Circle average and delocalization of random fields

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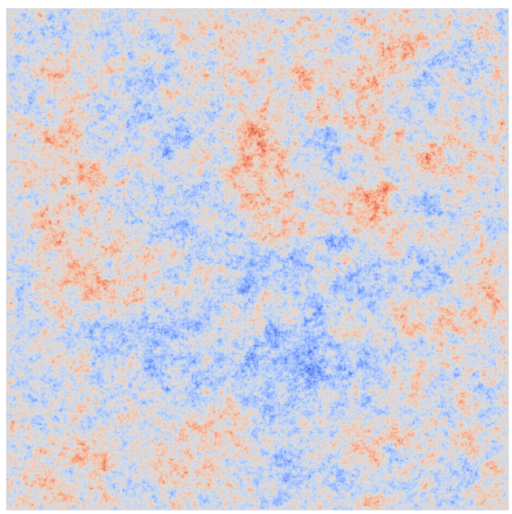

Abstract

The discrete and continuous gaussian free fields and notions of circle average are introduced. This motivates a look at random discrete circles. In the final section we briefly comment on whether circle average can be adapted with these ideas in mind, possibly facilitating a new approach to delocalization.

Acknowledgements

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Contents

The gaussian free field

The gaussian free field (GFF) is one of the central objects of this semester paper, being perhaps the prototypical example of a random surface and acting as a useful comparison for many other such models. It's not only a toy model however, but rather appears as the scaling limit of a wide variety of random height models. The main interest will be understanding the field by averaging over circles, or circle-like loops around a certain point. Some aspects of this averaging theory are more elegant in the continuous setting, so in the later sections we look at whether we can develop it further for the discrete case.

1.1 Discrete GFF

We begin by introducing the DGFF, using notation and conventions from Werner and Powell. [8]

We consider $D \subset \mathbb{Z}^d$ and will mainly consider the case $d = 2$ in this note. The boundary is $\partial D = \{x \in \mathbb{Z}^d : d(x, D) = 1\}$ where d is the typical 'shortest-path' distance. We have $\bar{D} = D \cup \partial D$, and collect the edges of $\bar{D}$ in E_D.

For a function on the vertices of D (equivalently a vector in $\mathbb{R}^D$), we have at the edge e connecting x and y, $|\nabla_e f| := |f(x) - f(y)|$.

Definition 1.1.1. *(Dirichlet energy)*
Let f be a function on the vertices of $\bar{D}$. The Dirichlet energy of f is the quantity

$$\mathcal{E}_D(f) = \sum_{e \in E_{\bar{D}}} |\nabla_e f|^2 \tag{1.1}$$

Intuitively, the Dirichlet energy measures the extent to which the function varies locally. Wild differences at neighboring vertices lead to a large energy.

To define a random surface, one of the simplest possible ideas is to take some $D \subset \mathbb{Z}^2$ and consider the graph of some random function on D. While one could in principle select the 'height' independently for each vertex in D, this would give little geometric structure. The DGFF instead takes the approach of penalizing large differences between neighboring points, specifically using the Dirichlet energy we just introduced.

Definition 1.1.2. *The DGFF in D with zero boundary condition is the random gaussian vector $(\Gamma_x)_{x \in D}$ with density function at $(\gamma_x)_x$ proportional to*

$$\exp\left(-\frac{1}{2}\frac{\mathcal{E}_D(\gamma)}{2d}\right) \tag{1.2}$$

A possible generalization of the above is motivated by noticing that

$$\begin{aligned}
\frac{1}{2}\frac{1}{2d}\mathcal{E}_D(\gamma) &= \sum_{e \in E_{\bar{D}}} \frac{1}{2d}\frac{|\nabla_e \gamma|^2}{2} && (1.3)\\
&= \frac{1}{2}\sum_{x,y \in D} p(x,y)V(\gamma_x - \gamma_y) + \sum_{x \in D, y \in \partial D} p(x,y)V(\gamma_x - \gamma_y) && (1.4)\\
&=: H_V(\gamma) && (1.5)
\end{aligned}$$

taking $p(\cdot,\cdot)$ to encode the transition probabilities of a symmetric random walk on D and $V : \mathbb{R} \to \mathbb{R}$ given by $V(x) = \frac{1}{2}x^2$. In the general Gibbs measure formulation, we consider densities proportional to $\exp(-\beta H_V(\gamma))$ for some interaction potential V and inverse temperature β. If one insists on $V \in C^2$ with $V(x) = V(-x)$ and $0 < c_- \leq V''(x) \leq c_+ < \infty$, we get the class of Ginzburg-Landau fields.

A big reason for the tractability of the DGFF is that with the $V(x) = \frac{1}{2}x^2$ interaction and $\gamma|_{\partial D} \equiv 0$, we get

$$H_V = \frac{1}{4}\sum_{x,y\in D} p(x,y)(\gamma_x - \gamma_y)^2 + \frac{1}{2}\sum_{x\in D, y\in\partial D} p(x,y)(\gamma_x - \gamma_y)^2 \tag{1.6}$$

$$= \frac{1}{2}\sum_{x,y\in D} \gamma_x(\delta_{x,y} - p(x,y))\gamma_y \tag{1.7}$$

$$= \frac{1}{2}\gamma^{tr}(\mathbf{1} - P)\gamma \tag{1.8}$$

with the last equation using the more compact matrix notation. From this representation we get a relation between the covariance and a random walk $(X_n)_n$ on the underlying lattice:

$$\mathrm{cov}(\Gamma_x, \Gamma_y) = \left((\mathbf{1} - P)^{-1}\right)_{x,y} \tag{1.9}$$

$$= \left(\sum_{n\geq 0} P^n\right)_{x,y} \tag{1.10}$$

$$= \sum_{n\geq 0} P_x(X_n = y, n < \tau_D) \tag{1.11}$$

$$= \sum_{n\geq 0} E_x[\mathbf{1}(X_n = y, n < \tau_D)] \tag{1.12}$$

$$=: G_D(x,y) \tag{1.13}$$

$$= E_x[N_y] \tag{1.14}$$

where N_y is the number of visits to y before exiting the domain D (at the exit time $\tau_D = \min\{n \in \mathbb{N} : X_n \notin D\}$) and G_D is the Green's function, an object with lots of interesting properties on its own, of which the discrete harmonicity (when viewed as a function $x \mapsto G_D(x,y)$) will be most useful to us. In the survey by Velenik [7] there is a brief remark on the extendability of this kind of representation to other potentials, using a (modified) continuous time random walk.

