

## graph theory notes\*

# Stiebitz's proof of Gallai's conjecture on the number of components in the high and low vertex subgraphs of critical graphs

Tibor Gallai conjectured the following in 1963 [1, 2] and Michael Stiebitz proved it in 1982 [3]. For a graph  $G$ , let  $\mathcal{L}(G)$  be the subgraph of  $G$  induced on the vertices of degree  $\delta(G)$  and let  $\mathcal{H}(G)$  be the subgraph of  $G$  induced on the vertices of degree larger than  $\delta(G)$ .

**Theorem** (Stiebitz). *If  $G$  is a color-critical graph with  $\delta(G) = \chi(G) - 1$ , then  $\mathcal{H}(G)$  has at most as many components as  $\mathcal{L}(G)$ .*

In fact, Stiebitz proved a stronger statement. Theorem follows immediately from Lemma 3 using  $X = V(\mathcal{L}(G))$ . The main induction step in the proof requires a non-trivial fact about bipartite graphs, we save the proof of this fact until later but state it here. A bipartite graph  $G$  with parts  $A$  and  $B$  has *positive surplus* with respect to  $A$  if  $|N(S)| > |S|$  for all  $\emptyset \neq S \subseteq A$ . Note that  $A$  could be empty here, in which case the graph has positive surplus vacuously.

**Lemma 1** (Stiebitz). *Let  $G$  be a bipartite graph with parts  $A \neq \emptyset$  and  $B$  such that  $G$  has positive surplus with respect to  $A$ . Then there is  $x \in A$  such that for any different  $y, z \in N(x)$ , the bipartite graph  $G'$  formed by contracting  $\{y, z\}$  and removing  $x$  has positive surplus with respect to  $A \setminus \{x\}$ .*

Lemma 1 is used in Claim 2 of the proof of Lemma 2. Note that following Lemma 2 is trivially true in the  $X = \emptyset$  case, we allow this to avoid having to handle a base case where  $G[X]$  has one component separately.

**Lemma 2** (Stiebitz). *Let  $G$  be a graph and  $X \subseteq V(G)$  such that*

- $d_G(x) \leq k - 1$  for all  $x \in X$ ; and
- $\chi(G - X) \leq k - 1$ ; and
- for each component  $C$  of  $G[X]$ , we have  $\chi(G - V(C)) \leq k - 1$ .

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Suppose  $G - X$  is the disjoint union of (possibly not connected) graphs  $M_1, \dots, M_{\ell+1}$  such that the bipartite graph  $\mathcal{B}$  formed by contracting each  $M_i$  and each component of  $G[X]$  has positive surplus with respect to the  $G[X]$  side. If  $f_i$  is a  $(k-1)$ -coloring of  $M_i$  for each  $i \in [\ell+1]$ , then there are permutations  $\pi_1, \dots, \pi_{\ell+1}$  of  $[k-1]$  such that the  $(k-1)$ -coloring of  $G - X$  given by  $(\pi_1 \circ f_1) \cup \dots \cup (\pi_{\ell+1} \circ f_{\ell+1})$  extends to a  $(k-1)$ -coloring of  $G$ .

*Proof.* Suppose the lemma is false and choose a counterexample  $G$  and nonempty  $X \subseteq V(G)$  so that  $|X|$  is as small as possible. So,  $G - X$  is the disjoint union of graphs  $M_1, \dots, M_{\ell+1}$  and we have  $(k-1)$ -colorings  $f_i$  of  $M_i$  for each  $i \in [\ell+1]$  so that no permutations allow us to extend to a  $(k-1)$ -coloring of  $G$ .

