
 and $\Delta(O_n)=\left\lceil \frac{n-1}{2} \right\rceil +n-2$

Landon Rabern

A prison problem

Some background

The Ore-degre

Rephrasing the problem

Solving the rephrased problem

A spectrum of generalization

Generalizing maximum degre The generalized

The lower bound on t_{ϵ} What about Δ_0 ?

Further improvement

The lower bound on t_{ϵ}

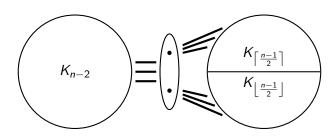


Figure: The graph O_n .

- $\chi(O_n) = n > \omega(O_n)$ and $\Delta(O_n) = \left\lceil \frac{n-1}{2} \right\rceil + n 2$
- $\mathcal{H}(O_n)$ is edgeless

Landon Rabern

A prison problem

background

Rephrasing

Solving the rephrased problem

A spectrum of generalization

Generalizing maximum degre The generalized

The lower bound on t_{ϵ} What about Δ_0 ?

Further improvement

The lower bound on t_{ϵ}

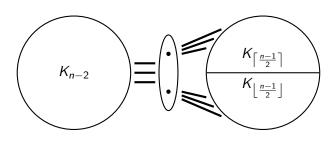


Figure: The graph O_n .

- $\chi(O_n)=n>\omega(O_n)$ and $\Delta(O_n)=\left\lceil \frac{n-1}{2} \right\rceil +n-2$
- $\mathcal{H}(O_n)$ is edgeless
- computing Δ_{ϵ} gives $t_{\epsilon} \geq rac{2}{\epsilon} + 1$

Landon Rabern

problem

The

Rephrasing

Solving the rephrased problem

A spectrum of generalizations

maximum degree The generalized bound

on t_{ϵ} What about Δ_0 ?

Further improvements

What about Δ_0 ?

 \bullet the above proofs only work for $\epsilon>0$

Landon Rabern

problem

backgroun

The Ore-degree

Rephrasing the problem

Solving the rephrased problem

A spectrum of generalizations

The generalized bound
The lower bound

on t_{ϵ} What about Δ_0 ?

Further improvement

What about Δ_0 ?

- \bullet the above proofs only work for $\epsilon>0$
- what happens when $\epsilon=0$?

Landon Rabern

problem

backgroun

The Ore-degree

Rephrasing the problem

Solving the rephrased

A spectrum of generalizations

The generalized bound
The lower bound on t_{ϵ} What about

Further

What about Δ_0 ?

- ullet the above proofs only work for $\epsilon>0$
- what happens when $\epsilon = 0$?
- the parameter Δ_0 has already been investigated by Stacho under the name Δ_2

 Δ_0 ?

Landon Rabern

A prison problem

The

Rephrasing

Solving the rephrased problem

A spectrum of generalizations

The generalized bound
The lower bound

on t_{ϵ} What about Δ_0 ?

Further improvement

What about Δ_0 ?

- ullet the above proofs only work for $\epsilon>0$
- what happens when $\epsilon = 0$?
- the parameter Δ_0 has already been investigated by Stacho under the name Δ_2

Definition (Stacho 2001)

For a graph G define

$$\Delta_0(G) := \max_{xy \in E(G)} \min\{d(x), d(y)\}.$$

Landon Rabern

A prison problem

The

Rephrasing

Solving th rephrased

A spectrum of generalizations

The generalized bound The lower bound on t_{ϵ}

What about Δ_0 ?

Further improvements

Facts about Δ_0

• greedy coloring (in any order) shows that every graph satisfies $\chi \leq \Delta_0 + 1$

Rephrasing the problem

Solving the rephrased problem

A spectrum of generalizations

The generalized bound
The lower bound on t_{ϵ}

What about Δ_0 ?

Further improvement

Facts about Δ_0

- greedy coloring (in any order) shows that every graph satisfies $\chi \leq \Delta_0 + 1$
- for any fixed $t\geq 3$, the problem of determining whether or not $\chi(G)\leq \Delta_0(G)$ for graphs with $\Delta_0(G)=t$ is NP-complete (Stacho 2001)

Landon Rabern

A prisor problem

Some

The

Rephrasing the problem

Solving th rephrased problem

A spectrum of generalization

maximum degre The generalized

The lower boun on t_{ϵ} What about Δ_{0} ?

