most low Alon-Tarsi notes

August 23, 2015

1 Introduction

We consider graphs with vertices labeled by natural numbers; that is, pairs (G, h) where G is a graph and $h\colon V(G) \to \mathbb{N}$. We say that (G, h) *is AT* if G is $(d_G - h)$-AT. When H is an induced subgraph of G, we simplify notation by referring to the pair (H, h) where we really mean $\left(H, h{\restriction}_{V(H)}\right)$.

2 Subgraphs, subdivisions and cuts

Definition 1. A graph G is *h-minimal* if G is connected and (H, h) is not AT for every proper induced subgraph H of G. A graph G is *h-greedy-minimal* if G is connected and (H, h) is not AT for every proper induced subgraph H of G where $h(v) = 0$ for all $v \in V(G) \setminus V(H)$. Note that if G is h-minimal then it is also h-greedy-minimal.

Lemma 2.1. *If G is connected and (G, h) is not AT, then G is h-greedy-minimal.*

Proof. If there were a proper induced subgraph H such that $\left(H, h{\restriction}_{V(H)}\right)$ is AT, then by ordering the vertices of each component of $G - V(H)$ by increasing distance to H and directing all edges away from H in this order we conclude that (G, h) is AT. $\square$

Lemma 2.2. *If (G', h') is formed from (G, h) by subdividing an edge e of G twice and having h' give zero on the two new vertices, then*

1. *if (G, h) is AT, then (G', h') is AT; and*
2. *if (G', h') is AT, then either (G, h) is AT or $(G - e, h)$ is AT.*

Proof. Suppose $e = xy$ and call the new vertices x' and y' so that G' contains the induced path $xx'y'y$. For (1), let D be an orientation of G showing that (G, h) is AT. By symmetry we may assume $xy \in E(D)$. Make an orientation D' of G' from D by replacing xy with the directed path $xx'y'y$. There is a natural parity preserving bijection between the spanning Eulerian subgraphs of D and D', so we conclude that (G', h') is AT.

For (2), let D' be an orientation of G' showing that (G', h') is AT. Suppose G' contains the directed path $xx'y'y$ or the directed path $yy'x'x$. By symmetry, we can assume it is $xx'y'y$. Then make an orientation D of G by replacing $xx'y'y$ with the directed edge xy. As above, we have a parity preserving bijection between the spanning Eulerian subgraphs of D and D', so we conclude that (G, h) is AT. Otherwise, no spanning Eulerian subgraph of D' contains a cycle passing through x' and y'. So, the spanning Eulerian subgraph counts of D' are the same as those of $D' - x' - y'$. But this gives an orientation of $G - e$ showing that $(G - e, h)$ is AT. □

Lemma 2.3. *Let $\{A_1, A_2\}$ be a separation of G such that $A_1 \cap A_2 = \{x\}$. If $G[A_i]$ is f_i-AT for $i \in [2]$, then G is f-AT where $f(v) = f_i(v)$ for $v \in V(A_i - x)$ and $f(x) = f_1(x) + f_2(x) - 1$. Going the other direction, if G is f-AT, then $G[A_i]$ is f_i-AT for $i \in [2]$ where $f_i(v) = f(v)$ for $v \in V(A_i - x)$ and $f_1(x) + f_2(x) \leq f(x) + 1$.*

Proof. For $i \in [2]$, choose an orientation D_i of A_i showing that A_i is f_i-AT. Together these give an orientation D of G and since no cycle has vertices in both $A_1 - x$ and $A_2 - x$, we have

$$\begin{aligned} EE(D) - EO(D) &= EE(D_1)EE(D_2) + EO(D_1)EO(D_2) - (EE(D_1)EO(D_2) + EO(D_1)EE(D_2)) \\ &= (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2)) \\ &\neq 0. \end{aligned}$$

Hence G is f-AT.

Now, suppose G is f-AT and choose an orientation D of G showing this. Put $D_i = D[A_i]$ for $i \in [2]$. Then, as above, we have $0 \neq EE(D) - EO(D) = (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2))$ and hence $EE(D_1) - EO(D_1) \neq 0$ and $EE(D_2) - EO(D_2) \neq 0$. Since the in-degree of x in D is the sum of the in-degree of x in D_1 and the in-degree of x in D_2, the lemma follows. □

Corollary 2.4. *Let G be an h-greedy-minimal graph. If (G, h) is AT and G has an induced path $x_1x_2x_3x_4$ such that $d_G(x_2) = d_G(x_3) = 2$ and $h(x_2) = h(x_3) = 0$, then*

$$\left((G - x_2 - x_3) + x_1x_4, h\restriction_{V(G)\setminus\{x_2,x_3\}}\right) \text{ is AT.}$$

