# graph theory notes*

# Stiebitz's proof of Gallai's conjecture on the number of components in the high and low vertex subgraphs of critical graphs

Tibor Gallai conjectured the following in 1963 [1, 2] and Michael Stiebitz proved it in 1982 [3]. For a graph $G$, let $\mathcal{L}(G)$ be the subgraph of $G$ induced on the vertices of degree $\delta(G)$ and let $\mathcal{H}(G)$ be the subgraph of $G$ induced on the vertices of degree larger than $\delta(G)$.

**Theorem** (Stiebitz)**.** *If $G$ is a color-critical graph with $\delta(G) = \chi(G) - 1$, then $\mathcal{H}(G)$ has at most as many components as $\mathcal{L}(G)$.*

**Lemma.** *Let $G$ be a connected graph and $\emptyset \neq X \subseteq V(G)$ such that*

- *$d_G(x) \leq k - 1$ for all $x \in X$; and*
- *for each component $C$ of $G - X$, we have $\chi(G - V(C)) \leq k - 1$; and*
- *$G[X]$ has $\ell$ components and $G - X$ has at least $\ell + 1$ components.*

*If $G - X$ is the disjoint union of (possibly not connected) graphs $M_1, \ldots, M_{\ell+1}$ and $f_i$ is a $(k-1)$-coloring of $M_i$ for each $i \in [\ell + 1]$, then there are permutations $\pi_1, \ldots, \pi_{\ell+1}$ of $[k - 1]$ such that the $(k - 1)$-coloring of $G - X$ given by $(\pi_1 \circ f_1) \cup \cdots \cup (\pi_{\ell+1} \circ f_{\ell+1})$ extends to a $(k - 1)$-coloring of $G$.*

*Proof.* Suppose the lemma is false and choose a counterexample $G$ and nonempty $X \subseteq V(G)$ so that $|X|$ is as small as possible. So, $G - X$ is the disjoint union of graphs $M_1, \ldots, M_{\ell+1}$ and we have $(k - 1)$-colorings $f_i$ of $M_i$ for each $i \in [\ell + 1]$ so that no permutations allow us to extend to a $(k - 1)$-coloring of $G$.

**Claim 1.** *Each component of $G[X]$ has edges to at least two of the $M_i$.* Suppose to the contrary that we have a component $C$ of $G[X]$ that has edges to at most one of the $M_i$. Then, since $G$ is connected, we must have $\ell \geq 2$. But now the hypotheses of the lemma are satisfied with $X' = X \setminus V(C)$ in place of $X$, so by minimality of $|X|$ we get permutations that allow us to extend to a $(k - 1)$-coloring of $G$, a contradiction.

**Claim 2.** *Each non-separating vertex in $G[X]$ has neighbors in at least two of the $M_i$.* Suppose to the contrary that we have a component $C$ of $G[X]$ and $x \in V(C)$ a

*clarifications, errors, simplifications ⇒ `landon.rabern@gmail.com` non-separating vertex that has neighbors in at most one of the $M_i$. Then, by Claim 1, we must have $|C| \geq 2$. But then $x$ has at most $k-2$ neighbors in $G - X$, so we can greedily complete any $(k-1)$-coloring of $G - X$ to $G - X'$ where $X' = X \setminus \{x\}$. So, the hypotheses of the lemma are satisfied with $X'$ in place of $X$. Again, by minimality of $|X|$, we get permutations that allow us to extend to a $(k-1)$-coloring of $G$, a contradiction.

**Claim 3.** *The lemma is true.* Pick a component $C$ in $G[X]$ and a non-separating vertex $x \in V(C)$. By Claim 2 and symmetry, we may assume that $x$ has neighbors $y_1, y_2$ in $M_1, M_2$ respectively. Let $G' = G - V(C)$ and $X' = X \setminus V(C)$. Then $G'$ is the disjoint union of the $\ell$ graphs $M_1 \cup M_2, M_3, \ldots, M_{\ell+1}$. Let $\tau$ be a permutation of $[k-1]$ such that $(\tau \circ f_2)(y_2) = f_1(y_1)$ and let $f_* = f_1 \cup (\tau \circ f_2)$. WHY $G'$ CONNECTED? By minimality of $|X|$, we can apply the lemma to $G'$ with $M_1 \cup M_2, M_3, \ldots, M_{\ell+1}$ and colorings $f_*, f_3, \ldots, f_{\ell+1}$ to get permutations $\pi_*, \pi_3, \ldots, \pi_{\ell+1}$ such that the $(k-1)$-coloring of $G' - X'$ given by $(\pi_* \circ f_*) \cup (\pi_3 \circ f_3) \cup \cdots \cup (\pi_{\ell+1} \circ f_{\ell+1})$ extends to a $(k-1)$-coloring of $G'$. But this is the same as the $(k-1)$-coloring $(\pi_* \circ f_1) \cup (\pi_* \circ \tau \circ f_2) \cup (\pi_3 \circ f_3) \cup \cdots \cup (\pi_{\ell+1} \circ f_{\ell+1})$, so using the permutations $\pi_*, \pi_* \circ \tau, \pi_3, \ldots, \pi_{\ell+1}$ we get a coloring of $G - X$ that extends to $G - V(C)$. But in this coloring, $y_1$ and $y_2$ receive the same color. This means that $x$ has $k - 1 - (d_G(x) - d_C(x)) + 1 \geq d_C(x) + 1$ colors available and each other vertex $v$ in $C$ has $k-1-(d_G(v)-d_C(v))+1 \geq d_C(v) \geq d_C(v)$ colors available. So, coloring $C$ greedily in order of decreasing distance from $x$ gives an extension to a $(k-1)$-coloring of $G$, a contradiction. $\square$

## References


[1] T. Gallai, *Kritische graphen I.*, Math. Inst. Hungar. Acad. Sci **8** (1963), 165–192 (in German).

[2] ———, *Kritische graphen II.*, Math. Inst. Hungar. Acad. Sci **8** (1963), 373–395 (in German).

[3] M. Stiebitz, *Proof of a conjecture of T. Gallai concerning connectivity properties of colour-critical graphs*, Combinatorica **2** (1982), no. 3, 315–323.