

# 1 Definitions

A *2-partition* of a set  $S$  is a partition of  $S$  into sets of size two and at most one set of size one.

## 2 Fixable graphs

For different colors  $a, b \in P$ , let  $S_{L,a,b}$  be all the vertices of  $G$  that have exactly one of  $a$  or  $b$  in their list; more precisely,  $S_{L,a,b} = \{v \in V(G) \mid |\{a, b\} \cap L(v)| = 1\}$ . If  $\mathcal{X}$  is a 2-partition of  $S_{L,a,b}$  and  $J \subseteq \mathcal{X}$ , let  $L_J$  be the list assignment formed from  $L$  by swapping  $a$  and  $b$  in  $L(v)$  for every  $v \in \bigcup J$ . If  $J = \{X\}$ , we also write  $L_X$  for  $L_J$ .

**Definition 1.**  $G$  is  $(L, P)$ -fixable if either

- (1)  $G$  has an  $L$ -edge-coloring; or
- (2) there are different colors  $a, b \in P$  such that for every 2-partition  $\mathcal{X}$  of  $S_{L,a,b}$  there exists  $J \subseteq \mathcal{X}$  so that  $G$  is  $(L_J, P)$ -fixable.

The meaning of (1) is clear. Intuitively, (2) says the following. There is some pair of colors,  $a$  and  $b$ , such that regardless of how the vertices of  $S_{L,a,b}$  are paired via Kempe chains for colors  $a$  and  $b$  (or not paired with any vertex of  $S_{L,a,b}$ ), we can swap the colors on some subset  $J$  of the Kempe chains so that the resulting partial edge-coloring is fixable.

We write  $L$ -fixable as shorthand for  $(L, \text{pot}(L))$ -fixable. When  $G$  is  $(L, P)$ -fixable, the choices of  $a, b$ , and  $J$  in each application of (2) determine a tree where all leaves have lists satisfying (1). The *height* of  $(L, P)$  is the minimum possible height of such a tree. We write  $h_G(L, P)$  for this height and let  $h_G(L, P) = \infty$  when  $G$  is not  $(L, P)$ -fixable.

**Lemma 2.1.** *If a multigraph  $M$  has a partial  $k$ -edge-coloring  $\pi$  such that  $M_\pi$  is  $(L_\pi, [k])$ -fixable, then  $M$  is  $k$ -edge-colorable.*

### 2.1 A necessary condition

Since the edges incident to a vertex  $v$  must all get different colors, if  $G$  is  $(L, P)$ -fixable, then  $|L(v)| \geq d_G(v)$  for all  $v \in V(G)$ .

By considering the maximum size of matchings in each color, we get a more interesting necessary condition. For each  $C \subseteq \text{pot}(L)$  and  $H \subseteq G$ , let  $H_{L,C}$  be the subgraph of  $H$  induced by the vertices  $v$  with  $L(v) \cap C \neq \emptyset$ . When  $L$  is clear from context, we write  $H_C$  for  $H_{L,C}$ . If  $C = \{\alpha\}$ , we write  $H_\alpha$  for  $H_C$ . For  $H \subseteq G$ , let

$$\psi_L(H) = \sum_{\alpha \in \text{pot}(L)} \left\lfloor \frac{|H_{L,\alpha}|}{2} \right\rfloor.$$

Each term in the sum gives an upper bound on the size of a matching in color  $\alpha$ . So  $\psi_L(H)$  is an upper bound on the number of edges in a partial  $L$ -edge-coloring of  $H$ . The pair  $(H, L)$  is *abundant* if  $\psi_L(H) \geq \|H\|$  and  $(G, L)$  is *superabundant* if for every  $H \subseteq G$ , the pair  $(H, L)$  is abundant.

**Lemma 2.2.** *If  $G$  is  $(L, P)$ -fixable, then  $(G, L)$  is superabundant.*

**Definition 2.**  $G$  is  $(L, P)$ -subfixable if either

- (1)  $G$  is  $(L, P)$ -fixable; or
- (2) there is  $xy \in E(G)$  and  $\tau \in L(x) \cap L(y)$  such that  $G - xy$  is  $L'$ -subfixable, where  $L'$  is formed from  $L$  by removing  $\tau$  from  $L(x)$  and  $L(y)$ .

