A better lower bound on average degree of k-list-critical graphs.

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1 Introduction

Main Theorem. *For $k \geq 7$, every non-complete k-list-critical graph has average degree at least*

$$k - 1 + \frac{(k-3)^2(2k-3)}{k^4 - 2k^3 - 11k^2 + 28k - 14}.$$

We stated the bounds for

Let $c_k^*(G)$ be the number of components of G containing a copy of K_{k-1}. Let $q_k(G)$ be the number of non-cut vertices in G that appear in copies of K_{k-1}. Let $\beta_k(G)$ be the independence number of the subgraph of G induced on the vertices of degree $k-1$. When k is defined in context, 'just write $c^*(G)$, $q(G)$ and $\beta(G)$. Sections 3 and 4 prove the following upper bounds on $q(\mathcal{L})$ and $\beta(\mathcal{L})$.

Lemma 1.1. *Let G be a non-complete k-list-critical graph where $k \geq 5$. Let $\mathcal{L}$ be the subgraph of G induced on $(k-1)$-vertices, $\mathcal{H}^-$ the subgraph of G induced on k-vertices, $\mathcal{H}$ the subgraph of G induced on k^+-vertices, $\mathcal{H}^+$ the subgraph of G induced on $(k+1)^+$-vertices. Then*

$$q(\mathcal{L}) \leq c^*(\mathcal{L}) + 4\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\|,$$

and if $k \geq 7$, then

$$q(\mathcal{L}) \leq 2c^*(\mathcal{L}) + 3\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\|.$$

Lemma 1.2. *Let G be a k-list-critical graph. Let $\mathcal{L}$ be the subgraph of G induced on $(k-1)$-vertices and $\mathcal{H}$ the subgraph of G induced on k^+-vertices. If $2 \leq \lambda \leq \frac{6(k-1)}{k}$, then*

$$\beta(\mathcal{L}) \leq \frac{2}{\lambda}\left\|\mathcal{H}\right\| + \frac{2\left\|G\right\| - (k-2)\left|G\right| - \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)\left|\mathcal{H}\right| - 1}{k-1}.$$

2 General bounds on average degree

Definition 1. A quadruple (p, h, z, f) of functions from $\mathbb{N}$ to $\mathbb{R}$ is *r-Gallai* if for every $k \geq r$ and Gallai tree $T \neq K_k$ with $\Delta(T) \leq k-1$, the following hold:

- if $K_{k-1} \subseteq T$, then $2\left\|T\right\| \leq (k-3+p(k))\left|T\right| + h(k)q(T) + z(k)\beta(T) + f(k)$; and
- if $K_{k-1} \not\subseteq T$, then $2\left\|T\right\| \leq (k-3+p(k))\left|T\right| + z(k)\beta(T)$.

Theorem 2.1. *Let (p, h, z, f) be 7-Gallai. If $k \geq 7$ and $2 \leq z(k) \leq \frac{6(k-1)}{k}$, then for any non-complete k-list-critical graph G,*

$$d(G) \geq k - 1 + \frac{2 - p(k) - \frac{z(k)}{k-1} + \frac{\frac{z(k)}{k-1} - (2h(k) + f(k))c^*(\mathcal{L})}{|G|}}{k + 1 + 3h(k) - p(k) - \frac{(k-2)z(k)}{2(k-1)}},$$

where $\mathcal{L}$ is the subgraph of G induced on $(k-1)$-vertices.

Proof. Let $\mathcal{L}$ be the subgraph of G induced on $(k-1)$-vertices, $\mathcal{H}^-$ the subgraph of G induced on k-vertices, $\mathcal{H}$ the subgraph of G induced on k^+-vertices, $\mathcal{H}^+$ the subgraph of G induced on $(k+1)^+$-vertices and $\mathcal{D}$ the components of $\mathcal{L}$ containing K_{k-1}. Plainly, the following bounds hold.

$$2\left\|G\right\| \geq k\left|G\right| - \left|\mathcal{L}\right| \tag{1}$$

$$2\left\|G\right\| \geq (k+1)\left|G\right| - \left|\mathcal{H}^-\right| - 2\left|\mathcal{L}\right| \tag{2}$$

$$2\left\|G\right\| \geq k\left|\mathcal{H}^-\right| + (k-1)\left|\mathcal{L}\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\| \tag{3}$$

$$\left\|\mathcal{H}, \mathcal{L}\right\| = (k-1)\left|\mathcal{L}\right| - 2\left\|\mathcal{L}\right\| \tag{4}$$

