

A common generalization of Hall's theorem and Vizing's edge-coloring theorem

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LBD Data

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November 6, 2014

Hall's theorem

- given finite sets A_1, A_2, \dots, A_n

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Hall's theorem

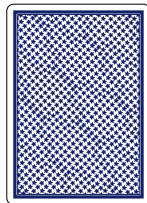
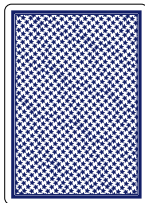
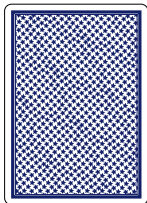
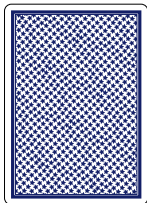
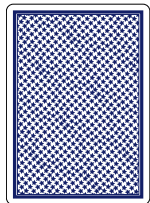
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- when can we pick an SDR?
- if k of the sets together have fewer than k elements, we can't
 - $A_1 = \{1, 2\}, A_2 = \{1, 2\}, A_3 = \{1, 2\}$
- **Hall's theorem: this is the only thing that can go wrong**

$$\text{SDR exists} \Leftrightarrow \left| \bigcup_{i \in I} A_i \right| \geq |I| \text{ for all } I \subseteq \{1, \dots, n\}$$

some card games

the simplest variation

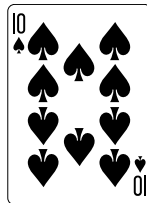
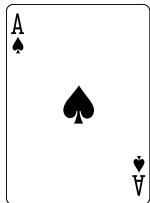
- Dealer vs. Player



some card games

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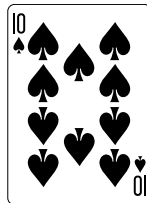
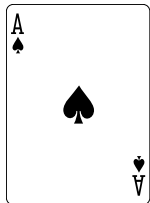
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- the deck has just many copies of the high spade cards



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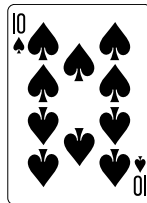
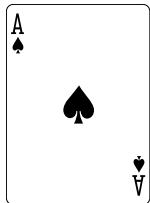
- Dealer vs. Player
- the deck has just many copies of the high spade cards
- Dealer makes 5 stacks of cards with no duplicates, all cards face-up



some card games

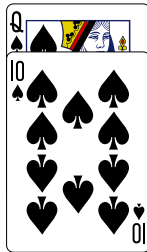
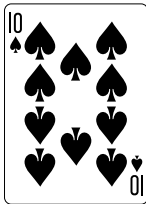
the simplest variation

- Dealer vs. Player
- the deck has just many copies of the high spade cards
- Dealer makes 5 stacks of cards with no duplicates, all cards face-up
- Player wins if he can pick a Royal Flush, one card from each stack



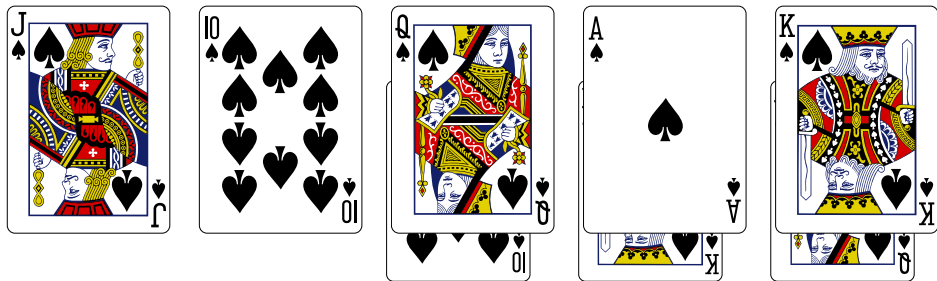
some card games

example, a Player win



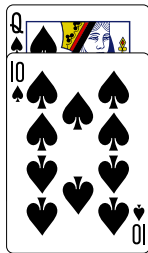
some card games

example, a Player win



some card games

example, a Dealer win



some card games

winning condition

- Player cannot win if there is a set of k stacks that together have fewer than k different cards

some card games

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some card games

winning condition

- Player cannot win if there is a set of k stacks that together have fewer than k different cards
- Hall's theorem says: **Player wins otherwise**



some card games

making things harder for Dealer

- this isn't a fun game, far too easy for Dealer to win

some card games

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Player's Move

Player can pick any card A from the deck and swap it for another card B in one stack (not containing A).

