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On optimal Loewner energy and quasiconformal deformation

Master Thesis

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Abstract

We study the Loewner energy I^L of Jordan curves γ and conformal weldings h, and consider two constrained optimization problems, prescribing respectively n marked points $z_1, \dots, z_n \in \hat{\mathbb{C}}$ that the curve is required to pass through, and n boundary correspondences $h(x_k) = y_k$. For the latter, less well-understood problem, we prove the existence of a solution. Finally, we study the residues of certain Schwarzians associated with the optimal γ^* and h^*, showing via quasiconformal deformations that, under differentiability assumptions, the residues equal $\frac{1}{2}$ times the corresponding first variations of the optimal Loewner energy.

Acknowledgments

I would like to express deep gratitude to my supervisor Yilin Wang. Our sessions at the blackboard made it possible for me to navigate a topic that bridges multiple fields and that seemed very daunting at first. The thing that will stay with me for the longest, however, is the care with which she guided me in thinking about my future and what it means to pursue creative work.

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Chapter 1

Introduction

Consider a Jordan curve $\gamma : [0, 1] \to \hat{\mathbb{C}}$ in the extended complex plane, tracing out a simple loop, i.e. starting and ending at the same point, $\gamma(0) = \gamma(1)$. One concrete visual example is the equator on the two-dimensional sphere. There are of course many other loops without self-crossing and in this thesis we study in detail some problems related to the Loewner energy of such curves, denoted $I^L(\gamma)$, a functional that measures roughly the deviation of such a loop from being a circle.

Before we go into further detail regarding the specific tasks that lie ahead, it seems prudent to take a step back and examine what exactly we are measuring with Loewner energy and in what sense it is an energy. In 1923, Loewner examined families of conformal maps related to slit domains of the unit disk. [6] Translating to the conformally equivalent setting of the upper halfplane, we consider a curve γ starting at zero and growing towards infinity. At any given point in time, it carves out a simply connected domain $H_t = \mathbb{H} \backslash \gamma[0, t]$ and then from the Riemann mapping theorem and a suitable normalization, we get a choice of conformal map $g_t : H_t \to \mathbb{H}$, with the expansion $g_t(z) = z + \frac{2t}{z} + O(|z|^{-2})$ at infinity.

This yields a family of maps $(g_t)_t$ and remarkably, these so-called mapping-out functions satisfy, for each z, an ODE of the form $\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}$, a description of how the individual z flow across time as the curve continues its growth towards infinity. What is more, the curve γ is encoded by ξ in the above ODE, called the Loewner driving function. In two papers from 2015 and 2016 Friz-Shekhar [3] and then independently Wang [17] used this representation to define the chordal Loewner energy of γ as the Dirichlet energy of the Loewner driving function, namely

$$I^C_{\mathbb{H};0,\infty}(\gamma) := \frac{1}{2} \int_0^\infty (\frac{d\xi_t}{dt})^2 dt. \tag{1.1}$$

To get from this chordal setting to loops, one exploits that for a Jordan curve γ, the segment $\gamma[\epsilon, 1]$ is a chord in the simply connected domain $\hat{\mathbb{C}} \backslash \gamma[0, \epsilon]$ and then by using a limiting procedure it is possible to define the loop Loewner energy [11]

$$I^L(\gamma) := \lim_{\epsilon \to 0} I^C_{\hat{\mathbb{C}} \backslash \gamma[0,\epsilon]}(\gamma[\epsilon, 1]), \tag{1.2}$$

putting us firmly back in the setting of the opening paragraph. This can be taken one step further however. Any such Jordan curve γ separates the extended complex plane $\hat{\mathbb{C}}$ into a bounded and unbounded component Ω and Ω^*. Up to Möbius automorphisms, the Riemann mapping theorem gives conformal maps $f : \mathbb{H} \to \Omega$ and $g : \mathbb{H}^* \to \Omega^*$ from the upper and lower halfplanes onto these respective components. Defining the conformal welding $h = g^{-1} \circ f|_{\mathbb{R}}$ one obtains a different encoding of the geometric information of the curve. One defines the Loewner energy of a welding as that of a representative curve γ_h, which has h as its conformal welding, namely $I^L(h) := I^L(\gamma_h)$. In conclusion, the Loewner energy is natural both for Jordan curves and for conformal weldings.

For a chord in the upper halfplane to have zero Loewner energy, we must set the driving function to zero, and this gives a curve that traces out the segment $i\mathbb{R}_+ \subset \mathbb{H}$. For loops, we end up with circles as the global minima and in the case of weldings, we get the identity welding pre- and post-composed by a Möbius map. These are the global minimizing objects for Loewner energy in their respective settings.

A very natural next step is to start putting some constraints on the set of curves or weldings being considered in the minimization.

A problem in this vein was considered in detail by Wang and collaborators in [7] [2]. Let $z_1, \ldots, z_n \in \hat{\mathbb{C}}$ be n distinct points and consider the set of Jordan curves passing through these points in that order. Insist furthermore that the curves are all homotopic relative to these n points, denoting this class by $\mathcal{L}(z, \tau) = \mathcal{L}(z_1, \ldots, z_n, \tau)$, where τ is a representative curve within the homotopy class. As soon as $n \geq 4$, it is not assured that the points all lie on some circle, and thus we have in general that the Loewner energy of the minimizing curve, if it exists, is strictly positive.

After establishing existence, uniqueness and some interesting geometric properties of the solution to the curve problem, Wang in 2025 [19] considered a similar setup for weldings. Let $x_1, y_1, \ldots x_n, y_n \in \hat{\mathbb{R}}$ be n pairs for which $x_i \neq x_j$, $y_i \neq y_j$ for $i \neq j$ and insist now that the welding map $h = g^{-1} \circ f|_{\mathbb{R}}$ satisfies $h(x_k) = y_k$, denoting this class by $\Phi_{x,y}$. In the same paper it is suggested that a solution should exist and be unique, but not proved.

Some interesting comments regarding the geometry of the solution, particularly the representative curve γ_h are made. There are also some hints regarding the structure of the Schwarzians $\mathcal{S}[f]$ and $\mathcal{S}[g]$ and how these should exhibit properties similar to $\mathcal{S}[f^{-1}]$ and $\mathcal{S}[g^{-1}]$ from the optimal solution to the curve problem.

1.1 Main results

This thesis studies the two optimization problems above, namely

$$\inf_{\gamma \in \mathcal{L}(z_1,\ldots,z_n,\tau)} I^L(\gamma), \qquad \inf_{h \in \Phi_{x,y}} I^L(h), \tag{1.3}$$

the existence and uniqueness of their solutions and the geometric properties thereof with particular emphasis on the Schwarzians of f^{-1}, g^{-1} for the curve and f, g for the welding. Recall the definition of the Schwarzian derivative of a holomorphic function f

$$\mathcal{S}[f](z) = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{1.4}$$

Using the geometric properties of the solution curves (or the representative curve in the case of weldings), one obtains by setting $F = f^{-1}$ on Ω and $F = g^{-1}$ on Ω^* that $\mathcal{S}[F]$ can be extended to all of $\hat{\mathbb{C}}$ and that it has the following simple pole structure

$$\mathcal{S}[F](z) = \sum_{k=1}^{n} \frac{\mathrm{Res}(\mathcal{S}[F], z_k)}{z - z_k}. \tag{1.5}$$

Similarly, for the welding, it will turn out that $\mathcal{S}[f]$ and $\mathcal{S}[g]$ can both be extended to all of $\hat{\mathbb{C}}$, albeit as different meromorphic functions, and that

$$\mathcal{S}[f](z) = \sum_{k=1}^{n} \frac{\mathrm{Res}(\mathcal{S}[f], x_k)}{z - x_k} \tag{1.6}$$

$$\mathcal{S}[g](z) = \sum_{k=1}^{n} \frac{\mathrm{Res}(\mathcal{S}[g], y_k)}{z - y_k}. \tag{1.7}$$

The main contribution of this thesis is to the understanding of the residues $\mathrm{Res}(\mathcal{S}[F], z_k)$, $\mathrm{Res}(\mathcal{S}[f], x_k)$ and $\mathrm{Res}(\mathcal{S}[g], y_k)$. We have the following results, the first of which was previously derived in [2].

Theorem 1.1 *Consider the Loewner energy optimization problems for curves in $\mathcal{L}(z_1, ..., z_n; \tau)$ giving rise to optimal value and curve*

$$I^L(z_1, ..., z_n) := I^L(\gamma^*). \tag{1.8}$$

Let F be the function associated to the optimal curve γ^* as above. Assuming the derivative exists, we have the following formula for the residues of the Schwarzian.

$$Res(\mathcal{S}[F], z_k) = \frac{1}{2}\partial_{z_k} I^L(z_1, ..., z_n) \tag{1.9}$$

For the welding optimization problem, we obtain:

Theorem 1.2 *Consider the Loewner energy optimization problem for weldings in $\Phi_{x,y}$ with optimum*

$$I^L(x, y) := I^L(h^*) \tag{1.10}$$

Let f and g be the functions associated to the solution h^*. Assuming the derivatives exist, we have the following formula for the residues:

$$Res(\mathcal{S}[f], x_k) = \frac{1}{2}\partial_{x_k} I^L(x, y) \tag{1.11}$$

$$Res(\mathcal{S}[g], y_k) = \frac{1}{2}\partial_{y_k} I^L(x, y). \tag{1.12}$$

To carry out the proofs we adapt a technique from Sung and Wang's work on quasiconformal deformations and how it relates to Loewner energy [13]. There it is shown that the infinitesimal change of the Loewner energy of a Jordan curve exposed to application of a quasiconformal map $\omega^{t\mu}$ with Beltrami differential $\|t\mu\|_\infty < 1$ can be related to an integral of the Schwarzians in the following way

$$\frac{d}{dt}|_{t=0} I^L(\omega^{t\mu}(\gamma)) = -\frac{4}{\pi}\mathrm{Re}\left[\int_\Omega \mathcal{S}[f^{-1}](z)\mu(z)d^2z + \int_{\Omega^*} \mathcal{S}[g^{-1}]\mu(z)d^2z\right], \tag{1.13}$$

a result that concretizes work by Takhtajan-Teo on variations of the universal Lioville action S_1, set in the context of universal Teichmüller space. [14]

The main idea to get from the variational formula (1.13) to the results on residues theorem 1.1 theorem 1.2 is to pick a simplifying quasiconformal deformation that allows one to analyze one residue at a time. On a general level, this is facilitated by a map that moves only the point associated with that one particular residue.

1.2 Outline

We begin in chapter 2 with the details on Loewner's equation, the Loewner transform and how this allows for the definition of Loewner energy of chords and loops as sketched in the above opening paragraphs.

In chapter 3 we recap some conformal geometry, the Schwarzian derivative and some important Riemann maps that are directly used in proving the simple pole structure and extendability results in theorem 1.1 and theorem 1.2. The class of conformal mappings are best understood as a subset of the quasiconformal maps and since quasiconformal deformation is the main ingredient in the new proof strategy for the main results, we devote them special attention. To unify the perspectives on curves and weldings, as well as use strong results on variation of Loewner energy, we also establish some Teichmüller theory.

In chapter 4 this bears fruit, as we get to use a theorem on first variation of the universal Liouville action, a functional with close ties to the Loewner energy, to understand how infinitesimal quasiconformal deformation of curves and weldings affects their Loewner energy. This is a key step to extend the proof strategy to cover the main welding result.

Then in chapter 5 we present the two optimization problems presented briefly above and discuss existence and uniqueness.

Finally in chapter 6 we put everything together and carry out the proofs of the results theorem 1.1 and theorem 1.2 using the quasiconformal deformation technique.

Chapter 2

Preliminaries

2.1 Loewner's theory

We follow the exposition in [1] closely, without giving proofs.

The starting point is sets $K \subset \mathbb{H}$ which are bounded and such that $\mathbb{H} \setminus K$ is simply connected, called compact $\mathbb{H}$-hulls. The typical look of such a set is one which shares part of its boundary with the real line and extends like a blob into the upper halfplane.

Associated to such subsets K there are so-called mapping out functions from the complement $H := \mathbb{H} \setminus K$ to $\mathbb{H}$, i.e. a conformal map $g_K : H \to \mathbb{H}$. One furthermore imposes that $g_K(z) - z \to 0$ as $|z| \to \infty$, i.e. that for z of great magnitude, the map looks almost like the identity. Then $g_K(z) - z$ is uniformly bounded on H, and for some $a_K \in \mathbb{R}$,

$$g_K(z) = z + \frac{a_K}{z} + O(|z|^{-2}) \qquad (|z| \to \infty). \tag{2.1}$$

The expansion above gives the number a_K, called the halfplane capacity, associated to K, which turns out to be a rough measure of the size of the set K. A nice result in this direction is that a_K is comparable to the Lebesgue measure of the union of balls centered at a point in K and tangent to the real line. [4]

Some care needs to be taken with this intuition of a_K as a measure of size however, as for every $R > 0$ and $\epsilon > 0$, it is possible to create a hull K for which $\text{diam}K > R$ and $a_K < \epsilon$, which can be seen by looking at the comparability we just stated and insisting that K stay very close to the real line at all times.

The next step is to consider not just one set K but a family $(K_t)_t$, each with its own mapping out function $g_t = g_{K_t}$. It is then natural to start asking questions about regularity of $(K_t)_t$ and how that relates to regularity of $(g_t)_t$.

For $s < t$ set the increment $K_{s,t} := g_{K_s}(K_t \setminus K_s)$. We say (K_t) has the local growth property if $\mathrm{rad}(K_{t,t+h}) \to 0$ as $h \downarrow 0$, uniformly for t in compact intervals.

One should take care to notice that $K_{s,t}$ is a "mapped out" version of a complement. Under the local growth condition there turns out to be a unique real-valued and continuous process $(\xi_t)_t$, where for each $t > 0$ there is a unique $\xi_t \in \mathbb{R}$ such that $\xi_t \in \overline{K_{t,t+h}}$ for all $h > 0$. Equivalently,

$$\{\xi_t\} = \bigcap_{h>0} \overline{K_{t,t+h}}. \tag{2.2}$$

Intuitively it is the mapped out version of the tip of instantaneous growth. We call this the Loewner transform or driving function of the family $(K_t)_t$.

