2.2

Limits.

We write $\lim_{x \to a} f(x) = L$

and say "the limit of f(x) as x approaches a, equals L"

if we can make the values of f(x) arbitrarily close to L by taking x to be sufficiently close to a. (but not equal to a)

$$\boxed{\lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

our vague notion of make h smaller and smaller becomes

$f(x) = x^3$

$$\lim_{h \to 0} \frac{(a+h)^3 - a^3}{h} = \lim_{h \to 0} \frac{a^3 + 3ah^2 + 3a^2h + h^3 - a^3}{h}$$

$$= \lim_{h \to 0} \frac{3ah^2 + 3a^2h + h^3}{h}$$

$$= \lim_{h \to 0} \frac{h(3ah + 3a^2 + h^2)}{h}$$

Limit never uses h=0, so ok.

$$= \lim_{h \to 0} 3ah + 3a^2 + h^2$$

$$= \lim_{h \to 0} \left(3a^2 + h(3a+h)\right)$$

$$= 3a^2 + \lim_{h \to 0} h(3a+h)$$

$$= 3a^2 \quad \text{since} \quad \lim_{h \to 0} h(3a+h) = 0.$$

can make this arbitrarily close to 0 by making h sufficiently close to 0.

To make this derivation precise, we need to talk about the domain of a function.

The domain of $f(x)$ is all the values we can plug into f that make sense.

ex1 $f(x) = \frac{1}{x-1}$ domain of f is all real numbers except 1 since $\frac{1}{1-1} = \frac{1}{0}$ is nonsense.

ex2 $f(x) = x^2$ domain is all real numbers

ex3 $f(x) = \sqrt{x}$, domain is all non-negative real numbes
$\sqrt{-1}$ = nonsense (for now).

ex 4

$f(x) = \frac{x-1}{x-1}$ $f(1) = \frac{0}{0}$ nonsense
all reals except 1.

but doesn't $f(x) = \frac{x-1}{x-1} = 1$?

domain of $f(x) = 1$ is all reals, so if the two functions $\frac{x-1}{x-1}$ and 1 are the same, how can they have ~~different~~ domains?

They aren't the same!

$\boxed{\frac{x-1}{x-1} = 1 \text{ is false}}$

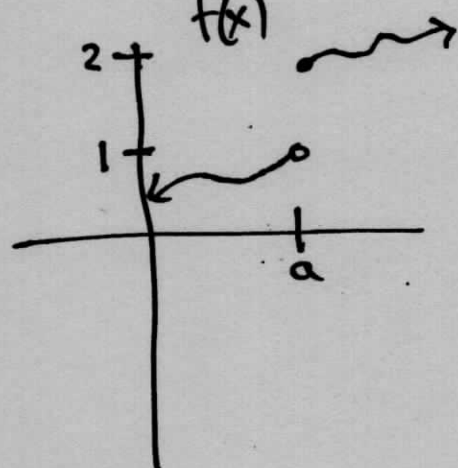
$\boxed{\frac{x-1}{x-1} = 1 \text{ when } x \neq 1}$ is true.

Direct substitution property of limits:

$\lim_{x \to a} f(x) = f(a)$ as long as a is in the domain of f and

f is "nice" near a.

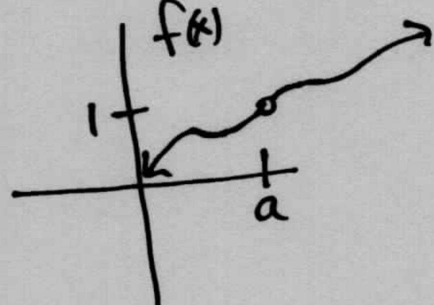
what does "nice" mean? what could go wrong.



if we come from the left f(x) goes to 1 as x goes to a.
but if we come from the right f(x) goes to 2.

$\boxed{\lim_{x \to a} f(x) \text{ is undefined}}$

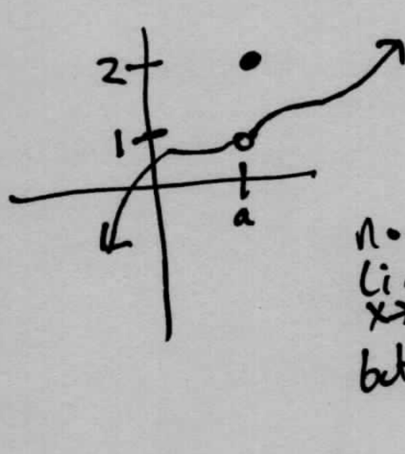
in particular not equal to f(a).