Superabundance is a necessary condition for subfixability because coloring an edge cannot make a non-abundant subgraph abundant. The conjectures in the rest of this paper may be easier to prove with subfixable in place of fixable. That would really be just as good since it would give the exact same results for edge coloring.

This may be useful. For a multigraph  $H$ , let  $\nu(H)$  be the number of edges in a maximum matching of  $H$ . For a list assignment  $L$  on  $H$ , let

$$\eta_L(H) = \sum_{\alpha \in \text{pot}(L)} \nu(H_\alpha).$$

Note that always  $\psi_L(H) \geq \eta_L(H)$ .

**Lemma 2.3** (Marcotte and Seymour). *Let  $T$  be a multitree and  $L$  a list assignment on  $V(T)$ . If  $\eta_L(H) \geq \|H\|$  for all  $H \subseteq T$ , then  $T$  has an  $L$ -edge-coloring.*

### 3 Swappable pairs

Suppose  $(G, L)$  is superabundant. We say that  $a, b \in \text{pot}(L)$  are *swappable* if  $(G, L_X)$  is superabundant for every  $X \subseteq S_{L,a,b}$  with  $|X| \leq 2$ .

**Lemma 3.1.** *Suppose  $(G, L)$  is superabundant. Then  $a, b \in \text{pot}(L)$  are swappable if for every  $H \subseteq G$ , at least one of the following holds:*

1.  $\psi_L(H) > \|H\|$ ; or,
2.  $|H_{L,a}|$  is odd; or,
3.  $|H_{L,b}|$  is odd.

Moreover, if (2) or (3) holds for  $G$ , then  $\psi_{L_X}(G) = \psi_L(G)$  for every  $X \subseteq S_{L,a,b}$  with  $|X| \leq 2$ .

*Proof.* Suppose not and choose  $X \subseteq S_{L,a,b}$  with  $|X| \leq 2$  such that  $(G, L_X)$  is not superabundant. Then we have  $H \subseteq G$  such that  $(H, L_X)$  is not abundant. Note that  $|H_{L,a}|$  and  $|H_{L_X,a}|$  differ by at most 2, so their contributions to  $\psi_L(H)$  and  $\psi_{L_X}(H)$  differ by at most 1; the same is true for  $|H_{L,b}|$  and  $|H_{L_X,b}|$ . If  $\psi_L(H) > \|H\|$ , then  $\psi_{L_X}(H) \geq \psi_L(H) - 1 \geq \|H\|$ , a contradiction. So (2) or (3) holds. The only way that we can have  $\psi_{L_X}(H) < \psi_L(H)$  is if  $\left\lfloor \frac{|H_{L_X,a}|}{2} \right\rfloor + \left\lfloor \frac{|H_{L_X,b}|}{2} \right\rfloor < \left\lfloor \frac{|H_{L,a}|}{2} \right\rfloor + \left\lfloor \frac{|H_{L,b}|}{2} \right\rfloor$ . Since  $|H_{L,b}| + |H_{L,a}| = |H_{L_X,b}| + |H_{L_X,a}|$ , this requires that both  $|H_{L,b}|$  and  $|H_{L,a}|$  are even; since (2) or (3) holds, this is impossible.  $\square$

## 4 Stars with one edge subdivided

We say a graph  $G$  is a  $\text{LongStar}_{r,s,t}$  if  $G$  is a star with one edge subdivided, where  $r$  is the center of the star,  $t$  the vertex at distance two from  $r$ , and  $s$  the intervening vertex.

We want to prove the following conjecture.

**Conjecture 4.1.** *Suppose  $G$  is a  $\text{LongStar}_{r,s,t}$ . If  $(G, L)$  is superabundant and  $|L(v)| \geq d_G(v)$  for all  $v \in V(G)$ , then  $G$  is  $L$ -subfixable if at least one of the following holds:*

- (a)  $|L(r)| > d_G(r)$ ; or
- (b)  $|L(s)| > d_G(s)$ ; or
- (c)  $\psi_L(G) > \|G\|$  (how do we do induction here?)

For a list assignment  $L$  on a  $\text{LongStar}_{r,s,t}$  graph  $G$ , create a bipartite graph  $B_L(G)$  with parts  $X_L(G) = \{uw \in E(G-t) \mid L(u) \cap L(w) \neq \emptyset\}$  and  $Y_L(G) = \{\alpha \in \text{pot}(L) \mid \nu((G-t)_\alpha) = 1\}$ , where  $uw \in X_L(G)$  is adjacent to  $\alpha \in Y_L(G)$  if and only if  $\alpha \in L(u) \cap L(w)$ . Put  $F_L(G) = L(r) \setminus \bigcup_{v \in N(r)} L(v)$ .