Proof. Suppose (G, h) is AT and G has such an induced path $x_1x_2x_3x_4$. Applying Lemma 2.2 part (2) shows that either $\left(G - x_2 - x_3, h\restriction_{V(G)\setminus\{x_2,x_3\}}\right)$ is AT or $\left((G - x_2 - x_3) + x_1x_4, h\restriction_{V(G)\setminus\{x_2,x_3\}}\right)$ is AT. But $G - x_2 - x_3$ is a proper induced subgraph of G, so the former cannot happen since G is h-greedy-minimal and $h(x_2) = h(x_3) = 0$. Hence $\left((G - x_2 - x_3) + x_1x_4, h\restriction_{V(G)\setminus\{x_2,x_3\}}\right)$ is AT. □

3 Extension lemma

This is a key lemma from [1], it generalizes a lemma from [2] from list coloring to Alon-Tarsi orientations. This is what i talked about in Baltimore. The basic idea is that in some cases we can pair off odd/even spanning Eulerian subgraphs via a parity reversing bijection.

Lemma 3.1. *Let G be a multigraph without loops and $f\colon V(G) \to \mathbb{N}$. If there are $F \subseteq G$ and $Y \subseteq V(G)$ such that:*

1. *any multiple edges in G are contained in $G[Y]$; and*

2. *$f(v) \geq d_G(v)$ for all $v \in V(G) \setminus Y$; and*

3. *$f(v) \geq d_{G[Y]}(v) + d_F(v) + 1$ for all $v \in Y$; and*

4. *For each component T of $G-Y$ there are different $x_1, x_2 \in V(T)$ where $N_T[x_1] = N_T[x_2]$ and $T - \{x_1, x_2\}$ is connected such that either:*

   (a) there are $x_1y_1, x_2y_2 \in E(F)$ where $y_1 \neq y_2$ and $N(x_i) \cap Y = \{y_i\}$ for $i \in [2]$; or

   (b) $|N(x_2) \cap Y| = 0$ and there is $x_1y_1 \in E(F)$ where $N(x_1) \cap Y = \{y_1\}$,

then G is f-AT.

Proof. Suppose not and pick a counterexample (G, f, F, Y) minimizing $|G - Y|$. If $|G - Y| = 0$, then $Y = V(G)$ and thus $f(v) \geq d_G(v) + 1$ for all $v \in V(G)$ by (3). Pick an acyclic orientation D of G. Then $EE(D) = 1$, $EO(D) = 0$ and $d^+_D(v) \leq d_G(v) \leq f(v) - 1$ for all $v \in V(D)$. Hence G is f-AT. So, we must have $|G - Y| > 0$.

Pick a component T of $G-Y$ and pick $x_1, x_2 \in V(T)$ as guaranteed by (4). First, suppose (4a) holds. Put $G' := (G - T) + y_1y_2$, $F' := F - T$, $Y' := Y$ and let f' be f restricted to $V(G')$. Then G' has an orientation D' where $f'(v) \geq d^+_{D'}(v) + 1$ for all $v \in V(D')$ and $EE(D') \neq EO(D')$, for otherwise (G', f', F', Y') would contradict minimality. By symmetry we may assume that the new edge y_1y_2 is directed toward y_2. Now we use the orientation of D' to construct the desired orientation of D. First, we use the orientation on $D' - y_1y_2$ on $G - T$. Now, order the vertices of T as $x_1, x_2, z_1, z_2, \ldots$ so that every vertex has at least one neighbor to the right. Orient the edges of T left-to-right in this ordering. Finally, we use y_1x_1 and x_2y_2 and orient all other edges between T and $G - T$ away from T. Plainly, $f(v) \geq d^+_D(v) + 1$ for all $v \in V(D)$. Since y_1x_1 is the only edge of D going into T, any Eulerian subgraph of D that contains a vertex of T must contain y_1x_1. So, any Eulerian subgraph of D either contains (i) neither y_1x_1 nor x_2y_2, (ii) both y_1x_1 and x_2y_2, or (iii) y_1x_1 but not x_2y_2. We first handle (i) and (ii) together. Consider the function h that maps an Eulerian subgraph Q of D' to an Eulerian subgraph $h(Q)$ of D as follows. If Q does not contain y_1y_2, let $h(Q) = \iota(Q)$ where $\iota(Q)$ is the natural embedding of $D' - y_1y_2$ in D. Otherwise, let $h(Q) = \iota(Q - y_1y_2) + \{y_1x_1, x_1x_2, x_2y_2\}$. Then h is a parity-preserving injection with image precisely the union of those Eulerian subgraphs of D in (i) and (ii). Hence if we can show that exactly half of the Eulerian subgraphs of D in (iii) are even, we will conclude $EE(D) \neq EO(D)$, a contradiction. To do so, consider an Eulerian subgraph A of D containing y_1x_1 and not x_2y_2. Since x_1 must have in-degree 1 in A, it must also have out-degree 1 in A. We show that A has a mate A' of opposite parity. Suppose $x_2 \notin A$ and $x_1z_1 \in A$; then we make A' by removing x_1z_1 from A and adding $x_1x_2z_1$. If $x_2 \in A$ and $x_1x_2z_1 \in A$, we make A' by removing $x_1x_2z_1$ and adding x_1z_1. Hence exactly half of the Eulerian subgraphs of D in (iii) are even and we conclude $EE(D) \neq EO(D)$, a contradiction.

