COMP0083 Convex Optimisation Coursework

Jan 9, 2023

Part 1: Questions with multiple answers

1.1

The function for (a) is a convex function:

$$\max\{ax + b, x^4 - 5, e^{x^2}\}$$

1.2

For the function:

$$f(x) = \begin{cases} -x, & \text{if } x \in]-1, 0] \\ x^2, & \text{if } x \geq 0 \end{cases}$$

the sub-differential is:

$$\partial f(x) = \begin{cases} -1, & \text{if } x \in]-1, 0[\\ [0, 1], & \text{if } x = 0 \\ 2x, & \text{if } x > 0 \end{cases}$$

corresponding to Figure (a).

1.3

For a function $f(x) = \langle Ax, x\rangle + \langle x, b\rangle + c$ where A is a square matrix not necessarily symmetric, the gradient is (a):

$$\nabla f(x) = A^* x + Ax + b$$

where A^* denotes the transpose of A.

1.4

The Fenchel conjugate of $f(x) = g(2x)$ is (a):

$$f^*(u) = g^*(u/2)$$

1.5

The solution to the dual problem of the ridge regression problem is (c):

$$\bar{u} = (\mathbf{K} + \lambda n \mathbf{Id})^{-1} y$$

Part 2: Theory on convex analysis and optimization

2.1

The Fenchel conjugate of $f(x)$ is defined as:

$$f^*(u) = \sup_{x \in \mathcal{X}} \{\langle x, u\rangle - f(x)\}$$

2.1.1

Given :

$$f(x) = \begin{cases} +\infty, & \text{if } x \leq 0 \\ -\log x, & \text{if } x > 0 \end{cases}$$

Substituting into the Fenchel conjugate definition:

$$f^*(u) = \sup_{x>0} \{\langle x, u\rangle + \log x\}$$

To find the supremum, we can take the partial derivative with respect to x, set to zero, and solve for x:

$$\frac{\partial}{\partial x}(ux + \log x) = u + \frac{1}{x} = 0$$

Thus, the supremum above is acheived when $x = \frac{-1}{u}$:

$$f^*(u) = u\left(\frac{-1}{u}\right) + \log\left(\frac{-1}{u}\right)$$

where $x > 0$ so $u < 0$. Simplifying, we have the Fenchel conjugate:

$$f(x) = \begin{cases} -(1 + \log(-u), & \text{if } u < 0 \\ +\infty, & \text{if } u \geq 0 \end{cases}$$

2.1.2

Given:

$$f(x) = x^2$$

Substituting into the Fenchel conjugate definition:

$$f^*(u) = \sup_{x \in \mathcal{X}} \{\langle x, u\rangle - x^2\}$$

We can compute the partial derivative with respect to x, set to zero, and solve for x:

$$\frac{\partial}{\partial x}(ux - x^2) = u - 2x = 0$$

Thus, the supremum is achieved when $x = \frac{u}{2}$:

$$f^*(u) = u\left(\frac{u}{2}\right) - \left(\frac{u}{2}\right)^2$$

Simplifying, we have the Fenchel conjugate:

$$f^*(u) = \frac{u^2}{4}$$

2.1.3

Given:

$$f(x) = i_{[0,1]}$$

Substituting into the Fenchel conjugate definition:

$$f^*(u) = \sup_{x \in \mathcal{X}} \{\langle x, u \rangle - i_{[0,1]}\}$$

Thus,

$$f^*(u) = \sup_{x \in [0,1]} \{\langle x, u \rangle\}$$

We can see that the Fenchel conjugate is:

$$f^*(u) = \max(0, u)$$

the ReLu function.

2.2.1

Given f a proper convex function, to prove by induction Jensen's inequality:

$$f\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq \sum_{i=1}^{n} \lambda_i f(x_i)$$

for all $x_1, ...x_n \in \mathcal{X}$ and for all $\lambda_1, ..., \lambda_n \in \mathbb{R}_+$ with $\sum_{i=1}^{n} \lambda_i = 1$, we start with the definition of convexity which states:

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

for all $x_1, x_2 \in \mathcal{X}$ and $\lambda_1, \lambda_2 \in \mathbb{R}_+$ with $\lambda_1 + \lambda_2 = 1$. This acts as a base case for Jensen's inequality when $n = 2$. Our inductive step will prove that the inequality continues to hold for $n + 1$. We begin with:

$$f\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) = f\left(\sum_{i=1}^{n} \lambda_i x_i + \lambda_{n+1} x_{n+1}\right)$$

where $\sum_{i=1}^{n+1} \lambda_i = 1$. We can insert the term $(1 - \lambda_{n+1})$:

$$f\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) = f\left((1 - \lambda_{n+1}) \sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} x_i + \lambda_{n+1} x_{n+1}\right)$$

