Solving a Stochastic Growth Model: VFI

Seungyoon Jeong

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Compute equilibria of the following growth model:

$$\max_{\{c_t,x_t,l_t,h_t\}} \mathbb{E}\sum_{t=0}^{\infty}\beta^t\left\{logc_t+\psi logl_t\right\}N_t \text{ subject to } c_t+x_t=k_t^{\theta}\left((1+\gamma_z)^t z_t h_t\right)^{1-\theta}$$

$$N_{t+1}k_{t+1}=[(1-\delta)k_t+x_t]N_t$$

$$logz_t=\rho logz_{t-1}+\epsilon_t,\ \epsilon\sim N(0,\sigma^2)$$

$$h_t+l_t=1$$

$$c_t,x_t\geq 0$$

$$\text{where } N_t=(1+\gamma_n)^t$$

First, let me detrend stochastic the technological progress first. Resource constraint can be written as

$$c_t+x_t=k_t^{\theta}\left((1+\gamma_z)^t z_t h_t\right)^{1-\theta}\Leftrightarrow c_t+x_t=(1+\gamma_z)^t\left(\frac{k_t}{(1+\gamma_z)^t}\right)^{\theta}(z_th_t)^{1-\theta}$$

Define $\hat{c}_t=\frac{c_t}{(1+\gamma_z)^t}$, $\hat{x}_t=\frac{x_t}{(1+\gamma_z)^t}$, $\hat{k}_t=\frac{k_t}{(1+\gamma_z)^t}$, and $\hat{\beta}=\beta(1+\gamma_n)$. Then rewrite our original model,

$$\max_{\{\hat{c}_t,\hat{k}_{t+1},h_t\}} \mathbb{E}\sum_{t=0}^{\infty}\hat{\beta}^t\left\{log(1+\gamma_z)^t\hat{c}_t+\psi log(1-h_t)\right\} \text{ subject to } \hat{c}_t+\hat{x}_t=\left(\hat{k}_t\right)^{\theta}\left(e^{z_t}h_t\right)^{1-\theta}$$

$$[(1+\gamma_n)(1+\gamma_z)]\,\hat{k}_{t+1}=(1-\hat{\delta})\hat{k}_t+\hat{x}_t$$

$$z_t=z_{t-1}+\epsilon_t,\ \epsilon\sim N(0,\sigma^2)$$

$$c_t,x_t\geq 0$$

$$\text{where } N_t=(1+\gamma_n)^t$$

write corresponding Bellman equation for the problem. By substituting in c_t, x_t, we have following Bellman's equation.

$$V(\hat{k}_t,z_t)=\max_{\hat{k}_{t+1},h_t}\left\{log\left(\left(\hat{k}_t\right)^{\theta}\left(e^{z_t}h_t\right)^{1-\theta}-(1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1}+(1-\delta)\hat{k}_t\right)+\psi log(1-h_t)+\hat{\beta}\mathbb{E}\left[V(\hat{k}_{t+1},z_{t+1})\right]\right\}$$

Then, F.O.C. are

$$[\boldsymbol{\hat{k}_{t+1}}]: \frac{-(1+\gamma_n)(1+\gamma_z)}{\left(\hat{k}_t\right)^{\theta}\left(e^{z_t}h_t\right)^{1-\theta}-(1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1}+(1-\delta)\hat{k}_t}+\hat{\beta}\mathbb{E}\left[\frac{\partial V(\hat{k}_{t+1},z_{t+1})}{\partial \hat{k}_{t+1}}\right]=0$$

$$[\boldsymbol{h_t}]: \frac{(1-\theta)\hat{k}_t^{\theta}e^{z_t(1-\theta)}h_t^{-\theta}}{\left(\hat{k}_t\right)^{\theta}\left(e^{z_t}h_t\right)^{1-\theta}-(1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1}+(1-\delta)\hat{k}_t}-\frac{\psi}{(1-h_t)}=0$$

$$[\boldsymbol{ENV}]: \frac{\partial V(\hat{k}_{t+1},z_{t+1})}{\partial \hat{k}_{t+1}}=\frac{\theta\hat{k}_{t+1}^{\theta-1}\left(e^{z_{t+1}}h_{t+1}\right)^{1-\theta}+1-\delta}{\left(\hat{k}_{t+1}\right)^{\theta}\left(z_{t+1}h_{t+1}\right)^{1-\theta}-(1+\gamma_n)(1+\gamma_z)\hat{k}_{t+2}+(1-\delta)\hat{k}_{t+1}}$$

Combining first F.O.C. and Envelope condition, we obtain an Euler Equation

$$\frac{(1+\gamma_n)(1+\gamma_z)}{\left(\hat{k}_t\right)^{\theta}\left(e^{z_t}h_t\right)^{1-\theta}-(1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1}+(1-\delta)\hat{k}_t}=\hat{\beta}\mathbb{E}_t\left[\frac{\theta\hat{k}_{t+1}^{\theta-1}\left(e^{z_{t+1}}h_{t+1}\right)^{1-\theta}+1-\delta}{\left(\hat{k}_{t+1}\right)^{\theta}\left(z_{t+1}h_{t+1}\right)^{1-\theta}-(1+\gamma_n)(1+\gamma_z)\hat{k}_{t+2}+(1-\delta)\hat{k}_{t+1}}\right]$$