Now suppose (4b) holds. Put $G' := G - T$, $F' := F - T$, $Y' := Y$ and define f' by $f'(v) = f(v)$ for all $v \in V(G' - y_1)$ and $f'(y_1) = f(y_1) - 1$. Then G' has an orientation D' where $f'(v) \geq d^+_{D'}(v) + 1$ for all $v \in V(D')$ and $EE(D') \neq EO(D')$, for otherwise (G', f', F', Y') would contradict minimality. We orient $G - T$ according to D, orient T as in the previous case, again use y_1x_1 and orient all other edges between T and $G - T$ away from T. Since we decreased $f'(y_1)$ by 1, the extra out edge of y_1 is accounted for and we have

$f(v) \geq d_D^+(v) + 1$ for all $v \in V(D)$. Again any additional Eulerian subgraph must contain y_1x_1 and since x_2 has no neighbor in $G - T$ we can use x_2 as before to build a mate of opposite parity for any additional Eulerian subgraph. Hence $EE(D) \neq EO(D)$ giving our final contradiction. □

4 Degree-AT graphs

A graph G is called *degree-AT* if (G, h) is *AT* where h is the constant zero function.

Lemma 4.1. *A connected graph G is degree-AT if it is not a Gallai tree.*

Proof. Suppose there exists a connected graph that is not a Gallai tree, but is also not degree-AT. Let G be such a graph with as few vertices as possible. Since G is not degree-AT, no induced subgraph H of G is degree-AT by Lemma 2.1. Hence, for any $v \in V(G)$ that is not a cutvertex, $G - v$ must be a Gallai tree by minimality of $|G|$.

If G has more than one block, then for endblocks B_1 and B_2, choose noncutvertices $w \in B_1$ and $x \in B_2$. By the minimality of $|G|$, both $G - w$ and $G - x$ are Gallai trees. Since every block of G appears either as a block of $G - w$ or as a block of $G - x$, every block of G is either complete or an odd cycle. Hence, G is a Gallai tree, a contradiction. So instead G has only one block, that is, G is 2-connected. Further, $G - v$ is a Gallai tree for all $v \in V(G)$.

Let v be a vertex of minimum degree in G. Since G is 2-connected, $d_G(v) \geq 2$ and v is adjacent to a noncutvertex in every endblock of $G - v$. If $G - v$ has a complete block B with noncutvertices x_1, x_2 where $v \leftrightarrow x_1$ and $v \not\leftrightarrow x_2$, then we can apply Lemma 3.1 with $Y = \{v\}$ and $F = vx_1$ to conclude that G is degree-AT, a contradiction. So, v must be adjacent to every noncutvertex in every complete endblock of $G - v$.

Suppose $d_G(v) \geq 3$. Then no endblock of $G - v$ can be an odd cycle of length at least 5 (there would be vertices of degree 3 but we'd have $d_G(v) \geq 4$). Let B be a smallest complete endblock of $G - v$. Then for a noncutvertex $x \in V(B)$, we have $d_G(x) = |B|$ and hence $d_G(v) \leq |B|$. If $G - v$ has at least two endblocks, then $2(|B| - 1) \leq |B|$ and hence $d_G(v) \leq |B| = 2$, a contradiction. Hence $G - v = B$ and v is joined to B, so G is complete, a contradiction.