some card games

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Dealer can (i) do nothing or (ii) swap A and B in one other stack.

some card games

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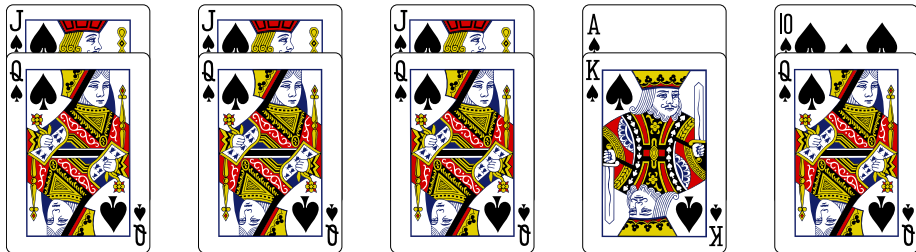
Dealer can (i) do nothing or (ii) swap A and B in one other stack.

Winning

Player wins if he can pick a Royal Flush at the start of one of his turns, otherwise Dealer wins.

some card games

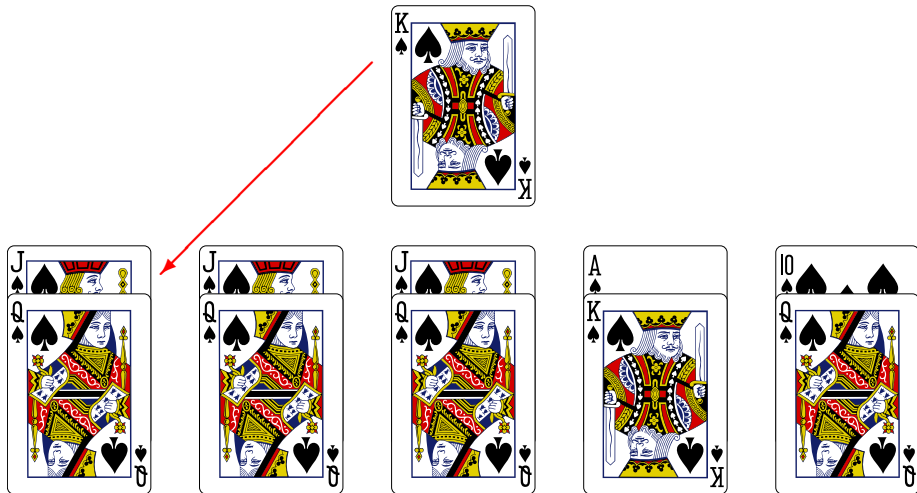
example, a Player win



some card games

example, a Player win

- Player picks a King from the deck and swaps it for a Queen in the first stack



some card games

example, a Player win

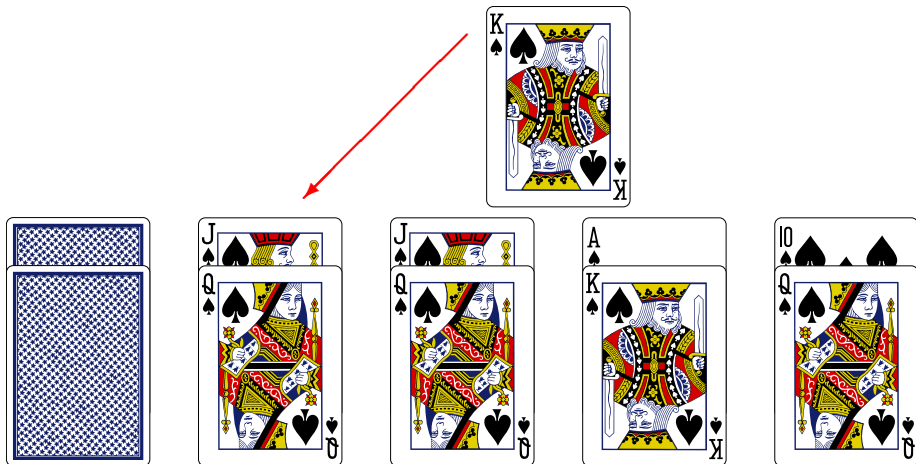
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some card games

