A common generalization of Hall's theorem and Vizing's edge-coloring theorem

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LBD Data

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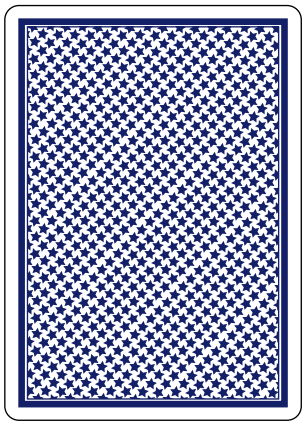
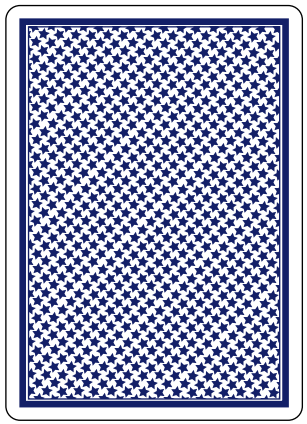
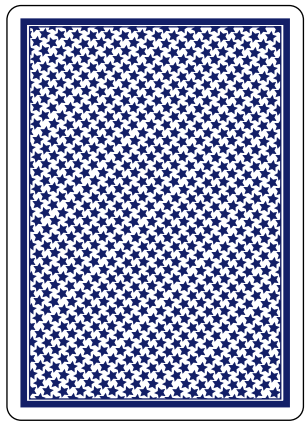
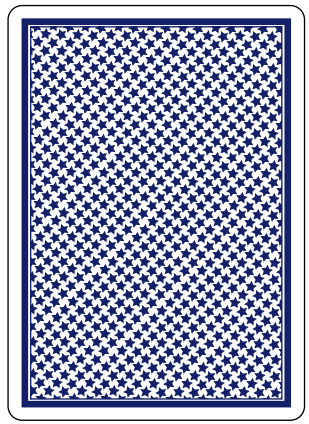
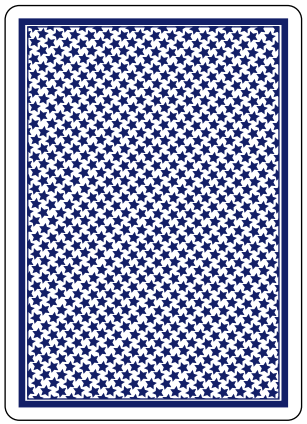
  - $A_1 = \{1, 2\}$, $A_2 = \{1, 2\}$, $A_3 = \{1, 2\}$
- Hall's theorem: this is the only thing that can go wrong

$$\text{SDR exists} \Leftrightarrow \left|\bigcup_{i \in I} A_i\right| \geq |I| \text{ for all } I \subseteq \{1, \ldots, n\}$$

some card games

the simplest variation

- two players, Dealer and Player

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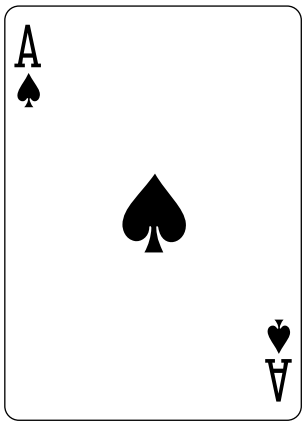
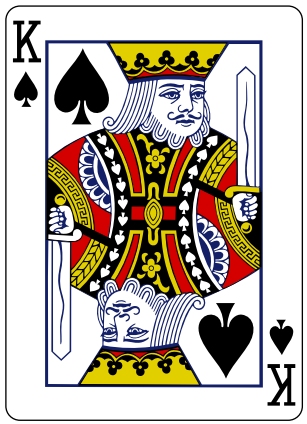
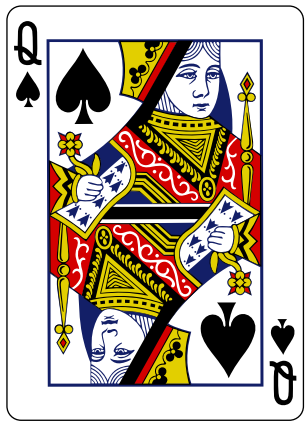
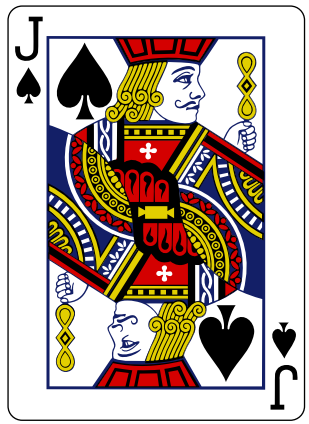
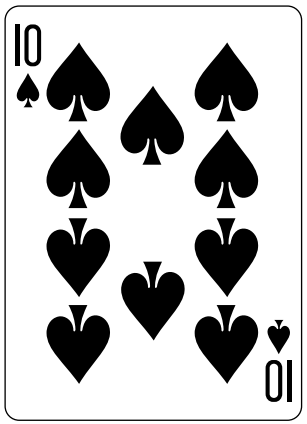

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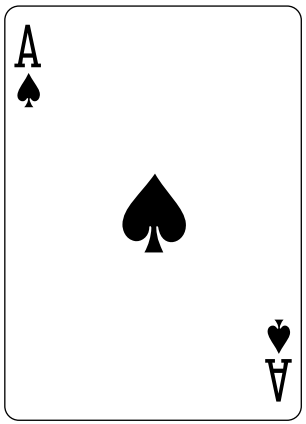

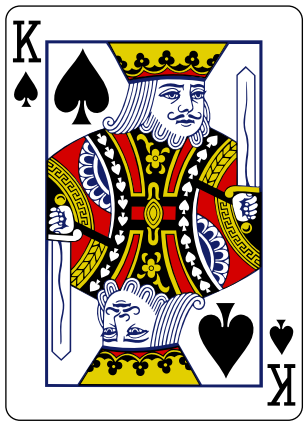

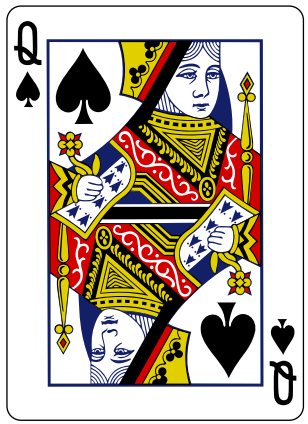

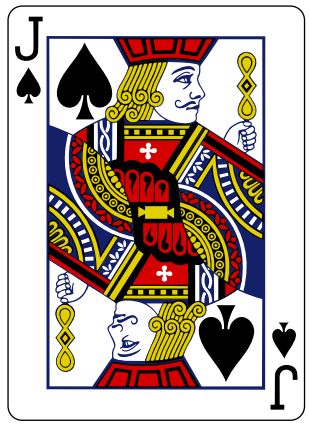

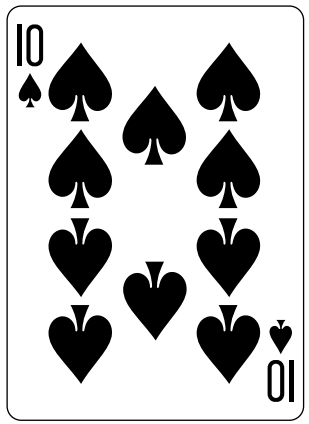

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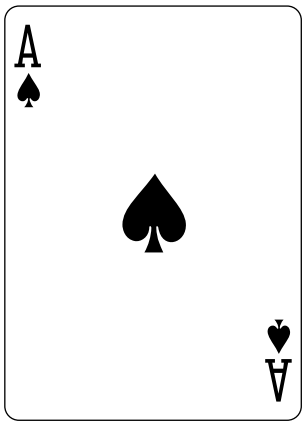
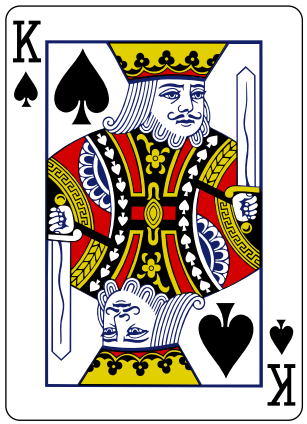
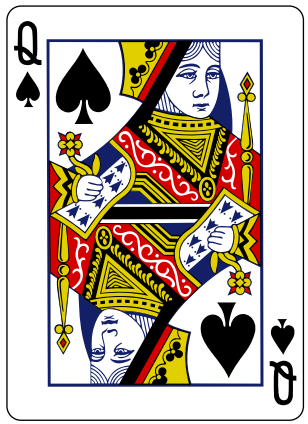
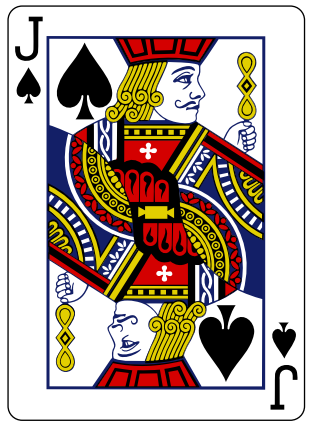
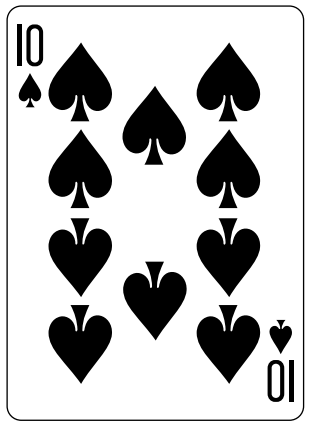




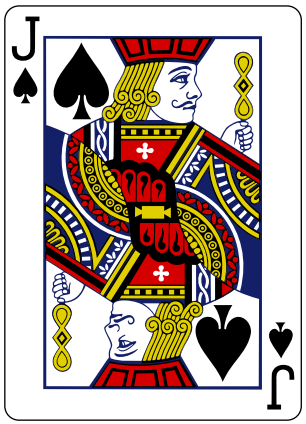
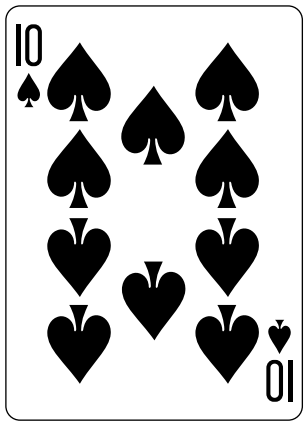
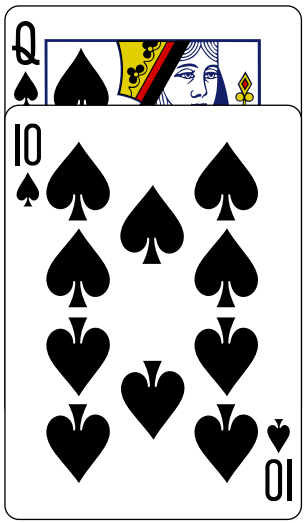