We may just as well define the DGFF as follows:

Definition 1.1.3. *The discrete gaussian free field with zero boundary condition is the centered gaussian vector* $(\Gamma_x)_{x\in D}$ *with covariance given by* $cov(\Gamma_x, \Gamma_y) = G_D(x, y)$.

In a later section we will use a special Markov property and a resulting decomposition of the Gaussian free field. For this we consider the finite subsets $A \subset D$ and $O = D\backslash A$. Given boundary data f on some ∂B, we call h_f the discrete harmonic extension, the function with $h_f|_{\partial B} = f$ and discrete harmonic $\Delta h_f = 0$ in B. If f is random, the same is true for h_f. With this in mind, we define

$$\Gamma_A = \Gamma \mathbf{1}_A + h_{\Gamma|\partial O}\mathbf{1}_O \tag{1.15}$$

$$\Gamma^A = \Gamma - \Gamma_A \tag{1.16}$$

Using the density function definition 1.1.2 and properties of the Dirichlet energy, one can show that Γ_A and Γ^A are independent and that Γ^A is a Gaussian free field in O with zero boundary conditions.

We have that

$$\begin{aligned} G_D(x, y) = E[\Gamma(x)\Gamma(y)] &= E[(\Gamma_A(x) + \Gamma^A(x))(\Gamma_A(y) + \Gamma^A(y))] &(1.17)\\ &= E[\Gamma_A(x)\Gamma_A(y)] + E[\Gamma^A(x)\Gamma^A(y)] &(1.18)\\ &= E[\Gamma_A(x)\Gamma_A(y)] + G_O(x, y) &(1.19)\end{aligned}$$

which gives a useful expression for the covariance of Γ_A.

For the discrete free field we will interchangeably use Γ_x, $\Gamma(x)$ and ϕ_x depending on what is most convenient. In the last section, we will see terms of the form $\phi_x - \Gamma_A(x)$, which could be read as $\Gamma(x) - \Gamma_A(x)$.

1.2 Continuous GFF

A first naive approach to defining the continuous (2D) Gaussian free field might set out to obtain a Gaussian process $(\Gamma_x)_{x\in D}$ indexed in the uncountably infinite D. This would mean that all finite dimensional distributions are Gaussian, i.e. that $(\Gamma_{x_1}, ..., \Gamma_{x_m})$ is Gaussian.

It should also be a scaling limit of the discrete gaussian free field. Defining $D_\delta = \delta\mathbb{Z}^2 \cap D$ as a discrete grid approximation to D, we can scale by δ^{-1} to blow up D_δ to $\delta^{-1}D_\delta \subset \mathbb{Z}^2$, use the definition of the discrete Gaussian free field there to get $(\Gamma_{\delta^{-1}D_\delta}(x))_x$ and then define $\Gamma_\delta(x) := \Gamma_{\delta^{-1}D_\delta}(\delta^{-1}x)$ for $x \in D_\delta$. We can extend Γ_δ to a piecewise constant function on D. Unfortunately, $G_{D_\delta}(x,x) := E[\Gamma_\delta(x)^2]$ grows on the order of $\log(\delta^{-1})$ as $\delta \to 0$, so taking this limit, we would obtain a Gaussian with infinite variance in the limit. This is seen from Green's function estimates for the random walk, see [3].

The approach taken instead is to directly define a stochastic process $(\Gamma_\mu)_\mu$ indexed by a certain collection of measures supported in D.

One crucial ingredient is the so called continuum Green's function, also denoted $G_D(\cdot,\cdot)$, which is determined by insisting on harmonicity of $G_D(\cdot,y)$ in $D\backslash\{y\}$ and that for $x \to \partial D$, $G_D(x,y) \to 0$.

The construction in $d = 2$ ($d \geq 3$ is treated similarly) is carried out by starting with $H_y(x) = \frac{1}{2\pi}\log(|x-y|^{-1})$ which is harmonic in $D\backslash\{y\}$. By subtracting the unique harmonic function in D with boundary values $H_y|_{\partial D}$ we obtain $G_D(\cdot,y)$.

It inherits some of the properties of its discrete counterpart, including symmetry, relation to Brownian motion (via occupation time rather than expected number of visits, as was the case for the discrete Green's function in relation to random walk) and, when viewed as an integral operator, being the inverse of the continuum Laplacian. This latter property is more cleanly expressed thanks to the normalization $\frac{1}{2\pi}$ used for H_y above.

With G_D defined, one can start by considering finite measures supported in D with the property that

$$\int_{D\times D} G_D(x,y)d\mu(x)d\mu(y) < \infty \tag{1.20}$$

and then using the vector space structure to obtain a space of signed measures, denoted here by $\mathcal{M}$. This space contains (among other things) measures of the intuitive form $d\mu = f dx$ for f continuous and compactly supported in D, and dx the Lebesgue measure.

Definition 1.2.1. *The continuous Gaussian free field in D is the centered Gaussian process $(\Gamma_\mu)_{\mu\in\mathcal{M}}$ with covariance*

$$E[\Gamma_\mu \Gamma_\nu] = \int_{D\times D} G_D(x,y)d\mu(x)d\nu(y) \tag{1.21}$$

where one can show that the right hand side does indeed define a covariance function.

Given $A \subset D$ and certain regularity conditions, it can be shown (with a rather large dose of technical details) that there is a way to decompose the continuum Gaussian free field in the following way:

$$\Gamma(\mu) = \Gamma_A(\mu) + \Gamma^A(\mu) \tag{1.22}$$

where Γ^A is a Gaussian free field in $D\backslash A$ with zero boundary conditions and Γ_A is equal to Γ in A (i.e. for measures supported in A) and is harmonic in $D\backslash A$, and in a sense the 'harmonic extension of $\Gamma|_{\partial(D\backslash A)}$'. We use quotation marks here, since Γ is not defined pointwise, see previous discussion. This is the continuum version of the Markov property.

1.3 Circle average

We now introduce a process derived from the gaussian free field, and one of the main starting points for what is to come in the next sections. We follow the presentation in [8]. Let $\lambda_{z_0,r}$ be the uniform measure to the ball $B(z_0,r) \subset \mathbb{C}$ and write $\lambda_r := \lambda_{0,r}$, $B_r = B(0,r)$. Using the harmonicity property of G_D ($x \mapsto G_D(x,y)$) we get

$$\int_{D\times D} G_D(x,y)d\lambda_r(x)d\lambda_r(y) = \int_{\partial B_r} G_D(0,x)d\lambda_r(x) < \infty \tag{1.23}$$

since $r > 0$ so one sees that $\lambda_{z_0,r} \in \mathcal{M}$, and it makes sense to consider $\gamma(r) := \Gamma(\lambda_r)$.