example, a Player win

- Player picks a King from the deck and swaps it for a Queen in the first stack
- Dealer can swap a King and Queen in one of the other stacks



some card games

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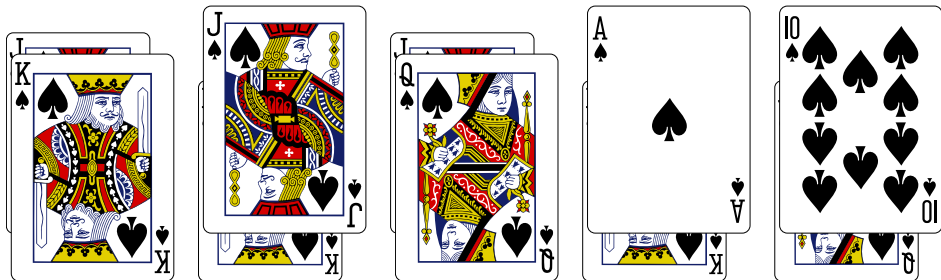
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some card games

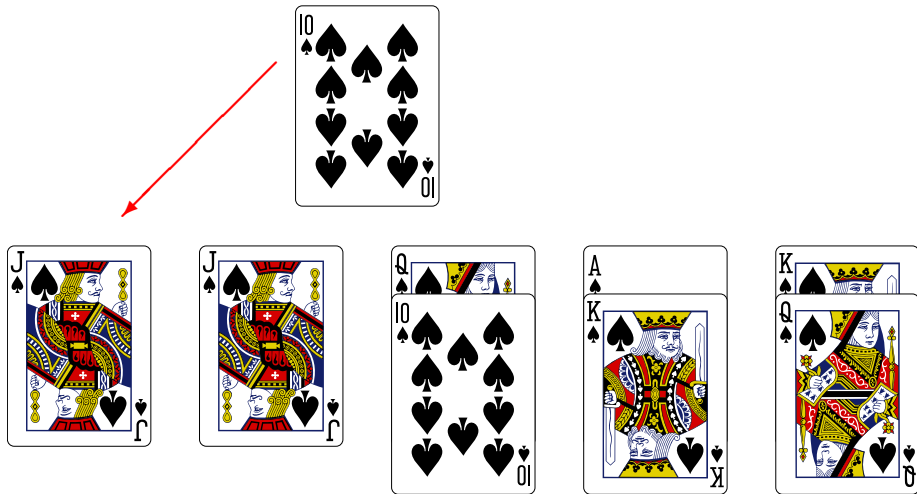
example, a Player win

- Player picks a King from the deck and swaps it for a Queen in the first stack
- Dealer can swap a King and Queen in one of the other stacks
- Player wins no matter what Dealer does



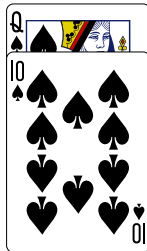
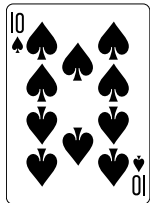
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example, a Dealer win



