FIXER-BREAKER AND SHORT TASHKINOV TREES

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1. Introduction

Suppose we want to k-color a graph G. If we already have a k-coloring of an induced subgraph H of G, we might try to extend this coloring to all of G. We can view this task as the problem of trying to list color $G-H$, where each vertex v in $G-H$ gets a list of colors formed from $\{1,\dots,k\}$ by removing all of the colors used on $N(v)$ in H. Such list coloring problems are interesting in their own right, outside the context of completing partial colorings [degree-choosable graphs, other examples here]. In many situations we cannot complete just any k-coloring of H to all of G. Instead, we may need to modify the k-coloring of H to get a coloring we can extend. Given rules for how we are allowed to modify the k-coloring of H, we can recast the task of modifying the k-coloring and then completing it to G as the problem of trying to list color $G-H$ where each vertex gets a list as before, but now we are allowed to modify these lists in a prescribed manner. Studying such list coloring/modifying problems in their own right has also proved useful.

As an example of this paradigm, the second author proved [6] a common generalization of Hall's marriage theorem and Vizing's theorem on edge-coloring. The present paper will generalize a special case of this result and put it into a broader context. An interesting caveat arises when investigating this list coloring/modifying paradigm. Since we often want to prove coloring results for all graphs having certain properties and not just some fixed graph, we only have partial control over the outcome of a recoloring of H. For example, if we swap colors red and green in a component C of the red-green subgraph (that is, we perform a Kempe change), we may succeed in making some desired vertex red, but if C is somewhat arbitrary, we cannot precisely control what happens to the colors of the other vertices. In the list modifying/coloring paradigm, we model this lack of control as a two-player game—we move by doing the part of the recoloring we desire and then the other player gets a turn to muck things up. In the original context where we want to color G, the opponent is the graph G; more precisely, the embedding of $G-H$ in G is one way to describe a strategy for the second player. The general paradigm that we described above is for vertex coloring. In the rest of the paper, we consider only the special case that is edge-coloring (or, equivalently, vertex coloring line graphs).

2. The basic game

The *basic game* is played on a multigraph G by *Fixer* and *Breaker*. To set up the game, we assign a list of colors $L(v)$ to each $v \in V(G)$. Put $\mathrm{Pot}(L) = \bigcup_{v\in V(G)} L(v)$. Fixer always gets the first move and he wins if and only if on some turn, before moving, he can construct a proper edge-coloring of G, say $\pi\colon E(G) \to \mathrm{Pot}(L)$ such that $\pi(xy) \in L(x) \cap L(y)$ for each

$xy \in E(G)$. It is important to notice that we are coloring the edges, even though the lists are assigned to the vertices.

Fixer's turn. Pick $\alpha \in \mathrm{Pot}(L)$ and $v \in V(G)$ with $\alpha \notin L(v)$ and set $L(v) := L(v) \cup \{\alpha\} - \beta$ for some $\beta \in L(v)$.

Breaker's turn. If Fixer modified $L(v)$ by inserting α and removing β, then Breaker can either do nothing or pick $w \in V(G - v)$ and modify its list by swapping α for β or β for α.

These options for Breaker's turn correspond to the outcomes of performing a Kempe change on a longest path colored with α and β beginning at vertex v, if graph G is embedded in some larger graph Q. Say w is the other endpoint of this path. If $w \notin V(G)$, then only $L(v)$ is modified; this corresponds to Breaker doing nothing. Otherwise, $L(w)$ is modified by removing one of α and β and adding the other. In Section 3, we further discuss this connection to completing edge-colorings.

If G fails to have an L-edge-coloring because of some property of L that Fixer cannot change, then not having this property is a necessary condition for Fixer to have a winning strategy. For example, since Fixer cannot change the list sizes, if he is going to have any chance of winning we must have $|L(v)| \geq d_G(v)$ for all $v \in V(G)$. In an L-edge-coloring of G, the edges of each subgraph H are partitioned into matchings, so if $\|H\|$ exceeds the sum of the sizes of the maximum matchings in H in each color, then G cannot have an L-edge-coloring. We show that if G has such a subgraph H, then Fixer cannot fix H. This gives a strong necessary condition.

For $C \subseteq \mathrm{Pot}(L)$ and $H \subseteq G$, let $H_{L,C}$ be the subgraph of H induced on the vertices v with $L(v) \cap C \neq \emptyset$. When L is clear from context, we may write H_C for $H_{L,C}$. If $C = \{\alpha\}$, we may write H_α for H_C. For $H \subseteq G$, put

$$\psi_L(H) = \sum_{\alpha \in \mathrm{Pot}(L)} \left\lfloor \frac{|H_{L,\alpha}|}{2} \right\rfloor.$$

Each term in the sum gives an upper bound on the size of a matching in color α. So $\psi_L(H)$ is an upper bound on the the number of edges in a partial L-edge-coloring of H. We say that (H, L) is *abundant* if $\psi_L(H) \geq \|H\|$ and that (H, L) is *superabundant* if for every $H' \subseteq H$, the pair (H', L) is abundant. If (H, L) is not superabundant, then Fixer cannot win on the first move since there is some subgraph he cannot properly color. We now show that if (H, L) is not superabundant, then Breaker can prevent Fixer from creating a superabundant position, and thus Fixer can never win.

Lemma 1. *If Fixer has a winning strategy in the basic game, then (G, L) is superabundant.*

Proof. Suppose (G, L) is not superabundant, say $H \subseteq G$ is such that (H, L) is not abundant. We show that no matter what move Fixer makes, Breaker has a response, giving lists L', such that (H, L') is not abundant. But then Fixer can never win since any state admitting a proper edge-coloring must be superabundant.

Suppose Fixer swaps α for β in $L(v)$ for some $v \in V(G)$. If Fixer is going to have any chance of increasing $\psi_L(H)$, we must have $v \in V(H)$ and both $|H_{L,\alpha}|$ and $|H_{L,\beta}|$ must be odd. Since $|H_{L,\alpha}|$ and $|H_{L,\beta}|$ have the same parity and $|\{\alpha,\beta\} \cap L(v)| = 1$, there must be $w \in V(H - v)$ such that $|\{\alpha,\beta\} \cap L(w)| = 1$. Breaker should swap α and β in $L(w)$. If L' is the resulting list assignment, then Breaker has ensured that both $|H_{L',\alpha}|$ and $|H_{L',\beta}|$ are again odd; so $\psi_{L'}(H) = \psi_L(H)$, and hence (H, L') is not abundant. □

We say that G is *f-fixable* for $f : V(G) \to \mathbb{N}$ if Fixer has a winning strategy in the basic game for every list assignment L where (G, L) is superabundant and $|L(v)| \geq f(v)$ for all $v \in V(G)$. By the above discussion, we know that if G is f-fixable, then $f(v) \geq d(v)$ for all $v \in V(G)$. For brevity's sake, we will say that f is *valid* if $f(v) \geq d(v)$ for all $v \in V(G)$.

3. Relationship to completing partial edge-colorings

Suppose G is a subgraph of a multigraph Q. Given a proper k-edge-coloring π of $Q' = Q - E(G)$, put $L_\pi(v) = [k] - \pi(E_{Q'}(v))$ for $v \in V(G)$. So, L_π is the list assignment given by starting with $[k]$ for $v \in V(G)$ and then removing all the colors π uses on edges incident to v. Suppose there exist α, β with $\alpha \in L_\pi(v)$ but $\beta \notin L_\pi(v)$. Then there is an edge colored β incident to v in Q'. Consider what happens to L_π when we swap α and β on a longest path P in Q' starting at v that alternates between colors β and α. When $w \in V(G)$ is not in P or is an internal vertex in P, there is no change to $L_\pi(w)$. The only changes are that $L_\pi(v)$ has gained β and lost α and if P ends at some vertex z, then $L_\pi(z)$ has either gained β and lost α or lost β and gained α. Therefore, making this color swap induces a Fixer move followed by a Breaker move in the basic game on G with lists L_π. Since Fixer can always make any of his legal moves in this manner, the existence of a winning strategy for Fixer on (G, L_π) implies that we can modify π and complete the edge-coloring to Q.

In translating to the basic game, we have lost properties of the embedding of Q in G that might help in completing the edge-coloring. For example, if Q is bipartite, then two-colored paths in Q' must start and end with the same or opposite color depending on whether the distance of the endpoints in G are at even or odd distance in G (so as not to form an odd cycle). So, Breaker playing with the strategy given by Q is a very weak opponent. This is why Kőnig's theorem on edge-coloring is so easy to prove.

Further, there are (prima facie helpful) properties we have lost that come just from the fact that G is embedded in *some* graph Q. For example, suppose $x, y \in V(G)$ both have blue in their lists. If we add a blue edge between x and y (and remove blue from their lists) before performing a swap of red for blue at some vertex, then we are guaranteed that x and y will have a color in common afterwards (it may be red instead of blue). In this way we can "protect" the property "x and y have a color in common". In the basic game, Fixer doesn't have this ability, since Breaker is free to change the list for one of x or y and not the other. It is possible to define a game where Fixer is directly given abilities of this sort, but we don't bother doing so because we can achieve this "protection" using a different advantage as follows.

Another interesting advantage that we get from the embedding is the fact that, for each $x \in V(G)$, there is at most one longest red-blue path containing x. In particular, if a longest red-blue path starts at x and ends at y, then no longest red-blue path starts at x and ends at z where $z \neq y$. We can partially model this in the game by adding a sort of memory where some information about Breaker's previous moves is stored. We call the memory a *chronicle*. In the next section, we'll define this *chronicled game*.

Making explicit the properties we use to extend edge-colorings can help to simplify and clarify proofs of edge-coloring results. That is one purpose served by the different games. One piece is still missing; we'd like a way to discuss the ability to "win" on (G, L) given the full power of being embedded. To this end, we say that Fixer has a winning strategy on (G, L) in the *master game* if for any multigraph Q with $G \subseteq Q$ and any edge-coloring π of $Q - E(G)$ where $L = L_\pi$, there exists an $|\text{im}(\pi)|$-edge-coloring of Q.

To gain further intuition for why requiring superabundance is natural, consider the following conjecture. If true, this would be very hard to prove since it implies Goldberg's conjecture. We state it for the basic game, though it is more likely to hold in a game where Fixer has more power (the "if" direction, that is); the "only if" direction for the basic game follows from Lemma 1.

Conjecture 2. *If $f(v) \geq d(v) + 1$ for all $v \in V(G)$, then G is f-fixable.*

Let's see how Conjecture 2 implies Goldberg's conjecture. Suppose G is not k-edge-colorable for some $k \geq \Delta(G) + 1$. Let $L(v) = [k]$ for all $v \in V(G)$. Now Fixer has no moves, so Fixer has no winning strategy. By Conjecture 2, there exists $H \subseteq G$ with

$$\|H\| > \sum_{\alpha \in \text{Pot}(L)} \left\lfloor \frac{|H_\alpha|}{2} \right\rfloor = \sum_{\alpha \in \text{Pot}(L)} \left\lfloor \frac{|H|}{2} \right\rfloor = k \left\lfloor \frac{|H|}{2} \right\rfloor .$$

That is, $\left\lfloor \frac{\|H\|-1}{\left\lfloor \frac{|H|}{2} \right\rfloor} \right\rfloor \geq k$. Hence, if G is not $(\Delta(G)+1)$-edge-colorable, then it is $w(G)$-edge-colorable where $w(G) = \max_{\substack{H \subseteq G \\ |H| \geq 2}} \left\lceil \frac{\|H\|}{\left\lfloor \frac{|H|}{2} \right\rfloor} \right\rceil$. This is Goldberg's Conjecture.

4. The chronicled game

The *chronicled game* is the same as the basic game except that we maintain a *chronicle* $\mathcal{C}$ which restricts Breaker's possible moves. (This chronicle is a way of requiring Breaker to make moves that are consistent with some embedding of G in a larger graph Q.) Suppose Fixer and Breaker play on (G, L). It will be convenient to add a "point at infinity" to the chronicle in order to model Breaker doing nothing in a uniform manner (this corresponds to a two-colored path leaving and never coming back). We call this point ∞. The chronicle is a multigraph with vertex set $V(G) \cup \{\infty\}$ that will be updated as the game progresses. Each edge of $\mathcal{C}$ will be labeled with a doubleton of colors $\{\alpha, \beta\} \subseteq \text{Pot}(L)$. At the start of the game $\mathcal{C}$ is edgeless.

Fixer's turn. Pick $v \in V(G)$ and different $\alpha, \beta \in \text{Pot}(L)$ with $|\{\alpha, \beta\} \cap L(v)| = 1$ and swap α and β at v.

Breaker's turn. If there is a $vx \in E(\mathcal{C} - \infty)$ labeled $\{\alpha, \beta\}$, then Breaker swaps α and β at x. If instead $v\infty \in E(\mathcal{C})$, then Breaker does nothing. Otherwise, Breaker can either do nothing, or pick $w \in V(G - v)$ with $|\{\alpha, \beta\} \cap L(w)| = 1$ such that no edge incident to w in $\mathcal{C}$ has label $\{\alpha, \beta\}$, and swap α and β at w.

Right after Breaker's move, the chronicle is updated as follows.

Chronicle update. Remove all edges of $\mathcal{C}$ whose label intersects $\{\alpha, \beta\}$ in exactly one color. If Breaker swapped α and β at z and there is no vz edge in $\mathcal{C}$ labeled $\{\alpha, \beta\}$, then add one. Otherwise, if Breaker did nothing and there is no $v\infty$ edge in $\mathcal{C}$ labeled $\{\alpha, \beta\}$, then add one.

Lemma 3. *Suppose G is a subgraph of a multigraph Q and π a k-edge-coloring of $Q - E(G)$. If Fixer has a winning strategy against Breaker in the chronicled game on (G, L_π), then Q is k-edge-colorable.*

Proof. Fixer can make any of his legal moves in the game by flipping a 2-colored path in $Q - E(G)$. Our only worry is that we have restricted Breaker too much. It will suffice to show that at the start of any of Fixer's turns the following hold. Suppose $xy \in E(\mathcal{C})$ where $x \neq \infty$ is labeled $\{\alpha, \beta\}$. If $y \neq \infty$, then there is a longest $\alpha - \beta$ path in $Q - E(G)$ with endpoints x and y. If $y = \infty$, then there is a longest $\alpha - \beta$ path in $Q - E(G)$ with endpoints x and z where $z \in V(Q) - V(G)$.

Say we are in round n. If $y \neq \infty$, put $z = y$. Since xz is labeled $\{\alpha, \beta\}$ by the chronicle update, there is a largest round k with $k < n$ in which Fixer swapped α and β at one of x or z and then Breaker swapped α and β at the other one. If instead $y = \infty$, then the other end of the path is some $z \in V(Q) - V(G)$. So there is a largest round k with $k < n$ where Fixer swapped α and β at x and then Breaker did nothing. In both the cases $y = \infty$ and $y \neq \infty$, no swap involving exactly one of α or β occurred at any round j with $k < j < n$ (if it had, then the chronicle update would have removed the edge xy labeled $\{\alpha, \beta\}$). The moves in round k imply that at the start of round k, there is a longest $\alpha - \beta$ path in $Q - E(G)$ with endpoints x and z. Since no two-colored path involving exactly one of α or β was swapped at any round j with $k < j < n$, at round n there is still a longest $\alpha - \beta$ path in $Q - E(G)$ with endpoints x and z. $\square$

We say that G is *f-fixable in the chronicled game* for $f \colon V(G) \to \mathbb{N}$ if Fixer has a winning strategy in the chronicled game for every list assignment L where (G, L) is superabundant and $|L(v)| \geq f(v)$ for all $v \in V(G)$.