some card games

example, a Player win

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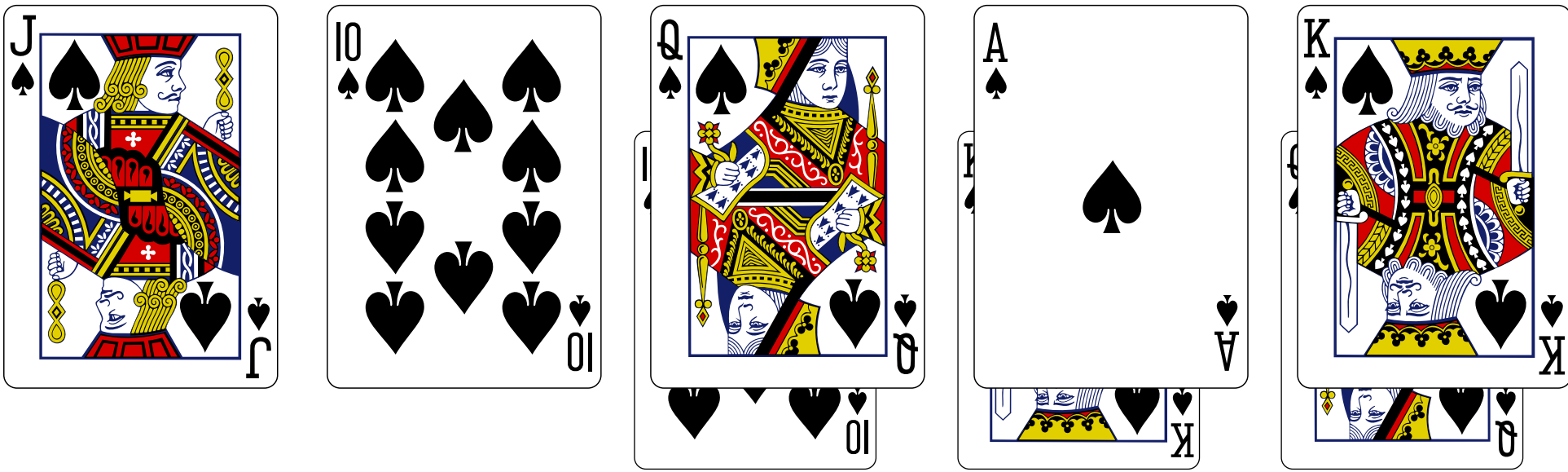
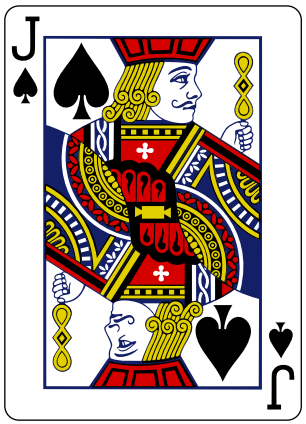
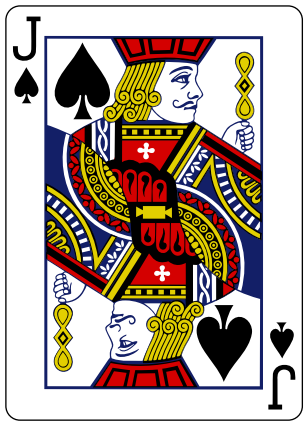
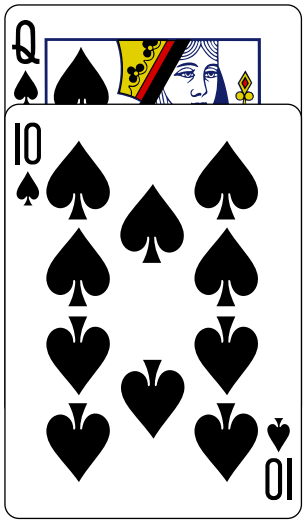


some card games

example, a Dealer win







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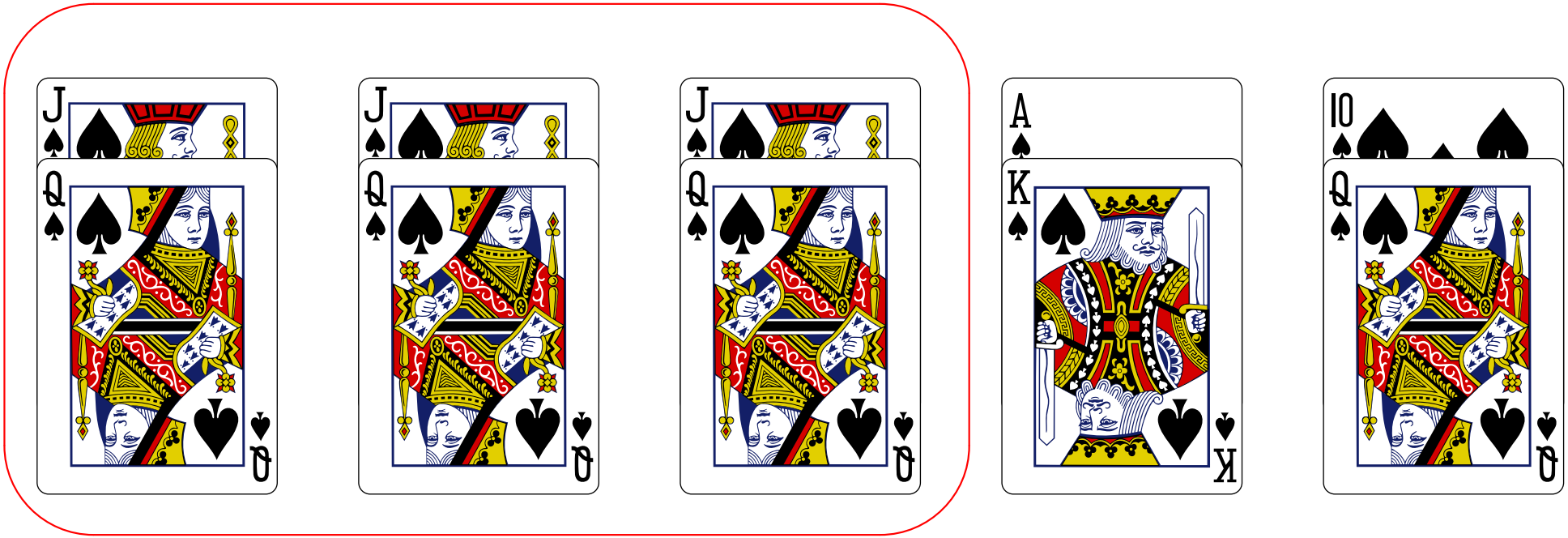

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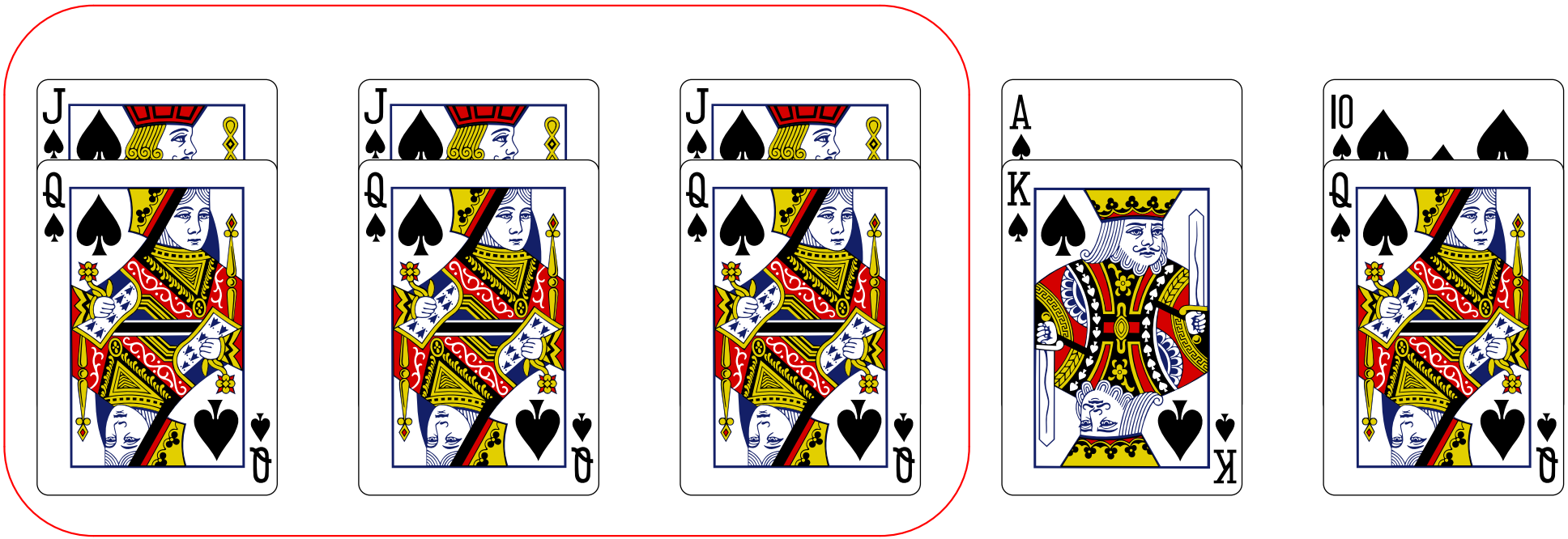
- Hall's theorem says: **Player wins otherwise**



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Player wins if he can pick a Royal Flush at the start of one of his turns, otherwise Dealer wins.

some card games

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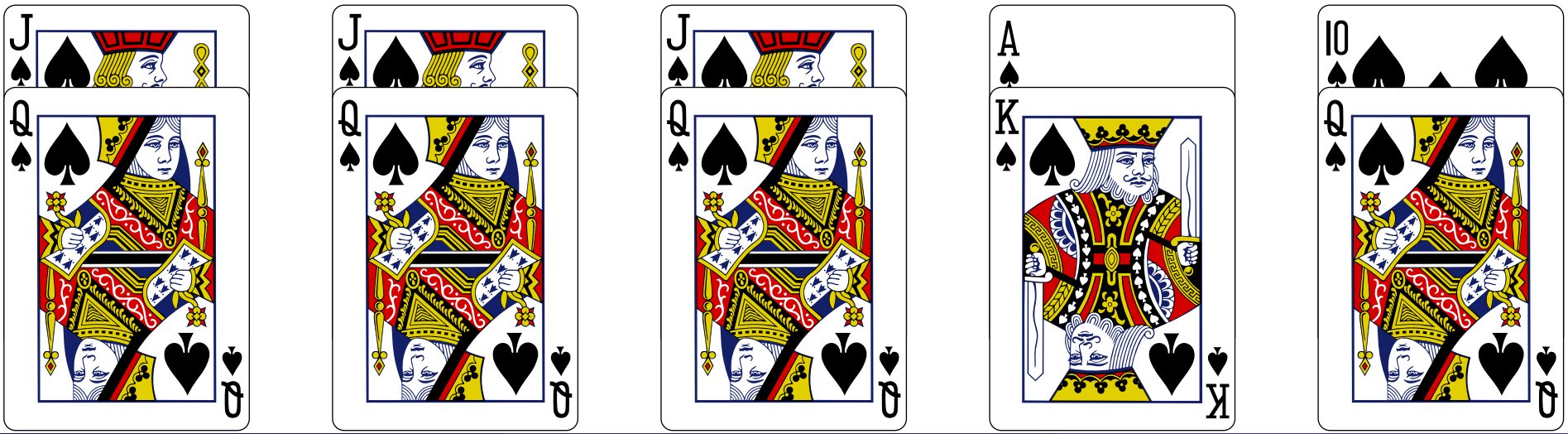

