Edge-coloring via fixable subgraphs

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1   Introduction

All multigraphs are loopless. Let G be a multigraph and L a list assignment on $V(G)$ and $\text{pot}(L) = \bigcup_{v \in V(G)} L(v)$. An L-*pot* is a set X containing $\text{pot}(L)$. An edge-coloring π of G such that $\pi(x) \in L(x) \cap L(y)$ for all $xy \in E(G)$ is called an L-*edge-coloring*.

2   Completing edge-colorings

Our goal is to convert a partial k-edge-coloring of a multigraph M into a (total) k-edge-coloring of M. For a partial k-edge-coloring π of M, let M_π be the subgraph of M induced on the uncolored edges and let L_π be the list assignment on the vertices of M_π given by $L_\pi(v) = [k] - \{\tau \mid \pi(vx) = \tau \text{ for some } vx \in E(M)\}$.

Kempe chains give a powerful technique for converting a partial k-edge-coloring into a total k-edge-coloring. The idea is to repeatedly exchange colors on two-colored paths until M_π has an edge-coloring ζ such that $\zeta(xy) \in L_\zeta(x) \cap L_\zeta(y)$ for all $xy \in E(M_\pi)$. In this sense the original list assignment L_π on M_π is *fixable*. In the next section, we give an abstract definition of this notion that frees us from the embedding in the ambient graph M. As we will see, computers enjoy this new freedom.

Let G be a multigraph, L a list assignment on $V(G)$ and P an arbitrary L-pot. Throughout this section, G, L and P will refer to these objects.

2.1   Fixable graphs

For different colors $a, b \in P$, let $S_{L,a,b}$ be all the vertices of G that have exactly one of a or b in their list; more precisely, $S_{L,a,b} = \{v \in V(G) \mid |\{a, b\} \cap L(v)| = 1\}$.

Definition 1. G is (L, P)-*fixable* if either

(1) G has an L-edge-coloring; or

(2) there are different $a, b \in P$ such that for every partition $X_1, \ldots, X_t$ of $S_{L,a,b}$ into sets of size at most two, there is $J \subseteq [t]$ so that G is (L', P)-fixable where L' is formed from L by swapping a and b in $L(v)$ for every $v \in \bigcup_{i \in J} X_i$.

We write L-fixable as shorthand for $(L, \text{pot}(L))$-fixable. When G is (L, P)-fixable, the choices of a, b and J in each application of (2) determine a tree where all leaves have lists satisfying (1). The *height* of (L, P) is the minimum possible height of such a tree. We write $h_G(L, P)$ for this height and let $h_G(L, P) = \infty$ when G is not (L, P)-fixable.

Lemma 2.1. *If a multigraph M has a partial k-edge-coloring π such that M_π is $(L_\pi, [k])$-fixable, then M is k-edge-colorable.*

Proof. Choose a partial k-edge-coloring π of M such that M_π is $(L_\pi, [k])$-fixable minimizing $h_{M_\pi}(L_\pi, [k])$. If $h_{M_\pi}(L_\pi, [k]) = 0$, then (1) must hold for M_π and L_π; that is, M_π has an edge-coloring ζ such that $\zeta(x) \in L_\pi(x) \cap L_\pi(y)$ for all $xy \in E(M_\pi)$. But that means that $\pi \cup \zeta$ is the desired k-edge-coloring of M.

So, we may assume that $h_{M_\pi}(L_\pi, [k]) > 0$. Let $a, b \in [k]$ be a choice in (2) that leads to a tree of height $h_{M_\pi}(L_\pi, [k])$. Let H be the subgraph of M induced on all edges e with $\pi(e) \in \{a, b\}$. Let S be the vertices in M_π with degree exactly one in H. Consider the component C_x in H for each $x \in S$. We have $|V(C_x) \cap S| \in \{1, 2\}$ and hence the components of H give a partition $X_1, \ldots, X_t$ of S into sets of size at most two. Moreover, exchanging colors a and b on C_x has the effect of swapping a and b in $L_\pi(v)$ for each $v \in V(C_x) \cap S$. Hence we can achieve the needing swapping of colors in the lists in (2) by exchanging colors on the components of H. By (2) there is $J \subseteq [t]$ so that M_π is $(L', [k])$-fixable where L' is formed from L_π by swapping a and b in $L_\pi(v)$ for every $v \in \bigcup_{i \in J} X_i$. Choose such a J that leads to a tree of height $h_{M_\pi}(L_\pi, [k])$. Let π' be the partial k-edge-coloring of M created from π by performing the color exchanges to create L' from L_π. Then $M_{\pi'}$ is $(L_{\pi'}, [k])$-fixable and $h_{M_{\pi'}}(L_{\pi'}, [k]) < h_{M_\pi}(L_\pi, [k])$, contradicting the minimality of $h_{M_\pi}(L_\pi, [k])$. $\qquad\square$

The definition of L-fixable was originally motivated as a generalization of the Fixer-Breaker game in [3] from complete graphs to arbitrary graphs. The direct generalization of that game gives us less power because it does not take the fact that two-colored paths cannot cross into account. Interpreted in the Fixer-Breaker game, the choice of partition in (2) is forcing Breaker to choose two-colored paths in a way that is consistent with being embedded in *some* graph. For stars the two games have identical winning conditions because the obvious necessary condition is sufficient, but in general the extra power does make more graphs fixable. However, it is more natural to phrase some proofs in terms of the weaker game, so we define that here.

