

COMP0083 Convex Optimisation Assignment

Jan 9, 2023

Part 1: Questions with multiple answers

1.1

The function for (a):

$$\max\{ax + b, x^4 - 5, e^{x^2}\}$$

is a convex function.

1.2

For the function:

$$f(x) = \begin{cases} -x & \text{if } x \in]-1, 0] \\ x^2 & \text{if } x \geq 0 \end{cases}$$

the sub-differential is:

$$\partial f(x) = \begin{cases} -1 & \text{if } x \in]-1, 0[\\ [0, 1] & \text{if } x = 0 \\ 2x & \text{if } x > 0 \end{cases}$$

corresponding to Figure (a).

1.3

For a function:

$$f(x) = \langle Ax, x \rangle + \langle x, b \rangle + c$$

where A is a square matrix not necessarily symmetric, the gradient is (a):

$$\nabla f(x) = A^*x + Ax + b$$

1.4

The Fenchel conjugate of $f(x) = g(2x)$ is (a):

$$f^*(u) = g^*(u/2)$$

1.5

The solution to the dual problem is (c):

$$\bar{u} = (\mathbf{K} + \lambda n \mathbf{Id})^{-1} y$$

Part 2: Theory on convex analysis and optimization

2.1

2.1.1

Given :

$$f(x) = \begin{cases} +\infty & \text{if } x \leq 0 \\ -\log x & \text{if } x > 0 \end{cases}$$

The Fenchel conjugate is defined:

$$f^*(u) = \sup_{x \in \mathcal{X}} \{ \langle x, u \rangle + \log x \}$$

To find the supremum, we can take the partial derivative with respect to x , set to zero, and solve for x :

$$\frac{\partial}{\partial x}(ux + \log x) = u + \frac{1}{x} = 0$$

Thus, the supremum above is solved when $x = \frac{-1}{u}$:

$$f^*(u) = u \left(\frac{-1}{u} \right) + \log \left(\frac{-1}{u} \right)$$

Simplifying, we have the Fenchel conjugate:

$$f^*(u) = -(1 + \log u)$$

2.1.2

Given:

$$f(x) = x^2$$

The Fenchel conjugate is defined:

$$f^*(u) = \sup_{x \in \mathcal{X}} \{ \langle x, u \rangle - x^2 \}$$

We can compute the partial derivative:

$$\frac{\partial}{\partial x}(ux - x^2) = u - 2x = 0$$

Thus, the supremum is solved when $x = \frac{u}{2}$:

$$f^*(u) = u \left(\frac{u}{2} \right) - \left(\frac{u}{2} \right)^2$$

Simplifying, we have the Fenchel conjugate:

$$f^*(u) = \frac{u^2}{4}$$

2.1.3

Given:

$$f(x) = i_{[0,1]}$$

The Fenchel conjugate is defined:

$$f^*(u) = \sup_{x \in \mathcal{X}} \{ \langle x, u \rangle - i_{[0,1]} \}$$

Thus,

$$f^*(u) = \sup_{x \in [0,1]} \{ \langle x, u \rangle \}$$

We can see the Fenchel conjugate is:

$$f^*(u) = \max(0, u)$$

2.2.1

Given f a proper convex function, to prove by induction Jensen's inequality:

$$f \left(\sum_{i=1}^n \lambda_i x_i \right) \leq \sum_{i=1}^n \lambda_i f(x_i)$$

for all $x_1, \dots, x_n \in \mathcal{X}$ and for all $\lambda_1, \dots, \lambda_n \in \mathbb{R}_+$ with $\sum_{i=1}^n \lambda_i = 1$, we start with the definition of convexity which states:

$$f(\lambda_1 x_1 + \lambda_2 x_2) \leq \lambda_1 f(x_1) + \lambda_2 f(x_2)$$

for all $x_1, x_2 \in \mathcal{X}$ and $\lambda_1, \lambda_2 \in \mathbb{R}_+$ with $\lambda_1 + \lambda_2 = 1$. In this base case $n = 2$. Our inductive step will prove that the inequality continues to hold for $n + 1$:

$$f \left(\sum_{i=1}^{n+1} \lambda_i x_i \right) = f \left(\sum_{i=1}^n \lambda_i x_i + \lambda_{n+1} x_{n+1} \right)$$