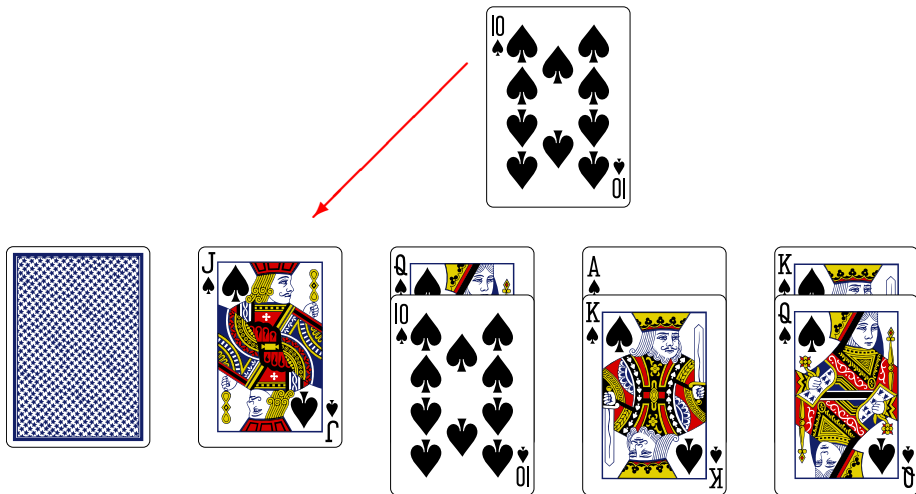
some card games

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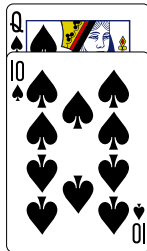
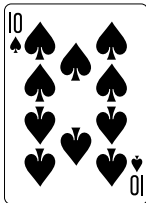
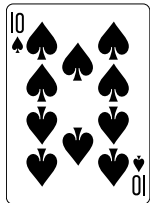
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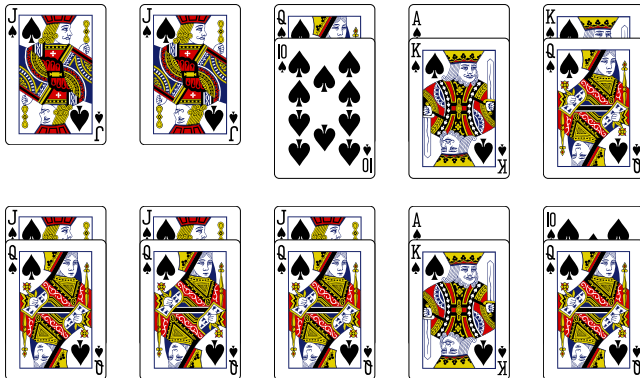
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example, a Dealer win



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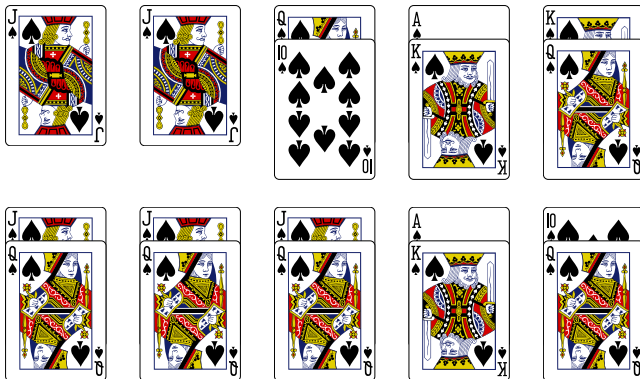
what was the difference?



some card games

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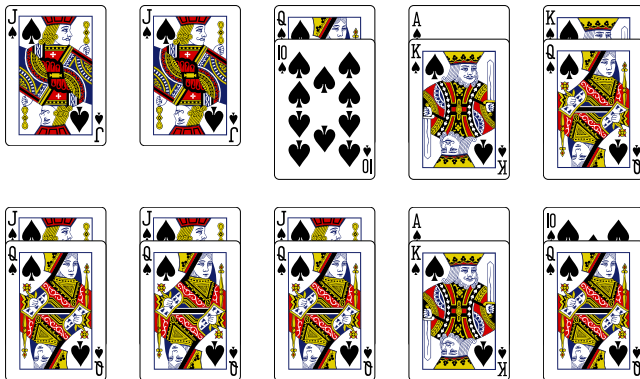
- in the top game, Dealer can prevent Player from increasing the number of different cards in the first two stacks



some card games

what was the difference?

- in the top game, Dealer can prevent Player from increasing the number of different cards in the first two stacks
- in the bottom game, Dealer cannot prevent Player from increasing the number of different cards in the first three stacks



some card games

necessary condition

- if the same card appears on three stacks, Player can force the addition of a new card to these stacks

some card games

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- it is not hard to show that this is essentially all Player can do

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Degree

The *degree* of a card C in a set of stacks S is the number of times C appears in S . We write $d_S(C)$ for this quantity.

some card games

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The *degree* of a card C in a set of stacks S is the number of times C appears in S . We write $d_S(C)$ for this quantity.