Definition 2. G is *weakly (L, P)-fixable* if either

(1) G has an L-edge-coloring; or

(2) there are different $a, b \in P$ and $v \in S_{L,a,b}$ such that for every $X \subseteq S_{L,a,b}$ with $|X| \leq 2$ and $v \in X$, it holds that G is (L', P)-fixable where L' is formed from L by swapping a and b in $L(v)$ for every $v \in X$.

Lemma 2.2. *If G is weakly (L, P)-fixable, then G is (L, P)-fixable.*

Proof. Clearly if (1) holds we are done. So, assume we have different $a, b \in P$ and $v \in S_{L,a,b}$ as in (2). Given a partition $X_1, \ldots, X_t$ of $S_{L,a,b}$ into sets of size at most two, let i be the index with $v \in X_i$. Then $J = \{i\}$ is the desired subset of $[t]$. $\qquad\square$

2.2 Some examples

We'd like a way to talk about subgraphs being reducible for k-edge-coloring. Lemma 2.1 gives us this with respect to a fixed partial k-edge-coloring π, but we want a condition independent of the particular coloring. Note that we have a lower bound on the sizes of the lists in L_π; specifically, if π is a partial k-edge-coloring of a multigraph M, then $|L_\pi(v)| \geq k + d_{M_\pi}(v) - d_M(v)$ for $v \in M_\pi$. Using this lower bound, we get our desired condition as follows.

Definition 3. If G is a graph and $f\colon V(G) \to \mathbb{N}$, then G is (f, k)-fixable if G is $(L, [k])$-fixable for every L with $|L(v)| \geq k + d_G(v) - f(v)$ for all $v \in V(G)$.

By Lemma 2.1, if G is (f, k)-fixable, then G cannot be a subgraph of a $(k+1)$-edge-critical graph M where $d_M(v) \leq f(v)$ for all $v \in V(G)$. So, now we can talk about a graph G with vertices labeled by f being k-fixable or not. The computer is extremely good at finding k-fixable graphs. Combined with discharging arguments, this gives a powerful method for proving (modulo trusting the computer) edge-coloring results for small Δ. We'll see some examples of such proofs later, for now Figure 1 shows some 3-fixable graphs. A gallery of hundreds more fixable graphs is available at `https://dl.dropboxusercontent.com/u/8609833/Web/GraphData/Fixable/index.html`.
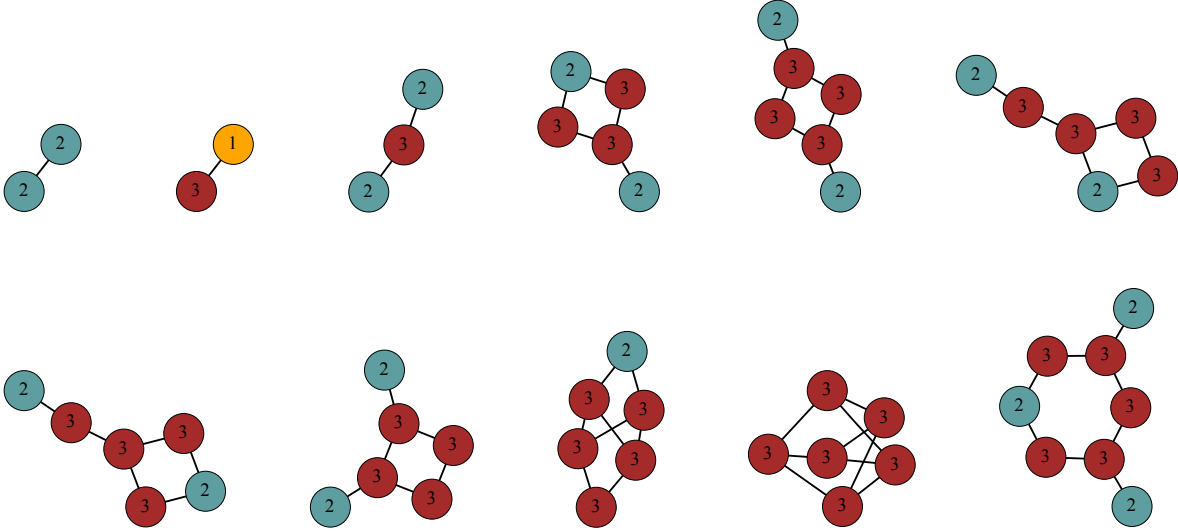

Figure 1: Some small 3-fixable graphs.

The penultimate graph in Figure 1 is an example of the more general fact that a k-regular graph with $f(v) = k$ for all v is k-fixable just is case it is k-edge-colorable. That the third graph in Figure 1 is reducible follows from Vizing's Adjacency Lemma.

2.3 A necessary condition

Since the edges incident to a given vertex must all get different colors, we have the following.