5. Kierstead-Tashkinov-Vizing assignments

Most edge-coloring results have been proved using a specific kind of superabundant pair (G, L) where superabundance can be proved via a special ordering. That is, the orderings given by the definition of Vizing fans, Kierstead paths, and Tashkinov trees. In this section, we show how superabundance easily follows from these orderings.

We first define a Tashkinov tree (Vizing fans and Kierstead paths are the special cases where the tree is a star and a path, respectively). Let Q be an edge-critical multigraph with $\chi'(Q) = k + 1$ for some $k \geq \Delta(G) + 1$. A tree $T \subseteq Q$ is *Tashkinov* if for some $xy \in E(T)$ there is a k-edge-coloring π of $G - xy$ and a total ordering '$<$' of $V(T)$ such that

(1) $x < y < z$ for all $z \in V(T - x - y)$; and
(2) $T[w \mid w \leq z]$ is a tree for all $z \in V(T)$; and
(3) for each $wz \in E(T - xy)$, there is $u < \max\{w, z\}$ such that $\pi(wz) \in \bar{\pi}(u)$.

By $\bar{\pi}(u)$ we mean the colors from $[k]$ not incident to u. We also say that the quadruple $(T, xy, \pi, <)$ is Tashkinov. When $xy \in E(G)$ and $X \subseteq V(G)$ with $x, y \in X$, we say that X is *elementary* with respect to a k-edge-coloring π of $G - xy$ if $\bar{\pi}(u) \cap \bar{\pi}(w) = \emptyset$ for all $u, w \in X$. The reason for interest in Tashkinov trees is that if G is edge-critical and π is a k-edge-coloring of $G - xy$, then every Tashkinov tree in G must be elementary with respect to π. For Vizing fans, this fact can be proved easily using Kempe changes. For Kierstead paths, the proof is a bit longer, but not hard. However, proving it for Tashkinov trees in general is tedious, to say the least.

If true, the following conjecture generalizes the fact that Tashkinov trees are elementary. Furthermore, the statement feels more natural, since the condition is both necessary and sufficient. Again, we state it for the basic game, but it is more likely to hold for games where Fixer has more power (the "if" direction that is; superabundance may no longer be necessary when Fixer has more power). Even proving the "if" direction for the master game would be quite interesting.

Conjecture 4. *If G is a tree and $f(v) \geq d(v) + 1$ for all $v \in V(G)$, then G is f-fixable.*

Let's see why Conjecture 4 implies that Tashkinov trees are elementary. First, we need some definitions. We say that a list assignment L on G is a *Kierstead-Tashkinov-Vizing* assignment (henceforth *KTV-assignment*) if for some $xy \in E(G)$, there is a total ordering '$<$' of $V(G)$ such that

(1) there is $\pi \colon E(G) \to \mathrm{Pot}(L)$ such that $\pi(uv) \in L(u) \cap L(v)$ for each $uv \in E(G - xy)$;
(2) $x < z$ for all $z \in V(G - x)$;
(3) $G[w \mid w \leq z]$ is connected for all $z \in V(G)$;
(4) for each $wz \in E(G - xy)$, there is $u < \max\{w, z\}$ such that $\pi(wz) \in L(u) - \{\pi(e) \mid e \in E(u)\}$;
(5) there are different $s, t \in V(G)$ such that $L(s) \cap L(t) - \{\pi(e) \mid e \in E(s) \cup E(t)\} \neq \emptyset$.

Lemma 5. *If L is a KTV-assignment on G, then (G, L) is superabundant.*

Proof. Let L be a KTV-assignment on G, and let $H \subseteq G$. We will show that (H, L) is abundant. Clearly it suffices to consider the case when H is an induced subgraph, so we assume this. Property (1) gives that $G - xy$ has an edge-coloring π, so $\psi_L(H) \geq \|H\| - 1$; also $\psi_L(H) \geq \|H\|$ if $\{x, y\} \not\subseteq V(H)$. Furthermore $\psi_L(H) \geq \|H\|$ if s and t from property (5) are both in $V(H)$, since then $\psi_L(H)$ gains 1 over the naive lower bound, due to the color in $L(s) \cap L(t)$. So $V(G) - V(H) \neq \emptyset$.

Now choose $z \in V(G) - V(H)$ that is smallest under $<$. Put $H' = G[w \mid w \leq z]$. By the minimality of z, we have $H' - z \subseteq H$. By property (2), $|H'| \geq 2$. By property (3), H' is connected and thus there is $w \in V(H' - z)$ adjacent to z. So, we have $w < z$ and $wz \in E(G) - E(H)$. Now $\pi(wz) \in L(w)$. By the definition of a KTV-assignment, property (4) implies that there exists u with $u < \max\{w, z\} = z$ and $\pi(wz) \in L(u) - \{\pi(e) | e \in E(u)\}$. Then $u \in V(H' - z) \subseteq V(H)$ and again we gain 1 over the naive lower bound on $\psi_L(H)$, due to the color in $L(u) \cap L(w)$. So $\psi_L(H) \geq \|H\|$. □

Lemma 6. *Assume that Conjecture 4 holds. Let Q be an edge-critical multigraph with $\chi'(Q) = k + 1$ for some $k \geq \Delta(G) + 1$. If $T \subseteq Q$ is a tree and $(T, xy, \pi, <)$ is Tashkinov, then $V(T)$ is elementary with respect to π.*

Proof. Let $T \subseteq Q$ be a tree such that $(T, xy, \pi, <)$ is Tashkinov. Put $H = Q - E(T)$ and $L(v) = [k] - \pi\left(E(v) - E(T)\right)$ for each $v \in V(T)$. Then $|L(v)| \geq k - (d(v) - d_T(v)) \geq d_T(v) + \Delta + 1 - d(v) \geq d_T(v) + 1$.

We play the Fixer/Breaker game on T with lists L. Fixer can make any of his legal moves in the game by flipping a 2-colored path in H. How doing so effects the rest of the lists corresponds to Breaker's response. Breaker has no more power than we allow him in the game (he may have less power, depending on Q); therefore, if Fixer could win the game, we could k-edge-color Q. Hence Fixer cannot win and thus, by Conjecture 4, it must be that (G, L) is not superabundant. Plainly, L satisfies properties (1)–(4) to be a KTV-assignment. If $V(T)$ is not elementary, then L satisfies property (5) as well and hence (G, L) is superabundant by Lemma 5, a contradiction. □

As we noted above, it is known that Tashkinov trees are elementary, whether or not Conjecture 4 is true. That proof is quite long and complicated; we might hope for a more natural proof in the generalized context of superabundance. For stars, there is a natural proof based on Hall's theorem, which we will see in Theorem 9.

6. Superabundance preserving moves

Suppose (G, L) is superabundant. Fixer would like to know what moves he can make without Breaker breaking superabundance. For different $\alpha, \beta \in \mathrm{Pot}(L)$, we say that α and β are *swappable* in (G, L) if whenever Fixer swaps α for β or β for α in some list and Breaker responds resulting in lists L', the pair (G, L') is superabundant.

Lemma 7. *Suppose (G, L) is superabundant. Then different $\alpha, \beta \in \mathrm{Pot}(L)$ are swappable if for every $H \subseteq G$, at least one of the following holds:*

(1) $\psi_L(H) > \|H\|$*; or,*
(2) $|H_{L,\alpha}|$ *is odd; or,*
(3) $|H_{L,\beta}|$ *is odd.*

Proof. Say Fixer swaps α in for β in $L(v)$. Let L' be the list assignment after Breaker's response. Choose $H \subseteq G$; we will show that (H, L') is abundant. Note that $|H_{L,\alpha}|$ and $|H_{L',\alpha}|$ differ by at most 2, so their contributions to $\psi_L(H)$ and $\psi_{L'}(H)$ differ by at most 1; the same is true for $|H_{L,\beta}|$ and $|H_{L',\beta}|$. We consider the three possibilities for H in the hypothesis. (1) If $\psi_L(H) > \|H\|$, then $\psi_{L'}(H) \geq \psi_L(H) - 1 \geq \|H\|$. So suppose (2) or (3) holds. The only way that we can have $\psi_{L'}(H) < \psi_L(H)$ is if $\left\lfloor \frac{|H_{L',\alpha}|}{2} \right\rfloor + \left\lfloor \frac{|H_{L',\beta}|}{2} \right\rfloor < \left\lfloor \frac{|H_{L,\alpha}|}{2} \right\rfloor + \left\lfloor \frac{|H_{L,\beta}|}{2} \right\rfloor$. Since $|H_{L,\beta}| + |H_{L,\alpha}| = |H_{L',\beta}| + |H_{L',\alpha}|$, this requires that both $|H_{L,\beta}|$ and $|H_{L,\alpha}|$ are even; since (2) or (3) holds, this is impossible. □

7. The game on stars

In [6], the second author proved that Conjecture 2 holds in stronger form when G is a star (in fact, more generally for multistars). For a graph G, let $\nu(G)$ be the number of edges in a maximum matching of G. For a list assignment L on G, put

$$\eta_L(G) = \sum_{\alpha \in \mathrm{Pot}(L)} \nu(G_\alpha).$$

If G has a proper edge-coloring from L, then $\eta_L(G) \geq \|G\|$. As a sort of partial converse, we will show that when G is a star and $\eta_L(G) \geq \|G\|$, we can color a subset of the edges with colors C so that no color in C appears in the lists on the uncolored edges. Then Fixer can just play on the uncolored edges, never performing swaps using colors from C, so this gives a way to reduce to a smaller game. The partial converse follows from Hall's theorem, but it is convenient in the proof to use the following intermediate lemma which was used by Borodin, Kostochka, and Woodall [1] in strengthening Galvin's Theorem about list edge-coloring of bipartite graphs [3].

Lemma 8. *Let G be a bipartite graph with nonempty parts X and Y. If $|X| \leq |Y|$ and Y has no isolated vertices, then G contains a nonempty matching M whose vertex set is $S \cup N(S)$ for some $S \subseteq Y$.*

Fixer's strategy on stars will be to perform a swap that increases $\eta_L(G)$ when $\eta_L(G) < \|G\|$ and to reduce to a smaller uncolored star when $\eta_L(G) \geq \|G\|$.

Theorem 9 (Rabern [6])**.** *If G is a star, then G is f-fixable for all valid f.*

Proof. Lemma 1 proves necessity of superabundance. Now we prove sufficiency.

Suppose the condition is not sufficient for Fixer to have a winning strategy in the basic game. Choose a counterexample G with lists L so as to minimize $|G|$ and subject to that, to maximize $\eta_L(G)$.

Let r be a vertex of maximum degree in G (r is unique when $\|G\| > 1$). Create a bipartite graph B with parts $X = \{uw \in E(G) \mid L(u) \cap L(w) \neq \emptyset\}$ and $Y := \{\alpha \in \mathrm{Pot}(L) \mid \nu(G_\alpha) = 1\}$, where $uw \in X$ is adjacent to $\alpha \in Y$ if and only if $\alpha \in L(u) \cap L(w)$. Informally, Y is the set of colors α that can be used on at least one edge, and X is the set of edges e with at least one color available on e, and a color α is adjacent to an edge e if α can be used on e.

[This would be a great place for a picture; maybe showing (G, L), then B, then M, then G' and the colors on $E(G) - E(G')$.]

Claim. *We have $\eta_L(G) < \|G\|$.*

Suppose $\eta_L(G) \geq \|G\|$. Since $|X| \leq \|G\| \leq \eta_L(G) = |Y|$ and Y has no isolated vertices, we can apply Lemma 8 to get a nonempty matching M whose vertex set is $S \cup N_B(S)$ for some $S \subseteq Y$. For each $\{uw, \alpha\} \in M$, use color α on edge uw. Let $G' = G - V(N_B(S) - r)$, that is, G with the colored edges deleted; let $L'(v) = L(v) - S$. Since $|G'| < |G|$, it suffices to show that $|L'(v)| \geq d_{G'}(v)$ for all v and that (G', L') is superabundant. Since $|L(r)| \geq d_G(r)$, and r lost exactly one color from its list for each incident edge colored, we have $|L'(r)| \geq d_{G'}(r)$. For all other $v \in V(G')$, we have $d_{G'}(v) = d_G(v)$. Further, since the colors S that we used appeared in no lists of vertices in $V(G')$, we have $L'(v) = L(v)$ for each $v \in V(G')$. Thus, for each $H \subseteq G'$, we see that (H, L') is abundant precisely because (H, L) is abundant. By the minimality of G, Fixer has a winning strategy on (G', L'), giving an edge-coloring of G'. Since the colors used on $E(G) - E(G')$ and on $E(G')$ form disjoint sets, the edge-colorings combine to give an edge-coloring of G.

Finish.

By the Claim, we have $\eta_L(G) < \|G\|$. Since $|L(r)| \geq d(r) = \|G\| > \eta_L(G)$, there exists $\alpha \in L(r) - Y$. So $\alpha \in L(r)$ and $|G_\alpha| = 1$. Since every subgraph of G that has edges contains r, Lemma 7 implies that α is swappable with β for all $\beta \in \mathrm{Pot}(L) - \alpha$.

Suppose there is $\beta \in Y$ such that $|N_B(\beta)| \geq 3$. Pick $rv \in N_B(\beta)$. Now Fixer should swap α for β in $L(v)$. Let L' be the list assignment after Breaker's response. Note that $L'(r) = L(r)$ since $\alpha, \beta \in L(r)$. Since Breaker can swap α for β in $L(w)$ for at most one w where $rw \in N_B(\beta)$, we have $\eta_{L'}(G) > \eta_L(G)$, which contradicts the maximality of $\eta_L(G)$.

Therefore, each $\beta \in Y$ has $|N_B(\beta)| \leq 2$. So each color in Y contributes at most one to $\psi_L(G)$. Since $|Y| = \eta_L(G) < \|G\| \leq \psi_L(G)$, there must be $\tau \notin Y$ such that $|G_\tau - r| \geq 2$. Now Fixer should swap τ for α in $L(r)$. Let L' be the list assignment after Breaker's response. Since Breaker can swap α for τ in $L(w)$ for at most one $w \in V(G_\tau - r)$, we again contradict the maximality of $\eta_L(G)$. □

8. Reduction in the chronicled game

In Section 3, we defined the chronicled game, where Fixer ostensibly has more power than in the basic game. Here we prove a lemma needed to prevent Breaker from destroying Fixer's progress in the form of a partial coloring. More precisely, if we can properly edge-color part of the graph such that Fixer has a winning strategy (in the chronicled game) on the rest of the graph with the reduced lists, then Fixer can win the chronicled game on the whole graph. We are unable to prove this result for the basic game, so this may be a good place to look if we are trying to prove that the basic game is strictly harder for Fixer.

In the case of stars we were able to do this sort of reduction because the colors in the partial edge-coloring we used did not appear elsewhere in the graph. For other graphs, say paths, a proper edge-coloring can use the same color on more than one edge, so we aren't hopeful that such a nice partial coloring can be found. And there lies the difficulty with proving the mentioned result for the basic game—if Fixer makes a move with a color that was used in the partial coloring, then Breaker can mess up the partial coloring. As we will see, Fixer has enough power in the chronicled game to successfully protect a partial coloring.