What does $\gamma(r)$ behave like as we vary $r > 0$? Consider $r < r'$ and another measure $\mu \in \mathcal{M}$ supported outside of the outer ball, i.e. $\mathrm{supp}(\mu) \subseteq D\backslash\overline{B(0,r')}$. Using again the harmonicity, we compute

$$E[(\gamma(r) - \gamma(r'))\Gamma(\mu)] = \int_{D\times D} (d\lambda_r(x) - d\lambda_{r'}(x))G_D(x,y)d\mu(y) \tag{1.24}$$

$$= \int_{\partial B_r\times D} d\lambda_r(x)G_D(x,y)d\mu(y) - \int_{\partial B_{r'}\times D} d\lambda_{r'}(x)G_D(x,y)d\mu(y) \tag{1.25}$$

$$= \int_D G_D(0,y)d\mu(y) - \int_D G_D(0,y)d\mu(y) = 0 \tag{1.26}$$

so $\gamma(r) - \gamma(r')$ is independent of $\Gamma(\mu)$ for such μ.

Next, computing the variance of the increment

$$E[(\gamma(r) - \gamma(r'))^2] = \int_{D\times D} d\lambda_r(x)G_D(x,y)d\lambda_r(y) + \int_{D\times D} d\lambda_{r'}(x)G_D(x,y)d\lambda_{r'}(y)+ \tag{1.27}$$

$$- 2\int_{D\times D} d\lambda_r(x)G_D(x,y)d\lambda_{r'}(y) \tag{1.28}$$

$$= \int_{\partial B_r} G_D(0,x)d\lambda_r(x) - \int_{\partial B_{r'}} G_D(0,x)d\lambda_{r'}(x) \tag{1.29}$$

$$= \int_{\partial B_r} G_{B_{r'}}(0,x)d\lambda_r(x) \tag{1.30}$$

where we use the Markov property introduced in the previous subsection. In dimension two this is equal to $\log(r'/r)$, so after choosing a reference radius r_0 and rescaling it as $(r_0e^{-u})_{u\geq 0}$ we get that

$$b_{r_0}(u) := \gamma(r_0e^{-u}) - \gamma(r_0) \quad u \geq 0 \tag{1.31}$$

has the finite dimensional distributions of a one-dimensional Brownian motion. If we take a countable collection of disjoint balls and consider such a circle average for each of them, then passing to modifications, we end up with a collection of independent Brownian motions.

An important recipe to make this work is the harmonicity property of G_D and the very nice fact that λ_r which is a measure on ∂B_r is the same as the harmonic measure, or hitting measure of a Brownian motion started at zero and stopped when exiting the ball. In the discrete setting, we have no 'perfect' grid circle, but if we take some notion of balls $(B_n)_{n\in\mathbb{N}}$, $B_0 = \{0\}$ (these could be good approximations of actual balls, or squares, and more generally any volume bounded by a closed simple curve will do), we can define

$$M_k := \sum_{x\in\partial B_{k-1}} H_{\partial B_{k-1}}(0,x)\phi_x = E[\phi_0|\mathcal{F}_{B_N\setminus B_{k-1}}] \tag{1.32}$$

with $\mathcal{F}_A = \sigma(\phi_x : x \in A)$ and where $H_{\partial B_k}(0,x) = P_0(X_{\tau_k} = x)$ is the probability that a random walk X started at zero hits x upon exit of B_k, where $\tau_k = \inf\{n : X_n \notin B_k\}$. Note the shifted index, which makes the definition $M_0 := \phi_0$ natural. The representation as a conditional expectation gives immediately that this is a martingale. If we wish to understand the variance of increments in the discrete case, we should probably specify our choice of balls $(B_n)_n$ and have some strategy to handle the discrete harmonic measure weights $(H_{\partial B_k}(0,x))_{x\in\partial B_k}$.

Since these weights are related to the random walk, one idea is to transfer some of this randomness into the balls $(B_n)_n$, hopefully yielding simpler weights, perhaps even uniform ones. What do we mean by 'transferring randomness' and how are the random balls to be selected? This interesting question led to some dead ends. In the next section we present the best formulation we have obtained until now.

Random discrete circles

2.1 Problem formulation

When taking the discrete 'circle average' we need to be aware of the discrete harmonic measure weights. The question we ask is whether it is possible to transfer the complexity in the weights into a randomness of the 'circles', allowing us to use simpler, perhaps even uniform weights.

Concretely, we look at a measure ν on subsets of D. We sample S from ν and consider the random arc ∂S and the following measures supported on it

$$\mu_H = \sum_{x \in \partial S} H_{\partial S}(0, x)\delta_x \tag{2.1}$$

$$\mu_U = \sum_{x \in \partial S} \frac{1}{|\partial S|}\delta_x \tag{2.2}$$

Note that

$$E[\phi_0 | \mathcal{F}_{D \setminus S}] = \sum_{x \in \partial S} H_{\partial S}(0, x)\phi_x \tag{2.3}$$

$$=: \int \phi d\mu_H \tag{2.4}$$

We ask the following question: Is it possible to construct a measure ν s.t.

$$E_\nu[\mu_H] = E_\nu[\mu_U] \tag{2.5}$$

$$\sum_C \nu(\partial S = C) \sum_{x \in C} \left(H_C(0,x) - \frac{1}{|C|} \right) \delta_x = 0 \tag{2.6}$$

i.e. averaging over ν, we can take uniform weights instead of discrete harmonic measure weights for the measure on the boundary?

2.2 Expectation equation

Writing $\nu(\partial S = C) =: p_C$ and $U_C(x) := \frac{1}{|C|}\mathbf{1}(x \in C)$ we may rewrite this as

$$\sum_{C \in \mathcal{L}} p_C \left[H_C(0,x) - U_C(x) \right] = 0 \quad \forall x \in D \subset \mathbb{Z}^2 \tag{2.7}$$

We will refer to $D_x^C := H_C(0,x) - U_C(x)$ as the discrepancy introduced by C at x. The equations above tell us that these discrepancies should cancel at all points $x \in D$.