**Claim 1.** *Each non-separating vertex in  $G[X]$  has neighbors in at least two of the  $M_i$ .* Suppose to the contrary that we have a component  $C$  of  $G[X]$  and  $x \in V(C)$  a non-separating vertex that has neighbors in at most one of the  $M_i$ . Let  $X' = X \setminus \{x\}$ . The hypotheses of the lemma are satisfied with  $X'$  in place of  $X$  and  $M'_i = G[V(M_i) \cup \{x\}]$  in place of  $M_i$  since  $\mathcal{B}$  remains the same and hence still has positive surplus. Since  $\mathcal{B}$  has positive surplus, we must have  $|C| \geq 2$  and hence  $x$  has at most  $k-2$  neighbors in  $G - X$ , so we can greedily complete the given  $(k-1)$ -coloring of  $G - X$  to  $G - X'$ . Applying minimality of  $|X|$  to this coloring of  $G - X'$ , we get permutations that allow us to extend to a  $(k-1)$ -coloring of  $G$ . But these same permutations also allow us to extend the given  $(k-1)$ -coloring of  $G - X$  to  $G$ , a contradiction.

**Claim 2.** *There is a component  $C$  of  $G[X]$  and  $i \neq j$  such that some non-separating  $x \in V(C)$  has neighbors in  $M_i$  and  $M_j$  and the bipartite graph formed from  $\mathcal{B}$  by contracting  $\{M_i, M_j\}$  and removing  $C$  has positive surplus.*

By Lemma 1,  $G[X]$  has a component  $C$  such that for any neighbors  $M_i, M_j$  of  $C$  in  $\mathcal{B}$ , the bipartite graph formed from  $\mathcal{B}$  by contracting  $\{M_i, M_j\}$  and removing  $C$  has positive surplus. Let  $x$  be a non-separating vertex in  $C$ . By Claim 1, there are different  $M_i, M_j$  in which  $x$  has neighbors, so  $C$  with  $x$  and  $i, j$  works.

**Claim 3.** *The lemma is true.* Let  $C, x, i, j$  be as in Claim 2, by symmetry we may assume  $i = 1$  and  $j = 2$ . Let  $G' = G - V(C)$  and  $X' = X \setminus V(C)$ . Then  $G' - X'$  is the disjoint union of the  $\ell$  graphs  $M_1 \cup M_2, M_3, \dots, M_{\ell+1}$ . Pick  $y_1 \in N(x) \cap V(M_1)$  and  $y_2 \in N(x) \cap V(M_2)$ . We permute the colors in the coloring of  $f_2$  so that  $y_1$  and  $y_2$  get the same color. This will save one color for  $x$  so that we can greedily color  $C$ , ending at  $x$ . Formally, let  $\tau$  be a permutation of  $[k-1]$  such that  $(\tau \circ f_2)(y_2) = f_1(y_1)$  and let  $f_* = f_1 \cup (\tau \circ f_2)$ . By Claim 2 and minimality of  $|X|$ , we can apply the lemma to  $G'$  with  $M_1 \cup M_2, M_3, \dots, M_{\ell+1}$  and colorings  $f_*, f_3, \dots, f_{\ell+1}$  to get permutations  $\pi_*, \pi_3, \dots, \pi_{\ell+1}$  such that the  $(k-1)$ -coloring of  $G' - X'$  given by  $(\pi_* \circ f_*) \cup (\pi_3 \circ f_3) \cup \dots \cup (\pi_{\ell+1} \circ f_{\ell+1})$  extends to a  $(k-1)$ -coloring of  $G'$ . But this is the same as the  $(k-1)$ -coloring  $(\pi_* \circ f_1) \cup (\pi_* \circ \tau \circ f_2) \cup (\pi_3 \circ f_3) \cup \dots \cup (\pi_{\ell+1} \circ f_{\ell+1})$ , so using the permutations  $\pi_*, \pi_* \circ \tau, \pi_3, \dots, \pi_{\ell+1}$  we get a coloring of  $G - X$  that extends to  $G - V(C)$ . In these colorings,  $y_1$  and  $y_2$  receive the same color. This means that  $x$  has  $k-1 - (d_G(x) - d_C(x)) + 1 \geq d_C(x) + 1$  colors available and each other vertex  $v$  in  $C$  has  $k-1 - (d_G(v) - d_C(v)) + 1 \geq d_C(v) \geq d_C(v)$  colors available. So, coloring  $C$  greedily in order of decreasing distance from  $x$  gives an extension to a  $(k-1)$ -coloring of  $G$ , a contradiction.  $\square$

Note that, like the previous one, the following lemma is trivially true in the  $X = \emptyset$  case, again we allow this to avoid having to handle a base case where  $G[X]$  is connected separately.