If $(K_t)_t$ has local growth, then $t \mapsto a_{K_t}$ is continuous and strictly increasing. One can reparametrize so that

$$a_{K_t} = 2t \qquad \text{for all } t \geq 0. \tag{2.3}$$

Under this parametrization then, introducing $\zeta(z) := \inf\{t : z \in K_t\}$, one has that $t \mapsto g_t(z)$ is differentiable on $[0, \zeta(z))$ and satisfies

$$\partial_t g_t(z) = \frac{2}{g_t(z) - \xi_t}. \tag{2.4}$$

So under the regularity conditions we have insisted on, ξ_t is a way to describe the evolution of $(K_t)_t$. It is quite natural to ask about the other direction, namely whether one can prescribe a certain driving function and recover the family $(K_t)_t$, and if this is possible, what the correspondence between the two representations look like, e.g. whether there is a bijection.

It turns out that for every choice of continuous function $\xi : [0, \infty) \to \mathbb{R}$, the ODE $\dot{z}_t = \frac{2}{z_t - \xi_t}$ has a maximal solution for each starting point $z \in \mathbb{H}$, with lifetime $\zeta(z) \in (0, \infty]$. To recover K_t, define

$$K_t := \{z \in \mathbb{H} : \zeta(z) \leq t\}, \tag{2.5}$$

then one ends up with an increasing family of compact $\mathbb{H}$-hulls, satisfying the local growth condition and such that $a_{K_t} = 2t$.

2.2 Loewner energy

The main functional of interest in this thesis is the Loewner energy, which can intuitively be described as a measure of the deviation from circularity, modulo Möbius transformations, of Jordan curves in $\hat{\mathbb{C}}$. The starting point for defining the Loewner energy however is the chordal setting.

Definition 2.1 *The chordal Loewner energy of a curve γ in $(D; a, b)$ is defined with reference to $\varphi(\gamma)$ in $(\mathbb{H}, 0, \infty)$, where $\varphi : D \to \mathbb{H}$ is conformal, as the Dirichlet energy of its driving function, namely*

$$I^C_{D;a,b}(\gamma) := I^C_{\mathbb{H};0,\infty}(\varphi(\gamma)) := \frac{1}{2}\int_{0,T}(\frac{d\xi_t}{dt})^2 dt \tag{2.6}$$

provided $t \mapsto \xi_t$ is absolutely continuous, and $I^C_{D;a,b}(\gamma) = \infty$ otherwise.

Different choices of $\varphi : D \to \mathbb{H}$ give different choices of ξ, but they differ only by scaling and their domain of definition, leaving the integral in the above definition invariant.

Given that $I^C_{D;a,b}(\gamma)$ can be finite for some well-behaved curves γ, it is natural to ask about the minimizer. The example of the imaginary axis in $\mathbb{H}$, which is the hyperbolic geodesic from 0 to ∞, together with the preservation of hyperbolic geodesics under conformal diffeomorphisms, gives immediately that η in $(D; a, b)$ that minimizes $I^C_{D;a,b}$ is precisely the hyperbolic geodesic, with the hyperbolic metric given with respect to D.

We are now in a position to define the Loewner energy of a Jordan curve $\gamma : [0, 1] \to \hat{\mathbb{C}}$. To set the stage, notice that $\hat{\mathbb{C}}\backslash\gamma[0, \epsilon]$ is simply connected. The curve segment $\gamma[\epsilon, 1]$ is a chord connecting $\gamma(\epsilon)$ with $\gamma(1)$ in this simply connected domain. For each $\epsilon > 0$, we have the well-defined quantity $I^C_{\hat{\mathbb{C}}\backslash\gamma[0,\epsilon]}(\gamma[\epsilon, 1])$. It turns out to be possible to take a well-defined limit [11]:

Definition 2.2 *The loop Loewner energy $I^L(\gamma)$ of a Jordan curve $\gamma : [0, 1] \to \hat{\mathbb{C}}$ is given by*

$$I^L(\gamma) := \lim_{\epsilon\to 0} I^C_{\hat{\mathbb{C}}\backslash\gamma[0,\epsilon]}(\gamma[\epsilon, 1]) \tag{2.7}$$

Here there is a choice of base point or root $\gamma(0)$ lurking in the background, but it was shown not to matter for the definition, i.e. there is the so-called root invariance of the loop Loewner energy. [11]

A key structural fact is that the loop Loewner energy behaves well under what one might call cutting and pasting: if one keeps part of a loop fixed and varies the complementary arc inside the resulting simply connected domain, the change in I^L is exactly the change of a chordal Loewner energy, something that will come in handy when we consider constrained minimization problems later.

Concretely, if $z_1, z_2 \in \gamma$ are distinct points, one may separate $\gamma = \gamma_1 \cup \gamma_2$, where γ_1 is the segment from z_1 to z_2 and γ_2 is the segment from z_2 to z_1.

Letting $\Omega_1 = \Omega \backslash \gamma_1$ we obtain a simply connected domain, and γ_2 is a chord in (Ω_1, z_2, z_1). Taking any other such chord η and constructing the Jordan curve $\tilde{\gamma} = \gamma_1 \cup \eta$, one then has

$$I^L(\tilde{\gamma}) - I^L(\gamma) = I^C_{\Omega_1;z_2,z_1}(\eta) - I^C_{\Omega_1;z_2,z_1}(\gamma_2) \tag{2.8}$$

From the definition it is apparent that Loewner energy is nonnegative. The zero energy loops can be characterized: For a circle $\gamma = T(\hat{\mathbb{R}}) \subset \hat{\mathbb{C}}$, the loop Loewner energy is zero $I^L(\gamma) = 0$. One can also show that if $I^L(\gamma) = 0$ then γ is a circle.

Chapter 3

Useful background

3.1 Conformal geometry

Let $\gamma_1, \gamma_2 : (-\varepsilon, \varepsilon) \to \mathbb{R}^2$ be two C^1 curves with $\gamma_1(0) = \gamma_2(0) = z_0$ and nonzero tangent vectors $v_j = \gamma_j'(0)$. The (oriented) angle between the curves at z_0 is the angle between v_1 and v_2 in $\mathbb{R}^2$ with its Euclidean inner product. For a map f defined in a neighborhood of z_0, we can ask whether for any such pair of curves, the angle between v_1 and v_2 equals the angle between $Df(z_0)v_1$ and $Df(z_0)v_2$. If this is so, we speak of an angle preserving map.

This can of course be extended to higher dimensional or even Riemannian settings, but the two-dimensional case is very rich and will be enough for what is to follow. The intuition for angle preservation motivates the following definition:

Definition 3.1 *Let $D, D' \subset \mathbb{C}$ be domains and $f : D \to D'$ be C^1. We say that f is conformal at $z_0 \in D$ if its differential*

$$Df(z_0) : \mathbb{R}^2 \to \mathbb{R}^2$$

is of the form $Df(z_0) = \lambda R$ with $\lambda > 0$ and $R \in SO(2)$. The map f is conformal on D if it is conformal at every point and $Df(z)$ is nonzero for all $z \in D$.

If we identify the plane $\mathbb{R}^2$ with $\mathbb{C}$ we can examine the complex analytic properties of such functions. It turns out that:

Theorem 3.2 *Let $D \subset \mathbb{C}$ be a domain and $f : D \to \mathbb{C}$ a C^1 map. For a point $z_0 \in D$, the following are equivalent:*

1. *f is conformal at z_0, i.e. $Df(z_0) = \lambda R$ with $\lambda > 0$ and $R \in SO(2)$.*
2. *f is complex differentiable at z_0 and $f'(z_0) \neq 0$.*

Note that antiholomorphic functions, where it is $\bar{f}$ that is holomorphic, preserve angles but not their orientation.

A subfamily of the conformal maps in two dimensions are the Möbius transformations:

Definition 3.3 *A Möbius transformation is a map $M : \mathbb{C} \to \mathbb{C}$ of the form*

$$M(z) = \frac{az+b}{cz+d}, \qquad a, b, c, d \in \mathbb{C},\ ad - bc \neq 0.$$

The Riemann sphere is the one-point compactification $\widehat{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. We can extend Möbius maps by defining

$$M\left(-\frac{d}{c}\right) = \infty, \qquad M(\infty) = \begin{cases} a/c, & c \neq 0, \\ \infty, & c = 0. \end{cases} \tag{3.1}$$

An important geometric fact is that Möbius maps preserve circles. More precisely, a circle on $\widehat{\mathbb{C}}$ will be transformed into (another) circle by such a map M.

Definition 3.4 *Let f be a holomorphic function on a domain $D \subset \mathbb{C}$. If we consider the open set*

$$D^{\times} := \{z \in D : f'(z) \neq 0\} \tag{3.2}$$

we can safely define the Schwarzian derivative of f as the meromorphic function $\mathcal{S}[f] : D^{\times} \to \mathbb{C}$ defined by

$$\mathcal{S}[f](z) = \left(\frac{f''(z)}{f'(z)}\right)' - \frac{1}{2}\left(\frac{f''(z)}{f'(z)}\right)^2 = \frac{f'''(z)}{f'(z)} - \frac{3}{2}\left(\frac{f''(z)}{f'(z)}\right)^2. \tag{3.3}$$

One can verify by a direct calculation that for a Möbius map M, it holds $\mathcal{S}[M] \equiv 0$. Conversely, if $\mathcal{S}[f] \equiv 0$ and f is locally injective, then f is Möbius. So one can say that roughly, the Schwarzian captures deviation from the family of Möbius maps. In light of the relationship between Möbius maps and circles expounded upon after the previous definition, we can update our intuition slightly: The Schwarzian detects maps that do not preserve circles.

There is the following Schwarzian chain rule

$$\mathcal{S}[f \circ g] = (\mathcal{S}[f] \circ g)(g')^2 + \mathcal{S}[g]. \tag{3.4}$$

A theorem we will use over and over again is the Riemann mapping theorem.

Theorem 3.5 *For $D \subset \mathbb{C}$ nonempty, simply connected, open and not all of $\mathbb{C}$, there exists a conformal map $f : D \to \mathbb{D}$, i.e. D is conformally equivalent to $\mathbb{D}$*

It follows that any two sets D and D' with the above properties are conformally equivalent, i.e. we can come up with a conformal map $g : D \to D'$. It is possible to classify all conformal maps $\mathbb{D} \to \mathbb{D}$ as being of the form $z \mapsto e^{i\theta} \frac{z-a}{1-\bar{a}z}$ for $a \in \mathbb{D}$ and $\theta \in \mathbb{R}$. It also allows us to see that the normalization $f(z_0) = 0$ for $z_0 \in D$ and $f'(z_0) > 0$ determines the map f uniquely.

The Riemann mapping theorem is very powerful in its generality, but tells us very little about what the Riemann map actually looks like. There are special examples where we can say more, e.g. for maps $\mathbb{H} \to \mathbb{D}$ or $\mathbb{D}^* \to \mathbb{D}$. A rather large class is that of the polygonal domains, i.e. domains bounded by a polygonal arc, with straight edges meeting with certain angles at junction points. Now let $D = P$ be a polygonal domain with interior angles $\pi\alpha_1, \pi\alpha_2, ..., \pi\alpha_n$.

Lemma 3.6 (Schwarz-Christoffel map) *For a polygonal domain with vertices $z_1, ..., z_n$ and interior angles $\pi\alpha_1, ..., \pi\alpha_n$, there is a conformal map $\mathbb{H} \to P$ of the form*

$$f(z) = A \int_0^z \prod_{k=1}^{n} (\xi - a_k)^{\alpha_k - 1} d\xi + B \tag{3.5}$$

where a_k is the preimage on the real line of the vertex $z_k = f(a_k)$.

A and B determine positioning and size of the polygon. The a_k need in general be numerically determined for a specific choice of polygon.

Proof For a polygonal domain P, the boundary ∂P is a Jordan curve, so by Caratheodory's theorem, the Riemann map f extends to the boundary. For this extended map, the vertices z_1, ..., z_n of the polygon have preimages $a_1, ..., a_n \in \partial\mathbb{H}$ and we will in fact assume to begin with that none of these points is the point at infinity. By a version of the Schwarz reflection principle, f is regular (holomorphic and $f'(z) \neq 0$) at all points of $\mathbb{R}$ except possibly the $a_1, ..., a_n$.

Reflect f across one segment $[a_k, a_{k+1}] \subset \mathbb{R}$. Reflecting across once more, possibly across another segment, will return to the same z, but give a different value of the extended f. Naming this double extension f_1 we find that it maps $\mathbb{H}$ to another polygon P' which is congruent to P, i.e. $f_1(z) = Af(z) + B$. Therefore

$$\frac{f_1''(z)}{f_1'(z)} = \frac{f''(z)}{f'(z)} \tag{3.6}$$

and we can unambiguously define a single-valued function g by $g(z) = \frac{f''(z)}{f'(z)}$ in all of $\mathbb{C}$. This is in contrast to f above which will in general give us different values depending on the exact inversion made (different congruent polygons).

Because of the regularity of f at all points except possibly the $a_1, ..., a_n$, these are also the only points where g can have singularities.

To study these singularities for given a_k, we study what happens to the expression $f(z) - f(a_k)$ as z approaches a_k on the real line. Since this is the preimage of the k'th vertex with interior angle $\pi\alpha_k$, this expression traces out two rays meeting at the origin with this interior angle. The function given by $h_k(z) = (f(z) - f(a_k))^{\frac{1}{\alpha_k}}$ will, in light of the geometric effect of $z \mapsto z^\beta$, trace out a straight line through the origin. By the symmetry principle h_k is regular at the point $z = a_k$, and $h_k(a_k) = 0$, so we may further factor it as $h_k(z) = (z - a_k)h_1(z)$ for h_1 regular at a_k. We get

$$f(z) = f(a_k) + (z - a_k)^{\alpha_k} h_1(z)^{\alpha_k} \tag{3.7}$$

and compute

$$\frac{f''(z)}{f'(z)} = -\frac{1 - \alpha_k}{z - a_k} + k(z) \tag{3.8}$$

for k regular at $z = a_k$.