This equation has a trivial solution:

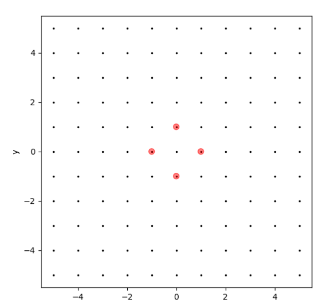


Figure 2.1: Trivial solution to 2.7

For the purpose of measuring the size of sets in D and analyzing different scales, we will make heavy use of the following sets:

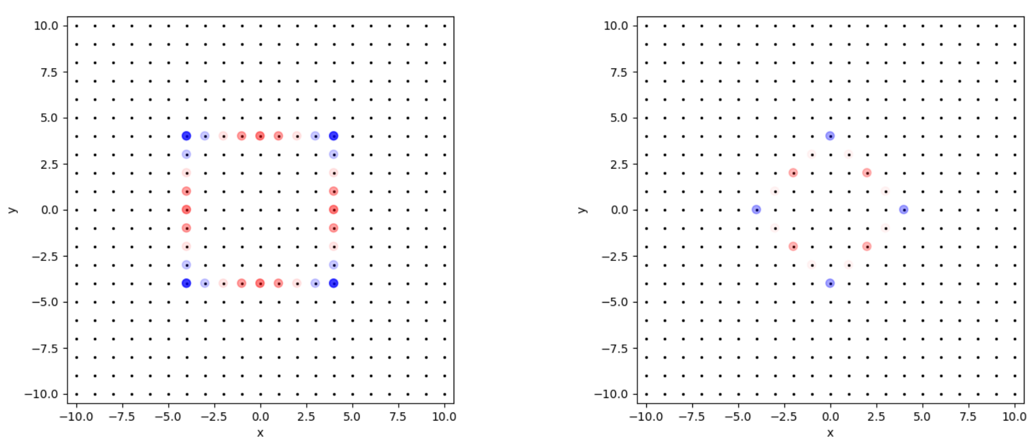
$$A_n = \{x \in \mathbb{Z}^2 : d(0, x) \leq n\} \tag{2.8}$$

These are the set of points reachable from 0 in n steps or less. In this notation, the trivial solution measure ν assigns $\nu(\partial S = A_1 \backslash \{0\}) = 1$. Note that $\partial A_n = A_{n+1} \backslash A_n$. We have the following:

Theorem 2.2.1. *Let $D \subset \mathbb{Z}^2$ be finite, with $A_1 \subset D$. Then the only solution to 2.7 is the trivial solution.*

Remark: A_1 has the structure of a rooted regular tree. If instead of $\mathbb{Z}^2$ we had considered a rooted regular tree $\mathcal{T}$ and the DGFF on it, we get one solution of this form for each level (distance to the root or zero vertex).

We first need a lemma on the discrete harmonic measure at corners of a polygon. For there to be nonzero mass, the point in the corner has to be reachable from zero, or more generally the interior of the polygon. For a square in $\mathbb{Z}^2$, this is not always possible, depending on the orientation of the square.



(a) Deep blue corner points: Zero harmonic measure

(b) Corner points are not deep blue: Nonzero harmonic measure assigned

Figure 2.2: Plotting the difference between the harmonic and uniform measures on arcs, with opacity corresponding to magnitude and color to the sign.

Let C be a loop forming the boundary of a polygon $\mathfrak{P}$, i.e. $C = \partial\mathfrak{P}$. Define the inner radius $r_-(C) = \max\{k : A_k \subseteq \mathfrak{P}\}$ and the outer radius $r_+(C) = \min\{k : \mathfrak{P} \subseteq A_k\}$.

We have that $C \cap \partial A_{r_+(C)-1}$ contains a corner point y of C, i.e. a point where the two edges connecting y to its neighbors in C do not form a straight line.

If both neighbors are in $\partial A_{r_+(C)-1}$, y is not a corner point. If none of the neighbors of y are in $\partial A_{r_+(C)-1}$, it is not reachable from zero, so the harmonic measure vanishes. Hence the discrepancy $D_y^C < 0$. If exactly one neighbor of y is in $\partial A_{r_+(C)-1}$, y is the corner of an edge lying partially in $\partial A_{r_+(C)-1}$. The claim is that even in this case we get a nonzero discrepancy in at least one of the corners. This seems hard to argue in the finite setting (where it is nevertheless borne out by simulation), but we show it in the $r_-(C) \to \infty$ limit. Making sense of this limit, means considering polygonal boundaries $(C_r)_r$ with larger and larger inner radii, s.t. they can be scaled back to the original shape, i.e. there is $(\lambda_r)_r$ decreasing and positive s.t. for all r in the sequence of radii we have

$$\lambda_r C_r \cap \mathbb{Z}^2 = C_{r_0} \tag{2.9}$$

We are really dealing with multiple polygons of the 'same shape' and are interested in the qualitative properties of the discrepancy observed at its corners. For clarity, we will index the polygons and their corner points themselves by r. So if y_{r_0} is a corner point, we have the relation $\lambda_r y_r = y_{r_0}$. When talking about the discrete harmonic measure in this context, it is implicit that it is derived from a random walk started at zero and stopped when exiting $\mathfrak{P}_r$.

Lemma 2.2.2. *Consider the reference polygon $\mathfrak{P}_{r_0}$ with boundary C_{r_0} and corner point $y_{r_0} \in \partial A_{r_+(C_{r_0})-1}$. Then $H_{C_r}(0, y_r)/U_{C_r}(y_r) \to 0$ as $r = r_- \to \infty$. In particular, the corner point introduces a discrepancy $D_{y_r}^{C_r} < 0$ in the $r_- \to \infty$ limit.*

Proof. Consider the Schwarz-Christoffel mapping $\mathbb{D} \to \mathfrak{P}_{\mathfrak{r}_0} \subset \mathbb{C}$, see [6]. These are a class of canonical conformal maps from the upper half plane or the unit disk to polygons. For a polygon with vertices $w_1, \ldots, w_N$ and interior angles

$$\theta_k = \pi(1 + \beta_k) \tag{2.10}$$

one has $f : \mathbb{D} \to \mathfrak{P}_{r_0}$ given by

$$f(z) = A + B \int^z \prod_{k=1}^{N} (\zeta - x_k)^{\beta_k} \, d\zeta, \tag{2.11}$$

where the x_k are the pre-images of the vertices, with A and B constants. A change of the lower integration limit can be absorbed into A.

