Lab 3: Fourier Series Analysis Using MATLAB

A. Lab Assignment

Problem A.1: Given the periodic signal $x_1(t)$, derive an expression for the Exponential Fourier Series Coefficients D_n.

$$x_1(t) = \cos\frac{3\pi}{10}t + \frac{1}{2}\cos\frac{\pi}{10}t,$$

Answer:

A.1 $x_1(t) = \cos\frac{3\pi}{10}t + \frac{1}{2}\cos\frac{\pi}{10}t$

$$= \left(\frac{1}{2}e^{\frac{3\pi}{10}jt} + \frac{1}{2}e^{\frac{-3\pi}{10}jt}\right) + \frac{1}{2}\left(\frac{1}{2}e^{\frac{\pi}{10}jt} + \frac{1}{2}e^{\frac{-\pi}{10}jt}\right)$$

$$= \frac{1}{2}e^{\frac{3\pi}{10}jt} + \frac{1}{2}e^{\frac{-3\pi}{10}jt} + \frac{1}{4}e^{\frac{\pi}{10}jt} + \frac{1}{4}e^{\frac{-\pi}{10}jt}$$

$\frac{3\pi/10}{\pi/10} = 3 \rightarrow$ rational ✓, $w_{01} = \frac{3\pi}{10}$, $w_{02} = \frac{\pi}{10} \rightarrow w_0 = \frac{\pi}{10}$, $T_0 = \frac{2\pi}{\pi/10} = 20$

$\sum_{n=-\infty}^{\infty} D_n e^{j\frac{\pi}{10}nt} \Rightarrow e^{\frac{3\pi}{10}jt} \rightarrow n=3,\ \boxed{D_3 = \frac{1}{2}}$ $\quad e^{\frac{-3\pi}{10}jt} \rightarrow n=-3,\ \boxed{D_{-3} = \frac{1}{2}}$

$e^{\frac{\pi}{10}jt} \rightarrow n=1,\ \boxed{D_1 = \frac{1}{4}}$ $\quad e^{\frac{-\pi}{10}jt} \rightarrow n=-1,\ \boxed{D_{-1} = \frac{1}{4}}$

$$D_n = \frac{1}{20}\int_{-10}^{10}\left[\frac{1}{2}e^{\frac{3\pi}{10}jt} + \frac{1}{2}e^{\frac{-3\pi}{10}jt} + \frac{1}{4}e^{\frac{\pi}{10}jt} + \frac{1}{4}e^{\frac{-\pi}{10}jt}\right]dt$$

$$= \frac{1}{20}\left[\frac{e^{(3-n)j\pi} - e^{-(3-n)j\pi}}{2j(3-n)\frac{\pi}{10}} + \frac{e^{(3+n)\pi j} - e^{-(3+n)\pi j}}{2j(3+n)\frac{\pi}{10}} + \frac{e^{j(1+n)\pi} - e^{-j(1+n)\pi}}{4j(1+n)\frac{\pi}{10}} + \frac{e^{j(1-n)\pi} - e^{-j(1-n)\pi}}{4j(1-n)\frac{\pi}{10}}\right]$$

$$= \frac{1}{2}\left[\sin((3-n)\pi) + \sin((3+n)\pi) + \frac{1}{2}\sin(1+n)\pi + \frac{1}{2}\sin(1-n)\pi\right]$$

Problem A.2: Repeat Problem A.1 for the periodic signals $x_2(t)$ and $x_3(t)$ shown in Figure 1.

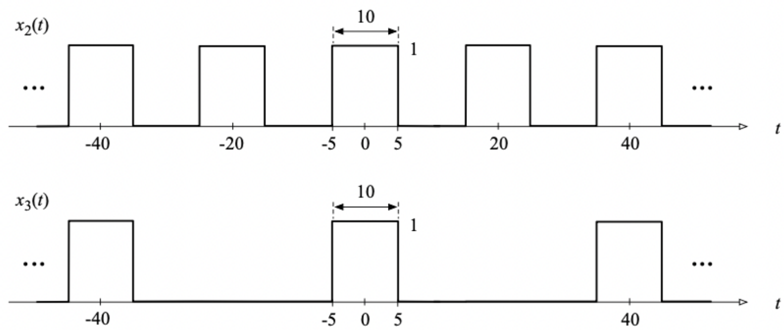


Figure 1: Periodic functions $x_2(t)$ and $x_3(t)$.

