survey on the Borodin-Kostochka conjecture

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1 Introduction

Conjecture 1 (Borodin and Kostochka [3]). *Every graph G with $\Delta(G) \geq 9$ satisfies $\chi(G) \leq \max\{\omega(G), \Delta(G) - 1\}$.*

2 Techniques

Catlin/Mozhan shuffling, independent transversals and strong coloing, d_1-choosables, mules, fractional coloring, probabilistic method, kernels, Kostochka's method

claw-free, doubly-critical edges, squares, vertex-transitive

3 Excluded induced subgraphs by d_1-choosability

A graph G is *d_r-choosable* if G can be L-colored from every list assingment L with $|L(v)| \geq d_G(v) - r$ for all $v \in V(G)$. Every graph is d_{-1}-choosable. The d_0-choosable graphs were classified by Borodin [2] and independently by Erdős, Rubin, and Taylor [10] as those graphs whose every block is either complete or an odd cycle (a connected such graph is a *Gallai tree*). Classifying the d_r-choosable graphs for any $r \geq 1$ appears to be a hard problem. However, we can get useful sufficient conditions for a graph to be d_1-choosable. For example, all of the graphs here are d_1-choosable (the vertex color indicates components of the complement): `https://landon.github.io/graphdata/borodinkostochka/offline/index.html`

Cranston and Rabern [7] classified all d_1-choosable graphs of the form $A \vee B$.

4 Properties of minimum counterexamples

In [8] Cranston and Rabern used the d_1-choosable graphs in Section 3 to prove properties of a minimum counterexample to the Borodin-Kostochka conjecture. For example, the following improves a lemma Reed used in his proof [16].

Lemma 2. *Let G be a minimum counterexample to the Borodin-Kostochka conjecture for fixed $\Delta(G) \geq 8$. If X is a $K_{\Delta(G)-1}$ in G, then every $v \in V(G - X)$ has at most one neighbor in X.*

In combination with more d_1-choosable graphs, Cranston and Rabern [8] used Lemma 2 to severely restrict the kinds of joins a minimum counterexample can contain.

Theorem 3. *Let G be a minimum counterexample to the Borodin-Kostochka conjecture for fixed $\Delta(G) \geq 9$. Let A and B be disjoint subgraphs of G with $|A| + |B| = \Delta(G)$ such that $|A|, |B| \geq 3$. If G contains all edges between A and B, then $A = K_1 + K_{|A|-1}$ and $B = K_1 + K_{|B|-1}$.*

Corollary 4. *If G is a minimum counterexample to the Borodin-Kostochka conjecture for fixed $\Delta(G) \geq 9$, then $K_3 \vee \overline{K_{\Delta(G)-3}} \not\subseteq G$.*

4.1 Lemma 2 in restricted classes of graphs

Definition 1. For $r \in \mathbb{N}$, a collection of graphs $\mathcal{C}$ is *r-permissible* if the following three conditions hold:

1. $\Delta(G) \leq r$ for all $G \in \mathcal{C}$; and
2. if $\Delta(G) \leq r$ and there is $e \in E(G)$ such that $G - e \in \mathcal{C}$, then $G \in \mathcal{C}$; and
3. if $G \in \mathcal{C}$ and $S \subseteq V(G)$ has $|S| \in \{r-1, r, r+1\}$ and $\|G[S]\| \geq \binom{|S|}{2} - 3$, then $G - S \in \mathcal{C}$.

For example, if $\mathcal{D}$ is a collection of graphs closed under vertex removal and edge addition, then the collection of graphs in $\mathcal{D}$ with maximum degree at most r is r-permissible. Not all r-permissible collections are if this form, later we will use the collection of graphs with maximum degree at most r where every vertex is in a clique on at least $\frac{2}{3}r + 1$ vertices.

Lemma 5. *Let $\mathcal{C}$ be an r-permissible collection of graphs where $r \geq 8$. Suppose there exist graphs $H \in \mathcal{C}$ with $\chi(H) = \Delta(H) = r$ and $\omega(H) < r$. Let $G \in \mathcal{C}$ be such an H minimizing $|G|$. If X is a $K_{\Delta(G)-1}$ in G, then every $v \in V(G - X)$ has at most one neighbor in X.*

5 Decompositions

5.1 Reed's decomposition

In [16], Reed proved the Borodin-Kostochka conjecture for graphs G with $\Delta(G) \geq 10^{14}$. A piece of that proof was a decomposition of G into dense chunks and one sparse chunk that also works for smaller $\Delta(G)$. The following tight form of this decomposition is given in [15]. Let $\mathcal{C}_t(G)$ be the maximal cliques in G having at least t vertices.