Hence, we must have $d_G(v) = 2$. Suppose $G - v$ has at least 2 endblocks. Then, it has exactly 2 and v is adjacent to one noncutvertex in each. Neither of the endblocks can be odd cycles of length at least 5 since then we could get a smaller counterexample by Lemma 2.2. Since v is adjacent to every noncutvertex in every complete endblock of $G - v$, both endblocks must be K_2. But then either $G = C_4$ (which is trivially degree-AT) or we can get a smaller counterexample by Lemma 2.2. So, $G - v$ must be 2-connected. Since $G - v$ is a Gallai tree, it is either complete or an odd cycle. If $G - v$ is not complete, we can get a smaller counterexample by Lemma 2.2. So, $G - v$ is complete and v is adjacent to every noncutvertex of $G - v$; that is, G is complete, a contradiction. □

5 When h is 1 for at most one vertex

For a graph G and $x \in V(G)$ let $h_x\colon V(G) \to \mathbb{N}$ be defined by $h_x(x) = 1$ and $h_x(v) = 0$ for all $v \in V(G - x)$. We classify the connected h_x-minimal graphs G such that (G, h_x) is AT for some $x \in V(G)$.

To start we will reduce to the case when G is 2-connected.

Lemma 5.1. *Let G be h_x-minimal for $x \in V(G)$ and let $\mathcal{B}$ be the set of blocks of G containing x. Then (G, h_x) is AT if and only if*

1. *$\mathcal{B}$ contains at least two degree-AT graphs; or*
2. *G is 2-connected and (G, h_x) is AT.*

Proof. Since G is h_x-minimal, no block outside of $\mathcal{B}$ is degree-AT. The lemma follows since if G is not 2-connected, then (G, h_x) is AT if and only if (1) holds by Lemma 2.3. $\square$

Lemma 5.2. *If G is a connected graph and $x \in V(G)$ with $d_G(x) = 2$, then (G, h_x) is AT if and only if $G - x$ is degree-AT.*

Proof. Let D be an orientation of G showing that (G, h_x) is AT. Then $d_D^-(x) = 2$ and hence no spanning Eulerian subgraph contains a cycle passing through x. Therefore, the Eulerian subgraph counts in $G - x$ are different and $G - x$ is degree-AT. The other direction is immediate from Lemma 2.1. $\square$

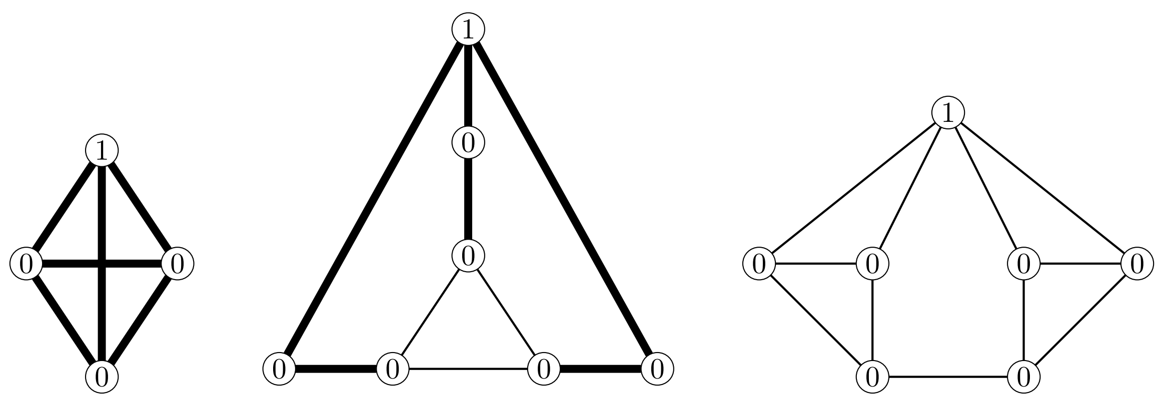

Figure 1: The seed blocks. Each bold edge can be removed without making the figure AT.

Lemma 2.2 part (2) suggests a way to construct G such that (G, h) is not AT from smaller graphs. Specifically, we have the following.

Corollary 5.3. *If e is an edge in G such that (G, h) is not AT and $(G - e, h)$ is not AT, then (G', h') is not AT where (G', h') is formed from (G, h) by subdividing e twice and having h' give zero on the two new vertices.*

Let $\mathcal{D}$ be the smallest collection of pairs (G, h) containing the pairs in Figure 1 that is closed under the operation in Corollary 5.3. Explicitly, $\mathcal{D}$ is all graphs that can be created from the graphs in Figure 1 by replacing each bold edge with an odd length path. The rightmost graph in Figure 1 (the Moser Spindle) has no bold edges.

For a connected graph G and endblock B of G, let x_B be the cutvertex of G contained in B.

