The list version of the Borodin-Kostochka Conjecture for graphs with large maximum degree

Ilkyoo Choi*, Hal Kierstead†, Landon Rabern‡, Bruce Reed§

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Abstract

Brooks' Theorem states that for a graph G with maximum degree $\Delta(G)$ at least 3, the chromatic number is at most $\Delta(G)$ when the clique number is at most $\Delta(G)$. Vizing proved that the list chromatic number is also at most $\Delta(G)$ under the same conditions. Borodin and Kostochka conjectured that a graph G with maximum degree at least 9 must be $(\Delta(G)-1)$-colorable when the clique number is at most $\Delta(G)-1$; this was proven for graphs with maximum degree at least 10^{14} by Reed. In this paper, we prove an analogous result for the list chromatic number; namely, we prove that a graph G with $\Delta(G) \geq 10^{20}$ is $(\Delta(G)-1)$-choosable when the clique number is at most $\Delta(G)-1$.

1 Introduction

Let K_n be the complete graph on n vertices and let E_n be the empty graph on n vertices. A *k-clique* is the vertex set of a complete graph on k vertices. For a graph G, let $\Delta(G)$ and $\omega(G)$ denote the maximum degree and the clique number, respectively. For a vertex v, the *neighborhood* of v, denoted $N(v)$, is the set of vertices of G that are adjacent to v, and the *degree* of v, denoted $d(v)$, is $|N(v)|$. For a subgraph H of G and a vertex v, let $N_H(v) = N(v) \cap H$ and $d_H(v) = |N_H(v)|$.

Given a graph G, a *proper coloring* is a function from $V(G)$ to a set of colors such that the two endpoints of each edge receive different colors. A *list assignment* L is a function on $V(G)$ such that $L(v)$ is the set of *available colors* for each $v \in V(G)$. Given a list assignment L, an *L-coloring* (or an *acceptable coloring* when the lists are clear from the context) is a proper coloring f such that $f(v) \in L(v)$ for each vertex v. A graph is *k-choosable* if it has an

*Department of Mathematics, University of Illinois at Urbana-Champaign, ichoi4@illinois.edu. Partially supported by NSF Grant DMS-0965587.

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L-coloring whenever $|L(v)| \geq k$ for each vertex v. The *list chromatic number* of G, denoted $\chi_l(G)$, is the minimum k such that G is k-choosable. A graph is *k-colorable* if it has an L-coloring where all the lists have the same k colors. The *chromatic number* of G, denoted $\chi(G)$, is the minimum k such that G is k-colorable. It follows that for every graph G, it must be the case that $\chi(G) \leq \chi_l(G)$.

It is a trivial fact that a graph G can be properly colored with $\Delta(G)+1$ colors. $\Delta(G)+1$ happens to be the least upper bound on $\omega(G)$. In 1941, Brooks [2] proved the following classical result that connects $\Delta(G)$, $\omega(G)$, and $\chi(G)$.

Theorem 1.1. *[2] For a graph G with $\Delta(G) \geq 3$, if $\omega(G) \leq \Delta(G)$, then $\chi(G) \leq \Delta(G)$.*

The condition on the maximum degree is tight, as the conclusion does not follow for odd cycles. Actually, in 1976, Vizing [9] showed that an analogous result holds for the list chromatic number under the same conditions.

Theorem 1.2. *[2] For a graph G with $\Delta(G) \geq 3$, if $\omega(G) \leq \Delta(G)$, then $\chi_l(G) \leq \Delta(G)$.*

Shortly after, in 1977, Borodin and Kostochka [1] conjectured a similar type of result when the upper bound on the clique number is one less. The condition on the maximum degree is tight, as there exist graphs with maximum degree less then 9 where the conclusion is not true. We state the contrapositive.

Conjecture 1.3. *[1] Every graph G satisfying $\chi(G) \geq \Delta(G) \geq 9$ contains a $K_{\Delta(G)}$.*

There are various partial results regarding this conjecture. Kostochka [5] proved the following result, which guarantees a clique of size almost the maximum degree.

Theorem 1.4. *[5] Every graph G satisfying $\chi(G) \geq \Delta(G)$ contains a $K_{\Delta(G)-28}$.*

A relaxation on the lower bound on the maximum degree condition allows a theorem by Mozhan in his thesis, which ensures a bigger, but still less than the maximum degree, clique.

Theorem 1.5. *Every graph G satisfying $\chi(G) \geq \Delta(G) \geq 31$ contains a $K_{\Delta(G)-3}$.*

By drastically increasing the lower bound on the maximum degree, Reed [8] finally shows the existence of a clique of size equal to the maximum degree using probabilistic arguments.

Theorem 1.6. *[8] Every graph G satisfying $\chi(G) \geq \Delta(G) \geq 10^{14}$ contains a $K_{\Delta(G)}$.*

In this paper, we address Conjecture 1.3 for the list chromatic number. We prove that the conjecture is true even for the list chromatic number when the maximum degree is sufficiently large. The main result in this paper is the following.

Theorem 1.7. *For a graph G with $\Delta(G) \geq 10^{20}$, if $\omega(G) \leq \Delta(G)-1$, then $\chi_l(G) \leq \Delta(G)-1$.*

Throughout this paper, unless specified otherwise, G will be a counterexample to Theorem 1.7 with the minimum number of vertices and maximum degree Δ. Let L be a list assignment where there is no acceptable coloring and each list has size exactly $\Delta - 1$. By the minimality of G, every proper subgraph of G is L-colorable. Moreover, each vertex of G must have degree either Δ or $\Delta - 1$. If there exists a vertex v of degree less than $\Delta - 1$, then we can obtain an L-coloring of G since a L-coloring on $G - v$ exists by the minimality of G, and then extend this L-coloring to G by L-coloring v.

We will prove that such a counterexample G cannot not exist by showing that an L-coloring actually exists when $\Delta \geq 10^{20}$. The proof will come in two steps. We will first discuss some tools in Section 2, and then reveal some properties of the list assignment L in Section 3. The first step (Section 4) is to construct a decomposition of G that will facilitate the second step. The second step (Section 5) is to show that G is actually L-colorable via a probabilistic argument involving the Lovász Local Lemma and Azuma's Inequality.

2 Tools

For a graph H and a function f on $V(H)$, the graph H is *f-choosable* if it has an L-coloring whenever $|L(v)| \geq f(v)$ for each vertex v. For an integer r, a graph is *d_r-choosable* if it is f-choosable where $f(v) = d(v) - r$. Every graph is d_{-1}-choosable.

Cranston and Rabern [3] carry out an in depth study of minimum counterexamples to Conjecture 1.3; one of their tools were studying d_1-choosable graphs. The following lemma is used in this paper. Recall that the *join* of two graphs G_1 and G_2, denoted $G_1 \vee G_2$, is the disjoint union of G_1 and G_2 with all possible edges between $V(G_1)$ and $V(G_2)$.

Lemma 2.1. *[3] If B is a graph with $\omega(B) \leq |B| - 2$, then $K_6 \vee B$ is d_1-choosable.*

G cannot have a d_1-choosable graph as an induced subgraph. Otherwise, if G has a d_1-choosable induced subgraph H, then we can give an L-coloring to $G - V(H)$ by the minimality of G, and then extend the L-coloring onto H. In particular, $K_6 \vee E_3$ cannot be an induced subgraph of G.

Given a graph H, a *matching* of H is a set of edges where no two edges share an endpoint. A matching M *saturates* a set S if every vertex in S is incident to one edge of M. Recall a classical result by Hall [4] that characterizes when a bipartite graph has a matching that saturates one part.

Theorem 2.2 (Hall's Theorem)**.** *[4] Let B be a bipartite graph with parts X and Y. A matching that saturates X exists if and only if $|N(S)| \geq |S|$ for all $S \subseteq X$.*

Let B be a bipartite graph with parts $V(G)$, $L(V(G))$ and each vertex in $v \in V(G)$ is adjacent to only the colors in $L(v)$. A matching in B corresponds to a partial L-coloring of G. A matching that saturates $V(G)$ corresponds to an L-coloring of G.

The next tool is the Lovász Local Lemma. This lemma is a very powerful tool since we can prevent every (bad) event in a certain set from happening by bounding the probability and the number of dependent events of each event.

Theorem 2.3 (Lovász Local Lemma). *Consider a set $\mathcal{E}$ of (bad) events where each E event satisfies the following.*

(i) $Pr(E) \leq p$

(ii) *E is mutually independent to a set of all but at most d other events.*

If $ep(d+1) \leq 1$, then with positive probability, none of the events in $\mathcal{E}$ occur.

The next probabilistic tool is Azuma's Inequality. Azuma's Inequality is a concentration type bound; in other words, Azuma's Inequality shows that with high probability, a random variable is close its mean. The inequality actually says that the probability that a random variable is far from its mean is very small.