Suppose Fixer and Breaker play the chronicled game on a graph G with list assignment L. We define the *length* of the game on G with L, written $\ell(G, L)$, as the maximum over all Breaker strategies of the minimum number of moves it takes Fixer to beat Breaker; when Fixer cannot win, we let $\ell(G, L) = \infty$.

Lemma 10. *Let G be a multigraph and L a list assignment on G. Suppose we have $xy \in E(G)$ and $\tau \in L(x) \cap L(y)$. Put $G' = G - xy$ and $L'(v) = L(v)$ for all $v \in V(G' - x - y)$ and $L'(v) = L(v) - \tau$ for $v \in \{x, y\}$. If Fixer has a winning strategy against Breaker in the chronicled game on G' with lists L', then Fixer has a winning strategy against Breaker in the chronicled game on G with lists L.*

Proof. Suppose the lemma is false and choose a counterexample so as to minimize $\ell(G', L')$. We give Fixer and Breaker the names "Fixer*" and "Breaker*" when they are playing on (G', L'). Suppose Fixer* is playing a winning strategy achieving $\ell(G', L')$. Consider Fixer*'s first move. Suppose Fixer* swapped α and β in $L'(v)$. We need to decide on a move for Fixer, playing on G with L.

Claim 1. $v \notin \{x, y\}$.

Suppose $v \in \{x, y\}$. By symmetry, we may assume $v = x$.

Subclaim 1a. *$\{\alpha, \beta\} \not\subseteq L(v) \cap L(y)$.*

Suppose $\{\alpha, \beta\} \subseteq L(v) \cap L(y)$. In this case Fixer doesn't even need to make a move. We must have $\tau \in \{\alpha, \beta\}$. By symmetry we may assume that $x = v$ and $\beta = \tau$. Change τ to be α in the statement of the lemma. The effect on L' is that both x and y have lost α and gained β. So, this is equivalent to Fixer* swapping α and β in $L'(x)$ and then Breaker* swapping α and β in $L'(y)$. But Fixer* is playing by a strategy that beats any move of Breaker* in at most $\ell(G', L') - 1$ more moves. By minimality of $\ell(G', L')$, Fixer has a winning strategy against Breaker in the chronicled game on G with L (since L didn't change), a contradiction.

Subclaim 1b. *$\{\alpha, \beta\} \not\subseteq L(v)$.*

Suppose $\{\alpha, \beta\} \subseteq L(v)$. We must have $\tau \in \{\alpha, \beta\}$. By symmetry we may assume that $\beta = \tau$. By Subclaim 1a, $\alpha \notin L(y)$. Fixer should swap α and τ in $L(y)$. Breaker's only responses are to do nothing, or to swap α and τ at some $w \notin \{x, y\}$. Say J is the list assignment on G after Breaker's response and J' the list assignment on G' (where we removed α now instead of τ). Now the J' lists look just as they would if Fixer* had swapped α and τ at x and then Breaker* had swapped α and τ at w. The only difference in J is that now x and y both have α instead of τ. So, $\ell(G', J') < \ell(G', L')$ and again Fixer has a winning strategy by minimality of $\ell(G', L')$.

Subclaim 1c. *Claim 1 is true.*

If $\tau \notin \{\alpha, \beta\}$, then Fixer wins by minimality of $\ell(G', L')$. So assume $\beta = \tau$. By Subclaim 1b, we have $\alpha \notin L(v)$. But then $\{\alpha, \beta\} \cap L'(v) = \emptyset$, so Fixer* couldn't have swapped α and β in $L'(v)$, a contradiction.

Fixer move. *Fixer swaps α and β in $L(v)$.*

Claim 2. *$\tau \in \{\alpha, \beta\}$ and Breaker swaps α and β at $w \in \{x, y\}$. By symmetry, we assume $\beta = \tau$ and $w = x$.*

Say J is the list assignment on G after Breaker's response and J' the list assignment on G'. If $\tau \notin \{\alpha, \beta\}$, or Breaker did nothing, or Breaker swapped α and β at $w \notin \{x, y\}$, then $\ell(G', J') < \ell(G', L')$. By minimality of $\ell(G', L')$, Fixer has a winning strategy against Breaker in the chronicled game on G with lists J. But combined with his first move, this is a winning strategy with lists L, a contradiction.

Claim 3. *$\alpha \notin L(y)$.*

Suppose $\alpha \in L(y)$. Say J is the list assignment on G after Breaker's response and J' the list assignment on G' (where we removed α now instead of τ). Then the only difference in J is that now x and y both have α instead of τ, the only difference in J' is that y has τ instead of α. So, as far as Fixer* is concerned, Breaker* swapped α and τ at y. But Fixer* is playing by a strategy that beats any move of Breaker* in at most $\ell(G', L') - 1$ more moves, so $\ell(G', J') < \ell(G', L')$ and again Fixer has a winning strategy by minimality of $\ell(G', L')$.

Claim 4. *Fixer wins.*

By Claim 3, $\alpha \notin L(y)$. We are going to have Fixer make another move in such a way that it still looks like one Fixer* move followed by a Breaker* move. Fixer should swap α and τ at y. The chronicle contains an edge labeled $\{\alpha, \tau\}$ incident with x, so Breaker cannot respond by swapping α and τ at x. So Breaker's only responses are to do nothing, or to swap α and τ at some $w \notin \{x, y\}$. Say J is the list assignment on G after Breaker's response and J' the list assignment on G' (where we removed α now instead of τ). Now the J' lists look just as they would if Breaker* had swapped α and τ at w. The only difference in J is that now x and y both have α instead of τ. So, $\ell(G', J') < \ell(G', L')$ and again Fixer has a winning strategy by minimality of $\ell(G', L')$. $\square$

Lemma 11. *Let G be a multigraph and L a list assignment on G. Suppose we have an edge-coloring π of $H \subseteq G$ where $\pi(xy) \in L(x) \cap L(y)$ for all $xy \in E(H)$. Put $G' = G - E(H)$ and $L'(v) = L(v) - \pi(E_H(v))$ for all $v \in V(G')$. If Fixer has a winning strategy against Breaker in the chronicled game on G' with lists L', then Fixer has a winning strategy against Breaker in the chronicled game on G with lists L.*

Proof. If not, we can take a counterexample minimizing $\|H\|$ and then apply Lemma 10 to get a smaller counterexample. $\square$

In the proof of Claim 4 of Lemma 10 we had Fixer play a two-move combination to force what he wanted using the chronicle. We can turn this idea into a very useful lemma where Fixer may need to perform a multi-move combination.

Lemma 12. *Let G be a multigraph, L a list assignment on G and $\alpha, \beta \in \mathrm{Pot}(L)$. Let $S \subseteq V(G)$ be those vertices v with $|\{\alpha, \beta\} \cap L(v)| = 1$. Then there is a graph A_S with vertex set S and $\Delta(A_S) \leq 1$ such that Fixer has a sequence of moves against Breaker in the chronicled game resulting in a list assignment where Fixer has chosen to swap α and β in all or none of the vertices in each component of A_S.*

Proof. For each $v \in S$, Fixer should swap α and β at v twice in a row. Now every $v \in S$ is incident to an edge in $\mathcal{C}$; that is, as long as Fixer only does swaps with α and β, Breaker's moves are already foretold in the chronicle. Now add an edge in A_S for each $xy \in \mathcal{C} - \infty$ labeled $\{\alpha, \beta\}$. The lemma follows. $\square$

9. Stars with one edge subdivided

Let G be the graph created from a star with at least 3 vertices by subdividing one edge. Let r be the center of the star, t the vertex at distance two from r and s the intervening vertex.

Theorem 13. *If f is valid and $f(r) \geq d(r) + 1$, then G is f-fixable in the chronicled game.*

Proof. Suppose the condition is not sufficient for Fixer to have a winning strategy in the chronicled game. Choose a counterexample G with lists L so as to minimize $|G|$ and subject to that, to maximize $\eta_L(G - t)$.

Create a bipartite graph B with parts $X = \{uw \in E(G - t) \mid L(u) \cap L(w) \neq \emptyset\}$ and $Y = \{\alpha \in \mathrm{Pot}(L) \mid \nu((G - t)_\alpha) = 1\}$, where $uw \in X$ is adjacent to $\alpha \in Y$ if and only if $\alpha \in L(u) \cap L(w)$. Put $F = L(r) - \bigcup_{v \in N(r)} L(v)$.

Claim 1. *If $\beta \in Y$ is swappable with $\gamma \in F$, then $|G_\beta - r - t| \leq 2$.*

Suppose $|G_\beta - r - t| \geq 3$. Pick $v \in V(G_\beta - r - t)$. Now Fixer should swap γ for β in $L(v)$. Let L' be the list assignment after Breaker's response. Note that $L'(r) = L(r)$ since $\gamma, \beta \in L(r)$. Since Breaker can swap γ for β in $L(w)$ for at most one $w \in V(G_\beta - r - t)$, we have $\eta_{L'}(G - t) > \eta_L(G - t)$, which contradicts the maximality of $\eta_L(G - t)$.

Claim 2. *If $\beta \notin Y$ is swappable with $\gamma \in F$, then $|G_\beta - r - t| \leq 1$.*

Suppose $|G_\beta - r - t| \ge 2$. Now Fixer should swap β for γ in $L(r)$. Let L' be the list assignment after Breaker's response. Since Breaker can swap γ for β in $L(w)$ for at most one $w \in V(G_\beta - r - t)$, we again contradict the maximality of $\eta_L(G-t)$.

Claim 3. *We have $\eta_L(G-t) \ge \|G-t\|$.*

Suppose $\eta_L(G-t) < \|G-t\|$. Then we have $|F| \ge 2$. If $F \not\subseteq L(t)$, pick $\gamma \in F - L(t)$; otherwise pick any $\gamma \in F$.

Subclaim 3a. *γ is swappable with $\beta \in \mathrm{Pot}(L) - \gamma$ unless $\gamma \notin L(t)$ and $L(s) \cap L(t) = \{\beta\}$. In particular, there is at most one color with which γ is not swappable.*

Suppose $\gamma \in L(t)$. Then $F \subseteq L(t)$ by our choice of γ. Let $H \subseteq G$. If $|H_{L,\gamma}|$ is even, then $r, t \in V(H)$ and hence $\psi_L(H) > \|H\|$. Therefore γ is swappable with any $\beta \in \mathrm{Pot}(L) - \gamma$ by Lemma 7.

Instead, suppose $\gamma \notin L(t)$. Now, the only subgraph with edges where $|H_{L,\gamma}|$ is even is $G[s,t]$, so if γ is not swappable with some $\beta \in \mathrm{Pot}(L) - \gamma$, then it must be $H = G[s,t]$ that fails all conditions of Lemma 7. Hence we have $L(s) \cap L(t) = \{\beta\}$. This proves the claim.

Subclaim 3b. *There is $\beta \in \mathrm{Pot}(L) - \gamma$ not swappable with γ. Moreover, if $\beta \in Y$, then $|G_\beta - r - t| \ge 3$ and otherwise $|G_\beta - r - t| \ge 2$. In particular, $\gamma \notin L(t)$ and $L(s) \cap L(t) = \{\beta\}$.*

Suppose γ is swappable with all $\beta \in \mathrm{Pot}(L) - \gamma$. Then, by Claim 1, the colors in Y each contribute at most one to $\psi_L(G-t)$. By Claim 2, the colors not in Y contribute nothing to $\psi_L(G-t)$. Hence $\psi_L(G-t) \le |Y| = \eta_L(G-t) < \|G-t\|$, a contradiction. So, there is $\beta \in \mathrm{Pot}(L) - \gamma$ not swappable with γ. Since Claim 1 and Claim 2 apply to all colors except β the second sentence follows since it just gives the bounds from Claim 1 and Claim 2 for β. The final sentence follows from Subclaim 3a.

Subclaim 3c. *If $\beta \in Y$ is not swappable with γ, then $|G_\beta - r - t| \le 3$.*

Suppose $|G_\beta - r - t| \ge 4$. By Subclaim 3a, $L(s) \cap L(t) = \{\beta\}$. Pick $v \in V(G_\beta - r - t - s)$. Now Fixer should add the edge st colored β and then swap γ for β in $L(v)$. Let L' be the list assignment after Breaker's response. Note that $L'(r) = L(r)$ since $\gamma, \beta \in L(r)$. Now Breaker can replace β in both $L(s)$ and $L(t)$ with γ and then swap γ for β in $L(w)$ for at most one $w \in V(G_\beta - r - t - s)$. Hence we have $\eta_{L'}(G-t) > \eta_L(G-t)$, which contradicts the maximality of $\eta_L(G-t)$.

Subclaim 3d. *If $\beta \notin Y$ is not swappable with γ, then $|G_\beta - r - t| \le 2$.*

Suppose $|G_\beta - r - t| \ge 3$. By Subclaim 3a, $L(s) \cap L(t) = \{\beta\}$. Now Fixer should add the edge st colored β and then swap β for γ in $L(r)$. Let L' be the list assignment after Breaker's response. Now Breaker can replace β in both $L(s)$ and $L(t)$ with γ and then swap γ for β in $L(w)$ for at most one $w \in V(G_\beta - r - t)$. Hence we again contradict the maximality of $\eta_L(G-t)$.

Subclaim 3e. *There is $\delta \in L(t) - L(s)$ such that $|G_\delta - t|$ is odd.*

By Subclaim 3b, we have $\beta \in \mathrm{Pot}(L) - \gamma$ not swappable with γ. By Claim 1, the colors in $Y - \beta$ contribute at most $|Y - \beta|$ to $\psi_L(G-t)$. By Subclaim 3c and Subclaim 3d, the total contribution of Y and β to $\psi_L(G-t)$ is at most $|Y| + 1$. Since nothing else contributes by Claim 2, we have $\psi_L(G-t) \le \eta_L(G-t) + 1 \le \|G\| - 1$. Since $\psi_L(G) \ge \|G\|$, there must be $\delta \in L(t) - L(s)$ such that $|G_\delta - t|$ is odd.

Subclaim 3f. *We may assume $\delta \in L(r) \cap L(t) - \bigcup_{v \in N(r)} L(v)$.*

Suppose not. Suppose $\delta \in Y$. Then, by Claim 1 and since $|G_\delta - t|$ is odd, we have $\delta \in L(u) \cap L(w)$ for $u, w \in N(r) - s$. Fixer should swap γ for δ in $L(t)$. If Breaker replaces any δ with γ, he increases $\eta_L(G - t)$, so Breaker must pass. Pick $\alpha \in F - \gamma$. Then, by Subclaim 3b applied to this new position, we must have $\alpha \notin L(t)$, so we can use α in place of γ and γ in place of δ, which proves the subclaim in this case.

Otherwise, by Claim 2, we must have $\delta \in L(u)$ for exactly one u in $N(r) - s$. Again, if Fixer swaps γ for δ in $L(t)$, Breaker must pass lest he increase $\eta_L(G - t)$. As before we can use α in place of γ and γ in place of δ.

Subclaim 3g. *Claim 3 is true.*

Fixer should swap δ for β in $L(s)$. First, suppose $\beta \in Y$. Then, by Subclaim 3b, we have $|G_\beta - r - t| \geq 3$. No matter what Breaker does, $\eta_L(G - t)$ has increased. Similarly, if $\beta \notin Y$, Subclaim 3b gives $|G_\beta - r - t| \geq 2$ and $\eta_L(G - t)$ has increased no matter Breaker's response.