Thus $z \mapsto g(z) + \frac{1-\alpha_k}{z-a_k}$ is regular at a_k and we can repeat for the other values of k to get

$$g_1(z) = g(z) + \sum_{k=1}^{n} \frac{1 - \alpha_k}{z - a_k} \tag{3.9}$$

where the so constructed g_1 is regular for all $a_1, ..., a_n$, and in fact for all of $\mathbb{C}$. From the earlier discussion we have that it is also bounded, and so Liouville's theorem gives us that it is constant.

To see that this constant is zero, we can use the assumption that none of the a_k is the point at infinity, to expand

$$f(z) = f(\infty) + c_1 z^{-1} + c_2 z^{-2} + ... \tag{3.10}$$

and so differentiating term-wise we see that f' and f'' have respectively second and third order zeros at $z = \infty$, so $g(\infty) = 0$. Furthermore, for all k, $z \mapsto (z - a_k)^{-1}$ vanishes as $z \to \infty$. We conclude that $g_1(\infty) = 0$.

We are left to integrate the expression

$$g(z) = \frac{f''(z)}{f'(z)} = -\sum_{k=1}^{n} \frac{1 - \alpha_k}{z - a_k} \tag{3.11}$$

which can be aided significantly by rewriting $\frac{f''}{f'} = (\ln f')'$ and produces the promised formula with constants to be determined. □

A direct generalization of the straight edges of a polygonal domain is to allow circular arcs. We will call the boundary piecewise circular and the domains they enclose will be of great importance in chapter 5. We can again talk about the Riemann map from the upper half-plane onto such a shape, and identify $a_1, ..., a_n$ as the preimages of the vertices, just as before. The circular arcs meet at the vertices and form interior angles $\pi\alpha_1, ..., \pi\alpha_n$.

Lemma 3.7 *Consider $f : \mathbb{H} \to \tilde{P}$ a Riemann map from the upper half-plane to the piecewise circular polygon. The Schwarzian $\mathcal{S}[f]$ of f satisfies the following relation:*

$$\mathcal{S}[f] = \frac{1}{2} \sum_{k=1}^{n} \frac{1 - \alpha_k^2}{(z - a_k)^2} + \sum_{k=1}^{n} \frac{\beta_k}{z - a_k} \tag{3.12}$$

for some real constants $\beta_k \in \mathbb{R}$.

Proof Since f is conformal in $\mathbb{H}$ and has non-vanishing derivative at all points in $\mathbb{R}$ except possibly $a_1, ..., a_n$, we have that $\mathcal{S}[f]$ is regular in $\overline{\mathbb{H}} \backslash \{a_1, ..., a_n\}$.

Now consider the circular arc $\gamma_k = f([a_k, a_{k+1}])$. There is a suitable Möbius transformation M_k that maps γ_k onto an interval in $\mathbb{R}$, in particular $M_k(\gamma_k) \subset \mathbb{R}$, and the real-valuedness extends to its Schwarzian. By Möbius invariance $\mathcal{S}[M_k \circ f] = \mathcal{S}[f]$, so we have that the real-valuedness holds also for $\mathcal{S}[f]$ on (a_k, a_{k+1}). Since k was arbitrary, $\mathcal{S}[f]$ is real-valued on $\mathbb{R} \backslash \{a_1, ..., a_n\}$.

The next step is to understand the singularities at the points $a_1, ..., a_n$. To do so we pick again an arbitrary $k \in \{1, ..., n\}$ and focus on the preimage of the k'th vertex $f(a_k)$. The claim is that for $0 < \alpha_k < 1$, there is a Möbius transformation T_k which takes $f(a_k)$ to zero and transforms the arcs $\gamma_{k-1} = f([a_{k-1}, a_k])$ and $\gamma_k = f([a_k, a_{k+1}])$ into straight lines. These straight lines will still meet at an angle of $\pi\alpha_k$ at the vertex $f(a_k)$, since T_k is conformal.

To see this, note that $f(a_k)$ can first be mapped to zero by an affine transformation. Then, for the straightening of the circular arcs, note that both of these

γ_{k-1} and γ_k lie on full circles C_{k-1} and C_k. If C_{k-1} and C_k are not tangent at $f(a_k)$, they have a second intersection point z_k. Mapping via Möbius T_k, $z_k \mapsto \infty$ and fixing 0 transforms both circles into straight lines. (If they are tangent, one obtains two parallel lines by a limiting version of the same construction.)

Since $\mathcal{S}[T_k \circ f] = \mathcal{S}[f]$ we may just as well assume, for purposes of understanding the Schwarzian, that f maps the vertex $f(a_k)$ to zero and that the arcs meeting there are straight lines. In that case f takes the form $f(z) = (z - a_k)^{\alpha_k} f_1(z)$ where $f_1(z)$ is regular at $z = a_k$, does not vanish there and is real for z real.

We compute

$$\mathcal{S}[f] = \frac{1}{2}\frac{1-\alpha_k^2}{(z-a_k)^2} + \frac{\beta_k}{z-a_k} + f_2(z) \tag{3.13}$$

with f_2 regular at $z = a_k$.

Here, due to real-valuedness of f_1 near a_k, we have that

$$\beta_k = \frac{1-\alpha_k^2}{\alpha_k}\frac{f_1'(a_k)}{f_1(a_k)} \tag{3.14}$$

is real valued.

Iterating this analysis for all choices of k and summing we get that

$$H := \mathcal{S}[f](z) - \frac{1}{2}\sum_{k=1}^{n}\frac{1-\alpha_k^2}{(z-a_k)^2} - \sum_{k=1}^{n}\frac{\beta_k}{z-a_k} \tag{3.15}$$

is regular at $a_1, ..., a_n$ and hence in view of the initial arguments, regular for all of $\mathbb{R}$. Note also that by real-valuedness of Schwarzian on $\mathbb{R}$ and of the parameters, this expression is also real valued on $\mathbb{R}$.

By the local expansion at each a_k, the singularities of $\mathcal{S}[f]$ are exactly cancelled, hence H is holomorphic near each a_k.

By the reflection argument on each interval (a_k, a_{k+1}), $\mathcal{S}[f]$ (and therefore H) extends holomorphically across $\mathbb{R} \setminus \{a_1, \dots, a_n\}$ and takes real values there. Thus H extends to an entire function on $\mathbb{C}$.

Since all prevertices a_k are finite, the point ∞ lies in the interior of one boundary interval of $\partial\mathbb{H}$ whose image under f is a single circular side of $\tilde{P}$.

Reflecting across the supporting circle of that side shows that f extends holomorphically across ∞ (as a point of $\widehat{\mathbb{C}}$).

Equivalently, $g(w) := f(1/w)$ is holomorphic near $w = 0$ with $g'(0) \neq 0$. Because $\iota(z) = 1/z$ is Möbius, we have

$$\mathcal{S}[f](z) = \mathcal{S}[g](1/z)\,(\iota'(z))^2 = \mathcal{S}[g](1/z)\,z^{-4}, \tag{3.16}$$

so $\mathcal{S}[f](z) = O(z^{-4})$ as $z \to \infty$.

Since the rational terms above are $O(1/z)$, it follows that $H(z) = O(1/z)$ as $z \to \infty$. Hence H is bounded on $\mathbb{C}$, and Liouville's theorem implies H is constant, say $H \equiv 0$.

Therefore

$$\mathcal{S}[f](z) = \frac{1}{2}\sum_{k=1}^{n} \frac{1-\alpha_k^2}{(z-a_k)^2} + \sum_{k=1}^{n} \frac{\beta_k}{z-a_k}. \tag{3.17}$$

The parameters $\beta_1, ..., \beta_n$ are not independent however. Using the Laurent expansion at infinity, we see that the coefficients of order zero to three vanish.

On the other hand, we can expand (3.17) in a Laurent series close to $z = \infty$. We have

$$\frac{1}{z-a_k} = \frac{1}{z}\frac{1}{1-a_k/z} = \frac{1}{z}\sum_{m\geq 0}(a_k z^{-1})^m \tag{3.18}$$

valid for $|z| > |a_k|$, which will be true for all k provided we are close to ∞. We also have

$$\frac{d}{dz}\frac{1}{z-a_k} = -\frac{1}{(z-a_k)^2} \tag{3.19}$$

and can get a series representation of the right hand side by arguing that the left hand differentiation can be applied term by term on the series representation computed in the preceding equation. This is a consequence of the uniform convergence of the series for $\frac{1}{z-a_k}$ on $|z| > \max|a_k|$.

Scaling by the appropriate factors to get the right expansion for (3.17) and matching the first coefficients, we obtain the relations

$$\sum_{k=1}^{n} \beta_k = 0 \tag{3.20}$$

$$\sum_{k=1}^{n} 2a_k\beta_k + 1 - \alpha_k^2 = 0 \tag{3.21}$$

$$\sum_{k=1}^{n} \beta_k a_k^2 + a_k(1 - \alpha_k^2) = 0 \tag{3.22}$$

which is slightly more than we set out to show. □

Remark: One comment regarding the parallels between lemma 3.6 and lemma 3.7 is that both derive a differential equation for f. In both cases geometric transformations applied to the polygon, normal or piecewise circular, leave certain expressions invariant, namely the pre-Schwarzian and the Schwarzian, and this is what is exploited. In lemma 3.6, we find ourselves in the fortunate position of being able to integrate this equation with little additional work.

Remark: A particular case of interest is that of a piecewiese circular polygon where the circular arcs meet at angles of π (the reason for this will become clear in the upcoming chapters). In this special and very important case, the Schwarzian differential equation reduces to $\mathcal{S}[f](z) = \sum_{k=1}^{n} \frac{\beta_k}{z-a_k}$.

3.2 Quasiconformal geometry

A generalization of the conformal maps is obtained by the following.

Definition 3.8 *A homeomorphism $f : \mathbb{C} \to \mathbb{C}$ is called quasiconformal if it is a $W^{1,2}_{loc}$ solution to*

$$\partial_{\bar{z}} f = \mu(z) \partial_z f \tag{3.23}$$

for some measurable function μ with $\|\mu\|_\infty < 1$.

We can similarly define quasiconformal maps on some domain $D \subset \mathbb{C}$.

The function μ is called the Beltrami coefficient of f, and the associated constant $K := \frac{1+\|\mu\|_\infty}{1-\|\mu\|_\infty} \geq \frac{1+|\mu(z)|}{1-|\mu(z)|} \geq 1$ designates the map f as a K-quasiconformal map. A 1-quasiconformal map is conformal. Note also that $\mu(z) \leq \frac{K-1}{K+1} < 1$ a.e. and that $J_f = \det Df = |f_z|^2 - |f_{\bar{z}}|^2 = |f_z|^2(1 - |\mu|^2) > 0$, so f is sense-preserving.

We will use repeatedly that we can find, given Beltrami differential, a corresponding quasiconformal map.

Theorem 3.9 *Let $D \subset \mathbb{C}$ be a domain and $\mu \in L^\infty(D)$ with $\|\mu\|_\infty < 1$. Then there exists a quasiconformal homeomorphism $f : D \to D'$ onto a domain $D' \subset \mathbb{C}$ whose Beltrami coefficient agrees with μ for almost every $z \in D$.*

The proof consists of first examining the case where $\mu \in C_0^\infty$ and getting a locally injective solution to the Beltrami equation. This solution is then upgraded to a globally injective one. Finally, the general case for $\mu \in L_1^\infty$ (subscript denoting the unit ball) is argued by means of approximation with $\mu_n \in C_0^\infty$.

Now for some proper simply connected domain D_1 there is a quasiconformal homeomorphism $f : D_1 \to f(D_1)$. The relevant topological properties are inherited in the image and so there is for any proper, simply connected domain D_2 a conformal map $g : f(D_1) \to D_2$, and then by invoking lemma 3.16 we see that $\mu_{g \circ f} = \mu_f$ so that we have existence of a quasiconformal homeomorphism $D_1 \to D_2$.

Quasiconformal maps, particularly those on $\mathbb{H}$ and extended to $f(\infty) = \infty$, have a close relationship with certain homeomorphisms $\hat{\mathbb{R}} \to \hat{\mathbb{R}}$.

Definition 3.10 *An increasing homeomorphism $h : \mathbb{R} \to \mathbb{R}$ (equivalently, an orientation-preserving homeomorphism of $\hat{\mathbb{R}}$ fixing ∞) is called k-quasisymmetric if there exists $k \geq 1$ such that for all $x \in \mathbb{R}$ and $t > 0$,*

$$\frac{1}{k} \leq \frac{h(x+t) - h(x)}{h(x) - h(x-t)} \leq k. \tag{3.24}$$

The set of quasisymmetric homeomorphisms is that for which an equality of this type holds globally for some $1 \leq k < \infty$. The smallest such k (for a given function h), is called the quasisymmetry constant of h.

The definition can be made more general, to work with homeomorphisms between metric spaces $(X, d_X) \to (Y, d_Y)$, more on this below.

It turns out the boundary map of quasiconformal homeomorphisms $\mathbb{H} \to \mathbb{H}$ is always a quasisymmetric map. A further reason for our interest in this class of functions is the following Beurling-Ahlfors extension. A good reference is [5].

Theorem 3.11 *Let $h : \mathbb{R} \to \mathbb{R}$ be a quasisymmetric homeomorphism. Then there exists a quasiconformal self-mapping of $\mathbb{H}$ with boundary values given by h and such that the maximal dilatation K depends only on k.*

We will not give a detailed proof of this theorem but will comment briefly on the technique used and the intuition behind it.

The proof is constructive, in the sense that it gives us a clear recipe for f, namely we start with the candidate

$$f(z) = \frac{1}{2}\int_0^1 (h(x+ty) + h(x-ty))dt + i\int_0^1 (h(x+ty) - h(x-ty))dt \tag{3.25}$$

and then show that it satisfies the properties we seek. A first sanity check is to look at $z \in \mathbb{R}$, where $y = 0$, and it is immediately clear that the candidate f agrees with h there.

