mixed Alon-Tarsi notes

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1 Introduction

We study graphs G that are f-AT where $f(v) = d_G(v)$ for most vertices v. It is convenient to be able to discuss graphs that are AT with a fixed number of vertices with $f(v) < d_G(v)$. The following definition achieves this.

Definition 1. For $M\colon \mathbb{N}_{>0} \to \mathbb{N}$, we say that a graph G is M-AT if G is $(d_G - h)$-AT for some $h\colon V(G) \to \mathbb{N}$ where $|h^{-1}(n)| = M(n)$ for all $n \in \mathbb{N}_{>0}$.

For example, when M is the constant zero function, the M-AT graphs are exactly the degree-AT graphs. In fact, the M-AT graphs inherit many of the nice properties of degree-AT graphs. For instance, the non-M-AT graphs are closed under taking induced subgraphs and the M-AT graphs are closed under the operation of subdividing an edge twice. In the next two lemmas, we formalize these properties.

Lemma 1.1. *G is M-AT if and only if G has an induced subgraph H that is M-AT.*

Proof. The forward direction is trivial. For the reverse, we have $h\colon V(H) \to \mathbb{N}$ where $|h^{-1}(n)| = M(n)$ for all $n \in \mathbb{N}_{>0}$ such that H is $(d_H - h)$-choosable (resp. $(d_H - h)$-paintable, $(d_H - h)$-AT). Extend h to $h'\colon V(G) \to \mathbb{N}$ by letting $h'(v) = 0$ for all $v \in V(G) \setminus V(H)$. By ordering the vertices of each component of $G - V(H)$ by increasing distance to H and directing all edges away from H in this order, we conclude that G is $(d_G - h')$-AT. Hence G is M-AT. □

Lemma 1.2. *For any G' formed from G by subdividing an edge twice, G is M-AT if and only if G' is M-AT.*

Proof. This is immediate since there is a parity preserving bijection between the spanning Eulerian subgraphs of G and G'. □

Lemma 1.3. *Let $\{A_1, A_2\}$ be a separation of G such that $A_1 \cap A_2 = \{x\}$. If $G[A_i]$ is f_i-AT for $i \in [2]$, then G is f-AT where $f(v) = f_i(v)$ for $v \in V(M_i - x)$ and $f(x) = f_1(x) + f_2(x) - 1$. Going the other direction, if G is f-AT, then $G[A_i]$ is f_i-AT for $i \in [2]$ where $f_i(v) = f(v)$ for $v \in V(M_i - x)$ and $f_1(x) + f_2(x) \le f(x) + 1$.*

Proof. For $i \in [2]$, choose an orientation D_i of A_i showing that A_i is f_i-AT. Together these give an orientation D of G and since no cycle has vertices in both $A_1 - x$ and $A_2 - x$, we have

$$\begin{aligned} EE(D) - EO(D) &= EE(D_1)EE(D_2) + EO(D_1)EO(D_2) - (EE(D_1)EO(D_2) + EO(D_1)EE(D_2)) \\ &= (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2)) \\ &\neq 0. \end{aligned}$$

Hence G is f-AT.

Now, suppose G is f-AT and choose an orientation D of G showing this. Put $D_i = D[A_i]$ for $i \in [2]$. Then, as above, we have $0 \neq EE(D) - EO(D) = (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2))$ and hence $EE(D_1) - EO(D_1) \neq 0$ and $EE(D_2) - EO(D_2) \neq 0$. Since the in-degree of x in D is the sum of the in-degree of x in D_1 and the in-degree of x in D_2, the lemma follows. □

We need to define a few terms. For $M\colon \mathbb{N}_{>0} \to \mathbb{N}$ and graph G, a *G-realization of M* is a function $h\colon V(G) \to \mathbb{N}$ where $|h^{-1}(n)| = M(n)$ for all $n \in \mathbb{N}_{>0}$. We say that a G-realization h of M is *admissible* if G is $(d_G - h)$-AT. A G-realization h of M is *minimal* if h is admissible and no proper induced subgraph H of G is $(d_H - h\restriction_{V(H)})$-AT. If h is a G-realization of M, put $\mathcal{L}_h(G) = G\left[h^{-1}(0)\right]$ and $\mathcal{H}_h(G) = G - V(\mathcal{L}_h(G))$.