Claim 4. *If $|C| \geq |N_B(C)|$ for $C \subseteq Y$, then $C \cap L(t) \neq \emptyset$.*

Suppose not. Let B' be the subgraph of B induced on $C \cup N_B(C)$. Then we may apply Lemma 8 to get a nonempty matching M of B' whose vertex set is $S \cup N_B(S)$ for some $S \subseteq C$. For each $\{uw, \alpha\} \in M$, color uw with α. Then since we used colors S, no edge in $X - N_B(S)$ has a color in S, so the graph $G' = G - V(N_B(S) - r)$ with lists $L'(v) = L(v) - S$ satisfies the hypotheses of the lemma. Since $|G'| < |G|$, Fixer has a winning strategy on G' with lists L'. But this strategy wins on G with L as well since $N_B(S)$ is colored with S, a contradiction.

Claim 5. *We have $|C| \leq |N_B(C)|$ for $C \subseteq Y$. In particular, $\eta_L(G - t) = \|G - t\|$ and $F \neq \emptyset$.*

Suppose not and choose $C \subseteq Y$ such that $|C| > |N_B(C)|$ so as to minimize $|C|$. For all $\tau \in C$, by minimality of $|C|$, we have $N_B(C - \tau) = N_B(C)$. Since $|N_B(C')| \geq |C|$ for every $C' \subseteq C - \tau$, Hall's theorem gives a nonempty matching M_τ whose vertex set is $(C - \tau) \cup N_B(C - \tau) = (C - \tau) \cup N_B(C)$. So, for every $\tau \in C$, we can color $N_B(C - \tau)$ using $C - \tau$ as in Claim 4; the key point is that each of these colorings colors the same edge set.

Put $R = C \cap L(t)$. By Claim 4, $R \neq \emptyset$. For $\tau \in R$, we have $|C - \tau| \geq |N_B(C - \tau)|$, so Claim 4 gives $|R| \geq 2$.

First, suppose $rs \in N_B(C)$. Pick $\tau \in R \cap L(s)$ if possible; otherwise pick $\tau \in R$ arbitrarily. For each $\{uw, \alpha\} \in M_\tau$, color uw with α. Put $G' = G - V(N_B(C) - r)$ and $L'(v) = L(v) - (C - \tau)$ for $v \in V(G')$. We claim that (G', L') is superabundant. Suppose otherwise that we have $H \subseteq G'$ such that (H, L') is not abundant. Since rs got colored, the only subgraph to worry about is $H = G[s, t]$. If $\tau \in L(s)$, then $\tau \in L'(s) \cap L'(t)$, so we are good. Otherwise, $R \cap L(s) = \emptyset$, so there must be some color $\delta \in L(s) \cap L(t) - C$; in particular, $\delta \in L'(s) \cap L'(t)$. Since (G', L') satisfies the hypotheses of the lemma and $|G'| < |G|$, Fixer has a winning strategy by minimality of $|G|$. But this strategy wins on G with L as well since $N_B(C - \tau)$ is colored with $C - \tau$, a contradiction.

Hence, we may assume that $rs \notin N_B(C)$. So, $R \cap L(s) = \emptyset$. Pick $\tau \in R$. For each $\{uw, \alpha\} \in M_\tau$, color uw with α. Put $G' = G - V(N_B(C) - r)$ and $L'(v) = L(v) - (C - \tau)$ for $v \in V(G')$. We claim that (G', L') is superabundant. Suppose otherwise that we have $H \subseteq G'$ such that (H, L') is not abundant. Since $\tau \notin L(s$, we must have $r, t \in V(H)$. Now $V(H_\tau - t) = \{r\}$ since $N_B(\tau) \subseteq N_B(C)$. So, when we add t back in, τ contributes one to $\psi_{L'}(H)$. But $(H-t, L')$ is abundant, so (H, L') is abundant, a contradiction. Since (G', L') satisfies the hypotheses of the lemma and $|G'| < |G|$, Fixer has a winning strategy by minimality of $|G|$. But this strategy wins on G with L as well since $N_B(C-\tau)$ is colored with $C-\tau$, a contradiction.

Claim 6. *$G-t$ has an L-edge-coloring π.*

By Claim 5 and Hall's theorem, B has a perfect matching which gives an L-edge-coloring of $G-t$.

Claim 7. *There is a color β such that $L(s) \cap L(t) = \{\beta\}$. Moreover, $\beta \in L(r)$.*

Otherwise, we L-edge-color $G-t$ by Claim 6 and then use one of the two colors in $L(s) \cap L(t)$ to color st, a contradiction.

Claim 8. *We have $F \cap L(t) = \emptyset$.*

Suppose otherwise that there is $\gamma \in F \cap L(t)$. Color the edges of $G-s-t$ via π and let L' be the resulting list assignment on rst. Then $\beta \in L'(r) \cap L'(s) \cap L'(t)$ and $\gamma \in' L(r) \cap L'(t)$. Hence (rst, L') is superabundant and hence Fixer has a winning strategy on rst with L' by Theorem 9. But then Lemma 11 shows that Fixer has a winning strategy on G with L, a contradiction.

Claim 9. *We have $L(r) \cap L(s) = \{\beta\}$.*

Suppose not and pick $\tau \in L(r) \cap L(s) - \beta$. Pick $\gamma \in F$.

Suppose τ appears in more than three lists. Then Fixer swaps γ for τ in $L(s)$. If Breaker does nothing, then Fixer colors $G-s-t$ from π, colors rs with γ and st with β to win. Hence Breaker must swap γ for τ at some $v \in N(r) - s$. But $\tau \in L(w)$ for some $w \in N(r) - s - v$, so $\eta_L(G-t)$ has increased, a contradiction.

Suppose β appears in more than three lists. Then Fixer swaps τ for β in $L(t)$. If Breaker does nothing or swaps τ for β somewhere, then Fixer can finish the coloring. Hence Breaker must swap β for τ at some $v \in N(r) - s$. But now applying the previous paragraph with the roles of β and τ reversed gives a contradiction.

Therefore, β and τ each contribute only one to $\psi_L(G)$. Since $\eta_L(G-t) = \|G-t\|$ by Claim 5, applying Claim 1 and Claim 2 shows that there must be $\delta \in L(t) - L(s)$ such that $|G_\delta - t|$ is odd. But now by the same argument as in Subclaim 3f, Fixer can achieve a position contradicting Claim 8.

Claim 10. *Fixer wins.*

Pick $\gamma \in F$. Since $L(r) \cap L(s) = \{\beta\}$ by Claim 9, there must be $\alpha \in L(r) \cap L(t)$ since rst is abundant and $L(s) \cap L(t) = \{\beta\}$ by Claim 7.

Pick $\tau \in L(s) - \beta$. Suppose τ appears in more than one list. Then Fixer should swap γ for τ in $L(s)$. Then $\eta_L(G-t)$ has increased unless Breaker swaps τ for γ in $L(r)$. But since τ is in more than one list, it must be in $L(v)$ for some $v \in N(r) - s$ and $\eta_L(G-t)$ has increased as well, a contradiction.

Suppose β appears in more than three lists. Then Fixer should swap τ for β in $L(t)$. If Breaker does nothing or swaps τ for β in any list other than $L(r)$, then Fixer can complete the coloring. So Fixer must swap τ for β in $L(r)$. But $\beta \in L(v)$ for some $v \in N(r) - s$, so applying the previous paragraph with the roles of β and τ reversed gives a contradiction.

Suppose α appears in more than three lists. Then Fixer should swap γ for α in $L(t)$. Now Breaker must pass lest he increase $\eta_L(G-t)$. But this is a position contradicting Claim 8.

Therefore, β and α each contribute only one to $\psi_L(G)$. Since $\eta_L(G-t) = \|G-t\|$ by Claim 5, applying Claim 1 and Claim 2 shows that there must be $\delta \in L(t) - L(s)$ such that $|G_\delta - t|$ is odd. But now by the same argument as in Subclaim 3f, Fixer can achieve a position contradicting Claim 8. □

This generalizes the "short Kierstead paths" of Kostochka and Stiebitz (see Theorem 3.3 in [8]). Parts (a), (b) and (c) of Theorem 3.3 in [8] are the special case of the following where T is P_4.

Corollary 14. *Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q) + 1$. Suppose $T \subseteq Q$ is a star with center r and one edge subdivided. If $d_Q(r) < \Delta(Q)$ and $(T, xy, \pi, <)$ is Tashkinov, then $V(T)$ is elementary with respect to π.*

Part (d) of Theorem 3.3 in [8] now follows from the following theorem.

Theorem 15. *If (G, L) is superabundant and $\psi_L(G) > \|G\|$, then Fixer has a winning strategy against Breaker in the chronicled game.*

Proof. Suppose the condition is not sufficient for Fixer to have a winning strategy in the chronicled game. Choose a counterexample G with lists L so as to minimize $|G|$ and subject to that, to maximize $\eta_L(G-t)$. By Theorem 13 we know that $|L(v)| = d(v)$.

Create a bipartite graph B with parts $X = \{uw \in E(G-t) \mid L(u) \cap L(w) \neq \emptyset\}$ and $Y = \{\alpha \in \text{Pot}(L) \mid \nu((G-t)_\alpha) = 1\}$, where $uw \in X$ is adjacent to $\alpha \in Y$ if and only if $\alpha \in L(u) \cap L(w)$. Put $F = L(r) - \bigcup_{v \in N(r)} L(v)$. Claim 1 and Claim 2 follow with the same proof as in Theorem 13.

Claim 1. *If $\beta \in Y$ is swappable with $\gamma \in F$, then $|G_\beta - r - t| \leq 2$.*

Claim 2. *If $\beta \notin Y$ is swappable with $\gamma \in F$, then $|G_\beta - r - t| \leq 1$.*

Claim 3. *We have $\eta_L(G-t) \geq \|G-t\|$.*

Suppose $\eta_L(G-t) < \|G-t\|$. Then we have $|F| \geq 1$. Pick $\gamma \in F$.

Subclaim 3a. *We have $|F| = 1$ and hence $\eta_L(G-t) = \|G-t\| - 1$.*

Otherwise, the proof of Claim 3 in Theorem 13 goes through, a contradiction.

Subclaim 3b. *If $\gamma \notin L(t)$, then γ is swappable with $\beta \in \text{Pot}(L) - \gamma$ unless $L(s) \cap L(t) = \{\beta\}$. In particular, if $\gamma \notin L(t)$, then there is at most once color with which γ is not swappable.*

The proof is the same as this case of Subclaim 3a in Theorem 13.

Subclaim 3c. *We may assume that $\gamma \in L(t)$.*

Suppose $\gamma \notin L(t)$. Then Claims 3b through 3e from Theorem 13 all go through. Hence we have $\delta \in L(t) - L(s)$ such that $|G_\delta - t|$ is odd. If $\delta \in L(r) \cap L(t) - \bigcup_{v \in N(r)} L(v)$, then $|F| \geq 2$ contradicting Claim 3a.

Suppose $\delta \in Y$. Then, by Claim 1 and since $|G_\delta - t|$ is odd, we have $\delta \in L(u) \cap L(w)$ for $u, w \in N(r) - s$. Fixer should swap γ for δ in $L(t)$. If Breaker replaces any δ with γ, he increases $\eta_L(G-t)$, so Breaker must pass. This gives the desired position.

Otherwise, by Claim 2, we must have $\delta \in L(u)$ for exactly one u in $N(r) - s$. Again, if Fixer swaps γ for δ in $L(t)$, Breaker must pass lest he increase $\eta_L(G-t)$. We again have the desired position.

Subclaim 3d. *We can finish this proof somehow.* □

This should also be true and makes our adjacency lemma a bit better.

Theorem 16. *If f is valid and $f(s) \geq d(s)+1$, then G is f-fixable in the chronicled game.*

Proof. □

10. Small fixable graphs

Lemma 17. *Let G be P_5. Then the following hold:*

(1) *G is f-fixable in the chronicled game for any valid f such that $f(v) \geq d_G(v)+1$ for at least two internal vertices v;*
(2) *if (G, L) is superabundant, $|L(v)| \geq d_G(v)+1$ for at least one internal vertex v and $\psi_L(G) > \|G\|$, then Fixer has a winning strategy against Breaker in the chronicled game.*

Proof. Let $P = v_1 \dots v_5$ and v_2 and v_3 are high. Suppose that (G, L) is superabundant.
Claim 1 If color 0 appears on all vertices and color 1 appears on v_2, v_3, then Fixer wins.
If 1 appears nowhere else, then Fixer colors v_1v_2 with 0 and v_2v_3 with 1. Fixer can complete the coloring, since $v_3v_4v_5$ is superabundant. Suppose instead that color 2 appear also on v_4 or v_5. Now Fixer has a Kierstead path, with color 0 on v_1v_2 and v_3v_4 and color 1 on v_2v_3.

Claim 2 If color 0 appears on all vertices and color 1 appears on 4 vertices, then Fixer wins.

By Claim 1, we need only consider the two cases where color 1 appears on v_1, v_2, v_4, v_5 and where color 1 appears on v_1, v_3, v_4, v_5. We will reduce the first case to the second and show how Fixer can win the second. In the first case, let 2 be another color on v_3. Fixer will swap 1 and 2. If Breaker does not pair v_2 and v_3, then Fixer swaps 1 for 2 at v_3. Now we have a Kierstead path with 0 on v_1v_2, 1 on v_2v_3, and 0 on v_3v_4. Hence, Breaker pairs v_2 and v_3. After Fixer swaps 2 for 1 at v_2 and 1 for 2 at v_3, we are in the second case, which we now consider.

Let color 2 be another color at v_2. Fixer will swap colors 0 and 2. There are 3 cases for the matching that Breaker chooses. It could include v_1v_3 or v_1v_4 or v_1v_5.

In the first case, Fixer swaps 2 for 0 at v_1 and v_3. Let 3 be another color at v_3. Fixer will swap 3 for 0. If Breaker does not pair v_2 and v_3, then Fixer swaps 0 for 3 at v_3 (and Breaker possibly swaps 3 for 0 at v_4 or v_5). Now Fixer has a Kierstead path with color 2 on v_1v_2, color 0 on v_2v_3, and color 1 on v_3v_4. So Breaker must pair v_2 and v_3 (for swapping 0 and 3). After Fixer swaps 3 for 0 at v_2 and 0 for 3 at v_3, he colors v_3v_4 with 0 and colors v_4v_5 with 1. Now Fixer can win on $v_1v_2v_3$, since it is superabundant (color 2 appears everywhere and color 1 appears at v_1 and v_3).

In the second case, Fixer swaps 2 for 0 at v_1 and v_4. Now we have a Kierstead path with color 2 on v_1v_2, color 0 on v_2v_3, and color 1 on v_3v_4. In the third case, Fixer swaps 2 for 0 at v_3 and v_4. Now we have a Kierstead path with color 0 on v_1v_2, color 3 on v_2v_3, and color 1 on v_3v_4.

Claim 3 If color 0 appears on all vertices and color 1 appears on v_1 and v_2, then Fixer wins.