For $z \in \mathbb{H}$, it is instructive to think about the structure of the candidate: There are two integrals (for the real and complex part respectively), and the real part is essentially a kind of averaging operation

$$\frac{1}{2}\int_0^1 (h(x+ty) + h(x-ty))dt = \frac{1}{2}\int_{-1}^1 h(x+ty)dt \tag{3.26}$$

$$= \frac{1}{2y}\int_{-y}^{y} h(x+u)du \tag{3.27}$$

that, as y grows larger and we move away from the boundary $\partial\mathbb{H} = \hat{\mathbb{R}}$ leads to a smoothing effect on the local oscillations of the real part.

A similar analysis gives that the imaginary part has a magnitude which can roughly be described as "the average of the symmetric increment of h":

$$\int_0^1 (h(x+ty) - h(x-ty))dt = \frac{1}{y}\int_0^y (h(x+s) - h(x-s))ds \tag{3.28}$$

which also has a smoothing effect, and importantly goes to infinity as $y \to \infty$.

This is not a of course not a proof of the dependence of the quasiconformal constant K on the quasisymmetric constant k, but makes the global dependence $K = K(k)$ believable.

3.2.1 Spaces of quasisymmetric maps

We collect here some useful results on quasisymmetric maps, specifically certain families of such maps with prescribed points. The main reference from which we adapt is [16].

To leverage results from that reference and tie them to quasisymmetric maps in definition 3.10, we work in the setting of homeomorphisms $\hat{\mathbb{R}} \to \hat{\mathbb{R}}$ and endow $\hat{\mathbb{R}}$ with the spherical metric d.

A homeomorphism $h : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$ is called η-quasisymmetric (with respect to the spherical metric d) if there exists a homeomorphism $\eta : [0, \infty) \to [0, \infty)$ such that for all distinct $x, a, b \in \hat{\mathbb{R}}$,

$$\frac{d(h(x), h(a))}{d(h(x), h(b))} \leq \eta\left(\frac{d(x, a)}{d(x, b)}\right). \tag{3.29}$$

Consider the control homeomorphism η and the family $\mathcal{F}$ of η-quasisymmetric homeomorphisms $\hat{\mathbb{R}} \to \hat{\mathbb{R}}$, for which it holds that all $f \in \mathcal{F}$ have $d(f(a), f(b)) < M$. Then [16] show that the family $\mathcal{F}$ is equicontinuous on $\hat{\mathbb{R}}$, compare their theorem 3.4.

They [16] also show, see their theorem 3.7, that for $f_n : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$ a sequence of η-quasisymmetric embeddings converging pointwise to a function $f : \hat{\mathbb{R}} \to \hat{\mathbb{R}}$, one has either that f is constant, or that f is an η–quasisymmetric embedding.

Moreover, the convergence $f_n \to f$ is uniform on every compact subset of $\hat{\mathbb{R}}$. Since $\hat{\mathbb{R}}$ is compact, the convergence is uniform on $\hat{\mathbb{R}}$.

From this we deduce the compactness under point constraints, in the following precise sense:

Lemma 3.12 *Fix a control function η and marked points $x_1, ..., x_n$ and $y_1, ..., y_n$, all in $\mathbb{R}$, where $x_i \neq x_j$ and $y_i \neq y_j$ when $i \neq j$, $n \geq 3$ (actually $n \geq 2$ will do). Let*

$$QS_\eta(x, y) = \{h : \hat{\mathbb{R}} \to \hat{\mathbb{R}} \ \eta\text{–}QS \text{ embedding} : \ h(x_k) = y_k, \ k = 1, \ldots, n\}. \tag{3.30}$$

Then $QS_\eta(x, y)$ is compact in the topology of uniform convergence on $\hat{\mathbb{R}}$.

The proof relies on Arzela-Ascoli theorem, referenced below in a slightly adapted form from [9].

Theorem 3.13 (Arzela-Ascoli) *Consider a compact Hausdorff space X and a metric space (Y, d), with $C(X, Y)$ the space of continuous functions between them, endowed with the topology of uniform convergence. A set $\mathcal{F} \subset C(X, Y)$ has compact closure in $C(X, Y)$ if and only if*

- *$\mathcal{F}$ is equicontinuous.*
- *$\mathcal{F}_a = \{f(a) : f \in \mathcal{F}\}$ has compact closure for each $a \in X$*

Indeed, pick $i \neq j$ and set $a := x_i$, $b := x_j$ and $M := d(y_i, y_j) > 0$. For every $h \in QS_\eta(x, y)$ we have $d(h(a), h(b)) = M$. Hence, by [16], the family $QS_\eta(x, y)$ is equicontinuous.

Because $\hat{\mathbb{R}}$ is compact, for each $a \in \hat{\mathbb{R}}$ the set $QS_\eta(x, y)_a \subset \hat{\mathbb{R}}$ has compact closure. Thus Arzela-Ascoli implies that $QS_\eta(x, y)$ has compact closure in $C(\hat{\mathbb{R}}, \hat{\mathbb{R}})$ (with the uniform topology).

Finally, $QS_\eta(x,y)$ is closed. Indeed, let $h_n \in QS_\eta(x,y)$ and assume $h_n \to h$ uniformly on $\hat{\mathbb{R}}$. Then $h_n \to h$ pointwise and $h(x_k) = \lim_{n\to\infty} h_n(x_k) = y_k$ for all k. By [16], h is either constant or an η-quasisymmetric embedding. Since $h(x_i) = y_i \neq y_j = h(x_j)$, h is not constant. Therefore $h \in QS_\eta(x,y)$.

We can conclude the compactness of $QS_\eta(x,y)$, so lemma 3.12 holds.

3.3 Teichmüller theory

Teichmüller theory will be important to describe and analyze Loewner energy, since problems in the latter area can be formulated into more abstract language using the former.

To get there, we first need to cover some basic theory on the universal Teichmüller space, namely the two main definitions, its structure, relation to curves and weldings, the universal Liouville action and finally some vector fields on it. We will then see how these vector fields are involved in a first variation formula for the action, and by extension, the Loewner energy.

3.3.1 Models of Teichmüller space

We follow [14] in introducing two models A and B of the universal Teichmüller space. In both descriptions, we start with the space of bounded Beltrami differentials $L^\infty(\mathbb{D}^*)$ and the unit ball $L^\infty(\mathbb{D}^*)_1$. Note that this identification is done over an implicit choice of coordinates, namely z and $\bar{z}$. The Beltrami differential itself is not an L^∞-function but its coefficient for given coordinate system is. Note also that the Beltrami differentials we consider here are defined on $\mathbb{D}^*$.

The next step is to use two different extensions of $\mu \in L^\infty(\mathbb{D}^*)$ to all of $\mathbb{D}$, namely for model A,

$$\mu_A(z) = \overline{\mu(1/\bar{z})}\frac{z^2}{\bar{z}^2} \quad \text{for } z \in \mathbb{D} \tag{3.31}$$

and for the model B,

$$\mu_B(z) = 0 \quad \text{for } z \in \mathbb{D}. \tag{3.32}$$

In each of these cases we then consider the unique quasiconformal map $\mathbb{C} \to \mathbb{C}$, subject to certain normalization constraints, denoted ω_μ for Model A and ω^μ for Model B, recalling theorem 3.9.

In model A the normalization constraint is that the map fixes -1, $-i$ and 1, whereas in model B we insist instead on normalization of the conformal part that is defined inside of $\mathbb{D}$, by assigning for $f = \omega^\mu|_\mathbb{D}$ the values $f(0) = 0$, $f'(0) = 1$ and $f''(0) = 0$.

What is the geometric effect on the resulting normalized maps ω_μ and ω^μ? The first claim is that the model A map actually fixes the domains $\mathbb{D}$ and $\mathbb{D}^*$, as well as the unit circle S^1.

The model B does in general not fix $\mathbb{D}$, $\mathbb{D}^*$ amd S^1, but instead takes S^1 to a quasicircle, more on this below.

With these normalized mappings in place we are finally in a position to define the universal Teichmüller space $T(1)$:

Definition 3.14 *(Universal Teichmüller space definition A) $T(1) = L^\infty(\mathbb{D}^*)_1/\sim$ where $\sim$ is the equivalence relation given by $\mu \sim \nu$ iff $\omega_\mu|_{S^1} = \omega_\nu|_{S^1}$.*

Definition 3.15 *(Universal Teichmüller space definition B) $T(1) = L^\infty(\mathbb{D}^*)_1/\sim$ where $\sim$ is the equivalence relation given by $\mu \sim \nu$ iff $\omega^\mu|_\mathbb{D} = \omega^\nu|_\mathbb{D}$.*

In both instances the equivalence relation $\sim$ (or perhaps more accurately $\sim_A$ and $\sim_B$) checks for equality of restricted versions of the resulting quasiconformal maps, either to the unit circle or to the conformal part inside the unit disk.

3.3.2 Group structure

Recall the section 3.2 on quasiconformal geometry. It turns out that the quasiconformal maps on $\mathbb{C}$ form a group. From the perspective of the Beltrami coefficient we have:

Lemma 3.16 *Let $f, g : \mathbb{C} \to \mathbb{C}$ be quasiconformal maps with Beltrami coefficients μ_f and μ_g respectively. Then $h = g \circ f$ is quasiconformal with Beltrami coefficient*

$$\mu_h = \frac{\mu_f + (\mu_g \circ f) \cdot \theta_f}{1 + \overline{\mu_f} \cdot (\mu_g \circ f) \cdot \theta_f} \tag{3.33}$$

where $\theta_f = \frac{\overline{f_z}}{f_z}$.

Proof Introduce $w = f(z)$. We have the chain rules

$$h_z = g_w(w, \bar{w}) w_z + g_{\bar{w}}(w, \bar{w})(\bar{w})_z \tag{3.34}$$

$$h_{\bar{z}} = g_w(w, \bar{w}) w_{\bar{z}} + g_{\bar{w}}(w, \bar{w})(\bar{w})_{\bar{z}} \tag{3.35}$$

and the relations $(\bar{w})_z = \bar{f}\bar{z}$ and $(\bar{w})_{\bar{z}} = \bar{f}_z$. By successively unrolling definitions of the Beltrami coefficients, we get

$$h_z = g_w \circ f f_z(1 + (\mu_g \circ f)\bar{\mu_f}\theta_f) \tag{3.36}$$
$$h_{\bar{z}} = g_w \circ f f_z(\mu_f + (\mu_g \circ f)\theta_f) \tag{3.37}$$

and taking the ratio we see that the $g_w \circ f f_z$ factor cancels, which gives the result. This expression is well-defined, since the denominator is never zero: $|\mu_f|, |\mu_g| < 1$ and $|\theta_f| = 1$. □

The previous result allows us to introduce the group structure for $L^\infty(\mathbb{D}^*)_1$, defining it via composition of the associated quasiconformal mappings.

$$\lambda = \nu * \mu^{-1}, \tag{3.38}$$
$$\omega_\lambda = \omega_\nu \circ \omega_\mu^{-1}, \tag{3.39}$$
$$\mu * \mu^{-1} = 0. \tag{3.40}$$

It projects to $T(1)$ via $[\nu] * [\mu] = [\nu * \mu]$.

The composition rule for the inverse is then given explicitly by (compare lemma 3.16)

$$\lambda = \left(\frac{\nu - \mu}{1 - \bar{\mu}\nu} \frac{(\omega_\mu)_z}{(\overline{\omega_\mu})_{\bar{z}}} \right) \circ \omega_\mu^{-1}. \tag{3.41}$$

3.3.3 Weldings and curves

Recall the definition of the Loewner energy $I^L(\gamma)$ of Jordan curves $\gamma \subset \hat{\mathbb{C}}$ from chapter 2. A subset of the Jordan curves are the quasicircles, realized as images of the unit circle under a quasiconformal map of the plane. Taking the model B quasiconformal map ω^μ and applying it we get a quasicircle $\omega^\mu(S^1)$. In the reference [5] it is shown that quasicircles can be realized by quasiconformal maps which are conformal in $\mathbb{D}$, which is precisely the setup in the model B.

Hence, there is a bijection between the universal Teichmüller space and the space of quasicircles modulo Möbius transformations (recall the normalization in model B).

It is a nontrivial theorem that the set of Jordan curves with finite Loewner energy coincides with the Weil-Petersson quasicircles. [18] In the Takhtajan–Teo Hilbert-manifold picture of $T(1)$, these correspond to $T_0(1)$, the connected component of the identity. [14]

For a Jordan curve γ separating the extended complex plane $\hat{\mathbb{C}}$ into two components Ω and Ω^*, the inside and outside of the loop with respect to the orientation respectively, there are Riemann maps $f : \mathbb{H} \to \Omega$ and $g : \mathbb{H}^* \to \Omega^*$, see theorem 3.5. By Caratheodory's theorem, for Jordan domains these extend homeomorphically to the boundary.

Defining $h = g^{-1} \circ f|_{\mathbb{R}}$, implicitly using the extensions of f and g to the boundary, we obtain the so-called conformal welding. From the Beurling-Ahlfors extension theorem 3.11, the subclass of quasisymmetric homeomorphisms stands in correspondence with quasiconformal maps, and hence with quasidiscs.

In the Teichmüller picture this enters through the model A map

$$[\mu] \mapsto \omega_\mu|_{S^1} \tag{3.42}$$

creating a bijection $T(1) \to \text{Möb}(S^1)\backslash Homeo_{qs}(S^1)$.

The Weil-Petersson class quasicircles have their matching WP quasisymmetric homeomorphisms under welding; in particular, one characterization on the homeomorphism side is absolute continuity with $\log h' \in H^{1/2}$. [12].

3.3.4 Universal Liouville action

The following functional on $T_0(1)$ is of intrinsic interest on its own, but we record the definition and the following theorem simply for what it will allow us to infer about Loewner energy.