If we define $\bar{x} = \sum_{i=1}^{n} \frac{\lambda_i}{1-\lambda_{n+1}} x_i$, we are back to our $n = 2$ base case, so we know that:

$$f\left((1 - \lambda_{n+1})\bar{x} + \lambda_{n+1} x_{n+1}\right) \leq (1 - \lambda_{n+1}) f\left(\bar{x}\right) + \lambda_{n+1} f\left(x_{n+1}\right)$$

Substituting back the definition for $\bar{x}$:

$$f\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) \leq (1 - \lambda_{n+1}) f\left(\sum_{i=1}^{n} \frac{\lambda_i}{1 - \lambda_{n+1}} x_i\right) + \lambda_{n+1} f\left(x_{n+1}\right)$$

We know that from the inequality in the n case:

$$f\left(\sum_{i=1}^{n}\frac{\lambda_i}{1-\lambda_{n+1}}x_i\right)=\frac{1}{1-\lambda_{n+1}}f\left(\sum_{i=1}^{n}\lambda_i x_i\right)\leq\frac{1}{1-\lambda_{n+1}}\sum_{i=1}^{n}\lambda_i f\left(x_i\right)=\sum_{i=1}^{n}\frac{\lambda_i}{1-\lambda_{n+1}}f\left(x_i\right)$$

Thus we can upper bound our previous inequality:

$$f\left(\sum_{i=1}^{n+1}\lambda_i x_i\right)\leq\left(1-\lambda_{n+1}\right)\sum_{i=1}^{n}\frac{\lambda_i}{1-\lambda_{n+1}}f\left(x_i\right)+\lambda_{n+1}f\left(x_{n+1}\right)=\sum_{i=1}^{n+1}\lambda_i f\left(x_i\right)$$

proving our inductive step for $n+1$ and Jensen's inequality as required. □

2.2.2

The characterisation for differential functions for convexity states:

$$\text{A function } f \text{ is convex} \leftrightarrow \langle\nabla f(x)-\nabla f(y), x-y\rangle\geq 0$$

$\forall x, y \in dom f$.
For $f(x)=-\log(x)$, $\nabla f(x)=\frac{-1}{x}$:

$$\langle\nabla f(x)-\nabla f(y), x-y\rangle=\langle\frac{1}{y}-\frac{1}{x}, x-y\rangle$$

If $x>y$:

$$x-y>0 \text{ and } \frac{1}{y}-\frac{1}{x}>0$$

so $\langle\frac{1}{y}-\frac{1}{x}, x-y\rangle>0$.
If $x<y$:

$$x-y<0 \text{ and } \frac{1}{y}-\frac{1}{x}<0$$

so $\langle\frac{1}{y}-\frac{1}{x}, x-y\rangle>0$.
If $x=y$:

$$x-y=0 \text{ and } \frac{1}{y}-\frac{1}{x}=0$$

so $\langle\frac{1}{y}-\frac{1}{x}, x-y\rangle=0$.
Thus, $\forall x, y \in dom f$, $\langle\nabla f(x)-\nabla f(y), x-y\rangle\geq 0$ so by the characterization for differentiable functions, $f(x)=-\log(x)$ is convex. □

2.2.3

Given Jensen's inequality in 2.2.1 for a convex function $f(x)$:

$$f\left(\sum_{i=1}^{n}\lambda_i x_i\right)\leq\sum_{i=1}^{n}\lambda_i f(x_i)$$

and having proved in 2.2.2 that $f(x) = -\log(x)$ is convex, we can write:

$$-\log\left(\sum_{i=1}^{n} \lambda_i x_i\right) \leq -\sum_{i=1}^{n} \lambda_i \log(x_i)$$

Rearranging:

$$\sum_{i=1}^{n} \lambda_i x_i \geq \exp\left(\sum_{i=1}^{n} \log(x_i^{\lambda_i})\right)$$

Choosing $\lambda_i = \frac{1}{n}$:

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \exp\left(\sum_{i=1}^{n} \log(x_i^{\frac{1}{n}})\right)$$

so $\sum_{i=1}^{n} \lambda_i = 1$ as required by Jensen's inequality.
The sum of logarithms is the logarithm of the products:

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \exp\left(\log((x_1 \cdots x_n)^{\frac{1}{n}})\right)$$

Thus we get our inequality:

$$\frac{1}{n}\sum_{i=1}^{n} x_i \geq \sqrt[n]{x_1 \cdots x_n}$$

for all $x_1, ..., x_n \in \mathbb{R}^+$ $\square$

2.3

Given a polytope $C = co(a_1, ..., a_m)$ in X, we know that any point $x \in C$ can be expressed:

$$x = \sum_{i=1}^{m} \lambda_i a_i$$

where $\sum_{i=1}^{m} \lambda_i = 1$. Knowing that f is a convex function on C, we can write Jensen's inequality:

$$f\left(\sum_{i=1}^{m} \lambda_i a_i\right) \leq \sum_{i=1}^{m} \lambda_i f(a_i)$$