If color 1 does not appear elsewhere, then Fixer colors v_1v_2 with 1 and can finish on $v_2v_3v_4v_5$ since it is superabundant. So assume that color 1 appears elsewhere. By Claim 1, we assume that it does not appear on v_3; by Claim 2, we assume that 1 does not appear on both v_4 and v_5.

Suppose that 1 appears on v_4. Since (G, L) is superabundant, some color 2 appears on at least two of v_3, v_4, v_5. If 2 appears on v_3, v_4, then we color v_1v_2 with 2, v_2v_3 with 0, v_3v_4 with 2, and v_4v_5 with 0. If instead 2 appears on v_4v_5, then Fixer colors v_4v_5 with 3. He can win on the smaller graph, since it is superabundant. Thus, 2 appears on v_3, v_5. Now Fixer will swap 1 and 2. If Breaker matches v_3 and v_5, then Fixer wins, since he can get 0 and 1 both appearing on every vertex. Similarly, if Breaker matches v_3 and v_4, then Fixer swaps 1 and 2 at v_3 and then finishes the coloring. If Breaker matches v_3 and v_1, then Fixer swaps 1 and 2 at v_3 and v_1; now we have a Kierstead path with 0 on v_1v_2, 2 on v_2v_3, and 0 on v_3v_4. Hence Breaker must match v_3 and v_2. If Breaker does not match v_1 and v_4, then Fixer swaps 3 for 2 at v_4. Now fixer colors v_1v_2 with 1, v_2v_3 with 0, v_3v_4 with 2, and v_4v_5 with 0. So Breaker matches v_1 and v_4. Now Fixer swaps 3 for 2 at v_1 and v_4. Finally, Fixer can win by Claim 2.

Suppose instead that 1 appears on v_5. Since (G, L) is superabundant, some other color 2 appears on at least two of v_3, v_4, v_5. If 2 appears on v_3 and v_4, the we color v_1v_2 with 1, v_2v_3 with 0, v_3v_4 with 2, and v_4v_5 with 0. So assume that 2 does not appear on v_3 and v_4. Now some other color 3 appears on at least two of v_2, v_3, v_4. Hence, Fixer can color v_1v_2 with 1, and reduce to the shorter path $v_2v_3v_4v_5$, since it will still be superabundant.

Claim 4 If color 0 appears on all vertices and color 1 appears on v_3 and v_4, then Fixer wins.

If color 1 does not appear on v_1 or v_2, then Fixer colors v_3v_4 with 2 and colors v_4v_5 with 0. He can complete the coloring on the shorter path $v_1v_2v_3$ since it is still superabundant. By Claim 1, we assume that color 1 does not appear on v_2, so assume that color 1 appears on v_1. If color 1 also appears on v_5, then Fixer wins by Claim 2, so assume that it doesn't. Since L is superabundant, some other color 2 appears on at least two vertices. If 2 appears on two vertices other than v_5, then we color v_4v_5 with 0. Fixer can complete the coloring on $v_1v_2v_3v_4$ since it is still superabundant. So assume that 2 appears on v_5.

If 2 also appears on v_4, then we color v_4v_5 with 2. Fixer can color the shorter path, since it is still superabundant. Thus we assume that color 2 appears on v_1, v_2, or v_3. We may assume that 2 appears on v_2 as follows. If not, then let 3 be another color at v_2. Now Fixer will swap 2 and 3. This will put color 2 on v_2 and some neighbor (in which case Fixer wins by Claim 1 or Claim 3), unless Breaker pairs them in the matching for 2 and 3. Thus, Breaker leave v_5 unpaired, so now color 2 is at v_2 and v_5. Now Fixer swaps 1 and 2. Regardless of which matching Breaker chooses, after Fixer swaps at v_2, it will have the same color as either v_1 or v_3, and the lists will still be superabundant. So Fixer can win by either Claim 1 or Claim 3.

Claim 5 If color 0 appears on all vertices, then Fixer wins.

By the Claims 1, 3, and 4 we may assume that no color 1 appears on adjacent vertices among v_1, v_2, v_3, v_4. Since L is superabundant, there exists colors 1 and 2 such that 1 appears on v_1 and v_3 and 2 appears on v_2 and v_4. Now fixer swaps 1 for 2 at v_2. Regardless of which matching Breaker chooses, the lists remain superabundant and now color 1 appears on v_2 and one of its neighbors. Hence, Fixer can win by Claim 1 or 3. □

Lemma 18. *C_4 is degree-fixable in the chronicled game.*

Proof. Suppose not. Let G be a C_4, say $v_1v_2v_3v_4v_1$. Let L be a superabundant list assignment on G with $|L(v)| \geq 2$ for all $v \in V(G)$ where Fixer has no winning strategy in the chronicled game. A color in $Pot(L)$ is a *quadruple* if it is available at all four vertices of

C_4; *triples*, *pairs*, and *singletons* are defined analogously. An *adjacent pair* is one that is available on the endpoints of an edge. Pairs that are not adjacent are *diagonal*. Among all choices for such a bad L, choose L to maximize the number quadruples, then triples, then adjacent pairs, then diagonal pairs. (We only use this extremal choice of L in Case 2.c.3.a, but it shortens that case, so I'm leaving it for now.) In the first few claims, we often use the following fact. If L has a quadruple 1 and Fixer can color the two edges of a matching with colors other than 1, then Fixer wins.

Claim 1. If L contains a quadruple then L has no diagonal pairs.
Proof. Suppose not. Let L be a bad superabundant list assignment with 1 as a quadruple and with the minimum number of diagonal pairs, including 2 on v_1 and v_3 (by symmetry).
Case 1. Suppose that L has no triples. By considering $H = C[v_1, v_2, v_4]$, we see that L must have another pair on $\{v_1, v_2, v_4\}$.
Case 1.a. First suppose that L also contains a diagonal pair 3 on v_2 and v_4. Fixer swaps 2 for 3 at v_2; Breaker must pass or Fixer will win immediately. So v_2 has an edge labeled $2, 3$ to ∞ in the chronicle. Now Fixer swaps 2 for 3 at v_4. By symmetry, Breaker swaps 3 for 2 at v_3. Now Fixer swaps 3 for 2 at v_2; Breaker passes (due to the chronicle), so Fixer wins.
Case 1.b. Suppose instead that L contains no other diagonal pair. The subgraph induced by $\{v_1, v_2, v_4\}$ implies that L contains a pair among these three vertices. Similarly for $\{v_2, v_3, v_4\}$. Since L has a diagonal pair only on v_1, v_3, L must have two adjacent pairs. By symmetry, assume that 3 appears on v_1 and v_2 and that 4 appears on v_2 and v_3. Choose $5 \in L(v_4) - 1$. Fixer swaps 2 for 5 at v_4. Regardless of Breaker's response, Fixer wins immediately.
Case 2. Suppose instead that L has a triple.
Case 2.a. Suppose that 3 appears on v_1, v_2, v_4. Fixer swaps 2 for 3 at v_2. If Breaker swaps 2 for 3 at v_4, then Fixer wins, since 1 and 2 are now quadruples. If Breaker swaps 3 for 2 at v_3, then Fixer uses 2 on v_1v_2 and 3 on v_3v_4. Finally, if Breaker passes, then Fixer uses 2 on v_2v_3 and 3 on v_1v_4.
Case 2.b Suppose that 3 appears on v_1, v_2, v_3. Let 4 be another color in $L(v_4)$. Now Fixer swaps 2 for 4 at v_4, threatening to win with 2 on v_1v_4 and 3 on v_2v_3. So Breaker swaps 4 for 2 at v_1. Now Fixer wins with 3 on v_1v_2 and 2 on v_3v_4.

Claim 2. If L contains a quadruple, then L contains no triple.
Proof. Suppose instead that L contains the quadruple 1, and the triple 2, by symmetry on v_1, v_2, v_3. Since L is superabundant (and contains no diagonal pair by Claim 1), some color 3 other than 1 and 2 is available on some edge e. If e is v_3v_4 or v_4v_1, then Fixer wins immediately. So assume that 3 appears on v_1 and v_2, and let 4 be another color in $L(v_4)$. Fixer swaps 4 for 3 at v_2; Breaker must pass or Fixer will win immediately. The resulting list assignment is superabundant with a diagonal pair, so Fixer wins by Claim 1.

Claim 3. L contains no quadruples.
Proof. Suppose that L contains the quadruple 1. Now consider the subgraphs induced by the four vertex subsets of size 3. Since L contains no triple and no diagonal pair (by Claims 2 and 1), three of the four edges have some available color other than 1. Thus, Fixer wins.

Claim 4. Fixer wins.

Proof. If L contains no triples, then (since L contains no quadruples, by Claim 3) some pair is available on each edge, and these pairs are disjoint, so Fixer wins. When L contains one triple, the situation is similar.

Case 1. L contains exactly 1 triple.

Suppose 1 is available on v_1, v_2, v_3, and L contains no other triples. Each of the edges v_3v_4 and v_4v_1 has a color available, say colors 2 and 3. If also a color other than 1 is available on either v_1v_2 or v_2v_3, then Fixer wins. Since L is superabundant, some pair is available on two of v_1, v_2, v_3. Since it is not an adjacent pair, it must be a diagonal pair 4 on v_1, v_3. Choose $5 \in L(v_2) - 1$. Fixer swaps 4 for 5 at v_2. However Breaker responds, Fixer wins immediately.

Case 2. L contains at least 2 triples.

Suppose that all triples are available only on v_1, v_2, v_3.

By symmetry among colors, assume colors 1 and 2 are available on v_1, v_2, v_3. By superabundance, some color is available on each of edges v_3v_4 and v_4v_1. Using these colors, together with colors 1 and 2, Fixer wins.

Case 2.a. Triples appear on v_1, v_2, v_3 and v_2, v_3, v_4 and v_3, v_4, v_1.

If a triple also is available on v_4, v_1, v_2, then Fixer wins immediately. Similarly, if L contains any adjacent pair, then Fixer wins immediately. Without loss of generality, assume that 1 is available on v_1, v_2, v_3 and 2 is available on v_2, v_3, v_4 and 3 is available on v_3, v_4, v_1. Since L is superabundant, it contains some other pair 4. As noted above, L contains no adjacent pair. So assume that 4 is a diagonal pair. We have two cases. If 4 is available on v_1, v_3, then Fixer swaps 4 for 2 at v_2. Now Fixer wins (whether or not Breaker swaps 2 for 4 at v_1) with 1 on v_1, v_2 and 4 on v_2, v_3 and 2 on v_3, v_4 and 3 on v_4, v_1. Assume instead that 4 is available on v_2, v_4. Now Fixer swaps 4 for 3 at v_1. Again Fixer wins (whether or not Breaker swaps 3 for 4 at v_2) with 1 on v_1v_2 and 2 on v_2v_3 and 3 on v_3v_4 and 4 on v_4v_1.

Case 2.b. Triples are available only on v_1, v_2, v_3 and v_2, v_3, v_4.

Without loss of generality, Color 1 is available on v_1, v_2, v_3 and color 2 is available on v_2, v_3, v_4. By superabundance, some color 3 is available on v_4, v_1. By superabundance, another pair 4 is available. First suppose 4 is a diagonal pair, by symmetry on v_1, v_3 (or 4 is a triple, on vertices including v_1, v_3). Fixer swaps 4 for 2 at v_2. Breaker cannot swap 4 for 2 at v_4, since Claim 3 implies that Fixer wins if L contains a quadruple. Regardless of whether or not Breaker swaps 2 for 4 at v_1, Fixer wins with 1 on v_1v_2 and 4 on v_2v_3 and 2 on v_3v_4 and 3 on v_4v_1. So instead 4 must be an adjacent pair or triple. If 4 is available on any edge other than v_4v_1, then Fixer wins immediately. So assume that 4 is available only on v_4, v_1. Fixer swaps 3 for 2 at v_2. Breaker cannot swap 3 for 2 at v_3, since then 3 is a quadruple. So Breaker must swap 2 for 3 at v_1. Now Fixer swaps 3 for 1 at v_3. Breaker cannot swap 3 for 1 at v_1, since then 3 would be a quadruple. Now Fixer wins (whether or not Breaker swaps 1 for 3 at v_4) with 1 on v_1v_2 and 3 on v_2v_3 and 2 on v_3v_4 and 4 on v_4v_1.

Case 2.c. Triples are available only on v_1, v_2, v_3 and v_3, v_4, v_1.

Without loss of generality, color 1 is available on v_1, v_2, v_3 and color 2 is available on v_3, v_4, v_1. Case 2.c.1: If 3 is available on v_2, v_4, then swap 2 for 3 at v_2. Breaker must respond, since otherwise 2 is a quadruple, and Fixer wins by Claim 3. By symmetry, Breaker swaps 3 for 2 at v_3. It is easy to check that the new list assignment is still superabundant with two triples, so it reduces to Case 2.b. or Case 2.a.

Case 2.c.2: If v_2 and v_4 each is available in an adjacent pair, then Fixer wins.
Case 2.c.3: Hence, by symmetry, assume that $L(v_2)$ contains a singleton 3 and v_2 is available in no adjacent pair (and no triple other than 1).
Case 2.c.3.a: If L contains a diagonal pair 4, then it must be available on v_1, v_3. Swap 4 for 3 at v_2. Breaker must respond; otherwise Fixer increased the number of triples (contradicting our extremal choice of L). So Breaker swaps 3 for 4 at v_1 (by symmetry). But now Fixer has increased the number of adjacent pairs. All that remains is to check that the lists are superabundant, which they are. So L contains no diagonal pairs.
Case 2.c.3.b: Since L is superabundant and contains no diagonal pair, and v_2 is available in no adjacent pair and no triple other than 1, two more adjacent pairs are available on v_3v_4 and v_4v_1 (possibly both pairs are available on the same edge).
Case 2.c.3.b.1: Colors 4 and 5 both are available on v_3v_4.
Fixer swaps 4 for 3 at v_2. Breaker must respond, so swaps 3 for 4 at v_3. Now Fixer swaps 2 for 4 at v_2. If Breaker swaps 4 for 2 at v_1, then Fixer wins with 1 on v_1v_2 and 2 on v_2v_3 and 5 on v_3v_4 and 4 on v_4v_1. If Breaker passes, then Fixer wins with 1 on v_1v_2 and 2 on v_2v_3 and 5 on v_3v_4 and 2 on v_4v_1. So Breaker swaps 4 for 2 at v_3. The resulting list assignment is superabundant, so it reduces to Case 1.b.
Case 2.c.3.b.2: Color 4 is available on v_3, v_4 and color 5 is available on v_4, v_1.
Fixer swaps 4 for 3 at v_2. Breaker must swap 3 for 4 at v_3 (otherwise Fixer wins with 1 on v_1v_2 and 4 on v_2v_3 and 2 on v_3v_4 and 5 on v_4v_1). Fixer swaps 4 for 2 at v_1, threatening to reduce to Case 2.b. Breaker can't swap 4 for 2 at v_3, since then 4 is a quadruple. Thus, Breaker swaps 2 for 4 at v_2; but this case again reduces to Case 2.b. $\square$

11. Adjacency lemmas

11.1. **Precursors.** Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q) + 1$ and $G \subseteq Q$. For a $\Delta(Q)$-edge-coloring π of $Q - E(G)$, put $L_\pi(v) = [\Delta(Q)] - \pi\left(E_Q(v) - E(G)\right)$ for all $v \in V(G)$. We say that G is a Ψ*-subgraph* of Q if there is a $\Delta(Q)$-edge-coloring π of $Q - E(G)$ such that each $H \subsetneq G$ is abundant. Put $E_L(H) = |\{\alpha \in \text{Pot}(L) \mid |H_{L,\alpha}| \text{ is even}\}|$ and $O_L(H) = |\{\alpha \in \text{Pot}(L) \mid |H_{L,\alpha}| \text{ is odd}\}|$. Plainly, $Pot(L) = E_L(G) + O_L(G)$.