Further improvements

A tempting thought

A tempting thought

There exists t such that every graph with $\Delta_0 \geq t$ satisfies $\chi \leq \max\{\omega, \Delta_0\}$.

The generalized bound
The lower bound

What about Δ_0 ?

Further improvements

A tempting thought

A tempting thought

There exists t such that every graph with $\Delta_0 \geq t$ satisfies $\chi \leq \max\{\omega, \Delta_0\}$.

• since $t_{\epsilon} \geq \frac{2}{\epsilon} + 1$, we see that $t_{\epsilon} \to \infty$ as $\epsilon \to 0$

Generalizing maximum degre The generalized

The lower bound on t_{ϵ} What about Δ_0 ?

Further improvement

A tempting thought

A tempting thought

There exists t such that every graph with $\Delta_0 \geq t$ satisfies $\chi \leq \max\{\omega, \Delta_0\}$.

- since $t_{\epsilon} \geq \frac{2}{\epsilon} + 1$, we see that $t_{\epsilon} \to \infty$ as $\epsilon \to 0$
- thus, t₀ does not exist and the tempting thought cannot hold

Generalizing maximum degree The generalized bound

on t_{ϵ} What about Δ_0 ?

Further improvement

A tempting thought

A tempting thought

There exists t such that every graph with $\Delta_0 \geq t$ satisfies $\chi \leq \max\{\omega, \Delta_0\}$.

- since $t_{\epsilon} \geq \frac{2}{\epsilon} + 1$, we see that $t_{\epsilon} \to \infty$ as $\epsilon \to 0$
- thus, t₀ does not exist and the tempting thought cannot hold
- there is a cute algorithmic way to see this assuming $P \neq NP$

maximum degree
The generalized
bound
The lower bound
on t_{ϵ} What about

Further improvement

A tempting thought

A tempting thought

There exists t such that every graph with $\Delta_0 \geq t$ satisfies $\chi \leq \max\{\omega, \Delta_0\}$.

- since $t_{\epsilon} \geq \frac{2}{\epsilon} + 1$, we see that $t_{\epsilon} \to \infty$ as $\epsilon \to 0$
- ullet thus, t_0 does not exist and the tempting thought cannot hold
- there is a cute algorithmic way to see this assuming $P \neq NP$
- we use Lovász's ϑ parameter which can be appoximated in polynomial time and has the property that $\omega(\mathcal{G}) \leq \vartheta(\mathcal{G}) \leq \chi(\mathcal{G})$

 Δ_0 ?

Landon Rabern

A prison problem

Some backgroun

The Ore-degree

Rephrasing the problem

Solving the rephrased problem

A spectrum of generalizations

The generalized bound
The lower bound

on t_{ϵ} What about Δ_0 ?

Further improvements

A polynomial-time algorithm

ullet assume the tempting thought holds for some $t\geq 3$

Landon Rabern

problem

backgroun

Ore-degree

Rephrasing the problem

rephrased problem

A spectrum of generalizations

The generalized bound
The lower bound

on t_{ϵ} What about Δ_0 ?

Further improvement

- assume the tempting thought holds for some $t \ge 3$
- ullet take any arbitrary graph with $\Delta_0 \geq t$

Landon Rabern

problem

The

Rephrasing

Solving th rephrased

A spectrum o generalization

The generalized bound
The lower bound

on t_{ϵ} What about Δ_0 ?

Further improvement

- assume the tempting thought holds for some $t \ge 3$
- ullet take any arbitrary graph with $\Delta_0 \geq t$
- first, compute Δ_0 in polynomial time

Landon Rabern

problem

The

Rephrasing the problem

Solving the rephrased problem

A spectrum of generalizations

The generalized bound
The lower bound

What about Δ_0 ?

Further improvements

- assume the tempting thought holds for some $t \ge 3$
- ullet take any arbitrary graph with $\Delta_0 \geq t$
- first, compute Δ_0 in polynomial time
- second, compute x such that $x \frac{1}{2} < \vartheta < x + \frac{1}{2}$ in polynomial time

The generalized bound
The lower bound

on t_{ϵ} What about Δ_0 ?