Lemma 2.3. *If G is (L, P)-fixable, then $|L(v)| \geq d_G(v)$ for all $v \in V(G)$.*

By considering the maximum size of matchings in each color, we get a more interesting necessary condition. For $C \subseteq \mathrm{pot}(L)$ and $H \subseteq G$, let $H_{L,C}$ be the subgraph of H induced on the vertices v with $L(v) \cap C \neq \emptyset$. When L is clear from context, we may write H_C for $H_{L,C}$. If $C = \{\alpha\}$, we may write H_α for H_C. For $H \subseteq G$, put

$$\psi_L(H) = \sum_{\alpha \in \mathrm{pot}(L)} \left\lfloor \frac{|H_{L,\alpha}|}{2} \right\rfloor.$$

Each term in the sum gives an upper bound on the size of a matching in color α. So $\psi_L(H)$ is an upper bound on the number of edges in a partial L-edge-coloring of H. We say that (H, L) is *abundant* if $\psi_L(H) \geq \|H\|$ and that (G, L) is *superabundant* if for every $H \subseteq G$, the pair (H, L) is abundant.

Lemma 2.4. *If G is (L, P)-fixable, then (G, L) is superabundant.*

Proof. Suppose to the contrary that G is (L, P)-fixable and there is $H \subseteq G$ such that (H, L) is not abundant. We show that for all different $a, b \in P$ there is a partition $X_1, \ldots, X_t$ of $S_{a,b}$ into sets of size at most two, such that for all $J \subseteq [t]$, the pair (H, L') is not abundant where L' is formed from L by swapping a and b in $L(v)$ for every $v \in \bigcup_{i \in J} X_i$. Since G can never be edge-colored from a list assignment that is not superabundant, this contradicts the (L, P)-fixability of G.

Pick different $a, b \in P$. Let $S = S_{L,a,b} \cap V(H)$ and let S_a be the $v \in S$ with $a \in L(v)$. Put $S_b = S \setminus S_a$. Swapping a and b will only effect the terms $\left\lfloor \frac{|S_a|}{2} \right\rfloor$ and $\left\lfloor \frac{|S_b|}{2} \right\rfloor$ in $\psi_L(H)$. So, if $\psi_L(H)$ is increased by the swapping, it must be that both $|S_a|$ and $|S_b|$ are odd and after swapping they are both even. Say $S_a = \{a_1, \ldots, a_p\}$ and $S_b = \{b_1, \ldots, b_q\}$. By symmetry, we may assume $p \leq q$. For $i \in [p]$, let $X_i = \{a_i, b_i\}$. Since both p and q are odd, $q - p$ is even, so we get a partition by, for each $j \in \left[\frac{q-p}{2}\right]$, letting $X_{p+j} = \{b_{p+2j-1}, b_{p+2j}\}$. For any $i \in [p]$, swapping a and b in $L(v)$ for every $v \in X_i$ maintains $|S_a|$ and $|S_b|$. For any $j \in \left[\frac{q-p}{2}\right]$, swapping a and b in $L(v)$ for every $v \in X_{p+j}$ maintains the parity of $|S_a|$ and $|S_b|$. So no choice of J can increase $\psi_L(H)$ and hence (H, L') is never abundant. $\square$

In particular, we have learned the following.

Corollary 2.5. *If G is (f, k)-fixable, then (G, L) is superabundant for every L with $L(v) \subseteq [k]$ and $|L(v)| \geq k + d_G(v) - f(v)$ for all $v \in V(G)$.*

Intuitively, superabundance requires the potential for a large enough matching in each color. If instead we require the existence of a large enough matching in each color, we get a stronger condition that has been studied before. For a multigraph H, let $\nu(H)$ be the number of edges in a maximum matching of H. For a list assignment L on H, put $\eta_L(H) = \sum_{\alpha \in \mathrm{pot}(L)} \nu(H_\alpha)$. Note that we always have $\psi_L(H) \geq \eta_L(H)$.

The following generalization of Hall's theorem was proved by Marcotte and Seymour [2] and independently by Cropper, Gyárfás and Lehel [1]. By a *multitree* we mean a tree that possibly has edges of multiplicity greater than one.

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Lemma 2.6 (Marcotte and Seymour)**.** *Let T be a multitree and L a list assignment on $V(T)$. If $\eta_L(H) \geq \|H\|$ for all $H \subseteq T$, then T has an L-edge-coloring.*

In [3], the second author proved that superabundance is also a sufficient condition for fixability when we restrict our graphs to be multistars. This immediately implies the fan equation. The proof uses Hall's theorem to reduce to a smaller star and one might hope we could do the same for arbitrary trees with Lemma 2.6 in place of Hall's theorem (thus giving a short proof that Tashkinov trees are elementary), but we haven't yet been able to make this work.

2.4 Fixability of stars

When G is a star, the conjunction of our two necessary conditions is sufficient. This generalizes Vizing fans [4]; in the next section we will define "Kierstead-Tashkinov-Vizing assignments" and show that they are always superabundant. In [3], the second author proved a common generalization of Theorem 2.7 and Hall's theorem, we reproduce the proof for the special case of edge-coloring.

Theorem 2.7. *If G is a multistar, then G is weakly L-fixable if and only if (G, L) is superabundant and $|L(v)| \geq d_G(v)$ for all $v \in V(G)$.*

Proof. Our strategy is simply to increase $\eta_L(G)$ if we can; if we cannot, then Hall's theorem allows us to reduce to a smaller graph. An easy way to describe this strategy is via a double induction as follows. Suppose the theorem is false and choose a counterexample (G, L) minimizing $\|G\|$ and subject to that maximizing $\eta_L(G)$.

