

Knapsack Problem (背包问题)

Introduction to Advanced Operations Research

Learn by yourself on the knapsack problem given below, and complete the exercise

0 – 1 Knapsack problem

Given a set of items, each with a weight and a value, we want to select some items and pack them into a knapsack with limited total weight. How can we determine these items so as to maximize the total value?

Mathematical model

Settings:

- collection of items $\{1, 2, \dots, n\}$;
- item i has a value v_i and a weight w_i for each $i = 1, \dots, n$;
- the maximum weight in the bag is x ;
- the objective is to maximize the total value of selected items.

On basis of the above settings, we can formulate the problem as below:

$$g(x) = \max_{y_i \in \{0,1\}, \forall i} \left\{ \sum_{i=1}^n v_i y_i : \text{such that } \sum_{i=1}^n w_i y_i \leq x \right\}, \quad (1)$$

where y_i indicates whether item i is selected ($y_i = 1$) or not ($y_i = 0$), and $g(x)$ denote the maximum value corresponding to x .

For simplicity, you can start from the assumptions below:

- w_i are all positive integers.
- x is also a positive integer.

Solution: Dynamic programming (DP)

The main idea is to inductively calculate $g(0), g(1), \dots, g(x)$. Firstly observe that $g(0) = 0$, i.e., if the bag allows only 0 maximum weight, then clearly no item can be selected hence the total value

is 0. Moreover, we have that

$$g(x) = \max_{w_i \leq x} [v_i + g(x - w_i)]$$

Then the desirable value $g(x)$ can be founded in an $O(nx)$ time.

Heuristic: Greedy approximation algorithm

1. sort the items in decreasing order of value per unit of weight, v_i/w_i :

$$v_1/w_1 \geq v_2/w_2 \geq \dots \geq v_n/w_n$$

2. select items into the sack starting from the one with highest v_i/w_i until the total weight exceeds x

$$\max_n \left\{ \sum_{i=1}^n v_i/w_i : \sum_{i=1}^n w_i \leq x \right\}.$$

Matlab Exercise

Code both DP algorithm and the greedy approximation algorithm, and report the performance of the latter on basis of numerical study. You need to figure out by yourself the exact description of task, detailed plan, and the target.