Since (p, h, z, f) is 7-Gallai,

$$2\left\|\mathcal{L}\right\| \leq (k-3+p(k))\left|\mathcal{L}\right| + f(k)\left|\mathcal{D}\right| + h(k)q(\mathcal{L}) + z(k)\beta(\mathcal{L}) \tag{5}$$

By Lemma 1.1,

$$q(\mathcal{L}) \leq 2\left|\mathcal{D}\right| + 3\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\|,$$

plugging this into (5) gives

$$2\left\|\mathcal{L}\right\| \leq (k-3+p(k))\left|\mathcal{L}\right| + 3h(k)\left|\mathcal{H}^-\right| + h(k)\left\|\mathcal{H}^+, \mathcal{L}\right\| + z(k)\beta(\mathcal{L}) + S_1, \tag{6}$$

where

$$S_1 := (2h(k) + f(k))\left|\mathcal{D}\right|.$$

Now using (1) and (6),

$$\begin{aligned}
2\left\|G\right\| &= 2\left\|\mathcal{H}\right\| + 2\left\|\mathcal{H}, \mathcal{L}\right\| + 2\left\|\mathcal{L}\right\| \\
&= 2\left\|\mathcal{H}\right\| + 2((k-1)\left|\mathcal{L}\right| - 2\left\|\mathcal{L}\right\|) + 2\left\|\mathcal{L}\right\| \\
&= 2\left\|\mathcal{H}\right\| + 2(k-1)\left|\mathcal{L}\right| - 2\left\|\mathcal{L}\right\| \\
&\geq 2\left\|\mathcal{H}\right\| + (k+1-p(k))\left|\mathcal{L}\right| - 3h(k)\left|\mathcal{H}^-\right| - h(k)\left\|\mathcal{H}^+, \mathcal{L}\right\| - z(k)\beta(\mathcal{L}) - S_1
\end{aligned} \tag{7}$$

Adding $h(k)$ times (3) to (7) gives

$$2\left\|G\right\| \geq \frac{2\left\|\mathcal{H}\right\| + (k+1+(k-1)h(k)-p(k))\left|\mathcal{L}\right| + (k-3)h(k)\left|\mathcal{H}^-\right| - z(k)\beta(\mathcal{L}) - S_1}{1+h(k)} \tag{8}$$

Lemma 1.2 gives

$$\beta(\mathcal{L}) \leq \frac{2}{z(k)}\left\|\mathcal{H}\right\| + \frac{2\left\|G\right\| - (k-2)\left|G\right| - \left(\frac{k}{2} + \frac{k-1}{z(k)}\right)\left|\mathcal{H}\right| - 1}{k-1}.$$

Plugging this into (8) yields

$$2\left\|G\right\| \geq \frac{(k+1+(k-1)h(k)-p(k))\left|\mathcal{L}\right| + (k-3)h(k)\left|\mathcal{H}^-\right| + \frac{(k-2)z(k)}{k-1}\left|G\right| + \left(\frac{kz(k)}{2(k-1)} + 1\right)\left|\mathcal{H}\right| + S_2}{1 + h(k) + \frac{z(k)}{k-1}}, \tag{9}$$

where

$$S_2 := \frac{z(k)}{k-1} - S_1.$$

Now using $\left|\mathcal{H}\right| = \left|G\right| - \left|\mathcal{L}\right|$ gives

$$2\left\|G\right\| \geq \frac{\left(k+(k-1)h(k)-p(k)-\frac{kz(k)}{2(k-1)}\right)\left|\mathcal{L}\right| + (k-3)h(k)\left|\mathcal{H}^-\right| + \left(\frac{(3k-4)z(k)}{2(k-1)} + 1\right)\left|G\right| + S_2}{1 + h(k) + \frac{z(k)}{k-1}}. \tag{10}$$

Now using (2) to get a lower bound on $|\mathcal{H}^-|$ gives

$$2\|G\| \geq \frac{\left(k-(k-5)h(k)-p(k)-\frac{kz(k)}{2(k-1)}\right)|\mathcal{L}|+\left((k+1)(k-3)h(k)+\frac{(3k-4)z(k)}{2(k-1)}+1\right)|G|+S_2}{1+(k-2)h(k)+\frac{z(k)}{k-1}}. \tag{11}$$