We prove part (a).

For a  $\text{LongStar}_{r,s,t}$  graph  $G$ , let  $\mathcal{L}(G)$  be all list assignments  $L$  such that  $(G, L)$  is superabundant and  $|L(r)| > d_G(r)$  and  $|L(v)| \geq d_G(v)$  for all  $v \in V(G-r)$ . Suppose there is a  $\text{LongStar}_{r,s,t}$  graph  $G$  and  $L \in \mathcal{L}(G)$  such that  $G$  is not  $L$ -subfixable. Choose such a  $G$  and  $L$  to minimize  $|G|$  and subject to that to maximize  $\eta_L(G-t)$ . Since  $G$  is not  $L$ -subfixable, for each pair of colors  $a, b \in \text{pot}(L)$  there is a 2-partition  $\mathcal{X}_{a,b}$  of  $S_{L,a,b}$  such that  $G$  is not  $L_J$ -subfixable for every  $J \subseteq \mathcal{X}_{a,b}$ . Let  $F = F_L(G)$ ,  $B = B_L(G)$  and  $Y = Y_L(G)$ .

**Lemma 4.2.** *Every  $\delta \in L(s) \cap L(t)$  appears an even number of times in  $L$ .*

*Proof.* If not, then we can color  $st$  with  $\delta$  and remove  $\delta$  from  $L(s)$  to get a list assignment  $L'$  on  $G-t$  such that  $(G-t, L')$  is superabundant. Now  $G-t$  is a star and hence  $L'$ -fixable.  $\square$

**Lemma 4.3.** *For any  $\beta \in \text{pot}(L) \setminus F$  that is swappable with some  $\gamma \in F$ , we have  $|G_{L,\beta} - r - t| \leq 2$ . Moreover, if  $\beta \notin Y$ , then  $|G_{L,\beta} - r - t| \leq 1$ .*

*Proof.* First, suppose  $\beta \in Y$  is swappable with  $\gamma \in F$  and  $|G_{L,\beta} - r - t| \geq 3$ . Pick  $v \in V(G_\beta - r - t)$ . Let  $X \in \mathcal{X}_{\beta,\gamma}$  with  $v \in X$ . Note that  $L_X(r) = L(r)$  since  $\gamma, \beta \in L(r)$ . Since  $|X \cap V(G_\beta - r - t)| \leq 2$ , we have  $\eta_{L_X}(G-t) > \eta_L(G-t)$ , which contradicts the maximality of  $\eta_L(G-t)$ .

Now, suppose  $\beta \in \text{pot}(L) \setminus (Y \cup F)$  is swappable with  $\gamma \in F$  and  $|G_{L,\beta} - r - t| \geq 2$ . Let  $X \in \mathcal{X}_{\beta,\gamma}$  with  $r \in X$ . Since  $|X \cap V(G_\beta - r - t)| \leq 1$ , we again contradict the maximality of  $\eta_L(G-t)$ .  $\square$

**Lemma 4.4.** *If every  $\beta \in \text{pot}(L) \setminus F$  is swappable with some  $\gamma \in F$ , then  $\eta_L(G-t) \geq \psi_L(G-t) \geq \|G-t\|$ .*

*Proof.* Suppose every  $\beta \in \text{pot}(L) \setminus F$  is swappable with some  $\gamma \in F$ . Then, by Lemma 4.3, the colors in  $Y$  each contribute at most one to  $\psi_L(G-t)$  and the colors not in  $Y$  contribute nothing to  $\psi_L(G-t)$ . Hence  $\psi_L(G-t) \leq |Y| = \eta_L(G-t)$ .  $\square$

**Lemma 4.5.** *If  $|F \cap L(t)| \geq 2$ , then every  $\beta \in \text{pot}(L) \setminus F$  is swappable with every  $\gamma \in F \cap L(t)$ .*

*Proof.* Suppose  $|F \cap L(t)| \geq 2$ . Then  $|L(r) \cap L(t)| \geq 2$ . Fix  $\gamma \in F \cap L(t)$ . Let  $H \subseteq G$ . If  $|H_{L,\gamma}|$  is even, then  $r, t \in V(H)$  and hence  $\psi_L(H) \geq \|H - t\| + 2 \geq \|H\| + 1$ . Therefore  $\gamma$  is swappable with  $\beta$  by Lemma 3.1.  $\square$