Reed's Decomposition. *Suppose G is a graph with $\Delta(G) \geq 8$ that contains no $K_{\Delta(G)}$ and has no d_1-choosable induced sugraph. If $\frac{\Delta(G)+5}{2} \leq t \leq \Delta(G) - 1$, then $\bigcup \mathcal{C}_t(G)$ can be partitioned into sets $D_1, \ldots, D_r$ such that for each $i \in [r]$ at least one of the following holds:*

1. *$D_i = C_i \in \mathcal{C}_t(G)$,*
2. *$D_i = C_i \cup \{x_i\}$ where $C_i \in \mathcal{C}_t(G)$ and $|N(x_i) \cap C_i| \geq t - 1$.*

5.2 Fajtlowicz's decomposition

In [11], Fajtlowicz proved that every graph has $\alpha(G) \geq \frac{2|G|}{\omega(G)+\Delta(G)+1}$. The proof of this result gives a decomposition which we state in the special case needed for the Borodin-Kostochka conjecture.

Fajtlowicz's Decomposition. *Suppose G is a vertex-critical graph with $\chi(G) = \Delta(G)$. Then $V(G)$ can be partitioned into sets M, T, and K such that*

1. *M contains a maximum independent set I of G; and*
2. *each $v \in T$ has $d_G(v) = \Delta(G)$, two neighbors in I and zero neighbors in $M \setminus I$; and*
3. *K can be covered by $\alpha(G)$ (or fewer) cliques; and*
4. *each $v \in K$ has exactly one neighbor in I and at most one neighbor in $M \setminus I$ (none if $d_G(v) < \Delta(G)$); and*
5. *the vertices in $M \setminus I$ can be ordered $v_1, \ldots, v_r$ such that for $i \in [r]$, either v_i has at least three neighbors in $I \cup \{v_1, \ldots, v_{i-1}\}$ or $d_G(v_i) < \Delta(G)$ and v_i has at least two neighbors in $I \cup \{v_1, \ldots, v_{i-1}\}$.*

Proof. Let I be a maximum independent set in G. Construct a maximal length sequence $I = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r$ such that for $j > 0$,

- every $v \in M_j$ with $d_G(v) = \Delta(G)$ either has at least three neighbors in M_{j-1} or at least two neighbors in $M_{j-1} \setminus I$; and
- every $v \in M_j$ with $d_G(v) = \Delta(G) - 1$ either has at least two neighbors in M_{j-1} or at least one neighbor in $M_{j-1} \setminus I$.

Now let $M = M_r$, let T be the vertices in $V(G) \setminus M$ with exactly two neighbors in I and let K be the vertices in $V(G) \setminus M$ with exactly one neighbor in I. The decomposition has the properties 1,2,4 and 5 since the sequence $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r$ was chosen to be maximal length. Property 3 follows since for each $v \in I$, the set of $x \in K$ adjacent to v must be a clique for otherwise we could get an independent set larger than I. $\square$

6 Results from strong coloring

In [15], using ideas from strong coloring [12, 1], Rabern showed that any counterexample to the Borodin-Kostochka conjecture must have some sparse neighborhood and large independence number.

Theorem 6. *Every graph G with $\chi(G) \geq \Delta(G) \geq 9$ such that every vertex is in a clique on $\frac{2}{3}\Delta(G) + 2$ vertices contains $K_{\Delta(G)}$.*

Theorem 7. *Every graph G with $\omega(G) < \Delta(G)$ such that $d(G[N(v)]) \geq \frac{2}{3}\Delta(G) + 4$ for each $v \in V(G)$ is $(\Delta(G) - 1)$-colorable.*

Theorem 8. *Every graph G satisfies $\chi(G) \leq \max\{\omega(G), \Delta(G) - 1, 4\alpha(G)\}$.*

In the next subsection, we improve Theorem 6 and Theorem 7 slightly.