Also, Labour-Consumption choices are governed by

$$\frac{\psi}{(1-h_t)}=\frac{(1-\theta)\hat{k}_t^{\theta}e^{z_t(1-\theta)}h_t^{-\theta}}{\left(\hat{k}_t\right)^{\theta}\left(e^{z_t}h_t\right)^{1-\theta}-(1+\gamma_n)(1+\gamma_z)\hat{k}_{t+1}+(1-\delta)\hat{k}_t}$$

Using the Euler equation, we can find a Steady State of this economy with following calibration

Parameter	Value
θ	0.35
β	0.9722
δ	0.0464
γ_z	0.016
γ_n	0.015
σ	0.5
ρ	0.2
ψ	2.24

Value Function Iteration

- For the Value Function Iteration, I constructed 1000 grids of Captial around steady states level of captial, and 2 grids for production shocks, as a simple case. It has been discretized using Tauchen-Hussey's Method.
- I first solved the static problem from Intratemporal equation with Bisection to find optimal level of labour choice over the multidimensional grid to speed up the algorithm.

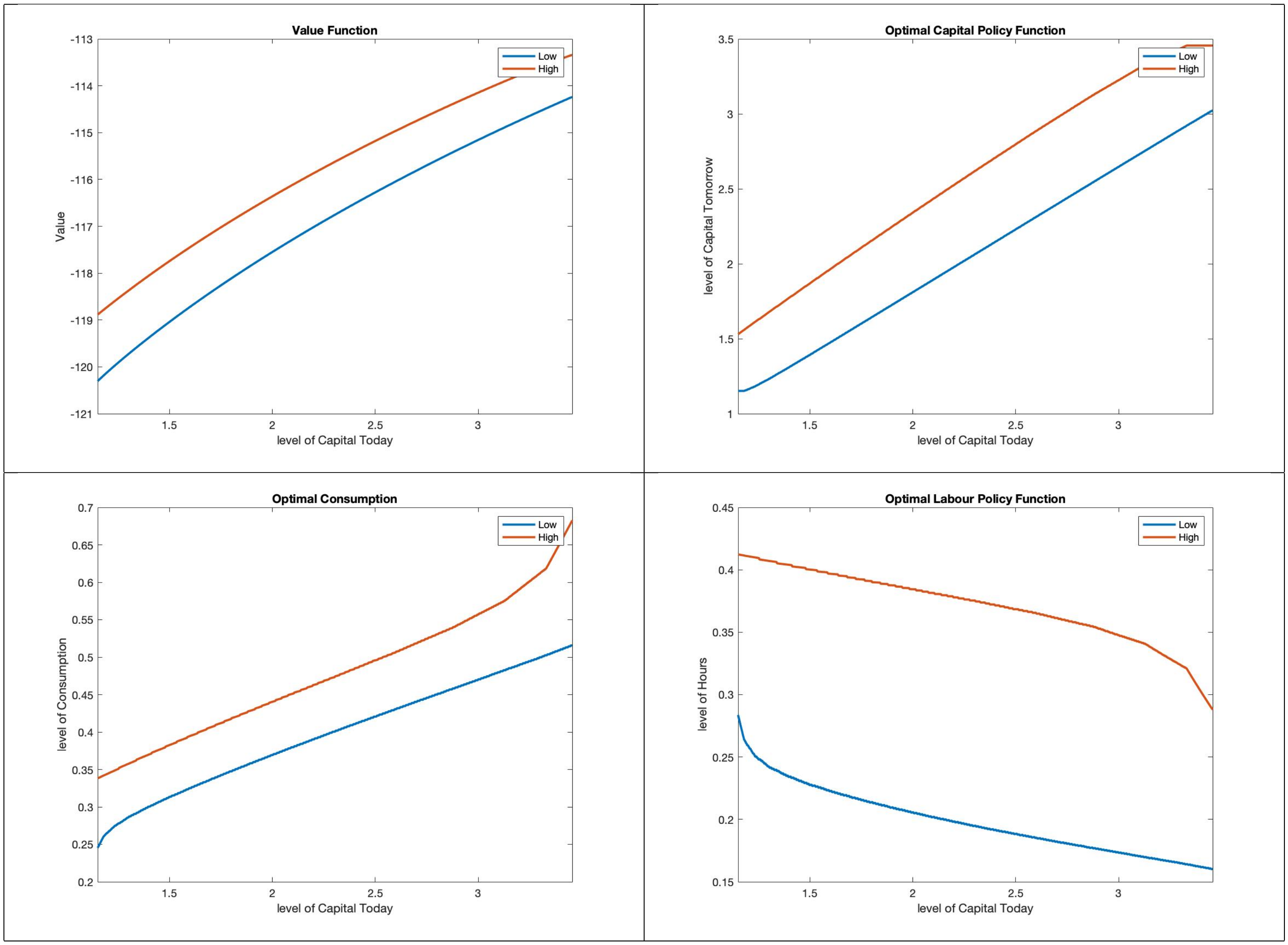

Figure 1: Result of Value Function Iteration