Necessary Condition

If Player can win, then for every set of stacks S we must have

$$\sum_{C \in \cup S} \left\lceil \frac{d_S(C)}{2} \right\rceil \geq |S|.$$

some card games

intuition

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some card games

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- in Hall's theorem, each C is 'worth' 1 in $\sum_{C \in \bigcup S} 1 = \left| \bigcup S \right| \geq |S|$
- Player can turn $2t + 1$ of the same card into $t + 1$ different cards, so C is 'worth' $\left\lceil \frac{d_S(C)}{2} \right\rceil$

some card games

Dealer's strategy

- given a set of stacks S with $\sum_{C \in \cup S} \left\lceil \frac{d_S(C)}{2} \right\rceil < |S|$

some card games

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some card games

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some card games

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some card games

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 - so, Dealer can swap A for B somewhere else, decreasing $\left\lceil \frac{d_S(B)}{2} \right\rceil$ and not increasing $\left\lceil \frac{d_S(A)}{2} \right\rceil$

some card games

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 - so, Dealer can swap A for B somewhere else, decreasing $\left\lceil \frac{d_S(B)}{2} \right\rceil$ and not increasing $\left\lceil \frac{d_S(A)}{2} \right\rceil$
 - Dealer has maintained $\sum_{C \in \cup S} \left\lceil \frac{d_S(C)}{2} \right\rceil < |S|$

some card games

winning condition

- **this necessary condition is also sufficient**

some card games

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Winning Condition

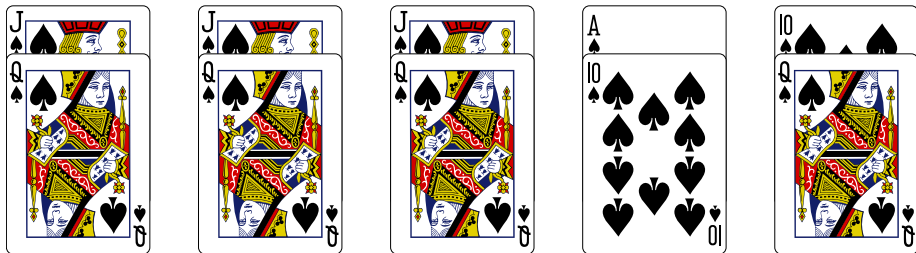
Player can win if and only if for every set of stacks S we have

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some card games

proof idea

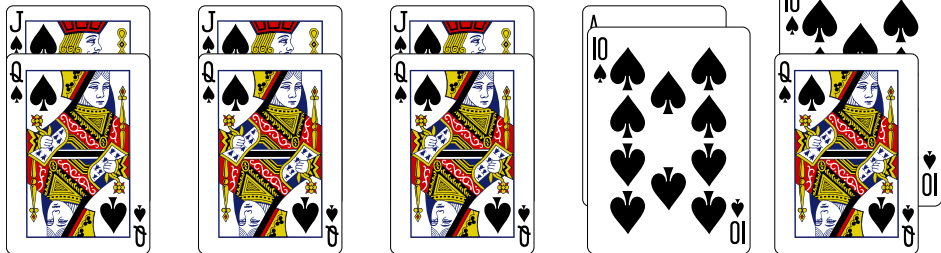
- 1 Player looks for a set of card types that give a system of distinct representatives of all the stacks containing them



some card games

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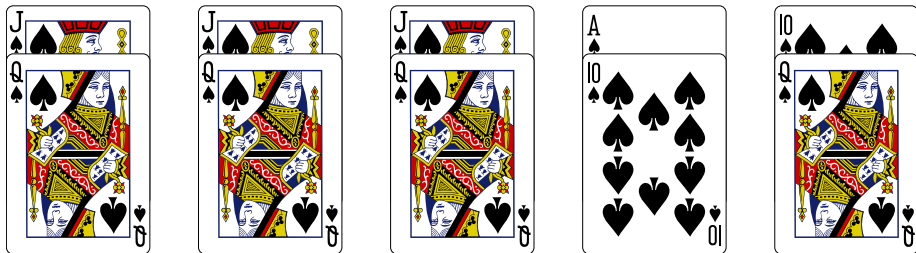
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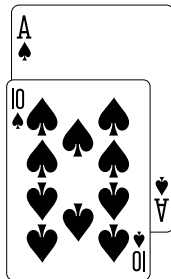
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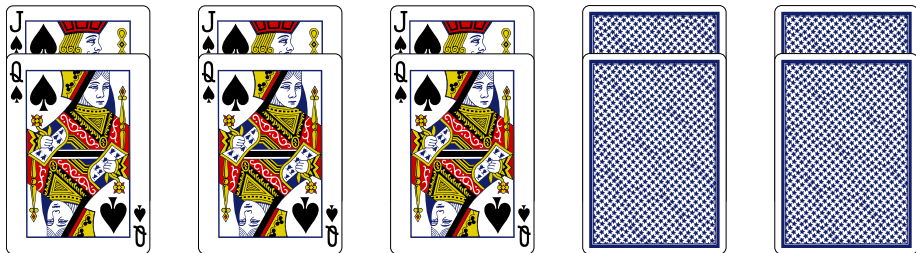
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some card games