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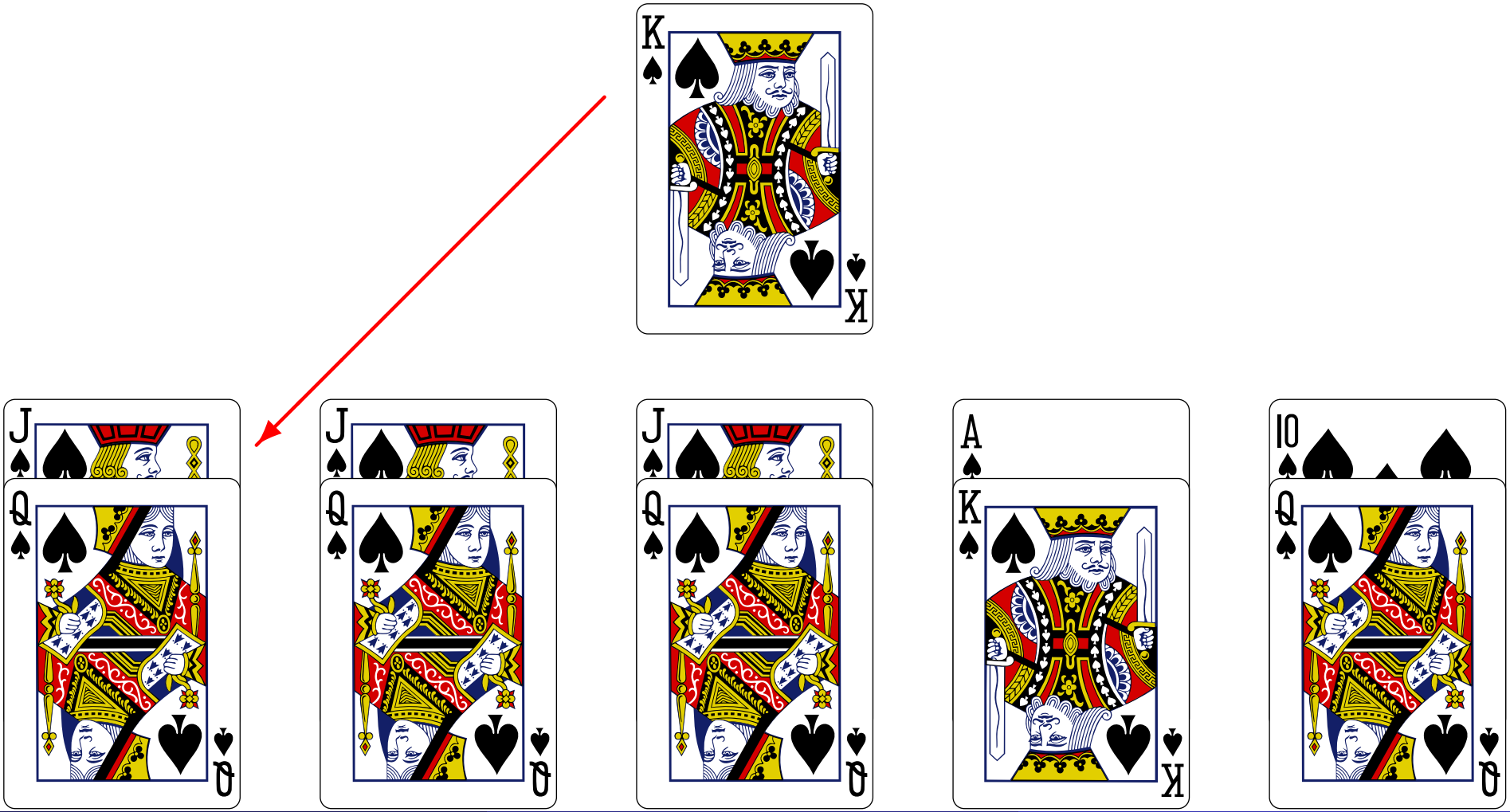

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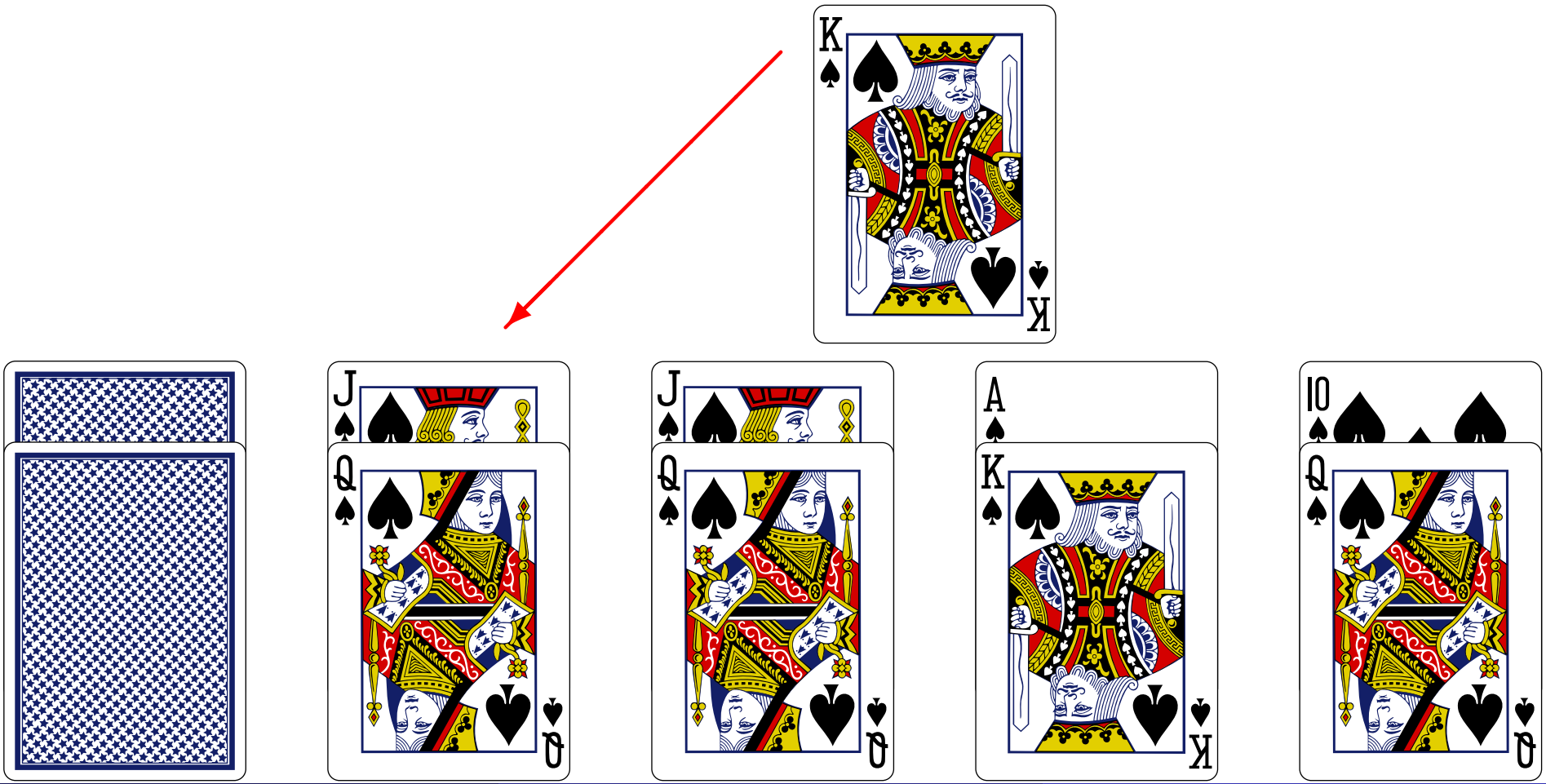

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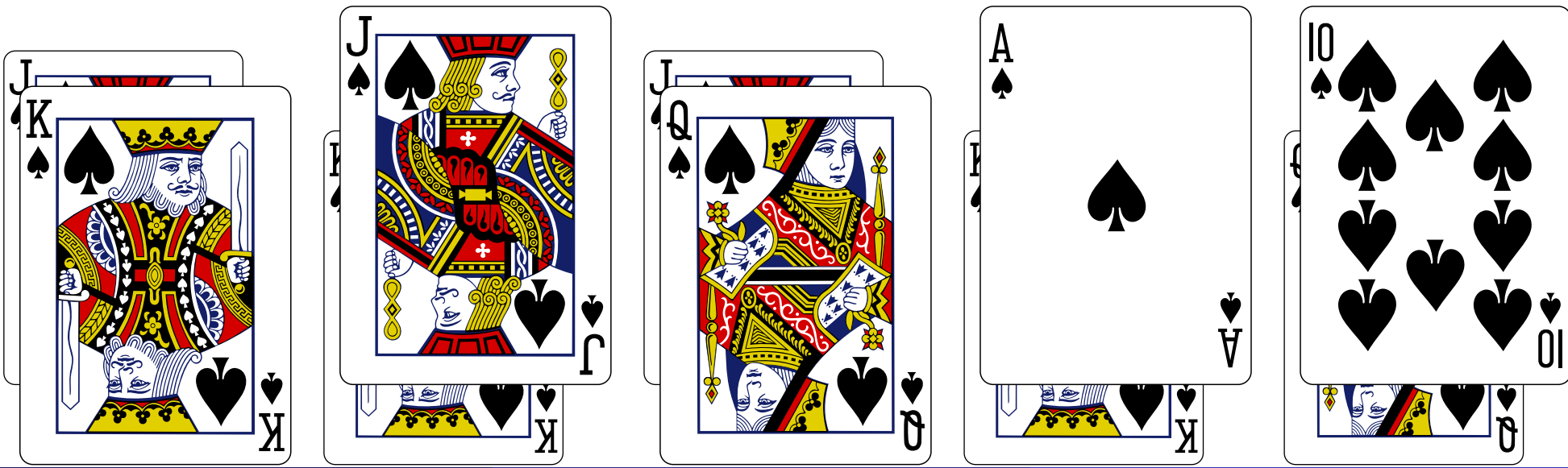
- Player wins no matter what Dealer does