Definition 3.17 *The universal Liouville action is the function $S_1 : T_0(1) \to \mathbb{R}$ defined by*

$$S_1([\mu]) := \int_{\mathbb{D}} \left| \frac{f''}{f'}(z) \right|^2 dz + \int_{\mathbb{D}^*} \left| \frac{g''}{g'}(z)dz \right|^2 dz - 4\pi \log |g'(\infty)| \tag{3.43}$$

Note the structure with two L^2-style integrals of pre-Schwarzians and a normalization term at the end.

The following theorem from [18] establishes a connection between the Loewner energy and the universal Liouville action:

Theorem 3.18 *For γ a bounded Jordan curve, it holds that the Loewner energy is finite $I^L(\gamma) < \infty$ if and only if γ is a Weil-Petersson quasicircle, and furthermore we have the formula*

$$I^L(\gamma) = \frac{S_1([\gamma])}{\pi} \tag{3.44}$$

3.3.5 Bers embedding and Bers vector fields

At the very real risk of needlessly bogging ourselves down in functional analysis that we will not directly use for the remainder, I will here try to give some explanation for the derivative in the variational formula we are about to see.

We start with the Bers embedding. Introduce

$$A_\infty(\mathbb{D}) := \{\varphi \text{ holomorphic in } \mathbb{D} : \|\varphi\|_\infty := \sup_{z\in\mathbb{D}} (1-|z|^2)^2|\varphi(z)| < \infty\}. \tag{3.45}$$

Using the Schwarzian derivative, the Bers embedding is the map

$$\beta : T(1) \longrightarrow A_\infty(\mathbb{D}) \qquad \beta([\mu]) := \mathcal{S}(w_\mu|_{\mathbb{D}}). \tag{3.46}$$

Just as the universal Teichmüller space can be modelled as equivalence classes of Beltrami differentials as we have seen in the two models A and model B, the tangent space at the origin is similarly modelled by the space of bounded harmonic Beltrami differentials in $\mathbb{D}^*$.

Using $d^2z = dx \wedge dy$, we first define the integrable holomorphic functions on $\mathbb{D}^*$:

$$A_1(\mathbb{D}^*) := \{\varphi \text{ holomorphic on } \mathbb{D}^* : \iint_{\mathbb{D}^*} |\varphi| d^2z < \infty\} \tag{3.47}$$

This set is used as a detector class to define the subspace of infinitesimally trivial Beltrami differentials, namely

$$\mathcal{N}(\mathbb{D}^*) := \{\mu \in L^\infty(\mathbb{D}^*) : \iint_{\mathbb{D}^*} \mu\varphi d^2z = 0 \text{ for all } \varphi \in A_1(\mathbb{D}^*)\}. \tag{3.48}$$

We use the following set of holomorphic functions

$$A_\infty(\mathbb{D}^*) := \{\varphi \text{ holomorphic in } \mathbb{D}^* : \|\varphi\|_\infty = \sup_{z\in\mathbb{D}^*} |(1-|z|^2)^2\varphi(z)| < \infty\}. \tag{3.49}$$

to construct the space of bounded harmonic Beltrami differentials in $\mathbb{D}^*$

$$\Omega^{-1,1}(\mathbb{D}^*) := \{\mu \in L^\infty(\mathbb{D}^*) : \mu(z) = (1-|z|^2)^2\overline{\varphi(z)}, \quad \varphi \in A_\infty(\mathbb{D}^*)\} \tag{3.50}$$

which turns out to be a Banach space.

There is a decomposition

$$L^\infty(\mathbb{D}^*) = \mathcal{N}(\mathbb{D}^*) \oplus \Omega^{-1,1}(\mathbb{D}^*) \tag{3.51}$$

which leads to an identification of the tangent space at the identity $T_0T(1) = L^\infty/\mathcal{N}(\mathbb{D}^*)$ with $\Omega^{-1,1}(\mathbb{D}^*)$.

By using the group structure introduced previously in section 3.3.2 we can represent tangent vectors at other points than the identity by some $\nu \in \Omega^{-1,1}(\mathbb{D}^*)$. We skip the fine details and continue the analysis at the origin.

Connecting the tangent space identification with the Bers picture, we note that the differential at the origin provides a linear map

$$D_0\beta : \Omega^{-1,1}(\mathbb{D}^*) \longrightarrow A_\infty(\mathbb{D}). \tag{3.52}$$

Given a tangent direction $\nu \in \Omega^{-1,1}(\mathbb{D}^*)$, we set

$$\varphi := D_0\beta(\nu) \in A_\infty(\mathbb{D}). \tag{3.53}$$

Intuitively, φ is the infinitesimal change of the Schwarzian $S(w_{\epsilon\nu}|_{\mathbb{D}})$ at $\epsilon = 0$, i.e.

$$\varphi = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} \beta([\epsilon\nu]) = \frac{d}{d\epsilon}\bigg|_{\epsilon=0} S(w_{\epsilon\nu}|_{\mathbb{D}}). \tag{3.54}$$

In particular, in Bers coordinates near the origin, the direction ν is represented by the vector φ in the model space $A_\infty(\mathbb{D})$.

Now we can define vector fields associated to ν by insisting that the pushforward under the appropriate chart equals this φ, i.e. with h_0 the coordinate chart at the origin

$$Dh_0(\frac{\partial}{\partial \epsilon_\nu}) = \varphi = D_0\beta(\nu) \tag{3.55}$$

on the chart U_0. Starting from $\frac{\partial}{\partial \epsilon_\nu}$ we get real vector fields

$$\frac{\partial}{\partial t_\nu} = \frac{\partial}{\partial \epsilon_\nu} + \frac{\partial}{\partial \overline{\epsilon_\nu}}, \tag{3.56}$$

with the relations

$$\frac{\partial}{\partial \epsilon_\nu} = \frac{1}{2}\left(\frac{\partial}{\partial t_\nu} - i\,\frac{\partial}{\partial t_{i\nu}}\right), \qquad \frac{\partial}{\partial \overline{\epsilon_\nu}} = \frac{1}{2}\left(\frac{\partial}{\partial t_\nu} + i\,\frac{\partial}{\partial t_{i\nu}}\right). \tag{3.57}$$

These are the fundamental building blocks needed to understand the derivative in the next section.

Chapter 4

Quasiconformal deformation

In this chapter we will introduce the technique of quasiconformal deformation, which takes perhaps its clearest expression in [13]. The general idea there can however just as well be expressed in the context of the universal Teichmüller space, and since a goal for the final chapters is to connect the welding and curve perspectives, it is necessary to introduce this more abstract point of view as well.

4.1 A variational formula for $T_0(1)$

In [15] it is proved:

Theorem 4.1 *We have the following formula for the first variation of the function S_1:*

$$\frac{\partial S_1}{\partial \epsilon_\eta}([\nu]) = 2\int_{\mathbb{D}^*} \mathcal{S}[g_\nu](z)\eta(z)d^2z \tag{4.1}$$

Due to theorem 3.18 this result can be rephrased in terms of the Loewner energy, and even proved by means of more direct methods. The following is a result from [13] which specializes to the case of curves (as representatives of points in $T_0(1)$) and with direction of the derivative expressed via a quasiconformal deformation given by an infinitesimal Beltrami differential, applied to this curve.

Theorem 4.2 *Let γ be a Weil-Petersson quasicircle. For $\mu \in L^\infty(\mathbb{C})$ a Beltrami differential with compact support in $\hat{\mathbb{C}}\backslash\gamma$ and for $0 < t \in \mathbb{R}$ small enough for $\|t\mu\|_\infty < 1$, consider any quasiconformal mapping $\omega^{t\mu} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$. Define the deformed curve $\gamma^t = \omega^{t\mu}(\gamma)$. Then we have the following first variation formula for the Loewner energy:*

$$\frac{d}{dt}|_{t=0} I^L(\gamma^t) = -\frac{4}{\pi} Re \left[\int_\Omega \mathcal{S}[f^{-1}](z)\mu(z) d^2z + \int_{\Omega^*} \mathcal{S}[g^{-1}](z)\mu(z) d^2z \right] \quad (4.2)$$

The apparent discrepancies between theorem 4.1 and theorem 4.2 are due to whether the Beltrami differential is taken to be supported on one or two sides of the curve as well as the difference between $\frac{d}{dt}$ with $\omega^{t\mu}$ and $\frac{\partial}{\partial \epsilon_\eta}$, where for the latter, some care needs to be taken to keep track of the fact that $\epsilon \in \mathbb{C}$. We now give an account of these differences and derive the latter from the former.

Claim: theorem 4.2 can be viewed as a restatement of theorem 4.1.

Indeed, the deformation induced by the infinitesimal Beltrami differential $t\mu$ corresponds to some Bers coordinate tangent vector $\eta \in \Omega^{-1,1}(\mathbb{D}^*)$, and the reference curve $\gamma^0 = \gamma$, being a quasicircle, corresponds to some $[\nu] \in T_0(1)$. In the geometrically more approachable formulation of theorem 4.2, $t\mu$ is in general taken to be supported on both connected components of $\hat{\mathbb{C}} \backslash \gamma$ (away from the curve itself). To make the connection to theorem 4.1 transparent, we begin with the special case where $t\mu$ is supported in Ω^*, the unbounded connected component of $\hat{\mathbb{C}} \backslash \gamma$.

Using the connection between Loewner energy and the universal Liouville action, as well as their real-valuedness, we get

$$\frac{d}{dt}|_{t=0} I^L(\omega^{t\mu}(\gamma)) = \frac{1}{\pi} \frac{\partial}{\partial t_\eta} S_1([\nu]) \quad (4.3)$$

$$= \frac{1}{\pi} \left(\frac{\partial}{\partial \epsilon_\eta} + \frac{\partial}{\partial \bar{\epsilon}_\eta}\right) S_1([\nu]) \quad (4.4)$$

$$= \frac{1}{\pi} \left(\frac{\partial}{\partial \epsilon_\eta} S_1([\nu]) + \overline{\frac{\partial}{\partial \epsilon_\eta} S_1([\nu])} \right) \quad (4.5)$$

$$= \frac{2}{\pi} \mathrm{Re} \left[\frac{\partial}{\partial \epsilon_\eta} S_1([\nu]) \right] \quad (4.6)$$

$$= \frac{4}{\pi} \mathrm{Re} \left[\int_{\mathbb{D}^*} \mathcal{S}[g_\nu](z)\eta(z) d^2z \right] \quad (4.7)$$

The final step is to perform a change of variables to integrate over the domain Ω^* rather than $\mathbb{D}^*$. For brevity, write $g = g_\nu$ and compute using the Schwarzian chain rule

$$\int_{\mathbb{D}^*} \mathcal{S}[g]\eta d^2 z = \int_{\Omega^*} \mathcal{S}[g] \circ g^{-1} \eta \circ g^{-1} |(g^{-1})'|^2 d^2 w \tag{4.8}$$

$$= \int_{\Omega^*} \mathcal{S}[g] \circ g^{-1} ((g^{-1})_w)^2 \eta \circ g^{-1} \frac{\overline{(g^{-1})_w}}{(g^{-1})_w} d^2 w \tag{4.9}$$

$$= -\int_{\Omega^*} \mathcal{S}[g^{-1}] g_* \eta d^2 w. \tag{4.10}$$

Since $\mu = g_* \eta$, we are done.

For the general case when $t\mu$ is supported in both Ω and Ω^* we can decompose $t\mu = t\mu_\Omega + t\mu_{\Omega^*}$, where both have support restricted according to their subscript. Due to linearity, it is enough to show that the case of μ_Ω, with support in Ω, can be reduced to the first case.

To this end, let $\mu = \mu_\Omega$ for the remainder of this argument, and note that for $\iota(z) = \frac{1}{z}$, we have $I^L(\iota(\gamma)) = I^L(\gamma)$ for any Jordan curve γ, due to Möbius invariance. We also have that $\iota(\Omega)$ is the unbounded component of $\hat{\mathbb{C}} \setminus \iota(\gamma)$ (since $0 \in \Omega$). Define

$$\tilde{g} = \iota \circ f \circ \iota : \mathbb{D}^* \to \iota(\Omega) \tag{4.11}$$

$$\tilde{\mu} = \iota_* \mu \tag{4.12}$$

$$\tilde{\omega}^{t\tilde{\mu}} = \iota \circ \omega^{t\mu} \circ \iota \tag{4.13}$$

We have, using again the Schwarzian chain rule

$$\frac{d}{dt}|_{t=0} I^L(\omega^{t\mu}(\gamma)) = \frac{d}{dt}|_{t=0} I^L(\tilde{\omega}^{t\tilde{\mu}}(\iota(\gamma))) \tag{4.14}$$

$$= -\frac{4}{\pi} \mathrm{Re} \left[\int_{\iota(\Omega)} \mathcal{S}[\tilde{g}^{-1}] \tilde{\mu} d^2 w \right] \tag{4.15}$$

$$= -\frac{4}{\pi} \mathrm{Re} \left[\int_{\Omega} \mathcal{S}[f^{-1}] \mu d^2 z \right] \tag{4.16}$$

Putting these ingredients together, we recover theorem 4.2 from theorem 4.1.

4.2 Adaptation to the welding

As we saw in the section on curves and weldings section 3.3.3 these objects can be unified under the umbrella of Teichmüller theory. The fact that the formula in theorem 4.2 can be seen as a consequence of theorem 4.1 suggests that one should be able to obtain a variational formula for weldings with little additional work. This is indeed the case.

One way to look at this is that the curve γ is the image of the unit circle under some Model B Teichmüller map ω^{ν}, i.e.

$$\gamma = \omega^{\nu}(S^1). \tag{4.17}$$

So $\gamma^t = \omega^{t\mu} \circ \omega^{\nu}(S^1)$ and so regarding S^1 as a canonical reference object against which one measures the curve γ^t, we might as well work with the function $\omega^{t\mu} \circ \omega^{\nu}$.