**Lemma 4.6.** *If  $\gamma \in F \setminus L(t)$  and  $L(s) \cap L(t) \neq \{\delta\}$ , then  $\gamma$  and  $\delta$  are swappable.*

*Proof.* The only subgraph  $H$  with edges where  $|H_{L,\gamma}|$  is even is  $G[s, t]$ , so if  $\gamma$  is not swappable with  $\delta$ , then it must be  $H = G[s, t]$  that fails all conditions of Lemma 3.1. Hence we have  $L(s) \cap L(t) = \{\delta\}$ .  $\square$

**Lemma 4.7.** *Suppose there is  $\gamma \in F \setminus L(t)$  and  $\delta \in L(t) \setminus L(s)$  such that  $|G_{L,\delta} - t|$  is odd. Then there is a list assignment  $L'$  and a color  $\delta'$  such that*

- $(G, L')$  is superabundant; and
- $\eta_{L'}(G - t) = \eta_L(G - t)$ ; and
- $\delta' \in F_{L'}(G) \cap L'(t)$ .

*Proof.* If  $\delta \in F$  then  $L$  works for  $L'$ , so we may assume  $\delta \notin F$ . By Lemma 4.6,  $\gamma$  and  $\delta$  are swappable.

First, suppose  $\delta \in Y$ . Then, by Lemma 4.3 and since  $|G_\delta - t|$  is odd, we have  $\delta \in L(u) \cap L(w)$  for exactly two  $u, w \in N(r) - s$ . Let  $X \in \mathcal{X}_{\delta,\gamma}$  with  $t \in X$ . We have  $L_X(r) = L(r)$  and thus if  $|X| = 2$ , then  $\eta_{L_X}(G - t) > \eta_L(G - t)$ , a contradiction. So,  $|X| = 1$  and  $L_X$  differs from  $L$  only on  $t$  where  $L_X$  has  $\gamma$  instead of  $\delta$ . Hence we can use  $L' = L_X$  and  $\delta' = \gamma$ .

Otherwise, by Lemma 4.3, we must have  $\delta \in L(u)$  for exactly one  $u$  in  $N(r) - s$ . Let  $X \in \mathcal{X}_{\delta,\gamma}$  with  $t \in X$ . As before, we conclude  $|X| = 1$  and again we can use  $L' = L_X$  and  $\delta' = \gamma$ .  $\square$

**Lemma 4.8.** *Let  $G$  be a bipartite graph with nonempty parts  $P$  and  $Q$ . If  $|P| \leq |Q|$  and  $Q$  has no isolated vertices, then  $G$  contains a nonempty matching  $M$  whose vertex set is  $S \cup N(S)$  for some  $S \subseteq Q$ .*

**Lemma 4.9.** *For every  $C \subseteq Y$  with  $|C| \geq |N_B(C)|$ , we have  $C \cap L(t) \neq \emptyset$*

*Proof.* Suppose not and let  $C \subseteq Y$  with  $|C| \geq |N_B(C)|$  and  $C \cap L(t) = \emptyset$ . Let  $B'$  be the subgraph of  $B$  induced on  $C \cup N_B(C)$ . Then we may apply Lemma 4.8 to get a nonempty matching  $M$  of  $B'$  whose vertex set is  $S \cup N_B(S)$  for some  $S \subseteq C$ . For each  $\{uw, \alpha\} \in M$ , color  $uw$  with  $\alpha$ . Let  $G' = G - V(N_B(S) - r)$  and define  $L'$  by  $L'(v) = L(v) \setminus S$  for  $v \in V(G')$ . Then  $L'(v) = L(v)$  for  $v \in V(G' - r)$ . So,  $(G', L')$  is superabundant and  $|L'(r)| > d_{G'}(r)$ . Therefore, we can apply minimality of  $|G|$  to  $G'$  to conclude that  $G'$  is  $L'$ -subfixable which implies that  $G$  is  $L$ -subfixable, a contradiction.  $\square$

**Lemma 4.10.** *For every  $C \subseteq Y$ , we have  $|C| \leq |N_B(C)|$ . In particular,  $\eta_L(G - t) \leq \|G - t\|$  and  $F \neq \emptyset$ .*