We can insert the term $(1 - \lambda_{n+1})$:

$$f \left(\sum_{i=1}^{n+1} \lambda_i x_i \right) = f \left((1 - \lambda_{n+1}) \sum_{i=1}^n \frac{\lambda_i}{1 - \lambda_{n+1}} x_i + \lambda_{n+1} x_{n+1} \right)$$

If we define $\bar{x} = \sum_{i=1}^n \frac{\lambda_i}{1-\lambda_{n+1}} x_i$ and we are back to our $n = 2$ base case, so we know from convexity:

$$f((1-\lambda_{n+1})\bar{x} + \lambda_{n+1}x_{n+1}) \leq (1-\lambda_{n+1})f(\bar{x}) + \lambda_{n+1}f(x_{n+1})$$

Rewriting this,

$$f\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) \leq (1-\lambda_{n+1})f\left(\sum_{i=1}^n \frac{\lambda_i}{1-\lambda_{n+1}} x_i\right) + \lambda_{n+1}f(x_{n+1})$$

We know that the first term on the right hand side:

$$f\left(\sum_{i=1}^n \frac{\lambda_i}{1-\lambda_{n+1}} x_i\right) = \frac{1}{1-\lambda_{n+1}} f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \frac{1}{1-\lambda_{n+1}} \sum_{i=1}^n \lambda_i f(x_i) = \sum_{i=1}^n \frac{\lambda_i}{1-\lambda_{n+1}} f(x_i)$$

Thus can upper bound our previous inequality:

$$f\left(\sum_{i=1}^{n+1} \lambda_i x_i\right) \leq \sum_{i=1}^n \frac{\lambda_i}{1-\lambda_{n+1}} f(x_i) + \lambda_{n+1}f(x_{n+1}) = \sum_{i=1}^{n+1} \frac{\lambda_i}{1-\lambda_{n+1}} f(x_i)$$

proving our inductive step and Jensen's inequality as required. \square

2.2.2

The characterisation of convexity states:

$$f \text{ is convex} \Leftrightarrow \langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0$$

$\forall x, y \in \text{dom} f$.

For $f(x) = -\log(x)$, $\nabla f(x) = \frac{-1}{x}$:

$$\langle \nabla f(x) - \nabla f(y), x - y \rangle = \left\langle \frac{1}{y} - \frac{1}{x}, x - y \right\rangle$$

If $x > y$ we have the following:

$$\begin{aligned} \frac{1}{y} &> \frac{1}{x} \\ x - y &> 0 \\ \frac{1}{y} - \frac{1}{x} &> 0 \end{aligned}$$

and

$$\left\langle \frac{1}{y} - \frac{1}{x}, x - y \right\rangle > 0$$

If $x < y$ we have the following:

$$\begin{aligned}\frac{1}{y} &< \frac{1}{x} \\ x - y &< 0 \\ \frac{1}{y} - \frac{1}{x} &< 0\end{aligned}$$

and

$$\left\langle \frac{1}{y} - \frac{1}{x}, x - y \right\rangle > 0$$

If $x = y$ we have the following:

$$\begin{aligned}\frac{1}{y} &= \frac{1}{x} \\ x - y &= 0 \\ \frac{1}{y} - \frac{1}{x} &= 0\end{aligned}$$

and

$$\left\langle \frac{1}{y} - \frac{1}{x}, x - y \right\rangle = 0$$

Thus, $\forall x, y \in \text{dom } f$, $\langle \nabla f(x) - \nabla f(y), x - y \rangle \geq 0$ so $f(x) = -\log(x)$ is convex. \square