Answer:

A2: $x_2(t)$: $T_0 = 20$, $\omega_0 = \frac{2\pi}{20} = \frac{\pi}{10}$

$$D_n = \frac{1}{20}\int_{-5}^{5} (1) e^{-jn\frac{\pi}{10}t}\, dt = \frac{1}{20}\left[\frac{-1}{jn\frac{\pi}{10}} e^{-jn\frac{\pi}{10}t}\right]_{-5}^{5} = \frac{1}{20}\left[\frac{-10}{jn\pi} e^{-jn\frac{\pi}{2}} + \frac{10}{jn\pi} e^{jn\frac{\pi}{2}}\right]$$

$$= \frac{1}{n\pi}\sin\left(\frac{\pi n}{2}\right)$$

$x_3(t) = T_0 = 40$, $\omega_0 = \frac{2\pi}{40} = \frac{\pi}{20}$

$$D_n = \frac{1}{40}\int_{-5}^{5} (1) e^{-jn\frac{\pi}{20}t}\, dt = \frac{1}{40}\left[\frac{-1}{jn\frac{\pi}{20}} e^{-jn\frac{\pi}{20}t}\right]_{-5}^{5} = \frac{1}{40}\left[\frac{-20}{jn\pi} e^{-jn\frac{\pi}{4}} + \frac{20}{jn\pi} e^{jn\frac{\pi}{4}}\right]$$

$$= \frac{1}{n\pi}\sin\left(\frac{\pi n}{4}\right)$$

Problem A.3: Now that you have an expression for D_n, write a MATLAB function that generates D_n for a user specific range of values of n.

Problem A.4: Generate and plot the magnitude and phase spectra of $x_1(t)$, $x_2(t)$ and $x_3(t)$ (using the stem command) from their respective D_n sets for the following index ranges: <u>Note: </u>You can use the MATLAB commands abs and angle to determine the magnitude and phase of a complex number.

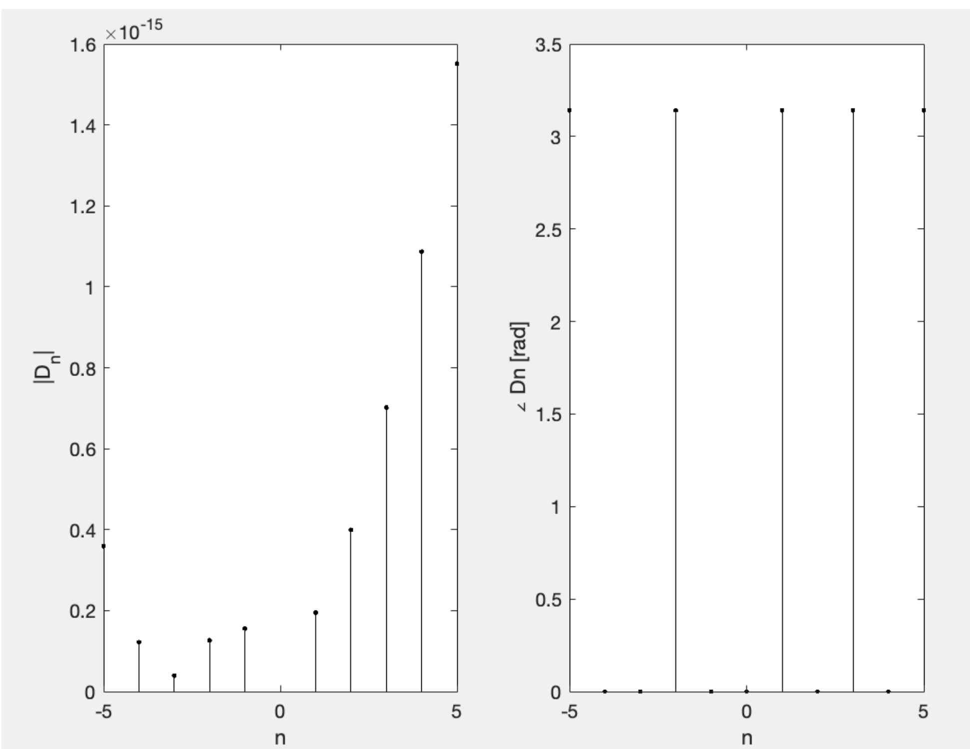
(a) $-5 \leq n \leq 5$;
(b) $-20 \leq n \leq 20$;
(c) $-50 \leq n \leq 50$;
(d) $-500 \leq n \leq 500$.