Because $\sum_{i=1}^{m} \lambda_i f(a_i)$ is a weighted average of $f(a_i)$'s we know that:

$$\sum_{i=1}^{m} \lambda_i f(a_i) \leq \max_i f(a_i)$$

the weighted average will always be less than or equal to the maximum $f(a_i)$ value. Thus, we know that:

$$f\left(\sum_{i=1}^{m} \lambda_i a_i\right) \leq \max_i f(a_i)$$

So all function evaluations of $f(x)$ for $x \in C$ are less than or equal to the maximum of the function evaluations at the verticies $a_1, ..., a_m$ and so the maximum of the convex function f on C is attained at one of the vertices $a_1, ..., a_m$. $\square$

2.4

To prove that the function $f(x, y) = \|x - 2y\|^2$ is convex, we will show that the Hessian is a positive semi-definite matrix. The Hessian is defined as:

$$\nabla^2 f(x, y) = \begin{bmatrix} \frac{\partial f(x,y)}{\partial^2 x} & \frac{\partial f(x,y)}{\partial y \partial x} \\ \frac{\partial f(x,y)}{\partial x \partial y} & \frac{\partial f(x,y)}{\partial^2 y} \end{bmatrix}$$

Calculating each term:

$$\nabla^2 f(x, y) = \begin{bmatrix} 2 & -4 \\ -4 & 8 \end{bmatrix}$$

we have our Hessian matrix for which the eigenvalues are greater than or equal to zero, so we see that the Hessian is positive semi-definite and so the function is jointly convex.

2.5

The conditions for the existence of minimizers is that f is closed and coercive. The conditions for the uniqueness of minimizers is that f is strictly convex. Thus the minimal sufficient conditions for the existence and uniqueness of minimizers for a convex function f, are that f is closed, coercive, and strictly convex.

2.6

We are considering the optimisation problem:

$$\min_{\|\mathbf{Ax}-\mathbf{b}\|_\infty \leq \epsilon} \frac{1}{2}\|\mathbf{x}\|^2$$

where $\epsilon > 0$, $\mathbf{A} \in \mathbb{R}^{n \times d}$, $\mathbf{b} \in \mathbb{R}^{n \times 1}$, and $\mathbf{x} \in \mathbb{R}^{d \times 1}$.

2.6.1

To compute the dual problem, we begin by reformulating the problem as:

$$\min_{\mathbf{x}} f(\mathbf{x}) + g(\mathbf{Ax})$$

where $f(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2$ and $g(\mathbf{Ax}) = i_{\frac{1}{\epsilon}\|\mathbf{Ax}-\mathbf{b}\|_\infty \leq 1}(\mathbf{Ax})$, the indicator function of the unit ball for $\frac{1}{\epsilon}\|\mathbf{Ax} - \mathbf{b}\|_\infty$.

Thus we have our primal problem:

$$\min_{\mathbf{x}} \frac{1}{2}\|\mathbf{x}\|^2 + i_{\frac{1}{\epsilon}\|\mathbf{Ax}-\mathbf{b}\|_\infty \leq 1}(\mathbf{Ax})$$

We know from the Fenchel-Rockafellar duality theory that the dual problem is formulated as:

$$\min_{\mathbf{u}} g^*(\mathbf{u}) + f^*(-\mathbf{A}^*\mathbf{u})$$

where $\mathbf{u} \in \mathbb{R}^{n \times 1}$, $\mathbf{A}^* \in \mathbb{R}^{d \times n}$ the transpose of $\mathbf{A}$, and g^* and f^* are the Fenchel conjugates of g and f respectively.

For the Fenchel conjugate of $g(\mathbf{Ax})$, we first recognize that the dual of the indicator function is the support function under $\frac{1}{\epsilon}\|\mathbf{Ax}-\mathbf{b}\|_\infty$ which is the definition for the dual norm $\frac{1}{\epsilon}\|\mathbf{Ax}-\mathbf{b}\|_1$.

Thus:

$$g^*(\mathbf{Ax}) = i^*_{\frac{1}{\epsilon}\|\mathbf{Ax}-\mathbf{b}\|_\infty \leq 1}(\mathbf{Ax}) = \frac{1}{\epsilon}\|\mathbf{Ax}-\mathbf{b}\|_1$$

.