Lemma 19. *Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q) + 1$. If $G \subseteq Q$ and π is a $\Delta(Q)$-edge-coloring of $Q - E(G)$ such that $\psi_L(G) \leq \|G\|$, then $|O_{L_\pi}(G)| \geq \sum_{v \in V(G)} \Delta(Q) - d_Q(v)$. Furthermore, if $\psi_L(G) < \|G\|$, then $|O_{L_\pi}(G)| > \sum_{v \in V(G)} \Delta(Q) - d_Q(v)$.*

Proof. The proof is a straightforward counting argument. For fixed degrees and list sizes, as $|O_L(G)|$ gets larger, $\psi_L(G)$ gets smaller (half as quickly). The details forthwith. Put $L = L_\pi$.
Since $\|G\| \geq \psi_L(G)$, we have

$$\|G\| \geq \sum_{\alpha \in \text{Pot}(L)} \left\lfloor \frac{|G_{L,\alpha}|}{2} \right\rfloor = \sum_{\alpha \in \text{Pot}(L)} \frac{|G_{L,\alpha}|}{2} - \sum_{\alpha \in O_L(H)} \frac{1}{2}. \tag{1}$$

Also,

$$\begin{aligned}
\sum_{\alpha\in \mathrm{Pot}(L)} \frac{|G_{L,\alpha}|}{2} &= \sum_{v\in V(G)} \frac{\Delta(Q)-(d_Q(v)-d_G(v))}{2} \\
&= \sum_{v\in V(G)} \frac{d_G(v)}{2} + \sum_{v\in V(G)} \frac{\Delta(Q)-d_Q(v)}{2} \\
&= \|G\| + \sum_{v\in V(G)} \frac{\Delta(Q)-d_Q(v)}{2}. \qquad (2)
\end{aligned}$$

Now we solve for $\|G\| - \sum_{\alpha\in\mathrm{Pot}(L)} \frac{|G_{L,\alpha}|}{2}$ in (1) and (2), set the expressions equal, and then simplify. The result is (3).

$$|O_L(G)| \geq \sum_{v\in V(G)} \Delta(Q) - d_Q(v). \qquad (3)$$

Finally, if the inequality in (1) is strict, then the inequality in (3) is also strict. □

Lemma 20. *Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q) + 1$. Suppose H is a Ψ-subgraph of Q where H is a star with one edge subdivided. Let r be the center of the star, t the vertex at distance two from r and s the intervening vertex. Then there is $X \subseteq N(r)$ with $V(H - r - t) \subseteq X$ such that*

$$\sum_{v \in X \cup \{t\}} (d_Q(v) + 1 - \Delta(Q)) \geq 0.$$

Moreover, if $\{r, s, t\}$ does not induce a triangle in Q, then

$$\sum_{v \in X \cup \{t\}} (d_Q(v) + 1 - \Delta(Q)) \geq 1.$$

Furthermore, if $d_Q(r) < \Delta(Q)$ or $d_Q(s) < \Delta(Q)$, then we can improve both lower bounds by 1.

Proof. Let G be a maximal Ψ-subgraph of Q containing H such that G is a star with one edge subdivided. Let π be a coloring of $Q - E(G)$ showing that G is a Ψ-subgraph and put $L = L_\pi$.

We first show that $|E_L(G)| \geq d_Q(r) - d_G(r) - 1$ if rst induces a triangle; otherwise, $|E_L(G)| \geq d_Q(r) - d_G(r)$. Suppose rst does not induce a triangle; for arbitrary $x \in N_Q(r) - V(G)$, let $\alpha = \pi(rx)$. If $\alpha \in O_L(G)$, then adding x to G gives a larger Ψ-subgraph of the required form; this contradicts the maximality of G. Hence $\alpha \in E_L(G)$. Therefore, $|E_L(G)| \geq d_Q(r) - d_G(r)$ as desired. If rst induces a triangle, then we lose one off this bound from the rt edge.

By Theorem 15, we have $\psi_L(G) \leq \|G\|$. By Lemma 19, we have $|O_L(G)| \geq \sum_{v \in V(G)} \Delta(Q) - d_Q(v)$. Suppose rst does not induce a triangle. Then

$$\begin{aligned}
\Delta(Q) &\geq \mathrm{Pot}(L) \\
&= |E_L(G)| + |O_L(G)| \\
&\geq d_Q(r) - d_G(r) + \sum_{v \in V(G)} \Delta(Q) - d_Q(v) \qquad (4) \\
&= \Delta(Q) - d_G(r) + \sum_{v \in V(G-r)} \Delta(Q) - d_Q(v) \\
&= \Delta(Q) + 1 \sum_{v \in V(G-r)} \Delta(Q) - 1 - d_Q(v).
\end{aligned}$$

Therefore, $\sum_{v \in V(G-r)} \Delta(Q) - 1 - d_Q(v) \leq -1$. Negating gives the desired inequality. If rst induces a triangle, we lose one off the bound. Theorem 13 and Theorem 16 give the final statement. □

11.2. **Improvement on Woodall's adjacency lemma.** Woodall [9] proved some beautiful adjacency lemmas generalizing lemmas of Sanders and Zhao [7] as well as lemmas of Luo and Zhang [4]. In their edge coloring book, Stiebitz et al. [8] restated Woodall's results and we follow their presentation. Let Q be an edge-critical graph and $xy \in E(Q)$. We would like to find as many Ψ-subgraphs of Q of the form zxy as possible, because we can apply Lemma 20 and get lots of information about the neighbors of x. If π is a Δ-edge-coloring of $Q - xy$, then every $\alpha \in \bar{\pi}(y)$ must appear on an edge $x_\alpha x$ (in general, we write x_τ for the end of the edge colored τ incident to x if there is one). But then $x_\alpha xy$ is a Ψ-subgraph. So, the number of Ψ-subgraphs of the form zxy is at least $|\bar{\pi}(x)|$. Woodall showed that there can be more, to state his result we need a couple definitions.

For $t \in \mathbb{N}$ and $xy \in E(Q)$, the *t-Kierstead set* of the pair (x, y) is

$$K_t(x,y) = \{z \in N(y) - x \mid d(x) + d(y) + d(z) \geq 2\Delta(Q) + 2 + t\}.$$

The *t-Kierstead number* of (x, y) is $\sigma_t(x, y) = |K_t(x, y)|$. For all of the zxy above, that we showed were Ψ-subgraphs, we have $z \in K_0(y, x)$ since, by Lemma 9, zxy is not superabundant. Hence $\sigma_0(y, x) \geq \Delta(Q) + 1 - d(y)$. Let $Z(x, y)$ be all the $z \in N(x) - y$ for which there is a coloring π of $G - xy$ such that $\pi(zx) \in \bar{\pi}(y)$. Then $Z(x, y) \subseteq K_0(y, x)$. Also, by Theorem 9, we have $d_Q(z) \geq 2\Delta(Q) + 2 - d_Q(x) - d_Q(y)$ for $z \in Z(x, y)$. Woodall proved the following.

Lemma 21. *Let Q be an edge-critical graph and $xy \in E(Q)$. Then $|Z(x, y)| \geq \Delta(Q) - \sigma_0(x, y) \geq \Delta(Q) + 1 - d(y)$. In particular, $\sigma_0(y, x) + \sigma_0(x, y) \geq \Delta(Q)$.*

We improve Lemma 21 in the case when $d(x)+d(y) \geq \Delta(G)+3$ and $\sigma_0(x, y) > \sigma_1(x, y)+1$.

Lemma 22. *Let Q be an edge-critical graph and $xy \in E(Q)$ with $d(x) + d(y) \geq \Delta(G) + 3$. Then $|Z(x, y)| \geq \Delta(Q) - 1 - \sigma_1(x, y)$. In particular, $\sigma_1(y, x) + \sigma_1(x, y) \geq \Delta(Q) - 1$.*

Proof. Note that $Z(x, y) \subseteq K_1(y, x)$ since $d(x)+d(y) \geq \Delta(G)+3$. Let π be a Δ-edge-coloring of $G - xy$. Then x_α exists for each $\alpha \in \bar{\pi}(y)$. Let $A = \{x_\alpha \mid \alpha \in \bar{\pi}(y)\}$. Then $A \subseteq Z(x, y)$.

Suppose $\alpha \in \pi(x) \cap \pi(y)$ is such that $y_\alpha \notin K_1(x, y)$. We will show that $x_\alpha \in Z(x, y)$. Now $|\bar{\pi}(x) \cup \bar{\pi}(y)| = 2\Delta(Q)+2-d_Q(x)-d_Q(y)$ and $|\bar{\pi}(y_\alpha)| = \Delta(Q)-d_Q(y_\alpha) \geq \Delta(Q)-(2\Delta(Q)+2-d_Q(x)-d_Q(y)) = d_Q(x)+d_Q(y)-2-\Delta(Q)$. Since $\alpha \notin \bar{\pi}(x) \cup \bar{\pi}(y)$ and $\alpha \notin \bar{\pi}(y_\alpha)$, there must be $\beta \in \bar{\pi}(y_\alpha) \cap (\bar{\pi}(x) \cup \bar{\pi}(y))$. Also, there is $\tau \in \bar{\pi}(y)$. Clearly, $\tau \neq \beta$ and $\tau \notin \bar{\pi}(x)$. If $\beta \in \bar{\pi}(y)$, then recoloring yy_α with β shows that $x_\alpha \in Z(x, y)$ and we are done. Otherwise, the $\tau - \beta$ path starting at x must end at y. Change colors on this path and then color yy_α with β to again show that $x_\alpha \in Z(x, y)$.

Putting this together with A, we see that the lemma holds if there is at most one $\alpha \in \bar{\pi}(x) \cap \pi(y)$ such that $y_\alpha \notin K_1(x, y)$. Suppose not and pick $\tau, \gamma \in \bar{\pi}(x) \cap \pi(y)$ such that $y_\tau, y_\gamma \notin K_1(x, y)$. Applying Theorem 9 shows that $|\bar{\pi}(x)|+|\bar{\pi}(y)|+|\bar{\pi}(y_\tau)|+|\bar{\pi}(y_\gamma)| \leq \Delta(Q)$. But that implies that $d_Q(x) + d_Q(y) \leq \Delta(Q) + 2$, contradicting our assumption. $\square$

We improve the adjacency lemma of Woodall [9] using Lemma 20 and Lemma 17. Let Q be edge-critical, $xy \in E(Q)$ and $z \in N(x) - y$. The goal is to show that each $z \in Z(x, y)$ has a lot of neighbors of high degree. To this end it will help to separate the vertices in $N(z) - \{x, y\}$ into two groups. Let $\Pi(x, y, z)$ be the set of all Δ-edge-colorings π of $G - xy$ such that $\pi(zx) \in \bar{\pi}(y)$. For $\pi \in \Pi(x, y, z)$, let $C_\pi(x, y, z) = (\bar{\pi}(x) \cup \bar{\pi}(y)) - \pi(zx)$. Then each $\beta \in C_\pi(x, y, z)$ must be incident to z since, by Theorem 9, zxy is not superabundant. Let $W_\pi(x, y, z) = \{z_\beta \mid \beta \in C_\pi(x, y, z)\}$ and let $U_\pi(x, y, z) = N(z) - \{x, y\} - W_\pi(x, y, z)$. To make stating the bounds easier, we put $a(y, z) = a(z, y) = 0$ if z is adjacent to y and

$a(y,z) = a(z,y) = 1$ otherwise. Also, put $b(x,z) = b(z,x) = 0$ if $d_Q(z) = d_Q(x) = \Delta(Q)$ and $b(x,z) = b(z,x) = 1$ otherwise. We allow a *partition* of a set to contain empty sets. By definition, we have $|W_\pi(x,y,z)| = 2\Delta(Q) - d_Q(x) - d_Q(y) + a(y,z)$. We'll prove two lemmas, the first tells us about $W_\pi(x,y,z)$ and the second tells us about $U_\pi(x,y,z)$.

Lemma 23. *Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q)+1$ and $xy \in E(Q)$. For each $z \in Z(x,y)$ and $\pi \in \Pi(x,y,z)$, we have a partition $\{M,S\}$ of $W_\pi(x,y,z)$ such that:*

1. *M consists of major vertices and $|M| \geq 2\Delta(Q) - d_Q(x) - d_Q(y) - 2 + a(y,z) + b(x,z)$; and*
2. *for each $v \in S$, we have $d_Q(v) \geq \Delta(Q) + |S| - 3 + b(x,z)$ and $d_Q(v) \geq 3\Delta(Q) - d_Q(x) - d_Q(y) - d_Q(z) + 1 + b(x,z)$.*

Proof. Pick $\beta \in C_\pi(x,y,z)$ to minimize $d_Q(z_\beta)$. If z_β is a major vertex, then $M = W_\pi(x,y,z)$ and we are done.

So, suppose z_β is not a major vertex. Since $z_\beta zxy$ is a Ψ-subgraph, we can apply Lemma 20 with z as the root to get $X \subseteq N(z)$ with $z_\beta, x \in X$ such that $\sum_{v\in X\cup\{y\}}(d_Q(v)+1-\Delta(Q)) \geq a(y,z)+b(x,z)$. But then $d_Q(x)+d_Q(y)+d_Q(z_\beta)+3-3\Delta(Q)+\sum_{v\in X-x-z_\beta}(d_Q(v)+1-\Delta(Q)) \geq a(y,z)+b(x,z)$ and hence $\sum_{v\in X-x-z_\beta}(d_Q(v)+1-\Delta(Q)) \geq 3\Delta(Q) - d_Q(x) - d_Q(y) - d_Q(z_\beta) - 3 + a(y,z) + b(x,z)$. Since only major vertices add to the sum, $X - x - z_\beta$ must contain at least $3\Delta(Q) - d_Q(x) - d_Q(y) - d_Q(z_\beta) - 3 + a(y,z) + b(x,z)$ major vertices. Let M be the major vertices in $W_\pi(x,y,z)$. Since z_β is not a major vertex, we have $|M| \geq 2\Delta(Q) - d_Q(x) - d_Q(y) - 2 + a(y,z) + b(x,z)$. This proves (1).

Now we prove (2). Let $S = W_\pi(x,y,z) - M$. Then, we must have $d_Q(z_\beta) \geq 3\Delta(Q) - d_Q(x) - d_Q(y) - 3 - M + a(y,z) + b(x,z) = \Delta(Q) + |S| - 3 + b(x,z)$. Now our minimality condition on $d(z_\beta)$ proves the first inequality. For the second we just apply Lemma 20 to the path $vzxy$. $\square$

For $z \in Z(x,y)$ and $\pi \in \Pi(x,y,z)$, let $\pi^* \in \Pi(x,z,y)$ be the Δ-edge coloring of $G - xz$ obtained from π by uncoloring xz and coloring xy with $\pi(xz)$.