Using (1) to get a lower bound on $|\mathcal{L}|$ and simplifying gives

$$\frac{2\|G\|}{|G|} \geq \frac{k^2+3(k-1)h(k)-kp(k)+1-\frac{k^2-3k+4}{2(k-1)}z(k)+\frac{S_2}{|G|}}{k+1+3h(k)-p(k)-\frac{(k-2)z(k)}{2(k-1)}}. \tag{12}$$

Now factoring out $k-1$ gives the desired bound. $\square$

A nearly identical argument, using the other inequality in Lemma 1.1, proves a bound that holds for $k \geq 5$.

Theorem 2.2. *Let (p,h,z,f) be 5-Gallai. If $k \geq 5$ and $2 \leq z(k) \leq \frac{6(k-1)}{k}$, then for any non-complete k-list-critical graph G,*

$$d(G) \geq k-1+\frac{2-p(k)-\frac{z(k)}{k-1}+\frac{\frac{z(k)}{k-1}-(h(k)+f(k))c^*(\mathcal{L})}{|G|}}{k+1+4h(k)-p(k)-\frac{(k-2)z(k)}{2(k-1)}},$$

where $\mathcal{L}$ is the subgraph of G induced on $(k-1)$-vertices.

When $k=4$, we cannot apply Lemma 1.1, but using $h(k)=0$ and running through the same argument proves the following bound for $k \geq 4$.

Theorem 2.3. *Let $(p,0,z,f)$ be 4-Gallai. If $k \geq 4$ and $2 \leq z(k) \leq \frac{6(k-1)}{k}$, then for any non-complete k-list-critical graph G,*

$$d(G) \geq k-1+\frac{2-p(k)-\frac{z(k)}{k-1}+\frac{\frac{z(k)}{k-1}-f(k)c^*(\mathcal{L})}{|G|}}{k+1-p(k)-\frac{(k-2)z(k)}{2(k-1)}},$$

where $\mathcal{L}$ is the subgraph of G induced on $(k-1)$-vertices.

When $z(k)<2$, using Lemma 1.2 worsens the lower bound, so we may as well use $z(k)=0$; that is, drop the $\beta(\mathcal{L})$ term entirely. Doing so in the above argument shows that Theorems 2.1, 2.2, 2.3 also hold for $z(k)=0$. This gives the bounds proved by discharging in Cranston and R. [1].

3 Gallai quadruples

Lemma 3.1 (Gallai [2])**.** $\left(\frac{k+1}{k-1},0,0,-2\right)$ *is 4-Gallai.*

Lemma 3.2 (Kostochka-Stiebitz [5])**.** $\left(\frac{4(k-1)}{k^2-3k+4},\frac{k^2-3k}{k^2-3k+4},0,\frac{-4(k^2-3k+2)}{k^2-3k+4}\right)$ *is 7-Gallai.*

Lemma 3.3 (Cranston-R. [1])**.** $\left(\frac{3k-5}{k^2-4k+5},\frac{k(k-3)}{k^2-4k+5},0,\frac{-2(k-1)(2k-5)}{k^2-4k+5}\right)$ *is 5-Gallai.*

Lemma 3.4 (R. [7])**.** $(1,0,2,0)$ *is 4-Gallai.*

For an endblock B of a Gallai tree T, let x_B be the cutvertex contained in B.

Lemma 3.5. *Let $z\colon \mathbb{N} \to \mathbb{R}$ such that $z(k)=0$ or $z(k) \geq 2$. For all $k \geq 5$ and Gallai trees T with $\Delta(T) \leq k-1$ and $K_{k-1} \not\subseteq T$, we have*

$$2\|T\| \leq \left(k-3+\frac{\max\{2,3-z(k)\}}{k-2}\right)|T|+z(k)\beta(T).$$

Proof. Suppose the lemma is false and choose a counterexample T minimizing $|T|$.
Claim 1. *T has at least two blocks.*

If T has only one block, then $2\,\|T\| \leq (k-3)\,|T|$.
Claim 2. *Each endblock of T is K_{k-2}.*