For the welding h on the other hand, we have the relation $h = \omega_{\nu}|_{S^1}$ (working here with S^1 as topologically equivalent to $\hat{\mathbb{R}}$). Now let ψ_1 be a quasisymmetric homeomorphism of S^1 and note that by the Beurling-Ahlfors extension theorem 3.11 there is a quasiconformal extension to $\mathbb{D}^*$ with Beltrami differential μ. Let ψ_t be the quasisymmetric homeomorphism of S^1 with Beltrami differential $t\mu$. Then

$$\begin{aligned} h^t &= \psi_t \circ h & (4.18)\\ &= \omega_{t\mu}|_{S^1} \circ \omega_{\nu}|_{S^1} & (4.19)\\ &= (\omega_{t\mu} \circ \omega_{\nu})|_{S^1} & (4.20) \end{aligned}$$

where the last line is essentially the multiplication defining the group structure of $T(1)$, see section 3.3.2. Here it is clear that S^1 is the natural reference object, being the domain of definition for the weldings.

To relate these two computations, we should check the Beltrami coefficients of the resulting functions of these two computations. To this end, and for notational simplicity, let $\mathrm{Bel}_z(f) = \mu_f(z)$.

Lemma 4.3 *For the choice $\tilde{\mu} = g_*\mu$ we have for any $z \in \mathbb{D}^*$*

$$Bel_z(\omega_{t\mu} \circ \omega_{\nu}) = Bel_z(\omega^{t\tilde{\mu}} \circ \omega^{\nu}) \tag{4.21}$$

where $g = \omega^{\nu} \circ \omega_{\nu}^{-1}$ from conformal welding.

Proof We use the composition formula for quasiconformal maps. For a given $z \in \mathbb{D}^*$

$$\mathrm{Bel}_z(\omega_{t\mu} \circ \omega_\nu) = \frac{\nu(z) + t\mu \circ \omega_\nu(z)\theta_\nu(z)}{1 + \overline{\nu(z)}t\mu \circ \omega_\nu(z)\theta_\nu(z)} \tag{4.22}$$
$$= \frac{\nu(z) + t(\omega_\nu)^*\mu(z)}{1 + \overline{\nu(z)}t(\omega_\nu)^*\mu(z)} \tag{4.23}$$

with $\theta_\nu = \frac{\overline{(\omega_\nu)_z}}{(\omega_\nu)_z}$. We will shortly use the notation $\theta^\nu = \frac{\overline{(\omega^\nu)_z}}{(\omega^\nu)_z}$. To compute the right hand side we make use of the contravariance of the pullback and

$$g_*\mu = (g^{-1})^*\mu \tag{4.24}$$
$$= (\omega_\nu \circ (\omega^\nu)^{-1})^*\mu \tag{4.25}$$
$$= ((\omega^\nu)^{-1})^*\omega_\nu^*\mu \tag{4.26}$$

so that

$$(\omega^\nu)^* g_*\mu = (\omega^\nu)^*((\omega^\nu)^{-1})^*\omega_\nu^*\mu \tag{4.27}$$
$$= \omega_\nu^*\mu \tag{4.28}$$

which allows us to compare left and right hand sides of the statement and compute

$$\mathrm{Bel}_z(\omega^{tg_*\mu} \circ \omega^\nu) = \frac{\nu(z) + tg_*\mu \circ \omega^\nu(z)\theta^\nu(z)}{1 + \overline{\nu(z)}tg_*\mu \circ \omega^\nu(z)\theta^\nu(z)} \tag{4.29}$$
$$= \mathrm{Bel}_z(\omega_{t\mu} \circ \omega_\nu) \tag{4.30}$$

□

So we can in some sense view the quasiconformal deformation of the curve as a postcomposition by a quasisymmetric homeomorphism on the space of weldings. It is essentially the same phenomenon and the same variational formula applies. Letting $\mathcal{C}$ be the set of normalized quasicircles, obtainable via mapping of S^1 by model B maps of the universal Teichmüller space and $\Phi = \text{Möb}(S^1)\backslash\text{Homeo}_{qs}(S^1)$ the set of quasisymmetric homeomorphisms modulo Möbius transformations. Let Weld be the map from a curve to its welding. Then, we can neatly summarize these relations in the following diagram:

Corollary 4.4 *The following diagram commutes:*

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "Weld"] \arrow[d, "\omega^{\epsilon g * \mu}"'] & \Phi \arrow[d, "\psi_{\epsilon\mu}"] \\
\mathcal{C} \arrow[r, "Weld"'] & \Phi
\end{tikzcd}$$

Remark: In the welding case, a deformation supported on the outside of the unit disk shows up as a postcomposition (this is the computation we made above), while a deformation supported on the inside will yield a precomposition by an inverse. An intuitive explanation for this is that we are relabeling the input vs output coordinates to the welding.

Chapter 5

Two optimization problems of the Loewner energy

We follow [19] and introduce two optimization problems for the Loewner energy. As we have seen in chapter 2, a Jordan curve γ has $I^L(\gamma) = 0$ iff γ is a generalized circle. We will now start adding constraints to this optimization problem, and analyze the properties of the minimizers.

5.1 Optimizing the curve

Just as before, for any Jordan curve γ, there is associated a bounded and unbounded component Ω, Ω^*, and conformal maps $f : \mathbb{H} \to \Omega$ and $g : \mathbb{H}^* \to \Omega^*$. It will be clear from context which curve γ we are referring to when working with these objects.

Let $z_1, ..., z_n \in \hat{\mathbb{C}}$ be n distinct points, and consider Jordan curves passing through these points in that order. There are many such curves, but they separate into relative homotopy classes (relative to the collection $z = (z_1, ..., z_n)$), with representative Jordan curves $\{\tau\}$.

Concretely, two Jordan curves $\gamma_1, \gamma_2 : S^1 \to \hat{\mathbb{C}}$ with markings $\gamma_i(p_k^i) = z_k$ are homotopic relative to $z_1, ..., z_n$ if there exists a continuous map $H : S^1 \times [0,1] \to \hat{\mathbb{C}}$ such that $H(\cdot, 0) = \gamma_1$ and $H(\cdot, 1) = \gamma_2$ where every $H(\cdot, s)$ for $s \in [0,1]$ is a Jordan curve with the same z-marking. We then write $\gamma_1 \sim \gamma_2$ relative to z.

We define the collection

$$\mathcal{L}(z, \tau) := \{\gamma \quad z\text{-marked Jordan curve} : \gamma \sim \tau \text{ relative to } z\} \tag{5.1}$$

and consider the optimization problem

$$\inf_{\gamma \in \mathcal{L}(z,\tau)} I^L(\gamma). \tag{5.2}$$

What can we say about existence and uniqueness of solutions to this problem and what qualitative properties do the solution curves have? It is clear from the constraints that curves must pass through the n points $z_1, ..., z_n$ and lie within the homotopy class of τ, but beyond that, what do the segments γ_k between points z_k and z_{k+1} look like?

The segment γ_k lives in a simply connected domain $\Omega_k = \hat{\mathbb{C}} \backslash (\gamma \backslash \gamma_k)$, which can be endowed with a hyperbolic metric.

Geodesy in the domain Ω_k turns out to be important for the characterisation of the minimizers.

Definition 5.1 *A Jordan curve $\gamma \in \mathcal{L}(z,\tau)$ has the geodesic property if for each $k \in \{1, \ldots, n\}$, the arc γ_k between z_k and z_{k+1} (with $z_{n+1} = z_1$) is the hyperbolic geodesic in the simply connected domain*

$$\Omega_k := \hat{\mathbb{C}} \setminus (\gamma \setminus \gamma_k). \tag{5.3}$$

Note that this definition is global in the sense that it is a statement about geodesy of γ_k in a domain Ω_k that clearly depends on the rest of the curve. It turns out that any minimizer of the problem 5.2 must abide by this property.

Theorem 5.2 *If γ minimizes I^L among curves in $\mathcal{L}(z,\tau)$, then it has the geodesic property.*

Proof Let γ be a minimizer and suppose it does not have the geodesic property, in particular there is some segment γ_k which is a chord in (Ω_k, z_k, z_{k+1}) but not a hyperbolic geodesic in Ω_k.

By cutting and pasting (recall eq. (2.8)) we replace γ by $\tilde{\gamma} = (\gamma \backslash \gamma_k) \cup \eta$, where η is the hyperbolic geodesic in Ω_k. It holds that $I^C_{\Omega_k;z_k,z_{k+1}}(\eta) \leq I^C_{\Omega_k;z_k,z_{k+1}}(\gamma_k)$ and with

$$I^L(\tilde{\gamma}) - I^L(\gamma) = I^C_{\Omega_k;z_k,z_{k+1}}(\eta) - I^C_{\Omega_k;z_k,z_{k+1}}(\gamma_k) < 0 \tag{5.4}$$

we obtain a contradiction. $\square$

So a minimizing curve γ must have the geodesic property. On the welding side (recall section 3.3.3) this takes the following form:

Lemma 5.3 *Let $\gamma \in \mathcal{L}(z, \tau)$ have the geodesic property. Then any welding homeomorphism h of γ is piecewise Möbius.*

Proof Associate $\mathbb{C} \backslash \mathbb{R}_-$ with Ω_k via conformal φ_k, with $\varphi_k(\mathbb{R}_+) = \gamma_k$. This is possible since by theorem 5.2 γ_k is the hyperbolic geodesic in Ω_k.

Then we have conformal $\varphi_k|_{\mathbb{H}} : \mathbb{H} \to \Omega$ and $\varphi_k|_{\mathbb{H}^*}\mathbb{H}^* \to \Omega^*$. Since f, g are unique up to Möbius transformations we have for some Möbius maps M_1 and M_2 that

$$f = \varphi_k|_{\mathbb{H}} \circ M_1 \tag{5.5}$$

$$g = \varphi_k|_{\mathbb{H}^*} \circ M_2 \tag{5.6}$$

On $f^{-1}(\gamma_k) = M_1^{-1} \circ \varphi_k|_{\mathbb{H}}^{-1}(\gamma_k) = M_1^{-1}(\mathbb{R}_+)$ the welding map

$$h = g^{-1} \circ f = M_2^{-1} \circ \varphi_k|_{\mathbb{H}^*}^{-1} \circ \varphi_k|_{\mathbb{H}} \circ M_1 \tag{5.7}$$

$$= M_2^{-1} \circ M_1 \tag{5.8}$$

using the continuity of φ_k across $\mathbb{R}_+$. So the welding h is piecewise Möbius.□

Theorem 5.4 *There is a unique minimizer of I^L in $\mathcal{L}(z, \tau)$. It is also the unique Jordan curve in $\mathcal{L}(z, \tau)$ with the geodesic property and finite Loewner energy.*

The existence part of the proof is similar to what we will see below for the welding. For the far trickier uniqueness part, we refer the reader to [2].

5.1.1 Schwarzian associated with the optimal curve

The Schwarzians of the maps f^{-1} and g^{-1} associated with the piecewise geodesic optimal curve in $\mathcal{L}(z, \tau)$ exhibit some interesting structural properties.

Theorem 5.5 *Let γ^* be the solution to the problem* (5.2), *and define the glued function*

$$F = \begin{cases} f^{-1} & \text{on } \Omega \\ g^{-1} & \text{on } \Omega^* \end{cases} \tag{5.9}$$

then the Schwarzian $\mathcal{S}[F]$ extends to a meromorphic function on $\hat{\mathbb{C}}$ and

$$\mathcal{S}[F] = \sum_{k=1}^{n} \frac{c_k}{z - z_k} \tag{5.10}$$

for some constants c_k.

Using the expression of the Schwarzian $\mathcal{S}[F]$, expanding in powers of $\frac{1}{z}$ and using that $\mathcal{S}[F](z) = O(z^{-4})$ at $z = \infty$ and matching coefficients, one gets

$$\sum_{k=1}^{n} c_k = \sum_{k=1}^{n} c_k z_k = \sum_{k=1}^{n} c_k z_k^2 = 0 \tag{5.11}$$

(argument is similar to the end of lemma 3.7).

To establish that the Schwarzian has simple poles, the proof carried out in [7] introduces the concept of a piecewise geodesic pair. This is a chord $\eta_1 \cup \eta_2$ in a domain $(D; a, b)$ with added data ξ, where $\eta_1 \cap \eta_2 = \{\xi\}$ and the two segments are the hyperbolic geodesics in $(D \backslash \eta_2, a, \xi)$ and $(D, \backslash \eta_1, \xi, b)$ respectively.

The canonical setting for this is geodesic pairs in $(\mathbb{D}, e^{i\theta}, -e^{i\theta})$ with common intersection point 0, from which one can translate to e.g. $(\mathbb{H}, 0, \infty)$ with intersection point re^{it} for $r \geq 0$ and $t \in (0, \pi)$, and beyond.

In the work [2] they carry out concrete computations for the setup in $(\mathbb{D}, e^{i\theta}, -e^{i\theta})$ with 0 and then transfer to the case of two neighboring segments γ_k, γ_{k+1} inside the domain $\hat{\mathbb{C}} \backslash (\gamma \backslash (\gamma_k \cup \gamma_{k+1}))$ to determine the simple pole structure. We omit the details but can look forward to determining a relation between the residues c_k and the Loewner energy of the optimal curve γ.

5.2 Optimizing the welding

The condition $I^L = 0$ translates to a generalized circle on the curve side, i.e. the image of $\hat{\mathbb{R}}$ under a Möbius map. For weldings, this translates to the equivalence class of the identity on $\hat{\mathbb{R}}$.

Recalling the inside and outside of the curve Ω and Ω^* with maps $f : \mathbb{H} \to \Omega$ and $g : \mathbb{H}^* \to \Omega^*$ and welding $h = g^{-1} \circ f$, the case $\gamma = \hat{\mathbb{R}}$ gives $h = \text{id}$ and for different choices of f, g we get a Möbius conjugate of h.

Now let $x_1, ..., x_n \in \hat{\mathbb{R}}$, both collections increasingly ordered, be markings on the domain and $y_1, ..., y_n \in \mathbb{R}$ the markings on the codomain of a prospective welding h, i.e. $h(x_k) = y_k$ (compare lemma 3.12 which will become relevant).