Definition 2. For $M\colon \mathbb{N}_{>0} \to \mathbb{N}$, a graph G is *minimal M-AT* if

- G admits a minimal G-realization of M; and
- G cannot be formed from an M-AT graph by subdividing an edge twice.

For example, when M is the constant zero function, there are only two minimal M-AT graphs: C_4 and K_4^-.

Lemma 1.4. *For $M\colon \mathbb{N}_{>0} \to \mathbb{N}$, let G be minimal M-AT. If h is a minimal G-realization of M, then*

1. *$\mathcal{L}_h(G)$ is a Gallai forest; and*
2. *no vertex in $\mathcal{L}_h(G)$ is a cutvertex in G.*

Proof. (1) follows immediately from minimality of h. For (2), suppose there is $x \in V(\mathcal{L}_h(G))$ that is a cutvertex in G. Let $\{A_1, A_2\}$ be a separation of G such that $A_1 \cap A_2 = \{x\}$. By Lemma 1.3, we know that $G[A_i]$ is f_i-AT for $i \in [2]$ where $f_i(v) = d_G(v) - h(v)$ for $v \in V(M_i - x)$ and $f_1(x) + f_2(x) \leq d_G(x) - h(x) + 1$. Since $h(x) = 0$, this last inequality is $f_1(x) + f_2(x) \leq d_G(x) + 1$ and therefore $f_i(x) \leq d_{G[A_i]}(x)$ for at least one $i \in [2]$. But then $G[A_i]$ is $(d_{G[A_i]} - h\restriction_{A_i})$-AT contradicting minimality of h. □

Since our goal is to classify the basic building blocks of M-AT graphs, we'd like to exclude cutvertices in $\mathcal{H}_h(G)$ as well, but unfortunately these can appear in minimal M-AT graphs. Fortunately, Lemma 1.3 tells us exactly how this can happen, so we can understand the structure of minimal AT-graphs by only considering those without cutvertices in $\mathcal{H}_h(G)$. By Lemma 1.4, this means we can restrict our attention to 2-connected minimal M-AT graphs.

2 General Lemma

This is a key lemma from [1], it generalizes a lemma from [2] from list coloring to Alon-Tarsi orientations. This is what i talked about in Baltimore. The basic idea is that in some cases we can pair off odd/even spanning Eulerian subgraphs via a parity reversing bijection.

Lemma 2.1. *Let G be a multigraph without loops and $f\colon V(G) \to \mathbb{N}$. If there are $F \subseteq G$ and $Y \subseteq V(G)$ such that:*

1. *any multiple edges in G are contained in $G[Y]$; and*
2. *$f(v) \geq d_G(v)$ for all $v \in V(G) \setminus Y$; and*
3. *$f(v) \geq d_{G[Y]}(v) + d_F(v) + 1$ for all $v \in Y$; and*
4. *For each component T of $G-Y$ there are different $x_1, x_2 \in V(T)$ where $N_T[x_1] = N_T[x_2]$ and $T - \{x_1, x_2\}$ is connected such that either:*
   - *(a) there are $x_1y_1, x_2y_2 \in E(F)$ where $y_1 \neq y_2$ and $N(x_i) \cap Y = \{y_i\}$ for $i \in [2]$; or*
   - *(b) $|N(x_2) \cap Y| = 0$ and there is $x_1y_1 \in E(F)$ where $N(x_1) \cap Y = \{y_1\}$,*

then G is f-AT.

Proof. Suppose not and pick a counterexample (G, f, F, Y) minimizing $|G - Y|$. If $|G - Y| = 0$, then $Y = V(G)$ and thus $f(v) \geq d_G(v) + 1$ for all $v \in V(G)$ by (3). Pick an acyclic orientation D of G. Then $EE(D) = 1$, $EO(D) = 0$ and $d^+_D(v) \leq d_G(v) \leq f(v) - 1$ for all $v \in V(D)$. Hence G is f-AT. So, we must have $|G - Y| > 0$.