*Proof.* Suppose not and choose  $C \subseteq Y$  such that  $|C| > |N_B(C)|$  so as to minimize  $|C|$ . For all  $\tau \in C$ , by minimality of  $|C|$ , we have  $N_B(C - \tau) = N_B(C)$ . Since  $|N_B(C')| \geq |C|$  for every  $C' \subseteq C - \tau$ , Hall's theorem gives a nonempty matching  $M_\tau$  whose vertex set is  $(C - \tau) \cup N_B(C - \tau) = (C - \tau) \cup N_B(C)$ . So, for every  $\tau \in C$ , we can color  $N_B(C - \tau)$  using  $C - \tau$  as in Lemma 4.9; the key point is that each of these colorings colors the same edge set.

Put  $R = C \cap L(t)$ . By Lemma 4.9,  $R \neq \emptyset$ . For  $\tau \in R$ , we have  $|C - \tau| \geq |N_B(C - \tau)|$ , so Lemma 4.9 gives  $|R| \geq 2$ .

First, suppose  $rs \in N_B(C)$ . Pick  $\tau \in R \cap L(s)$  if possible; otherwise pick  $\tau \in R$  arbitrarily. For each  $\{uw, \alpha\} \in M_\tau$ , color  $uw$  with  $\alpha$ . Put  $G' = G - V(N_B(C) - r - t)$  and  $L'(v) = L(v) \setminus (C - \tau)$  for  $v \in V(G')$ . Then  $L'(v) = L(v)$  for  $v \in V(G' - r)$ . Then  $(G', L')$  is superabundant and  $|L'(r)| > d_{G'}(r)$ . If we can now color  $st$  with a color different than  $rs$  received, then by minimality of  $|G|$  we conclude that  $G'$  is  $L'$ -subfixable which implies that  $G$  is  $L$ -subfixable, a contradiction. If we cannot color  $st$ , then  $R \cap L(s) \neq \emptyset$  and hence  $\tau \in L(s) \cap L(t)$  and  $\tau$  is not used on  $rs$ , so we can color  $st$  with  $\tau$ , a contradiction.

Hence, we may assume that  $rs \notin N_B(C)$ . So,  $R \cap L(s) = \emptyset$ . Pick  $\tau \in R$ . For each  $\{uw, \alpha\} \in M_\tau$ , color  $uw$  with  $\alpha$ . Put  $G' = G - V(N_B(C) - r)$  and  $L'(v) = L(v) \setminus (C - \tau)$  for  $v \in V(G')$ . We claim that  $(G', L')$  is superabundant. Suppose otherwise that we have  $H \subseteq G'$  such that  $(H, L')$  is not abundant. Since  $\tau \notin L(s)$ , we must have  $r, t \in V(H)$ . Now  $V(H_\tau - t) = \{r\}$  since  $N_B(\tau) \subseteq N_B(C)$ . So, when we add  $t$  back in,  $\tau$  contributes one to  $\psi_{L'}(H)$ . But  $(H - t, L')$  is abundant, so  $(H, L')$  is abundant, a contradiction. Since  $(G', L')$  is superabundant and  $|G'| < |G|$ , by minimality we conclude that  $G'$  is  $L'$ -subfixable which implies that  $G$  is  $L$ -subfixable, a contradiction.  $\square$

**Lemma 4.11.**  $\eta_L(G - t) \geq \|G - t\|$

*Proof.* Suppose  $\eta_L(G - t) < \|G - t\|$ . Then we have  $|F| \geq 2$ .

**Claim 0.** *There is  $\beta \in \text{pot}(L) \setminus F$  that is not swappable with any  $\gamma \in F$ .*

Since  $\eta_L(G - t) < \|G - t\|$ , this is immediate from Lemma 4.4.

**Claim 1.** *There is  $\gamma \in F \setminus L(t)$ . In particular,  $V(G_\gamma) = \{r\}$ .*

Immediate from Lemma 4.5.

**Claim 2.**  *$\gamma$  is swappable with every color in  $\text{pot}(L) \setminus (F \cup \{\beta\})$ . Moreover,  $L(s) \cap L(t) = \{\beta\}$ .*

The only subgraph with edges where  $|H_{L,\gamma}|$  is even is  $G[s, t]$ , so if  $\gamma$  is not swappable with some color  $c \in \text{pot}(L) - F$ , then it must be  $H = G[s, t]$  that fails all conditions of Lemma 3.1. Hence we have  $L(s) \cap L(t) = \{c\}$ . So, there is only one such  $c$  and by Claim 0, it is  $\beta$ .