Theorem 2.4 (Azuma's Inequality). *Let X be a random variable determined by n trials $T_1, \ldots, T_n$. If for each i and any two possible sequences of outcomes $t_1, \ldots, t_{i-1}, t_i$ and $t_1, \ldots t_{i-1}, t'_i$ the following holds:*

$$|E(X|T_1 = t_1 \ldots, T_i = t_i) - E(X|T_1 = t_1, \ldots, T_i = t'_i)| \leq c_i$$

then $Pr(|X - E(X)| > t) \leq 2e^{-t^2/\left(2\sum c_i^2\right)}$.

Azuma's Inequality typically comes after showing that the expectation of a random variable is high. This way, we can guarantee that a certain structure occurs enough times.

3 Properties of the list assignment L

Recall that G is a counterexample to Theorem 1.7 with the minimum number of vertices. The lemmas in this section will reveal some aspects of the list assignment L of the vertices of cliques of G. In particular, the available colors of vertices in $(\Delta - 1)$-cliques are analyzed. Recall that Lemma 2.1 shows that G cannot have $K_6 \vee B$ as an induced subgraph whenever $\omega(B) \leq |B| - 2$.

Definition 3.1. Given a partial L-coloring f of G, an uncolored vertex v of degree $\Delta + 1 - i$ in G is *safe* if there exists a subset Z of $N(v)$ with $3 - i$ vertices such that for every vertex z in Z, either $f(z) \in f(N(v) - Z)$ or $f(z) \notin L(v)$.

Given a partial L-coloring f on G, at most $\Delta - 2$ colors can appear in the neighborhood of a safe vertex by definition. Since each vertex of G has $\Delta - 1$ available colors, there is always a color in $L(v)$ for a safe vertex v that is not used on $N(v)$, and therefore can be used on v. This will be the general idea of the proofs regarding colorings in this paper: ensure a partial L-coloring of G except the safe vertices, and then extend the partial L-coloring to an L-coloring of G.

We will first show that the lists of all but at most one vertex in a clique of G have many colors in common. This lemma will be used in section 5 when we show that there exists an L-coloring of G. Given a partial L-coloring f on G, let $L_f(v)$ denote the remaining available colors on v; in other words, $L_f(v) = L(v) - \{f(u) : u \in N(v) \text{ and } f(u) \text{ is defined}\}$.

Lemma 3.2. *If C is a clique of G, then there exists $C' \subset C$ such that $|C'| = |C| - 1$ and $|L(x) \cap L(y)| \geq |C| - 3$ for all $x, y \in C'$.*

Proof. Let f be an L-coloring of $G - C$, which exists by the minimality of G. For $v \in C$, since v has at most $\Delta - (|C| - 1)$ neighbors outside C, it follows that $|L_f(v)| \geq |C| - 2$. Since G is a minimum counterexample, a system of distinct representatives for $L_f(v)$ where $v \in C$ does not exist. Thus, by Hall's theorem, there exists a subset F of C such that the union of the $L_f(v)$ for v in F has size less than $|F|$. Since each L_f list has size at least $|C| - 2$, we know that $|F|$ has size at least $|C| - 1$. If $|F|$ has size $|C| - 1$, then every vertex in F has the same L_f list, which is of size $|C| - 2$ so we are done. Otherwise, F is C, and the union of the L_f lists for the vertices in C has at most $|C| - 1$ elements. We can assume there are two vertices x and y with distinct lists as otherwise we are done. Now the union of these L_f lists has size exactly $|C| - 1$ and every vertex in C has at most one color missing from $L_f(x) \cup L_f(y)$. □

Now we show two lemmas that analyze the distribution of colors in the list of available colors on vertices in $(\Delta - 1)$-cliques of G. We first prove a lemma that will be used heavily in the second lemma.

Lemma 3.3. *For a $(\Delta - 1)$-clique C of G and a vertex $w \notin C$ such that $|N(w) \cap C| \geq 5$, if f is a partial L-coloring on $G - C - w$, then $L_f(u) = L_f(v)$ for all $u, v \in N(w) \cap C$.*

Proof. Let $A = N(w) \cap C$. Since w cannot be adjacent to every vertex of C, there must be a vertex x in $C - A$. Let f be an L-coloring of $G - C - w$, which exists by the minimality of G. Since each vertex in A has at most one neighbor not in $C \cup \{w\}$, each L_f list has size at least $\Delta - 2$ for a vertex in A. By similar logic, $|L_f(w)| \geq 4$ and $|L_f(x)| \geq \Delta - 3$.

We will show that the L_f list is the same for all vertices in A. Assume for the sake of contradiction that there exist $u, v \in A$ such that $L_f(u) \neq L_f(v)$. If there exists a color $c \in L_f(w) \cap L_f(x)$, then by using c on both w and x, the L-coloring f on $G - C - w$ can be extended to G by Hall's Theorem, which is a contradiction. Thus, $L_f(w) \cap L_f(x) = \emptyset$, which implies that $|L_f(w) \cup L_f(x)| \geq \Delta + 1$. If $c \in (L_f(w) \cup L_f(x)) - L_f(A)$, then by coloring w and x using c and an arbitrary color, f can be extended to G by Hall's Theorem, which is again a contradiction. This implies that $|L_f(A)| \geq \Delta + 1$. If we cannot extend f to G by coloring w and x arbitrarily from their respective L_f lists, then there must exist a nonempty $T \subseteq A$ such that $|L_f(T)| < |T|$. Since $|L_f(A)| \geq \Delta + 1$, it must be that $|T| = \Delta - 3$ and $|L_f(T)| = \Delta - 4$. By coloring w and x using a color in $(L_f(w) \cup L_f(x)) - L_f(T)$ and another color, f can be extended to G by Hall's Theorem, which is a contradiction. □

Lemma 3.4. *For a $(\Delta - 1)$-clique C of G and a vertex $w \notin C$ such that $|N(w) \cap C| \geq 5$, the following holds:*

(i) each vertex in $N(w) \cap C$ has degree Δ;

(ii) there exists a set S of $\Delta - 2$ colors that are in $L(v)$ for every $v \in N(w) \cap C$;

(iii) each vertex $y \notin C \cup \{w\}$ has at most 4 neighbors in $N(w) \cap C$;

(iv) for each $v \in N(w) \cap C$, the color in $L(v) - S$ appears in the lists of at most 5 vertices in $N(w) \cap C$.

Proof. of (i) and (ii). Let f be an L-coloring of $G - C - w$, which exists by the minimality of G. Assume for the sake of contradiction that there exists a vertex $v \in N(w) \cap C$ with degree $\Delta - 1$, which implies $|L(v)| = |L_f(v)| = \Delta - 1$. By Lemma 3.3, the L_f lists are all the same for vertices in $N(w) \cap C$. Therefore, the size of the L_f lists must all be $\Delta - 1$ for vertices in $N(w) \cap C$. Then, by similar logic to the proof of Lemma 3.3, we can extend f to G, which is a contradiction. This proves both (i) and (ii). □

Proof. of (iii). Assume for the sake of contradiction that there exists a vertex $y \notin C \cup \{w\}$ that has at least 5 neighbors in $N(w) \cap C$. Let $x \in C - N(w)$ and let $z \in C - N(w) - x$ be a vertex not adjacent to y. Such a z must exist since if not, then y and w must be adjacent to every vertex in C except x. Now, x, w, y form an independent set since adding any edge would create a Δ-clique. This implies that G has $E_3 \vee K_6$ as an induced subgraph; this is a contradiction to Lemma 2.1 since $E_3 \vee K_6$ is d_1-choosable.

Let f be an L-coloring of $G - C - w - y$, which exists by the minimality of G. Since y and w each have at least 5 neighbors in C, it follows that $|L_f(y)| \geq 4$ and $|L_f(w)| \geq 4$. By similar logic, $|L_f(x)| \geq \Delta - 3$ and $|L_f(z)| \geq \Delta - 3$. Let $v, u \in N(w) \cap N(y) \cap C$ so that $|L_f(v)| = |L(v)| = \Delta - 1$. Whenever y is L-colored, that partial coloring g on G is an L-coloring on $G - C - w$. Recall that by Lemma 3.3, vertices in $N(y) \cap N(w) \cap C$ must have the same L_g list. In particular, $L_g(v) = L_g(u)$.

Assume $L_f(w) \cap L_f(x) \neq \emptyset$ and $L_f(y) \cap L_f(z) \neq \emptyset$. If $|(L_f(w) \cap L_f(x)) \cup (L_f(y) \cap L_f(z))| \geq 2$, then we can find two different colors c and c' where we can color w, x with c and y, z with c'. We can now color every vertex in $C - N(y) \cap N(w)$ first since each vertex is adjacent to two uncolored vertices (y and w). We can then color the vertices in $N(y) \cap N(w) \cap C$ to complete an L-coloring of G since every vertex in $N(y) \cap N(w) \cap C$ is safe.