Lemma 24. *Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q)+1$ and $xy \in E(Q)$. For each $z \in Z(x,y)$ and $\pi \in \Pi(x,y,z)$, we have*

1. *$|U_\pi(x,y,z) - K_0(x,z)| \leq |U_\pi(x,y,z) - K_1(x,z)| \leq |U_{\pi^*}(x,z,y) \cap K_0(x,y)|$; and*
2. *$|U_\pi(x,y,z) - K_0(x,z)| \leq |U_{\pi^*}(x,z,y) \cap K_1(x,y)| \leq |U_{\pi^*}(x,z,y) \cap K_0(x,y)|$; and*
3. *if $b(x,z) = 1$, then either*
   - (a) *$|U_\pi(x,y,z) - K_1(x,z)| \leq |U_{\pi^*}(x,z,y) \cap K_1(x,y)|$; or*
   - (b) *$d_Q(x) = 3$, $d_Q(y) = d_Q(z) = \Delta(Q)$, $a(y,z) = 1$, the unique $u \in U_\pi(x,y,z)$ has $d_Q(u) \geq \Delta(Q) - 1$ and $U_{\pi^*}(x,z,y) \cap K_1(x,y) = \emptyset$.*

Proof. Since the proofs of (1) and (2) are both easier versions of the proof of (3), we omit them. Half of the inequalities in (1) and (2) just follow from $K_1(x,z) \subseteq K_0(x,z)$ and $K_1(x,y) \subseteq K_0(x,y)$.

Suppose $b(x,z) = 1$ and (3b) does not hold. We will prove (3a) by showing that h given by $h(u) = y_{\pi(zu)}$ maps $U_\pi(x,y,z) - K_1(x,z)$ into $U_{\pi^*}(x,z,y) \cap K_1(x,y)$.

Since at most one edge of any given color is incident to y, we see that h is injective. Fix $u \in U_\pi(x,y,z) - K_1(x,z)$. Then $h(u) \in U_{\pi^*}(x,z,y)$ since $\pi(zu)$ is missing at neither z or x

and the change to π^* has no effect on this fact. So, it will suffice to show that $h(u) \in K_1(x,y)$. Suppose not. Let $\beta = \pi(zu)$ and $\alpha = \pi(xz)$, then $h(u) = y_\beta$.

Since $y_\beta \notin K_1(x,y)$, we have $d_Q(y_\beta) \le 2\Delta(Q) - d_Q(x) - d_Q(y) + 2$. Since $b(x,z) = 1$, at most one of x or z is major.

Consider the path $P = uzxyy_\beta$. Uncolor all the edges of P, let π' be the resulting edge-coloring and put $L = L_{\pi'}$. We will show that (P,L) is superabundant and either at most one of x,y,z is major or $\psi_L(P) > \|P\|$. Since at least one of x,y,z is not major, this contradicts Lemma 17. We have $\beta \in L(u)$, $\alpha,\beta \in L(z)$, $\alpha \in L(x)$, $\alpha,\beta \in L(y)$ and $\beta \in L(y_\beta)$. Hence every subgraph of P with at most two edges is abundant. Since $d_Q(u) \le 2\Delta(Q) - d_Q(x) - d_Q(z)+2$ and $d_Q(y_\beta) \le 2\Delta(Q) - d_Q(x) - d_Q(y)+2$, we have $|L(u)| \ge d_Q(x)+d_Q(z)-\Delta(Q)-1$ and $|L(y_\beta)| \ge d_Q(x)+d_Q(y)-\Delta(Q)-1$. Now $|L(x) \cup L(y)| \ge 2\Delta(Q)+3-d_Q(x)-d_Q(y)$, so $L(y_\beta)$ has at least two colors in $L(x) \cup L(y)$, one of which is β, call the other one τ. Similarly, $L(u)$ has at least two colors in $L(x) \cup L(z)$, one of which is β, call the other one γ.

Using τ and γ (one or both of which could be α), we see that (P,L) is superabundant. If $\tau \neq \gamma$, then $\psi_L(P) > \|P\|$ and we are done. If at most one of x,y,z is major, we are done, so either both x and y are major or both y and z are major.

Suppose x and y are major. Then $d_Q(y_\beta) = 2$ and hence $|L(y_\beta)| = \Delta(Q) - 1$. If $L(y_\beta)$ has at least 3 colors in $L(x) \cup L(z)$, then $\psi_L(P) > \|P\|$, so we must have $\Delta(Q) - 1 + 2\Delta(Q) + 3 - d_Q(x) - d_Q(z) \le \Delta(Q) + 2$ which gives $d_Q(z) \ge \Delta(Q)$, a contradiction.

Hence it must be that y and z are major. Since $L(u)$ and $L(y_\beta)$ have exactly two colors in common, we have $|L(u) \cup L(y_\beta)| = |L(u)|+|L(y_\beta)|-2 \ge 2d_Q(x)+d_Q(y)+d_Q(z)-2\Delta(Q)-4 = 2d_Q(x)-4$. Since $\beta \notin L(x)$, there is at most one color in $L(x)$ that is in $L(u) \cup L(y_\beta)$ (it is τ whether or not $\tau = \alpha$). So, $2d_Q(x)-4+\Delta(Q)+2-d_Q(x) \le |L(u) \cup L(y_\beta)|+|L(x)| \le \Delta(Q)+1$ which gives $d_Q(x) \le 3$. Then $|U_\pi(x,y,z)| = d_Q(z) - 2 + a(y,z) - |W_\pi(x,y,z)| = \Delta(Q) - 2 + a(y,z) - (2\Delta(Q) - d_Q(x) - d_Q(y) + 1) = a(y,z)$. So $a(y,z) = 1$. We conclude that (3b) holds, a contradiction. ☐

For $p \in \mathbb{N}$, $xy \in E(Q)$ and $z \in Z(x,y)$ put $t_p(x,y,z) = 1$ if $y \in K_p(x,z)$ and $t_p(x,y,z) = 0$ otherwise. Since $|K_p(x,z)| + |K_q(x,y)| = |K_p(x,z) \cap W_\pi(x,y,z)| + |K_p(x,z) \cap U_\pi(x,y,z)| + |K_q(x,y) \cap W_{\pi^*}(x,z,y)| + |K_q(x,y) \cap U_{\pi^*}(x,z,y)| + t_p(x,y,z) + t_q(x,z,y)$, Lemma 24 has the following immediate consequence.

Corollary 25. *Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q)+1$ and $xy \in E(Q)$. For each $z \in Z(x,y)$ and $\pi \in \Pi(x,y,z)$, we have*

1. $\sigma_0(x,z)+\sigma_0(x,y) \ge |K_0(x,z) \cap W_\pi(x,y,z)|+|K_0(x,y) \cap W_{\pi^*}(x,z,y)|+|U_\pi(x,y,z)|+ t_0(x,y,z) + t_0(x,z,y)$*; and*
2. $\sigma_1(x,z)+\sigma_0(x,y) \ge |K_1(x,z) \cap W_\pi(x,y,z)|+|K_0(x,y) \cap W_{\pi^*}(x,z,y)|+|U_\pi(x,y,z)|+ t_1(x,y,z) + t_0(x,z,y)$*; and*
3. $\sigma_0(x,z)+\sigma_1(x,y) \ge |K_0(x,z) \cap W_\pi(x,y,z)|+|K_1(x,y) \cap W_{\pi^*}(x,z,y)|+|U_\pi(x,y,z)|+ t_0(x,y,z) + t_1(x,z,y)$*; and*
4. *if $b(x,z) = 1$ and $d_Q(x) > 3$, then $\sigma_1(x,z) + \sigma_1(x,y) \ge |K_1(x,z) \cap W_\pi(x,y,z)| + |K_1(x,y) \cap W_{\pi^*}(x,z,y)| + |U_\pi(x,y,z)| + t_1(x,y,z) + t_1(x,z,y)$.*

Corollary 26. *Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q)+1$ and $xy \in E(Q)$. For each $z \in Z(x,y)$ and $\pi \in \Pi(x,y,z)$ we have:*

(1) $W_\pi(x,y,z) \subseteq K_0(x,z)$. *In particular,* $|K_0(x,z) \cap W_\pi(x,y,z)| \geq 2\Delta(Q) - d_Q(x) - d_Q(y) + a(y,z)$*; and*
(2) *if* $\Delta(Q) \geq 5$ *and* $d(x) + d(z) \geq \Delta(Q) + 3$*, then* $|K_1(x,z) \cap W_\pi(x,y,z)| \geq 2\Delta(Q) - d_Q(x) - d_Q(y) - 1 + a(y,z)$.

Proof. First we prove (1). By Lemma 23, for each $v \in W_\pi(x,y,z)$ we have $d_Q(v) \geq 3\Delta(Q) - d_Q(x) - d_Q(y) - d_Q(z) + 1 + b(x,z)$. So, if $d_Q(x) + d_Q(z) + d_Q(v) < 2\Delta(Q) + 2$, then $d_Q(y) = \Delta(Q)$ and $b(x,z) = 0$, but then $d_Q(v) < 2$ which is impossible since Q is edge-critical. Hence $v \in K_0(x,z)$.

Now we prove (2). Let $\{M, S\}$ be the partition of $W_\pi(x,y,z)$ given by Lemma 23. Now if $v \in W_\pi(x,y,z)$ and $d_Q(v) = \Delta(Q) - p$ then $v \in K_1(x,z)$ unless $d(x) + d(z) + \Delta(Q) - p \leq 2\Delta(Q) + 2$ and hence $d(x) + d(z) \leq \Delta(Q) + 2 + p$. Since we assumed $d(x) + d(z) \geq \Delta(Q) + 3$, we have $M \subseteq K_1(x,z)$. So, if $|M| \geq 2\Delta(Q) - d_Q(x) - d_Q(y) - 1 + a(y,z)$ we are done. Otherwise, $|M| = 2\Delta(Q) - d_Q(x) - d_Q(y) - 2 + a(y,z)$ and $b(x,z) = 0$. Also, we are done unless $S \cap K_1(x,z) = \emptyset$. Since $|W_\pi(x,y,z)| = 2\Delta(Q) - d_Q(x) - d_Q(y) + a(y,z)$ and M and S partition $W_\pi(x,y,z)$, we have $|S| = 2 - a(y,z)$. Then by part (2) of Lemma 23, we have $d_Q(v) \geq \Delta(Q) - 1 - a(y,z)$ for $v \in S$. Since $S \cap K_1(x,z) = \emptyset$, we must have $d(x) + d(z) = \Delta(Q) + 3 + a(y,z)$. But $b(x,z) = 0$, so that gives $2\Delta(Q) = \Delta(Q) + 3 + a(y,z)$ and hence $\Delta(Q) \leq 4$, contradicting our assumption. $\square$

Corollary 27. *Let* Q *be an edge-critical graph with* $\chi'(Q) = \Delta(Q) + 1$ *and* $xy \in E(Q)$*. For each* $z \in Z(x,y)$ *we have:*

(1) $\sigma_0(x,z) + \sigma_0(x,y) \geq 2\Delta(Q) - d_Q(x)$*; and*
(2) *if* $\Delta(Q) \geq 5$ *and* $d_Q(x) + d_Q(y) \geq \Delta(Q) + 3$*, then*
    (a) $\sigma_0(x,z) + \sigma_1(x,y) \geq 2\Delta(Q) - d_Q(x) - 2 + a(y,z) + t_1(x,y,z)$*; and*
    (b) *if* $b(x,z) = 1$ *and* $d_Q(x) + d_Q(z) \geq \Delta(Q) + 3$*, then* $\sigma_1(x,z) + \sigma_1(x,y) \geq 2\Delta(Q) - d_Q(x) - 5 + 3a(y,z) + 2t_1(x,y,z)$.

Proof. Pick $\pi \in \Pi(x,y,z)$. Then $|U_\pi(x,y,z)| = d_Q(z) - 2 + a(y,z) - |W_\pi(x,y,z)|$.

For (1), Corollary 25 and Corollary 26 give $\sigma_0(x,z) + \sigma_0(x,y) \geq |K_0(x,z) \cap W_\pi(x,y,z)| + |K_0(x,y) \cap W_{\pi^*}(x,z,y)| + |U_\pi(x,y,z)| + t_0(x,y,z) + t_0(x,z,y) = |W_\pi(x,y,z)| + |W_{\pi^*}(x,z,y)| + d_Q(z) - 2 + a(y,z) - |W_\pi(x,y,z)| + t_0(x,y,z) + t_0(x,z,y) = |W_{\pi^*}(x,z,y)| + d_Q(z) - 2 + a(y,z) + t_0(x,y,z) + t_0(x,z,y) = 2\Delta(Q) - d_Q(x) - 2 + 2a(y,z) + t_0(x,y,z) + t_0(x,z,y)$. Since we always have $d_Q(x) + d_Q(y) + d_Q(z) \geq 2\Delta(Q) + 2$ for $z \in Z(x,y)$, we see that $t_0(x,y,z) = t_0(x,z,y) = 1$ if $a(y,z) = 0$. Therefore, $\sigma_0(x,z) + \sigma_0(x,y) \geq 2\Delta(Q) - d_Q(x)$.

For (2a), Corollary 25 and Corollary 26 give $\sigma_0(x,z) + \sigma_1(x,y) \geq |K_0(x,z) \cap W_\pi(x,y,z)| + |K_1(x,y) \cap W_{\pi^*}(x,z,y)| + |U_\pi(x,y,z)| + t_0(x,y,z) + t_1(x,z,y) = |W_\pi(x,y,z)| + |K_1(x,y) \cap W_{\pi^*}(x,z,y)| + d_Q(z) - 2 + a(y,z) - |W_\pi(x,y,z)| + t_0(x,y,z) + t_1(x,z,y) = |K_1(x,y) \cap W_{\pi^*}(x,z,y)| + d_Q(z) - 2 + a(y,z) + t_0(x,y,z) + t_1(x,z,y) \geq 2\Delta(Q) - d_Q(x) - 3 + a(z,y) + a(y,z) + t_0(x,y,z) + t_1(x,z,y)$. Now $a(z,y) = a(y,z)$ and $t_0(x,y,z) = 1$ if $a(y,z) = 0$, so this gives $\sigma_0(x,z) + \sigma_1(x,y) \geq 2\Delta(Q) - d_Q(x) - 2 + a(y,z) + t_1(x,z,y)$. Since $t_1(x,z,y) = t_1(x,y,z)$, we are done.

For (2b), first assume $d_Q(x) > 3$. Then applying Corollary 26 gives $\sigma_1(x,z) + \sigma_1(x,y) \geq |K_1(x,z) \cap W_\pi(x,y,z)| + |K_1(x,y) \cap W_{\pi^*}(x,z,y)| + |U_\pi(x,y,z)| + t_1(x,y,z) + t_1(x,z,y) \geq (2\Delta(Q) - d_Q(x) - d_Q(z) - 1 + a(z,y)) + (2\Delta(Q) - d_Q(x) - d_Q(y) - 1 + a(y,z)) + (d_Q(z) - 2 + a(y,z) - |W_\pi(x,y,z)|) + t_1(x,y,z) + t_1(x,z,y) = (2\Delta(Q) - d_Q(x) - d_Q(z) - 1 + a(y,z)) +$

$(-2+a(y,z))+(d_Q(z)-2+a(y,z))+t_1(x,y,z)+t_1(x,z,y) = 2\Delta(Q)-d_Q(x)-5+3a(y,z)+2t_1(x,y,z)$.

