Design of a 3D Lissajous Trajectory for Six-Axis IMU Excitation

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September 3, 2019

1 Introduction

This supplementary material describes how we designed our 3D Lissajous trajectory which is used in three of our past works to sufficiently excite the IMU to enable quality state estimation.

Our procedure is as follows: we design a general Lissajous trajectory $\boldsymbol{p}_s : [0, T_s] \to \mathbb{R}^3$, where $T_s \in \mathbb{R}^+$ is the period of the trajectory. This trajectory does not have a smooth ramp-up and ramp-down, so we then reparameterize the trajectory to $\boldsymbol{p}_t = \boldsymbol{p}_s \circ s : [0, T_t] \to \mathbb{R}^3$ so that $s : [0, T_t] \to [0, T_s]$ has continuous derivatives up to the third order. This is done so that the trajectory has continuous jerk, which means that the angular velocity command, which is a function of the jerk, will not have discontinuities.

2 Trajectory Before Reparameterization

The trajectory is given:

$$\boldsymbol{p}_s(s) = \begin{pmatrix} A_x(1 - \cos(2\pi \cdot n_x \cdot s/T_s)) \\ A_y \sin(2\pi \cdot n_y \cdot s/T_s) \\ A_z \sin(2\pi \cdot n_z \cdot s/T_s) \end{pmatrix}. \tag{1}$$

The designer of the trajectory specifies 7 parameters relating to the trace: the overall period T_s, three amplitudes A_x, A_y and A_z, and the number of cycles n_x, n_y, and n_z per period T_s for each coordinate axis.

3 Reparameterization

We now design the reparameterization $s(t)$. The function $\dot{s}(t)$ will have a smooth ramp-up at the beginning of the trajectory, when $t < t_r$, followed by a section where it is equal to unity, when $t_r < t < T_t - t_r$, followed by a smooth ramp-down at the end of the trajectory, when $T_t - t_r < t < T_t$. The parameter t_r

is chosen by the designer of the trajectory, and T_t will is calculated, as we will show, as a function of T_s and t_r.

We denote the ramp-up function $s_r(t)$. Its derivative $\dot{s}(t)$ is designed to be a polynomial function of degree 7. However, we constrain $\dot{s}_r(0) = \ddot{s}_r(0) = \dddot{s}_r(0) = 0$, so the zeroth, first, and second order coefficients are zero:

$$\dot{s}(t) = c_7 t^7 + c_6 t^6 + c_5 t^5 + c_4 t^4 + c_3 t^3. \tag{2}$$

There are five unknown coefficients, so we need five more constraints to determine $\dot{s}_r(t)$. The next constraint is that $\frac{d^4 s_r}{dt^4}(0) = 0$, which forces $c_3 = 0$:

$$\dot{s}_r(t) = c_7 t^7 + c_6 t^6 + c_6 t^5 + c_4 t^4. \tag{3}$$

Hence, we need four more constraints. These constraints are $\dot{s}_r(t_r) = 1$, $\ddot{s}_r(t_r) = 0$, $\dddot{s}_r(t_r) = 0$, and $\frac{d^4 s_r}{dt^4}(t_r) = 0$. From these constraints, we have the system:

$$\begin{pmatrix} t_r^7 & t_r^6 & t_r^5 & t_r^4 \\ 7t_r^6 & 6t_r^5 & 5t_r^4 & 4t_r^3 \\ 42t_r^5 & 30t_r^4 & 20t_r^3 & 12t_r^2 \\ 210t_r^4 & 120t_r^3 & 60t_r^2 & 24t_r \end{pmatrix} \begin{pmatrix} c_7 \\ c_6 \\ c_5 \\ c_4 \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}. \tag{4}$$

Solving this equation yields $c_7 = -20/t_r^7$, $c_6 = 70/t_r^6$, $c_5 = -84/t_r^5$, and $c_4 = 35/t_r^4$. Hence,

$$s_r(t) = -\frac{5}{2t_r} t^8 + \frac{10}{t_r} t^7 - \frac{14}{t_r} t^6 + \frac{7}{t_r} t^5. \tag{5}$$

The reparameterization is:

$$s(t) = \begin{cases} s_r(t) & 0 \le t \le t_r \\ s_r(t_r) + t - t_r & t_r < t \le T_t - t_r \\ T_s - s_r(T_t - t) & T_t - t_r < t \le T_t \end{cases}, \tag{6}$$

where

$$T_s = \int_0^{T_t} \dot{s}(t) dt = 2 s_r(t_r) + T_t - 2t_r. \tag{7}$$

In the expression above, $T_t - 2t_r$ is the duration over which $\dot{s}(t)$ is zero. Hence, we have the relationship necessary to determine T_t as function of T_s and t_r.

If wish to run multiple cycles of the Lissajous trajectory between the ramp-up and ramp-down, we have the ability to specify this as well, with only a slight change in the approach. This modification is easy to make and it is not discussed.

4 Compound Trajectories

In order for the velocity and drag coefficients to be observable, the quadrotor must reach a sufficiently high linear velocity. At the same time, in order to observe the moment of inertia, the quadrotor must achieve a significant angular acceleration. We have observed that, with a single Lissajous trajectory, it is difficult to achieve both of these requirements without saturating the motors. However, by "summing" two such trajectories, this goal becomes more feasible. In our case, we sum low-frequency, high-velocity trajectory with a higher frequency, low velocity trajectory. The first trajectory helps with observability of the linear velocity, while the second helps with the observability of the moment of inertia. The trajectories are summed by simply adding their reference positions, velocities, accelerations and jerks.