We next parametrize the unit circle by $z = e^{i\theta}$, $\theta \in [0, 2\pi)$ and endow it with the uniform probability measure $d\mu = \frac{d\theta}{2\pi}$. The pushforward $\tilde{\mu} = f_* \mu$ gives a measure on the boundary of the polygon and is described by

$$\begin{aligned}
\tilde{\mu}(B) &= \mu \circ f^{-1}(B) && (2.12)\\
&= \int_{\mathbb{D} \cap f^{-1}(B)} \frac{d\theta}{2\pi} && (2.13)\\
&= \int_B \frac{1}{2\pi |f'(f^{-1}(s))|} ds && (2.14)
\end{aligned}$$

so pushing the uniform measure forward we obtain density $\frac{1}{2\pi |f'(e^{i\theta})|}$ w.r.t. ds at the point $f(e^{i\theta})$.

Approaching the θ corresponding to the corner point y in the sense $e^{i\theta} = x_j$ for the j s.t. x_j is the preimage of the corner y_{r_0} of the polygon, we have

$$\begin{aligned}
\lim_{\phi \to \theta} f'(e^{i\phi}) &= \lim_{\phi \to \theta} B \prod_{k=1}^{N} (e^{i\phi} - x_k)^{\beta_k} && (2.15)\\
&= \infty && (2.16)
\end{aligned}$$

as $\beta_j < 0$ (determined by the interior turning angle). This uses that the considered corner point is at the outer level, i.e. $y_{r_0} \in \partial A_{r_+(C_{r_0})-1}$. Therefore the density of the pushforward measure is zero at this corner point.

For radius r_- of the polygon $\mathfrak{P}_r$ going to infinity, the discrete harmonic measure is well approximated by (continuous) harmonic measure $\omega_{\partial\mathfrak{P}_{r_0}}(0,\cdot)$, measuring probable exit sites for a Brownian motion in $\mathfrak{P}_{r_0} \subset \mathbb{C}$, started at zero. This follows from the convergence of the two-dimensional random walk to two-dimensional Brownian motion.

It is a standard result that two-dimensional Brownian motion is conformally invariant in the sense that its image under a conformal map is a time changed two-dimensional Brownian motion. The time change does not affect the harmonic measure. [5]

The harmonic measure on the boundary of the unit ball agrees with the uniform measure. By using conformal invariance and considering the conformal Schwarz-Christoffel map $f : \mathbb{D} \to \mathfrak{P}_{r_0}$ we see

$$\omega_{\partial\mathfrak{P}_{r_0}}(0,\cdot) = f_*\mu = \tilde{\mu} \tag{2.17}$$

so we can conclude that in the limit, the discrepancy at corners holds.

□

As an instructive example, we consider the regular octagon, with interior angles $\theta = \frac{(8-2)\pi}{8} = \frac{3\pi}{4}$ and exterior angles $\pi - \theta = \frac{\pi}{4}$.

A natural choice for the pre-vertices on the unit circle are the roots of unity $z_k = e^{2\pi i k/8}$ for $k = 0, 1, \ldots, 7$.

Since $z^8 - 1 = \prod_{k=0}^{7}(z - z_k)$, one may choose $f'(z) = \left(1 - z^8\right)^{-1/4}$ so that

$$f(z) = \int_0^z \left(1 - \zeta^8\right)^{-1/4} d\zeta. \tag{2.18}$$

For the pushforward $\tilde{\mu} = f_*\mu$ to the boundary of the octagon we have

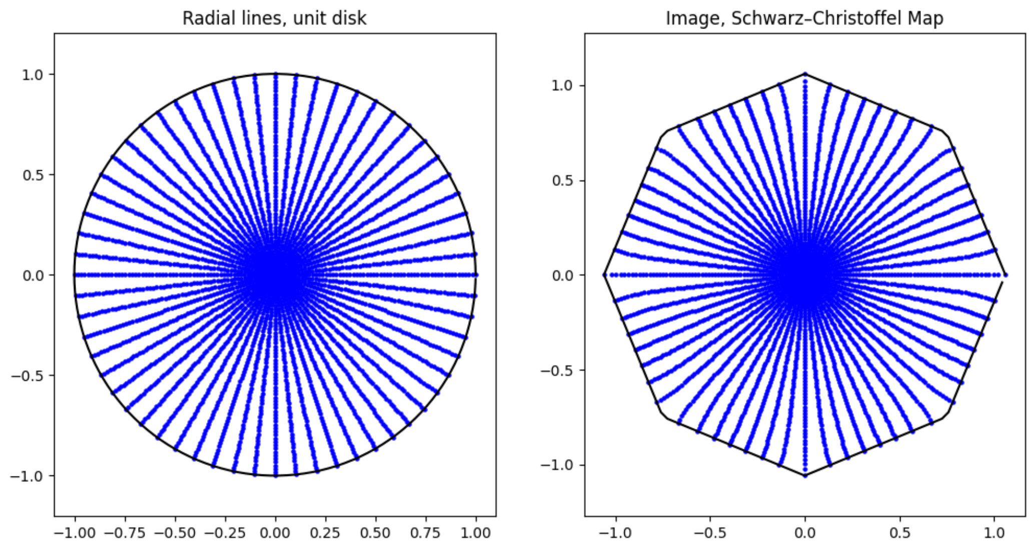


Figure 2.3

$$\tilde{\mu}(B) = \mu \circ f^{-1}(B) \tag{2.19}$$
$$= \int_{\mathbb{D}\cap f^{-1}(B)} \frac{d\theta}{2\pi} \tag{2.20}$$
$$= \int_B \frac{1}{2\pi|f'(f^{-1}(s))|} ds \tag{2.21}$$

with density $\frac{1}{2\pi|f'(e^{i\theta})|}$ at the point $f(e^{i\theta})$. Note that the octagon has circumference slightly larger than 2π, so to get a probability density we would need to correct for this.

We compute:

$$\left|f'(e^{i\theta})\right| = \left|1 - e^{8i\theta}\right|^{-1/4} = \left(\left[(1-\cos(8\theta))^2 + \sin(8\theta)^2\right]^{\frac{1}{2}}\right)^{-\frac{1}{4}} \tag{2.22}$$
$$= ((4\sin^2(4\theta))^{\frac{1}{2}})^{-\frac{1}{4}} \tag{2.23}$$
$$= 2^{-\frac{1}{4}}\sin(4\theta)^{-\frac{1}{4}} \tag{2.24}$$

so the density w.r.t. ds at $f(e^{i\theta})$ is

$$\frac{d\tilde{\mu}}{ds}(f(e^{i\theta})) = \frac{2^{\frac{1}{4}}\sin(4\theta)^{\frac{1}{4}}}{2\pi} \tag{2.25}$$

The first corner is at $\theta = 0$, the second one at $\theta = \frac{2\pi}{8} = \frac{\pi}{4}$. These are assigned density zero. At the midpoint of the first edge, where $\theta = \frac{\pi}{8}$ the above density is maximized.