**Claim 3.** *If  $\beta \in Y$ , then  $|G_\beta - r - t| = 3$ .*

If  $\beta \in Y$  and  $|G_\beta - r - t| \leq 2$ , then the argument in Lemma 4.4 gives  $\psi_L(G - t) < \|G - t\|$ , a contradiction.

So, suppose  $\beta \in Y$  and  $|G_\beta - r - t| \geq 4$ . Pick  $v_1, v_2, v_3 \in V(G_\beta - r - t - s)$ . Then there is  $i \in [3]$  and  $X \in \mathcal{X}_{\beta,\gamma}$  with  $v_i \in X$  such that  $X \cap \{s, t\} = \emptyset$ . Note that  $L_X(r) = L(r)$ ,  $L_X(s) = L(s)$  and  $L_X(t) = L(t)$ . Since the only subgraph with edges where  $|H_{L,\gamma}|$  is even is  $G[s, t]$ , the argument in Lemma 3.1 shows that  $(G, L_X)$  is superabundant. Now  $\{\beta, \gamma\} \subseteq L(v_1) \cup L(v_2) \cup L(v_3)$ , so  $\eta_{L_X}(G - t) > \eta_L(G - t)$ , which contradicts the maximality of  $\eta_L(G - t)$ .

**Claim 4.** *If  $\beta \notin Y$ , then  $|G_\beta - r - t| = 2$ .*

If  $\beta \notin Y$  and  $|G_\beta - r - t| \leq 1$ , then the argument in Lemma 4.4 gives  $\psi_L(G - t) < \|G - t\|$ , a contradiction.

So, suppose  $\beta \notin Y$  and  $|G_\beta - r - t| \geq 3$ . Pick  $v_1, v_2 \in V(G_\beta - r - t - s)$  and let  $v_3 = r$ . Then there is  $i \in [3]$  and  $X \in \mathcal{X}_{\beta, \gamma}$  with  $v_i \in X$  such that  $X \cap \{s, t\} = \emptyset$ . Since the only subgraph with edges where  $|H_{L, \gamma}|$  is even is  $G[s, t]$ , the argument in Lemma 3.1 shows that  $(G, L_X)$  is superabundant. Now  $\{\beta, \gamma\} \cap L(r) \subseteq L(v_1) \cup L(v_2)$ , so  $\eta_{L_X}(G - t) > \eta_L(G - t)$ , which contradicts the maximality of  $\eta_L(G - t)$ .

**Claim 5.** *There is  $\delta \in L(t) \setminus L(s)$  such that  $|G_\delta - t|$  is odd. Moreover,  $\delta$  and  $\gamma$  are swappable.*

By Lemma 4.3 and Claim 2, the colors in  $Y - \beta$  contribute at most  $|Y - \beta|$  to  $\psi_L(G - t)$ . By Claim 3 and Claim 4, the total contribution of  $Y$  and  $\beta$  to  $\psi_L(G - t)$  is at most  $|Y| + 1$ . Since nothing else contributes by Lemma 4.3, we have  $\psi_L(G - t) \leq \eta_L(G - t) + 1 \leq \|G\| - 1$ . Since  $\psi_L(G) \geq \|G\|$ , there must be  $\delta \in L(t) \setminus L(s)$  such that  $|G_\delta - t|$  is odd. The final statement follows by Claim 2 since  $\delta \neq \beta$  because  $\beta \in L(s)$ .

**Claim 6.** *We may assume  $\delta \in F \cap L(t)$ .*

Suppose  $\delta \notin F \cap L(t)$ . First, suppose  $\delta \in Y$ . Then, by Lemma 4.3 and since  $|G_\delta - t|$  is odd, we have  $\delta \in L(u) \cap L(w)$  for exactly two  $u, w \in N(r) - s$ . Let  $X \in \mathcal{X}_{\delta, \gamma}$  with  $t \in X$ . We have  $L_X(r) = L(r)$  and thus if  $|X| = 2$ , then  $\eta_{L_X}(G - t) > \eta_L(G - t)$ , a contradiction. So,  $|X| = 1$  and  $L_X$  differs from  $L$  only on  $t$  where  $L_X$  has  $\gamma$  instead of  $\delta$ . If  $F \subseteq L_X(t)$ , then we get a contradiction by the arguments in Lemma 4.4 and Claim 1. So, we may choose  $\alpha \in F \setminus L_X(t)$ . Now we can use  $\alpha$  in place of  $\gamma$  and  $\gamma$  in place of  $\delta$ , which proves the claim in this case.