The Fenchel conjugate of $f(\mathbf{x})$:

$$f^*(\mathbf{x}) = \frac{1}{2}\|\mathbf{x}\|^2$$

We can substitute:

$$\min_{\mathbf{u}} \frac{1}{\epsilon}\|\mathbf{u}-\mathbf{b}\|_1 + \frac{1}{2}\|-\mathbf{A}^*\mathbf{u}\|^2$$

Simplifying, we have our dual problem:

$$\min_{\mathbf{u}} \frac{1}{2}\mathbf{u}^*\mathbf{A}\mathbf{A}^*\mathbf{u} + \frac{1}{\epsilon}\|\mathbf{u}-\mathbf{b}\|_1$$

2.6.2

To determine if strong duality holds, we use the qualification condition:

$$\mathbf{0}_n \in int\big(dom(g) - \mathbf{A}dom(f)\big)$$

that $\mathbf{0}_n = \mathbf{0} \in \mathbb{R}^{n\times 1}$ is in the interior of the intersection of $dom(g)$, the domain of g, with $\mathbf{A}dom(f)$, $\mathbf{A}$ applied to the domain of f.

We know that for $f(x) = \frac{1}{2}\|\mathbf{x}\|^2$:

$$\mathbf{0}_d \in int\big(dom(f)\big)$$

where $\mathbf{0}_d = \mathbf{0} \in \mathbb{R}^{d\times 1}$. Thus, knowing that $\mathbf{A} \in \mathbb{R}^{n\times d}$ is a linear operator:

$$\mathbf{0}_n \in int\big(\mathbf{A}dom(f)\big)$$

Moreover, for $g(\mathbf{Ax}) = i_{\frac{1}{\epsilon}\|\mathbf{Ax}-\mathbf{b}\|_\infty \leq 1}(\mathbf{Ax})$:

$$\mathbf{0}_n \in dom(g)$$

if $\|\mathbf{b}\|_\infty \leq \epsilon$ and,

$$\mathbf{0}_n \in int\big(dom(g)\big)$$

if $\|\mathbf{b}\|_\infty < \epsilon$.

Under the condition that $\|\mathbf{b}\|_\infty < \epsilon$, strong duality holds. This is because it will be the case that $\mathbf{0}_n \in int\big(\mathbf{A}dom(f)\big)$ and $\mathbf{0}_n \in int\big(dom(g)\big)$ and so the same will hold for their intersection, $\mathbf{0}_n \in int\big(dom(g) - \mathbf{A}dom(f)\big)$ as required.

2.6.3

The KKT conditions:

$$\mathbf{x} \in \partial f^*(-\mathbf{A}^*\mathbf{u}) \text{ and } \mathbf{A}\mathbf{x} \in \partial g^*(\mathbf{u})$$

and equivalently,

$$-\mathbf{A}^*\mathbf{u} \in \partial f(\mathbf{x}) \text{ and } \mathbf{u} \in \partial g(\mathbf{A}\mathbf{x})$$

though we will use the formulation involving ∂f^* and ∂g^*, the subgradients of f^* and g^* respectively.

We know that:

$$\partial f^*(-\mathbf{A}^*\mathbf{u}) = \partial \frac{1}{2}\| - \mathbf{A}^*\mathbf{u}\|^2 = -\mathbf{A}^*\mathbf{u}$$

Moreover:

$$\partial g^*(\mathbf{u}) = \partial \frac{1}{\epsilon}\|\mathbf{u} - \mathbf{b}\|_1$$

and so

$$\left(\partial \frac{1}{\epsilon}\|\mathbf{u} - \mathbf{b}\|_1\right)_i = \begin{cases} \frac{1}{\epsilon}, & \text{if } |u_i - b_i| > 0 \\ -\frac{1}{\epsilon}, & \text{if } |u_i - b_i| < 0 \\ [-\frac{1}{\epsilon}, \frac{1}{\epsilon}], & \text{if } |u_i - b| = 0 \end{cases}$$

where $(\partial\|\mathbf{u} - \mathbf{b}\|_1)_i$ is the i^{th} element of $\partial\|\mathbf{u} - \mathbf{b}\|_1 \in \mathbb{R}^{n\times 1}$, for $i = 1, ..., n$.

Thus, our KKT conditions state that:

$$\mathbf{x} = -\mathbf{A}^*\mathbf{u}$$

and

$$(\mathbf{A}\mathbf{x})_i \in \begin{cases} \frac{1}{\epsilon}, & \text{if } |u_i - b_i| > 0 \\ -\frac{1}{\epsilon}, & \text{if } |u_i - b_i| < 0 \\ [-\frac{1}{\epsilon}, \frac{1}{\epsilon}], & \text{if } |u_i - b| = 0 \end{cases}$$

where $(\mathbf{A}\mathbf{x})_i$ is the i^{th} element of $\mathbf{A}\mathbf{x} \in \mathbb{R}^{n\times 1}$ for $i = 1, ..., n$.