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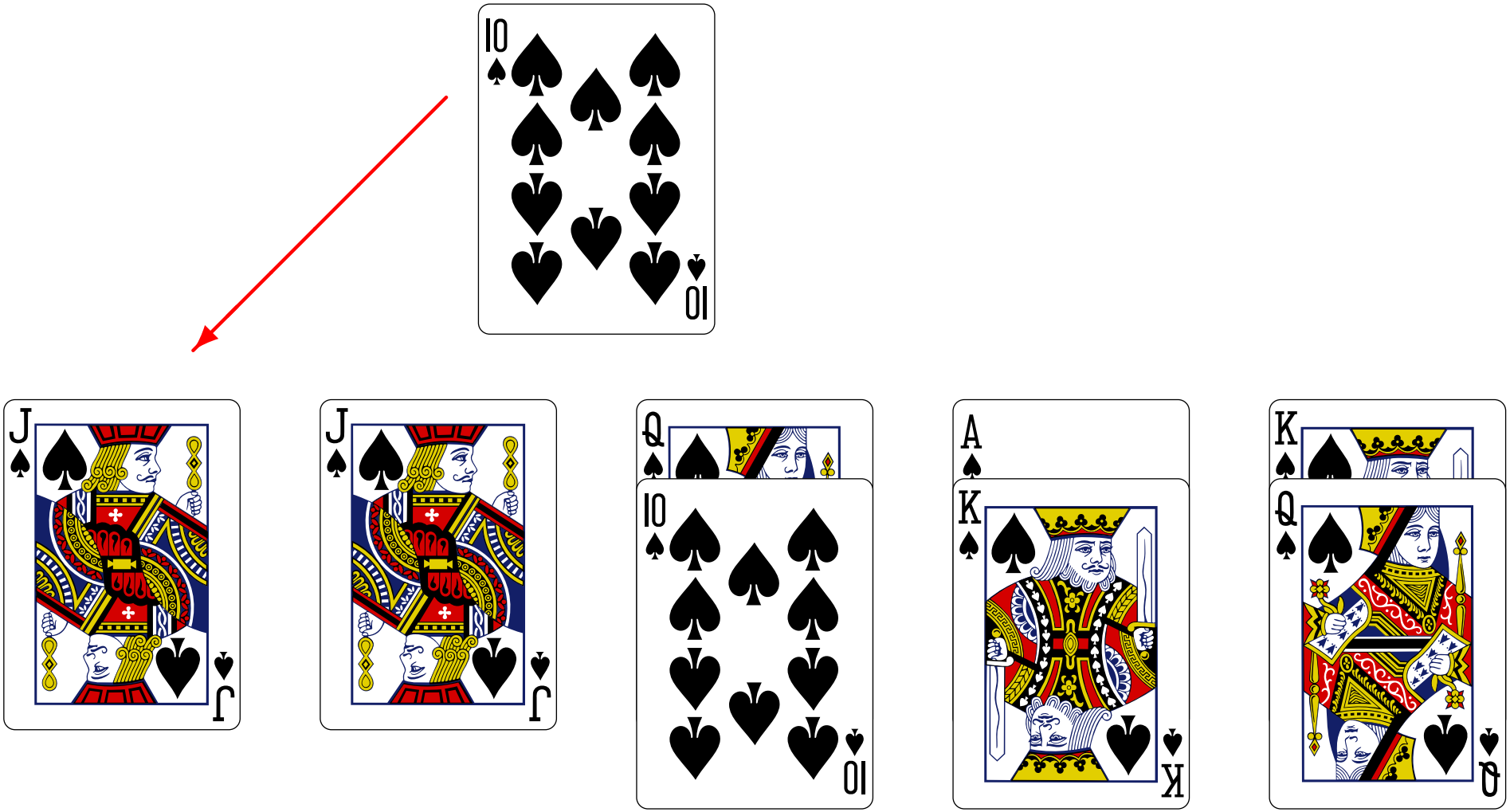

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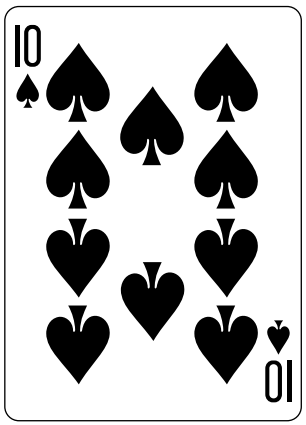

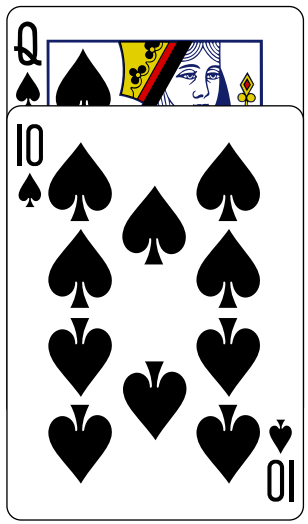

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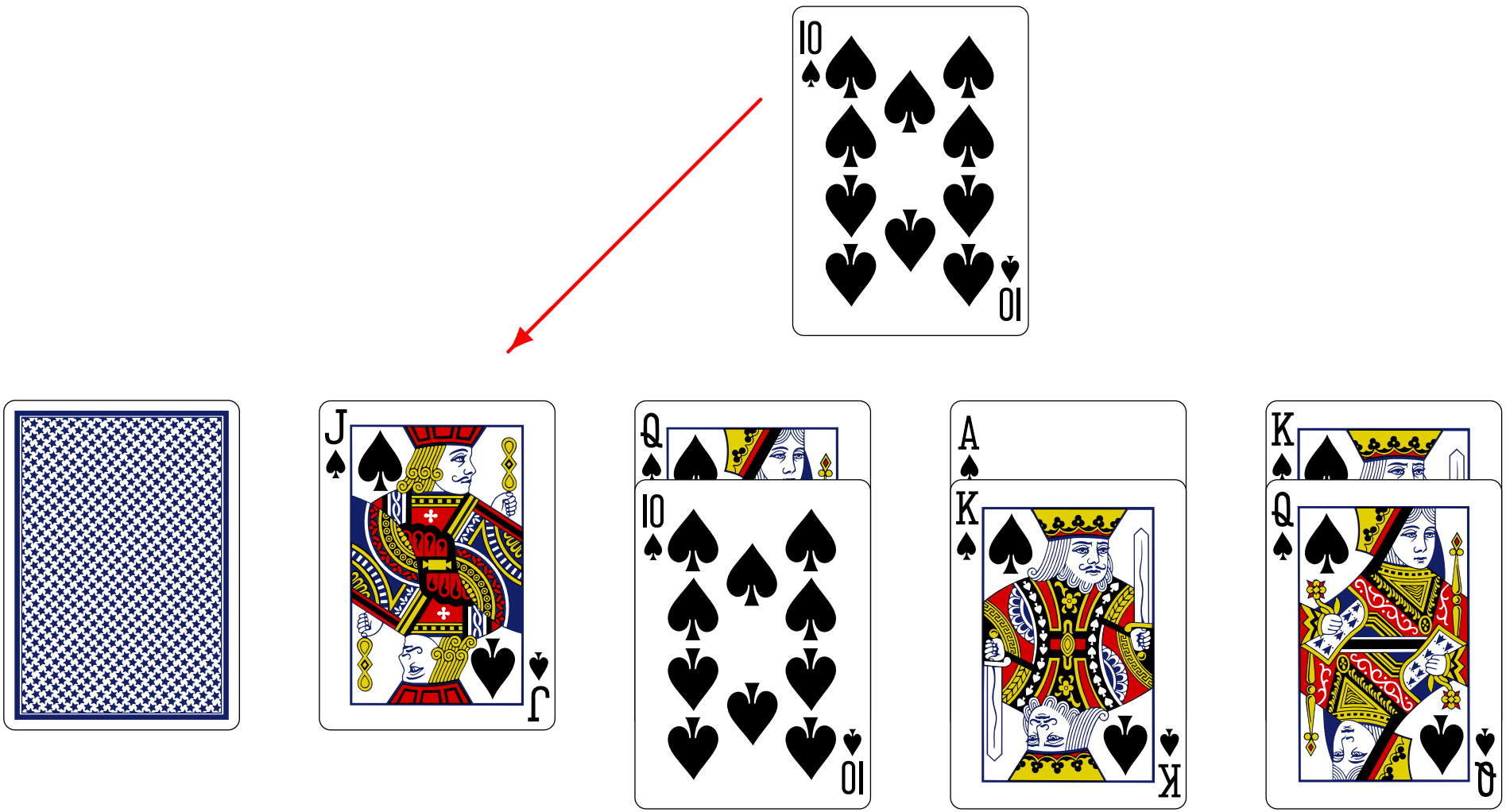

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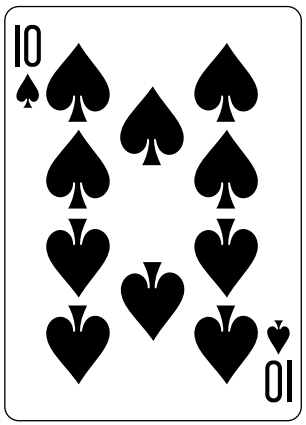
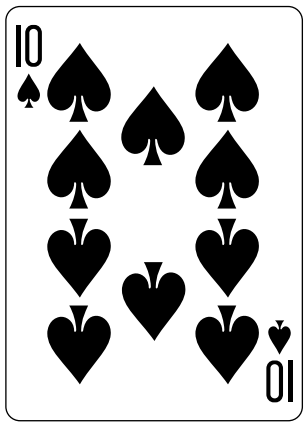
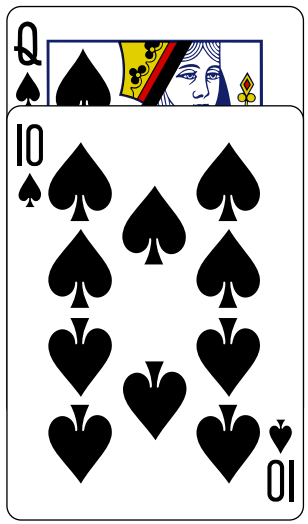

some card games

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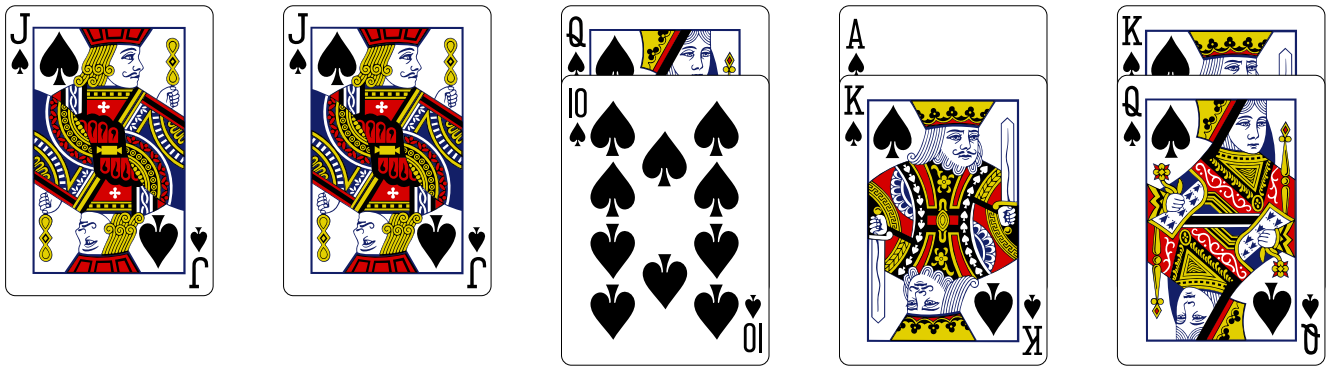

some card games

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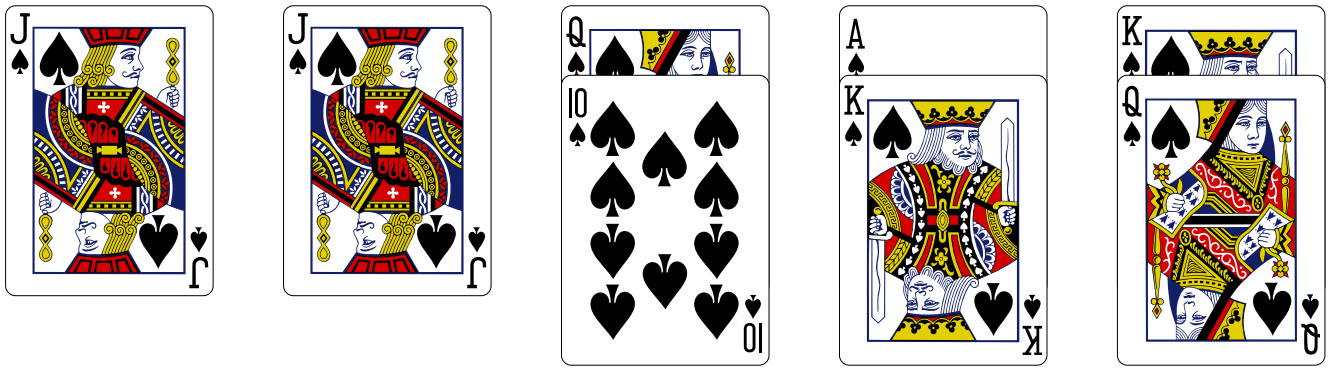

