Conjectures that should be true*

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1 Edges in list-critical graphs

A graph G is *k-list-critical* if G is not $(k-1)$-choosable, but every proper subgraph of G is $(k-1)$-choosable. Replace '$(k-1)$-' with 'online $(k-1)$-' and 'k-' with '*online* k-' in the previous sentence and read it.

Conjecture 1. *Every incomplete k-list-critical graph has average degree at least*

$$k - 1 + \frac{k-3}{(k-1)^2}.$$

Background. The connected graphs in which each block is a complete graph or an odd cycle are called *Gallai trees*. Gallai [11] proved that in a k-critical graph, the vertices of degree $k-1$ induce a disjoint union of Gallai trees. The same is true for k-list-critical graphs [1, 10]. This quickly implies a lower bound on the average degree of k-list-critical graphs of

$$k - 1 + \frac{k-3}{k^2-3}.$$

In [21], R. improved this to

$$k - 1 + \frac{k-3}{k^2-2k+2}$$

using a lemma from Kierstead and R. [14] that generalizes a kernel technique of Kostochka and Yancey [15]. As noted at the end of [21], a small improvement to the argument would yield Conjecture 1. □

Conjecture 2. *Every incomplete online k-list-critical graph G has*

$$2\,\|G\| \geq (k-1)\,|G| + k - 3.$$

Background. Dirac [9] proved this for k-critical graphs. Kostochka and Stiebitz [16] proved it for k-list-critical graphs. Their proof does not seem to generalize. When $|G|$ is large compared with k, the conjecture holds by Gallai-type bounds on the average degree of online k-list-critical graphs [13, 2]. □

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1.1 The $\frac{5}{6}$ bound

Conjecture 3. *Every vertex-transitive graph has* $\chi \leq \max\left\{\omega, \left\lceil \frac{5\Delta+3}{6} \right\rceil\right\}$.

Background. In [3], the following was proved

- the conjecture holds for the fractional chromatic number χ^*,
- the conjecture holds with $\left\lceil \frac{5\Delta+3}{6} \right\rceil$ replaced by $\epsilon(\Delta+1)$ for some $\epsilon < 1$,
- the conjecture holds both if Reed's ω, Δ, χ conjecture and the strong 2Δ-colorability conjecture hold for vertex-transitive graphs (only strong $\frac{5}{2}\Delta$-colorability is required),

Does the conjecture hold for Cayley graphs? □

Conjecture 4. *Every line graph (of a multigraph) has* $\chi \leq \max\left\{\omega, \left\lceil \frac{5\Delta+3}{6} \right\rceil\right\}$.

Background. In [18] this was proved with $\left\lceil \frac{5\Delta+3}{6} \right\rceil$ replaced by $\frac{7\Delta+10}{8}$. Conjecture 14 in [18] that implies this conjecture is now known to be false. □

2 Planar graphs

Conjecture 5. *Every planar graph with no K_5-subdivision is 2-fold 9-colorable.*

Background. In [4], Cranston and R. gave a short proof of this conjecture with K_5-minors excluded instead of K_5-subdivisions (which also follows from the Four Color Theorem). Hajós conjectured that every graph is $(k-1)$-colorable unless it contains a subdivision of K_k. This is known to be true for $k \leq 4$ and false for $k \geq 7$. The cases $k = 5$ and $k = 6$ remain unresolved. □

3 Maximum degree, clique number and colorings

3.1 Around Borodin-Kostochka

Conjecture 6. *Every graph with $\chi \geq \Delta \geq 8$ contains a $K_3 \vee H$ where H is some graph on $\Delta - 3$ vertices.*

Background. By results in [8], for $\Delta \geq 9$ the existence of $K_3 \vee H$ implies the existence of K_Δ. So, this (seemingly weaker) conjecture for $\Delta \geq 9$ implies the Borodin-Kostochka conjecture. The one known connected counterexample to the Borodin-Kostochka conjecture for $\Delta = 8$ is a 5-cycle with each vertex blown up to a triangle. This graph is not a counterexample to Conjecture 6. □

Conjecture 7. *Every graph with $\chi \geq \Delta$ contains $K_{\Delta-3}$.*

Background. Results in [5] show that this holds with $K_{\Delta-4}$ instead of $K_{\Delta-3}$. Moreover, [5] proves the conjecture for all but $\Delta \in \{6, 8, 9, 11, 12\}$. Reed's conjecture [22] that every graph satisfies $\chi \leq \left\lceil \frac{\omega+\Delta+1}{2} \right\rceil$ implies this conjecture with $K_{\Delta-2}$ instead of $K_{\Delta-3}$. □

Conjecture 8. *Every graph with $\chi \geq \Delta$ either contains K_Δ or contains a $K_{\Delta-4}$ with all Δ-vertices.*

Background. Results in [5] show that this holds with $K_{\Delta-5}$ instead of $K_{\Delta-4}$. For $\Delta \leq 7$, the conjecture holds by [19, 17]. Also by [5], it holds when $\Delta = 3r + 1$ for $r \geq 3$. □

Conjecture 9. *Every graph with $\Delta \geq 8$ and $\omega < \Delta$ is 2-fold $(2\Delta - 1)$-colorable.*

Background. The one known connected counterexample to the Borodin-Kostochka conjecture for $\Delta = 8$ is a 5-cycle with each vertex blown up to a triangle. This graph is not a counterexample to Conjecture 9. □

Conjecture 10. *Every graph with $\theta \geq 10$ and $\omega \leq \frac{\theta}{2}$ is $\left\lfloor \frac{\theta}{2} \right\rfloor$-choosable.*

Background. Here θ is the *Ore degree* give by $\theta(G) := \max_{xy \in E(G)} d(x) + d(y)$. This conjecture holds for ordinary coloring [12, 19, 17, 20]. In [14], the conjecture is proved for $\theta \geq 18$ for both list-coloring and online list-coloring. Further lowering of θ would follow from improved bounds on average degree of list-critical graphs [13]. □

Conjecture 11. *Every claw-free graph with $\Delta \geq 9$ and $\omega < \Delta$ is $(\Delta - 1)$-choosable.*

Background. In [7], this was proved for ordinary coloring. In [6], the conjecture was proved for $\Delta \geq 69$. Also, [6] proved that the full conjecture follows from the line-graph case. □

Conjecture 12. *There is a polynomial time graph algorithm that finds either a $(\Delta - 1)$-coloring or a $K_{\Delta-3}$.*

Background. In [5], the following was proved

- the conjecture holds with $K_{\Delta-3}$ replaced by $K_{\Delta-4}$,
- the conjecture holds for $\Delta \geq 25$ (the proof uses algorithmic versions of the local lemma),
- the conjecture holds when $\Delta = 3r + 1$.

□

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