Suppose T has an endblock B that is not K_{k-2}. Then removing $V(B) \setminus \{x_B\}$ from T to get T' and applying minimality of $|T|$ gives

$$2\,\|B\| > \left(k-3+\frac{\max\{2, 3-z(k)\}}{k-2}\right)(|B|-1)\,.$$

This is a contradiction unless $k=5$ and $B=K_3$, but then $B=K_{k-2}$, a contradiction.
Claim 3. *If B is an endblock of T, then $d_T(x_B)=k-1$.*

Suppose B is an endblock of T with $d_T(x_B) < k-1$. Then $B=K_{k-2}$ by Claim 2 and hence $d_T(x_B)=k-2$. Removing $V(B)$ from T to get T^* and applying minimality of $|T|$ gives the contradiction

$$(k-2)(k-3)+6 > \left(k-3+\frac{\max\{2, 3-z(k)\}}{k-2}\right)(k-1)\,.$$

Claim 4. *T does not exist.*

By the previous claims, we know that every endblock T is a K_{k-2} that shares a vertex with an odd cycle. Pick and endblock B that is the end of a longest path in the block-tree of T. Let C be the odd cycle sharing x_B with B. Since B is the end of a longest path in the block-tree, there is a neighbor y of x_B on C such that $d_T(y)=2$ or y is contained in another endblock A (which must be a K_{k-2}). First, suppose $d_T(y)=2$. Removing $V(B)\cup\{y\}$ from T to get T' and applying minimality of $|T|$ gives the contradiction (since $\beta(T') < \beta(T)$)

$$(k-2)(k-3)+6 > \left(k-3+\frac{\max\{2, 3-z(k)\}}{k-2}\right)(k-1)+z(k)(\beta(T)-\beta(T')).$$

Hence y is contained in another K_{k-2} endblock A. Removing $V(B)\cup V(A)$ from T to get T^* and applying minimality of $|T|$ gives the contradiction (since $\beta(T^*) < \beta(T)$)

$$2(k-2)(k-3)+6 > \left(k-3+\frac{\max\{2, 3-z(k)\}}{k-2}\right)(2(k-2))+z(k)(\beta(T)-\beta(T^*)).$$

□

Lemma 3.6. *Let $p\colon \mathbb{N}\to\mathbb{R}_{\geq 0}$, $f\colon \mathbb{N}\to\mathbb{R}$, $h\colon \mathbb{N}\to\mathbb{R}_{\geq 0}$, $z\colon \mathbb{N}\to\mathbb{R}_{\geq 0}$ such that $z(k)=0$ or $z(k)\geq 2$. For all $k\geq 5$ and Gallai trees $T\neq K_k$ with $\Delta(T)\leq k-1$ and $K_{k-1}\subseteq T$, we have*

$$2\,\|T\| \leq (k-3+p(k))\,|T| + f(k) + h(k)q(T) + z(k)\beta(T)$$

whenever p, f, h and z satisfy all of the following conditions:

(1) $f(k) \geq (k-1)(1-p(k)-h(k))$; *and*

(2) $p(k) \geq \frac{3-z(k)}{k-2}$; *and*

(3) $p(k) \geq h(k)+5-k$; *and*

(4) $p(k) \geq \frac{2+h(k)}{k-2}$; *and*

(5) $(k-1)p(k)+(k-3)h(k)+z(k) \geq k+1$.

Proof. Suppose the lemma is false and choose a counterexample T minimizing $|T|$.
Claim 1. *T has at least two blocks.*

Otherwise, $T=K_{k-1}$ and (1) gives a contradiction.
Claim 2. *Each endblock of T is K_{k-2} or K_{k-1}.*

Suppose T has an endblock B that is not K_{k-2} or K_{k-1}. Then removing $V(B) \setminus \{x_B\}$ from T to get T' and applying minimality of $|T|$ gives

$$2\,\|B\| > (k-3+p(k))\,(|B|-1) + h(k)(q(T)-q(T')) + z(k)(\beta(T)-\beta(T')).$$

If $B = K_2$, then $q(T') \leq q(T)+1$, otherwise $q(T') = q(T)$. For $B = K_2$, we have to contradiction (to (3))

$$2 > (k-3+p(k)) - h(k).$$

Suppose $B = K_t$ for $4 \leq t \leq k-3$. Then we have the contradiction

$$t(t-1) > (k-3+p(k))\,(t-1)\,.$$

Finally, suppose B is an odd cycle of length ℓ. Then, we have

$$2\ell > (k-3+p(k))\,(\ell-1)\,.$$

This simplifies to

$$\ell < 1 + \frac{2}{k-5+p(k)}.$$

Since $k-5+p(k) \geq 1$ when $k \geq 6$, this implies that $k = 5$. Using (4), we conclude $\ell = 3$, but then $B = K_{k-2}$, a contradiction.