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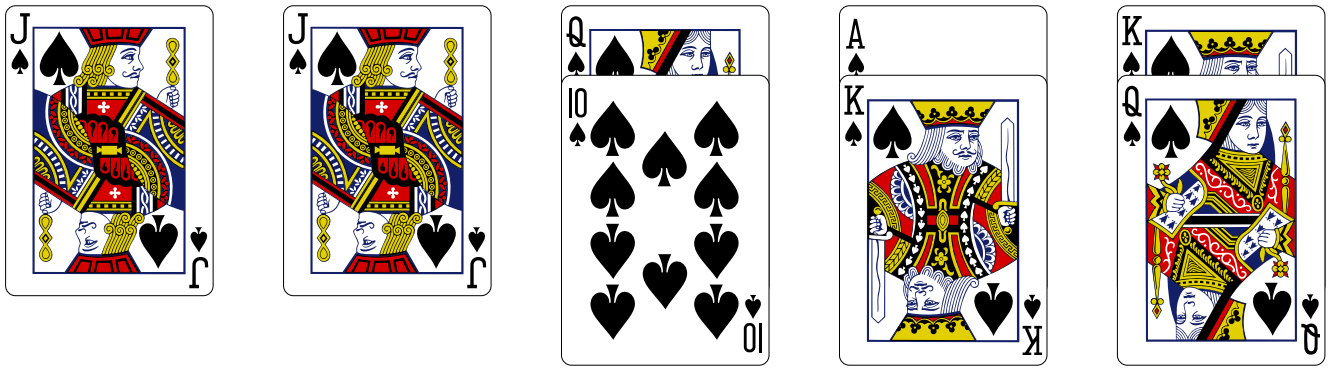
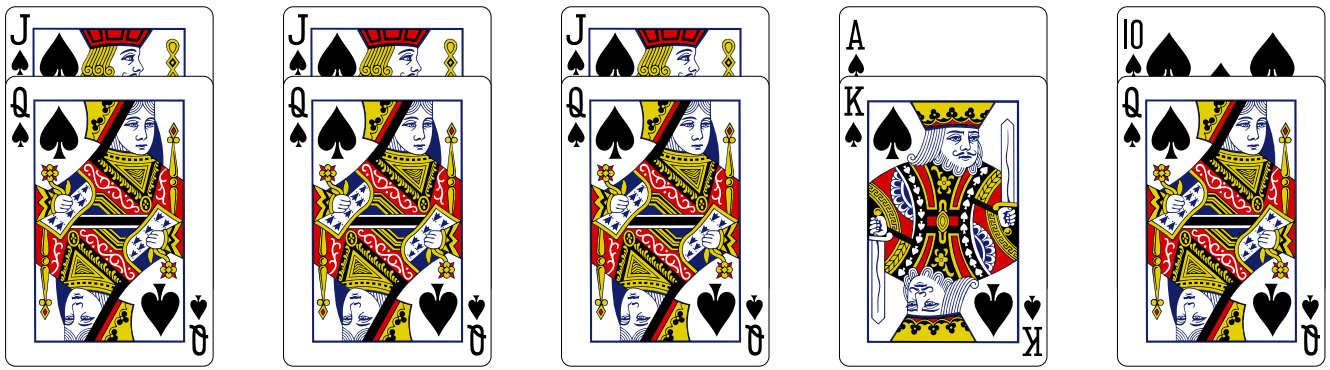
- in the top game, Dealer can prevent Player from increasing the number of different cards in the first two stacks
- in the bottom game, Dealer cannot prevent prevent Player from increasing the number of different cards in the first three stacks





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- Player can turn $2t+1$ of the same card into $t+1$ different cards, so C is 'worth' $\left\lceil \frac{d_S(C)}{2} \right\rceil$

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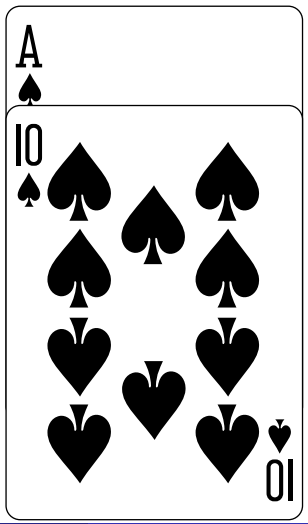
1. Player looks for a set of card types that give a system of distinct representatives of all the stacks containing them




some card games

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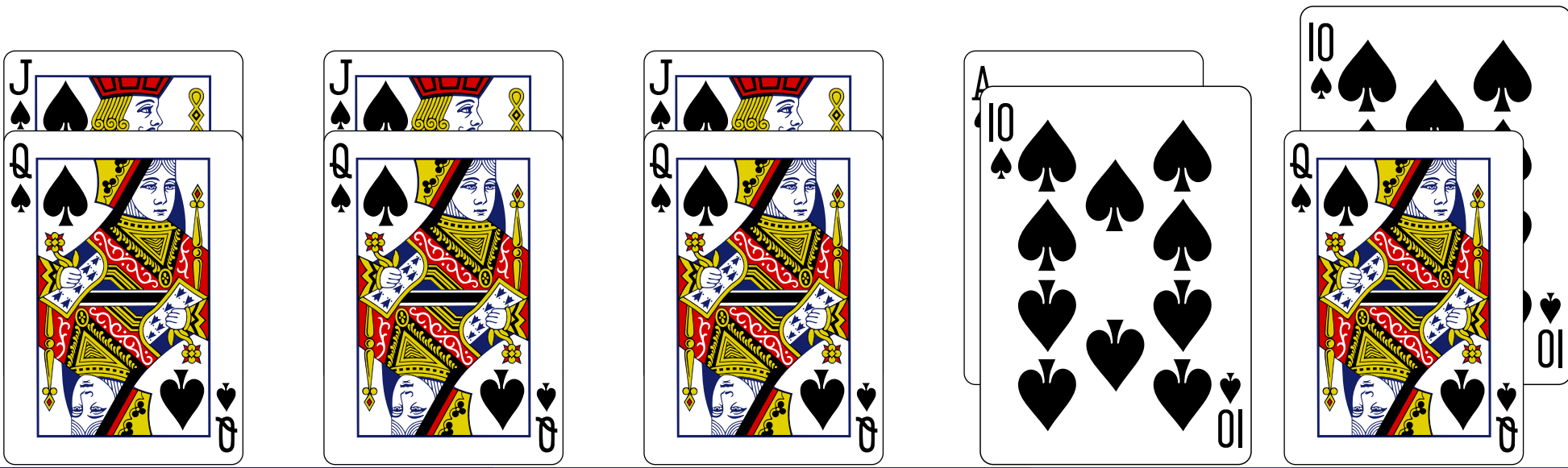

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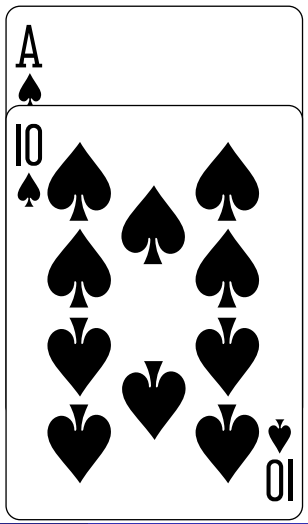


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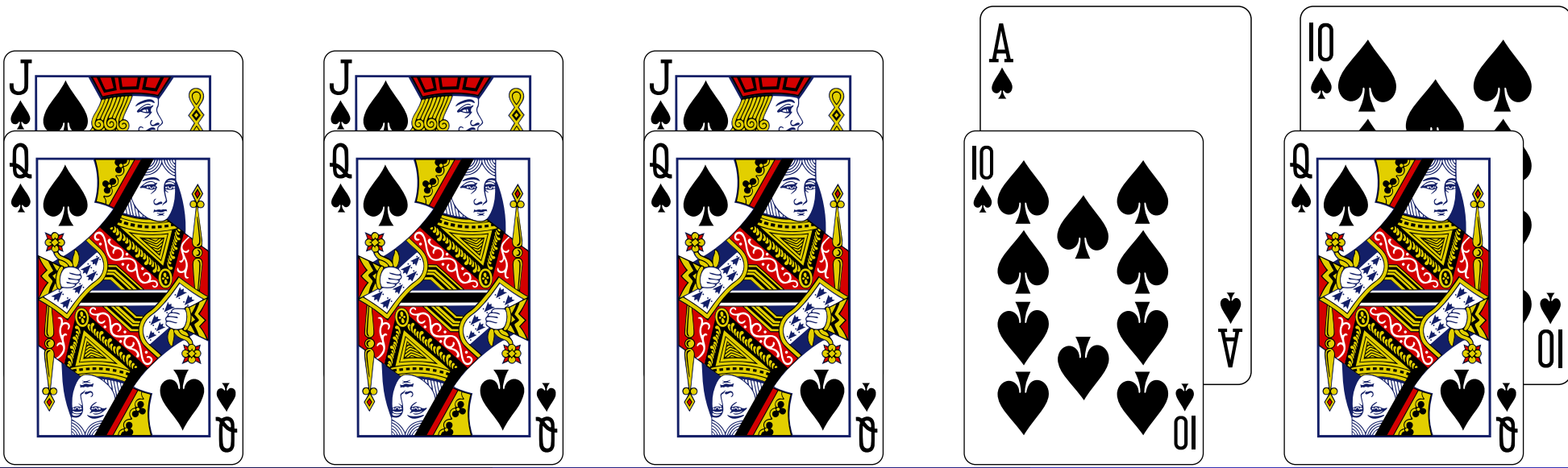

