notes on the Borodin-Kostochka conjecture

February 28, 2017

1 Introduction

The goal here is to prove Borodin and Kostochka's conjecture from 1977. If proving the full conjecture is unfeasable, we aim to prove the conjecture for large classes of graphs.

Conjecture 1 (Borodin and Kostochka [2])**.** *Every graph G with $\Delta(G) \geq 9$ satisfies $\chi(G) \leq \max\{\omega(G), \Delta(G) - 1\}$.*

2 $4K_1$-free graphs

Let's use $4K_1$-free graphs as a test case for the various methods of attack on the Borodin-Kostochka conjecture. The results in Section 6 prove the conjecture in this case for $\Delta(G) \geq 13$. We want to get this down to $\Delta(G) \geq 9$. Here are a few known facts that should help. The results in Section 5 can be used to prove the following.

Lemma 2. *If G is a minimum counterexample to the Borodin-Kostochka conjecture for $4K_1$-free graphs, then $\omega(G) \leq \Delta(G) - 2$.*

3 Excluded induced subgraphs by d_1-choosability

A graph G is *d_r-choosable* if G can be L-colored from every list assingment L with $|L(v)| \geq d_G(v) - r$ for all $v \in V(G)$. Every graph is d_{-1}-choosable. The d_0-choosable graphs were classified by Borodin [1] and independently by Erdős, Rubin, and Taylor [8] as those graphs whose every block is either complete or an odd cycle (a connected such graph is a *Gallai tree*). Classifying the d_r-choosable graphs for any $r \geq 1$ appears to be a hard problem. However, we can get useful sufficient conditions for a graph to be d_1-choosable. For example, all of the graphs here are d_1-choosable (the vertex color indicates components of the complement): `https://landon.github.io/graphdata/borodinkostochka/offline/index.html`

4 Decompositions

4.1 Reed's decomposition

In [13], Reed proved the Borodin-Kostochka conjecture for graphs G with $\Delta(G) \geq 10^{14}$. A piece of that proof was a decomposition of G into dense chunks and one sparse chunk that also works for smaller $\Delta(G)$. The following tight form of this decomposition is proved in [12]. Let $\mathcal{C}_t(G)$ be the maximal cliques in G having at least t vertices.

Reed's Decomposition. *Suppose G is a graph with $\Delta(G) \geq 8$ that contains no $K_{\Delta(G)}$ and has no d_1-choosable induced sugraph. If $\frac{\Delta(G)+5}{2} \leq t \leq \Delta(G) - 1$, then $\bigcup \mathcal{C}_t(G)$ can be partitioned into sets $D_1, \ldots, D_r$ such that for each $i \in [r]$ at least one of the following holds:*

1. $D_i = C_i \in \mathcal{C}_t(G)$,
2. $D_i = C_i \cup \{x_i\}$ *where* $C_i \in \mathcal{C}_t(G)$ *and* $|N(x_i) \cap C_i| \geq t - 1$.

4.2 Fajtlowicz's decomposition

In [9], Fajtlowicz proved that every graph has $\alpha(G) \geq \frac{2|G|}{\omega(G)+\Delta(G)+1}$. The proof of this result gives a decomposition which we state in the special case needed for the Borodin-Kostochka conjecture.

Fajtlowicz's Decomposition. *Suppose G is a vertex-critical graph with $\chi(G) = \Delta(G)$. Then $V(G)$ can be partitioned into sets M, T, and K such that*

1. *M contains a maximum independent set I of G; and*
2. *each $v \in T$ has $d_G(v) = \Delta(G)$, two neighbors in I and zero neighbors in $M \setminus I$; and*
3. *K can be covered by $\alpha(G)$ (or fewer) cliques; and*
4. *each $v \in K$ has exactly one neighbor in I and at most one neighbor in $M \setminus I$ (none if $d_G(v) < \Delta(G)$); and*
5. *the vertices in $M \setminus I$ can be ordered $v_1, \ldots, v_r$ such that for $i \in [r]$, either v_i has at least three neighbors in $I \cup \{v_1, \ldots, v_{i-1}\}$ or $d_G(v_i) < \Delta(G)$ and v_i has at least two neighbors in $I \cup \{v_1, \ldots, v_{i-1}\}$.*

Proof. Let I be a maximum independent set in G. Construct a maximal length sequence $I = M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r$ such that for $j > 0$,

- every $v \in M_j$ with $d_G(v) = \Delta(G)$ either has at least three neighbors in M_{j-1} or at least two neighbors in $M_{j-1} \setminus I$; and
- every $v \in M_j$ with $d_G(v) = \Delta(G) - 1$ either has at least two neighbors in M_{j-1} or at least one neighbor in $M_{j-1} \setminus I$.

Now let $M = M_r$, let T be the vertices in $V(G) \setminus M$ with exactly one neighbor in I and let K be the vertices in $V(G) \setminus M$ with exactly two neighbors in I. The decomposition has the properties 1,2,4 and 5 since the sequence $M_0 \subsetneq M_1 \subsetneq \cdots \subsetneq M_r$ was chosen to be maximal length. Property 3 follows since for each $v \in I$, the set of $x \in X$ adjacent to v must be a clique for otherwise we could get an independent set larger than I. $\square$

5 Properties of minimum counterexamples

In [6] Cranston and R. used the d_1-choosable graphs in Section 3 to prove properties of a minimum counterexample to the Borodin-Kostochka conjecture. Almost all of the proofs there (specifically, the proofs only involving edge addition and not vertex set contraction) work for minimum counterexamples within a given collection of graphs that is closed under taking induced subgraphs and adding edges. Call such a collection of graphs *permissible*. For example, the following improves a lemma Reed used in his proof [13].