Claim 3. *T has at most one K_{k-1} endblock.*

Suppose T has at least two K_{k-1} endblocks. Let B be one of them. Then removing $V(B)$ from T leaves a graph T' with $K_{k-1} \subseteq T'$. So, we may apply minimality of $|T|$ to get

$$(k-1)(k-2)+2 > (k-3+p(k))\,(k-1) + h(k)(q(T)-q(T')) + z(k)(\beta(T)-\beta(T')).$$

Now $\beta(T') < \beta(T)$ and $q(T') \leq q(T)-(k-2)+1$, so we have the contradiction (to (5))

$$k+1 > (k-1)p(k) + (k-3)h(k) + z(k).$$

Claim 4. *If B is an endblock of T, then $d_T(x_B) = k-1$.*

Suppose B is an endblock of T with $d_T(x_B) < k-1$. Then $B = K_{k-2}$ by Claim 2. Removing $V(B)$ from T leaves a graph T' with $K_{k-1} \subseteq T'$. So, we may apply minimality of $|T|$ to get

$$(k-2)(k-3)+2 > (k-3+p(k))\,(k-2) + h(k)(q(T)-q(T')) + z(k)(\beta(T)-\beta(T')).$$

We have $q(T') \leq q(T)+1$, so this is gives the contradiction (to (4))

$$2 > (k-1)p(k) - h(k).$$

Claim 5. *T does not exist.* □

4 Bounding $q(\mathcal{L})$

This section is devoted to extracting the reusable Lemma 4.1 from the proof of Kierstead and R. [3].

Definition 2. A graph G is *AT-reducible* to H if H is a nonempty induced subgraph of G which is f_H-AT where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not AT-reducible to any nonempty induced subgraph, then it is *AT-irreducible.*

Lemma 4.1. *Let G be a non-complete AT-irreducible graph with $\delta(G) = k-1$ where $k \geq 5$. Let $\mathcal{L}$ be the subgraph of G induced on $(k-1)$-vertices, $\mathcal{H}^-$ the subgraph of G induced on k-vertices, $\mathcal{H}$ the subgraph of G induced on k^+-vertices, $\mathcal{H}^+$ the subgraph of G induced on $(k+1)^+$-vertices and $\mathcal{D}$ the components of $\mathcal{L}$ containing K_{k-1}. Then*

$$q(\mathcal{L}) \leq |\mathcal{D}| + 4\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\|,$$

and if $k \geq 7$, then

$$q(\mathcal{L}) \leq 2\,|\mathcal{D}| + 3\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\|.$$

Observation. The hypotheses of Lemma 4.1 are satisfied by non-complete k-critical, k-list-critical, online k-list-critical and k-AT-critical graphs.

The proof of Lemma 4.1 requires the following four lemmas from [3].

Lemma 4.2. *Let G be a graph and $f\colon V(G) \to \mathbb{N}$. If $\|G\| > \sum_{v \in V(G)} f(v)$, then G has an induced subgraph H such that $d_H(v) > f(v)$ for each $v \in V(H)$.*

Proof. Suppose not and choose a counterexample G minimizing $|G|$. Then $|G| \geq 3$ and we have $x \in V(G)$ with $d_G(x) \leq f(x)$. But now $\|G - x\| > \sum_{v \in V(G-x)} f(v)$, contradicting minimality of $|G|$. $\square$

Let $\mathcal{T}_k$ be the Gallai trees with maximum degree at most $k-1$, excepting K_k. For a graph G, let $W^k(G)$ be the set of vertices of G that are contained in some K_{k-1} in G.

Lemma 4.3. *Let $k \geq 5$ and let G be a graph with $x \in V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *$G - x$ has t components $H_1, H_2, \ldots, H_t$, and all are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k - 1$ for all $v \in V(G - x)$; and*
4. *$\left|N(x) \cap W^k(H_i)\right| \geq 1$ for $i \in [t]$; and*
5. *$d_G(x) \geq t + 2$.*

Then G is f-AT where $f(x) = d_G(x) - 1$ and $f(v) = d_G(v)$ for all $v \in V(G - x)$.