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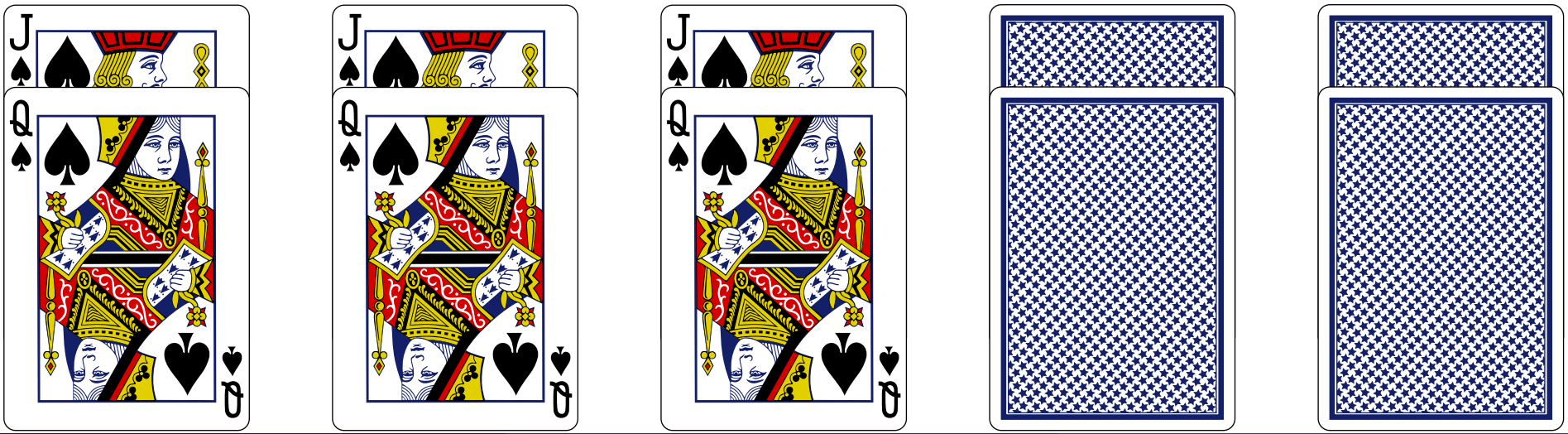
2. Player calls those stacks done and never plays with those card types again



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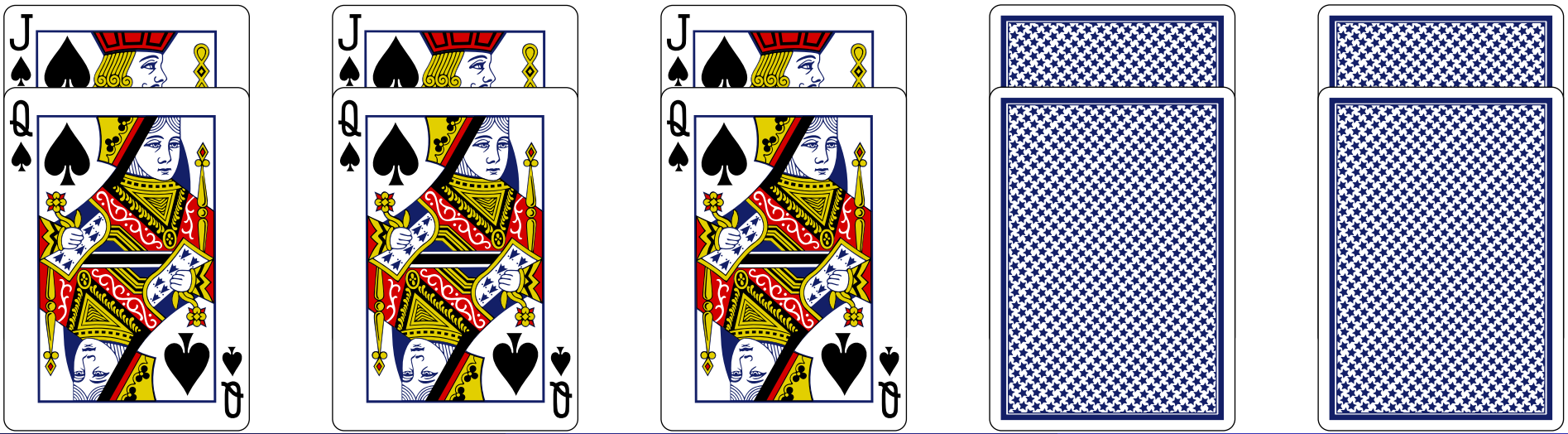

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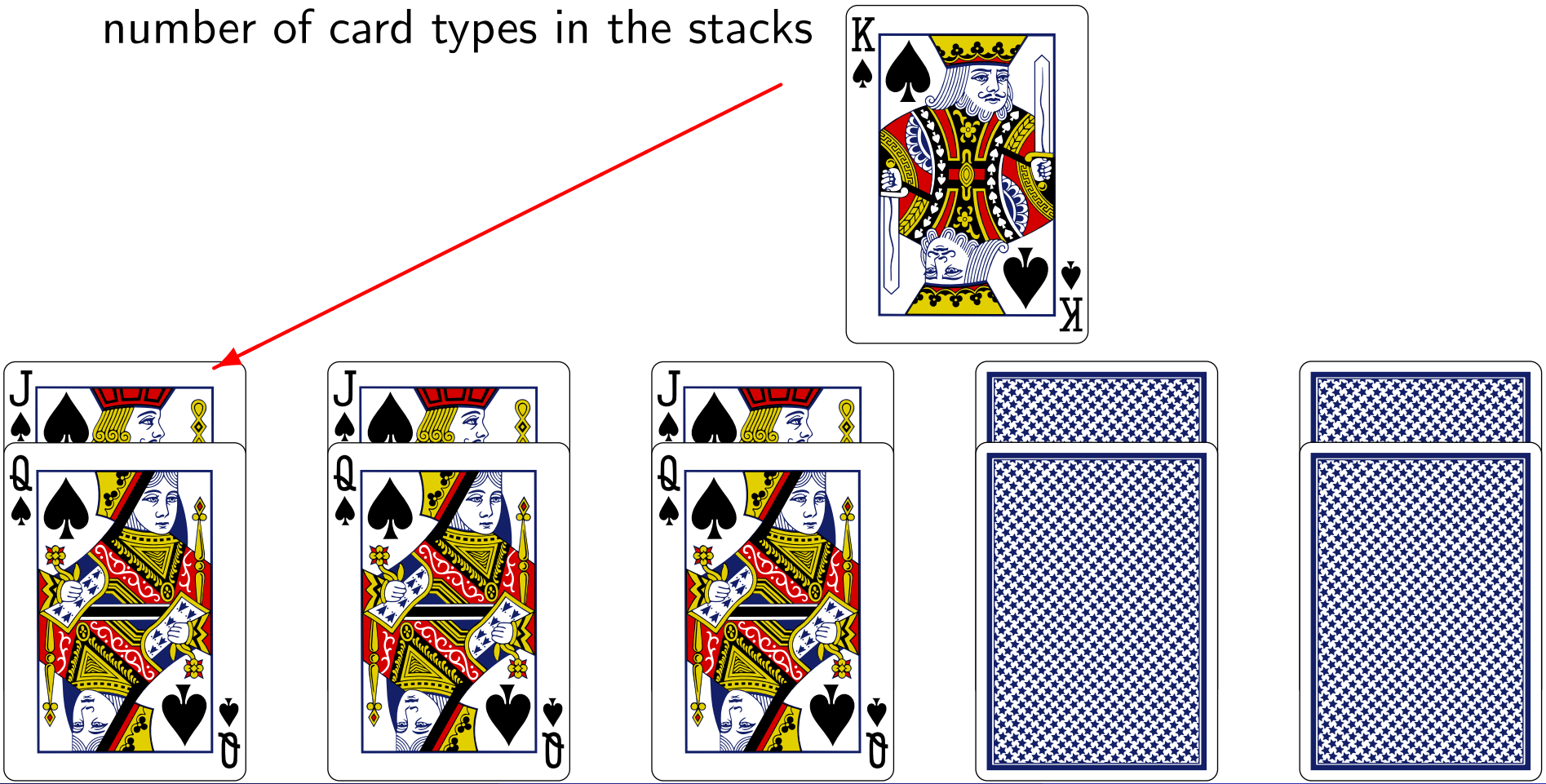

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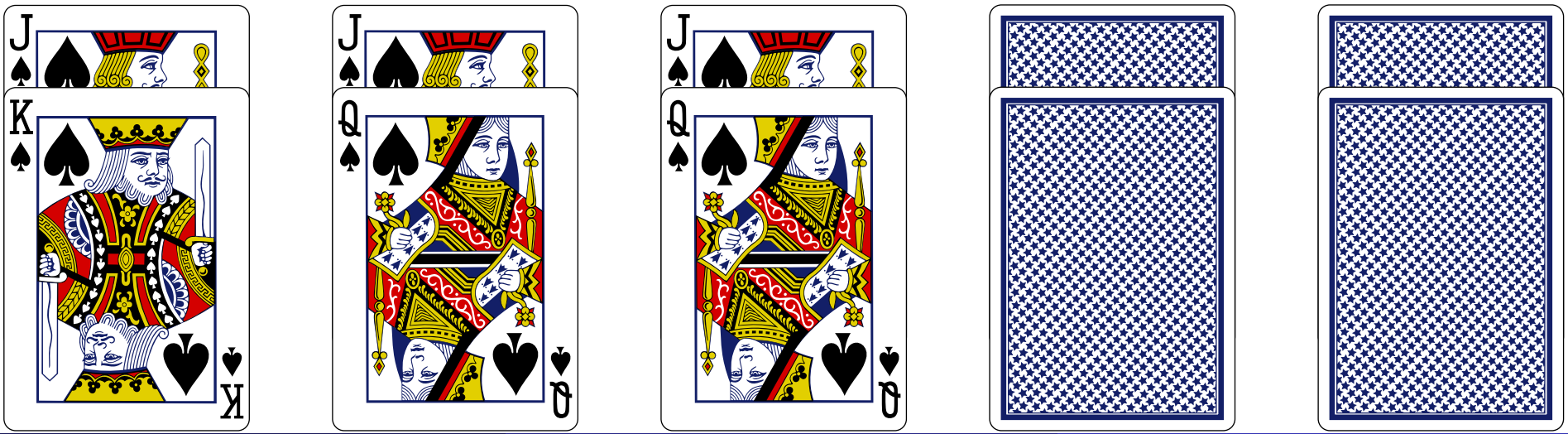