2.6.4

To derive a rate of convergence on the primal iterates from the applications of FISTA (Fast Iterative Shrinkage Threshold Algorithm) on the dual problem when strong duality holds, the distance to the primal solution is bounded by the dual objective values:

$$\frac{2}{\mu}\|\mathbf{x}_k - \bar{\mathbf{x}}\|^2 \leq \Psi(\mathbf{u}_k) - \Psi(\bar{\mathbf{u}})$$

where $\bar{\mathbf{x}} \in \mathbb{R}^{d\times 1}$ and $\bar{\mathbf{u}} \in \mathbb{R}^{n\times 1}$ are minimizers for the primal and dual solution respectively, $\mathbf{x}_k \in \mathbb{R}^{d\times 1}$ and $\mathbf{u}_k \in \mathbb{R}^{n\times 1}$ are respectively the primal and dual solutions for the k^{th} step of FISTA, and μ is the coefficient of strong convexity for f. Moreover, $\Psi(\mathbf{u}) = \frac{1}{2}\mathbf{u}^*\mathbf{A}\mathbf{A}^*\mathbf{u} + \frac{1}{\epsilon}\|\mathbf{u} - \mathbf{b}\|_1$, our dual objective.

We also know for FISTA that:

$$\Psi(\mathbf{u}_k) - \Psi(\bar{\mathbf{u}}) \leq \frac{\|\mathbf{u}_0 - \bar{\mathbf{u}}\|^2}{2\gamma\left(t_{k-1}\right)^2}$$

where $\mathbf{u}_0 \in \mathbb{R}^{n\times 1}$ is the initial starting point for FISTA, and $0 < \gamma \leq \frac{1}{L}$ and t_{k-1} are both scalars dictating step sizes in FISTA (t_{k-1} is for step $k-1$).

Thus:

$$\|\mathbf{x}_k - \bar{\mathbf{x}}\|^2 \leq \frac{\mu\|\mathbf{u}_0 - \bar{\mathbf{u}}\|^2}{4\gamma\left(t_{k-1}\right)^2}$$

Simplifying:

$$\|\mathbf{x}_k - \bar{\mathbf{x}}\| \leq \sqrt{\frac{\mu}{4\gamma}}\frac{\|\mathbf{u}_0 - \bar{\mathbf{u}}\|}{t_{k-1}}$$

Two choices for t_k in FISTA are:

- $t_k = \frac{1+\sqrt{1+4(t_{k-1})^2}}{2}$ where $\frac{1}{t_{k-1}} \leq \frac{2}{k+1}$
- $t_k = \frac{k+a}{a}$ with $a \geq 2$ where $\frac{1}{t_{k-1}} \leq \frac{a}{k+1}$

In both cases the rate of convergence of the primal iterates when applying FISTA on the dual problem:

$$\|\mathbf{x}_k - \bar{\mathbf{x}}\| \leq \sqrt{\frac{a^2\mu}{4\gamma}}\frac{\|\mathbf{u}_0 - \bar{\mathbf{u}}\|}{k+1}$$

where $a = 2$ if $t_k = \frac{1+\sqrt{1+4(t_{k-1})^2}}{2}$. In other words, the rate is $\mathcal{O}(\frac{1}{k})$.

Part 3: Solving the Lasso problem

Our problem:

$$\min_{\mathbf{x}\in\mathbb{R}^d} \frac{1}{2n}\|\mathbf{A}\mathbf{x}-\mathbf{y}\|^2 + \lambda\|x\|_1$$

where $\mathbf{A}\in\mathbb{R}^{n\times d}$, $\mathbf{x}\in\mathbb{R}^{d\times 1}$, $\mathbf{y}\in\mathbb{R}^{n\times 1}$, n is the number of data points, and d is the number of dimensions.

Equivalently, our problem:

$$\min_{\mathbf{x}\in\mathbb{R}^d} \frac{1}{2n}\sum_{i=1}^{n}\left(\left\langle \mathbf{a}^i, \mathbf{x}\right\rangle - y_i\right)^2 + \lambda\|x\|_1$$

where $\mathbf{a}^i\in\mathbb{R}^{1\times d}$ is the i^{th} row of $\mathbf{A}$, $i=1,...,n$.

3.1

The Proximal Stochastic Gradient Algorithm:

$$\mathbf{x}^{k+1} = prox_{\gamma_k\lambda\|\cdot\|_1}\left(\mathbf{x}^k - \gamma_k\left(\left\langle \mathbf{a}^{i_k}, \mathbf{x}^k\right\rangle - y_{i_k}\right)\mathbf{a}^{i_k}\right)$$

where:

$$\gamma_k = \frac{n}{\|A\|^2\sqrt{k+1}}$$

and

$$prox_{\gamma_k\lambda\|\cdot\|_1}(x) = soft_{\gamma_k\lambda}(x) = \begin{cases} 0, & \text{if } |x|\le\gamma_k\lambda \\ x-\gamma_k\lambda, & \text{if } x>\gamma_k\lambda \\ x+\gamma_k\lambda, & \text{if } x<-\gamma_k\lambda \end{cases}$$

and i_k is sampled uniformly from $\{1,...,n\}$ at each step k.