Consider the set of quasisymmetric, increasing homeomorphisms of $\hat{\mathbb{R}}$ with these point prescriptions, i.e.

$$\Phi_{x,y} := \{h \in \mathrm{Homeo}_{qs}(\hat{\mathbb{R}}) : h(x_k) = y_k, \quad k = 1, ..., n\} \tag{5.12}$$

Define the Loewner energy of $h \in \Phi_{x,y}$ as the Loewner energy of a Jordan curve γ_h with associated welding h.

To see that this makes sense, we first have to argue that for given h, we can always find γ_h, and secondly, that I^L is constant on elements of $\{\gamma_h\}$ for given fixed h. The first assurance is furnished by the solution to the conformal welding problem, and the second by its particular form: Just as the Loewner energy is Möbius-invariant, so we have a family of $\{\gamma_h\}$ differing by Möbius maps. Furthermore, in the case that γ_h is not Weil-Petersson, $I^L(h) = I^L(\gamma_h) = \infty$.

We consider the following minimization problem:

$$\inf_{h \in \Phi_{x,y}} I^L(h) \tag{5.13}$$

We minimize over a set of homeomorphisms $\Phi_{x,y}$ which has in and of itself a clear geometrical structure (in the slightly naive sense that we can draw pictures of it), but the objective function is really defined in terms of the energy of a curve γ_h. Since the welding and the curve share a complicated relationship, a direct translation to the previous optimization problem we studied does not seem to be the most efficient route. We will however see that there are clear parallels and unifications possible between both problems, and the universal Teichmüller space will be the main avenue for this realization.

The first natural task is to ascertain the existence and uniqueness of the optimization problem in eq. (5.13).

Theorem 5.6 (Existence of optimal welding) *Consider the problem in eq.* (5.13)*, with x, y and $\Phi_{x,y}$ as above. There exists a solution $h = h^* \in \Phi_{x,y}$.*

Before moving on to the proof, we list some needed lemmas.

Lemma 5.7 *$I^L(h) \leq M$ implies that h is η_M-quasisymmetric.*

Proof Since $I^L(h) < \infty$ it is the welding of a Weil-Petersson quasicircle, and in Prop. 2.9 of [11] it is shown that it is in fact a K-quasicircle, with K depending only on $I^L(h)$, hence on the bound M.

But for a K-quasicircle the associated conformal welding is η_K-quasisymmetric, and since K depends on M we may as well write η_M. □

We also assume the following:

Lemma 5.8 *Let (h_n) be a sequence of weldings in $\Phi_{x,y}$ converging uniformly to some h in $\Phi_{x,y}$, all η-quasisymmetric. Then after normalizing by Möbius maps so that the representative curves γ_n and γ have the first three junction points fixed, say to $-1, 0, 1$, it holds that $\gamma_n \to \gamma$ uniformly.*

The following is from [11] lemma 2.12.

Lemma 5.9 *Let (γ_n) be a sequence of simple loops in $\hat{\mathbb{C}}$ with two point normalization $\gamma_n(0) = z_0$, $\gamma_n(e^{i\pi}) = z_2$. If there exists a simple loop γ such that $\gamma_n \to \gamma$ uniformly, then we have the lower semicontinuity*

$$\liminf_{n\to\infty} I^L(\gamma_n) \geq I^L(\gamma) \tag{5.14}$$

Proof (of theorem 5.6)

For convenience, we make use of the notation $m = \inf_{h\in\Phi_{x,y}} I^L(h)$.

We start by arguing that $m < \infty$, as this implies that there is an approximating sequence (h_n) with $\lim_n I^L(h_n) = m$ for which we have eventually that $I^L(h_n) \leq M$, say for large enough $n > N$. Indeed, smooth $h \in \Phi_{x,y}$ are in the WP class and so have finite Loewner energy.

Without loss of generality, assume $I^L(h_n) \leq M$ for all $n \geq 1$, i.e. for the entirety of the chosen approximating sequence. By lemma 5.7, this means that all h_n are η_M-quasisymmetric, i.e. $h_n \in QS_\eta(x, y)$.

From lemma 3.12 we have that h_n is a sequence in a compact subset of $\Phi_{x,y}$, with a subsequence $(h_{n_k})_k$ converging uniformly to some $h^* \in QS_\eta(x, y)$.

By lemma 5.8 we have $\gamma_{h_{n_k}} \to \gamma_{h^*}$ uniformly.

By the lower semicontinuity of Loewner energy lemma 5.9 we have

$$m = \lim_k I^L(h_{n_k}) \geq \liminf_k I^L(h_{n_k}) \tag{5.15}$$

$$\geq I^L(h^*) \tag{5.16}$$

$$\geq m \tag{5.17}$$

which gives existence of the optimal welding h^* for which $I^L(h^*) = m$. □

Just as it is possible to say something about the welding associated with the optimal curve in eq. (5.2), we can say something about the curve associated with the optimal welding from eq. (5.13).

Definition 5.10 *We say that a Jordan curve $\gamma \subset \hat{\mathbb{C}}$ is piecewise circular if it can be written as a concatenation of circular arcs $\gamma = \gamma_1 \cup ... \cup \gamma_n$, where γ_k is the image of $\mathbb{R}_+$ under a Möbius map of $\hat{\mathbb{C}}$.*

In [19] it is shown that the curves associated with the optimal welding satisfy this property.

Lemma 5.11 *Let $h \in \Phi_{x,y}$ be an optimal welding. Then the associated curve γ_h is piecewise circular and in $C^{1,1}$.*

5.2.1 Schwarzians associated with the optimal welding

The optimal welding h^* has an associated representative γ_{h^*} with maps $f : \mathbb{H} \to \Omega$ and $g : \mathbb{H}^* \to \Omega^*$, just as before.

In the curve optimization case we could essentially glue together the maps f, g and then get the structure of the Schwarzian of the inverse, recall theorem 5.5. Here we get instead two different Schwarzians, of the maps themselves, not the inverse, that can be extended to all of $\hat{\mathbb{C}}$, but yield different such extensions.

Theorem 5.12 *The Schwarzian of f, $\mathcal{S}[f]$ can be meromorphically extended to all of $\hat{\mathbb{C}}$. We have the following structure*

$$\mathcal{S}[f](z) = \sum_{k=1}^{n} \frac{C_k^f}{z - x_k} \tag{5.18}$$

A similar statement, with different residues C_k^g and x_k replaced by y_k is true for the Schwarzian $\mathcal{S}[g]$.

Proof This is a direct application of lemma 3.7, noting that since the minimizer $I^L(h) < \infty$, the representative piecewise circular curve γ must have arcs meeting at angles of π at the junction points. Then the remark right after lemma 3.7 forces the poles to be simple. $\square$

5.2.2 Uniqueness of the optimal welding

A result that we set out to prove but were unable to obtain deals with the uniqueness of the optimal welding in eq. (5.13).

In analogy with theorem 5.4 it seems reasonable to use some structural property within the class of minimizers, and what we have to work with to that end is lemma 5.11 on the piecewise circularity of the associated curve.

From piecewise circularity one can say a lot about the structure of the Schwarzians $\mathcal{S}[f]$ and $\mathcal{S}[g]$. The question then becomes: How can this information be used in conjunction with the boundary markings of the class $\Phi_{x,y}$?

We tried an approach of having the markings $x_k \mapsto y_k$ enter via an interpretation of the Schwarzian chain rule in a distributional sense, i.e. trying to give some interpretation of $\mathcal{S}[h]$ in terms of $\mathcal{S}[f]$ and $\mathcal{S}[g]$.

We were not able to remove enough degrees of freedom. Future work will likely show this uniqueness to be true, in line with the remark made in [19].

Chapter 6

Residue formulas

We have introduced the two optimization problems eq. (5.2) eq. (5.13) for Loewner energy and seen how their solutions give rise to characteristic Schwarzians theorem 5.5 theorem 5.12. It turns out that we can say far more about the residues of these functions, relating them to the value of the optimization problem.

The perspective we take in examining both of these problems will be furnished by the quasiconformal deformation technique we previously introduced in chapter 4.

6.1 Basic setup

To ground the discussion, we first revisit the case of quasiconformal deformation of a curve γ (and implicitly the welding, recall end of section 4), namely $\gamma^t = \omega^{t\mu}(\gamma)$ for $\omega^{t\mu} : \hat{\mathbb{C}} \to \hat{\mathbb{C}}$ a quasiconformal map with Beltrami differential $t\mu$ compactly supported away from the curve, with $\omega^0 = \mathrm{id}$. Define the vector field of infinitesimal variation by

$$v = v_\mu = \frac{d}{dt}|_{t=0}\omega^{t\mu} \tag{6.1}$$

which describes how the family $(\omega^{t\mu})$ moves the points of $\hat{\mathbb{C}}$ at $t = 0$. It is clear that, up to choice of the exact representative quasiconformal maps $\omega^{t\mu}$, v depends on μ. In fact, we have the following explicit relation.

Lemma 6.1 *Let $v = v_\mu$ be the vector field of infinitesimal variation for a Beltrami differential μ as above. Then it holds*

$$\bar{\partial} v = \mu \tag{6.2}$$

Proof We compute

$$\frac{d}{dt}(t\mu) = \frac{d}{dt}\left(\frac{\bar{\partial}\omega^{t\mu}}{\partial\omega^{t\mu}}\right) \tag{6.3}$$
$$= \frac{\bar{\partial}v}{\partial\omega^0} - \frac{\bar{\partial}\omega^0(\partial v)}{(\partial\omega^0)^2} \tag{6.4}$$
$$= \bar{\partial}v \tag{6.5}$$

using that $\omega^0 = \text{id}$. □

This relation is useful for the main reason that the vector field v is sometimes a lot easier to specify, and furthermore, as we will later see, that this specification need only be explicitly stated in a very local way. To translate from a question about μ to v, the following lemma will be very useful, see [10].

Lemma 6.2 *Let v and μ be as above. Then it holds*

$$-\frac{4}{\pi}Re\left[\int_{\Omega}\mathcal{S}[f^{-1}](z)\mu(z)d^2z\right] = -\frac{2}{\pi}Im\left[\int_{\partial\Omega}v(z)\mathcal{S}[f^{-1}](z)dz\right] \tag{6.6}$$

and there is a similar statement for (g, Ω^*).

Note that one integral is with $d^2z = dx \wedge dy$ over a domain and the other is a complex line integral.

Proof Using lemma 6.1 and the holomorphicity of the Schwarzian

$$\int_{\Omega}\mu\mathcal{S}[f^{-1}]d^2z = \int_{\Omega}\bar{\partial}\left(v\mathcal{S}[f^{-1}]\right)\frac{d\bar{z}\wedge dz}{2i} \tag{6.7}$$
$$= \frac{1}{2i}\int_{\Omega}d\left(v\mathcal{S}[f^{-1}]dz\right) \tag{6.8}$$
$$= \frac{1}{2i}\int_{\partial\Omega}v\mathcal{S}[f^{-1}]dz \tag{6.9}$$

The factor in front of the integral on the last line explains the switch from real to imaginary part. Essentially the same calculation can be carried out for (g, Ω^*). □

There is an intimate relationship between complex line integrals and residues.

Theorem 6.3 (Residue formula) *Let φ be a meromorphic function on a domain and bounded by γ a positively oriented closed simple curve, with a finite set of* *isolated singularities $\{z_k\}$ on the inside of γ. Then we have the following formula for the line integral*

$$\int_\gamma \varphi(z)dz = 2\pi i \sum_k Res(\varphi, z_k) \tag{6.10}$$

In the special case we will be considering, the poles lie at the junctions of, hence on, the curve γ (being the curve we are directly optimizing for, or in the welding optimization, the representative curve).

Denote by PV the Cauchy principal value, defined by

$$PV \int_{-R}^{R} \varphi(x)\, dx := \lim_{\epsilon \to 0^+} \int_{[-R,R] \setminus \bigcup_{j=1}^{n} (x_j - \epsilon, x_j + \epsilon)} \varphi(x)\, dx, \tag{6.11}$$

provided this limit exists. We will make use of the following:

Lemma 6.4 *Let φ be meromorphic in $\overline{\mathbb{H}}$ and suppose that it has singularities on $\mathbb{R}$, with simple poles at $x_1, ..., x_n$ and no interior poles. Fix $R > \max_j |x_j|$ and for $\epsilon > 0$ set*

$$D_{R,\epsilon} := \{z \in \overline{\mathbb{H}} : |z| \leq R\} \setminus \bigcup_{j=1}^{n} \{z \in \overline{\mathbb{H}} : |z - x_j| \leq \epsilon\}. \tag{6.12}$$

Then, with $\partial D_{R,\epsilon}$ positively oriented, the limit of integrals

$$\lim_{\epsilon \to 0} \int_{\partial D_{R,\epsilon}} \varphi(z)dz \tag{6.13}$$

evaluates to

$$\int_{\{|z|=R,\, Im(z) \geq 0\}} \varphi(z)dz + PV \int_{-R}^{R} \varphi(x)dx - i\pi \sum_{j=1}^{n} Res(\varphi, x_j). \tag{6.14}$$

6.2 Differentiating the optimization problem

We will shortly present theorems relating the accessory parameters to certain derivatives of the optimal values of the Loewner energy. There is an immediate subtlety whenever one tries to differentiate not a function itself, but rather the result of some optimization involving the function. In the concrete setting we are in we have from eq. (5.2) eq. (5.13) (provided we admit the unproven uniqueness of the welding solution)

$$I^L(z_1, ..., z_n) := \inf_{\gamma \in \mathcal{L}(z_1,...,z_n;\tau)} I^L(\gamma) \tag{6.15}$$

$$I^L(x, y) := \inf_{h \in \Phi_{x,y}} I^L(h). \tag{6.16}$$

When we write $\partial_{z_k} I^L(z_1, ..., z_n)$ or $\partial_{x_k} I^L(x, y)$ we are varying within a class of minimizers, so the points z_k or x_k or y_k are moving, but so must the curve or welding, in a non-obvious global way. We will now try to argue that it is actually enough to focus on how the points are moving.