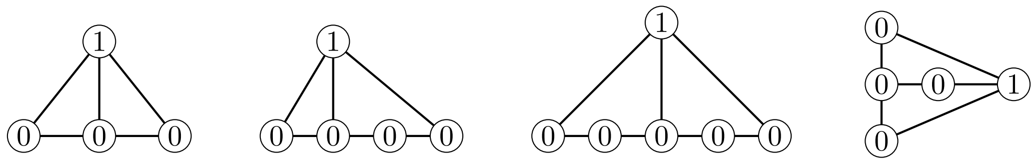

Figure 2: These are AT.

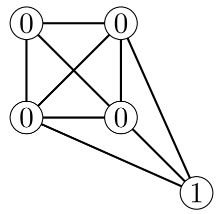

Figure 3: This is AT.

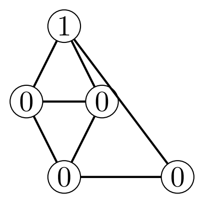

Figure 4: This is AT.

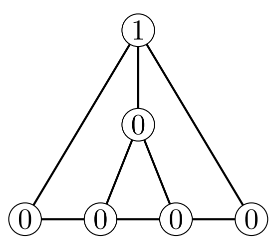

Figure 5: This is AT.

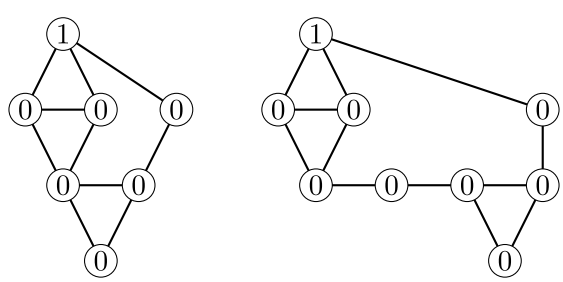

Figure 6: These are AT.

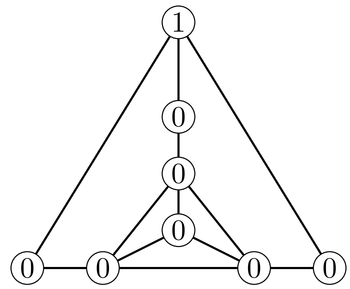

Figure 7: This is AT.

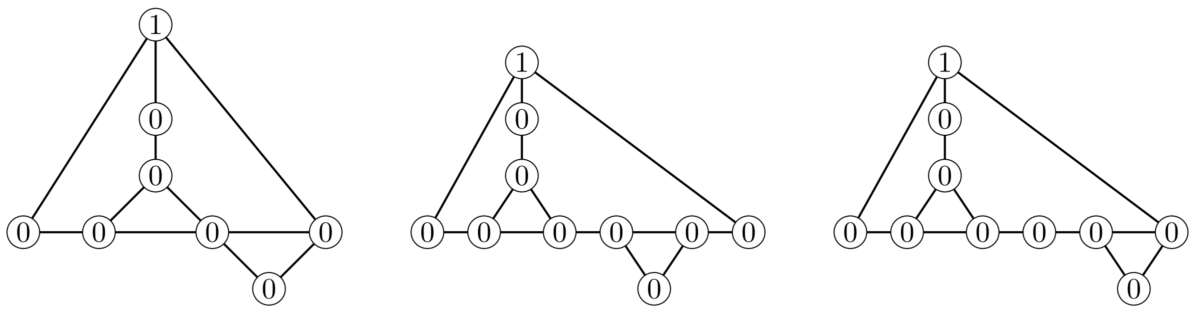

Figure 8: These are AT.

Lemma 5.4. *Let G be a connected graph and $v \in V(G)$ a cutvertex of G. If $G - v$ has t components, then there are endblocks $B_1, \dots, B_t$ and an induced subdivision of $K_{1,t}$ where the root is v and the leaves are $x_{B_1}, \dots, x_{B_t}$.*

Proof. Pick endblocks $B_1, \dots, B_t$, one in each component of $G - v$. Now the desired induced subdivision of $K_{1,t}$ is the union of shortest paths from x_{B_1} to x_{B_i} for $2 \leq i \leq t$. □

Lemma 5.4 will be really useful in applying the following lemma. Note that we can always extend the induced subdivision of $K_{1,3}$ or induced path we get one vertex into each endblock.

