graph theory notes*

Stiebitz's proof of Gallai's conjecture on the number of components in the high and low vertex subgraphs of critical graphs

Tibor Gallai conjectured the following in 1963 [1, 2] and Michael Stiebitz proved it in 1982 [3]. For a graph G, let $\mathcal{L}(G)$ be the subgraph of G induced on the vertices of degree $\delta(G)$ and let $\mathcal{H}(G)$ be the subgraph of G induced on the vertices of degree larger than $\delta(G)$.

Theorem (Stiebitz)**.** *If G is a color-critical graph with $\delta(G) = \chi(G) - 1$, then $\mathcal{H}(G)$ has at most as many components as $\mathcal{L}(G)$.*

In fact, Stiebitz proved a stronger statement. Theorem follows immediately from Lemma using $X = V(\mathcal{L}(G))$.

Lemma. *Let G be a connected graph and $\emptyset \neq X \subseteq V(G)$ such that*

- *$d_G(x) \leq k - 1$ for all $x \in X$; and*
- *for each component C of $G - X$, we have $\chi(G - V(C)) \leq k - 1$; and*
- *$G[X]$ has ℓ components and $G - X$ has at least $\ell + 1$ components.*

If $G - X$ is the disjoint union of (possibly not connected) graphs $M_1, \ldots, M_{\ell+1}$ and f_i is a $(k-1)$-coloring of M_i for each $i \in [\ell + 1]$, then there are permutations $\pi_1, \ldots, \pi_{\ell+1}$ of $[k - 1]$ such that the $(k - 1)$-coloring of $G - X$ given by $(\pi_1 \circ f_1) \cup \cdots \cup (\pi_{\ell+1} \circ f_{\ell+1})$ extends to a $(k - 1)$-coloring of G.

Proof. Suppose the lemma is false and choose a counterexample G and nonempty $X \subseteq V(G)$ so that $|X|$ is as small as possible. So, $G - X$ is the disjoint union of graphs $M_1, \ldots, M_{\ell+1}$ and we have $(k - 1)$-colorings f_i of M_i for each $i \in [\ell + 1]$ so that no permutations allow us to extend to a $(k - 1)$-coloring of G.

Claim 1. *Each component of $G[X]$ has edges to at least two of the M_i.* Suppose to the contrary that we have a component C of $G[X]$ that has edges to at most one of the M_i. Then, since G is connected, we must have $\ell \geq 2$. But now the hypotheses of the lemma are satisfied with $X' = X \setminus V(C)$ in place of X, so by minimality of $|X|$ we get permutations that allow us to extend to a $(k - 1)$-coloring of G, a contradiction.

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Claim 2. *Each non-separating vertex in $G[X]$ has neighbors in at least two of the M_i.* Suppose to the contrary that we have a component C of $G[X]$ and $x \in V(C)$ a non-separating vertex that has neighbors in at most one of the M_i. Then, by Claim 1, we must have $|C| \geq 2$. But then x has at most $k-2$ neighbors in $G-X$, so we can greedily complete any $(k-1)$-coloring of $G-X$ to $G-X'$ where $X' = X \setminus \{x\}$. So, the hypotheses of the lemma are satisfied with X' in place of X. Again, by minimality of $|X|$, we get permutations that allow us to extend to a $(k-1)$-coloring of G, a contradiction.

Claim 3. *If $G[X]$ has at least two components, then $G - V(C)$ is connected for every component C of $G-X$.* Suppose $G[X]$ has at least two components and let C be a component of $G-X$. If $G-V(C)$ is disconnected, then with $X' = V(C)$ in place of X, the hypotheses of the lemma are satisfied. By minimality of $|X|$, we get permutations that allow us to extend to a $(k-1)$-coloring of G, a contradiction.

Claim 4. *The lemma is true.* Pick a component C in $G[X]$ and a non-separating vertex $x \in V(C)$. By Claim 2 and symmetry, we may assume that x has neighbors y_1, y_2 in M_1, M_2 respectively. Let $G' = G - V(C)$ and $X' = X \setminus V(C)$. Then G' is the disjoint union of the ℓ graphs $M_1 \cup M_2, M_3, \ldots, M_{\ell+1}$. Let τ be a permutation of $[k-1]$ such that $(\tau \circ f_2)(y_2) = f_1(y_1)$ and let $f_* = f_1 \cup (\tau \circ f_2)$.

Suppose G' is connected. By minimality of $|X|$, we can apply the lemma to G' with $M_1 \cup M_2, M_3, \ldots, M_{\ell+1}$ and colorings $f_*, f_3, \ldots, f_{\ell+1}$ to get permutations $\pi_*, \pi_3, \ldots, \pi_{\ell+1}$ such that the $(k-1)$-coloring of $G' - X'$ given by $(\pi_* \circ f_*) \cup (\pi_3 \circ f_3) \cup \cdots \cup (\pi_{\ell+1} \circ f_{\ell+1})$ extends to a $(k-1)$-coloring of G'. But this is the same as the $(k-1)$-coloring $(\pi_* \circ f_1) \cup (\pi_* \circ \tau \circ f_2) \cup (\pi_3 \circ f_3) \cup \cdots \cup (\pi_{\ell+1} \circ f_{\ell+1})$, so using the permutations $\pi_*, \pi_* \circ \tau, \pi_3, \ldots, \pi_{\ell+1}$ we get a coloring of $G-X$ that extends to $G - V(C)$.

If G' is not connected, then $X = V(C)$ by Claim 3. So, $\ell = 1$ and f_* is a $(k-1)$-coloring of $G - V(C)$.

In these colorings, y_1 and y_2 receive the same color. This means that x has $k - 1 - (d_G(x) - d_C(x)) + 1 \geq d_C(x) + 1$ colors available and each other vertex v in C has $k - 1 - (d_G(v) - d_C(v)) + 1 \geq d_C(v) \geq d_C(v)$ colors available. So, coloring C greedily in order of decreasing distance from x gives an extension to a $(k-1)$-coloring of G, a contradiction. $\square$

References

[1] T. Gallai, *Kritische graphen I.*, Math. Inst. Hungar. Acad. Sci **8** (1963), 165–192 (in German).

[2] ______, *Kritische graphen II.*, Math. Inst. Hungar. Acad. Sci **8** (1963), 373–395 (in German).

[3] M. Stiebitz, *Proof of a conjecture of T. Gallai concerning connectivity properties of colour-critical graphs*, Combinatorica **2** (1982), no. 3, 315–323.