3.2

The Randomized Coordinate Proximal Gradient Algorithm:

$$x_j^{k+1} = \begin{cases} soft_{\gamma_j\lambda}(x_j^k - \frac{\gamma_j}{n}\left\langle a_j, \mathbf{A}\mathbf{x}^k - y\right\rangle), & \text{if } j=j_k \\ x_j^k, & \text{otherwise} \end{cases}$$

where we define $\mathbf{a}_j\in\mathbb{R}^{n\times 1}$ as the j^{th} column of $\mathbf{A}$, $i=1,...,d$.

$$\gamma_j = \frac{n}{\|\mathbf{a}_j\|^2}$$

and j_k is sampled uniformly from $\{1,...,d\}$ at each step k.

3.3

The randomly initialised vector passed to the algorithms:

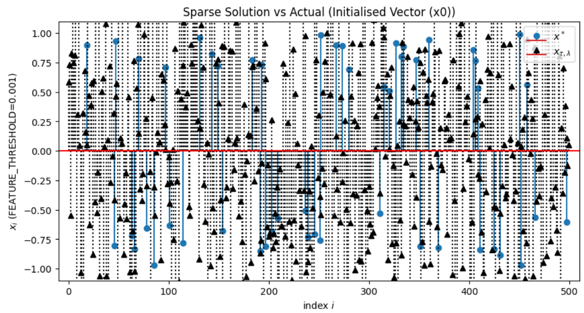
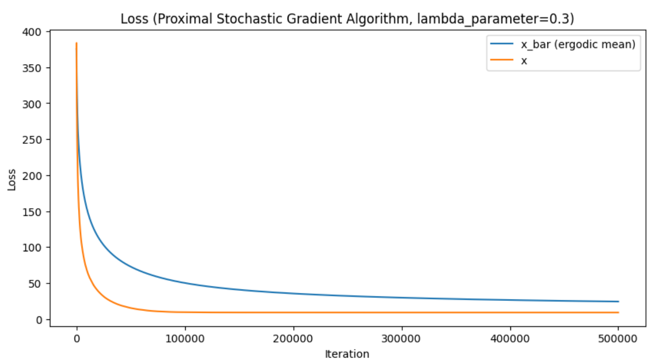


Figure 1: x_0 vs Sparse Vector

Plotting the objective function values vs the number of iterations for both the algorithms:



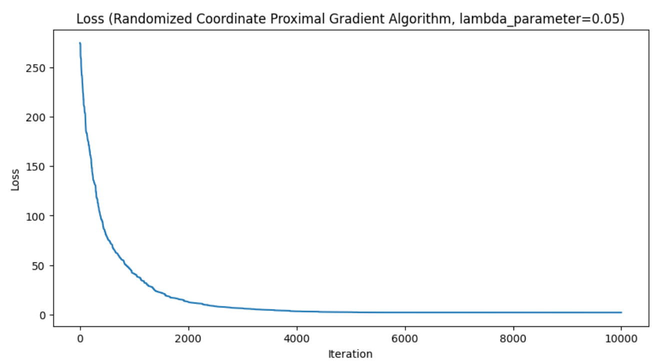
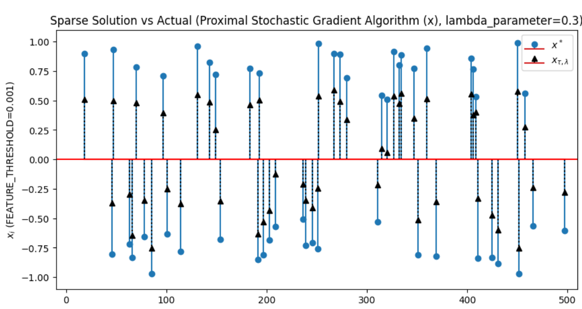


Figure 2: PSGA Loss

Figure 3: RCPGA Loss

The corresponding solution compared to the actual sparse vector for both the algorithms:



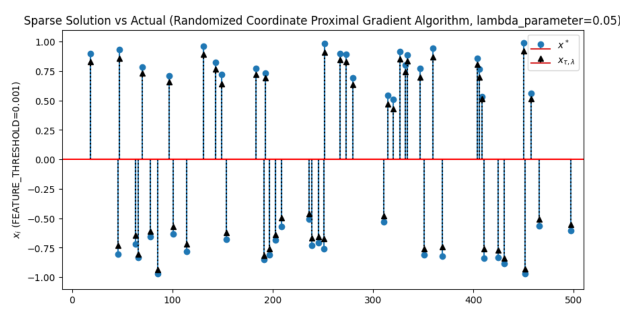


Figure 4: PSGA Solution vs Sparse Vector

Figure 5: RCPGA Solution vs Sparse Vector

We can see that both algorithms are able to recover a sparse solution, however RCPGA was able to converge in much fewer iterations compared to PSGA.