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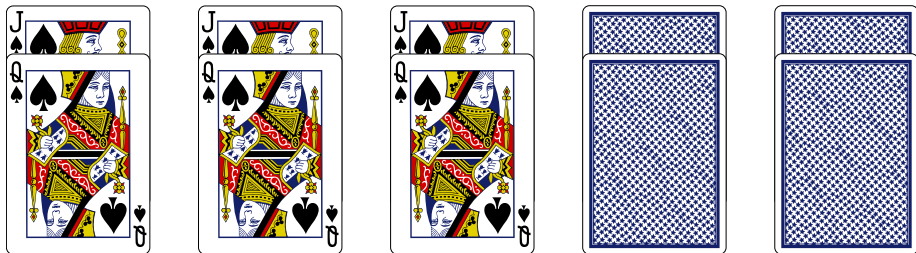
- 1 Player looks for a set of card types that give a system of distinct representatives of all the stacks containing them
- 2 Player calls those stacks done and never plays with those card types again



some card games

proof idea

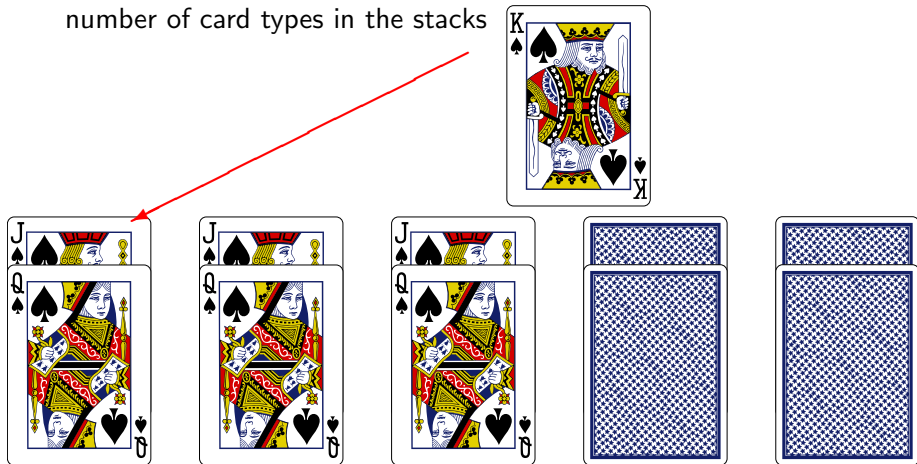
- ③ if no such set of card types exists, then Hall's theorem shows that there is at least one card appearing on none of the remaining stacks



some card games

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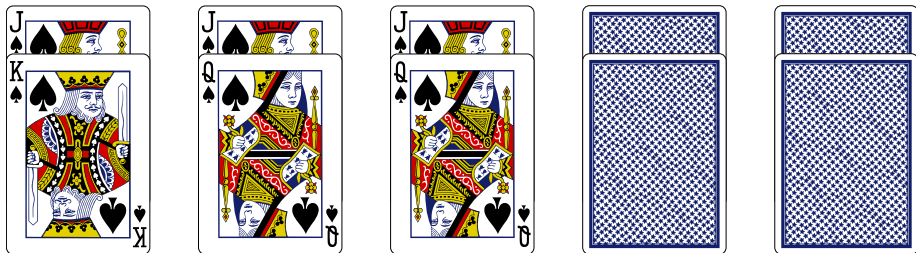
- 3 if no such set of card types exists, then Hall's theorem shows that there is at least one card appearing on none of the remaining stacks
- 4 but then some card appears at least thrice, so Player can increase the number of card types in the stacks