6.1 An improvement

The proof of Theorem 6 in [15] uses techniques developed for strong coloring. Here we show how to use the best known bounds for strong coloring directly and hence get a small improvement.

For a positive integer r, a graph G with $|G| = rk$ is called *strongly r-colorable* if for every partition of $V(G)$ into parts of size r there is a proper coloring of G that uses all r colors on each part. If $|G|$ is not a multiple of r, then G is strongly r-colorable iff the graph formed by adding $r\left\lceil \frac{|G|}{r} \right\rceil - |G|$ isolated vertices to G is strongly r-colorable. The *strong chromatic number* $s\chi(G)$ is the smallest r for which G is strongly r-colorable.

Note that a strong r-coloring of G with respect to a partition $V_1, \ldots, V_k$ of $V(G)$ with $|V_i| = r$ must partition $V(G)$ into r independent transversals of $V_1, \ldots, V_k$. In [17], Szabó and Tardos constructed partitioned graphs with part sizes $2\Delta - 1$ that have no independent transversal. So we must have $s\chi(G) \geq 2\Delta(G)$. That the upper bound $s\chi(G) \leq 2\Delta(G)$ holds is the *strong coloring conjecture*. Haxell [12] proved that $s\chi(G) \leq 3\Delta(G) - 1$.

Theorem 9. *Every graph G with $\chi(G) \geq \Delta(G) \geq 8$ such that every vertex is in a clique on $\frac{2}{3}\Delta(G) + 1$ vertices contains $K_{\Delta(G)}$.*

Proof. Suppose the theorem does not hold and let G be a counterexample minimizing $|G|$. Apply Reed's decomposition with $t := \left\lceil \frac{2}{3}\Delta(G) \right\rceil + 1$ to get sets $D_1, \ldots, D_r$ with $\bigcup_{i \in [r]} D_i = V(G)$. By Lemma 5, $|D_i| \leq \Delta(G) - 1$ for all $i \in [r]$. Create G' from G by removing all the edges in $G[D_i]$ for each $i \in [r]$. By Haxell's bound, $s\chi(G') \leq 3\Delta(G') - 1 \leq 3(\Delta(G) - (t - 1)) - 1 \leq \Delta(G) - 1$. This gives a $(\Delta(G) - 1)$-coloring of G, a contradiction. □

If the strong coloring conjecture holds, we get the following improvement.

Conjecture 10. *Every graph G with $\chi(G) \geq \Delta(G) \geq 8$ such that every vertex is in a clique on $\frac{\Delta(G)+5}{2}$ vertices contains $K_{\Delta(G)}$.*

7 Results from kernel methods

In [13], Kierstead and Rabern proved a general lemma that allows the user to get list colorings for free from large independent sets. Specialized to the Borodin-Kostochka conjecture, this becomes.

Kernel Magic. *Suppose G is a vertex-critical graph with $\chi(G) = \Delta(G)$. For every induced subgraph H of G and independent set I in H, we have*

$$\sum_{v \in I} d_H(v) < \sum_{v \in V(H)} \Delta(G) + 2 - d_G(v).$$

Applied with $H = G$, this gives:

Corollary 11. *If G is a vertex-critical graph with $\chi(G) = \Delta(G)$, then $\alpha(G) < \frac{2|G|}{\Delta(G)-1}$.*

8 Mozhan partitions

Extending ideas of Mozhan [14], Cranston and Rabern [9] proved the following.

Theorem 12. *If G is a vertex-critical graph with $\chi(G) = \Delta(G) \geq 13$, then $\omega(G) \geq \Delta(G) - 3$.*

9 Vertex-transitive graphs

In [5] Cranston and Rabern used Reed's decomposition and the ideas in Sections 6 and 8 to prove the Borodin-Kostochka conjecture for vertex-transitive graphs with $\Delta(G) \geq 13$. It would be interesting to improve this to $\Delta(G) \geq 9$.

Theorem 13. *Every vertex-transitive graph G with $\Delta(G) \geq 13$ satisfies $\chi(G) \leq \max\{\omega(G), \Delta(G) - 1\}$.*

10 Claw-free graphs

In [6], Cranston and Rabern proved the Borodin-Kostochka conjecture for claw-free graphs using some of the d_1-choosable graphs in Section 3 combined with the structure theorem for quasi-line graphs of Chudnovsky and Seymour [4].

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