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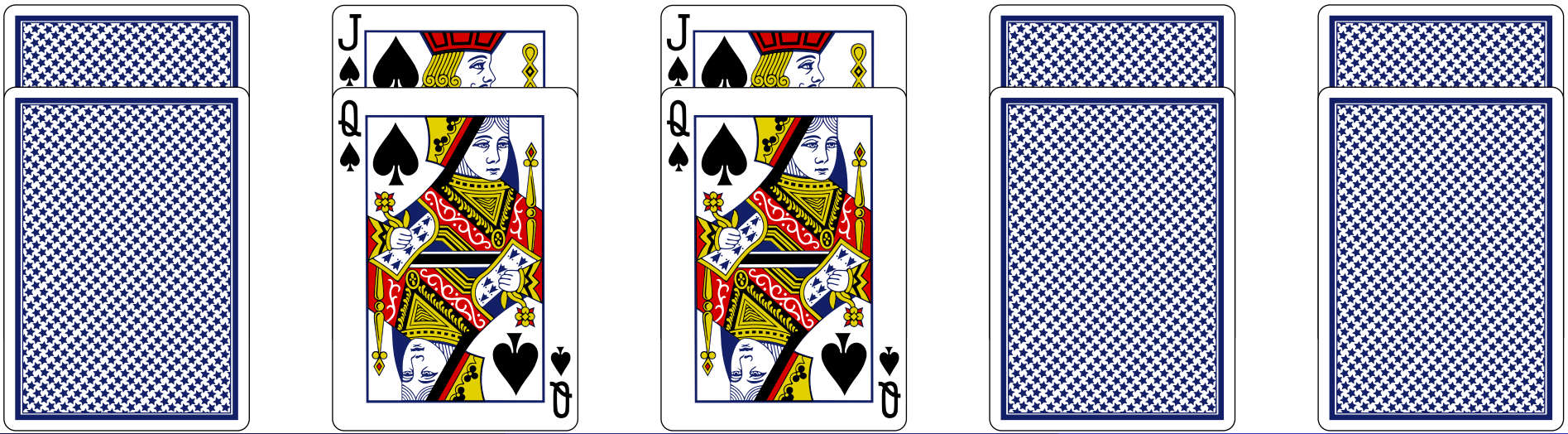

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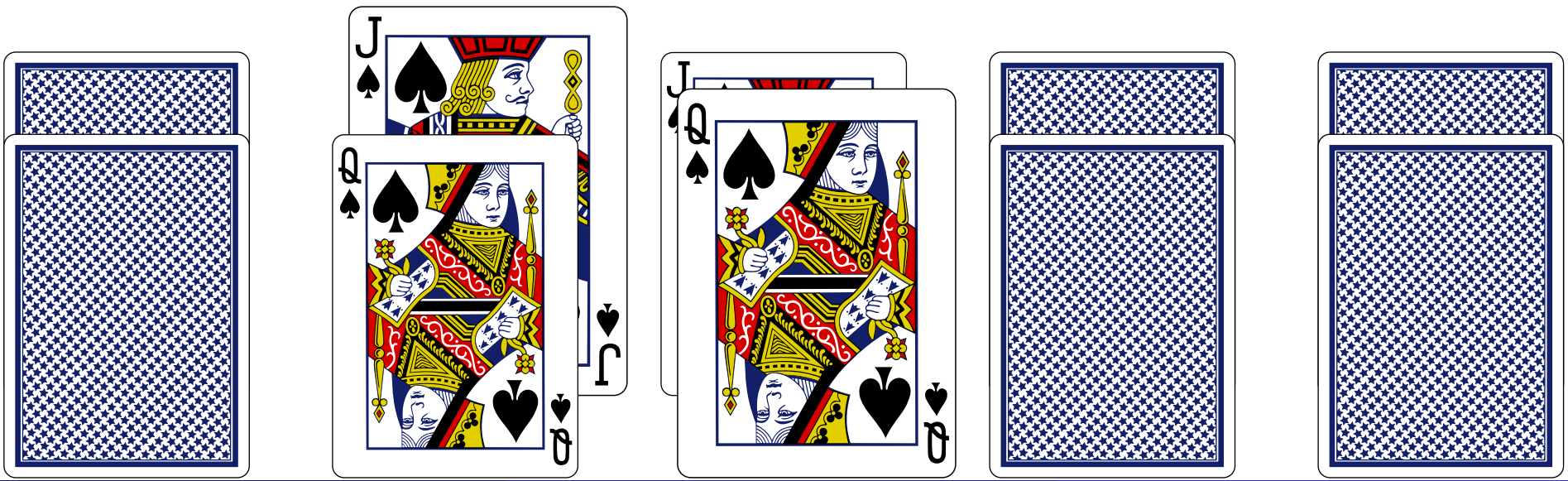

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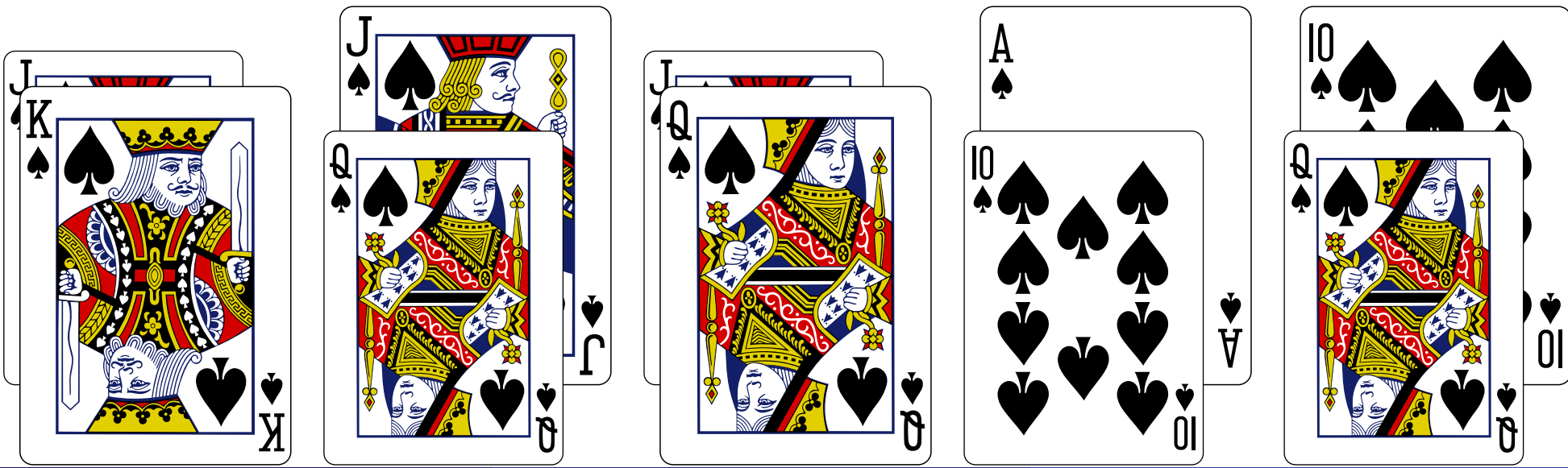

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making it harder for Player

- allow Dealer to make more swaps in response to Player's move

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  - Player's moves are useless

edge coloring

setup

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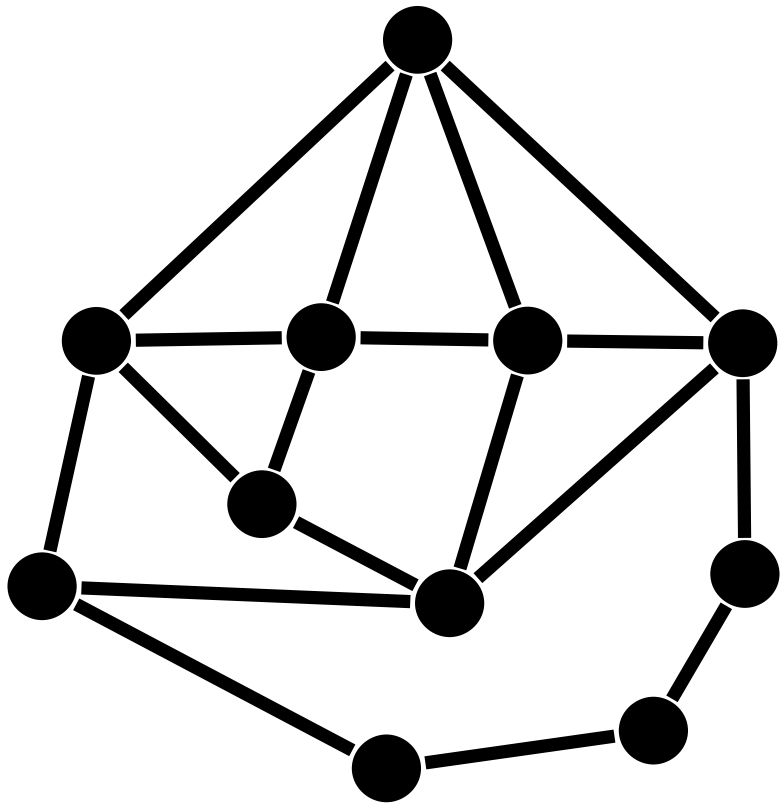
Vizing's theorem

Any simple graph can be edge-colored using at most one more color than its maximum degree.

edge coloring

proof of Vizing's theorem

- proceed by induction on the number of vertices



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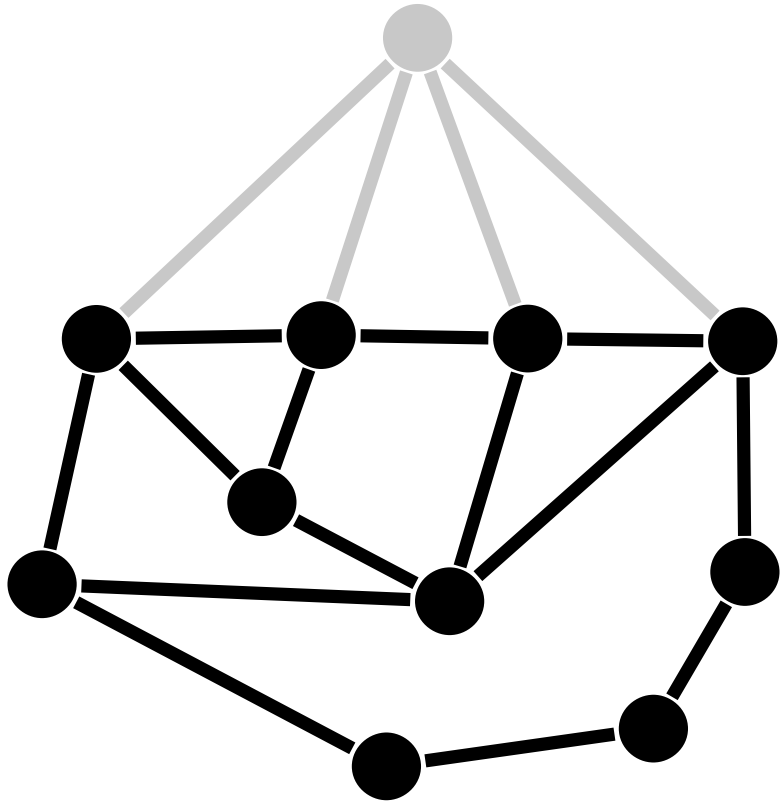
- proceed by induction on the number of vertices
- remove a vertex and edge-color the rest with one more color than its maximum degree