2.2.3

Given Jensen's inequality in 2.2.1 for a convex function $f(x)$:

$$f\left(\sum_{i=1}^n \lambda_i x_i\right) \leq \sum_{i=1}^n \lambda_i f(x_i)$$

and having proved in 2.2.2 that $f(x) = -\log(x)$ is convex, we can write:

$$-\log\left(\sum_{i=1}^n \lambda_i x_i\right) \leq -\sum_{i=1}^n \lambda_i \log(x_i)$$

Rearranging:

$$\sum_{i=1}^n \lambda_i x_i \geq \exp\left(\sum_{i=1}^n \log(x_i^{\lambda_i})\right)$$

Choosing $\lambda_i = \frac{1}{n}$:

$$\frac{1}{n} \sum_{i=1}^n x_i \geq \exp\left(\sum_{i=1}^n \log(x_i^{\frac{1}{n}})\right)$$

The sum of logarithms is the logarithm of the products:

$$\frac{1}{n} \sum_{i=1}^n x_i \geq \exp \left(\log((x_1 \cdots x_n)^{\frac{1}{n}}) \right)$$

Thus we get our inequality:

$$\frac{1}{n} \sum_{i=1}^n x_i \geq \sqrt[n]{x_1 \cdots x_n}$$

□

2.3

Given a polytope $C = co(a_1, \dots, a_m)$ in X , we know that any point $x \in C$ can be expressed:

$$x = \sum_{i=1}^m \lambda_i a_i$$

where $\sum_{i=1}^m \lambda_i = 1$. Knowing that f is a convex function on C , we can write Jensen's inequality for :

$$f \left(\sum_{i=1}^m \lambda_i a_i \right) \leq \sum_{i=1}^m \lambda_i f(a_i)$$

where $\sum_{i=1}^m \lambda_i = 1$.

Because $\sum_{i=1}^m \lambda_i f(a_i)$ is a weighted average of $f(a_i)$'s we know that:

$$\sum_{i=1}^m \lambda_i f(a_i) \leq \max_{\{a_i\}_{i=1}^m} f(a_i)$$

the weighted average will always be less than or equal to the maximum $f(a_i)$ value. Thus, we know that:

$$f \left(\sum_{i=1}^m \lambda_i a_i \right) \leq \max_{\{a_i\}_{i=1}^m} f(a_i)$$

So the maximum of the convex function f on C is attained at one of the vertices a_1, \dots, a_m . □

2.4

To prove that the function $f(x, y) = \|x - 2y\|^2$ is convex, we will show that the Hessian is a positive semi-definite matrix. The Hessian is defined as:

$$\nabla^2 f(x, y) = \begin{bmatrix} \frac{\partial f(x, y)}{\partial^2 x} & \frac{\partial f(x, y)}{\partial y \partial x} \\ \frac{\partial f(x, y)}{\partial x \partial y} & \frac{\partial f(x, y)}{\partial^2 y} \end{bmatrix}$$

Calculating each term:

$$\nabla^2 f(x, y) = \begin{bmatrix} 2 & 4 \\ 4 & 8 \end{bmatrix}$$

we see that the Hessian is positive semi-definite and so the function is jointly convex.

2.5

The conditions for the existence of minimizers is that f is closed and coercive. The conditions for the uniqueness of minimizers is that f is strictly convex. Thus for the existence and uniqueness of minimizers for a convex function f , we require that f is closed, coercive, and strictly convex.

2.6

We are considering the optimisation problem:

$$\min_{\|\mathbf{Ax}-\mathbf{b}\|_\infty \leq \epsilon} \frac{1}{2} \|\mathbf{x}\|^2$$

where $\epsilon > 0$, $\mathbf{A} \in \mathbb{R}^{n \times d}$, $\mathbf{b} \in \mathbb{R}^{n \times 1}$, and $\mathbf{x} \in \mathbb{R}^{d \times 1}$.