Further improvements

- assume the tempting thought holds for some $t \ge 3$
- ullet take any arbitrary graph with $\Delta_0 \geq t$
- first, compute Δ_0 in polynomial time
- second, compute x such that $x \frac{1}{2} < \vartheta < x + \frac{1}{2}$ in polynomial time
- if $x \ge \Delta_0 + \frac{1}{2}$, then $\chi \ge \vartheta > \Delta_0$ and hence $\chi = \Delta_0 + 1$

Further

A polynomial-time algorithm

- assume the tempting thought holds for some $t \ge 3$
- ullet take any arbitrary graph with $\Delta_0 \geq t$
- first, compute Δ_0 in polynomial time
- second, compute x such that $x \frac{1}{2} < \vartheta < x + \frac{1}{2}$ in polynomial time
- if $x \ge \Delta_0 + \frac{1}{2}$, then $\chi \ge \vartheta > \Delta_0$ and hence $\chi = \Delta_0 + 1$
- if $x < \Delta_0 + \frac{1}{2}$, then $\omega \le \vartheta < \Delta_0 + 1$, and hence $\omega \le \Delta_0$

 Δ_0 ?

Solving the rephrased

A spectrum of generalizations

Generalizing maximum degree The generalized bound

on t_{ϵ} What about Δ_0 ?

Further improvement

- assume the tempting thought holds for some $t \ge 3$
- ullet take any arbitrary graph with $\Delta_0 \geq t$
- first, compute Δ_0 in polynomial time
- second, compute x such that $x \frac{1}{2} < \vartheta < x + \frac{1}{2}$ in polynomial time
- if $x \ge \Delta_0 + \frac{1}{2}$, then $\chi \ge \vartheta > \Delta_0$ and hence $\chi = \Delta_0 + 1$
- if $x < \Delta_0 + \frac{1}{2}$, then $\omega \le \vartheta < \Delta_0 + 1$, and hence $\omega \le \Delta_0$
- now, $\chi \leq \max\{\omega, \Delta_0\} \leq \Delta_0$

Landon Rabern

problem

The

Rephrasing the problem

Solving the rephrased problem

A spectrum of generalizations Generalizing maximum degree The generalized

The lower bound on t_{ϵ} What about Δ_0 ?

Further improvements

- assume the tempting thought holds for some $t \ge 3$
- ullet take any arbitrary graph with $\Delta_0 \geq t$
- first, compute Δ_0 in polynomial time
- second, compute x such that $x \frac{1}{2} < \vartheta < x + \frac{1}{2}$ in polynomial time
- if $x \ge \Delta_0 + \frac{1}{2}$, then $\chi \ge \vartheta > \Delta_0$ and hence $\chi = \Delta_0 + 1$
- if $x < \Delta_0 + \frac{1}{2}$, then $\omega \le \vartheta < \Delta_0 + 1$, and hence $\omega \le \Delta_0$
- now, $\chi \leq \max\{\omega, \Delta_0\} \leq \Delta_0$
- we just gave a polynomial time algorithm to determine whether or not $\chi \leq \Delta_0$ for graphs with $\Delta_0 \geq t$

Further improvement

A polynomial-time algorithm

- assume the tempting thought holds for some $t \ge 3$
- ullet take any arbitrary graph with $\Delta_0 \geq t$
- ullet first, compute Δ_0 in polynomial time
- second, compute x such that $x \frac{1}{2} < \vartheta < x + \frac{1}{2}$ in polynomial time
- if $x \ge \Delta_0 + \frac{1}{2}$, then $\chi \ge \vartheta > \Delta_0$ and hence $\chi = \Delta_0 + 1$
- if $x < \Delta_0 + \frac{1}{2}$, then $\omega \le \vartheta < \Delta_0 + 1$, and hence $\omega \le \Delta_0$
- now, $\chi \leq \max\{\omega, \Delta_0\} \leq \Delta_0$
- we just gave a polynomial time algorithm to determine whether or not $\chi \leq \Delta_0$ for graphs with $\Delta_0 \geq t$
- this is impossible unless *P*=*NP*

 Δ_0 ?