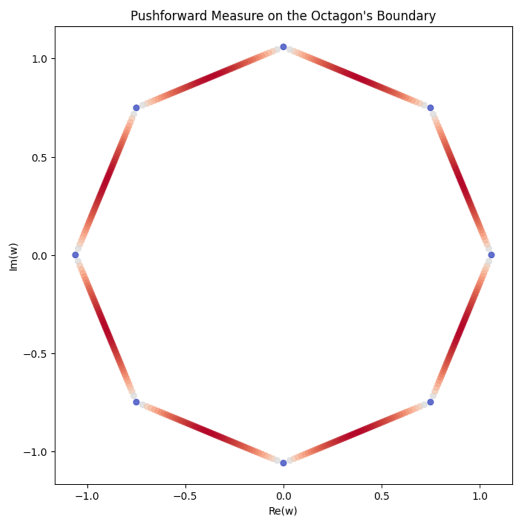


Figure 2.4: Pushforward of the uniform measure to the octagonal boundary

Octagons with two alternating sidelengths could in principle be handled similarly, but there might not be a clean closed form solution available.

We move on to a proof of the theorem:

Proof. Since $D \subset \mathbb{Z}^2$ is finite, we have the containment $D \subset A_N$ for some $N \in \mathbb{N}$. If there is a measure ν on loops in D supported on more than one element s.t. equation 2.7 holds we have a nontrivial solution, and then clearly the same measure considered as a measure on loops supported in A_N is a nontrivial solution. The strategy is to show that no such nontrivial solution can exist for the problem considered in any A_n.

We take the base case to be $n = 2$, where the possible collections of loops in the support and the resulting linear systems can be enumerated, with no available solution to any of the linear systems.

Assume the problem is not solvable in A_n, and suppose it were in A_{n+1}. Then the solution measure ν gives nonzero mass to loops (partially) in $A_{n+1} \backslash A_n$. Take any a loop C in the support of ν and partially in $A_{n+1} \backslash A_n$. It must have at least one corner point in $x \in A_{n+1} \backslash A_n$. If this were the only loop in the support of ν containing x, the equation 2.7 does not hold, since by lemma 2.2.2 we have an uncompensated discrepancy. So there must be some other loop C' in its support, containing x, but for which it is not a corner point. This gives another corner point x' of C' which will lie on a direct line between x and one of the corners of $A_{n+1} \backslash A_n$. Iterating, we get in the end a loop C'' in the support of ν containing a corner point of $A_{n+1} \backslash A_n$, say y, which is a corner point of any loop in A_{n+1}. Since again by 2.2.2, the discrepancy at y is nonzero for any loop in A_{n+1}, we conclude that 2.7 is not solvable in A_{n+1} for any n and hence in no finite $D \subset \mathbb{Z}^2$.

$\square$

A question motivated by the remark just after 2.2.1: Can we classify and determine connectivity conditions for graphs where there exist multiple solutions to the equation 2.7?

2.3 Asymptotic solution

With no interesting exact solutions in $\mathbb{Z}^2$, we turn to approximate ones. Can we construct a sequence ν_n solving the following problem

$$\left(\sum_{C \in \mathcal{L}} \nu_n(\partial S = C) \left[H_C(0,x) - U_C(x) \right] \right) = o(n^{-(2+\epsilon)}) \quad \forall x \in D \subset \mathbb{Z}^2 \quad (2.26)$$

for some $\epsilon > 0$?

2.3.1 Octagonal system

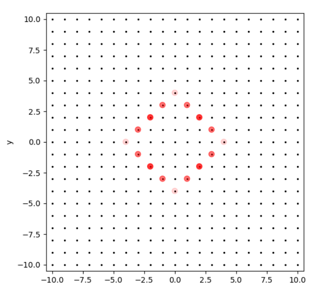
Definition 2.3.1. *The octagonal system truncated at n is the collection of octagonal (or possibly square) shapes formed by vertices in $\mathbb{Z}^2$ given by*

$$\mathcal{O}_n = \{O_{i,j} : 0 \leq i \leq j \leq n\} \tag{2.27}$$

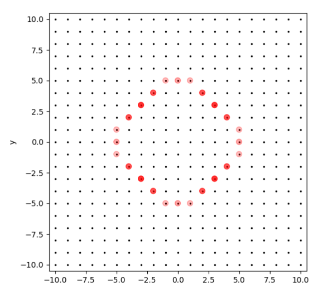
where $O_{i,j}$ is the unique $\pi/2$-rotationally symmetric subset of $\mathbb{Z}^2$ s.t. its intersection with the first quadrant $O_{i,j} \cap \mathbb{N}^2$ is equal to

$$\left([0,j]^2 \cap \mathbb{N}^2\right) \setminus \{(j-k, j-l) : 0 \leq k \leq j-i, 0 \leq l \leq j-i-k\} \tag{2.28}$$

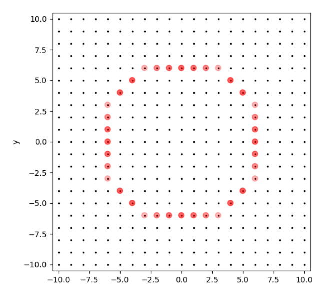
This last expression might seem convoluted, but all it says is that the intersection with the first quadrant $\{(x, y) \in \mathbb{Z}^2 : x, y \geq 0\}$ is given by the quadratic block $[0, j]^2 \cap \mathbb{N}^2$ where we cut away a right triangle with two sides of length $j - i$ in the corner (j, j). A special case is the square $O_{j,j}$.



(a) $(i, j) = (0, 3)$



(b) $(i, j) = (1, 4)$



(c) $(i, j) = (3, 5)$

Figure 2.5: Resulting boundary $\partial O_{i,j}$ for different (i, j), opacity corresponding to harmonic measure

Definition 2.3.2. *The (full) octagonal system is*

$$\mathcal{O} := \bigcup_{n\geq 0} \mathcal{O}_n \tag{2.29}$$

$$= \bigcup_{n\geq 0} \{O_{i,j} : i \leq j \leq n\} \tag{2.30}$$

Question: Is there a sequence of measures (ν_n) with ν_n supported on $\mathcal{O}_n$ s.t. 2.26 holds?

