Solving a Stochastic Growth Model: Linear-Quadratic

Seungyoon Jeong

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Compute equilibria of the following growth model:

$$\max_{\{c_t,x_t,l_t,h_t\}} \mathbb{E}\sum_{t=0}^{\infty}\beta^t\left\{logc_t+\psi logl_t\right\}N_t \text{ subject to } c_t+x_t=k_t^{\theta}\left((1+\gamma_z)^t z_t h_t\right)^{1-\theta}$$
$$N_{t+1}k_{t+1}=[(1-\delta)k_t+x_t]N_t$$
$$logz_t=\rho logz_{t-1}+\epsilon_t,\ \epsilon\sim N(0,\sigma^2)$$
$$h_t+l_t=1$$
$$c_t,x_t\geq 0$$
$$\text{where } N_t=(1+\gamma_n)^t$$

First, let me detrend stochastic the technological progress first. Resource constraint can be written as

$$c_t+x_t=k_t^{\theta}\left((1+\gamma_z)^t z_t h_t\right)^{1-\theta}\Leftrightarrow c_t+x_t=(1+\gamma_z)^t\left(\frac{k_t}{(1+\gamma_z)^t}\right)^{\theta}(z_t h_t)^{1-\theta}$$

Define $\hat{c}_t=\frac{c_t}{(1+\gamma_z)^t}$, $\hat{x}_t=\frac{x_t}{(1+\gamma_z)^t}$, $\hat{k}_t=\frac{k_t}{(1+\gamma_z)^t}$, and $\hat{\beta}=\beta(1+\gamma_n)$. Then rewrite our original model,

$$\max_{\{\hat{c}_t,\hat{k}_{t+1},h_t\}} \mathbb{E}\sum_{t=0}^{\infty}\hat{\beta}^t\left\{log(1+\gamma_z)^t\hat{c}_t+\psi log(1-h_t)\right\} \text{ subject to } \hat{c}_t+\hat{x}_t=\left(\hat{k}_t\right)^{\theta}\left(e^{z_t}h_t\right)^{1-\theta}$$
$$[(1+\gamma_n)(1+\gamma_z)]\,\hat{k}_{t+1}=(1-\hat{\delta})\hat{k}_t+\hat{x}_t$$
$$z_t=z_{t-1}+\epsilon_t,\ \epsilon\sim N(0,\sigma^2)$$
$$c_t,x_t\geq 0$$
$$\text{where } N_t=(1+\gamma_n)^t$$

write corresponding Bellman equation for the problem. By substituting in c_t, x_t, we have following Bellman's equation.

$$V(\hat{k}_t,z_t)=\max_{\hat{k}_{t+1},h_t}\left\{log\left(\left(\hat{k}_t\right)^{\theta}\left(e^{z_t}h_t\right)^{1-\theta}-(1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1}+(1-\delta)\hat{k}_t\right)+\psi log(1-h_t)+\hat{\beta}\mathbb{E}\left[V(\hat{k}_{t+1},z_{t+1})\right]\right\}$$

then, F.O.C. are

$$[\hat{\boldsymbol{k}}_{\boldsymbol{t+1}}] : \frac{-(1+\gamma_n)(1+\gamma_z)}{\left(\hat{k}_t\right)^{\theta}(e^{z_t}h_t)^{1-\theta} - (1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1} + (1-\delta)\hat{k}_t} + \hat{\beta}\mathbb{E}\left[\frac{\partial V(\hat{k}_{t+1}, z_{t+1})}{\partial \hat{k}_{t+1}}\right] = 0$$

$$[\boldsymbol{h_t}] : \frac{(1-\theta)\hat{k}_t^{\theta} e^{z_t(1-\theta)} h_t^{-\theta}}{\left(\hat{k}_t\right)^{\theta}(e^{z_t}h_t)^{1-\theta} - (1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1} + (1-\delta)\hat{k}_t} - \frac{\psi}{(1-h_t)} = 0$$

$$[\boldsymbol{ENV}] : \frac{\partial V(\hat{k}_{t+1}, z_{t+1})}{\partial \hat{k}_{t+1}} = \frac{\theta\hat{k}_{t+1}^{\theta-1}\left(e^{z_{t+1}}h_{t+1}\right)^{1-\theta} + 1 - \delta}{\left(\hat{k}_{t+1}\right)^{\theta}(z_{t+1}h_{t+1})^{1-\theta} - (1+\gamma_n)(1+\gamma_z)\hat{k}_{t+2} + (1-\delta)\hat{k}_{t+1}}$$