Now assume $L_f(w) \cap L_f(x) = L_f(y) \cap L_f(z) = \{c\}$ for some color c. In this case, $|(L_f(w) - L_f(x)) \cup (L_f(x) - L_f(w)) - L_f(v)| \geq 1$, which implies that there is a color c' in either $L_f(w)$ or $L_f(x)$ that is not in $L_f(v)$. So now we can color y and z with c and color either w or x with c' first. We can color uncolored vertices in $C - N(y)$ since each vertex is adjacent to two uncolored vertices (u and v) We can then finish off the L-coloring by coloring each vertex in $N(y)$ since each vertex in $N(y) \cap C$ is safe.

The remaining case is when $L_f(w) \cap L_f(x) = \emptyset$ or $L_f(y) \cap L_f(z) = \emptyset$. Without loss of generality, assume $L_f(w) \cap L_f(x) = \emptyset$, which implies $|L_f(w) \cup L_f(x) - L_f(v)| \geq \Delta + 1 - (\Delta - 1) = 2$. If $L_f(y) \cap L_f(z) = \emptyset$ as well, then by the same logic $|L_f(y) \cup L_f(z) - L_f(v)| \geq 2$. Now, color y or z with a color $c' \in L_f(y) \cup L_f(z) - L_f(v)$, and color w or x with a color c that is not in $L_f(v) \cup \{c'\}$. Such color c exists since $|L_f(w) \cup L_f(x) - L_f(v)| \geq 2$.

If $L_f(y) \cap L_f(z) \neq \emptyset$, then there exists a color c' in $L_f(y) \cap L_f(z)$. Now color y and z with c', and color w or x with a color c that is not in $L_f(v) \cup \{c'\}$. Such color c exists since $|L_f(w) \cup L_f(x) - L_f(v)| \geq 2$.

Either way, we can color every vertex in $C - u - v$ since each vertex is adjacent to two uncolored vertices (u and v). We can then color u since both u is adjacent to an uncolored vertex (v) and either there exists a color in $N(u)$ that is not in the list of u or there exists a repeated color in $N(u)$. Now since v is safe, we can L-color v, and we extended a partial L-coloring f of $G - C - w - y$ to G. This completes the proof of (*iii*). $\square$

Proof. of (*iv*). Assume for the sake of contradiction that a color in $L(v) - S$ appears in a set P of at least 6 vertices. Consider the set Q of neighbors of vertices in P that are not in $C \cup \{w\}$. Note that the neighbors of vertices in Q that are in P partition P since each vertex in P has exactly one neighbor not in $C \cup \{w\}$. By (*iii*), since a vertex outside of $C \cup \{w\}$ has at most 4 neighbors in $N(w) \cap C$, there must be at least 2 vertices in Q. Also, Q must be an independent set. Otherwise, if there is an edge with two endpoints in Q, then the endpoints of this edge will receive different colors in an L-coloring f of $G - C - w$, which exists by the minimality of G. Now, the vertices in P could not have had the same L_f list, which is a contradiction to Lemma 3.3.

For any edge e with both endpoints in Q, the graph $G - C - w + e$ has fewer vertices than G, maximum degree Δ, and clique number at most $\Delta - 1$. By the minimality of G, there must be an L-coloring of $G - C - w + e$, which is a contradiction since the two endpoints of e will receive different colors. Thus, adding an edge e with endpoints in Q must create a Δ-clique in $G - C - w$.

If $v \in Q$ has $d_P(v) \geq 3$, this is impossible since v cannot have $\Delta - 2$ neighbors outside P. This implies that $|Q| \geq 3$ and $d_P(v) \leq 2$ for all $v \in Q$. Note that three vertices $x, y, z \in Q$ must have at least $\Delta - 3$ common neighbors not in $C \cup \{w\}$. These common neighbors and x, y, z induce a copy of $E_3 \vee K_6$, which is d_1-choosable, which is a contradiction. $\square$

Definition 3.5. Let C be a $(\Delta - 1)$-clique of G and let a vertex $w \notin C$ be such that $|N(w) \cap C| \geq 6$. The *core* of $N(w) \cap C$ is the set of $\Delta - 2$ colors S that are in $L(v)$ for every $v \in N(w) \cap C$. For a vertex $v \in N(w) \cap C$, the *special color* of v is the color in $L(v) - S$, and the *external neighbor* of v is the one vertex that is adjacent to v that is not in $C \cup \{w\}$.

4 A Decomposition of G

In this section, we will construct a decomposition of G that will be very helpful in the next section. Here is a definition and a couple lemmas from [3].

Definition 4.1. Given a graph H and a list assignment L on H, let the *pot* of H, denoted $Pot(L)$, be $\bigcup_{v \in V(H)} L(v)$.

Lemma 4.2 (Small Pot Lemma)**.** *[3] Let H be a graph and $f : V(H) \to \mathbb{N}$ with $f(v) < |H|$ for all $v \in V(H)$. If H is not f-choosable, then H has a list assignment L where $|L(v)| = f(v)$ for each vertex v such that $|Pot(L)| < |H|$.*

Lemma 4.3. *For any graph B with $\delta(B) \geq \frac{|B|}{2} + 1$ and $\omega(B) \leq |B| - 2$, the graph $K_1 \vee B$ is d_1-choosable.*

Proof. By the Small Pot Lemma, it suffices to prove that all list assignments L on $K_1 \vee B$ with $|L(v)| = f(v) - 1$ for each vertex v with $|Pot(L)| \leq |B|$ are L-colorable. Let L be such a list assignment on $K_1 \vee B$.

First, suppose B contains disjoint nonadjacent pairs $\{x_1, y_1\}$ and $\{x_2, y_2\}$. Since $|L(x_i)| + |L(y_i)| \geq |B| + 2$, we have $|L(x_i) \cap L(y_i)| \geq 2$ for each i. Color x_1 and y_1 with $c_1 \in L(x_1) \cap L(y_1)$ and color x_2 and y_2 with $c_2 \in L(x_2) \cap L(y_2) - c_1$. By the minimum degree condition on B, each component of $B - \{x_1, y_1, x_2, y_2\}$ has a vertex joined to $\{x_1, y_1\}$ or $\{x_2, y_2\}$. Hence we can complete the coloring to all of B and then to the K_1. Thus L is good.

So, we may assume there are no disjoint nonadjacent pairs. Now let K be a maximum clique in B. Then we know $|K| \leq |B| - 2$ so we can pick $x, y \in B - K$. The only possibility is that there is $z \in K$ such that both x and y are joined to $K - z$. Since K is maximum x is not adjacent to y and hence B is a $K_{|B|-3} \vee E_3$. By Lemma 2.1, $|B| \leq 4$. Since $d_B(y) = |B| - 3$, this violates our minimum degree condition on B. □

Now we actually construct a decomposition using a definition on page 158 of [6].

Definition 4.4. A vertex v of G is *d-sparse* if the subgraph induced by its neighborhood contains fewer than $\binom{\Delta}{2} - d\Delta$ edges. Otherwise, v is *d-dense*.

Lemma 4.5. *We can partition $V(G)$ into $S, D_1, \ldots, D_l$ so that*

(i) each vertex of S is d-sparse;

(ii) each D_i contains a vertex w_i such that $D_i - w_i$ is a clique of size at least $\Delta - 8\Delta^{9/10} + 1$;

(iii) no vertex outside of D_i has more than $\frac{3\Delta}{4}$ neighbors in D_i and w_i has at least $\frac{3\Delta}{4}$ neighbors in D_i.

Proof. Let $C_1, \ldots, C_s$ be the maximal cliques in G with at least $\frac{3\Delta}{4} + 1$ vertices. Suppose $|C_i| \leq |C_j|$ and $C_i \cap C_j \neq \emptyset$. Then $|C_i \cap C_j| \geq |C_i| + |C_j| - (\Delta + 1) \geq 6$. It follows from Lemma 2.1 that $|C_i - C_j| \leq 1$. Now suppose C_i intersects C_j and C_k. By the above, $|C_i \cap C_j| \geq \frac{3\Delta}{4}$. Hence $|C_i \cap C_j \cap C_k| \geq \frac{\Delta}{2} \geq 6$. By Lemma 2.1 we see that $\omega(G[C_i \cup C_j \cup C_k]) \geq |C_i \cup C_j \cup C_k| - 1$ which is impossible since each of C_i, C_j, C_k are maximal. Hence $\bigcup_{i \in [s]} C_i$ can be partitioned into sets $F_1, \ldots, F_r$ so that each F_j is either one of the C_i or one of the C_i and an extra vertex w_i with at least $\frac{3\Delta}{4}$ neighbors in C_i.

Put $d = \Delta^{9/10}$ and let $D_1, \ldots, D_l$ be all the F_j such that some vertex in F_j is d-dense and let S be $V(G) - \bigcup_{i \in [l]} D_i$. Then (iii) follows by construction. It remains to check (i) and (ii).