2.6.1

To compute the dual problem, we begin by reformulating the problem as:

$$\min_{\mathbf{x}} f(\mathbf{x}) + g(\mathbf{Ax})$$

where $f(x) = \frac{1}{2} \|\mathbf{x}\|^2$ and $g(\mathbf{Ax}) = i_{[0,1]} \left(\frac{1}{\epsilon} \|\mathbf{Ax} - \mathbf{b}\|_\infty \right)$.

Thus we have our primal problem:

$$\min_{\mathbf{x}} \frac{1}{2} \|\mathbf{x}\|^2 + i_{[0,1]} \left(\frac{1}{\epsilon} \|\mathbf{Ax} - \mathbf{b}\|_\infty \right)$$

We know from the Fenchel-Rockafellar duality theory that the dual problem is formulated as:

$$\min_{\mathbf{u}} g^*(\mathbf{u}) + f^*(-\mathbf{A}^* \mathbf{u})$$

where $\mathbf{u} \in \mathbb{R}^{n \times 1}$, $\mathbf{A}^* \in \mathbb{R}^{d \times n}$ the transpose of \mathbf{A} , and g^* and f^* are the Fenchel conjugates of g and f respectively.

The Fenchel conjugates are:

$$g^*(\mathbf{Ax}) = i_{[0,1]}^* \left(\frac{1}{\epsilon} \|\mathbf{Ax} - \mathbf{b}\|_\infty \right) = \frac{1}{\epsilon} \|\mathbf{Ax} - \mathbf{b}\|_1$$

$$f^*(\mathbf{x}) = \frac{1}{2} \|\mathbf{x}\|^2$$

We can substitute:

$$\min_{\mathbf{u}} \frac{1}{\epsilon} \|\mathbf{u} - \mathbf{b}\|_1 + \frac{1}{2} \|\mathbf{u} - \mathbf{A}^* \mathbf{u}\|^2$$

Simplifying, we have our dual problem:

$$\min_{\mathbf{u}} \frac{1}{2} \mathbf{u}^* \mathbf{A} \mathbf{A}^* \mathbf{u} + \frac{1}{\epsilon} \|\mathbf{u} - \mathbf{b}\|_1$$

2.6.2

To determine if strong duality holds, we use the qualification condition:

$$\mathbf{0}_n \in \text{int}(\text{dom}(g) - \mathbf{A}\text{dom}(f))$$

that $\mathbf{0}_n = \mathbf{0} \in \mathbb{R}^{n \times 1}$ is in the interior of the intersection of $\text{dom}(g)$, the domain of g , with $\mathbf{A}\text{dom}(f)$, \mathbf{A} applied to the domain of f .

We know that for $f(x) = \frac{1}{2}\|\mathbf{x}\|^2$:

$$\mathbf{0}_d \in \text{int}(\text{dom}(f))$$

where $\mathbf{0}_d = \mathbf{0} \in \mathbb{R}^{d \times 1}$. Thus, knowing that $\mathbf{A} \in \mathbb{R}^{n \times d}$ is a linear operator:

$$\mathbf{0}_n \in \text{int}(\mathbf{A}\text{dom}(f))$$

Moreover, for $g(\mathbf{u}) = i_{[0,1]}(\frac{1}{\epsilon}\|\mathbf{u} - \mathbf{b}\|_\infty)$:

$$\mathbf{0}_n \in \text{dom}(g)$$

if $\|\mathbf{b}\|_\infty \leq \epsilon$. Therefore:

$$\mathbf{0}_n \in \text{int}(\text{dom}(g))$$

if $\|\mathbf{b}\|_\infty < \epsilon$.