Lemma 5.5. *Let G be h_x-minimal for $x \in V(G)$ with $d_G(x) \geq 3$. If (G, h_x) is not AT, then every induced subdivision of $K_{1,3}$ in G contains at most two vertices in $N(x)$. In particular, every induced path in G contains at most two vertices in $N(x)$.*

Proof. This is immediate from Lemma 2.2 and the graphs in Figure 2. □

Definition 2. For $a_1, a_2, a_3 \in \mathbb{N}$, let T_{a_1,a_2,a_3} be the graph consisting of

- a triangle $z_1z_2z_3$; and
- disjoint paths $z_iP_iw_i$ where P_i has length a_i for $i \in [3]$; and
- a vertex x adjacent to w_1, w_2, w_3.

Lemma 5.6. (T_{a_1,a_2,a_3}, h_x) *is AT when* $|\{i \in [3] \mid a_i \text{ is even }\}| \in \{1, 2\}$.

Proof. This follows from Lemma 2.2 and the fact that Figure 4 and Figure 5 are AT. □

Lemma 5.7. *Let H be formed from T_{a_1,a_2,a_3} by adding a new vertex with neighborhood $\{z_1, z_2, z_3\}$. Then (H, h_x) is AT.*

Proof. This follows from Lemma 5.6 and Lemma 2.1 using Lemma 2.2 and the fact that Figure 3 is AT and Figure 7 is AT. □

Lemma 5.8. *Let H be formed from T_{a_1,a_2,a_3} by adding a new vertex adjacent to two consecutive vertices on P_1. Then (H, h_x) is AT.*

Proof. This follows from Lemma 5.6 and Lemma 2.1, using Lemma 2.2 and the fact that the graphs in Figure 6 are AT and the graphs in Figure 8 are AT. □

Lemma 5.9. *Let G be h_x-minimal for $x \in V(G)$. If G is 2-connected, then (G, h_x) is AT if and only if*

1. *$d_G(x) \geq 3$; and*
2. *G is not complete; and*
3. *$(G, h_x) \notin \mathcal{D}$.*

Proof. Suppose the lemma is false and choose a counterexample G minimizing $|G|$. If $d_G(x) \leq 2$, then (G, h_x) is not AT by Lemma 5.2 since G is h_x-minimal. So, we must have $d_G(x) \geq 3$. Since (G, h_x) is not AT if $(G, h_x) \in \mathcal{D}$ by construction, it must be that $(G, h_x) \notin \mathcal{D}$ and (G, h_x) is not AT.

Claim 0. *$G - x$ is a Gallai tree and x is adjacent to a noncutvertex in every endblock of $G - x$.* This follows since G is h_x-minimal and 2-connected.

Claim 1. *$G - x - v$ has at most two components for any $v \in V(G - x)$.* Suppose $G - x$ has a cutvertex v such that $G - x - v$ has at least three components. Then, by Lemma 5.4, $G - x$ contains an induced $K_{1,3}$ violating Lemma 5.5.

Claim 2. *x is not adjacent to any cutvertex v of $G - x$* Using Lemma 5.4, we get an induced path from x_B to x_D containing v, where B and D are different endblocks violating Lemma 5.5.

Claim 3. *$G - x$ does not contain an induced path $v_1v_2v_3v_4$ such that $d_G(v_2) = d_G(v_3) = 2$.* If it did, then we could get a smaller counterexample by applying Lemma 2.2 part (1).

Claim 4. *Every block of $G - x$ is complete.* Suppose $G - x$ has a block B that is an odd cycle $v_1v_2 \cdots v_tv_1$ with $t \geq 5$.

Subclaim 4a. *B contains at most two cutvertices of $G - x$.* Otherwise there are $a, b, c \in [t]$ such that $v_a, \ldots, v_b, \ldots v_c$ contains exactly three cutvertices v_a, v_b and v_c. Apply Lemma 5.4 to the component of $G - \{x, v_1, \ldots, v_{a-1}, v_{b+1}, \ldots, v_t\}$ containing v_a, v_b, v_c with $v = v_b$ to get an induced $K_{1,3}$ violating Lemma 5.5.

Subclaim 4b. *B contains at most one cutvertex of $G - x$* Otherwise, by Subclaim 4a, B has exactly two cutvertices v_a and v_b. By Claim 3, x is adjacent to a noncutvertex $v \in V(B)$. Consider the induced path given by applying Lemma 5.4 to v_a. If this path does not contain v, then have it go the other way around B. Now we have an induced path violating Lemma 5.5.

Subclaim 4c. *Claim 4 is true.* By Claim 3, x must be adjacent to at least every other noncutvertex of B. So, if $G - x = B$, we immediately violate Lemma 5.5. If instead, $G - x$

has another endblock B' then we can pick two neighbors of x in B and one neighbor of x in B' all on an induced path in $G - x$, violating Lemma 5.5.