Combining first F.O.C. and Envelope condition, we obtain an Euler Equation

$$\frac{(1+\gamma_n)(1+\gamma_z)}{\left(\hat{k}_t\right)^{\theta}(e^{z_t}h_t)^{1-\theta} - (1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1} + (1-\delta)\hat{k}_t} = \hat{\beta}\mathbb{E}_t\left[\frac{\theta\hat{k}_{t+1}^{\theta-1}\left(e^{z_{t+1}}h_{t+1}\right)^{1-\theta} + 1 - \delta}{\left(\hat{k}_{t+1}\right)^{\theta}(z_{t+1}h_{t+1})^{1-\theta} - (1+\gamma_n)(1+\gamma_z)\hat{k}_{t+2} + (1-\delta)\hat{k}_{t+1}}\right]$$

Also, Labour-Consumption choices are governed by

$$\frac{\psi}{(1-h_t)} = \frac{(1-\theta)\hat{k}_t^{\theta} e^{z_t(1-\theta)} h_t^{-\theta}}{\left(\hat{k}_t\right)^{\theta}(e^{z_t}h_t)^{1-\theta} - (1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1} + (1-\delta)\hat{k}_t}$$

Using the Euler equation, we can find a Steady State of this economy with following calibration

Parameter	Value
θ	0.35
β	0.9722
δ	0.0464
γ_z	0.016
γ_n	0.015
σ	0.5
ρ	0.2
ψ	2.24

a) Linear Quadratic Approximation

Here, I assumed my main setup is with return function depending on hours, capital today, and capital tomorrow. The rest two setup has been attached on last page.

Return function depending on hours, capital today, and capital tomorrow

Step 1. Calculate the Steady State level of variable with nonlinear solver.

With my parameterization, I obtained

Capital	Working hours	Investment	Consumption	Innovation
2.3037	0.2922	0.1789	0.4231	1.0

Table 1: Steady States of the Economy

Step 2. Express the return function with Linear-Quadratic

Given my setup, now problem turn into

$$r\left(X_t = \begin{bmatrix} \hat{k}_t \\ \hat{z}_t \\ 1 \end{bmatrix}, u_t = \begin{bmatrix} \hat{k}_{t+1} \\ h_t \end{bmatrix}\right) = \log\left(\left(e^{\hat{z}_t}\hat{k}_t\right)^{\theta}(h_t)^{1-\theta} - (1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1} + (1-\delta)\hat{k}_t\right) + \psi\log(1-h_t)$$

$$\text{s.t.} \quad \begin{bmatrix} \hat{k}_{t+1} \\ \hat{z}_{t+1} \\ 1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & \rho & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{A} \begin{bmatrix} \hat{k}_t \\ \hat{z}_t \\ 1 \end{bmatrix} + \underbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}}_{B} \begin{bmatrix} k_{t+1} \\ h_t \end{bmatrix} + \underbrace{\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}}_{C} \varepsilon_{t+1}$$

Next, for convenience I applied Kydland and Prescott's Method* (please see Appendix on last page) to obtain R, Q,and W by implementing second order linearization around steady state. So, now we can fully express above problem into following set-up.

$$\max_{\{u_t\}_{t=0}^{\infty}} \mathbb{E}_0 \left[\sum_{t=0}^{\infty} \hat{\beta}^t \left\{X_t' Q X + u_t' R u_t + 2X_t' W u_t \mid X_0\right\}\right] \quad \text{s.t. } X_{t+1} = AX_t + Bu_t + C\epsilon_{t+1}$$

$$X_0 \text{ is given.}$$

That is,

$$r(X_t, u_t) = X_t' \underbrace{\begin{bmatrix} -3.0812 & -1.0719 & 2.4701 \\ -1.0719 & -0.1271 & 0.4624 \\ 2.4701 & 0.4624 & -2.4005 \end{bmatrix}}_{Q} X + u_t' \underbrace{\begin{bmatrix} -2.9706 & 3.8570 \\ 3.8570 & -9.1390 \end{bmatrix}}_{R} u_t + 2X_t' \underbrace{\begin{bmatrix} 3.0104 & -3.6683 \\ 1.1271 & -0.4348 \\ -2.4375 & 2.2357 \end{bmatrix}}_{W} u_t$$