One can form the matrix $D = (D_x^C)_{x,C}$ for $C \in \partial\mathcal{O}_n$ and $x \in \{(x_1, x_2) : 0 \leq x_2 \leq x_1 \leq n{+}1\}$. This collection of points suffices due to the symmetry of the considered octagonal shapes. Let also $p = (p_C)_C$. Adding the constraint $\sum_C p_C = 1$ as a row to D and solving the system $Dp = 0$ 'as well as possible' under the additional constraints $p_C \geq 0$ for all C, one obtains a probability measure ν_n supported on $\mathcal{O}_n$.

What does as well as possible mean? One possibility is treating the equations corresponding to $x = (x_1, x_2)$ with $x_1 \leq n$ as well as $\sum_C p_C = 1$ as hard constraints and then solving the equations for those x with $x_1 = n + 1$ in the least squares sense. After testing this for some small values of n and taking into account that the general structure of D does not change (it is always a very sparse system), this seems possible. What this means is that the discrepancies are pushed out to the outermost points (and exist only there). Recall that in the proof of theorem 2.2.1, we used that such discrepancies must exist at the outermost layer, but gave no information about the situation in the interior of the volume D. It remains an open question what can be said of the decay rate of the residuals.

2.3.2 Lower and upper truncated octagonal system

We can call $\mathcal{O}_n$ the upper truncated octagonal system. If one wants to steadily increase the minimum radius of octagons, adding a lower trunctation might make sense:

$$\mathcal{O}_{m_1,m_2} := \mathcal{O}_{m_2} \backslash \{O \in \mathcal{O} : O \subseteq A_{m_1}\} \tag{2.31}$$

for $m_1 < m_2$ and where A_m are the set of points reachable from zero in m steps introduced previously.

We can reformulate the question from the previous subsection: Is there a 'reasonable' $g : \mathbb{N} \to \mathbb{N}$ and a sequence of measures (ν_n) with ν_n is supported on $\mathcal{O}_{n,g(n)}$ s.t. 2.26 holds? What decay rate of the residuals can be achieved in this setup?

2.4 Another candidate: Evolving sets

Since the discrete gaussian free field is intimately connected to the random walk, it seems sensible to use some randomness related to this related process when designing the random arcs. A model we will consider is the evolving sets process, see [4].

The setup is an irreducible, aperiodic Markov chain X on a countable state space V, stationary distribution $\pi(\cdot)$ and transition probabilities $p(\cdot, \cdot)$ with $\sum_x \pi(x)p(x, y) = \pi(y)$. Letting $Q(x, y) = \pi(x)p(x, y)$ we can further define $Q(A, B) = \sum_{a \in A, b \in B} Q(a, b)$. Given this data, we are in a position to define a random process $(S_n)_n$ valued in subsets of the state space V.

Definition 2.4.1. *(Evolving sets) For given deterministic nonempty $S_0 \subset V$, the evolving set process $(S_n)_{n \geq 0}$ is constructed by choosing $(U_n)_{n \geq 1}$ iid uniform random variables on $[0, 1]$ and defining iteratively for $n \geq 0$*

$$S_{n+1} = \{y : Q(S_n, y) \geq U_{n+1}\pi(y)\} \tag{2.32}$$

Note that this is a Markov chain on the space of subsets of V and that $P(y \in S_{n+1} | S_n = S) = \frac{Q(S,y)}{\pi(y)}$. This process is very interesting in its own right and there are a number of other useful relations that can be derived, finding uses to examine mixing. [4]

We will specialize to consider $V = \mathbb{Z}^2$ and transition probabilities given by a random walk with holding probability $\frac{1}{2}$. We should be a bit careful when handling the stationary measure in this case since we have a countably infinite state space and no concentration. We can however modify the definition of evolving set to iteratively construct

$$S_{n+1} = \{y : \sum_{x \in S_n} p(x,y) \geq U_{n+1}\} \tag{2.33}$$

This implies a set of transition probabilities on the collection of subsets of $\mathbb{Z}^2$, which we may denote by

$$\tilde{p}(S,T) := P(S_{n+1} = T | S_n = S) = P(S_1 = T | S_0 = S) \tag{2.34}$$

Lemma 2.4.2. *The evolving set process is connected to the underlying random walk via*

$$p^n(x,y) = P(y \in S_n | S_0 = \{x\}) \tag{2.35}$$

Proof. $n = 0$ is clear. Assuming this holds for $n \geq 0$, we compute for $n+1$:

$$\begin{aligned} p^{n+1}(x,y) &= \sum_z p^n(x,z)p(z,y) && (2.36)\\ &= \sum_z P(z \in S_n | S_0 = \{x\})p(z,y) && (2.37)\\ &= E[\sum_{z \in S_n} p(z,y) | S_0 = z] && (2.38)\\ &= E[P(y \in S_{n+1} | S_n) | S_0 = \{x\}] && (2.39)\\ &= P(y \in S_{n+1} | S_0 = x) && (2.40) \end{aligned}$$

□

The martingale property of $(|S_n|)_n$ will be of use later:

$$E[|S_{n+1}||S_n] = \sum_x P(x \in S_{n+1}|S_n) \tag{2.41}$$
$$= \sum_x \sum_{z \in S_n} p(z,x) \tag{2.42}$$
$$= \sum_{z \in S_n} \sum_x p(z,x) \tag{2.43}$$
$$= \sum_{z \in S_n} 1 = |S_n| \tag{2.44}$$

Lemma 2.4.3. *For the particular choice of a regular random walk X on $V = \mathbb{Z}^2$ and $S_0 = \{0\}$, the state space of the evolving set process is the full octagonal system with the empty set appended.*

$$\mathcal{O} \cup \{\emptyset\} \tag{2.45}$$

Proof. This will follow by induction on n and the case $n = 0$ is clear since $S_0 = \{0\} = O_{0,0}$.

Assume S_n is supported on $\mathcal{O}_n$. Then on sets of the form $\{S_n = O_{i,j}\}$, $i \leq j \leq n$, the possible number of neighbors in S_n of a point in $\mathbb{Z}^2$ ranges from 0 to 4. The uniform random variable U_{n+1} determines a threshhold m (a.s. greater than zero) s.t. all points with number of neighbors greater or equal than $m = m(U_{n+1})$ are included in the set S_{n+1}. We get a map $m \mapsto S_{n+1}(m)$ given by

$$1 \mapsto O_{i+1,j+1} \tag{2.46}$$
$$2 \mapsto O_{(i+1)\wedge j,j} \tag{2.47}$$
$$3 \mapsto O_{i-1,j} \text{ for } i \geq 1 \quad O_{0,j-1}, \text{ for } i = 0 \tag{2.48}$$
$$4 \mapsto O_{i,j-1} \tag{2.49}$$

with the convention that $O_{i,-1} = \emptyset$. So S_{n+1} is in $\{\emptyset\} \cup \mathcal{O}_{n+1}$. □

Since $P(S_n \neq \emptyset) = P(0 \in S_n)$ and $p^n(0,0) \to 0$ as $n \to \infty$, we have $S_n \to \emptyset$ a.s..