Let z be the center of the multistar G. Create a bipartite graph B with parts $U = \{zw \in E(G) \mid L(z) \cap L(w) \neq \emptyset\}$ and $Y := \{\alpha \in \mathrm{pot}(L) \mid \nu(G_\alpha) = 1\}$, where $zw \in U$ is adjacent to $\alpha \in Y$ if and only if $\alpha \in L(z) \cap L(w)$. Informally, Y is the set of colors α that can be used on at least one edge, and U is the set of edges e with at least one color available on e, and a color α is adjacent to an edge e if α can be used on e.

First, suppose $|Y| < \|G\|$. Since $|L(z)| \geq d_G(z) = \|G\|$, we have $\tau \in L(z)$ with $\nu(G_{L,\tau}) = 0$. Suppose there is $\beta \in Y$ such that $|N_B(\beta)| \geq 3$. Pick $zw \in N_B(\beta)$. Since G is not L-fixable, there is $X \subseteq S_{L,\tau,\beta}$ with $|X| \leq 2$ and $w \in X$ such that G not L'-fixable where L' is formed from L by swapping τ and β in $L(v)$ for every $v \in X$. Since $|N_B(\beta)| \geq 3$, we have $\nu(G_{L',\beta}) = 1$ and $\nu(G_{L',\tau}) = 1$ and thus $\eta_{L'}(G) > \eta_L(G)$. Since (G, L') is still superabundant, this violates maximality of $\eta_L(G)$. Hence, we must have $|N_B(\beta)| \leq 2$ for each $\beta \in Y$. So, each color in Y contributes at most one to $\psi_L(G)$. Since $|Y| < \|G\| \leq \psi_L(G)$, there must be $\gamma \notin Y$ such that $|G_\gamma - z| \geq 2$. Since G is not L-fixable, there is $X \subseteq S_{L,\tau,\gamma}$ with $|X| \leq 2$ and $z \in X$ such that G is not L'-fixable where L' is formed from L by swapping τ and γ in $L(v)$ for every $v \in X$. Since $\nu(G_{L,\tau}) = 0$ and $\nu(G_{L,\gamma}) = 0$ and $\nu(G_{L,\gamma}) = 1$, we have $\eta_{L'}(G) > \eta_L(G)$. Since (G, L') is still superabundant, this violates maximality of $\eta_L(G)$.

Hence we must have $|Y| \geq \|G\|$. In particular, $|N_B(Y)| \leq |Y|$ so we may choose $\emptyset \neq C \subseteq Y$ minimal such that $|N_B(C)| \leq |C|$. We claim that $|N_B(C)| = |C|$ and there is a perfect matching M of C into $N_B(C)$. Since $|N_B(C)| \geq 1$, this is true if $|C| = 1$. Otherwise, for any $\tau \in C$, minimality of C shows that $|N_B(C \setminus \{\tau\})| > |C| - 1$. Hence $|N_B(C)| = |C|$ and Hall's theorem gives the desired perfect matching of C into $N_B(C)$. For each $\{zw, \alpha\} \in M$,

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use color α on edge zw. Form G' from G by removing all the colored edges and then discarding any isolated vertices. For $v \in V(G')$, let $L'(v) = L(v) \setminus C$. Since G is not L-fixable and C and $\mathrm{pot}(L')$ are disjoint it must be that G' is not L'-fixable. Since each vertex lost exactly one color from its list for each incident edge we have $|L'(v)| \geq d_{G'}(v)$ for all $v \in V(G')$. Also, for each $H \subseteq G'$, we have $\psi_{L'}(H) = \psi_L(H) - (\|G[V(H)]\| - \|H\|) \geq \|G[V(H)]\| - (\|G[V(H)]\| - \|H\|) = \|H\|$ since H is abundant. But $\|G'\| < \|G\|$, so by minimality of $\|G\|$, G' is L'-fixable, a contradiction. $\qquad\square$

2.5 Kierstead-Tashkinov-Vizing assignments

Many edge-coloring results have been proved using a specific kind of superabundant pair (G, L) where superabundance can be proved via a special ordering. That is, the orderings given by the definition of Vizing fans, Kierstead paths, and Tashkinov trees. In this section, we show how superabundance easily follows from these orderings.

We say that a list assignment L on G is a *Kierstead-Tashkinov-Vizing* assignment (henceforth *KTV-assignment*) if for some $xy \in E(G)$, there is a total ordering '$<$' of $V(G)$ such that

1. there is an edge-coloring π of $G - xy$ such that $\pi(uv) \in L(u) \cap L(v)$ for each $uv \in E(G - xy)$;

2. $x < z$ for all $z \in V(G - x)$;

3. $G[w \mid w \leq z]$ is connected for all $z \in V(G)$;

4. for each $wz \in E(G - xy)$, there is $u < \max\{w, z\}$ such that $\pi(wz) \in L(u) - \{\pi(e) \mid e \in E(u)\}$;

5. there are different $s, t \in V(G)$ such that $L(s) \cap L(t) - \{\pi(e) \mid e \in E(s) \cup E(t)\} \neq \emptyset$.

