SPARSE GRAPHS ADMIT HOMOMORPHISMS INTO ODD CYCLES

LANDON RABERN

Abstract.

1. Introduction

All graphs under consideration are nonempty finite simple graphs. For graphs G and H, we indicate the existence of a homomorphism from G to H or lack thereof by $G \to H$ and $G \not\to H$, respectively. We write $H \trianglelefteq G$ to indicate that H is an induced subgraph of G, when we want the containment to be proper, we write $H \lhd G$.

2. Potential functions

Kostochka and Yancey [2] used "potential functions" to great effect in proving lower bounds on the number of edges in critical graphs. Here we generalize this idea and prove some basic facts.

Definition 1. For positive integers a and b, the (a, b)-*potential function* is the function from graphs to $\mathbb{Z}$ given by $\rho_{a,b}(G) := a\,|G| - b\,\|G\|$. Additionally, put

$$\hat{\rho}_{a,b}(G) := \min_{H \trianglelefteq G} \rho_{a,b}(H).$$

The invariant $\hat{\rho}_{a,b}(G)$ is a measure of the sparseness of G, the larger $\hat{\rho}_{a,b}(G)$ is, the sparser G is. For example, if $\hat{\rho}_{a,b}(G) \geq 0$, then $\operatorname{mad}(G) \leq \frac{2a}{b}$ where $\operatorname{mad}(G)$ is the maximum average degree of G.

For any fixed graph T, we are interested in proving results of the form: any sufficiently sparse graph admits a homomorphism into T. To do so, it will be useful to get the benefits of having a minimum counterexample without being bound to a fixed inductive context. To achieve this, we use *mules* as introduced in [1, 3].

2.1. **Mules.**

Definition 2. If G and H are graphs, an *epimorphism* is a graph homomorphism $f\colon G \twoheadrightarrow H$ such that $f(V(G)) = V(H)$. We indicate this with the arrow $\twoheadrightarrow$.

Definition 3. Let G be a graph. A graph A is called a *child* of G if $A \neq G$ and there exists $H \trianglelefteq G$ and an epimorphism $f\colon H \twoheadrightarrow A$.

Note that the child-of relation is a strict partial order on the set of (finite simple) graphs $\mathcal{G}$. We call this the *child order* on $\mathcal{G}$ and denote it by '$\prec$'. By definition, if $H \lhd G$ then $H \prec G$.

Date: September 5, 2014.

$$\begin{tikzcd} H \arrow[r, hook, "\iota"] \arrow[d, two heads, "h"'] & G \arrow[d, two heads, "h'"] \\ Q \arrow[r, hook, "\iota"'] & G_h \end{tikzcd}$$

FIGURE 1. The commutative diagram for G_h.

Lemma 1. *The ordering $\prec$ is well-founded on $\mathcal{G}$; that is, every nonempty subset of $\mathcal{G}$ has a minimal element under $\prec$.*

Proof. Let $\mathcal{T}$ be a nonempty subset of $\mathcal{G}$. Pick $G \in \mathcal{T}$ minimizing $|V(G)|$ and then maximizing $|E(G)|$. Since any child of G must have fewer vertices or more edges (or both), we see that G is minimal in $\mathcal{T}$ with respect to $\prec$. □

Definition 4. Let $\mathcal{T}$ be a collection of graphs. A minimal graph in $\mathcal{T}$ under the child order is called a *$\mathcal{T}$-mule.*

2.2. Basic facts.

For a graph T together with positive integers a, b and c, let $\mathcal{C}_{T,a,b,c}$ be the set of graphs G such that $G \not\to T$ and $\hat{\rho}_{a,b}(G) \geq c$.

Lemma 2. *Let G be a $\mathcal{C}_{T,a,b,c}$-mule. If $H \lhd G$, then $H \to T$.*

Proof. Since $\hat{\rho}_{a,b}(H) \geq \hat{\rho}_{a,b}(G) \geq c$ and $H \prec G$, we must have $H \to T$ since G is a $\mathcal{C}_{T,a,b,c}$-mule. □

Definition 5. Let H be an induced subgraph of a graph G and $h\colon H \twoheadrightarrow Q$ an epimorphism onto some graph Q. Let G_h be the image of the natural extension of h to an epimorphism h' defined on G; that is, G_h and h' are such that the diagram in Figure 1 commutes (where ι indicates the inclusion map).