Claim 5. *If x is adjacent to a noncutvertex in a block, then x is adjacent to all noncutvertices in that block. In particular, x is adjacent to every noncutvertex in every endblock of $G - x$.* Suppose $G - x$ has a block B with noncutvertices v_1, v_2 where $x \leftrightarrow v_1$ and $x \not\leftrightarrow v_2$. By Claim 4, B is complete, so we can apply Lemma 3.1 with $Y = \{x\}$ and $F = xv_1$ to conclude that (G, h_x) is AT, a contradiction.

Claim 6. *$G - x$ has at least two endblocks.* If not, then $G - x$ is complete by Claim 0 and Claim 4. But then G is complete by Claim 5, a contradiction.

Claim 7. *The endblocks of $G - x$ are all K_2, except possibly one K_3.* By Claim 4, every endblock is complete. Suppose $G - x$ has an endblock $B = K_t$ for $t \geq 4$. Then by Claim 2, Claim 5 and Claim 6, G has an induced Figure 3, impossible. So every endblock of $G - x$ is K_2 or K_3. Suppose $G - x$ has two K_3 endblocks B_1 and B_2. Then $G[\{x\} \cup V(B_i)]$ is degree-AT for $i \in [2]$. If there is no edge between B_1 and B_2, then, by Lemma 5.1, G contains an induced subgraph H such that (H, h_x) is AT, a contradiction. If there is an edge between B_1 and B_2, then by Claim 1, G is the rightmost graph in Figure 1, a contradiction.

Claim 8. *Every noncutvertex of $G - x$ is adjacent to x.*

Suppose $G - x$ has a noncutvertex v with $v \not\leftrightarrow x$. Then $G - v$ is 2-connected and h_x-minimal, so by minimality of $|G|$, we conclude that $d_{G-v}(x) \leq 2$, $G - v$ is complete, or $(G - v, h_x) \in \mathcal{D}$. The first three clearly cannot occur, so we have $(G - v, h_x) \in \mathcal{D}$.

Subclaim 8a. *$G - v$ has an induced path $v_1v_2v_3v_4$ such that $d_G(v_2) = d_G(v_3) = 2$.*

Otherwise, $G - v$ is one of the graphs in Figure 1. But $G - v$ cannot be the leftmost, middle, or rightmost graph in Figure 1 because then G would contain the graph in Figure 3, Figure 7, and Figure 5 as an induced subgraph, respectively.

Subclaim 8b. *The block B containing v is K_3.*

By Claim 4, B is complete. Some neighbor w of v must have gone from degree 3 to degree 2. Since v is only adjacent to vertices in B, the only way for this to happen is if $B = K_3$.

Subclaim 8c. *$G - v$ is the result of applying the operation in Corollary 5.3 to a graph F in Figure 1 one time.* None of the graphs in Figure 1 have an induced path $v_1v_2v_3v_4$ such that $d_G(v_2) = d_G(v_3) = 2$.

Subclaim 8d. *F is not the rightmost graph in Figure 1.* For this one,removing any edge leaves an AT graph, so Corollary 5.3 cannot be applied.

Subclaim 8e. *F is not the middle graph in Figure 1.* For this one, removing any edge in the triangle leaves an AT graph, so Corollary 5.3 cannot be applied to those edges. But then G is one of the graphs in Figure 8, impossible.

Subclaim 8f. *Claim 8 is true.* By the previous subclaims, F must be the leftmost graph in Figure 1. For this one, removing any of the edges not incident to the vertex labeled 1 leaves an AT graph, so Corollary 5.3 cannot be applied to those edges. But then G contains an induced even subdivision of Figure 4 or the leftmost graph in Figure 6, impossible.

Claim 9. *Every internal block of $G - x$ consists entirely of cutvertices.* Suppose otherwise that we have an internal block B of $G - x$ containing a noncutvertex v. By Claim 8, $x \leftrightarrow v$. Note that by Lemma 2.2 and Figure 4, we get that Figure 5 is AT with either or both of the bottom left and bottom right edge subdivided once. But G contains at least one of these with edges subdivided twice some number of times as an induced subgraph, a contradiction.

Claim 10. *x's neighbors are precisely the noncutvertices in the endblocks of $G-x$.* By Claim 5, x is adjacent to all these vertices. By Claim 9 and Claim 2, x is not adjacent to any other vertex.