For a graph G, $\{X, Y\}$ a partition of $V(G)$ and $k \geq 4$, let $\mathcal{B}_k(X, Y)$ be the bipartite graph with one part Y and the other part the components of $G[X]$. Put an edge between $y \in Y$ and a component T of $G[X]$ iff $N(y) \cap W^k(T) \neq \emptyset$. The next lemma tells us that we have a reducible configuration if this bipartite graph has minimum degree at least three.

Lemma 4.4. *Let $k \geq 7$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*
4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $\delta(\mathcal{B}) \geq 3$.*

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.

We also have the following version with asymmetric degree condition on $\mathcal{B}$. The point here is that this works for $k \geq 5$. The consequence is that we trade a bit in our bound for the proof to go through with $k \in \{5, 6\}$.

Lemma 4.5. *Let $k \geq 5$ and let G be a graph with $Y \subseteq V(G)$ such that:*

1. *$K_k \not\subseteq G$; and*
2. *the components of $G - Y$ are in $\mathcal{T}_k$; and*
3. *$d_G(v) \leq k - 1$ for all $v \in V(G - Y)$; and*
4. *with $\mathcal{B} := \mathcal{B}_k(V(G - Y), Y)$ we have $d_{\mathcal{B}}(y) \geq 4$ for all $y \in Y$ and $d_{\mathcal{B}}(T) \geq 2$ for all components T of $G - Y$.*

Then G has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for all $v \in V(G' - Y)$.

Proof of Lemma 4.1. Put $W := W^k(\mathcal{L})$ and $L' := V(\mathcal{L}) \setminus W$. Define an auxiliary bipartite graph F with parts A and B where:

1. $B = V(\mathcal{H}^-)$ and A is the disjoint union of the following sets A_1, A_2 and A_3,
2. $A_1 = \mathcal{D}$ and each $T \in \mathcal{D}$ is adjacent to all $y \in B$ where $N(y) \cap W^k(T) \neq \emptyset$,
3. For each $v \in L'$, let $A_2(v)$ be a set of $|N(v) \cap B|$ vertices connected to $N(v) \cap B$ by a matching in F. Let A_2 be the disjoint union of the $A_2(v)$ for $v \in L'$,
4. For each $y \in B$, let $A_3(y)$ be a set of $d_{\mathcal{H}}(y)$ vertices which are all joined to y in F. Let A_3 be the disjoint union of the $A_3(y)$ for $y \in B$.

Define $f\colon V(F) \to \mathbb{N}$ by $f(v) = 1$ for all $v \in A_1 \cup A_2 \cup A_3$ and $f(v) = 3$ for all $v \in B$. First, suppose $\|F\| > \sum_{v \in V(F)} f(v)$. Then by Lemma 4.2, F has an induced subgraph Q such that $d_Q(v) > f(v)$ for each $v \in V(Q)$. In particular, $V(Q) \subseteq B \cup A_1$ and $d_Q(v) \geq 4$ for $v \in B \cap V(Q)$ and $d_Q(v) \geq 2$ for $v \in A_1 \cap V(Q)$. Put $Y := B \cap V(Q)$ and let X be $\bigcup_{T \in V(Q) \cap A_1} V(T)$. Now $Z := G[X \cup Y]$ satisfies the hypotheses of Lemma 4.5, so Z has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for $v \in X$. Since $Y \subseteq B$ and $X \subseteq V(\mathcal{L})$, we have $f(v) = k - 1 + d_{G'}(v) - d_G(v)$ for all $v \in V(G')$. Hence, G is AT-reducible to G', a contradiction. Therefore $\|F\| \leq \sum_{v \in V(F)} f(v) = 3\,|B| + |\mathcal{D}| + |A_2| + |A_3|$. By Lemma 4.3, for each $y \in B$ we have $d_F(y) \geq k-1$. Hence $\|F\| \geq (k-1)\,|B|$. This gives $(k-4)\,|B| \leq |\mathcal{D}| + |A_2| + |A_3|$. Now the first inequality in the lemma follows since $B = V(\mathcal{H}^-)$, $|A_3| = \sum_{v \in V(\mathcal{H}^-)} d_{\mathcal{H}}(v)$ and

$$\begin{aligned}|A_2| &= -q(\mathcal{L}) + \|\mathcal{H}, \mathcal{L}\| \\ &= -q(\mathcal{L}) + k\left|\mathcal{H}^-\right| + \left\|\mathcal{H}^+, \mathcal{L}\right\| - \sum_{v \in V(\mathcal{H}^-)} d_{\mathcal{H}}(v).\end{aligned}$$