Lemma 3. *Let G be a $\mathcal{C}_{T,a,b,c}$-mule and Q an arbitrary graph. If $H \unlhd G$ with $H \neq Q$ such that $H \twoheadrightarrow Q$, then $\rho_{a,b}(H) > \hat{\rho}_{a,b}(Q)$.*

Proof. Suppose to the contrary that there is $H \unlhd G$ with $H \neq Q$ such that $H \twoheadrightarrow Q$ and $\rho_{a,b}(H) \leq \hat{\rho}_{a,b}(Q)$. Let h be an epimorphism from H onto Q. Since G is a $\mathcal{C}_{T,a,b,c}$-mule, G_h cannot be a child of G. But we have an epimorphism h' from G onto G_h and $G_h \neq G$ since $H \neq Q$, so it must be that $G_h \notin \mathcal{C}_{T,a,b,c}$. Since $G \to G_h$ and $G \not\to T$, we must have $G_h \not\to T$. Therefore $\hat{\rho}_{a,b}(G_h) < c$. Pick $W \unlhd G_h$ with $\rho_{a,b}(W) < c$. Since $W \not\subseteq G$, we must have $V(W) \cap V(Q) \neq \emptyset$. Hence $\rho_{a,b}\left(G\left[(V(W) - V(Q)) \cup V(H)\right]\right) \leq \rho_{a,b}(W) - \hat{\rho}_{a,b}(Q) + \rho_{a,b}(H) \leq \rho_{a,b}(W) - \hat{\rho}_{a,b}(Q) + \hat{\rho}_{a,b}(Q) = \rho_{a,b}(W) < c$, a contradiction since $\hat{\rho}_{a,b}(G) \geq c$. □

We can easily weaken the condition $H \twoheadrightarrow Q$ to $H \to Q$.

Corollary 4. *Let G be a $\mathcal{C}_{T,a,b,c}$-mule and Q an arbitrary graph. If $H \unlhd G$ with $H \neq Q$ such that $H \to Q$, then $\rho_{a,b}(H) > \hat{\rho}_{a,b}(Q)$.*

Proof. Let h be a homomorphism from H into Q and put $Q' := \operatorname{im}(h)$. Then $H \neq Q'$ and $H \twoheadrightarrow Q'$, so Lemma 3 gives $\rho_{a,b}(H) > \hat{\rho}_{a,b}(Q') \geq \hat{\rho}_{a,b}(Q)$. □

We have the following basic bound on the potential of non-complete subgraphs of G.

Corollary 5. *Let G be a $\mathcal{C}_{T,a,b,c}$-mule. If $H \trianglelefteq G$ is not complete and $\chi(H) \leq \frac{2a}{b}$, then $\rho_{a,b}(H) > a$.*

Proof. Suppose $\chi(H) = k \leq \frac{2a}{b}$. Then there is an epimorphism from H onto K_k given by contracting all color classes in a k-coloring of H. Since $H \neq K_k$, Lemma 3 gives $\rho_{a,b}(H) > \hat{\rho}_{a,b}(K_k)$. But $\hat{\rho}_{a,b}(K_k) = \min_{t \in [k]} at - b\binom{t}{2} = a$ since $k \leq \frac{2a}{b}$, so we have the desired bound. □

There is room for improvement in the proof of Lemma 3; in particular, the bound

$$\rho_{a,b}\left(G\left[(V(W) - V(Q)) \cup V(H)\right]\right) \leq \rho_{a,b}(W) - \hat{\rho}_{a,b}(Q) + \rho_{a,b}(H)$$

can be improved in many cases. Put $X := V(W) - V(Q)$ and $Y := h^{-1}(V(W) \cap V(Q))$. We did not count any of the edges between X and $H - Y$ in our estimate, so in fact we can improve the upper bound to

$$\rho_{a,b}(W) - \hat{\rho}_{a,b}(Q) + \rho_{a,b}(H) - b\left|E_G(X, H - Y)\right|.$$

Furthermore, we gain when a vertex in X has more than $|V(W) \cap V(Q)|$ neighbors in Y because multi-edges get replaced by single edges. Taking this into account, improves the upper bound to

$$\rho_{a,b}(W) - \hat{\rho}_{a,b}(Q) + \rho_{a,b}(H) - b\left|E_G(X, H - Y)\right| - b\sum_{v \in X} \max\left\{0, |N_G(v) \cap Y| - |V(W) \cap V(Q)|\right\}.$$

Finally, we may be able to improve on the use of $\hat{\rho}_{a,b}(Q)$ when $|V(W) \cap V(Q)| > 1$. There is a tension between this final improvement and the previous two. For many Q in the wild, $\rho_{a,b}(Q') > \rho_{a,b}(K_1)$ for all $Q' \trianglelefteq Q$ with $|Q'| > 1$. When this occurs, our upper bound is improved by 1 unless $|V(W) \cap V(Q)| = 1$ and $|E(X, H - Y)| = 0$ and $|N(v) \cap Y| \leq 1$ for all $v \in X$. To apply this observation, we need some control over W. We can get this control when $Q = T$, we will be able to conclude that $(V(W) - V(Q)) \cup V(H) = V(G)$.