Landon Rabern

problem

The

Rephrasing the problem

Solving the rephrased problem

A spectrum of generalizations

The generalized bound
The lower bound

What about Δ_0 ?

Further improvements

What we can prove about Δ_0

Theorem (Rabern 2010)

Every graph with $\Delta \geq 3$ satisfies

$$\chi \leq \max \left\{ \omega, \Delta_0, rac{5}{6}(\Delta+1)
ight\}.$$

Generalizing maximum degre The generalized bound

The lower boun on t_{ϵ} What about Δ_0 ?

Further improvements

What we can prove about Δ_0

Theorem (Rabern 2010)

Every graph with $\Delta \geq 3$ satisfies

$$\chi \leq \max \left\{ \omega, \Delta_0, rac{5}{6}(\Delta+1)
ight\}.$$

• the proof uses a recoloring algorithm similar to the above

Landon Rabern

The generalized

What about

 Δ_0 ?

What we can prove about Δ_0

Theorem (Rabern 2010)

Every graph with $\Delta \geq 3$ satisfies

$$\chi \leq \max \left\{ \omega, \Delta_0, rac{5}{6}(\Delta+1)
ight\}.$$

- the proof uses a recoloring algorithm similar to the above
- actually, all the above results about Δ_{ϵ} follow from this result

Landon Rabern

A prison problem

Some backgroun

The Ore-degree

Rephrasing the problem

Solving the rephrased problem

A spectrum of generalization

Further improvements

In joint work with Kostochka and Stiebitz similar techniques were used to improve the bounds further. Highlights:

Landon Rabern

problem

The

Rephrasing

Solving the rephrased

A spectrum

Further improvements

In joint work with Kostochka and Stiebitz similar techniques were used to improve the bounds further. Highlights:

Theorem (Kostochka, Rabern and Stiebitz 2010)

Every graph with $\theta \geq 8$, except O_5 , satisfies $\chi \leq \max\left\{\omega, \left\lfloor \frac{\theta}{2} \right\rfloor\right\}$.

Further improvements

In joint work with Kostochka and Stiebitz similar techniques were used to improve the bounds further. Highlights:

Theorem (Kostochka, Rabern and Stiebitz 2010)

Every graph with $\theta \geq 8$, except O_5 , satisfies $\chi \leq \max\left\{\omega, \left\lfloor \frac{\theta}{2} \right\rfloor\right\}$.

Theorem (Kostochka, Rabern and Stiebitz 2010)

Every graph satisfies

$$\chi \leq \max \left\{ \omega, \Delta_0, rac{3}{4}(\Delta+2)
ight\}.$$

Landon Rabern

problem

The

Rephrasing the problem

Solving the rephrased

problem

generalizatio

Further improvements

Conjecture

Every graph satisfies

$$\chi \leq \max\left\{\omega, \Delta_0, \frac{2\Delta+5}{3}
ight\}.$$

The examples O_n above show that this would be tight.

> Landon Rabern

Further improvements



M. Grötschel, L. Lovász, and A. Schrijver.

The ellipsoid method and its consequences in combinatorial optimization. Combinatorica, 1(2):169-197, 1981.

H.A. Kierstead and A.V. Kostochka.

Ore-type versions of Brooks' theorem.

J. Combin. Theory Ser. B, 99(2):298-305, 2009.



A.V. Kostochka, L. Rabern, and M. Stiebitz.

Graphs with chromatic number close to maximum degree. Discrete Math, Forthcoming.



B. Rabern.

Reformulation as a prison problem.





An improvement on Brooks' theorem.

Submitted.



L. Rabern.

On hitting all maximum cliques with an independent set. J. Graph Theory, 66(1):32-37, 2011.



I Rahern

Δ-Critical graphs with small high vertex cliques.





New upper bounds for the chromatic number of a graph. J. Graph Theory, 36(2):117-120, 2001.



M. Stiebitz.

Proof of a conjecture of T. Gallai concerning connectivity properties of colour-critical graphs. Combinatorica, 2(3):315-323, 1982.