notes for planar 5-AT

November 30, 2015

1 orientation tools

Let G be a graph and $\leq$ a total order on $V(G)$. An orientation of G is *even* if the number of directed edges vw with $v \leq w$ is even; otherwise, the orientation is *odd*. Let $\alpha \colon V(G) \to \mathbb{N}$. An orientation X of G is an *α-orientation* if $d^+_X(v) = \alpha(v)$ for all $v \in V$. Let $D_\alpha(G)$ be the set of α-orientations of G. We partition $D_\alpha(G)$ into even α-orientations $DE_\alpha(G)$ and odd α-orientations $DO_\alpha(G)$. For $X, Y \in D_\alpha(G)$, let $X \oplus Y$ be the spanning subgraph of X with edge set

$$\{x_1x_2 \in E(X) \mid x_2x_1 \in E(Y)\}.$$

Then $X \oplus Y$ is a spanning Eulerian subgraph of X. We say that a spanning Eulerian subgraph of X is *even* if it has an even number of edges and *odd* otherwise. Let $EL(X)$ be the set of spanning Eulerian subgraphs of X. We partition $EL(X)$ into even spanning Eulerian subgraphs $EE(X)$ and odd spanning Eulerian subgraphs $EO(X)$.

Lemma 1.1. *Let $X \in D_\alpha(G)$. For each $S \in EL(X)$ there is a unique $X_S \in D_\alpha(G)$ such that $S = X \oplus X_S$. Moreover, S is odd when X and X_S have opposite parity and even otherwise. Therefore, if X is even, then $|EE(X)| = |DE_\alpha(G)|$ and $|EO(X)| = |DO_\alpha(G)|$. If X is odd, then $|EE(X)| = |DO_\alpha(G)|$ and $|EO(X)| = |DE_\alpha(G)|$. So, up to sign, we always have*

$$|EE(X)| - |EO(X)| = |DE_\alpha(G)| - |DO_\alpha(G)|.$$

Since Lemma 1.1 was for any $X \in D_\alpha(G)$, we have the following.

Corollary 1.2. *If $X, Y \in D_\alpha(G)$ then, up to sign, we have*

$$|EE(X)| - |EO(X)| = |EE(Y)| - |EO(Y)|.$$

It will be useful to investigate α-orientations further. First, a basic fact about Eulerian graphs.

Lemma 1.3. *If D is an Eulerian directed graph, then D is an edge-disjoint union of directed cycles.*

Proof. If D is not edgeless, it must have a directed cycle, remove it and apply induction. $\square$

One important thing to note about Lemma 1.3 is there may be multiple different decompositions of D into directed cycles. Following Felsner [1], we say that $vw \in E(G)$ is *α-rigid* if vw is oriented the same way in every α-orientation of G.

Lemma 1.4. *If $X, Y \in D_\alpha(G)$ with $x_1x_2 \in E(X)$ and $x_2x_1 \in E(Y)$, then there is a directed cycle C in X containing x_1x_2 such that Y contains the directed cycle made from C by reversing all edges.*

Proof. Since $X \oplus Y$ is Eulerian, it is an edge-disjoint union of directed cycles. Let C be the directed cycle containing x_1x_2. $\square$

From Lemma 1.4 we have the following.

Corollary 1.5. *An edge e of G is α-rigid if and only if no α-orientation of G has a directed cycle containing e.*

A graph G is *α-AT* if there is an α-orientation X of G with $EE(X) \neq EO(X)$. Note that by Lemma 1.1, if G is α-AT then $EE(X) \neq EO(X)$ for every $X \in D_\alpha(G)$. It is useful to see how α-AT behaves when we remove edges.

Lemma 1.6. *For any α-orientation of G and $vw \in E(G)$ with $v \leq w$, we have*

$$\begin{aligned}
|D_\alpha(G)| &= |D_{\alpha-1_v}(G)| + |D_{\alpha-1_w}(G)|, \text{ and} \\
|DE_\alpha(G)| &= |DO_{\alpha-1_v}(G)| + |DE_{\alpha-1_w}(G)|, \text{ and} \\
|DO_\alpha(G)| &= |DE_{\alpha-1_v}(G)| + |DO_{\alpha-1_w}(G)|.
\end{aligned}$$

Lemma 1.7. *Suppose G is α-AT and $vw \in E(G)$ with $v \leq w$. If vw is α-rigid (say always directed from v to w), then $G - vw$ is $(\alpha - 1_v)$-AT. Otherwise, $G - vw$ is either $(\alpha - 1_v)$-AT or $(\alpha - 1_w)$-AT.*

Proof. First, suppose vw is α-rigid. Let X be an α-orientation of G. Then vw is not contained in any $S \in EL(X)$ and hence removing it does not change parities. So, $G - vw$ is $(\alpha - 1_v)$-AT.

Now, suppose vw is not α-rigid. By Lemma 1.6, we have

$$0 \neq |DE_\alpha(G)| - |DO_\alpha(G)| = |DO_{\alpha-1_v}(G)| - |DE_{\alpha-1_v}(G)| + |DE_{\alpha-1_w}(G)| - |DO_{\alpha-1_w}(G)|.$$

Hence either $|DO_{\alpha-1_v}(G)| - |DE_{\alpha-1_v}(G)| \neq 0$ or $|DE_{\alpha-1_w}(G)| - |DO_{\alpha-1_w}(G)| \neq 0$. By Lemma 1.1, $G - vw$ is either $(\alpha - 1_v)$-AT or $(\alpha - 1_w)$-AT. $\square$

We will use the following to reverse an edge on a triangle cutset when the inductive hypothesis directs the triangle cyclically.

Lemma 1.8. *Suppose G is α-AT and X is an α-orientation of G. If Z is an induced subgraph of X such that $EE(Z) = EO(Z)$, then X has an induced cycle $C \not\subseteq Z$ containing an edge of Z.*

Proof. Otherwise, every spanning Eulerian subgraph of X is the edge-disjoint union of a spanning Eulerian subgraph of Z and a spanning Eulerian subgraph of $X - E(Z)$. But then $EE(Z) = EO(Z)$ implies $EE(X) = EO(X)$, a contradiction. $\square$

References

[1] Stefan Felsner, *Lattice structures from planar graphs*, Electron. J. Combin **11** (2004), no. 1, R15.