Lemma 2.8. *If L is a KTV-assignment on G, then (G, L) is superabundant.*

Proof. Let L be a KTV-assignment on G, and let $H \subseteq G$. We will show that (H, L) is abundant. Clearly it suffices to consider the case when H is an induced subgraph, so we assume this. Property (1) gives that $G - xy$ has an edge-coloring π, so $\psi_L(H) \geq \|H\| - 1$; also $\psi_L(H) \geq \|H\|$ if $\{x, y\} \not\subseteq V(H)$. Furthermore $\psi_L(H) \geq \|H\|$ if s and t from property (5) are both in $V(H)$, since then $\psi_L(H)$ gains 1 over the naive lower bound, due to the color in $L(s) \cap L(t)$. So $V(G) - V(H) \neq \emptyset$.

Now choose $z \in V(G) - V(H)$ that is smallest under $<$. Put $H' = G[w \mid w \leq z]$. By the minimality of z, we have $H' - z \subseteq H$. By property (2), $|H'| \geq 2$. By property (3), H' is connected and thus there is $w \in V(H' - z)$ adjacent to z. So, we have $w < z$ and $wz \in E(G) - E(H)$. Now $\pi(wz) \in L(w)$. By the definition of a KTV-assignment, property (4) implies that there exists u with $u < \max\{w, z\} = z$ and $\pi(wz) \in L(u) - \{\pi(e) \mid e \in E(u)\}$. Then $u \in V(H' - z) \subseteq V(H)$ and again we gain 1 over the naive lower bound on $\psi_L(H)$, due to the color in $L(u) \cap L(w)$. So $\psi_L(H) \geq \|H\|$. $\qquad\square$

3 Applications of small k-fixable graphs

3.1 Impoved lower bound on the average degree of 3-critical graphs

3.2 Impoved lower bound on the average degree of 4-critical graphs

3.3 The conjecture of Hilton and Zhao for $\Delta = 4$

Let $\mathcal{H}_4$ denote the class of connected graphs with maximum degree 4, minimum degree 3, each vertex adjacent to at least two 4-vertices, and each 4-vertex adjacent to exactly two 4-vertices.
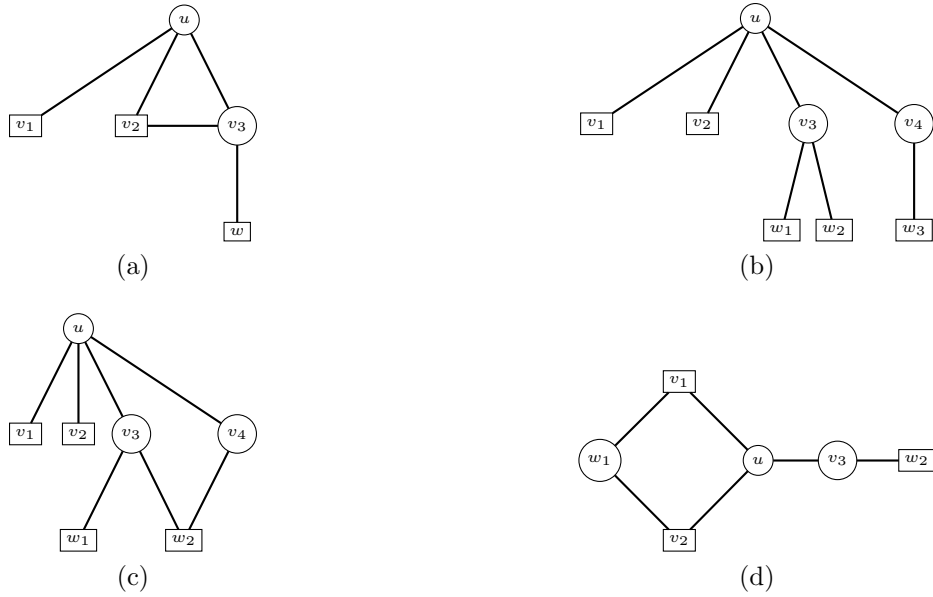

Figure 2: Vertices drawn as circles have degree 4 in G and vertices drawn as rectangles have degree 3 in G.

Lemma 3.1. *If G is a graph in $\mathcal{H}_4$ and G contains none of the four configurations in Figure 2 (not necessarily induced), then G is $K_5 - e$.*

Proof. Let G be a graph in $\mathcal{H}_4$. Note that every 4-vertex in G has exactly two 3-neighbors and two 4-neighbors. Let u denote a 4-vertex and let $v_1, \ldots, v_4$ denote its neighbors, where $d(v_1) = d(v_2) = 3$ and $d(v_3) = d(v_4) = 4$. When vertices x and y are adjacent, we write $x \leftrightarrow y$. We assume that G contains none of the configurations in Figure 2 and show that G must be $K_5 - e$.

First suppose that u has a 3-neighbor and a 4-neighbor that are adjacent. By symmetry, suppose that v_2 is adjacent to v_3. Since Figure 2(a) is forbidden, we have $v_3 \leftrightarrow v_1$. Now consider v_4. If v_4 has a 3-neighbor distinct from v_1 and v_2, then we have a copy of Figure 2(d). Hence $v_4 \leftrightarrow v_1$ and $v_4 \leftrightarrow v_2$. If $v_3 \leftrightarrow v_4$, then G is $K_5 - e$. Suppose not, and let x be a 4-neighbor of v_4. Since G has no copy of Figure 2(d), x must be adjacent to v_1 and v_2. This

is a contradiction, since now v_1 and v_2 are 3-vertices, but each have at least four neighbors. Hence, we conclude that each of v_1 and v_2 is non-adjacent to each of v_3 and v_4.