A slight modification of the law will ensure that we don't see this collapse:

Definition 2.4.4. *(Size-biased evolving sets) Let $\tilde{p}(\cdot,\cdot)$ be the transition probabilities for the evolving set process. The size-biased evolving set process is described by the transformed transition probabilities*

$$p^{\bullet}(S,T) = \frac{|T|}{|S|}\tilde{p}(S,T) \tag{2.50}$$

This makes sense since

$$\begin{aligned}
\sum_T p^{\bullet}(S,T) &= \sum_T \frac{|T|}{|S|}\tilde{p}(S,T) && (2.51)\\
&= \frac{1}{|S|}\sum_{n\geq 0} nP(|S_1| = n | S_0 = S) && (2.52)\\
&= \frac{1}{|S|}E[|S_1||S_0 = S] && (2.53)\\
&= 1 && (2.54)
\end{aligned}$$

using the martingale property of $(|S_n|)_n$.

We see immediately that for nonempty S, $p^{\bullet}(S,\emptyset) = 0$ and so the resulting process, which we will denote by $(S_n^{\bullet})_n$, never collapses to $\emptyset$.

Questions:

1. Taking ν_n to be the measure of evolving or size-biased evolving sets, do we solve equation 2.26?

2. We can for a given sampled S_n or $S_n^{\bullet}$ decompose into shells $S = \bigsqcup_{k\geq 0} C_k$ and pick a shell C_j with probability proportional to its size. Any shell (except $\{0\}$) is the boundary of some octagon in $\mathcal{O}_n$. The sampling of shells in this way induces a measure on octagons. What can be said about this measure and equation 2.26?

Delocalization

One phenomenon we touched briefly upon when introducing the continuous Gaussian free field is the blow up of the discrete Green's function. In a domain $D_N = [-N, N]^2$ growing with N, estimates for the random walk give

$$G_{D_N}(0,0) = O(\log N), \tag{3.1}$$

see again [3]. Since this is the variance of the field at the origin, we get a problem in the infinite volume and this is referred to as delocalization. Even though, as previously remarked, there might be random walk representations for other models of random fields, this rather direct method of proof that $\mathrm{Var}^{D_N}(\phi_0) \to \infty$ seems very particular to the gaussian free field. Using Brascamp-Lieb inequalities one can widen the range of interaction potentials for which the behavior of this variance can be understood. [7] [2]

One long-term goal that I was introduced to during my work on this semester project is to find an alternative proof strategy of delocalization that does not rely on asymptotic estimates of the Green's function. I will now quickly sketch some motivating ideas behind what we might call the martingale increments approach to delocalization, with some computations borrowed from [1] that I think shed some light on why this approach could be fruitful.

Let $(B_n)_{n\geq 0}$ be an appropriately shaped family of balls, $B_0 = \{0\}$ and $D_N = B_N$. Consider a discrete Gaussian free field $(\phi_x)_{x\in D_N}$ with zero boundary conditions and recall the discrete version of circle average with the associated backwards martingale

$$M_k := \sum_{x \in \partial B_{k-1}} H_{\partial B_{k-1}}(0,x)\phi_x = E[\phi_0 | \mathcal{F}_{D_N \setminus B_{k-1}}] \quad 1 \le k \le N+1 \tag{3.2}$$

We set $M_0 := \phi_0$ and note that $M_{N+1} = 0$ due to the boundary condition. For added clarity, we will write $\mathrm{Var}^{B_k}(\cdot)$ to denote the variance with respect to the measure on a zero boundary gaussian free field in B_k. It is important to remember that $\mathrm{Var}^{B_k}(\phi_x)$ and $\mathrm{Var}^{B_j}(\phi_x)$ are not only not the same (as numbers), but that the ϕ_x means different things here becuase the fields are different.

We compute

$$\begin{aligned}
\mathrm{Var}^{D_N}(\phi_0) &= \mathrm{Var}^{D_n}(M_0) && (3.3)\\
&= \mathrm{Var}^{D_N}\left(\sum_{k=0}^{N} M_k - M_{k+1}\right) && (3.4)\\
&= \sum_{k=0}^{N} \mathrm{Var}^{D_N}(M_k - M_{k+1}) && (3.5)
\end{aligned}$$

due to orthogonality of martingale increments. Defining $A_k = D_N \setminus B_k$ and recalling the definition of Γ_A and Γ^A in 1.15 we can gain a new perspective on the individual terms:

$$\begin{aligned}
\mathrm{Var}^{D_N}(M_k - M_{k+1}) &= \mathrm{Var}^{D_N}(M_k - \Gamma_{A_k}(0)) && (3.6)\\
&= \mathrm{Var}^{D_N}\left(\sum_{x \in \partial B_{k-1}} H_{\partial B_{k-1}}(0,x)\phi_x - \sum_{x \in \partial B_k} H_{\partial B_k}(0,x)\Gamma_{A_k}(x)\right) && (3.7)\\
&= \mathrm{Var}^{D_N}\left(\sum_{x \in \partial B_{k-1}} H_{\partial B_{k-1}}(0,x)(\phi_x - \Gamma_{A_k}(x))\right) && (3.8)\\
&= \mathrm{Var}^{B_k}(M_k) && (3.9)
\end{aligned}$$

where in the second equality we use that $0 \notin A_k$ and Γ_{A_k} is the harmonic extension from ∂B_k to B_k with boundary data given by the field. In the third equality, we extend once more (from ∂B_{k-1}) with data given by the previous harmonic extension, obtaining the original extension. Since Γ^{A_k} is a zero boundary Gaussian free field in B_k we get the last equality.

The terms $\mathrm{Var}^{B_k}(M_k)$ seem interesting, since they measure the variance of a harmonic circle average on a circle that is 'hugged' very tightly by the boundary of the domain of the field. This somehow seems more local than the asymptotic Green's function approach and one is led to wonder whether this can be used as a more robust tool for delocalization proofs. We gather these thoughts in the following concluding questions:

1. Is it possible to argue without using Green's function estimates that $\mathrm{Var}^{B_k}(M_k) = O(1/|\partial B_k|)$?

2. Is it possible to forge a connection between this and the previous sections, using some notion of random discrete circle (changing the terms at the level of the B_k's) to remove dependence on the harmonic measure?

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