Under the condition that $\|\mathbf{b}\|_\infty < \epsilon$, strong duality holds. This is because it will be the case that $\mathbf{0}_n \in \text{int}(\mathbf{A}\text{dom}(f))$ and $\mathbf{0}_n \in \text{int}(\text{dom}(g))$ and so the same will hold for their intersection, $\mathbf{0}_n \in \text{int}(\text{dom}(g) - \mathbf{A}\text{dom}(f))$ as required.

2.6.3

The KKT conditions:

$$\mathbf{x} \in \partial f^*(-\mathbf{A}^*\mathbf{u}) \text{ and } \mathbf{Ax} \in \partial g^*(\mathbf{u})$$

where ∂f^* and ∂g^* are the subgradients of f^* and g^* respectively.

We know that:

$$\partial f^*(-\mathbf{A}^*\mathbf{u}) = \partial \frac{1}{2}\|-\mathbf{A}^*\mathbf{u}\|^2 = -\mathbf{A}^*\mathbf{u}$$

Moreover:

$$\partial g^*(\mathbf{u}) = \partial \frac{1}{\epsilon}\|\mathbf{u} - \mathbf{b}\|_1$$

and so

$$(\partial \|\mathbf{u} - \mathbf{b}\|_1)_i = \begin{cases} \frac{1}{\epsilon}, & \text{if } |u_i - b_i| > 0 \\ -\frac{1}{\epsilon}, & \text{if } |u_i - b_i| < 0 \\ [-\frac{1}{\epsilon}, \frac{1}{\epsilon}], & \text{if } |u_i - b_i| = 0 \end{cases}$$

where $(\partial \|\mathbf{u} - \mathbf{b}\|_1)_i$ is the i^{th} element of $\partial \|\mathbf{u} - \mathbf{b}\|_1 \in \mathbb{R}^{n \times 1}$

Thus, our KKT conditions state that:

$$\mathbf{x} = -\mathbf{A}^* \mathbf{u}$$

and

$$(\mathbf{Ax})_i \in \begin{cases} \frac{1}{\epsilon}, & \text{if } |u_i - b_i| > 0 \\ -\frac{1}{\epsilon}, & \text{if } |u_i - b_i| < 0 \\ [-\frac{1}{\epsilon}, \frac{1}{\epsilon}], & \text{if } |u_i - b_i| = 0 \end{cases}$$

where $(\mathbf{Ax})_i$ is the i^{th} element of $\mathbf{Ax} \in \mathbb{R}^{n \times 1}$

2.6.4

To derive a rate of convergence on the primal iterates from the applications of FISTA (Fast Iterative Shrinkage Threshold Algorithm) on the dual problem when strong duality holds, the distance to the primal solution is bounded by the dual objective values:

$$\frac{2}{\mu} \|\mathbf{x}_k - \bar{\mathbf{x}}\|^2 \leq \Psi(\mathbf{u}_k) - \Psi(\bar{\mathbf{u}})$$

where $\bar{\mathbf{x}} \in \mathbb{R}^{d \times 1}$ and $\bar{\mathbf{u}} \in \mathbb{R}^{n \times 1}$ are minimizers for the primal and dual solution respectively, $\mathbf{x}_k \in \mathbb{R}^{d \times 1}$ and $\mathbf{u}_k \in \mathbb{R}^{n \times 1}$ are respectively the primal and dual solutions for the k^{th} step of FISTA, and μ is the coefficient of strong convexity for f . For our problem, $\mu = 1$. Moreover, $\Psi(\mathbf{u}) = \frac{1}{2} \mathbf{u}^* \mathbf{A} \mathbf{A}^* \mathbf{u} + \frac{1}{\epsilon} \|\mathbf{u} - \mathbf{b}\|_1$, our dual objective.

We also know for FISTA that (proof in lecture):

$$\Psi(\mathbf{u}_k) - \Psi(\bar{\mathbf{u}}) \leq \frac{\|\mathbf{u}_0 - \bar{\mathbf{u}}\|^2}{2\gamma (t_{k-1})^2}$$

where $\mathbf{u}_0 \in \mathbb{R}^{n \times 1}$ is the initial starting point for FISTA, and $0 < \gamma \leq \frac{1}{L}$ and t_{k-1} are both scalars dictating step sizes in FISTA (t_{k-1} is for step $k-1$). L is the Lipschitz smoothness of f . For our problem, $L = 1$.