Suppose $k \geq 7$. Define $f\colon V(F) \to \mathbb{N}$ by $f(v) = 1$ for all $v \in A_2 \cup A_3$ and $f(v) = 2$ for all $v \in B \cup A_1$. First, suppose $\|F\| > \sum_{v \in V(F)} f(v)$. Then by Lemma 4.2, F has an induced subgraph Q such that $d_Q(v) > f(v)$ for each $v \in V(Q)$. In particular, $V(Q) \subseteq B \cup A_1$ and $\delta(Q) \geq 3$. Put $Y := B \cap V(Q)$ and let X be $\bigcup_{T \in V(Q) \cap A_1} V(T)$. Now $Z := G[X \cup Y]$ satisfies the hypotheses of Lemma 4.4, so Z has an induced subgraph G' that is f-AT where $f(y) = d_{G'}(y) - 1$ for $y \in Y$ and $f(v) = d_{G'}(v)$ for $v \in X$. Since $Y \subseteq B$ and $X \subseteq V(\mathcal{L})$, we have $f(v) = k - 1 + d_{G'}(v) - d_G(v)$ for all $v \in V(G')$. Hence, G is AT-reducible to G', a contradiction.

Therefore $\|F\| \leq \sum_{v \in V(F)} f(v) = 2(|B| + |\mathcal{D}|) + |A_2| + |A_3|$. By Lemma 4.3, for each $y \in B$ we have $d_F(y) \geq k - 1$. Hence $\|F\| \geq (k-1)\,|B|$. This gives $(k-3)\,|B| \leq 2\,|\mathcal{D}| + |A_2| + |A_3|$. Now the second inequality in the lemma follows as before. $\square$

5 Bounding $\beta(\mathcal{L})$

This section is devoted to extracting the reusable Lemma 5.1 from the proof of R. [**?**].

Definition 3. A graph G is *OC-reducible* to H if H is a nonempty induced subgraph of G which is online f_H-choosable where $f_H(v) := \delta(G) + d_H(v) - d_G(v)$ for all $v \in V(H)$. If G is not OC-reducible to any nonempty induced subgraph, then it is *OC-irreducible.*

Lemma 5.1. *Let G be an OC-irreducible graph with $\delta(G) = k - 1$. Let $\mathcal{L}$ be the subgraph of G induced on $(k-1)$-vertices and $\mathcal{H}$ the subgraph of G induced on k^+-vertices. If $2 \leq \lambda \leq \frac{6(k-1)}{k}$, then*

$$\beta(\mathcal{L}) \leq \frac{2}{\lambda}\,\|\mathcal{H}\| + \frac{2\,\|G\| - (k-2)\,|G| - \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|\mathcal{H}| - 1}{k-1}.$$

Observation. The hypotheses of Lemma 5.1 are satisfied by k-critical, k-list-critical and online k-list-critical graphs.

The proof of Lemma 5.1 requires the following lemma from Kierstead and R. [4] that generalizes a kernel technique of Kostochka and Yancey [6].

Definition. The *maximum independent cover number* of a graph G is the maximum $\mathrm{mic}(G)$ of $\|I, V(G) \setminus I\|$ over all independent sets I of G.

Kernel Magic. *Every OC-irreducible graph G with $\delta(G) = k-1$ satisfies*

$$2\|G\| \geq (k-2)|G| + \mathrm{mic}(G) + 1.$$

Theorem 5.2. *[Lowenstein, et al.] If G is a connected graph then*

$$\alpha(G) \geq \frac{2}{3}|G| - \frac{1}{4}\|G\| - \frac{1}{3}.$$

Corollary 5.3. *If G is a connected graph then*

$$\alpha(G) \geq \frac{2}{3}|G| - \frac{1}{3}\|G\|.$$

Proof. By Theorem 5.2,

$$\alpha(G) \geq \frac{2}{3}|G| - \frac{1}{3}\|G\| + \frac{1}{12}\|G\| - \frac{1}{3},$$

so, the corollary holds if $\frac{1}{12}\|G\| \geq \frac{1}{3}$. If not, then $\|G\| < 4$, so G is K_1, K_2, P_3 or K_3 which all satisfy the desired bound. $\square$