Claim 11. *$G-x$ has at least three endblocks.* If not, then by Claim 6, $G-x$ has exactly two endblocks B_1 and B_2. Since $d_G(x) \geq 3$, Claim 7, Claim 9 and Claim 10 show that $G-x$ is a triangle $w_1w_2w_3$ with a path $w_1y_1y_2\dots y_t$ emanating from w_1. Since (G,h_x) is not AT, Lemma 2.2 and Figure 4 show that t must be even. But then $(G,h_x) \in \mathcal{D}$ since G is formed from the leftmost graph in Figure 1 by applying Corollary 5.3 some number of times, a contradiction.

Claim 12. *Every endblock of $G-x$ is K_2.* Suppose $G-x$ has a K_3 endblock B.

Subclaim 12a. *The component of $G-N(x_B)$ containing x is not degree-AT.* Since G is h_x-minimal, this follows by Lemma 5.1.

Subclaim 12b. *The component of $G-N(x_B)$ containing x is triangle-free.* If not, then by Claim 11, $G-N(x_B)$ contains an induced subgraph containing x that is a cycle $y_1\dots y_ty_1$ plus the edge y_1y_3 with $t \geq 4$. If t is even, then $G-N(x_B)$ contains an induced even cycle with at most one chord which is degree-AT, otherwise $G-N(x_B)$ contains the induced even cycle $y_1y_3\dots y_ty_1$ which is also degree-AT; this contradicts Subclaim 12a.

Subclaim 12c. *The other block D containing x_B is K_2.* If not, then G must have either an induced even subdivision of the leftmost graph in Figure 6 or an induced even subdivision of Figure 4 (both path length parities are covered).

Subclaim 12d. *Claim 12 is true.* By Subclaim 12b and 12c, $G-x$ has exactly one K_3 internal component, call it Q, and the rest of the internal components are K_2. Moreover, Q intersects D and in particular, a shortest path from Q to x passing through B has length three. By Claim 1 and Claim 11, $G-x$ has exactly three endblocks $B_1 = B, B_2$ and B_3. For $i \in [3]$, let ℓ_i be the length of the shortest path from Q to x passing through B_i. We know $\ell_1 = 3$. If both ℓ_2 and ℓ_3 are even, then G would have an induced even subdivision of Figure 5, a contradiction. So, at least one of ℓ_2, ℓ_3 are odd and hence G contains an induced even subdivision of Figure 4, a contradiction.

Claim 13. *All internal blocks of $G-x$ are K_2 or K_3.* If not, then by Claim 4, $G-x$ has a K_t block with $t \geq 4$. But this is impossible by Lemma 5.7 since G is h_x-minimal.

Claim 14. *$G-x$ has at least four endblocks.* If not, then by Claim 11, $G-x$ has exactly three endblocks B_1, B_2 and B_3, all of them K_2 by Claim 12. By Claim 1, Claim 9 and Claim 13, $G = T_{a_1,a_2,a_3}$ for some $a_1, a_2, a_3 \in \mathbb{N}$. By Lemma 5.6, either all or none of a_1, a_2, a_3 are even. If all, then $(G,h_x) \in \mathcal{D}$ since G is formed from the leftmost graph in Figure 1 by applying Corollary 5.3 some number of times, a contradiction. If none, then $(G,h_x) \in \mathcal{D}$ since G is formed from the middle graph in Figure 1 by applying Corollary 5.3 some number of times, a contradiction.

Claim 15. *The lemma is true.* If not, then by Claim 1, Claim 9, Claim 13 and Claim 14, G contains an induced subgraph that violates h_x-minimality by Lemma 5.8. □

Putting Lemma 5.1 and Lemma 5.9 together we get the following classification of h_x-minimal graphs.

Theorem 5.10. *Let G be h_x-minimal for $x \in V(G)$ and let $\mathcal{B}$ be the set of blocks of G containing x. Then (G,h_x) is AT if and only if*

1. $\mathcal{B}$ *contains at least two degree-AT graphs; or*

2. G *is 2-connected,* G *is not complete,* $(G, h_x) \notin \mathcal{D}$ *and* $d_G(x) \geq 3$.

6 Choosability and Paintability

The same classification holds. Just need to show none of the bad ones are choosable. Also, show that when we move up to two vertices x, y with $h(x) = h(y) = 1$, AT, choosability and paintability all separate.

7 Applications

When this is in a critical graph, we get h_x-minimality for free. If x has $t \geq 5$ neighbors in one component of the low vertex subgraph, then those neighbors are all in one K_t block. If $t = 4$, then the neighbors are either a K_4 block or we create the Moser Spindle.

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