Thus:

$$\|\mathbf{x}_k - \bar{\mathbf{x}}\|^2 \leq \frac{\|\mathbf{u}_0 - \bar{\mathbf{u}}\|^2}{4\gamma (t_{k-1})^2}$$

Simplifying:

$$\|\mathbf{x}_k - \bar{\mathbf{x}}\| \leq \frac{\|\mathbf{u}_0 - \bar{\mathbf{u}}\|}{2\sqrt{\gamma} t_{k-1}}$$

Two choices for t_k in FISTA are:

- $t_k = \frac{1 + \sqrt{1 + 4(t_{k-1})^2}}{2}$ where $\frac{1}{t_{k-1}} \leq \frac{2}{k+1}$
- $t_k = \frac{k+a}{a}$ with $a \geq 2$ where $\frac{1}{t_{k-1}} \leq \frac{a}{k+1}$

where $t_0 = 1$

In both cases the rate of convergence of the primal iterates when applying FISTA on the dual problem:

$$\|\mathbf{x}_k - \bar{\mathbf{x}}\| \leq \frac{a\|\mathbf{u}_0 - \bar{\mathbf{u}}\|}{\sqrt{\gamma}(k+1)}$$

where $a = 2$ if $t_k = \frac{1+\sqrt{1+4(t_{k-1})^2}}{2}$. In other words, the rate is $\mathcal{O}(\frac{1}{k})$.

Part 3: Solving the Lasso problem

Our problem:

$$\min_{\mathbf{x} \in \mathbb{R}^d} \frac{1}{2n} \|\mathbf{A}\mathbf{x} - \mathbf{y}\|^2 + \lambda \|\mathbf{x}\|_1$$

where $\mathbf{A} \in \mathbb{R}^{n \times d}$, $\mathbf{x} \in \mathbb{R}^{d \times 1}$, $\mathbf{y} \in \mathbb{R}^{n \times 1}$, n is the number of data points, and d is the number of dimensions.

Equivalently, our problem:

$$\min_{\mathbf{x} \in \mathbb{R}^d} \frac{1}{2n} \sum_{i=1}^n (\langle \mathbf{a}^i, \mathbf{x} \rangle - y_i)^2 + \lambda \|\mathbf{x}\|_1$$

where $\mathbf{a}^i \in \mathbb{R}^{1 \times d}$ is the i^{th} row of \mathbf{A} , $i = 1, \dots, n$.

3.1

The Proximal Stochastic Gradient Algorithm:

$$\mathbf{x}^{k+1} = \text{prox}_{\gamma_k \lambda \|\cdot\|_1} (\mathbf{x}^k - \gamma_k (\langle \mathbf{a}^{i_k}, \mathbf{x}^k \rangle - y_{i_k}) \mathbf{a}^{i_k})$$

where:

$$\gamma_k = \frac{n}{\|\mathbf{A}\|^2 \sqrt{k+1}}$$

and

$$\text{prox}_{\gamma_k \lambda \|\cdot\|_1}(x) = \text{soft}_{\gamma_k \lambda}(x) = \begin{cases} 0, & \text{if } |x| \leq \gamma_k \lambda \\ x - \gamma_k \lambda, & \text{if } x > \gamma_k \lambda \\ x + \gamma_k \lambda, & \text{if } x < -\gamma_k \lambda \end{cases}$$

and i_k is sampled uniformly from $\{1, \dots, n\}$ at each step k .