Proof of Lemma 5.1. Fix λ with $2 \leq \lambda \leq \frac{6(k-1)}{k}$. Let M be the maximum of $\|I, V(G) \setminus I\|$ over all independent sets I of G with $I \subseteq \mathcal{H}$. Since the vertices in $\mathcal{L}$ with $k-1$ neighbors in $\mathcal{L}$ have no neighbors in $\mathcal{H}$,

$$\mathrm{mic}(G) \geq M + (k-1)\beta(\mathcal{L}). \tag{13}$$

Claim 1. *If C is a component of $G[\mathcal{H}]$, then*

$$k\alpha(C) \geq \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|C| - \left(\frac{2(k-1)}{\lambda}\right)\|C\|.$$

First, suppose $\|C\| < |C|$. Then $\|C\| = |C| - 1$ and C is a tree. If $|C| \geq 2$, then

$$\begin{aligned}
k\alpha(C) &\geq k\frac{|C|}{2} \\
&\geq \left(\frac{k}{2} - \frac{k-1}{\lambda}\right)|C| + \frac{2(k-1)}{\lambda} \\
&= \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|C| - \left(\frac{2(k-1)}{\lambda}\right)(|C| - 1) \\
&= \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|C| - \left(\frac{2(k-1)}{\lambda}\right)\|C\|.
\end{aligned}$$

If instead, $|C| = 1$, then $k\alpha(C) = k \geq \left(\frac{k}{2} + \frac{k-1}{\lambda}\right) = \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|C| - \left(\frac{2(k-1)}{\lambda}\right)\|C\|$ since $\lambda \geq 2$.

So, we may assume $\|C\| \geq |C|$. Applying Corollary 5.3, we conclude

$$\begin{aligned}
k\alpha(C) &\geq \frac{2k}{3}|C| - \frac{k}{3}\|C\| \\
&= \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|C| - \left(\frac{2(k-1)}{\lambda}\right)\|C\| + \left(\frac{k}{6} - \frac{k-1}{\lambda}\right)|C| - \left(\frac{k}{3} - \frac{2(k-1)}{\lambda}\right)\|C\| \\
&= \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|C| - \left(\frac{2(k-1)}{\lambda}\right)\|C\| + \left(\frac{k-1}{\lambda} - \frac{k}{6}\right)|C| \\
&\geq \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|C| - \left(\frac{2(k-1)}{\lambda}\right)\|C\|,
\end{aligned}$$

where in the final inequality we used $\lambda \leq \frac{6(k-1)}{k}$.

Claim 2. *Lemma 5.1 is true.*

Summing the bound in Claim 1 over all components of $G[\mathcal{H}]$ and plugging into (13) gives

$$\operatorname{mic}(G) \geq \left(\frac{k}{2} + \frac{k-1}{\lambda}\right)|\mathcal{H}| - \left(\frac{2(k-1)}{\lambda}\right)\|\mathcal{H}\| + (k-1)\beta(\mathcal{L}). \tag{14}$$

Applying Kernel Magic using (14) and solving for $\beta(\mathcal{L})$ proves the claim. $\square$

References

[1] D. Cranston and L. Rabern, *Edge lower bounds for list critical graphs, via discharging*, arXiv:1602.02589 (2016).

[2] T. Gallai, *Kritische Graphen I.*, Publ. Math. Inst. Hungar. Acad. Sci **8** (1963), 165–192 (in German).

[3] H.A. Kierstead and L. Rabern, *Improved lower bounds on the number of edges in list critical and online list critical graphs*, arXiv:1406.7355 (2014).

[4] ———, *Extracting list colorings from large independent sets*, arXiv:1512.08130 (2015).

[5] A.V. Kostochka and M. Stiebitz, *A new lower bound on the number of edges in colour-critical graphs and hypergraphs*, Journal of Combinatorial Theory, Series B **87** (2003), no. 2, 374–402.

[6] A.V. Kostochka and M. Yancey, *Ore's conjecture on color-critical graphs is almost true*, J. Combin. Theory Ser. B **109** (2014), 73–101. MR 3269903

[7] Landon Rabern, *A better lower bound on average degree of 4-list-critical graphs*, arXiv:1602.08532 (2016).