Part 4: Support Vector Machines

We are given the primal problem:

$$\min_{\mathbf{w}\in\mathcal{H}} \frac{1}{\lambda n}\sum_{i=1}^{n}(1 - y_i\langle\mathbf{w}, \Lambda(\mathbf{x}_i)\rangle)_+ + \frac{1}{2}\|\mathbf{w}\|^2$$

where $\mathcal{H}$ is a Hilbert space.
The primal problem can be expressed the form:

$$\min_{w\in\mathcal{H}} f(\mathbf{w}) + g(\mathbf{A}\mathbf{w})$$

where:

$$f(\mathbf{w}) = \frac{1}{2}\|\mathbf{w}\|^2$$

and

$$g(\mathbf{A}\mathbf{w}) = \frac{1}{\lambda n}\sum_{i=1}^{n}(1 - y_i\langle\mathbf{w}, \Lambda(\mathbf{x}_i)\rangle)_+$$

where $\mathbf{A}_i = \Lambda(\mathbf{x}_i) \in \mathcal{H}$ our Hilbert space.
This corresponds to the dual problem of the form:

$$\min_{\mathbf{u}\in\mathbb{R}^n} f^*(-\mathbf{A}^*\mathbf{u}) + g^*(\mathbf{u})$$

where:

$$f^*(-\mathbf{A}^*\mathbf{u}) = \frac{1}{2}\|-\mathbf{A}^*\mathbf{u}\|^2 = \frac{1}{2}\mathbf{u}^*\mathbf{A}\mathbf{A}^*\mathbf{u}$$

and using the kernel defined by $\mathcal{H}$, we can substitute $\mathbf{A}\mathbf{A}^*$ with the gram matrix $\mathbf{K}$:

$$f^*(-\mathbf{A}^*\mathbf{u}) = \frac{1}{2}\mathbf{u}^*\mathbf{K}\mathbf{u}$$

Moreover,

$$g^*(\mathbf{u}) = -\langle\mathbf{y}, \mathbf{u}\rangle + \sum_{i=1}^{n} i_{\left[0,\frac{1}{\lambda n}\right]}(\mathbf{u}_i)$$

Defining $\alpha_i = u_i y_i$ and know that $y_i y_i = 1$ because $y_i \in \{-1, 1\}$ we can see that:

$$f^*(-\mathbf{A}^*\mathbf{u}) = \frac{1}{2}\alpha^*\mathbf{K}_y\alpha$$

and

$$g^*(\mathbf{u}) = -\langle\mathbf{1}_n, \alpha\rangle + \sum_{i=1}^{n} i_{\left[0,\frac{1}{\lambda n}\right]}(\alpha_i)$$

where $(\mathbf{K}_y)_{i,j} = y_i\mathbf{K}_{i,j}y_j$.
Combining, we have our dual problem:

$$\min_{\alpha\in\mathbb{R}^n} \frac{1}{2}\alpha^*\mathbf{K}_y\alpha - \langle\mathbf{1}_n, \alpha\rangle + \sum_{i=1}^{n} i_{\left[0,\frac{1}{\lambda n}\right]}(\alpha_i)$$

4.1

The Fast Iterative Shrinkage Threshold Algorithm (FISTA):

$$\alpha_{k+1} = prox_{\gamma g^*(\cdot)}\left(\nu_k + \gamma \mathbf{A}_y \nabla f^*(-\mathbf{A}_y^* \alpha_k)\right)$$

$$\nu_{k+1} = \alpha_k + \frac{t_k - 1}{t_{k+1}}(\alpha_{k+1} - \alpha_k)$$

where we choose $t_k = \frac{1+\sqrt{1+4(t_{k-1})^2}}{2}$ for our implementation.
For our dual problem:

$$\nabla f^*(-\mathbf{A}_y^* \alpha_k) = -\mathbf{A}_y^* \alpha_k$$

and

$$prox_{\gamma g^*(\cdot)}(\omega) = \begin{cases} 0, & \text{if } \omega < 0 \\ \omega, & \text{if } 0 \leq \omega \leq \frac{1}{\gamma n} \\ \frac{1}{\gamma n}, & \text{if } \frac{1}{\gamma n} < \omega \end{cases}$$

Implementing FISTA, we can plot the dual objective function:

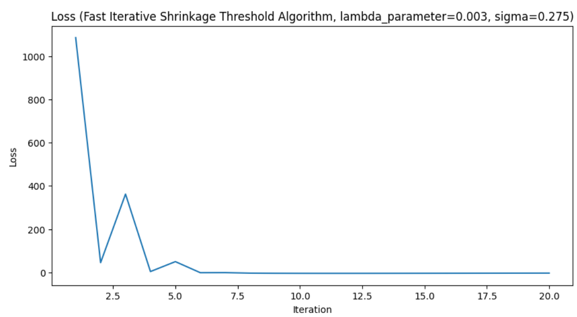


Figure 6: FISTA Loss

4.2

We can also implement the Randomised Coordinate Projected Gradient Algorithm on the dual problem and plot the dual objective function:

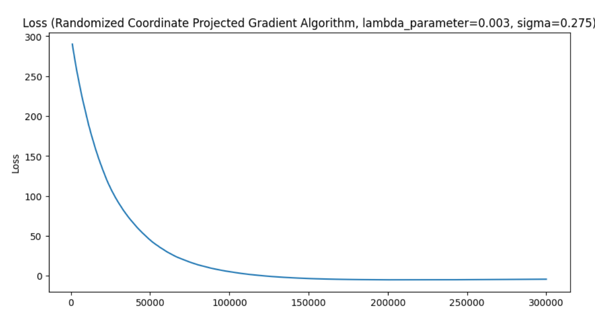


Figure 7: RCPGA Loss

4.3

We can plot the decision boundary for the randomly initialised α vector as well as the two classes:

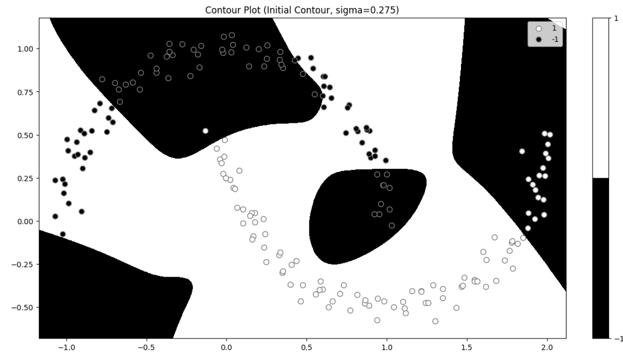


Figure 8: Initial Contour Plot

Similarly for each algorithm, we can plot the decision boundaries from the learned α:

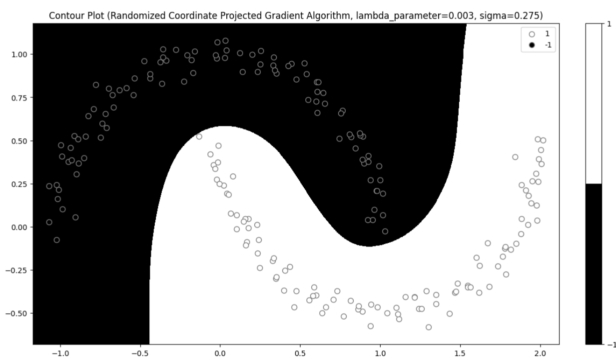


Figure 9: RCPGA Contour Plot

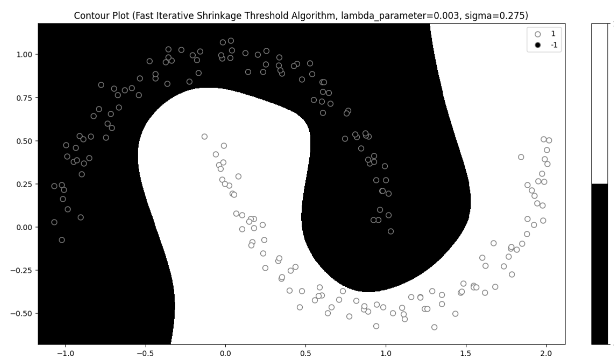


Figure 10: FISTA Contour Plot

4.4

Comparing the two algorithms, we can see that both are able to converge to α vectors producing decision boundaries that are successful at separating the training points for the two classes. It is difficult to determine which of the two decision boundaries are more suitable for the training data, without more context about the data itself. However considering the loss function plots for both algorithms, we can see that FISTA is able to converge in around 20 iterations whereas RCPGA required on the order of 3×10^5 iterations. Although each FISTA step involves two steps (to calculate α_{k+1} and ν_{k+1}), this is still a much faster algorithm than RCGPA. However, we can also see that the objective function for FISTA with respect to the algorithm iteration is much less smooth compared to RCGPA's loss curve. Depending on use case, this may or may not be a problem. Overall, FISTA is able to produce quality decision boundaries much faster than RCGPA. Moreover, during the parameter tuning process of λ and σ, RCPGA was much more robust, converging to good decision boundaries most of the time when compared to FISTA, which required more tuning of parameters to converge to a decision boundary that perfectly separated the two classes of the training data.