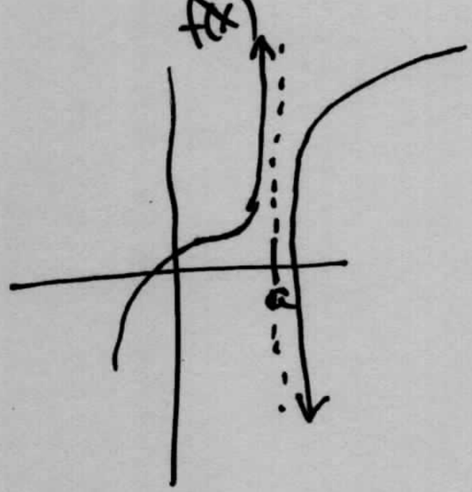
here as we approach a from left f(x) goes to 1.
same for approach from right.

$\lim_{x \to a} f(x) = 1$

but a is not in domain of f, that is, f(a) is not defined, so we cannot substitute



now
$\lim_{x \to a} f(x) = 1$
but $f(a) = 2$.



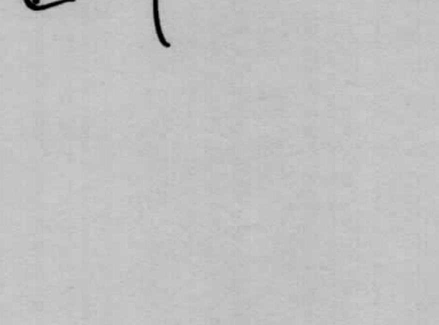
$\lim_{x \to a} f(x)$ does not exist

These 4 pictures are everything that can go wrong. Each is a "discontinuity" in f(x).

If f(x) has none of these issues, we say

f(x) is continuous at a.

Another way to say it: f(x) is continuous if we can draw its graph without ~~picking up~~ taking the chalk off the board.



~~2.6~~

The <u>derivative</u> of a function ~~f~~ at a, written $f'(a)$

is $f'(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$ (if this limit exists)

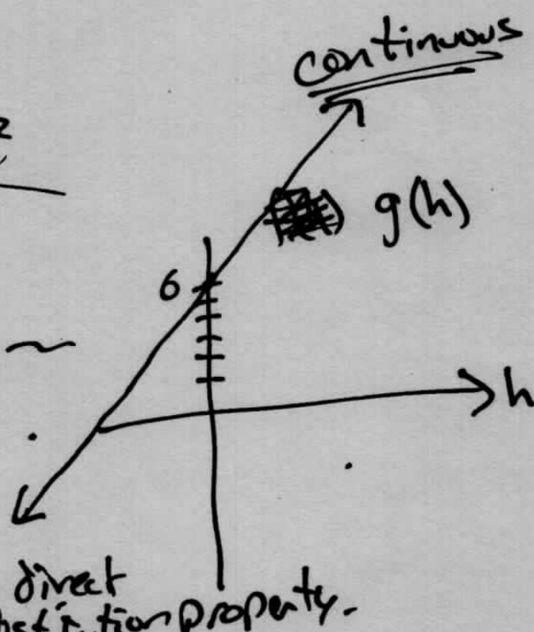
this is just the slope of the tangent line at $x = a$.

$\boxed{f'(a) \text{ is just notation for the slope of the tangent line at } x=a.}$

<u>ex</u> ~~7~~ compute $f'(3)$ where $f(x) = x^2$

$$f'(3) = \lim_{h\to 0} \frac{f(3+h) - f(3)}{h}$$
$$= \lim_{h\to 0} \frac{(3+h)^2 - 3^2}{h}$$
$$= \lim_{h\to 0} \frac{3^2 + 2\cdot 3\cdot h + h^2 - 3^2}{h}$$
$$= \lim_{h\to 0} \frac{6h + h^2}{h}$$
$$= \lim_{h\to 0} \frac{h(6+h)}{h}$$
$$= \lim_{h\to 0} 6+h$$
$$= 6+0$$
$$= 6$$

why?



so has direct substitution property.

2.7 Derivative as function.

For a function $f(x)$, let

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h}$$

ex

$f(x) = x^2$

$$
\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h} \\
&= \lim_{h\to 0} \frac{2xh + h^2}{h} \\
&= \lim_{h\to 0} \frac{h(2x+h)}{h} \\
&= \lim_{h\to 0} 2x + h. \\
&= 2x,
\end{aligned}
$$

$$\boxed{f'(x) = 2x}$$