Otherwise, by Lemma 4.3, we must have  $\delta \in L(u)$  for exactly one  $u$  in  $N(r) - s$ . Let  $X \in \mathcal{X}_{\delta, \gamma}$  with  $t \in X$ . As before, we conclude  $|X| = 1$  and again we can choose  $\alpha \in F \setminus L_X(t)$  and use  $\alpha$  in place of  $\gamma$  and  $\gamma$  in place of  $\delta$ .

**Claim 7.** *The lemma is true.*

Pick  $\tau \in L(s) - \beta$ . We claim that  $\delta$  and  $\tau$  are swappable. We know  $\tau \notin L(t)$  since  $L(s) \cap L(t) = \{\beta\}$ . Suppose  $\tau \notin L(r)$ . Then, by Lemma 4.3,  $\tau$  appears only in  $L(s)$  and we conclude that  $\delta$  and  $\tau$  are swappable. So, instead suppose  $\tau \in L(r)$ . Then there is at most one  $v \in N(r) - s$  with  $\tau \in L(v)$  by Lemma 4.3. If  $\delta$  and  $\tau$  are not swappable, then some subgraph of  $G[r, s, t]$  must fail all conditions in Lemma 3.1 since  $\tau$  appears an odd number of times in  $G[v, r, s, t]$ . But this is impossible since  $\psi_L(G[r, s, t]) \geq 3$ . Hence  $\delta$  and  $\tau$  are swappable.

Suppose  $\tau$  appears only on  $L(s)$ . Then  $\{s, t\} \in \mathcal{X}_{\delta, \tau}$  and we get  $\eta_{L_{\{s, t\}}}(G - t) > \eta_L(G - t)$ , a contradiction. So,  $\tau \in L(r)$ . There is at most one  $v \in N(r) - s$  with  $\tau \in L(v)$  by Lemma 4.3. Suppose there is such a  $v$ . Then  $\mathcal{X}_{\delta, \tau}$  is one of  $\{\{v, t\}, \{s\}\}$ ,  $\{\{v\}, \{s, t\}\}$  or  $\{\{v, s\}, \{t\}\}$ . For the first two, using the singleton set for  $X$  gives increases  $\eta_{L_X}(G - t) > \eta_L(G - t)$ , a contradiction. For the third using  $X = \{t\}$  gives  $L_X(s) \cap L_X(t) = \{\beta, \tau\}$ . But now  $\gamma$  is swappable with every color in  $\text{pot}(L) \setminus F$  which is impossible as in Claim 0.

Hence  $\tau$  must appear only on  $L(r)$  and  $L(s)$ . That means that for any induced subgraph  $H$  of  $G$  containing  $r, s, t$  we have  $\psi_L(H) \geq 3 + \psi_L(H - s - t) \geq 3 + \|H - s - t\| = \|H\| + 1$ . But now  $\delta$  is swappable with every color in  $\text{pot}(L) - F$  by Lemma 3.1 since any subgraph  $H$  with edges containing an even number of  $\delta$ 's must contain  $r, s, t$ , but then  $\psi_L(H) > \|H\|$ . Since  $\delta \in F$ , this contradicts Claim 0.  $\square$

**Lemma 4.12.**  $G - t$  has an  $L$ -edge-coloring  $\pi$ .

*Proof.* By Lemma 4.10, Lemma 4.11 and Hall's theorem,  $B$  has a perfect matching which gives an  $L$ -edge-coloring of  $G - t$ .  $\square$

**Theorem 4.13.** *Conjecture 4.1(a) is true.*

*Proof.* **Claim 1.** *There is a color  $\beta \in L(r)$  such that  $L(s) \cap L(t) = \{\beta\}$ .*

Otherwise, we  $L$ -edge-color  $G - t$  by using Lemma 4.12 and then use one of the two colors in  $L(s) \cap L(t)$  to color  $st$ , a contradiction.