Now consider the 3-neighbors of v_3 and v_4. If they have zero 3-neighbors in common, then we have a copy of Figure 2(b). If they have one 3-neighbor in common, then we have a copy of Figure 2(c). Otherwise they have two 3-neighbors in common, so we have a copy of Figure 2(d). □
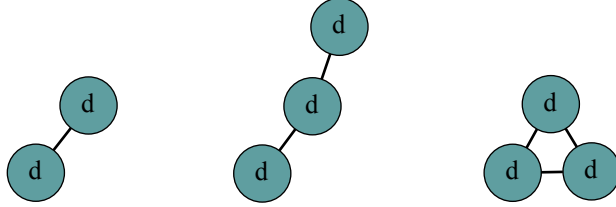
4   Superabundance sufficiency and adjacency lemmas

Figure 3: The fixable graphs on at most 3 vertices.

Definition 4. If G is a graph and $f\colon V(G) \to \mathbb{N}$ with $f(v) \geq d_G(v)$ for all $v \in V(G)$, then G is f-fixable if G is (L, P)-fixable for every L with $|L(v)| \geq f(v)$ for all $v \in V(G)$ and every L-pot P such that (G, L) is superabundant.

Since $f(v) \geq d_G(v)$, it is convenient to express the values of f as $d + k$ for a non-negative integer k; this means $f(v) = d_G(v) + k$. When $k = 0$, we just write d as in Figure 3.

Looking at the trees in Figures 4, 5 and 6 we might conjecture that a tree is f-fixable as long as there is at most one internal vertex labeled "d". This conjecture continues to hold for many more examples.

Conjecture 4.1. *A tree T is f-fixable if $f(v) = d_T(v)$ for at most one non-leaf v of T.*

Note that by Lemma 2.8, this would imply that Tashkinov trees are elementary under the same degree constraints. Can this be proved in the simpler case of Kierstead paths? For paths of length 4, this was done by Kostochka and Stiebitz, in the next section we present a generalization of this result to stars with one edge subdivided. One nice feature of the superabundance formulation is that since there is no need for an ordering like with Tashkinov trees, we can easily formulate results about graphs with cycles. The most general thing we might think is true is the following.

Conjecture 4.2 (false). *A multigraph G is f-fixable if $f(v) > d_G(v)$ for all $v \in V(T)$.*

This is very strong and implies Goldberg's conjecture. Unfortunately, it is false, we can make a counterexample on a 5-cycle. [SHOW HOW. No change like $f(v) > d_G(v) + 100$ will help.]
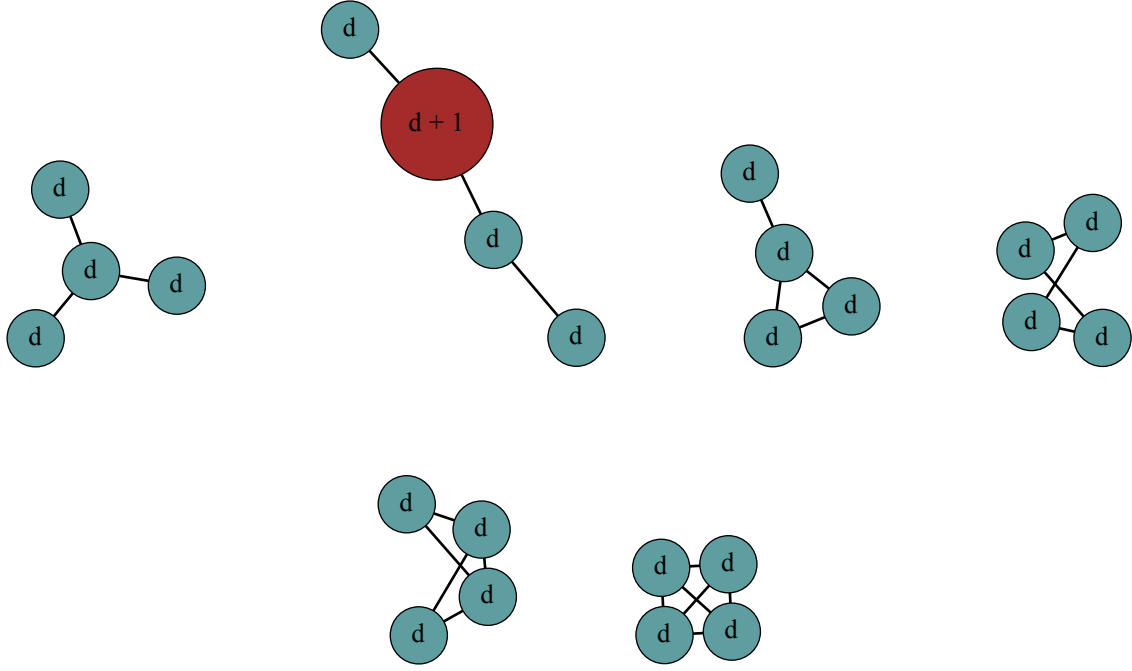
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Figure 4: The fixable graphs on 4 vertices.