Definition 6. Put $\tilde{\rho}_{a,b}(G) := \min\left\{\rho_{a,b}(H) \mid H \trianglelefteq G \text{ with } |G| \geq 2\right\}$.

Lemma 6. *Let G be a 2-connected $\mathcal{C}_{T,a,b,c}$-mule where $\tilde{\rho}_{a,b}(T) > \hat{\rho}_{a,b}(T)$ and $\hat{\rho}_{a,b}(T) \geq b + c - 1$. If $H \triangleleft G$ with $H \not\twoheadrightarrow T$, then $\rho_{a,b}(H) > \hat{\rho}_{a,b}(T) + 1$.*

Proof. Suppose to the contrary that we have such an H with $\rho_{a,b}(H) \leq \hat{\rho}_{a,b}(T) + 1$. Since $H \triangleleft G$, Lemma 2 shows that $H \to T$. Applying Corollary 4 gives $\rho_{a,b}(H) > \hat{\rho}_{a,b}(T)$. The same holds for any induced subgraph of H, so $\hat{\rho}_{a,b}(H) > \hat{\rho}_{a,b}(T)$ and hence $\hat{\rho}_{a,b}(H) = \hat{\rho}_{a,b}(T) + 1$.

Since G is 2-connected, H has at least two vertices x, y with neighbors outside H. If x is adjacent to y, put $H' := H$, otherwise put $H' := H + xy$. We have $\hat{\rho}_{a,b}(H') \geq \hat{\rho}_{a,b}(H) - b \geq \hat{\rho}_{a,b}(T) + 1 - b \geq c$. Since $H' \prec G$, we must have $H' \to T$ as G is a $\mathcal{C}_{T,a,b,c}$-mule.

Let h be a homomorphism from H' into T. Then h is a homomorphism from H into T with the property that $h(x) \neq h(y)$. Put $Q := \operatorname{im}(h)$. Now $H \neq Q$ since $H \not\twoheadrightarrow T$ and h is an epimorphism from H onto Q. Since G is a $\mathcal{C}_{T,a,b,c}$-mule, G_h cannot be a child of G. But we have an epimorphism h' from G onto G_h and $G_h \neq G$ since $H \neq Q$, so it must be that $G_h \notin \mathcal{C}_{T,a,b,c}$. Since $G \to G_h$ and $G \not\to T$, we must have $G_h \not\to T$. Therefore $\hat{\rho}_{a,b}(G_h) < c$. Pick $W \trianglelefteq G_h$ with $\rho_{a,b}(W) < c$. Since $W \not\subseteq G$, we must have $V(W) \cap V(Q) \neq \emptyset$. Hence $\rho_{a,b}\left(G\left[(V(W) - V(Q)) \cup V(H)\right]\right) \leq \rho_{a,b}(W) - \hat{\rho}_{a,b}(T) + \rho_{a,b}(H) \leq c$. Suppose $(V(W) - V(Q)) \cup V(H) \neq V(G)$. Then Lemma 2 shows that $G\left[(V(W) - V(Q)) \cup V(H)\right] \to T$. Applying Corollary 4 gives $\rho_{a,b}\left(G\left[(V(W) - V(Q)) \cup V(H)\right]\right) > \hat{\rho}_{a,b}(T)$ and hence $\hat{\rho}_{a,b}(T) < c$, a contradiction.

So, we have $(V(W) - V(Q)) \cup V(H) \neq V(G)$. Since $\tilde{\rho}_{a,b}(T) > \hat{\rho}_{a,b}(T)$, if $|V(W) \cap V(Q)| > 1$, then our above estimate on $\rho_{a,b}(G)$ is decreased by one, giving $\hat{\rho}_{a,b}(G) < c$, a contradiction. So we must have $V(W) \cap V(Q) = \{z\}$ for some z. By the discussion before this lemma, we must also have $|E(G - H, H - h^{-1}(z))| = 0$. But $h(x) \neq h(y)$, so by symmetry, we may assume that $h(y) \neq h(z)$. Therefore $y \in V(H - h^{-1}(z))$ and y has a neighbor in $G - H$, a contradiction. $\square$

We need a lemma from Kostochka and Yancey [2].