3.2

The Randomized Coordinate Proximal Gradient Algorithm:

$$x_j^{k+1} = \begin{cases} \text{soft}_{\gamma_j \lambda}(x_j^k - \frac{\gamma_j}{n} \langle \mathbf{a}_j, \mathbf{A}\mathbf{x}^k - \mathbf{y} \rangle), & \text{if } j = j_k \\ x_j^k, & \text{otherwise} \end{cases}$$

where we define $\mathbf{a}_j \in \mathbb{R}^{n \times 1}$ as the j^{th} column of \mathbf{A} , $i = 1, \dots, d$.

$$\gamma_j = \frac{n}{\|\mathbf{a}_j\|^2}$$

and j_k is sampled uniformly from $\{1, \dots, d\}$ at each step k .

3.3

Plotting the objective function values vs the number of iterations for both the algorithms:

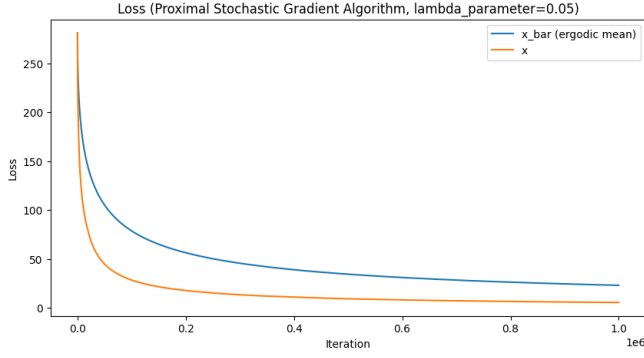


Figure 1: PSGA

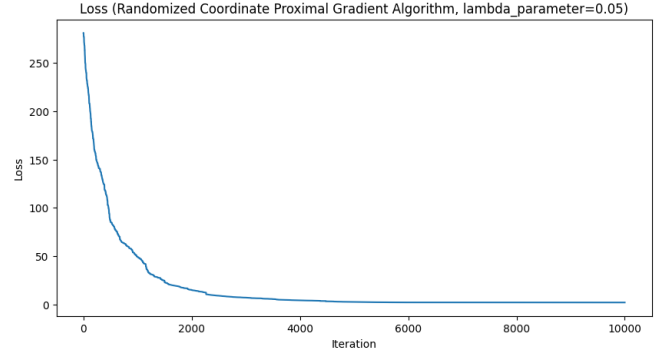


Figure 2: RCPGA

The corresponding solution compared to the actual sparse vector for both the algorithms:

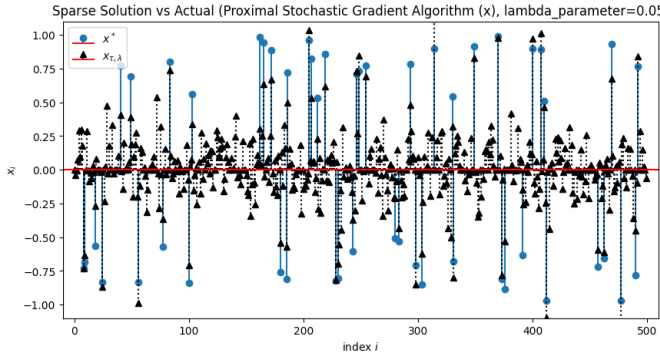


Figure 3: PSGA

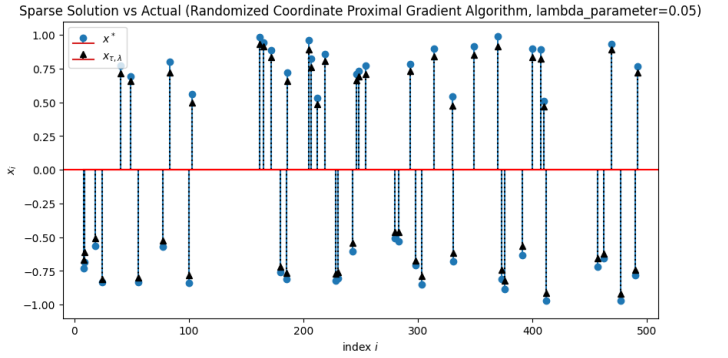


Figure 4: RCPGA

Part 4: Support Vector Machines

We are given the primal problem:

$$\min_{w \in \mathcal{H}} \frac{\lambda}{n} \sum_{i=1}^n (1 - y_i \langle w, \Lambda(x_i) \rangle)_+ + \frac{\lambda}{2} \|w\|^2$$