We show that if $v \in V(G)$ is d-dense, then it is in a $(\Delta - 8d + 2)$-clique. Since we know that any $v \in S$ is either in no $(\frac{3\Delta}{4} + 1)$-clique (and hence in no $(\Delta - 8d + 2)$-clique) or is d-sparse, (i) follows. Also, since each F_j contains a d-dense vertex, (ii) follows as well.

So, suppose $v \in V(G)$ is d-dense but in no $(\Delta - 8d + 2)$-clique. Then applying Lemma 4.3 repeatedly, we get a sequence $y_1, \ldots, y_{8d} \in N(v)$ such that

$$|N(y_i) \cap (N(x) - \{y_1, \ldots, y_{i-1}\})| \leq \frac{1}{2}(\Delta + 1 - i).$$

Hence the number of non-edges in v's neighborhood is at least

$$\frac{1}{2}\sum_{i=1}^{8d}(\Delta - i) > d\Delta.$$

$\square$

Definition 4.6. For convenience, let $K_i = \begin{cases} C_i & \text{if } D_i = C_i \\ C_i \cap N(w_i) & \text{if } D_i = C_i \cup \{w_i\} \end{cases}$

Definition 4.7. Also for convenience, partition the set of C_i into the following three sets (if some C_i can be either (ii) or (iii), then just choose an arbitrary one):

(i) $\mathcal{P}_1$: the set of C_i such that $|C_i| \le \Delta - 2$;

(ii) $\mathcal{P}_2$: the set of C_i such that $|C_i| = \Delta - 1$ and every vertex outside C_i has at most $\Delta^{0.29}$ neighbors in C_i;

(iii) $\mathcal{P}_3$: the set of C_i such that $|C_i| = \Delta - 1$ and some vertex outside C_i has more than $\Delta^{0.29}$ neighbors inside C_i.

Now we prove a structural lemma that will be crucial in the following sections. We use a lemma from [3] to prove the lemma needed.

Lemma 4.8. *[3] Let H be a d_0-choosable graph such that $F := K_1 \vee H$ is not d_1-choosable and let L be a bad d_1-assignment on F minimizing $|Pot(L)|$. If some nonadjacent pair in H has intersecting lists, then $|Pot(L)| \le |H| - 1$.*

Lemma 4.9. *Each $v \in C_i$ of G has at most one neighbor outside of C_i with more than 4 neighbors in C_i, and no such neighbor if v has degree $\Delta - 1$.*

Proof. Suppose there exists $v \in C_i$ with two neighbors $w_1, w_2 \in V(G) - C_i$, each with 5 or more neighbors in C_i. Put $Q := G[\{w_1, w_2\} \cup C_i - v]$, so that v is joined to Q and hence $K_1 \vee Q$ is an induced subgraph of G. We will show that $K_1 \vee Q$ must be d_1-choosable. Note that Q is d_0-choosable since it contains a K_4 without one edge. Let L be a bad d_1-assignment on $K_1 \vee Q$ minimizing $|Pot(L)|$.

First, suppose there are different $z_1, z_2 \in C_i$ such that $\{w_1, z_1\}$ and $\{w_2, z_2\}$ are independent. By the Small Pot Lemma 4.2, $|Pot(L)| \le |Q|$. Thus $|L(w_1)| + |L(z_1)| \ge 4 + |Q| - 3 > |Pot(L)|$ and therefore w_1 and w_2 have intersecting lists. Applying Lemma 4.8 shows that $|Pot(L)| \le |Q| - 1$.

Now $|L(w_j)| + |L(z_j)| \ge 4 + |Q| - 3 \ge |Pot(L)| + 2$. Hence $|L(w_j) \cap L(z_j)| \ge 2$. Pick $x \in N(w_1) \cap \{C_i - v - z_2\}$. Then after coloring each pair $\{w_1, z_1\}$ and $\{w_2, z_2\}$ with a different color, we can finish the coloring because we saved a color for x and two colors for v.

By maximality of C_i, neither w_1 nor w_2 can be adjacent to all of C_i hence it must be the case that there is $y \in C_i$ such that w_1 and w_2 are joined to $C_i - y$. If w_1 and w_2 aren't adjacent, then G contains $K_6 \vee E_3$ contradicting Lemma 2.1. Hence C_i intersects the larger clique $\{w_1, w_2\} \cup C_i - \{y\}$, this is impossible by the definition of C_i.

When v is low, an argument similar to the above shows that there can be no z_1 in C_i with $\{w_1, z_1\}$ independent, and hence $C_i \cup \{w_1\}$ is a clique contradicting maximality of C_i. $\square$

5 An L-Coloring of G

Now we will show that there exists an L-coloring of G by randomly choosing a color among the available colors for each vertex in a subgraph of G. If there exists an edge where both endpoints get the same color, then we erase both colors to guarantee we have a partial L-coloring on G. We will show that with positive probability, this partial L-coloring is good in the sense that we can extend the partial L-coloring greedily to the entire graph G.

The random coloring procedure we explained is called the *naive coloring procedure*, which is defined below. The subgraph of G that we will apply the naive coloring procedure to $G - \bigcup_{C_i \in \mathcal{P}_3} C_i$.

Definition 5.1. The *naive coloring procedure* is the following:

(i) For each vertex, choose a color in its list uniformly at random and use it on the vertex.

(ii) Uncolor any vertex that receives the same color as one of its neighbors.

Using the Lovász Local Lemma, we will show that with positive probability, the naive coloring procedure will produce a coloring in which none of the bad events happen. The bad events are defined in a way that if none of them happen, then we can extend the partial L-coloring to an L-coloring of G in a greedy fashion.

For $C_i \in \mathcal{P}_3$, let w_i be a vertex outside of C_i with the maximum number of neighbors in C_i and let K'_i be the neighbors of w_i that are in C_i.

Definition 5.2. The *bad events* are the following events:

(i) For $C_i \in \mathcal{P}_1$, let $\mathcal{E}_{1,i}$ be the event that K_i does not contain two uncolored safe vertices.

(ii) For $C_i \in \mathcal{P}_2$, let $\mathcal{E}_{2,i}$ be the event that K_i does not contain two uncolored safe vertices.

(iii) For $C_i \in \mathcal{P}_3$, let $\mathcal{E}_{3,i}$ be the event that K'_i does not contain two uncolored safe vertices.

(iv) For a sparse vertex v, let $\mathcal{S}_v$ be the event that v is not safe.

To apply the Lovász Local Lemma, we need to bound the dependencies among the events and bound the probability of each event. A bad event associated with C_i depends only on the colors of vertices that are distance at most 2 away from C_i. This implies that if two cliques have a path of length at most 4 connecting them, then the associated events are not (mutually) independent. This implies that a bad event is mutually independent to all but at most Δ^5 events since each clique has less than Δ vertices. The task of proving that the probability of each (bad) event is at most Δ^{-6} will be done in the following subsections.

Assuming none of the bad events happen, we will obtain an L-coloring of G in the following way: Apply the naive coloring procedure to $G - \bigcup_{C_i \in \mathcal{P}_3} C_i$ to obtain a partial L-coloring of G. First color all the uncolored vertices in the dense sets in $\mathcal{P}_1 \cup \mathcal{P}_2$ that are not the two uncolored safe vertices. This is possible since each vertex we are coloring in this phase is adjacent to the two uncolored (safe) vertices. Now, finish the coloring on $G - \bigcup_{C_i \in \mathcal{P}_3} C_i$

by coloring the sparse vertices, which are all safe, and the two uncolored safe vertices in each dense set in $\mathcal{P}_1 \cup \mathcal{P}_2$. By following the procedure in subsection 5.3, two uncolored safe vertices exist in each dense set in $\mathcal{P}_3$. At this point, all the uncolored vertices are either safe or adjacent to at least two uncolored safe vertices. Now, we can extend the partial L-coloring of G to an L-coloring of G by coloring the uncolored vertices that are adjacent to at least two uncolored safe vertices first, and then coloring the remaining uncolored vertices, which are all safe.

5.1 $Pr(\mathcal{E}_{1,i}) \leq \Delta^{-6}$

Let C'_i be a subset of C_i with one less vertex where every two vertices in C'_i have at least $|C_i| - 3$ colors in common in their lists; such a C'_i exists by Lemma 3.2. Note that $|C'_i| = |C_i| - 1 \geq \Delta - 8\Delta^{9/10} \geq 0.92\Delta$ for $\Delta \geq 10^{20}$. Let $\mathcal{T}_i$ be vertices in a maximum set of disjoint P_3 where the center vertex is in C'_i and each of the other two vertices is not in C'_i and has at most 4 neighbors in C'_i.

Claim 5.3. *There are at least* 0.1314Δ *such* P_3.