Lemma 7 (Kostochka and Yancey [2]). *Let S be a finite set, $\ell \geq 2$ an integer and $f \colon S \to \mathbb{N}_{\geq 1}$ such that $\sum_{v \in S} f(v) \geq \ell$. Then, for any $i \in \left[\frac{\ell}{2}\right]$, there is a graph H with $V(H) = S$ and $||H|| = i$ such that for any independent set I in H with $|I| \geq 2$, we have*

$$\sum_{v \in S - M} f(v) \geq i.$$

Lemma 8. *Let G be a 2-connected $\mathcal{C}_{T,a,b,c}$-mule where $\hat{\rho}_{a,b}(T) \geq c$ and $\tilde{\rho}_{a,b}(T) \leq 2\hat{\rho}_{a,b}(T) + 2 - b - c$. If $H \triangleleft G$ with $|H| > 1$ and $H \not\to T$ such that $|E(H, G - H)| \geq 2\left\lfloor \frac{\tilde{\rho}_{a,b}(T) - 1}{b} \right\rfloor$, then $\rho_{a,b}(H) > \tilde{\rho}_{a,b}(T)$.*

Proof. Suppose not and choose $H \triangleleft G$ with $|H| > 1$ and $H \not\to T$ minimizing $\rho_{a,b}(H)$. Since $H \triangleleft G$, Lemma 2 shows that $H \to T$. Applying Corollary 4 gives $\rho_{a,b}(H) > \hat{\rho}_{a,b}(T)$. The same holds for any induced subgraph of H, so $\hat{\rho}_{a,b}(H) > \hat{\rho}_{a,b}(T)$.

Let H' be a graph formed from H by adding $i := \left\lfloor \frac{\hat{\rho}_{a,b}(H) - c}{b} \right\rfloor$ edges (**IN A GOOD WAY**). Then $\hat{\rho}_{a,b}(H') \geq \hat{\rho}_{a,b}(H) - ib \geq c$. Since $H' \prec G$, we must have $H' \to T$ as G is a $\mathcal{C}_{T,a,b,c}$-mule.

Let h be a homomorphism from H' into T. Then h is a homomorphism from H into T with special properties we will use later. Put $Q := \mathrm{im}(h)$. Now $H \neq Q$ since $H \not\to T$ and h is an epimorphism from H onto Q. Since G is a $\mathcal{C}_{T,a,b,c}$-mule, G_h cannot be a child of G. But we have an epimorphism h' from G onto G_h and $G_h \neq G$ since $H \neq Q$, so it must be that $G_h \notin \mathcal{C}_{T,a,b,c}$. Since $G \to G_h$ and $G \not\to T$, we must have $G_h \not\to T$. Therefore $\hat{\rho}_{a,b}(G_h) < c$. Pick $W \trianglelefteq G_h$ with $\rho_{a,b}(W) < c$. Since $W \not\subseteq G$, we must have $V(W) \cap V(Q) \neq \emptyset$. Hence $\rho_{a,b}\left(G\left[(V(W) - V(Q)) \cup V(H)\right]\right) \leq \rho_{a,b}(W) - \hat{\rho}_{a,b}(T) + \rho_{a,b}(H) < \rho_{a,b}(H) + c - \hat{\rho}_{a,b}(T) \leq \rho_{a,b}(H)$. This contradicts the minimality of $\rho_{a,b}(H)$ unless $(V(W) - V(Q)) \cup V(H) \neq V(G)$. Also, if $|V(W) \cap V(Q)| > 1$, then we have $\rho_{a,b}\left(G\left[(V(W) - V(Q)) \cup V(H)\right]\right) \leq \rho_{a,b}(W) - \tilde{\rho}_{a,b}(T) + \rho_{a,b}(H) < c$, a contradiction.

Hence $(V(W) - V(Q)) \cup V(H) = V(G)$ and $V(W) \cap V(Q) = \{z\}$ for some z. By the discussion before this lemma and the estimate $\rho_{a,b}(G) < c - \hat{\rho}_{a,b}(T) + \rho_{a,b}(H) - b\left|E(G - H, H - h^{-1}(z))\right|$, we must have $b\,|E(G-H, H-h^{-1}(z))| < \rho_{a,b}(H) - \hat{\rho}_{a,b}(T)$. When forming H', we added edges to H in a way that guarantees $|E(G-H, H-h^{-1}(z))| \geq i$, so we must have $\rho_{a,b}(H) > \hat{\rho}_{a,b}(T)+ib = \hat{\rho}_{a,b}(T)+b\left\lfloor\frac{\hat{\rho}_{a,b}(T)-c}{b}\right\rfloor \geq \hat{\rho}_{a,b}(T)+\hat{\rho}_{a,b}(H)+1-b-c \geq 2\hat{\rho}_{a,b}(T)+2-b-c \geq \tilde{\rho}_{a,b}(T)$, a contradiction. □

References

[1] Daniel W. Cranston and Landon Rabern. Conjectures equivalent to the Borodin-Kostochka conjecture that appear weaker. *Arxiv preprint,* `http://arxiv.org/abs/1203.5380`, 2012.

[2] A. Kostochka and M. Yancey. Ore's Conjecture on color-critical graphs is almost true. *Arxiv preprint,* `http://arxiv.org/abs/1209.1050`, 2012.

[3] L. Rabern. *Coloring graphs from almost maximum degree sized palettes.* PhD thesis, Arizona State University, 2013.