We can express in the form:

$$\min_{w \in \mathcal{H}} g(w) + f(w)$$

where:

$$g(w) = \frac{\lambda}{n} \sum_{i=1}^n (1 - y_i \langle w, \Lambda(x_i) \rangle)_+$$

and

$$f(w) = \frac{\lambda}{2} \|w\|^2$$

The corresponding dual problem:

$$\min_{\alpha \in \mathbb{R}^n} \frac{1}{2} \langle \mathbf{K}_y \alpha, \alpha \rangle - \langle \mathbf{1}_n, \alpha \rangle + \sum_{i=1}^n i_{[0, \frac{1}{\lambda n}]}(\alpha_i)$$

We can express in the form:

$$\min_{\alpha \in \mathbb{R}^n} g^*(\alpha) + f^*(\alpha)$$

4.1

4.2

4.3

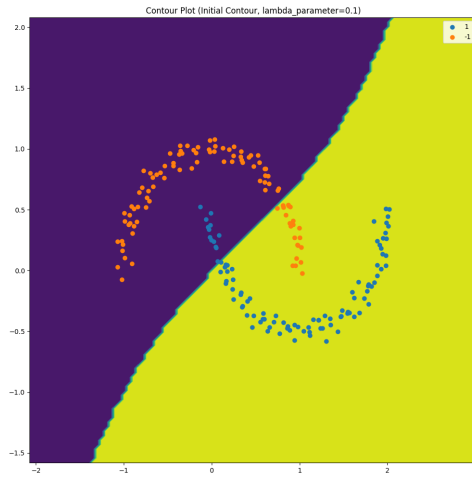


Figure 5: $\alpha^{(0)}$ Contour Plot

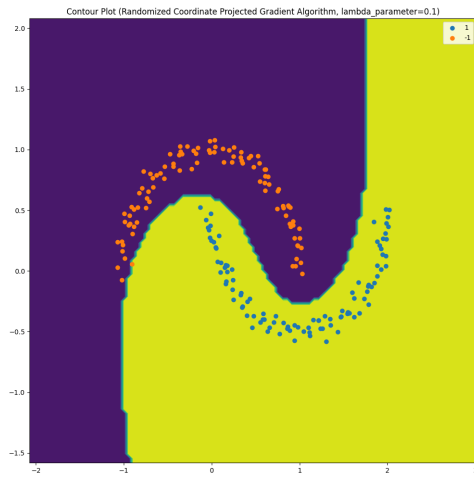


Figure 6: RCPGA Contour Plot

Figure 7: FISTA Contour Plot

4.4

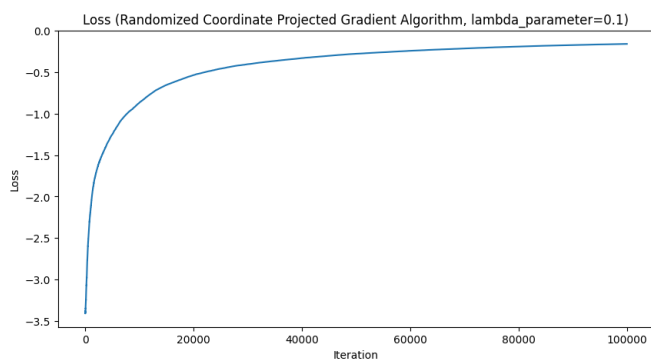


Figure 8: RCPGA Loss

Figure 9: FISTA Loss