$$g(X_t, u_t, \epsilon_{t+1}) = AX_t + Bu_t + C\epsilon_{t+1}$$

Then, we map this problem into undiscounted problem using below transformations

$$\tilde{X}_t = \beta^{t/2} X_t$$

$$\tilde{u}_t = \beta^{t/2}(u_t + R^{-1}W'X_t)$$

$$\tilde{A} = \sqrt{\beta}(A - BR^{-1}W')$$

$$\tilde{B} = \sqrt{\beta}B$$

$$\tilde{Q} = Q - WR^{-1}W'$$

Step 3. Get a policy function by using convergence of Ricatti Equation

Algorithm is as follow: Given initial P_0 and F_0

a) Update P_n and F_n with updating rule

$$P_{n+1} = \tilde{Q} + \tilde{A}' P_n \tilde{A} - \tilde{A}' P_n \tilde{B} \left(R + \tilde{B}' P_n \tilde{B} \right)^{-1} \tilde{B}' P_n \tilde{A}$$

$$F_{n+1} = \left(R + \tilde{B}' P_n \tilde{B} \right)^{-1} \tilde{B}' P_n \tilde{A}$$

b) Iterate until they satisfies its convergence critiria at the same time

c) Then, set F=$\tilde{F}_n$+R^{-1}W' and P=P_n

Finally, optimal policy function is

$$u_t = -F \begin{bmatrix} \tilde{k}_t \\ \tilde{z}_t \\ 1 \end{bmatrix}$$

Below, I plot the optimal capital and labour policy functions.

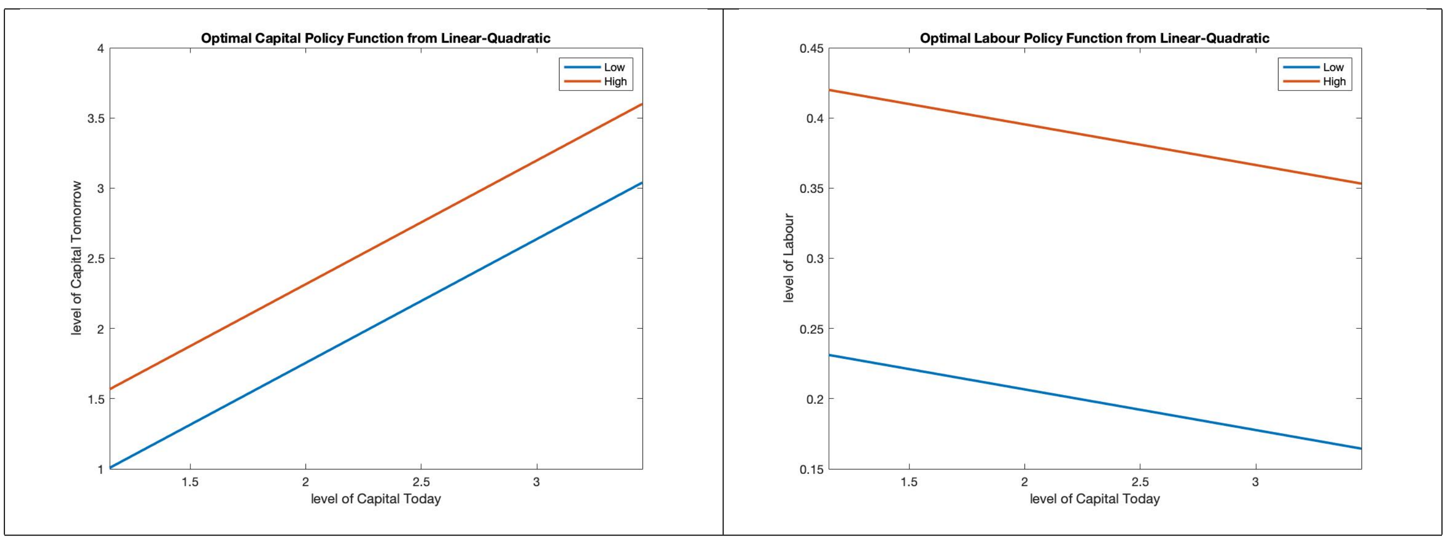

Figure 1: Result of Linear Quadratic Method

b) Acceleration with Vaughan's Algorithm

To improve slow convergence of Ricatti equation, we can also implement Vaughan's method. Steps are the same as above until getting undiscounted problem. Then, define $\mathbb{H}$ as

$$\mathbb{H} = \begin{bmatrix} \tilde{A}^{-1} & \tilde{A}^{-1}\tilde{B}R^{-1}\tilde{B}' \\ \tilde{Q}\tilde{A}^{-1} & \tilde{Q}\tilde{A}^{-1}\tilde{B}R^{-1}\tilde{B}' + \tilde{A}' \end{bmatrix}$$

