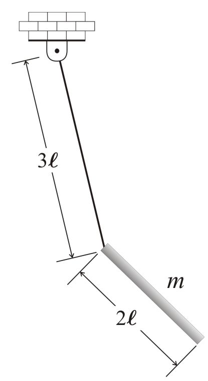
AER301H1 F

# Dynamics

## Problem Set IV

1. Consider a uniform rod of length $2\ell$ and mass $m$ attached by one end to a massless string of length $3\ell$. The motion is planar. Determine the equations of motion in matrix form.

   This problem was first addressed by Daniel Bernoulli, then Euler followed by Daniel's father Johann.




2. Prove the *Beltrami identity*,

$$\frac{d}{dx}\left(y'\frac{\partial F}{\partial y'} - F\right) + \frac{\partial F}{\partial x} = 0$$

   where $y(x)$ is the extremal of

$$I = \int_{x_1}^{x_2} F(y, y', x)dx$$

3. Let $\mathbf{J}$ represent the inertia matrix of a rigid body.

   ($a$) Prove the "triangle inequalities":

$$J_{11} + J_{22} > J_{33}$$
$$J_{22} + J_{33} > J_{11}$$
$$J_{33} + J_{11} > J_{22}$$

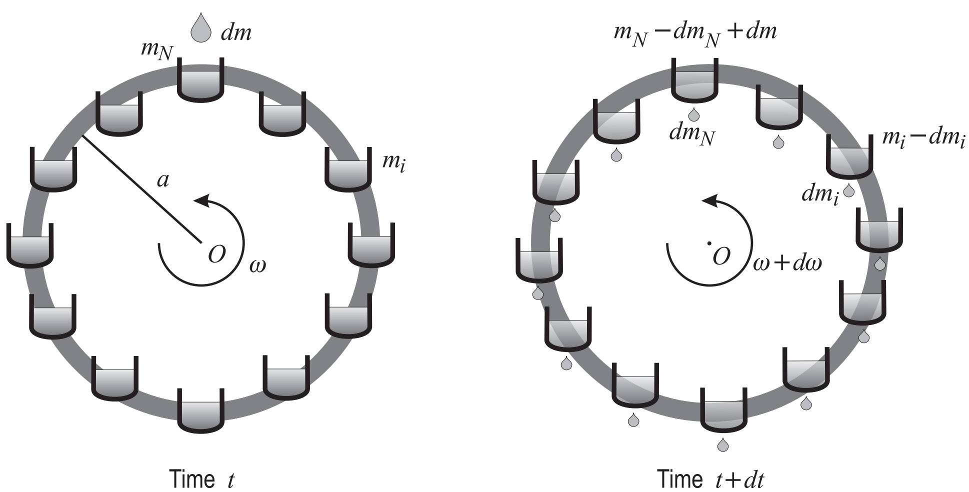
($b$) In addition, show that

$$J_{11}^2 > 4J_{23}^2$$
$$J_{22}^2 > 4J_{31}^2$$
$$J_{33}^2 > 4J_{12}^2$$

**4.** Consider a system as shown above where a number of buckets are attached, each by a frictionless pin, to a wheel of diameter $a$ which can rotate in the vertical plane. The system under consideration consists of the masses $m_i$ of water in each of the revolving buckets and the elemental mass $dm$ of water that is about to be added to the top bucket. The total mass of the water in the buckets is

$$m = \sum_{i=1}^{N} m_i$$

and is assumed constant so that the amounts $dm_i$ leaked from each bucket total $dm$. Let $\omega$ denote the angular speed of the wheel. Take the buckets to be point masses and ignore the mass of the wheel itself. None of the water leaks from one bucket into another.




**Part I.** A viscous frictional torque of the form

$$\tau_f = -c_d\omega$$

works on the wheel. Assuming that the rate at which water leaks out of each bucket is proportional to the head of water in the bucket or more simply the mass of the water, *i.e.*, $\dot{m}_i = km_i$, show that

$$\dot{\omega} = -(k+c)\omega - by$$

where $c \triangleq c_d/ma^2$ and $b \triangleq g/a^2$.

**Part II.** The center of mass of the system in the vertical plane is $(y_{\odot}, z_{\odot})$ which is not fixed. (Note $y$ is the horizontal coordinate and $z$ the vertical.) Show too then that

$$
\begin{aligned}
\dot{y}_{\odot} &= -k y_{\odot} - \omega z_{\odot} \\
\dot{z}_{\odot} &= k(a - z_{\odot}) + \omega y_{\odot}
\end{aligned}
$$

For convenience, recast the above three equations, respectively, in dimensionless form as

$$
\begin{aligned}
\varpi' &= -s_1 \varpi + s_2 \eta \\
\eta' &= \varpi - \eta - \varpi\zeta \\
\zeta' &= -\zeta + \varpi\eta
\end{aligned}
$$

Identify all the quantities given that dimensionless time is $\tau = kt$ and $(\cdot)' = d(\cdot)/d\tau$.

**Part III.** Unfortunately, these equations must be solved numerically. Take $s_1 = 3.5714$ and $s_2 = 51.0204$. Each of you, however, will have different initial conditions. Assume $\eta(0) = -0.60407$ and $\zeta(0) = 0.93000$ but for your $\varpi(0)$, consult the Attachment. Plot each variable against dimensionless time for $\tau \in [0, 600]$. (Make sure your integrator can handle the numerical sensitivity.)

**Part IV.** For the same initial conditions over the same time interval, plot $\varpi$ *vs* $\eta$, $\varpi$ *vs* $\zeta$ and $\zeta$ *vs* $\eta$.

**DUE: 15H10, 22 NOVEMBER 2016**

*You may, of course, hand it in sooner but not later without penalty! Any solution submitted after this time and date will be subject to a 10% penalty per school (ahem, skule) day.*

*Problem 4 is weighted triple (or treble) the each of the other problems.*

*Let it be known that only a subset of the solutions to this Problem Set will be marked. (All students will have solutions to the same problems graded.) Marks will nevertheless be deducted if solutions to the "ungraded" problems are not submitted.*