4.1 Stars with one edge subdivided

The following generalizes the "Short Kierstead Paths" of Kostochka and Stiebitz. Parts (a) and (b) are a special case of Conjecture 4.1. At present, we feel the proof we have is too involved and ugly to present, but it will be instructive to see how we can extract a fan-equation-like formula from Theorem 4.3.

Theorem 4.3. *Let G be a star with one edge subdivided where r is the center of the star, t the vertex at distance two from r and s the intervening vertex. Then G is L-fixable when L is superabundant, $|L(v)| \geq d_G(v)$ for all $v \in V(G)$ and at least one of the following holds:*

(a) $|L(r)| > d_G(r)$; or

(b) $|L(s)| > d_G(s)$; or

(c) $\psi_L(G) > \|G\|$.

Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q) + 1$ and $G \subseteq Q$. For a $\Delta(Q)$-edge-coloring π of $Q - E(G)$, put $L_\pi(v) = [\Delta(Q)] - \pi(E_Q(v) - E(G))$ for all $v \in V(G)$. We say that G is a Ψ-*subgraph* of Q if there is a $\Delta(Q)$-edge-coloring π of $Q - E(G)$ such that each $H \subsetneq G$ is abundant. Put $E_L(H) = |\{\alpha \in \mathrm{pot}(L) \mid |H_{L,\alpha}|$ is even$\}|$ and $O_L(H) = |\{\alpha \in \mathrm{pot}(L) \mid |H_{L,\alpha}|$ is odd$\}|$. Plainly, $\mathrm{pot}(L) = E_L(G) + O_L(G)$.
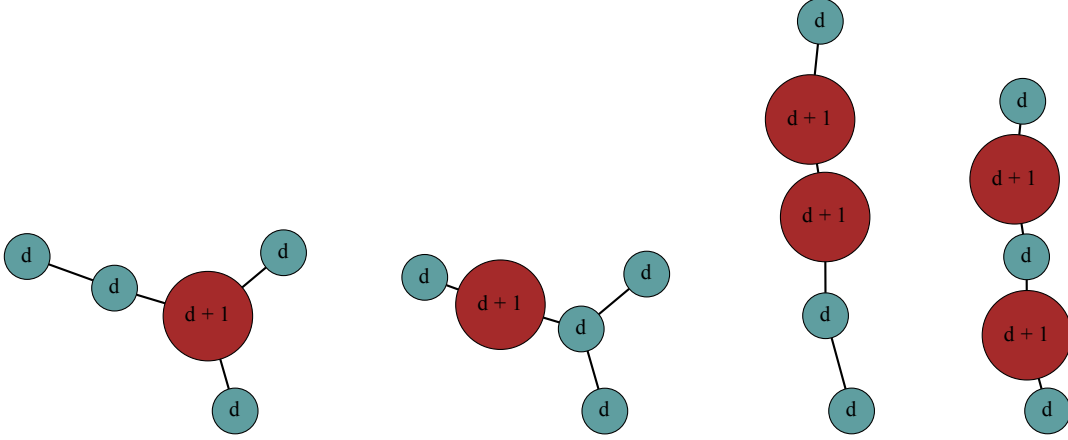
Figure 5: The fixable trees with maximum degree at most 3 on 5 vertices.

Lemma 4.4. *Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q) + 1$. If $G \subseteq Q$ and π is a $\Delta(Q)$-edge-coloring of $Q - E(G)$ such that $\psi_L(G) \leq \|G\|$, then $|O_{L_\pi}(G)| \geq \sum_{v \in V(G)} \Delta(Q) - d_Q(v)$. Furthermore, if $\psi_L(G) < \|G\|$, then $|O_{L_\pi}(G)| > \sum_{v \in V(G)} \Delta(Q) - d_Q(v)$.*

Proof. The proof is a straightforward counting argument. For fixed degrees and list sizes, as $|O_L(G)|$ gets larger, $\psi_L(G)$ gets smaller (half as quickly). The details forthwith. Put $L = L_\pi$.

Since $\|G\| \geq \psi_L(G)$, we have

$$\|G\| \geq \sum_{\alpha \in \text{pot}(L)} \left\lfloor \frac{|G_{L,\alpha}|}{2} \right\rfloor = \sum_{\alpha \in \text{pot}(L)} \frac{|G_{L,\alpha}|}{2} - \sum_{\alpha \in O_L(H)} \frac{1}{2}. \tag{1}$$

Also,

$$\sum_{\alpha \in \text{pot}(L)} \frac{|G_{L,\alpha}|}{2} = \sum_{v \in V(G)} \frac{\Delta(Q) - (d_Q(v) - d_G(v))}{2}$$

$$= \sum_{v \in V(G)} \frac{d_G(v)}{2} + \sum_{v \in V(G)} \frac{\Delta(Q) - d_Q(v)}{2}$$

$$= \|G\| + \sum_{v \in V(G)} \frac{\Delta(Q) - d_Q(v)}{2}. \tag{2}$$

Now we solve for $\|G\| - \sum_{\alpha \in \text{pot}(L)} \frac{|G_{L,\alpha}|}{2}$ in (1) and (2), set the expressions equal, and then simplify. The result is (3).