If $d_Q(x) \le 3$, (2b) follows from VAL. $\square$

Now some less general formulations.

Corollary 28. *Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q)+1 \ge 6$ and $xy \in E(Q)$. If $d_Q(x) < \Delta(Q)$, then for each $z \in Z(x,y) - N(y)$ we have:*

1. *$\sigma_0(x,z)+\sigma_0(x,y) \ge 2\Delta(Q) - d_Q(x)$; and*
2. *if $d_Q(x)+d_Q(y) \ge \Delta(Q)+3$, then*
   - (a) *$\sigma_0(x,z)+\sigma_1(x,y) \ge 2\Delta(Q)-d_Q(x)-1$; and*
   - (b) *if $d_Q(x)+d_Q(z) \ge \Delta(Q)+3$, then $\sigma_1(x,z)+\sigma_1(x,y) \ge 2\Delta(Q)-d_Q(x)-2$.*

When $d_Q(x)+d_Q(y) = \Delta(Q)+2$ or $d_Q(x)+d_Q(z) = \Delta(Q)+2$, we already have very good control of the degrees of vertices at distance at most two. The following is what we get when this doesn't happen.

Corollary 29. *Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q)+1 \ge 6$ and $xy \in E(Q)$. If $d_Q(x) < \Delta(Q)$ and $d_Q(x)+d_Q(y) \ge \Delta(Q)+3$, then for each $z \in Z(x,y) - N(y)$ with $d_Q(x)+d_Q(z) \ge \Delta(Q)+3$, we have:*

1. *$\sigma_0(x,z)+\sigma_0(x,y) \ge 2\Delta(Q)-d_Q(x)$; and*
2. *$\sigma_0(x,z)+\sigma_1(x,y) \ge 2\Delta(Q)-d_Q(x)-1$; and*
3. *$\sigma_1(x,z)+\sigma_0(x,y) \ge 2\Delta(Q)-d_Q(x)-1$; and*
4. *$\sigma_1(x,z)+\sigma_1(x,y) \ge 2\Delta(Q)-d_Q(x)-2$.*

We get more information about $z \in Z(x,y)$ from Lemma 23.

Corollary 30. *Let Q be an edge-critical graph with $\chi'(Q) = \Delta(Q)+1$ and $xy \in E(Q)$. If $d_Q(x) < \Delta(Q)$, then for each $z \in Z(x,y) - N(y)$ there is $A \subseteq N(z) - x$ with $|A| \ge 2\Delta(Q) - d_Q(x) - d_Q(y) + 1$ such that all but at most one vertex in A is major and if there is one that is not, it has degree $\Delta(Q)-1$.*

Let Q be an edge-critical graph and $x \in V(Q)$. For $t \in \mathbb{N}$, put $p_t(x) = d_Q(x) - 1 - \Delta(Q) + \min_{y \in N(x)} \sigma_t(x,y)$. This first one is all Woodall [9] used to prove that edge-critical graphs have average degree at least $\frac{2}{3}(\Delta+1)$.

Corollary 31. *If Q is an edge-critical graph and $x \in V(Q)$, then x has at least $d_Q(x) - p_0(x) - 1$ neighbors z with $\sigma_0(x,z) \ge \Delta(Q) - p_0(x) - 1$.*

We get an improvement of similar form.

Corollary 32. *Let Q be an edge-critical graph with $\Delta(Q) \ge 5$ and $x \in V(Q)$ with $d_Q(x) < \Delta(Q)$. Further, suppose $d_Q(x)+d_Q(y) \ge \Delta(Q)+3$ for all $y \in N(x)$. For any $y_0 \in N(x)$ minimizing $\sigma_0(x,y_0)$ and $y_1 \in N(x)$ minimizing $\sigma_1(x,y_1)$, we have*

1. *x has at least $d_Q(x) - p_1(x) - 2$ neighbors z with $\sigma_0(x,z) \ge \Delta(Q) - p_1(x) - 3 + a(y_1,z) + t_1(x,y_1,z)$; and*
2. *x has at least $d_Q(x) - p_0(x) - 1$ neighbors z with $\sigma_1(x,z) \ge \Delta(Q) - p_0(x) - 3 + a(y_0,z) + t_1(x,y_0,z)$; and*

(3) *x has at least $d_Q(x) - p_1(x) - 2$ neighbors z with $\sigma_1(x, z) \geq \Delta(Q) - p_1(x) - 6 + 3a(y_1, z) + 2t_1(x, y_1, z)$.*

The following is what we use when $d_Q(x) + d_Q(y) = \Delta(Q) + 2$.

Lemma 33. *[Zhang [10]] For all $xy \in E(Q)$, we have*

(1) *$d_Q(x) + d_Q(y) \geq \Delta(Q) + 2$; and*
(2) *vertices at distance one from $\{x, y\}$ have degree $\Delta(Q)$; and*
(3) *If $d_Q(x) + d_Q(y) = \Delta(Q) + 2$, then*
    (a) *vertices at distance two from $\{x, y\}$ have degree at least $\Delta(Q) - 1$; and*
    (b) *if u is at distance two from $\{x, y\}$ and $d_Q(u) = \Delta(Q) - 1$, then $\{x, y\} = \{v_2, v_\Delta\}$ where $d_Q(v_2) = 2$, $d_Q(v_\Delta) = \Delta(Q)$ and u is at distance 2 from v_Δ and distance 3 from v_2.*

It seems like Conjecture 34 in the next section could be strengthened where we have one fewer internal vertex with $f(v) \geq d_G(v) + 1$, but also $\psi_L(G) > \|G\|$.

12. A stronger conjecture and free strengthening

For $h \in \mathbb{N}$, we say that G is *h-defective* if G is f-fixable in the chronicled game for every valid f where $|\{v \in V(G) \mid f(v) > d(v) \geq 2\}| \leq h$. For example, Theorem 9 shows that stars are 0-defective and Theorems 13 and 16 together show that stars with one edge subdivided are 1-defective. These results and many computer simulations motivate the following conjecture. For a tree T, let $\iota(T)$ be the number of internal vertices in T.

Conjecture 34. *Any tree T is $(\iota(T) - 1)$-defective.*

This implies that a Tashkinov tree T in a simple class 2 graph G is elementary when $d_G(v) < \Delta(G)$ for all but at most one internal vertex of T. No restrictions are placed on the leaves, so they could each have degree $\Delta(G)$. We'll see that to prove this result, it would suffice to do so for trees where the leaves all have degree less than $\Delta(G)$. Is this conjecture for class 2 graphs even true when T is a path?

Conjecture 34 strengthens Conjecture 4 for the chronicled game in two ways. First, Conjecture 4 requires the leaves of the tree to have lists of size at least 2 whereas Conjecture 34 does not. We will show how to get this improvement from Conjecture 4 and conclude that if Conjecture 4 is true, then any tree T is $\iota(T)$-defective in the chronicled game. The second improvement is replacing $\iota(T)$ with $\iota(T) - 1$.

Let $\mathcal{T}$ be a hereditary collection of trees. A *bundle* on $\mathcal{T}$ is a collection $\{f_T\}_{T \in \mathcal{T}}$ where each f_T is a function from $V(T)$ to $\mathbb{N}$ such that for all $T', T \in \mathcal{T}$ with $T' \subseteq T$, we have $f_{T'}(v) \leq f_T(v) - (d_T(v) - d_{T'}(v))$ for all $v \in V(T')$. A bundle $\{f_T\}_{T \in \mathcal{T}}$ is *fixable* if T is f_T-fixable in the chronicled game for every $T \in \mathcal{T}$.

Lemma 35. *Let $\mathcal{T}$ be a hereditary collection of trees and $\{f_T\}_{T \in \mathcal{T}}$ a fixable bundle on $\mathcal{T}$ where, for each $T \in \mathcal{T}$ we have $f_T(v) \leq 2$ for all leaves v of T. Then the bundle formed by, for each $T \in \mathcal{T}$, setting $f_T(v) = 1$ for all leaves v of T is fixable.*

Proof. First, since a leaf in a tree is either not present or present with the same degree in any given subtree, we do actually get a bundle $\{g_T\}_{T \in \mathcal{T}}$ this way. Suppose the bundle is not fixable and choose $T \in \mathcal{T}$ minimal such that T is not g_T-fixable in the chronicled game. Then we have a superabundant list assignment L on T with $|L(v)| \geq g_T(v)$ for all $v \in V(T)$ where Fixer has no winning strategy in the chronicled game. By assumption, there must be a leaf x of T where $|L(x)| < f_T(x) = 2$ and hence $|L(x)| = 1$. Let w be x's neighbor. Since (xw, L) is abundant, we have $\alpha \in L(x) \cap L(w)$. Let L' be the list assignment on $T - x$ formed from L by removing α from $L(w)$.

Suppose $(T - x, L')$ is not superabundant and choose non-abundant $T' \subseteq T$. Then $w \in V(T')$ and $\psi_{L'}(T') = \|T'\| - 1$. But $\psi_L(T' + x) \geq \|T'\| + 1$, so adding x back adds two to ψ, but this is impossible since $L(x)$ has only one color.

Since $T - x \in \mathcal{T}$, by the definition of bundle, we have $g_{T-x}(v) \leq g_T(v) - (d_T(v) - d_{T-x}(v))$ for all $v \in V(T - x)$. Minimality of T shows that $T - x$ is g_{T-x}-fixable in the chronicled game, but since $|L'(v)| \geq g_{T-x}(v)$ for all $v \in V(T - x)$, this means that Fixer has a winning strategy in the chronicled game on $T - x$ with lists L'. Applying Lemma 11 now gives a winning strategy for Fixer on T with L, a contradiction. $\square$

For an example, let's see how Conjecture 4 can be strengthened. Suppose $\mathcal{T}$ is the collection of all trees and for each $T \in \mathcal{T}$, let $f_T(v) = d_T(v) + 1$ for all $v \in V(T)$. Then $\{f_T\}_{T \in \mathcal{T}}$ is clearly a bundle. By Conjecture 4, it is fixable. Consider the bundle $\{g_T\}_{T \in \mathcal{T}}$ where $g_T(v) = d_T(v) + 1$ for all internal vertices of T and $g_T(v) = d_T(v)$ for all leaves of T. Then Lemma 35 shows that $\{g_T\}_{T \in \mathcal{T}}$ is fixable as well.

13. Kierstead paths in the chronicled game

Theorem 36. *Let L be a KTV-assignment on a multigraph G with respect to $x_1x_2 \in E(G)$, coloring π, and ordering $<$ induced by a spanning path $x_1x_2 \cdots x_n$ in G. If $|L(v)| \geq d_G(v)+1$ for all $v \in V(G)$, then Fixer has a winning strategy against Breaker in the chronicled game on (G, L).*

Proof. Suppose not and choose a counterexample G minimizing $\|G\|$. Suppose there is $uw \in E(G) - E(x_1x_2 \cdots x_n)$. Define L' on $G - uw$ by $L'(v) := L(v)$ for $v \notin \{u, w\}$ and $L'(v) = L(v) - \pi(uw)$ for $v \in \{u, w\}$. Then L' is a KTV-assignment on $G - uw$. By minimality of $\|G\|$, Fixer has a winning strategy on $G - uw$; hence Fixer has a winning strategy on G by Lemma 10, a contradiction.

Hence G is the path $x_1x_2 \cdots x_n$. Let τ be the color guaranteed by property (5) of KTV-assignments. If there is $\alpha \in L(x_s) \cap L(x_t) - \{\pi(e) \mid e \in E(s) \cup E(t)\}$ for $1 \leq s < t < n$, then Fixer gets a winning strategy on G by coloring $x_{n-1}x_n$ using $\pi(x_{n-1}x_n)$ and applying minimality and Lemma 10. In particular, we must have $\tau \in L(x_n)$. By letting Fixer play for awhile if needed, we can assume that $\max\{j \in [n-1] \mid \tau \in L(x_j)\}$ is as large as Fixer can get it while maintaining a KTV-assignment. If $j = n-1$, then Fixer gets a winning strategy by coloring $x_{n-1}x_n$ with τ and then applying minimality using $\pi(x_{n-1}x_n)$ for property (5), and finally applying Lemma 10. Hence $j < n - 1$. Moreover, by property (4) of KTV-assignments, $\tau \notin L(x_s)$ for $s \in [j - 1]$.

Since $|L(x_{j+1})| \geq d_G(x_{j+1}) + 1$, we can pick $\gamma \in L(x_{j+1}) - \{\pi(e) \mid e \in E(x_{j+1})\}$. By property (4) of KTV-assignments, $\gamma \notin L(x_s)$ for $s \in [j]$.

Let $S \subseteq V(G)$ be those vertices v with $|\{\tau, \gamma\} \cap L(v)| = 1$. Let A_S be as in Lemma 12. From the above, we know that $S \subseteq \{x_j, \ldots, x_n\}$. Fixer can swap τ for γ at x_{j+1} without breaking any τ-edges or γ-edges by using A_S and fixing anything that Breaker breaks until Breaker runs out of options (we should extract a general lemma here). But then we still have a KTV-assignment and either j has increased or we moved τ from x_n to a lower vertex and Fixer wins. $\square$

14. Hall's theorem on trees

The following generalization of Hall's theorem was proved by Marcotte and Seymour [5] and independently by Cropper, Gyárfás and Lehel [2]. By a *multitree* we mean a tree that possibly has edges of multiplicity greater than one.

Lemma 37. *Let T be a multitree and L a list assignment on $V(T)$. If $\eta_L(H) \geq \|H\|$ for all $H \subseteq T$, then T has an edge-coloring $\pi\colon E(T) \to \mathrm{Pot}(L)$ such that $\pi(xy) \in L(x) \cap L(y)$ for each $xy \in E(T)$.*

When T is a star, this is Hall's theorem. In the proof of Theorem 9, we used Hall's theorem to reduce to a smaller game when $\eta_L(G) \geq \|G\|$. For more general trees, it seems reasonable that we might be able to use Lemma 37 instead (haven't been able to get it to work yet).

15. The fix number

For $k \in \mathbb{N}$, we say that G is k-fixable (in the chronicled game) if G is f-fixable where $f(v) = k$ for all $v \in V(G)$. The *fix number* of G, writen $\chi_{\text{fix}}(G)$, is the least k for which G is k-fixable. Similarly, we define the *chronicled fix number* of G, written $\chi_{\text{cfix}}(G)$, as the least k for which G is k-fixable in the chronicled game. For all G, we have $\Delta(G) \leq \chi_{\text{cfix}}(G) \leq \chi_{\text{fix}}(G)$. We haven't even proved that there is a finite upper bound, but the following should be true.

Conjecture 38. *Every multigraph G satisfies $\chi_{\mathit{cfix}}(G) \leq \chi_{\mathit{fix}}(G) \leq \Delta(G) + 1$.*

This implies Goldberg. The point is that requiring superabundance forbids overfull subgraphs when the lists are all the same. For now let's focus on the chronicled game.

For a graph G and $p \in \mathbb{N}$, put $d_p(v) = d_G(v) + p$ for all $v \in V(G)$. We say that G is k-fix-critical if G is not k-fixable in the chronicled game, but every proper subgraph of G is. If G is k-fix-critical where $k = \Delta(G) + 1 + p$, then no subgraph of G is d_p-fixable in the chronicled game. Hmm, maybe not, what if the strategy to win on $G - E(H)$ breaks superabundance on H? Can color all but one edge.

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