Lemma 3. *Let $\mathcal{A}$ be a permissible collection of graphs for which the Borodin-Kostochka conjecture does not hold. Let $G \in \mathcal{A}$ be a counterexample with the minimum number of vertices (of graphs in $\mathcal{A}$).*

1. *If X is a $K_{\Delta(G)-1}$ in G, then every $v \in V(G - X)$ has at most one neighbor in X; and*
2. *Let A and B be disjoint subgraphs of G with $|A| + |B| = \Delta(G)$ such that $|A|, |B| \geq 4$. If G contains all edges between A and B, then $A = K_1 + K_{|A|-1}$ and $B = K_1 + K_{|B|-1}$.*

6 Counterexamples have some sparse neighborhoods

In [12], R. showed that any counterexample to the Borodin-Kostochka conjecture must have some sparse neighborhoods and large independence number (increasing with $\Delta(G)$). For example,

Lemma 4. *If G is a counterexample to the Borodin-Kostochka conjecture, then*

1. *there exists $v \in V(G)$ such that v is not contained in any clique with at least $\frac{2}{3}\Delta(G)+2$ vertices; and*
2. *there exists $v \in V(G)$ such that $G[N(v)]$ has average degree at most $\frac{2}{3}\Delta(G) + 3$; and*
3. *$\alpha(G) \geq \frac{\Delta(G)}{4}$; and*
4. *$|G| \geq 16\Delta(G)^2 - 528\Delta(G) + 3527$.*

7 Results from kernel methods

In [10], Kierstead and R. proved a general lemma that allows the user to get list colorings for free from large independent sets. Specialized to the Borodin-Kostochka conjecture, this becomes.

Kernel Magic. *Suppose G is a vertex-critical graph with $\chi(G) = \Delta(G)$. For every induced subgraph H of G and independent set I in H, we have*

$$\sum_{v \in V(I)} d_H(v) < \sum_{v \in V(H)} \Delta(G) + 2 - d_G(v).$$

Applied with $H = G$, this gives:

Corollary 5. *If G is a vertex-critical graph with $\chi(G) = \Delta(G)$, then $\alpha(G) < \frac{2|G|}{\Delta(G)}$.*

8 Mozhan partitions

Extending ideas of Mozhan [11], Cranston and R. [7] proved the following.

Theorem 6. *If G is a vertex-critical graph with $\chi(G) = \Delta(G) \geq 13$, then $\omega(G) \geq \Delta(G) - 3$.*

9 Vertex-transitive graphs

In [4] Cranston and R. used Reed's decomposition and the ideas in Sections 6 and 8 to prove the Borodin-Kostochka conjecture for vertex-transitive graphs with $\Delta(G) \geq 13$. It would be interesting to improve this to $\Delta(G) \geq 9$.

Theorem 7. *Every vertex-transitive graph G with $\Delta(G) \geq 13$ satisfies $\chi(G) \leq \max\{\omega(G), \Delta(G) - 1\}$.*

10 Claw-free graphs

In [5], Cranston and R. proved the Borodin-Kostochka conjecture for claw-free graphs using some of the d_1-choosable graphs in Section 3 combined with the structure theorem for quasi-line graphs of Chudnovsky and Seymour [3].

References

[1] O.V. Borodin. Criterion of chromaticity of a degree prescription. In *Abstracts of IV All-Union Conf. on Th. Cybernetics*, pages 127–128, 1977. 1

[2] O.V. Borodin and A.V. Kostochka. On an upper bound of a graph's chromatic number, depending on the graph's degree and density. *Journal of Combinatorial Theory, Series B*, 23(2-3):247–250, 1977. 1

[3] M. Chudnovsky and P. Seymour. The structure of claw-free graphs. *Surveys in combinatorics*, 327:153–171, 2005. 4

[4] Daniel W Cranston and Landon Rabern. A note on coloring vertex-transitive graphs. *arXiv preprint arXiv:1404.6550*, 2014. 4

[5] D.W. Cranston and L. Rabern. Coloring claw-free graphs with $\Delta - 1$ colors. *Arxiv preprint arXiv:1206.1269*, 2012. 4

[6] D.W. Cranston and L. Rabern. Conjectures equivalent to the Borodin-Kostochka conjecture that appear weaker. *Arxiv preprint arXiv:1203.5380*, 2012. 3

[7] D.W. Cranston and L. Rabern. Graphs with $\chi = \Delta$ have big cliques. *arXiv preprint* `http://arxiv.org/abs/1305.3526`, 2013. 4

[8] P. Erdős, A.L. Rubin, and H. Taylor. Choosability in graphs. In *Proc. West Coast Conf. on Combinatorics, Graph Theory and Computing, Congressus Numerantium*, volume 26, pages 125–157, 1979. 1

[9] S. Fajtlowicz. Independence, clique size and maximum degree. *Combinatorica*, 4(1):35–38, 1984. 2

[10] H.A. Kierstead and L. Rabern. Extracting list colorings from large independent sets. *arXiv:1512.08130*, 2015. 3

[11] N.N. Mozhan. Chromatic number of graphs with a density that does not exceed two-thirds of the maximal degree. *Metody Diskretn. Anal.*, 39:52–65 (in Russian), 1983. 4

[12] L. Rabern. Coloring graphs with dense neighborhoods. *Journal of Graph Theory*, 2013. 2, 3

[13] B. Reed. A strengthening of Brooks' theorem. *Journal of Combinatorial Theory, Series B*, 76(2):136–149, 1999. 2, 3