Pick a component T of $G-Y$ and pick $x_1, x_2 \in V(T)$ as guaranteed by (4). First, suppose (4a) holds. Put $G' := (G - T) + y_1y_2$, $F' := F - T$, $Y' := Y$ and let f' be f restricted to $V(G')$. Then G' has an orientation D' where $f'(v) \geq d^+_{D'}(v) + 1$ for all $v \in V(D')$ and $EE(D') \neq EO(D')$, for otherwise (G', f', F', Y') would contradict minimality. By symmetry we may assume that the new edge y_1y_2 is directed toward y_2. Now we use the orientation of D' to construct the desired orientation of D. First, we use the orientation on $D' - y_1y_2$ on $G - T$. Now, order the vertices of T as $x_1, x_2, z_1, z_2, \ldots$ so that every vertex has at least one neighbor to the right. Orient the edges of T left-to-right in this ordering. Finally, we use y_1x_1 and x_2y_2 and orient all other edges between T and $G - T$ away from T. Plainly, $f(v) \geq d^+_D(v) + 1$ for all $v \in V(D)$. Since y_1x_1 is the only edge of D going into T, any Eulerian subgraph of D that contains a vertex of T must contain y_1x_1. So, any Eulerian subgraph of D either contains (i) neither y_1x_1 nor x_2y_2, (ii) both y_1x_1 and x_2y_2, or (iii) y_1x_1 but not x_2y_2. We first handle (i) and (ii) together. Consider the function h that maps an Eulerian subgraph Q of D' to an Eulerian subgraph $h(Q)$ of D as follows. If Q does not contain y_1y_2, let $h(Q) = \iota(Q)$ where $\iota(Q)$ is the natural embedding of $D' - y_1y_2$ in D. Otherwise, let $h(Q) = \iota(Q - y_1y_2) + \{y_1x_1, x_1x_2, x_2y_2\}$. Then h is a parity-preserving injection with image precisely the union of those Eulerian subgraphs of D in (i) and (ii). Hence if we can show that exactly half of the Eulerian subgraphs of D in (iii) are even, we will conclude $EE(D) \neq EO(D)$, a contradiction. To do so, consider an Eulerian subgraph A of D containing y_1x_1 and not x_2y_2. Since x_1 must have in-degree 1 in A, it must also have out-degree 1 in A. We show that A has a mate A' of opposite parity. Suppose $x_2 \notin A$ and $x_1z_1 \in A$; then we make A' by removing x_1z_1 from A and adding $x_1x_2z_1$. If $x_2 \in A$ and $x_1x_2z_1 \in A$, we make A' by removing $x_1x_2z_1$ and adding x_1z_1. Hence exactly half of the Eulerian subgraphs of D in (iii) are even and we conclude $EE(D) \neq EO(D)$, a contradiction.

Now suppose (4b) holds. Put $G' := G - T$, $F' := F - T$, $Y' := Y$ and define f' by $f'(v) = f(v)$ for all $v \in V(G' - y_1)$ and $f'(y_1) = f(y_1) - 1$. Then G' has an orientation D' where $f'(v) \geq d^+_{D'}(v) + 1$ for all $v \in V(D')$ and $EE(D') \neq EO(D')$, for otherwise (G', f', F', Y') would contradict minimality. We orient $G - T$ according to D, orient T as in the previous case, again use y_1x_1 and orient all other edges between T and $G - T$ away from T. Since we decreased $f'(y_1)$ by 1, the extra out edge of y_1 is accounted for and we have $f(v) \geq d^+_D(v) + 1$ for all $v \in V(D)$. Again any additional Eulerian subgraph must contain y_1x_1 and since x_2 has no neighbor in $G - T$ we can use x_2 as before to build a mate of opposite parity for any additional Eulerian subgraph. Hence $EE(D) \neq EO(D)$ giving our final contradiction. □

3 When M is zero, except M(1) = 1

Through this section, let G be a connected graph, $x \in V(G)$ and $f \colon V(G) \to \mathbb{N}$ where $f(x) = d_G(x) - 1$ and $f(v) = d_G(v)$ for all $v \in V(G - x)$. First, some basic properties.