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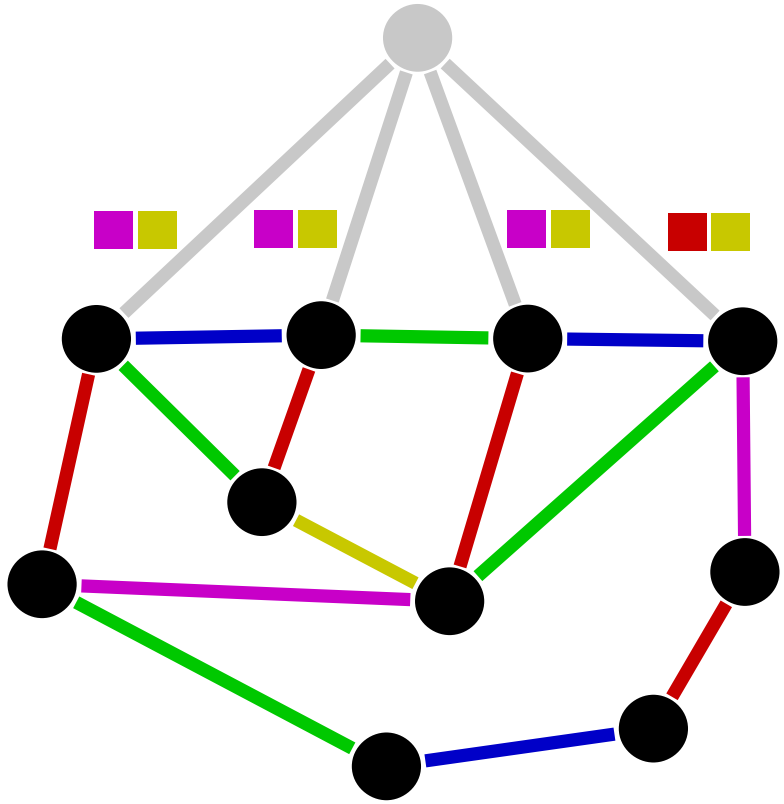

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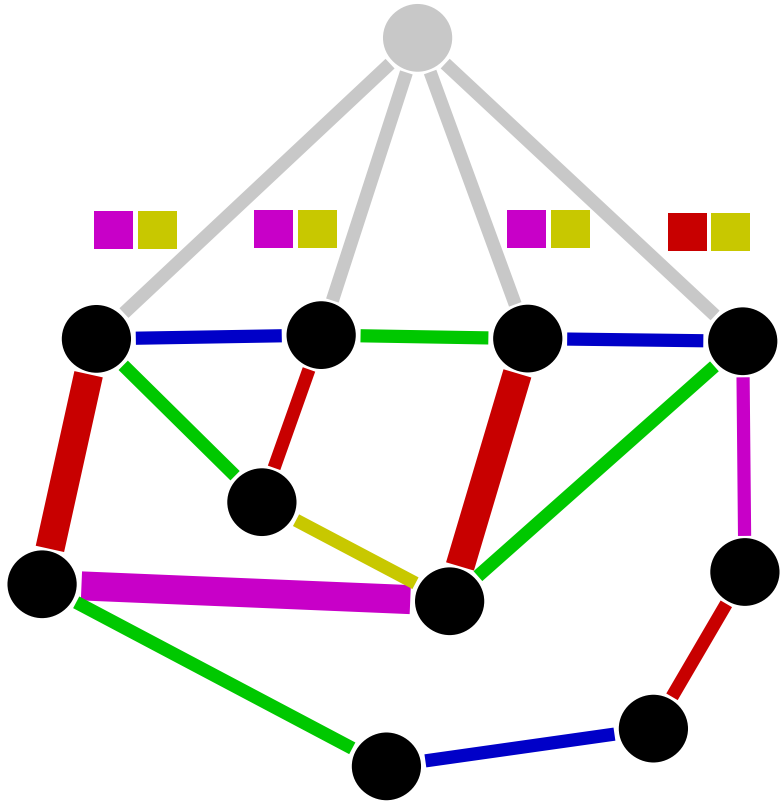

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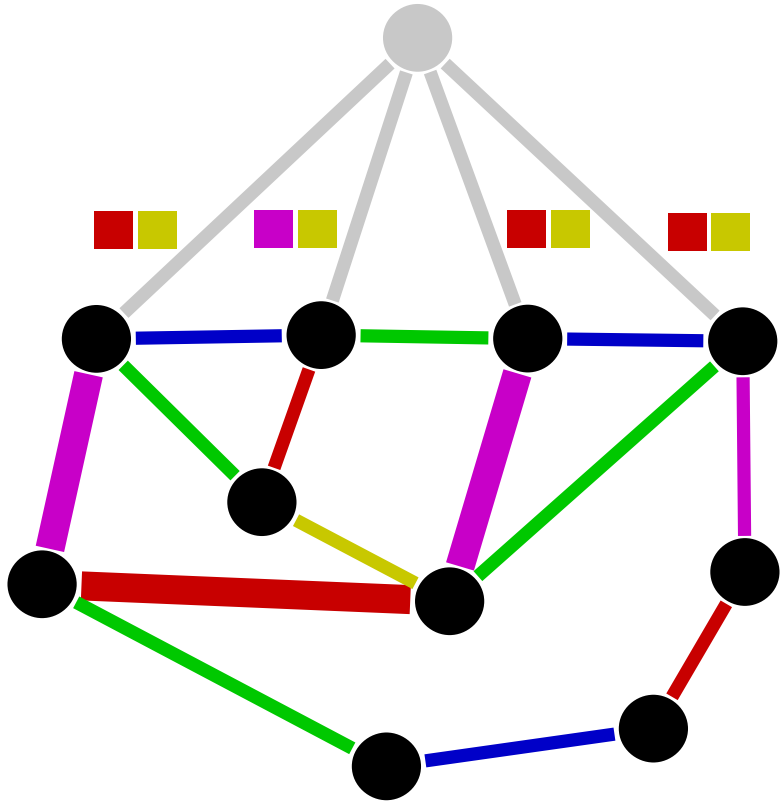

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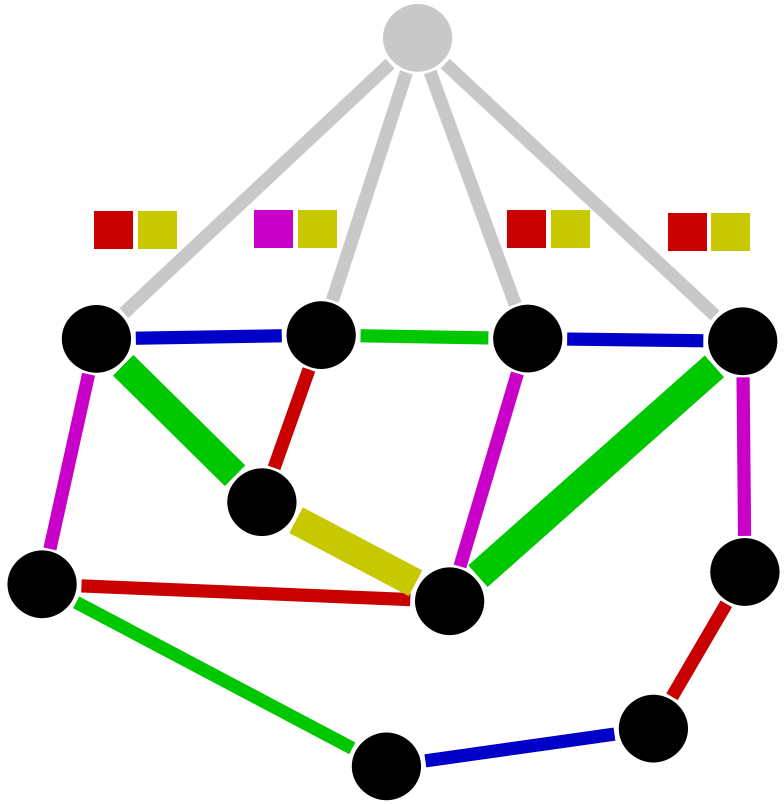

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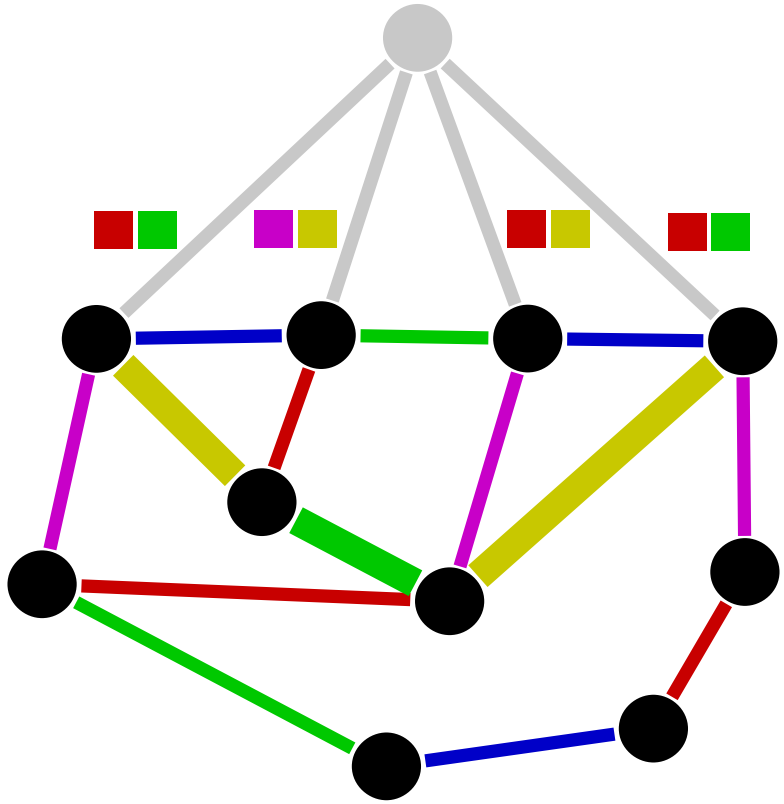

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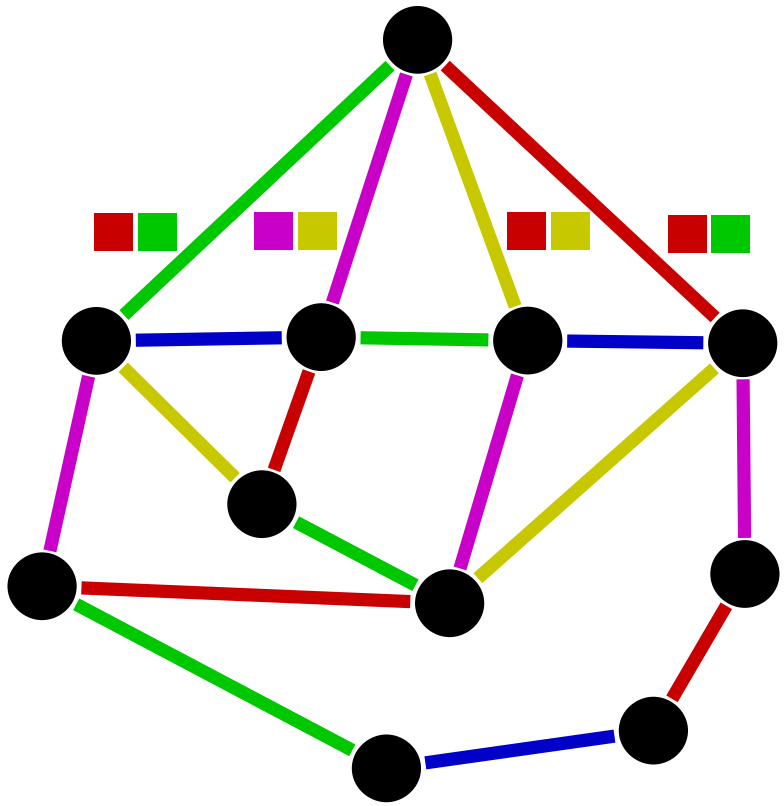

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