**Lemma 3** (Stiebitz). *Let  $G$  be a connected graph and  $X \subseteq V(G)$  such that*

- $d_G(x) \leq k - 1$  for all  $x \in X$ ; and
- $\chi(G - X) \leq k - 1$ ; and
- for each component  $C$  of  $G[X]$ , we have  $\chi(G - V(C)) \leq k - 1$ ; and
- $G[X]$  has  $\ell$  components and  $G - X$  has at least  $\ell + 1$  components.

*Then  $G$  is  $(k - 1)$ -colorable.*

*Proof.* Suppose not and choose a counterexample minimizing  $|X|$ . If  $|X| = 0$ , the lemma is trivially true, so we must have  $|X| \geq 1$ . Let  $\mathcal{B}$  be the bipartite graph formed by contracting each component of  $G - X$  and each component of  $G[X]$ . If  $\mathcal{B}$  has positive surplus with respect to the  $G[X]$  side, then applying Lemma 2 to any  $(k - 1)$ -coloring of  $G - X$  gives a  $(k - 1)$ -coloring of  $G$ . So, we may assume that  $\mathcal{B}$  does not have positive surplus. That is, for some  $t$ ,  $G[X]$  has a set of  $t$  components  $\{C_1, \dots, C_t\}$  which together have neighbors in at most  $t$  components of  $G - X$ . But then the other  $\ell - t$  components of  $G[X]$  together have neighbors in at least  $\ell + 1 - t$  components of  $G - X$  since  $G$  is connected. Let  $X' = X \setminus \bigcup_{i \in [t]} V(C_i)$ . Then the hypotheses of the lemma are satisfied with  $X'$  in place of  $X$ , so minimality of  $|X|$  shows that  $G$  is  $(k - 1)$ -colorable, a contradiction.  $\square$

## Bipartite graphs

A bipartite graph  $G$  with parts  $A$  and  $B$  is a *2-forest* with respect to  $A$  if  $G$  is a forest where each vertex in  $A$  has degree 2 in  $G$ . Plainly, any 2-forest has positive surplus. We first need a few lemmas about bipartite graphs with positive surplus. It is well-known that the edge-minimal positive surplus bipartite graphs are exactly the 2-forests (see [?]). More precisely,

**Lemma 4.** *Let  $G$  be a bipartite graph with parts  $A \neq \emptyset$  and  $B$  such that  $G$  has positive surplus with respect to  $A$ . If  $G - e$  does not have positive surplus for each  $e \in E(G)$ , then  $G$  is a 2-forest with respect to  $A$ .*

The next lemma says that a positive surplus bipartite graph always has a special vertex in  $A$  such that removing most of its incident edges leaves a positive surplus graph.

**Lemma 5** (Stiebitz). *Let  $G$  be a bipartite graph with parts  $A \neq \emptyset$  and  $B$  such that  $G$  has positive surplus with respect to  $A$ . Then there is  $x \in A$  such that removing any set of  $d_G(x) - 2$  edges incident to  $x$  leaves a bipartite graph having positive surplus with respect to  $A$ .*

*Proof.* Suppose not and choose a counterexample  $G$  minimizing  $||G||$ . First, suppose there is  $\emptyset \neq A' \subsetneq A$  such that  $|N(A')| \leq |A'| + 1$ . Let  $G' = G[A' \cup N(A')]$ . Then  $G'$  has positive surplus with respect to  $A'$  and hence by minimality of  $||G||$ , there is  $x \in A'$  such that removing any set of  $d_G(x) - 2$  edges incident to  $x$  leaves a bipartite graph having positive surplus with respect to  $A'$ . If  $W \subseteq A \setminus A'$ , then  $|W| + |A'| + 1 \leq |N_G(W) \cup N_G(A')| \leq$