Then, implement Eigenvalue decomposition on the $\mathbb{H}$ with adjustment of position to locate eigenvalue inside of unit circle as Γ

$$\mathbb{H} = \underbrace{\begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}}_{V} \begin{bmatrix} \Gamma & 0 \\ 0 & \Gamma^{-1} \end{bmatrix} \begin{bmatrix} V_{11} & V_{12} \\ V_{21} & V_{22} \end{bmatrix}^{-1}$$

So, we can obtain $P = V_{21}V_{11}^{-1}$ and finally F is

$$F = \left(R + \tilde{B}'P\tilde{B}\right)^{-1} \tilde{B}'P\tilde{A} + R^{-1}W'$$

I report my result, but actually F I obtained from Vaughan's method is exactly the same as I got in LQ, so the optimal policy function is the same.

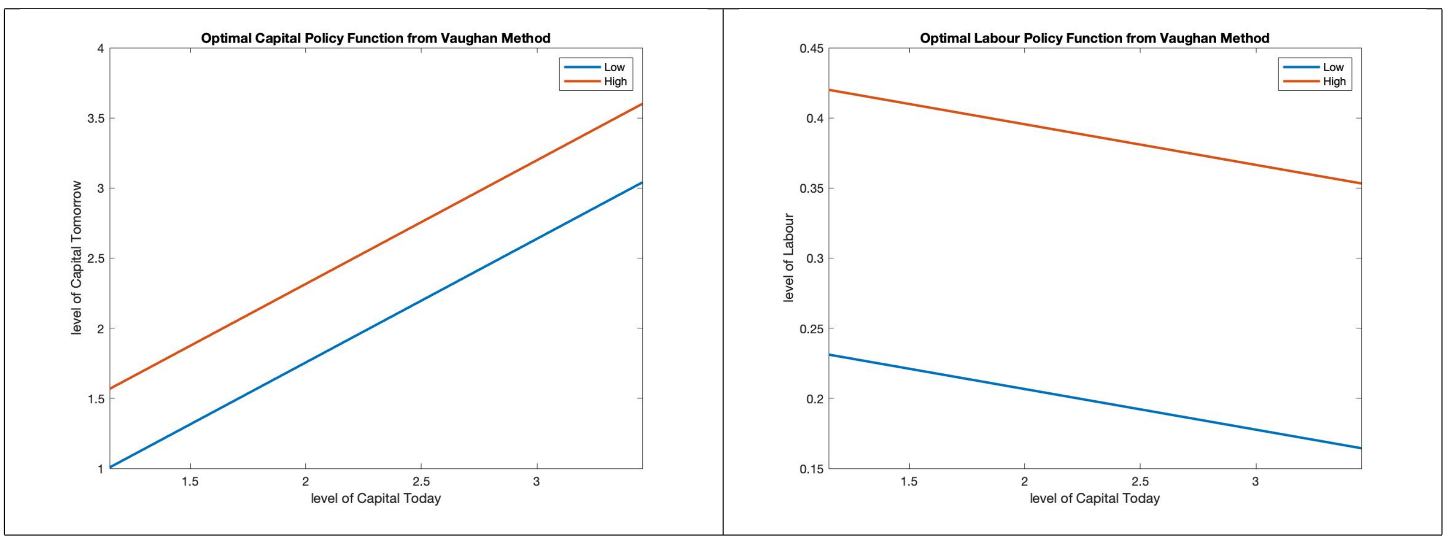

Figure 2: Result of Vaughan Method

Comments

I have two comments in a comparison of three method.

1. Accuracy

Remembering the Value Function Iteration is global method, while the two other methods are local method (approximation around steady-state), when I compare the three capital policy functions and labour policy function, I found they match exactly the same around the steady state. However, as x values being far away from the steady state, I can see some mismatches in the optimal policy function.

2. Speed

I briefly report speeds of three method

	Value Function Iteration	Linear-Quadratic	Vaughan's Method
Time	3 mins 9.262 sec	3.861 sec	3.792 sec

Table 2: Speed Comparison of Three method

For the VFI, convergence obtained in 1818 th iteration. Note that I used 1000 grids on Capital and two grid on productivity.