Proof. Consider a maximal set of P_3. Let A be the central vertices and let B be the endpoints of these P_3. Each vertex in B has at most 3 neighbors in $C'_i - A$ and by Lemma 4.9 and maximality, each vertex in $C'_i - A$ has at most 2 neighbors in $G - C'_i - B$. Thus, $6|A| = 3|B| \geq ||C'_i - A, B|| \geq |C'_i| - |A|$. Hence, $|A| \geq \frac{|C'_i|}{7} \geq \frac{0.92\Delta}{7} \geq 0.13142\Delta$. □

Consider a set T_i of 0.1314Δ such P_3. For some fixed P_3, we want to bound the probability that the center vertex c is uncolored and safe, and the colors used on the two end vertices, a and b, are used on none of the rest of T_i. To do so, we distinguish three cases.

Case 1. When $|L(a) \cap L(c)| < \frac{2}{3}\Delta$ and $|L(b) \cap L(c)| \geq \frac{2}{3}\Delta$.

For $\alpha \in L(a) - L(c)$, $y \in C'_i - T_i - N(b)$, $\beta \in L(b) \cap L(y)$, $z \in C'_i - T_i$, and $\gamma \in L(c) \cap L(z)$, where α, β, γ are all different and y, z, c are all different, let $A_{\alpha,\beta,\gamma,y,z}$ be the event that all of the following holds:

(i) α is used on a and none of the rest of $N(a) \cup T_i$;

(ii) β is used on b and y and none of the rest of $N(b) \cup N(y) \cup T_i$;

(iii) γ is used on c and z and none of the rest of T_i.

Then for $\Delta \geq 10^{20}$, by calculation 6.1,

$$\begin{aligned} Pr(A_{\alpha,\beta,\gamma,y,z}) &\geq (\Delta - 1)^{-5}\left(1 - \frac{1}{\Delta - 1}\right)^{|T_i \cup N(a)| + |T_i \cup N(b) \cup N(y)| + |T_i|} \\ &\geq \Delta^{-5}\left(1 - \frac{1}{\Delta - 1}\right)^{3.3942\Delta} \geq \Delta^{-5}e^{-3.3943} \end{aligned}$$

The $A_{\alpha,\beta,\gamma,y,z}$ are disjoint for different sets of indices. Since

$$|L(a) - L(c)| \geq \frac{\Delta}{3} - 1 \geq 0.3332\Delta$$

, we have 0.3332Δ choices for α. For y, we have at least

$$|C_i'| - |T_i \cap C_i'| - |N(b) \cap C_i'| - 1 \geq 0.92\Delta - 0.1314\Delta - 5 \geq 0.7885\Delta$$

choices. For each y, we have

$$0.92\Delta - 2 + \frac{2}{3}\Delta - (\Delta - 1) - 1 = 0.92\Delta - \frac{\Delta}{3} - 2 \geq 0.5866\Delta$$

choices for β since $|L(y) \cap L(c)| \geq |C_i| - 3$. There are

$$|C_i'| - |T_i| - 2 \geq 0.92\Delta - 0.1314\Delta - 2 \geq 0.7885\Delta$$

choices for z. Since $|L(z) \cap L(c)| \geq |C_i| - 3$, there are $0.92\Delta - 4 \geq 0.9199\Delta$ choices for γ.

Thus, the probability that $A_{\alpha,\beta,\gamma,y,z}$ holds for some choice of indices is at least

$$\Delta^{-5}e^{-3.3943} \cdot 0.3332\Delta \cdot (0.7885\Delta)^2 \cdot 0.5866\Delta \cdot 0.9199\Delta \geq 0.00375$$

Case 2. When $|L(a) \cap L(c)| < \frac{2}{3}\Delta$ and $|L(b) \cap L(c)| < \frac{2}{3}\Delta$.

For $\alpha \in L(a) - L(c)$, $\beta \in L(b) - L(c)$, $z \in C_i' - T_i$, and $\gamma \in L(c) \cap L(z)$, where α, β, γ are all different and $z \neq c$, let $A_{\alpha,\beta,\gamma,z}$ be the event that all of the following holds:

(*i*) α is used on a and none of the rest of $N(a) \cup T_i$;

(*ii*) β is used on b and none of the rest of $N(b) \cup T_i$;

(*iii*) γ is used on c and z and none of the rest of T_i.

Then for $\Delta \geq 10^{20}$, by calculation 6.1,

$$\begin{aligned} Pr(A_{\alpha,\beta,\gamma,z}) &\geq (\Delta - 1)^{-4} \left(1 - \frac{1}{\Delta - 1}\right)^{|T_i \cup N(a)| + |T_i \cup N(b)| + |T_i|} \\ &\geq \Delta^{-4} \left(1 - \frac{1}{\Delta - 1}\right)^{2.3942\Delta} \geq \Delta^{-4}e^{-2.3943} \end{aligned}$$

The $A_{\alpha,\beta,\gamma,z}$ are disjoint for different sets of indices. Similarly to Case 1, there are at least 0.3332Δ choices for α, at least 0.3332Δ choices for β, and at least 0.9199Δ choices for γ for each of the at least 0.7885Δ choices for z.

Thus, the probability that $A_{\alpha,\beta,\gamma,z}$ holds for some choice of indices is at least

$$\Delta^{-4}e^{-2.3943} \cdot (0.3332\Delta)^2 \cdot 0.7885\Delta \cdot 0.9199\Delta \geq 0.00734$$

Case 3. When $|L(a) \cap L(c)| \geq \frac{2}{3}\Delta$ and $|L(b) \cap L(c)| \geq \frac{2}{3}\Delta$.

For $x \in C_i' - T_i - N(a)$, $\alpha \in L(a) \cap L(c)$, $y \in C_i' - T_i - N(b)$, $\beta \in L(b) \cap L(y)$, $z \in C_i' - T_i$, and $\gamma \in L(c) \cap L(z)$, where α, β, γ are all different and x, y, z, c are all different, let $A_{\alpha,\beta,\gamma,x,y,z}$ be the event that all of the following hold:

(*i*) α is used on a and x and none of the rest of $N(a) \cup N(x) \cup T_i$;

(*ii*) β is used on b and y none of the rest of $N(b) \cup N(y) \cup T_i$;

(*iii*) γ is used on c and z and none of the rest of T_i.

Then for $\Delta \geq 10^{20}$, by calculation 6.1,

$$\begin{aligned} Pr(A_{\alpha,\beta,\gamma,x,y,z}) &\geq (\Delta-1)^{-6}\left(1-\frac{1}{\Delta-1}\right)^{|T_i \cup N(a) \cup N(x)|+|T_i \cup N(b) \cup N(y)|+|T_i|} \\ &\geq \Delta^{-6}\left(1-\frac{1}{\Delta-1}\right)^{4.3942\Delta} \geq \Delta^{-6}e^{-4.3943} \end{aligned}$$

The $A_{\alpha,\beta,\gamma,x,y,z}$ are disjoint for different sets of indices. Similarly to Case 1, there are at least 0.5866Δ choices for α for each of the at least 0.7885Δ choices for x, at least 0.5866Δ choices for β for each of the at least 0.7885Δ choices for y, and at least 0.9199Δ choices for γ for each of the at least 0.7885Δ choices for z.

Thus, the probability that $A_{\alpha,\beta,\gamma,x,y,z}$ holds for some choice of indices is at least

$$\Delta^{-6}e^{-4.3943} \cdot (0.7885\Delta)^3 \cdot (0.5866\Delta)^2 \cdot 0.9199\Delta \geq 0.00191$$

Since we have 0.1314Δ triples, the expected number of uncolored safe vertices X_i is at least $0.00191 \cdot 0.1314\Delta \geq 2.5 \cdot 10^{-4}\Delta$.

Now we use Azuma's Inequality to show that the probability that X_i deviates from the expected value is at most Δ^{-6}. Let the conditional expected value of X_i change by at most c_v when changing the color of v.

If $v \in T_i \cup C'_i$, then $c_v \leq 2$ since changing the color on v affects X_i by at most 2 for any given assignment of colors to the remaining vertices. Thus, the sum of the c_v^2 is at most $4|T_i \cup C'_i| \leq 4(0.1314\Delta + \Delta - 2) \leq 8\Delta$.

If $v \in V(G) - T_i - C'_i$, then changing the color of v from α to β will only affect X_i if some neighbor of v that is in $T_i \cup C'_i$ receives either α or β. This occurs with probability at most $\frac{2d_v}{\Delta-1}$, where d_v is the number of neighbors of v that are in $T_i \cup C'_i$. Therefore, by changing the color of v, the conditional expectation of X_i changes by at most $c_v = \frac{4d_v}{\Delta-1}$. Since the d_v sum is at most $\Delta(\Delta - 2 + 0.2628\Delta) \leq 2\Delta^2$, the sum of these c_v is at most $\frac{4}{\Delta}2\Delta^2 = 8\Delta$. As each c_v is at most 4, we see that the sum of c_v^2 is at most 32Δ.

Hence, the sum of all the c_v is at most 40Δ. Applying Azuma's Inequality yields $Pr(\mathcal{E}_{1,i}) \leq \Delta^{-6}$ for sufficiently large Δ. See calculation 6.2.