$|N_G(W) \setminus N_G(A')| + |N_G(A')| \leq |N_G(W) \setminus N_G(A')| + |A'| + 1$  since  $G$  has positive surplus. Therefore  $|N_G(W) \setminus N_G(A')| \geq |W|$  for each  $W \subseteq A \setminus A'$ . Since  $G$  is a counterexample, we have  $y, z \in N(x)$  such that the graph  $H$  made by removing all edges incident to  $x$  except  $xy, xz$  does not have positive surplus with respect to  $A$ . That is, we have  $S \subseteq A$  such that  $|N_H(S)| \leq |S|$ . Let  $S' = S \cap A'$  and  $W = S \setminus S'$ . Since  $H[A' \cup N(A')]$  has positive surplus with respect to  $A'$ , we have  $|N_H(S')| \geq |S'| + 1$ . But also,  $|N_H(W) \setminus N_H(A')| = |N_G(W) \setminus N_G(A')| \geq |W|$ , so  $|N_H(S)| = |N_H(S')| + |N_H(W) \setminus N_H(A')| \geq |S'| + 1 + |W| = |S| + 1$ , a contradiction.

So, for every  $\emptyset \neq A' \subsetneq A$  we must have  $|N(A')| \geq |A'| + 2$ . Pick  $x \in A$  arbitrarily. Since  $G$  is a counterexample, we have  $y, z \in N(x)$  such that the graph  $H$  made by removing all edges incident to  $x$  except  $xy, xz$  does not have positive surplus with respect to  $A$ . Since  $d_G(x) \geq 3$ , there is  $w \in N_G(x) \setminus \{y, z\}$ . Suppose the bipartite graph  $G - xw$  has positive surplus with respect to  $A$ . Then, by minimality of  $\|G\|$ , there is  $u \in A$  such that removing any set of  $d_{G-xw}(u) - 2$  edges incident to  $u$  from  $G - xw$  leaves a bipartite graph having positive surplus with respect to  $A$ . If  $u \neq x$ , then  $d_{G-xw}(u) - 2 = d_G(u) - 2$ , so  $u$  works as the special vertex for  $G$ , a contradiction. So,  $u = x$  and thus  $H$  has positive surplus, a contradiction.

Therefore, there is  $\emptyset \neq S \subseteq A$  such that  $|N_{G-xw}(S)| \leq |S|$ . But  $|N_{G-xw}(S)| \geq |N_G(S)| - 1$ , so if  $S \neq A$ , then  $|N_{G-xw}(S)| \geq |S| + 2 - 1$ . So, we must have  $S = A$  and  $|N(A)| = |A| + 1$  as well as  $|N_{G-xw}(A)| = |N_G(A)| - 1$  and hence  $d_G(w) = 1$ . This was for arbitrary  $x \in A$ , so we conclude that every  $x \in A$  has a neighbor in  $B$  of degree 1 and hence all but at most one vertex in  $B$  has degree 1 in  $G$ . But then every vertex in  $A$  has degree 2 in  $G$  and hence will work for the special vertex, giving the final contradiction.  $\square$

*Proof of Lemma 1.* By Lemma 5, there is  $x \in A$  such that for any different  $y, z \in N(x)$ , the bipartite graph  $H$  formed from  $G$  by removing all edges incident to  $x$  except  $xz, yz$  has positive surplus with respect to  $A$ . But then, by Lemma 4,  $H$  contains a 2-forest  $F$  with respect to  $A$  containing the edges  $xy$  and  $xz$ . Consider the graph  $F'$  formed from  $F$  by contracting  $\{y, z\}$  and removing  $x$ . Then  $F'$  is a 2-forest with respect to  $A$  since the degree of vertices in  $A$  don't change and  $x$  is a separating vertex in  $F$ . Now  $G'$  contains  $F'$  and hence has positive surplus, so we are done.  $\square$

## References

- [1] T. Gallai, *Kritische graphen I.*, Math. Inst. Hungar. Acad. Sci **8** (1963), 165–192 (in German).
- [2] ———, *Kritische graphen II.*, Math. Inst. Hungar. Acad. Sci **8** (1963), 373–395 (in German).
- [3] M. Stiebitz, *Proof of a conjecture of T. Gallai concerning connectivity properties of colour-critical graphs*, Combinatorica **2** (1982), no. 3, 315–323.