$$|O_L(G)| \geq \sum_{v \in V(G)} \Delta(Q) - d_Q(v). \tag{3}$$

Finally, if the inequality in (1) is strict, then the inequality in (3) is also strict. □
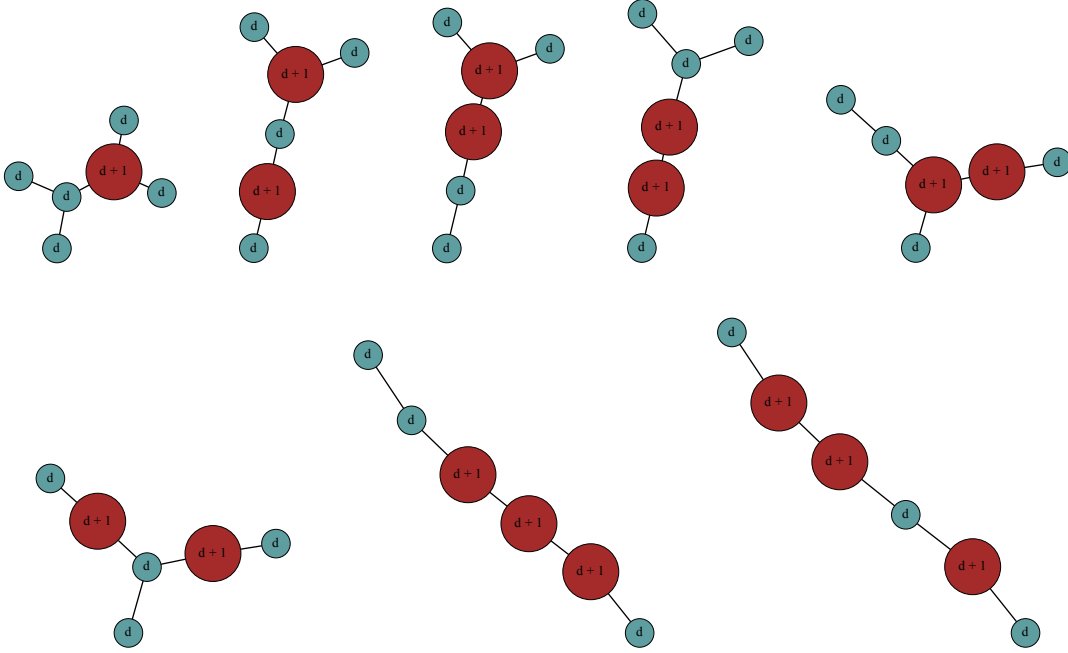
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Figure 6: The fixable trees with maximum degree at most 3 on 6 vertices.

Lemma 4.5. *Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q) + 1$. Suppose H is a Ψ-subgraph of Q where H is a star with one edge subdivided. Let r be the center of the star, t the vertex at distance two from r and s the intervening vertex. Then there is $X \subseteq N(r)$ with $V(H - r - t) \subseteq X$ such that*

$$\sum_{v \in X \cup \{t\}} (d_Q(v) + 1 - \Delta(Q)) \geq 0.$$

Moreover, if $\{r, s, t\}$ does not induce a triangle in Q, then

$$\sum_{v \in X \cup \{t\}} (d_Q(v) + 1 - \Delta(Q)) \geq 1.$$

Furthermore, if $d_Q(r) < \Delta(Q)$ or $d_Q(s) < \Delta(Q)$, then we can improve both lower bounds by 1.

Proof. Let G be a maximal Ψ-subgraph of Q containing H such that G is a star with one edge subdivided. Let π be a coloring of $Q - E(G)$ showing that G is a Ψ-subgraph and put $L = L_\pi$.

We first show that $|E_L(G)| \geq d_Q(r) - d_G(r) - 1$ if rst induces a triangle; otherwise, $|E_L(G)| \geq d_Q(r) - d_G(r)$. Suppose rst does not induce a triangle; for arbitrary $x \in N_Q(r) - V(G)$, let $\alpha = \pi(rx)$. If $\alpha \in O_L(G)$, then adding x to G gives a larger Ψ-subgraph of the required form; this contradicts the maximality of G. Hence $\alpha \in E_L(G)$. Therefore, $|E_L(G)| \geq d_Q(r) - d_G(r)$ as desired. If rst induces a triangle, then we lose one off this bound from the rt edge.

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By Theorem 4.3(c), we have $\psi_L(G) \leq \|G\|$. By Lemma 4.4, we have $|O_L(G)| \geq \sum_{v \in V(G)} \Delta(Q) - d_Q(v)$. Suppose rst does not induce a triangle. Then

$$
\begin{aligned}
\Delta(Q) &\geq \mathrm{pot}(L) \\
&= |E_L(G)| + |O_L(G)| \\
&\geq d_Q(r) - d_G(r) + \sum_{v \in V(G)} \Delta(Q) - d_Q(v) \\
&= \Delta(Q) - d_G(r) + \sum_{v \in V(G-r)} \Delta(Q) - d_Q(v) \\
&= \Delta(Q) + 1 \sum_{v \in V(G-r)} \Delta(Q) - 1 - d_Q(v).
\end{aligned}
\tag{4}
$$

Therefore, $\sum_{v \in V(G-r)} \Delta(Q) - 1 - d_Q(v) \leq -1$. Negating gives the desired inequality. If rst induces a triangle, we lose one off the bound. Theorem 4.3(a,b) gives the final statement. $\quad\square$

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