5.2 $Pr(\mathcal{E}_{2,i}) \leq \Delta^{-6}$

This subsection is similar to the previous subsection, except a linear (in terms of Δ) number of P_3 is not guaranteed. Let C'_i be a subset of C_i with one less vertex where every two vertices in C'_i have at least $|C_i| - 3$ colors in common in their lists; such a C'_i exists by Lemma 3.2. Let $\mathcal{T}_i$ be vertices in a maximum set of disjoint P_3 where the center vertex is in C'_i and each of the other two vertices is not in C_i' and has at most 4 neighbors in C_i'. Since at most one of the two endpoints can have more than 4 neighbors in C_i by Lemma 4.9, it follows that $|\mathcal{T}_i| \geq \frac{\Delta-1}{\Delta^{0.29}+4}$. By Calculation 6.3, the number of P_3 is at least $0.9999\Delta^{0.71}$. Consider a set T_i of $0.9999\Delta^{0.71}$ P_3 that are in $\mathcal{T}_i$.

For some such fixed path, we want to bound the probability that the center vertex c is uncolored and safe, and the colors used on the two end vertices, a and b, are used on none of the rest of T_i. To do so, we distinguish three cases.

Case 1. When $|L(a) \cap L(c)| < \frac{2}{3}\Delta$ and $|L(b) \cap L(c)| \geq \frac{2}{3}\Delta$.

For $\alpha \in L(a) - L(c)$, $y \in C_i' - T_i - N(b)$, $\beta \in L(b) \cap L(y)$, $z \in C_i' - T_i$, and $\gamma \in L(c) \cap L(z)$, where α, β, γ are all different and y, z, c are all different, let $A_{\alpha,\beta,\gamma,y,z}$ be the event that all of the following hold:

- *(i)* α is used on a and none of the rest of $N(a) \cup T_i$;
- *(ii)* β is used on b and y and none of the rest of $N(b) \cup N(y) \cup T_i$;
- *(iii)* γ is used on c and z and none of the rest of T_i.

Then for $\Delta \geq 10^{20}$, by calculation 6.4,

$$
\begin{aligned}
Pr(A_{\alpha,\beta,\gamma,y,z}) &\geq (\Delta-1)^{-5}\left(1-\frac{1}{\Delta-1}\right)^{|T_i \cup N(a)|+|T_i \cup N(b) \cup N(y)|+|T_i|} \\
&\geq \Delta^{-5}\left(1-\frac{1}{\Delta-1}\right)^{3\Delta+2.9998\Delta^{0.71}} \geq \Delta^{-5}e^{-3.1}
\end{aligned}
$$

The $A_{\alpha,\beta,\gamma,y,z}$ are disjoint for different sets of indices. Since

$$|L(a) - L(c)| \geq \frac{\Delta}{3} - 1 \geq 0.3332\Delta$$

, we have 0.3332Δ choices for α. For y, we have at least

$$|C_i'| - |T_i \cap C_i'| - |N(b) \cap C_i'| \geq 0.92\Delta - 0.9999\Delta^{0.71} - 5 \geq 0.9199\Delta$$

choices. For each y, we have about

$$\frac{2}{3}\Delta + 0.92\Delta + 1 - 3 - (\Delta - 1) - 1 = 0.92\Delta - \frac{\Delta}{3} - 2 \geq 0.5866\Delta$$

choices for β. There are

$$|C_i'| - |T_i| - 2 \geq 0.92\Delta - 0.9999\Delta^{0.71} - 2 \geq 0.9199\Delta$$

choices for z. Since $|L(z) \cap L(c)| \geq |C_i| - 3$, there are $0.92\Delta - 4 \geq 0.9199\Delta$ choices for γ.

Thus, the probability that $A_{\alpha,\beta,\gamma,y,z}$ holds for some choice of indices is at least

$$\Delta^{-5}e^{-3.1}\cdot 0.3332\Delta\cdot(0.9199\Delta)^3\cdot 0.5866\Delta \geq 0.00685$$

Case 2. When $|L(a)\cap L(c)| < \frac{2}{3}\Delta$ and $|L(b)\cap L(c)| < \frac{2}{3}\Delta$.

For $\alpha \in L(a)-L(c)$, $\beta \in L(b)-L(c)$, $z \in C_i'-T_i$, and $\gamma \in L(c)\cap L(z)$, where α,β,γ are all different and $c \neq z$, let $A_{\alpha,\beta,\gamma,z}$ be the event that all of the following hold:

(i) α is used on a and none of the rest of $N(a)\cup T_i$;

(ii) β is used on b and none of the rest of $N(b)\cup T_i$;

(iii) γ is used on c and z and none of the rest of T_i.

Then for $\Delta \geq 10^{20}$, by calculation 6.4,

$$\begin{aligned} Pr(A_{\alpha,\beta,\gamma,z}) &\geq (\Delta-1)^{-4}\left(1-\frac{1}{\Delta-1}\right)^{|T_i\cup N(a)|+|T_i\cup N(b)|+|T_i|} \\ &\geq \Delta^{-4}\left(1-\frac{1}{\Delta-1}\right)^{2\Delta+2.9998\Delta^{0.71}} \geq \Delta^{-4}e^{-2.1} \end{aligned}$$

The $A_{\alpha,\beta,\gamma,z}$ are disjoint for different sets of indices. Similarly to Case 1, there are at least 0.3332Δ choices for α, at least 0.3332Δ choices for β, and at least 0.9199Δ choices for γ for each of the at least 0.9199Δ choices for z.

Thus, the probability that $A_{\alpha,\beta,\gamma,z}$ holds for some choice of indices is at least

$$\Delta^{-4}e^{-2.1}\cdot(0.3332\Delta)^2\cdot(0.9199\Delta)^2 \geq 0.01150$$

Case 3. When $|L(a)\cap L(c)| \geq \frac{2}{3}\Delta$ and $|L(b)\cap L(c)| \geq \frac{2}{3}\Delta$.

For $x \in C_i'-T_i-N(a)$, $\alpha \in L(a)\cap L(c)$, $y \in C_i'-T_i-N(b)$, $\beta \in L(b)\cap L(y)$, $z \in C_i'-T_i$, and $\gamma \in L(c)\cap L(z)$, where α,β,γ are all different and c,z,y,x are all different, let $A_{\alpha,\beta,\gamma,x,y,z}$ be the event that all of the following hold:

(i) α is used on a and x and none of the rest of $N(a)\cup N(x)\cup T_i$;

(ii) β is used on b and y and none of the rest of $N(b)\cup N(y)\cup T_i$;

(iii) γ is used on c and z and none of the rest of T_i.

Then for $\Delta \geq 10^{20}$, by calculation 6.4,

$$\begin{aligned} Pr(A_{\alpha,\beta,\gamma,x,y,z}) &\geq (\Delta-1)^{-6}\left(1-\frac{1}{\Delta-1}\right)^{|T_i\cup N(a)\cup N(x)|+|T_i\cup N(b)\cup N(y)|+|T_i|} \\ &\geq \Delta^{-6}\left(1-\frac{1}{\Delta-1}\right)^{4\Delta+2.9998\Delta^{0.71}} \geq \Delta^{-6}e^{-4.1} \end{aligned}$$

The $A_{\alpha,\beta,\gamma,x,y,z}$ are disjoint for different sets of indices. Similarly to Case 1, there are at least 0.5866Δ choices for α for each of the at least 0.9199Δ choices for x, at least 0.5866Δ choices for β for each of the at least 0.9199Δ choices for y, and at least 0.9199Δ choices for γ for each of the at least 0.9199Δ choices for z.

Thus, the probability that $A_{\alpha,\beta,\gamma,x,y,z}$ holds for some choice of indices is at least

$$\Delta^{-6}e^{-4.1} \cdot (0.9199\Delta)^4 \cdot (0.5866\Delta)^2 \geq 0.00408$$

Since we have $0.9999\Delta^{0.71}$ triples, the expected number of uncolored safe vertices X_i is at least $4.0 \cdot 10^{-3}\Delta^{0.71}$.

Now we use Azuma's Inequality to show that the probability that X_i deviates from the expected value is at most Δ^{-6}. Let the conditional expected value of X_i change by at most c_v when changing the color of v.

If $v \in T_i \cup C_i'$, then $c_v \leq 2$ since changing the color on v affects X_i by at most 2 for any given assignment of colors to the remaining vertices. Thus, the sum of the c_v^2 is at most $4|T_i \cup C_i'| \leq 4(\Delta - 2 + 0.9999\Delta^{0.71}) \leq 4.001\Delta$.