**Claim 2.** *We have  $F \cap L(t) = \emptyset$ .*

Suppose otherwise that there is  $\gamma \in F \cap L(t)$ . Color the edges of  $G - s - t$  via  $\pi$  in Lemma 4.12 and let  $L'$  be the resulting list assignment on  $rst$ . Then  $\beta \in L'(r) \cap L'(s) \cap L'(t)$  and  $\gamma \in L'(r) \cap L'(t)$ . Hence  $(G[r, s, t], L')$  is superabundant and thus  $G[r, s, t]$  is  $L'$ -subfixable. But then  $G$  is  $L$ -subfixable, a contradiction.

**Claim 3.** *We have  $L(r) \cap L(s) = \{\beta\}$ .*

Suppose not and pick  $\tau \in L(r) \cap L(s) - \beta$ . Pick  $\gamma \in F$ .

Suppose  $\tau$  appears in at least four lists. Let  $X \in \mathcal{X}_{\gamma, \tau}$  with  $s \in X$ . If  $|X| = 1$ , then  $G$  is colorable from  $L_X$  by coloring  $G - s - t$  from  $\pi$ , coloring  $rs$  with  $\gamma$  and  $st$  with  $\beta$ . Hence  $X = \{s, v\}$  for some  $v \in N(r) - s$ . But  $\tau \in L(w)$  for some  $w \in N(r) - s - v$ , so  $\eta_{L_X}(G - t) > \eta_L(G - t)$ , a contradiction.

Suppose  $\beta$  appears in at least four lists. Let  $X \in \mathcal{X}_{\beta, \tau}$  with  $t \in X$ . If  $|X| = 1$  or...

Then Fixer swaps  $\tau$  for  $\beta$  in  $L(t)$ . If Breaker does nothing or swaps  $\tau$  for  $\beta$  somewhere, then Fixer can finish the coloring. Hence Breaker must swap  $\beta$  for  $\tau$  at some  $v \in N(r) - s$ . But now applying the previous paragraph with the roles of  $\beta$  and  $\tau$  reversed gives a contradiction.

Therefore,  $\beta$  and  $\tau$  each contribute only one to  $\psi_L(G)$ . Since  $\eta_L(G - t) = \|G - t\|$  by Claim 6, applying Claim 1 and Claim 2 shows that there must be  $\delta \in L(t) - L(s)$  such that  $|G_\delta - t|$  is odd. But now by the same argument as in Subclaim 3f, Fixer can achieve a position contradicting Claim 9.

**Claim 4.** *The theorem is true.*

Pick  $\gamma \in F$ . Since  $L(r) \cap L(s) = \{\beta\}$  by Claim 10, there must be  $\alpha \in L(r) \cap L(t)$  since  $(G[r, s, t], L)$  is abundant and  $L(s) \cap L(t) = \{\beta\}$  by Claim 8.

Pick  $\tau \in L(s) - \beta$ . Suppose  $\tau$  appears in more than one list. Then Fixer should swap  $\gamma$  for  $\tau$  in  $L(s)$ . Then  $\eta_L(G - t)$  has increased unless Breaker swaps  $\tau$  for  $\gamma$  in  $L(r)$ . But since  $\tau$  is in more than one list, it must be in  $L(v)$  for some  $v \in N(r) - s$  and  $\eta_L(G - t)$  has increased as well, a contradiction.

Suppose  $\beta$  appears in more than three lists. Then Fixer should swap  $\tau$  for  $\beta$  in  $L(t)$ . If Breaker does nothing or swaps  $\tau$  for  $\beta$  in any list other than  $L(r)$ , then Fixer can complete the coloring. So Fixer must swap  $\tau$  for  $\beta$  in  $L(r)$ . But  $\beta \in L(v)$  for some  $v \in N(r) - s$ , so applying the previous paragraph with the roles of  $\beta$  and  $\tau$  reversed gives a contradiction.

Suppose  $\alpha$  appears in more than three lists. Then Fixer should swap  $\gamma$  for  $\alpha$  in  $L(t)$ . Now Breaker must pass lest he increase  $\eta_L(G - t)$ . But this is a position contradicting Claim 9.

Therefore,  $\beta$  and  $\alpha$  each contribute only one to  $\psi_L(G)$ . Since  $\eta_L(G - t) = \|G - t\|$  by Claim 6, applying Claim 1 and Claim 2 shows that there must be  $\delta \in L(t) - L(s)$  such that  $|G_\delta - t|$  is odd. But now by the same argument as in Subclaim 3f, Fixer can achieve a position contradicting Claim 9.  $\square$