We have from [8] a treatment of the case where X is a choice set (corresponding here to curves or weldings) and $f : X \times [-\epsilon, \epsilon] \to \mathbb{R}$ the objective function for an optimization problem with value function $V(t) = \inf_{x \in X} f(x, t)$. Moving along the one-dimensional path parametrized by t (in our cases this would be a particular path along which a junction point $z_k = z_k(t)$ or boundary marking $x_k = x_k(t)$ depends on t) we have the optimal choice correspondence $X^*(t) = \{x \in X : f(x, t) = V(t)\}$.

Lemma 6.5 *Take $t_0 \in (-\epsilon, \epsilon)$ and $x^* \in X^*(t_0)$ and suppose that $\partial_t f(x^*, t_0)$ exists. Then if V is left-hand differentiable at t_0,*

$$V'(t_0-) \geq \partial_t f(x^*, t_0) \tag{6.17}$$

and if V is right-hand differentiable at t_0,

$$V'(t_0+) \leq \partial_t f(x^*, t_0). \tag{6.18}$$

In particular, if V is differentiable at t_0, $V'(t_0) = \partial_t f(x^*, t_0)$.

Proof For any $t' \in [-\epsilon, \epsilon]$, since $x^* \in X^*(t_0)$, we have $f(x^*, t_0) = V(t_0)$, and thus

$$V(t') = \inf_{x \in X} f(x, t') \leq f(x^*, t'), \tag{6.19}$$

i.e.

$$V(t') - V(t_0) \leq f(x^*, t') - f(x^*, t_0). \tag{6.20}$$

If $t' > t_0$, dividing by $t' - t_0 > 0$ gives

$$\frac{V(t') - V(t_0)}{t' - t_0} \leq \frac{f(x^*, t') - f(x^*, t_0)}{t' - t_0}, \tag{6.21}$$

and taking $t' \downarrow t_0$ yields $V'(t_0+) \leq \partial_t f(x^*, t_0)$.

If $t' < t_0$, dividing by $t' - t_0 < 0$ gives

$$\frac{V(t') - V(t_0)}{t' - t_0} \geq \frac{f(x^*, t') - f(x^*, t_0)}{t' - t_0}, \tag{6.22}$$

and taking $t' \uparrow t_0$ yields $V'(t_0-) \geq \partial_t f(x^*, t_0)$. □

To use lemma 6.5 we need to correctly identify V and f in our setup. In both the curve and welding optimization problems, the value function V is very straightforward, but the objective $f(x, t)$ is less so. As they write in [8], there is up until this point no special use of any structure of the choice set X. In our setting, the set that is optimized over will be varying with the parameter t and we need a way to take this into account when we make a choice for f.

As the title of this chapter betrays, we will ultimately be interested in residues, in particular the residues of the Schwarzians associated with uniformizing maps f and g that come at the optimal curves or weldings. In [7] the connection between residues in the curve case and variation of the corresponding junction point of the curve is established. Because of this, and since we will try to make similar inroads towards the problem of the residues in the welding case, the natural variations to consider are those that move points (like z_k, x_k or y_k).

At some optimal curve γ^* or optimal welding h^*, playing the role of x^* from lemma 6.5, with junction points $\{z_j\}$ and boundary markings $\{(x_j, y_j)\}$ respectively, we can move to some other (in general non-optimal within its class) curve or welding γ^t or h^t by means of quasiconformal deformation or pre- or post-composition

$$\gamma^t = \omega^{t\mu}(\gamma^*) \tag{6.23}$$
$$h^t := \omega_{t\mu}|_{\mathbb{R}} \circ h^* \text{ or} \tag{6.24}$$
$$h^t = h^* \circ (\omega_{t\mu})^{-1}|_{\mathbb{R}} \tag{6.25}$$

We take the choice set X to be either $\mathcal{L}(z_1, ..., z_n, \tau)$ or $\Phi_{x,y}$ as base or references, and then transform bijectively for each small enough t:

$$\mathcal{L}(z_1, ..., z_n, \tau) \to \{\omega^{t\mu}(\gamma) : \gamma \in \mathcal{L}(z, \tau)\} \tag{6.26}$$

and

$$\Phi_{x,y} \to \{\omega_{t\mu}|_{\mathbb{R}} \circ h : h \in \Phi_{x,y}\} \text{ or} \tag{6.27}$$
$$\Phi_{x,y} \to \{h \circ (\omega_{t\mu}|_{\mathbb{R}})^{-1} : h \in \Phi_{x,y}\} \tag{6.28}$$

So concretely $X^*(t) = \{x \in X : f(x,t) = V(t)\}$ becomes

$$\{\tilde{\gamma} \in \mathcal{L}(z_1, ..., z_n; \tau) : I^L(\omega^{t\mu}(\tilde{\gamma})) = \inf_{\gamma \in \mathcal{L}(z(t), \tau)} I^L(\gamma)\} \tag{6.29}$$

and

$$\{\tilde{h} \in \Phi_{x,y} : I^L(\tilde{h} \circ (\omega_{t\mu})^{-1}|_{\mathbb{R}}) = \inf_{h \in \Phi_{x(t),y(t)}} I^L(h)\} \text{ or} \tag{6.30}$$
$$\{\tilde{h} \in \Phi_{x,y} : I^L(\omega_{t\mu}|_{\mathbb{R}} \circ \tilde{h}) = \inf_{h \in \Phi_{x(t),y(t)}} I^L(h)\} \tag{6.31}$$

i.e. for two different values of t, say t_1 and t_2, $X^*(t_1)$ and $X^*(t_2)$, will, while being different, be subsets of the same index set X. This makes it possible to compare curves or weldings from different classes and apply lemma 6.5. We are aware that this step needs further tightening, but see it as a genuine attempt to remove the dependence of finding a variation within the class of minimizers.

6.3 Residues in the curve case

From [2] we have the following theorem for which we will present an alternative proof. In the introduction this is theorem 1.1.

Theorem 6.6 *Consider the Loewner energy optimization problem for curves in* $\mathcal{L}(z_1, ..., z_n; \tau)$ *eq.* (5.2) *giving rise to optimal value and curve*

$$I^L(z_1, ..., z_n) = I^L(\gamma). \tag{6.32}$$

Let $f : \mathbb{H} \to \Omega$, *and* $g : \mathbb{H}^* \to \Omega^*$ *be the associated conformal maps and set* $F = f^{-1}$ *on* Ω *and* $F = g^{-1}$ *on* Ω^* *as in theorem 5.5. Then the Schwarzian* $\mathcal{S}[F]$ *extends to a meromorphic function on all of* $\hat{\mathbb{C}}$ *and, assuming the derivative exists, we have the following formula for the residues:*

$$Res(\mathcal{S}[F], z_k) = \frac{1}{2}\partial_{z_k} I^L(z_1, ..., z_n) \tag{6.33}$$

Proof The meromorphic extendability to $\hat{\mathbb{C}}$ was already stated in theorem 5.5, so we deal only with the formula for the residues.

We have $\partial_{z_k} = \frac{1}{2}(\partial_{x_k} - i\partial_{y_k})$. Geometrically, we have a good understanding of both ∂_{x_k} and ∂_{y_k} in the sense that their actions on $I^L(z_1, ..., z_n) = I^L(\gamma)$ can be sketched in the same picture of the plane as γ, being the infinitesimal change in Loewner energy of the minimal curve where the junction point $z_k = x_k + iy_k$ is moved in the $(1, 0)$ and $(0, 1)$ directions of the plane respectively.

We thus prescribe $v_{k,x}(z_j) = \delta_{k,j}$ and $v_{k,y}(z_j) = i\delta_{k,j}$ and compute

$$\partial_{z_k} I^L(z_1, ..., z_n) = \frac{1}{2}(\partial_{x_k} - i\partial_{y_k}) I^L(z_1, ..., z_n) \tag{6.34}$$

$$= \frac{1}{2}\left(\frac{d}{dt}\bigg|_{t=0} I^L(\omega^{t\mu_{k,x}}(\gamma)) - i\frac{d}{dt}\bigg|_{t=0} I^L(\omega^{t\mu_{k,y}}(\gamma)) \right) \tag{6.35}$$

using lemma 6.5. Expanding using theorem 4.2 and lemma 6.2 we obtain

$$= -\frac{1}{\pi}\Big(\mathrm{Im}\left[\int_{\partial\Omega} v_{k,x}(z)\mathcal{S}[f^{-1}](z)dz \right] \tag{6.36}$$

$$+ \mathrm{Im}\left[\int_{\partial\Omega^*} v_{k,x}(z)\mathcal{S}[g^{-1}](z)dz \right] \tag{6.37}$$

$$- i\mathrm{Im}\left[\int_{\partial\Omega} v_{k,y}(z)\mathcal{S}[f^{-1}](z)dz \right] \tag{6.38}$$

$$- i\mathrm{Im}\left[\int_{\partial\Omega^*} v_{k,y}(z)\mathcal{S}[g^{-1}](z)dz \right] \Big) \tag{6.39}$$

which by lemma 6.4 is equal to

$$= -\frac{1}{\pi}\Big(- \operatorname{Im}\left[2\pi i \operatorname{Res}(\mathcal{S}[F], z_k)\right] \tag{6.40}$$

$$+ i\operatorname{Im}\left[2\pi i i \operatorname{Res}(\mathcal{S}[F], z_k)\right]\Big) \tag{6.41}$$

$$= 2\Big(\operatorname{Im}\left[i\operatorname{Res}(\mathcal{S}[F], z_k)\right] \tag{6.42}$$

$$+ i\operatorname{Im}\left[\operatorname{Res}(\mathcal{S}[F], z_k)\right]\Big) \tag{6.43}$$

$$= 2\operatorname{Res}(\mathcal{S}[F], z_k) \tag{6.44}$$

□

allowing us to conclude.

Note that the proof we have just given is different from that in [2].

6.4 Residues in the welding case

We have arrived at theorem 1.2 from the introduction.

Theorem 6.7 *Consider the Loewner energy optimization problem for weldings in* $\Phi_{x,y}$ *eq.* (5.13) *with optimum*

$$I^L(x,y) = I^L(h) \tag{6.45}$$

and representative curve $\gamma = \gamma_h$. *Just as before, let* $f : \mathbb{H} \to \Omega$ *and* $g : \mathbb{H}^* \to \Omega^*$ *be two conformal maps with Schwarzians* $\mathcal{S}[f]$ *and* $\mathcal{S}[g]$. *Then these Schwarzians can be extended to (two different) functions* $\hat{\mathbb{C}} \to \hat{\mathbb{C}}$ *and, assuming the derivatives exist, we have the following formula for the residues:*

$$Res(\mathcal{S}[f], x_k) = \frac{1}{2}\partial_{x_k} I^L(x,y) \tag{6.46}$$

$$Res(\mathcal{S}[g], y_k) = \frac{1}{2}\partial_{y_k} I^L(x,y) \tag{6.47}$$

Proof Just as in the case of the curve, the simple pole structure and extendability of the Schwarzians have been established, see theorem 5.12, so we can direct our full attention to the formula for the residues.

We carry out the proof for x_k where we will be working with f and $\mathbb{H}$, noting that the steps are very similar for y_k and essentially involves only switching to g and $\mathbb{H}^*$.

We prescribe that the vector field v_k on $\overline{\mathbb{H}}$ have $v_k(x_j) = \delta_{k,j}$, be real-valued on $\mathbb{R}$ and let $\mu_k = \bar{\partial} v_k$ and $\omega^{t\mu_k}$ be the normalized solution to the Beltrami equation in $\hat{\mathbb{C}}$, taking μ_k to be supported in $\mathbb{H}$. It acts on x_j like $x_j(t) := \omega^{t\mu_k}(x_j) = x_j + tv_k(x_j) + o(t) = x_j + t\delta_{k,j} + o(t)$.

Now $h_t := h \circ (\omega^{t\mu_k})^{-1}|_{\mathbb{R}}$ has

$$h_t(x_j(t)) = h \circ (\omega^{t\mu_k})^{-1}(x_j(t)) = h(x_j) = y_j \tag{6.48}$$

i.e. $h_t \in \Phi_{x(t),y}$. Use the lemma lemma 6.5 to transform from a derivative of the value function to one of the objective

$$\begin{aligned}
\partial_{x_k} I^L(x,y) &= \frac{d}{dt}|_{t=0} I^L(h_t) &(6.49)\\
&= -\frac{4}{\pi}\mathrm{Re}\left[\int_{\mathbb{H}} \mu_k \mathcal{S}[f] d^2 z\right] &(6.50)\\
&= -\frac{2}{\pi}\mathrm{Im}\left[\int_{\hat{\mathbb{R}}} v_k \mathcal{S}[f] dz\right] &(6.51)
\end{aligned}$$

using theorem theorem 4.2, the discussion at the end of chapter 4 and lemma lemma 6.2. But from the special residue formula lemma 6.4 we have

$$\int_{\hat{\mathbb{R}}} v_k \mathcal{S}[f] dz = \mathrm{PV}\int_{\hat{\mathbb{R}}} v_k \mathcal{S}[f] dz - i\pi \sum_{j=1}^{n} \mathrm{Res}(v_k \mathcal{S}[f], x_j) \tag{6.52}$$

and since both v_k and $\mathcal{S}[f]$ are real valued on $\mathbb{R}$ lemma 3.7 we have

$$\begin{aligned}
-\frac{2}{\pi}\mathrm{Im}\left[\int_{\hat{\mathbb{R}}} v_k \mathcal{S}[f] dz\right] &= -\frac{2}{\pi}(-\pi)\sum_{j=1}^{n} \mathrm{Res}(v_k \mathcal{S}[f], x_j) &(6.53)\\
&= 2\mathrm{Res}(\mathcal{S}[f], x_k). &(6.54)
\end{aligned}$$

which allows us to conclude. As stated in the beginning of the proof, the Schwarzian $\mathcal{S}[g]$ is handled in the same way. $\square$

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