If $v \in V(G) - T_i - C_i'$, then changing the color of v from α to β will only affect X_i if some neighbor of v that is in $T_i \cup C_i'$ receives either α or β. This occurs with probability at most $\frac{2d_v}{\Delta-1}$, where d_v is the number of neighbors of v that are in $T_i \cup C_i'$. Therefore, by changing the color of v, the conditional expectation of X_i changes by at most $c_v = \frac{4d_v}{\Delta-1}$. Since the d_v sum is at most $2(\Delta - 1 - 0.9999\Delta^{0.71}) + (\Delta - 1)0.9999\Delta^{0.71} \leq \Delta^{1.71}$ (see calculation 6.5), the sum of these c_v is at most $\frac{4}{\Delta}\Delta^{1.71}$. As each c_v is at most 4, we see that the sum of c_v^2 is at most $4\Delta^{0.71}$.

Hence, the sum of all the c_v is at most $4.001\Delta + 4\Delta^{0.71} \leq 4.01\Delta$ for $\Delta \geq 10^{20}$. Applying Azuma's Inequality yields $Pr(\mathcal{E}_{2,i}) \leq \Delta^{-6}$ for sufficiently large Δ. See Calculation 6.6.

5.3 $Pr(\mathcal{E}_{3,i}) \leq \Delta^{-6}$

Recall that a C_i corresponding to this case has a vertex w_i outside of C_i that has at least $\Delta^{0.29}$ neighbors inside C_i and $K_i' = N(w_i) \cap C_i$.

Now uncolor each w_i that is colored. If there exist two vertices in $x, y \in K_i'$ with different special colors such that the external neighbors of x, y are both in $\bigcup_{C_i \in \mathcal{P}_3}(C_i - K_i')$, then color x, y with their special colors; this is possible since none of the neighbors of x, y are colored yet. Note that such K_i' contains at least two uncolored safe vertices, namely, the vertices that do not contain the special colors of x, y in their lists.

Let $\mathcal{K}$ be the set of remaining K_i'. Let T_i be a maximum set of vertices in $K_i' \in \mathcal{K}$ such that every vertex in T_i has a different special color, and each vertex in T_i has its external neighbor in $G - \left(\bigcup_{C_i \in \mathcal{P}_3} C_i\right)$. Note that the external neighbors of T_i must all be distinct. Partition $\mathcal{K}$ into two sets $\mathcal{K}_1$ and $\mathcal{K}_2$ so that for $K_i' \in \mathcal{K}$, the set K_i' is in $\mathcal{K}_1$ if and only if $|T_i| \geq \frac{\Delta^{0.29}}{5} - 40$.

Claim 5.4. *There exists $Z \subset V(G)$ where $G[Z]$ is a 1-factor consisting of one edge from $G[K_i']$ for each $K_i' \in \mathcal{K}_2$.*

Proof. For each $K_i' \in \mathcal{K}_2$, vertices of at most one special color have their external neighbors in $\bigcup_{C_i \in \mathcal{P}_3}(C_i - K_i')$. Since there are at least $\frac{\Delta^{0.29}}{5}$ special colors by Lemma 3.4, for each $K_i' \in \mathcal{K}_2$, there exists at least 40 vertices with different special colors that have their external neighbors in $\bigcup_{K_i' \in \mathcal{K}_2} K_i'$; let R_i be 40 of these vertices for each $K_i' \in \mathcal{K}_2$. Choose two vertices from each R_i uniformly at random. We will apply the local lemma. Let E_e be the (bad) event that both endpoints of an edge e with endpoints in different K_i' is chosen. Thus, $Pr(E_e) \leq \left(\frac{2}{39}\right)^2$. E_{e_1} is mutually independent from E_{e_2} unless e_1 and e_2 have at least one endpoint in the same K_i'. Thus, E_e is mutually independent to all but at most 80 other events. Since $e\left(\frac{2}{39}\right)^2 81 < 1$, we are done. □

For each vertex in Z, color it with its special color; this is possible since no vertex in Z has a colored neighbor. Now each $K_i' \in \mathcal{K}_2$ has at least two uncolored safe vertices. We finish this section by finally showing that $Pr(\mathcal{E}_{3,i}) \leq \Delta^{-6}$.

Claim 5.5. *With high probability, for each $K_i' \in \mathcal{K}_1$, there are at least two vertices in K_i' where the special color of each vertex is available, and their external neighbors are colored.*

Proof. We will actually show that two vertices we are looking for are in T_i. Let $\mathcal{U}_i$ be the external neighbors of $\mathcal{T}_i$, which is a maximum set of vertices in T_i that satisfy the following conditions:

(i) the external neighbor of $x \in \mathcal{T}_i$ retains a color that is not the special color of x;

(ii) every external neighbor of a vertex in $\mathcal{T}_i$ has a distinct color.

We will first show that the expectation of $|\mathcal{T}_i|$ is high, and then we will show that $|\mathcal{T}_i|$ is concentrated around its expectation.

The probability that an external neighbor y of $x \in \mathcal{T}_i$ will not receive the special color of x is at least $1 - \frac{1}{\Delta-1}$, and the probability that the at most $\Delta - 1$ neighbors of y do not receive the color y received is at least $\left(1 - \frac{1}{\Delta-1}\right)^{\Delta-1}$, which is at least $\frac{1}{e}$. Since $|T_i| \geq \frac{\Delta^{0.29}}{5} - 40$, it follows that $E[|\mathcal{T}_i|] \geq \left(\frac{\Delta^{0.29}}{5} - 40\right) \cdot \frac{1}{e}$.

Now we use Azuma's Inequality to show that the probability that $|\mathcal{T}_i|$ deviates from the expected value is at most Δ^{-6}. Let the conditional expected value of $|\mathcal{T}_i|$ change by at most c_v when changing the color of v.

If $v \in \mathcal{U}_i - C_i$, then $c_v \leq 2$ since changing the color on v affects $|\mathcal{T}_i|$ by at most 2 for any given assignment of colors to the remaining vertices. Thus, the sum of the c_v^2 is at most $4|\mathcal{T}_i|$, which is about $\frac{4\Delta^{0.29}}{5}$.

If $v \in V(G) - \mathcal{U}_i - C_i$, then changing the color of v from α to β will only affect $|\mathcal{T}_i|$ if some neighbor of v that is in $\mathcal{U}_i$ receives either α or β. This occurs with probability at most $\frac{2d_v}{\Delta-1}$, where d_v is the number of neighbors of v that are in $\mathcal{U}_i$. Therefore, by changing the color of v, the conditional expectation of $|\mathcal{T}_i|$ changes by at most $c_v = \frac{4d_v}{\Delta-1}$. Since the d_v sum is at most $\left(\frac{\Delta^{0.29}}{5} - 40\right)(\Delta - 1)$, the sum of these c_v is at most $\frac{4\Delta^{0.29}}{5}$. As each c_v is at most 4, we see that the sum of c_v^2 is at most $\frac{16\Delta^{0.29}}{5}$.

Hence, the sum of all the c_v is at most $4\Delta^{0.29}$. Applying Azuma's Inequality yields $Pr(\mathcal{E}_{3,i}) \leq \Delta^{-6}$ for sufficiently large Δ. See Calculation 6.8. $\square$

Color these two vertices with their special colors for each $K_i' \in \mathcal{K}_1$.

This guarantees that every K_i' corresponding to this subsection has at least two vertices that are colored with their special colors. Now, the uncolored vertices within each K_i' that do not have the colored vertices in their lists are the safe vertices we are looking for.

5.4 $Pr(\mathcal{S}_v) \leq \Delta^{-6}$

Recall that v has at least $\Delta^{1+\frac{9}{10}}$ nonadjacent pairs of vertices in its neighborhood. Let $A = \{x \in N(v) : |L(x) \cap L(v)| \geq \frac{2}{3}\Delta\}$ and $B = N(v) - A$. Note that for $x, y \in A$, we have $|L(x) \cap L(y)| \geq \frac{\Delta}{3}$ and for $x \in B$ we have $|L(x) - L(v)| \geq \frac{\Delta}{3}$. Let b be the number of nonadjacent pairs in $N(v)$ that intersect B, so that $G[A]$ contains at least $\Delta^{1+\frac{9}{10}} - b$ nonadjacent pairs and $b \leq |B|\Delta$. Let A_v be the random variable that counts the number of colors that appear at least twice in $N(v)$. Let B_v be the random variable that counts the number of colors that appear in $N(v)$ that are not in the list of $L(v)$. Let $Z_v = A_v + B_v$ so that $E[Z_v] = E[A_v] + E[B_v]$. We will prove that $E[Z_v]$ is high, and then use Azuma's Inequality to prove that with high probability, Z_v is concentrated around its mean.

For A_v, let $x, y \in A$ be nonadjacent. We will actually calculate the number of colors that appear exactly twice in $N(v)$. Since $|L(x) \cap L(y)| \geq \frac{\Delta}{3}$, the probability that x, y get the same color and retain it and this color is not used on the rest of $N(v)$ is at least $\frac{\Delta}{3}(\Delta - 1)^{-2}(1 - \frac{1}{\Delta - 1})^{|N(v) \cup N(x) \cup N(y)|} \geq \Delta^{-1}3^{-5}$. Thus, $E[A_v] \geq (\Delta^{1+\frac{9}{10}} - b)\Delta^{-1}3^{-5}$.