some card games

proof idea

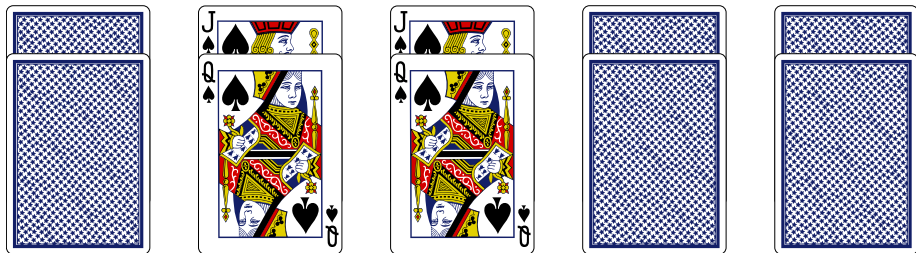
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some card games

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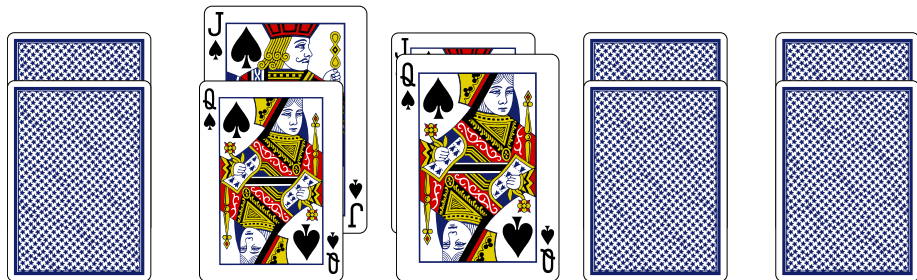
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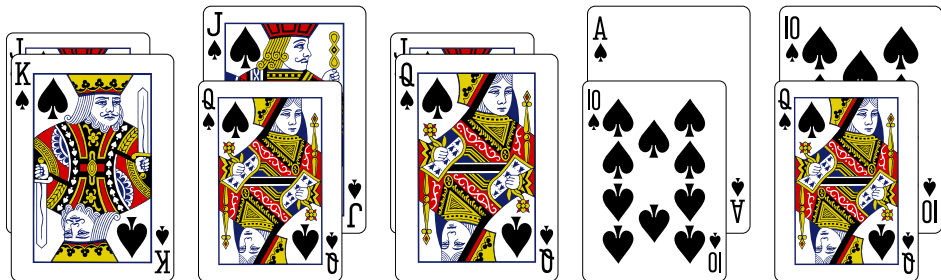
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some card games

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A generalization of Hall's theorem

making it harder for Player

- allow Dealer to make more swaps in response to Player's move

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- for each $t \geq 1$, the t -game allows Dealer to make up to t swaps

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- for each $t \geq 1$, the t -game allows Dealer to make up to t swaps

Winning Condition

Player can win in the t -game if and only if for every set of stacks S we have

$$\sum_{C \in \cup S} \left\lceil \frac{d_S(C)}{t+1} \right\rceil \geq |S|.$$

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- Hall's theorem is the winning condition in the $(k-1)$ -game when there are k total stacks:
 - $1 \leq d_S(C) \leq k$, so $\left\lceil \frac{d_S(C)}{t+1} \right\rceil = 1$

A generalization of Hall's theorem

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$$\sum_{C \in \bigcup S} \left\lceil \frac{d_S(C)}{t+1} \right\rceil \geq |S|.$$

- Hall's theorem is the winning condition in the $(k-1)$ -game when there are k total stacks:
 - $1 \leq d_S(C) \leq k$, so $\left\lceil \frac{d_S(C)}{t+1} \right\rceil = 1$
 - so, the sum equals $|\bigcup S|$

A generalization of Hall's theorem

making it harder for Player

- allow Dealer to make more swaps in response to Player's move
- for each $t \geq 1$, the t -game allows Dealer to make up to t swaps

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 - Player's moves are useless

edge coloring

setup

- assign colors to the edges of a graph so that incident edges get different colors

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- how few colors can we use?