Lemma 3.1. *The following facts hold.*

1. *If G is f-AT, then $d_G(x) \geq 2$; and*

2. *If $d_G(x) \geq 2$ and $G - x$ has a degree-AT component, then G is f-AT; and*

3. *If G is f-AT and $d_G(x) = 2$, then $G - x$ has a degree-AT component.*

Proof. (1) is immediate since x must have in-degree at least two. We get (2) by orienting all edges incident to x into x. For (3), both edges incident to x must be oriented into x to get in-degree two, so the spanning Eulerian subgraphs counts in G and $G - x$ are the same. □

Because of Lemma 3.1, we will henceforth assume that $d_G(x) \geq 3$ and $G - x$ has no degree-AT components.

Lemma 3.2. *If $G - x$ has at least two components A_1 and A_2 such that $G[V(A_i) \cup \{x\}]$ is degree-AT for all $i \in [2]$, then G is f-AT.*

Proof. Immediate by Lemma 1.3 and Lemma 1.1. □

Lemma 3.3. *If G is f-AT and $G - x$ has a component A such that $G[V(A) \cup \{x\}]$ is not degree-AT, then $G - V(A)$ is also f-AT.*

Proof. Suppose $G - x$ has a component A such that $G[V(A) \cup \{x\}]$ is not degree-AT but $G - V(A)$ is not f-AT. Let D be an orientation of G showing that G is f-AT. Let $D_1 = D[V(A) \cup \{x\}]$ and $D_2 = D[V(G) \setminus V(A)]$. Then, as in Lemma 1.3, we have $EE(D) - EO(D) = (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2))$ and hence both $EE(D_1) - EO(D_1) \neq 0$ and $EE(D_2) - EO(D_2) \neq 0$. Since $G - A$ is not f-AT, it must be that x has in-degree at most 1 in D_2. But then x has in-degree at least 1 in D_1 (since it has in-degree at least 2 in D). But D shows that G is f-AT, so every vertex in A has in-degree at least 1 in D_1 as well. Since $EE(D_1) - EO(D_1) \neq 0$, this shows that $G[V(A) \cup \{x\}]$ is degree-AT, a contradiction. □

Because of Lemma 3.1, Lemma 3.2 and Lemma 3.3, we will henceforth assume that $G - x$ is connected and not degree-AT; that is, $G - x$ is a Gallai tree. Now we show that we can assume G is 2-connected; in particular, x is adjacent to a noncutvertex in every endblock of $G - x$.

Lemma 3.4. *If G is f-AT and G is not 2-connected, then some proper subgraph of G is f-AT.*

Proof. Suppose G is f-AT and there is a cutvertex $v \in V(G-x)$. Then $G-v$ has a component A such that x is not adjacent to any vertex in A. Suppose $G - A$ is not f-AT. Let D be an orientation of G showing that G is f-AT. Let $D_1 = D[V(A) \cup \{v\}]$ and $D_2 = D[V(G) \setminus V(A)]$. Then, as in Lemma 1.3, we have $EE(D) - EO(D) = (EE(D_1) - EO(D_1))(EE(D_2) - EO(D_2))$ and hence both $EE(D_1) - EO(D_1) \neq 0$ and $EE(D_2) - EO(D_2) \neq 0$. Since $G - A$ is not f-AT, it must be that v has in-degree 0 in D_2. But then x has in-degree at least 1 in D_1 (since it has in-degree at least 1 in D). But D shows that G is f-AT, so every vertex in A has in-degree at least 1 in D_1 as well. Since $EE(D_1) - EO(D_1) \neq 0$, this shows that $G[V(A) \cup \{v\}]$ is degree-AT, a contradiction (since $G[V(A) \cup \{v\}]$ is a Gallai tree). □

There should be a more general lemma we can prove that implies Lemma 3.3 and Lemma 3.4 since the proofs are nearly identical.

Lemma 3.5. *If $G - x$ has a complete block B with noncutvertices w and z such that $x \leftrightarrow w$ and $x \not\leftrightarrow z$, then G is f-AT.*

Proof. Immediate from Lemma 2.1 where $Y = \{x\}$, F is just the edge xw and we use part 4b. □

References

[1] Hal Kierstead and Landon Rabern, *Improved lower bounds on the number of edges in list critical and online list critical graphs*, arXiv preprint arXiv:1406.7355 (2014).

[2] A.V. Kostochka and M. Stiebitz, *A new lower bound on the number of edges in colour-critical graphs and hypergraphs*, Journal of Combinatorial Theory, Series B **87** (2003), no. 2, 374–402.