For B_v, let $x \in B$. Since $|L(x) - L(v)| \geq \frac{\Delta}{3}$, the probability that x gets a color not in $L(v)$ and retains it and is not used on the rest of $N(v)$ is at least $\frac{\Delta}{3}\frac{1}{\Delta-1}(1-(\Delta-1)^{-1})^{|N(v) \cup N(x)|} \geq 3^{-4}$. Thus $E[B_v] \geq \frac{|B|}{3^4} \geq \frac{b}{3^4\Delta}$. Hence, $E[Z_v] \geq \Delta^{\frac{9}{10}}3^{-5}$.

Now we use Azuma's Inequality to show that the probability that Z_v deviates from the expected value is at most Δ^{-6}. Let the conditional expected value of Z_w change by at most c_w when changing the color of w.

Changing the color of w from α to β will only affect Z_v if some neighbor of w that is in $N(v)$ receives either α or β. This occurs with probability at most $\frac{2d_w}{\Delta-1}$, where d_w is the number of neighbors of w that are in $N(v)$. Therefore, by changing the color of w, the conditional expectation of Z_v changes by at most $c_w = \frac{4d_w}{\Delta-1}$. Since the d_w sum is at most Δ^2, the sum of these c_w is at most 5Δ. As each c_w is at most 5, we see that the sum of c_w^2 is at most 25Δ.

Hence, the sum of all the c_w is at most 25Δ. Applying Azuma's Inequality yields $Pr(\mathcal{S}_v) \leq \Delta^{-6}$ for sufficiently large Δ. See Calculation 6.9.

6 Calculations

Typed up calculations.

Calculation 6.1.

$$\left(1-\frac{1}{\Delta-1}\right)^{2.3942\Delta} \geq e^{-2.3943}$$

Proof.

$$\left(1-\frac{1}{\Delta-1}\right)^{2.3942\Delta} \geq \left(1-\frac{1}{\Delta-1}\right)^{2.3943(\Delta-2)} \geq e^{-2.3943}$$

□

Calculation 6.2. Azuma's Inequality for $\mathcal{E}_{1,i}$,

$$2\exp\left(\frac{-\left(2.5\cdot 10^{-4}\Delta-2\right)^2}{80\Delta}\right) \leq \Delta^{-6}$$

Proof.

$$\begin{aligned}
\Delta \geq 10^{12} \quad &\Rightarrow \quad 10^8 \cdot 200\ln\Delta \leq 2.5\Delta \\
&\Leftrightarrow \quad 500\Delta\ln\Delta \leq \left(2.5\cdot 10^{-4}\right)^2\Delta^2 \\
&\Rightarrow \quad (80\ln 2)\Delta + 6\cdot 80\Delta\ln\Delta \leq \left(2.5\cdot 10^{-4}\right)^2\Delta^2 + 4 - 4\left(2.5\cdot 10^{-4}\right)\Delta \\
&\Leftrightarrow \quad 2\Delta^6 \leq \exp\left(\frac{\left(2.5\cdot 10^{-4}\Delta-2\right)^2}{80\Delta}\right)
\end{aligned}$$

□

Calculation 6.3. For number of P_3 for $\mathcal{E}_{2,i}$, when proving

$$\frac{\Delta-1}{\Delta^{0.29}+4} \geq 0.9999\Delta^{1-0.29}$$

Proof. $\frac{\Delta-1}{\Delta+4\Delta^{0.71}}$ is an increasing function. For $\Delta = 10^{20}$,

$$0.9999 \leq \frac{10^{20}-1}{10^{20}+4\cdot 10^{20\cdot 0.7}} = \frac{\Delta-1}{\Delta+4\Delta^{0.71}} \Leftrightarrow \frac{\Delta-1}{\Delta^{0.29}+4} \geq 0.9999\Delta^{0.71}$$

□

Calculation 6.4.

$$\left(1-\frac{1}{\Delta-1}\right)^{2\Delta+2.9998\Delta^{0.71}} \geq e^{-2.1}$$

Proof.

$$\left(1-\frac{1}{\Delta-1}\right)^{2\Delta+2.9998\Delta^{0.71}} \geq \left(1-\frac{1}{\Delta-1}\right)^{2.1(\Delta-2)} \geq e^{-2.1}$$

□

Calculation 6.5.

$$2(\Delta - 1 - 0.9999\Delta^{0.71}) + (\Delta - 1)0.9999\Delta^{0.71} \leq \Delta^{1.71}$$

Proof.

$$\begin{aligned}
\Delta \geq 10^{20} \quad &\Rightarrow \quad 0.0001 \cdot (10^{20})^{0.71} \geq 15848931924 \\
&\Rightarrow \quad 2 \leq 0.0001\Delta^{0.71} \\
&\Leftrightarrow \quad 2\Delta + 0.9999\Delta^{1.71} \leq \Delta^{1.71} \\
&\Rightarrow \quad 2(\Delta - 1 - 0.9999\Delta^{0.71}) + (\Delta - 1)0.9999\Delta^{0.71} \leq 2\Delta + 0.9999\Delta^{1.71}
\end{aligned}$$

□

Calculation 6.6. Azuma's Inequality for $\mathcal{E}_{2,i}$,

$$2\exp\left(\frac{-\left(4.0 \cdot 10^{-3} \cdot \Delta^{0.71} - 2\right)^2}{8.02\Delta}\right) \leq \Delta^{-6}$$

Proof.

$$\begin{aligned}
\Delta \geq 10^{20} \quad &\Rightarrow \quad 50\Delta \ln \Delta \leq 16 \cdot 10^{-6}\Delta^{1.42} \\
&\Rightarrow \quad (8.02 \ln 2)\Delta + 6 \cdot 8.02\Delta \ln \Delta \leq \left(4.0 \cdot 10^{-3}\right)^2 \Delta^{1.42} + 4 - 4\left(4.0 \cdot 10^{-3}\right)\Delta^{0.71} \\
&\Leftrightarrow \quad 2\Delta^6 \leq \exp\left(\frac{\left(4.0 \cdot 10^{-3}\Delta^{0.71} - 2\right)^2}{8.02\Delta}\right)
\end{aligned}$$

□

Calculation 6.7.

$$\frac{\Delta^{0.29}}{5} - 40 \geq 1.9993\Delta^{0.29}$$

Proof.

$$1.9993 \leq \frac{1}{5} - \frac{40}{\left(10^{20}\right)^{0.29}} \Leftrightarrow \frac{\Delta^0.29}{5} - 40 \geq 1.9993\Delta^{0.29}$$

□

Calculation 6.8. Azuma's Inequality for $\mathcal{E}_{3,i}$, Claim 5.5,

$$2\exp\left(\frac{-\left(\frac{0.1999\Delta^{0.29}}{e} - 2\right)^2}{8\Delta^d}\right) \leq \Delta^{-6}$$

Proof.

$$\begin{aligned}
\Delta^{0.29} \geq \frac{50e^2 \ln \Delta}{0.1999^2} \quad &\Leftrightarrow \quad 50\Delta^{0.29} \ln \Delta \leq 0.1999^2 \left(\frac{\Delta^{0.29}}{e}\right)^2 \\
&\Rightarrow \quad (8 \ln 2)\Delta^{0.29} + 6 \cdot 8\Delta^{0.29} \ln \Delta \leq \left(0.1999\frac{\Delta^{0.29}}{e} - 2\right)^2 \\
&\Leftrightarrow \quad 2\Delta^6 \leq \exp\left(\frac{\left(\frac{0.1999\Delta^{0.29}}{e} - 2\right)^2}{8\Delta^{0.29}}\right)
\end{aligned}$$

□

Calculation 6.9. For $\mathcal{S}_v$,

$$2 \exp\left(\frac{-\left(3^{-5}\Delta^{\frac{9}{10}} - 2\right)^2}{50\Delta}\right) \leq \Delta^{-6}$$

Proof.

$$\begin{aligned}
50\Delta \leq 3^{-10}\Delta^{9/5} \quad &\Rightarrow \quad \frac{\Delta^{\frac{9}{10}}}{25} + 120\Delta + 300\Delta \ln \Delta \leq 3^{-10}\Delta^{\frac{9}{5}} \\
&\Rightarrow \quad (50 \ln 2)\Delta + 6 \cdot 50\Delta \ln \Delta \leq \left(3^{-5}\Delta^{\frac{9}{10}} - 2\right)^2 \\
&\qquad\qquad = 3^{-10}\Delta^{\frac{9}{5}} - 4 \cdot 3^{-5}\Delta^{\frac{9}{10}} + 4 \\
&\Leftrightarrow \quad 2\Delta^6 \leq \exp\left(\frac{\left(3^{-5}\Delta^{\frac{9}{10}} - 2\right)^2}{50\Delta}\right)
\end{aligned}$$

□

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