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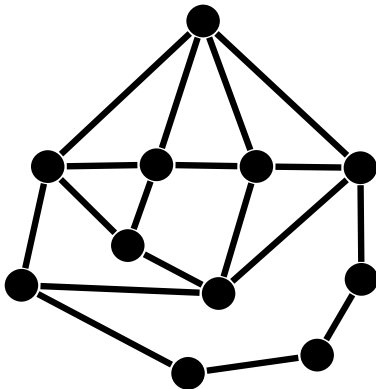
Vizing's theorem

Any simple graph can be edge-colored using at most one more color than its maximum degree.

edge coloring

proof of Vizing's theorem

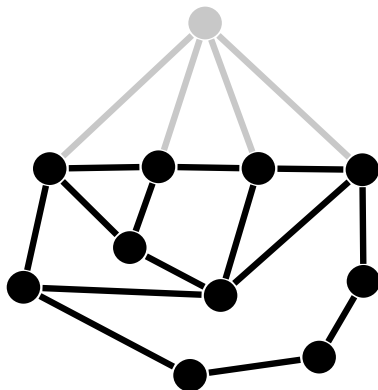
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edge coloring

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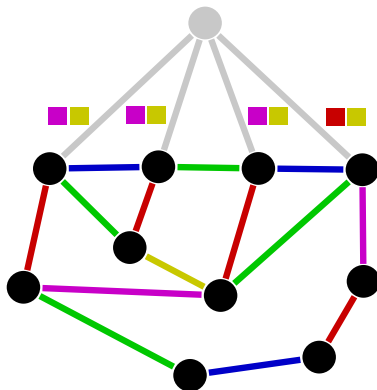
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edge coloring

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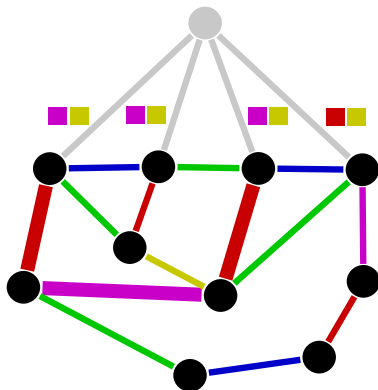
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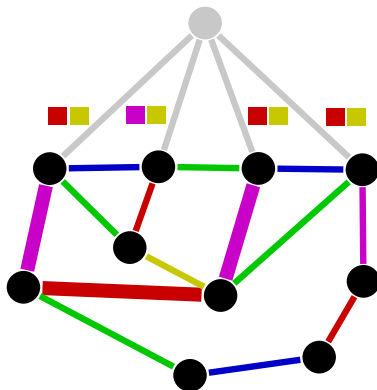
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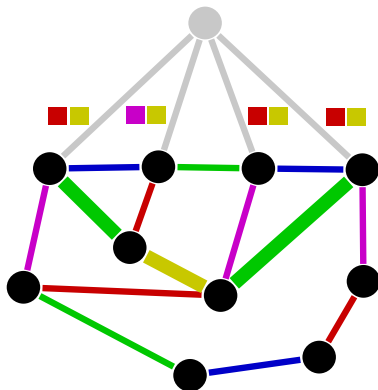
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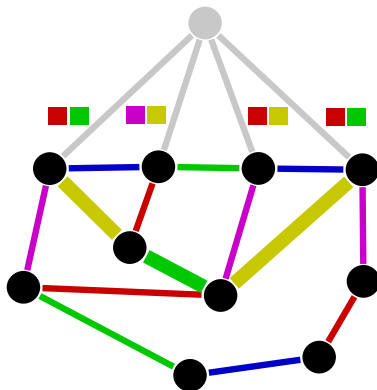
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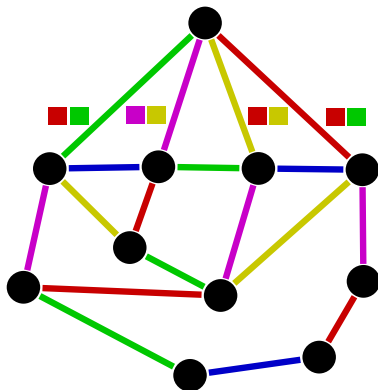
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proof of Vizing's theorem

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- so, we have the desired winning condition

$$\sum_{C \in \cup S} \frac{d_S(C)}{2} \geq |S|$$

summary

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 - a more general game unifies much of edge-coloring theory