(a)

Code: **$x_1(t)$**

```
clf;
n = (-5:5);
D_n = 1./2.*((1./(pi.*n)).*sin((3-n).*pi )) + (1./pi.*n).*sin((3+n).*pi)
    + (1./(2.*n.*pi).*sin((1+n).*pi)) + (1./(2.*n.*pi).*sin((1-n).*pi)) ;
subplot(1,2,1); stem(n,abs(D_n),'.k');
xlabel('n'); ylabel('|D_n|');
subplot(1,2,2); stem(n,angle(D_n),'.k');
xlabel('n'); ylabel('\angle Dn [rad]');
```

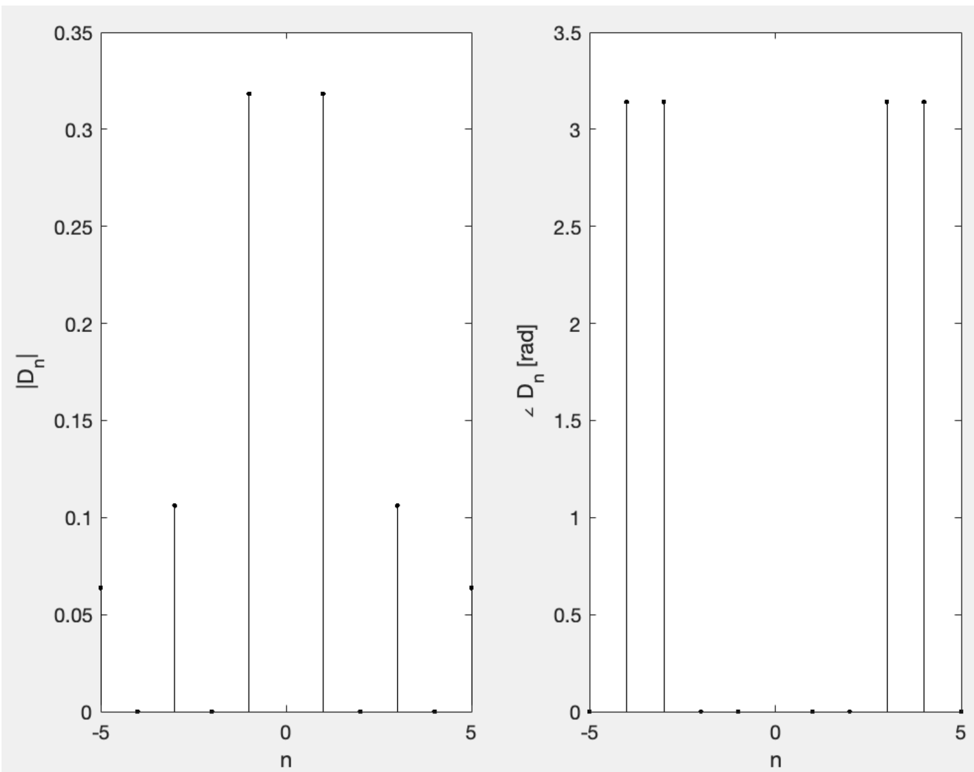
Plot: **$x_1(t)$**



Code: **x_2(t)**

```
clf;
n = (-5:5);
D_n = (1./(n.*pi).*sin((n.*pi)./2));
subplot(1,2,1); stem(n,abs(D_n),'.k');
xlabel('n'); ylabel('|D_n|');
subplot(1,2,2); stem(n,angle(D_n),'.k');
xlabel('n'); ylabel('\angle D_n [rad]');
```

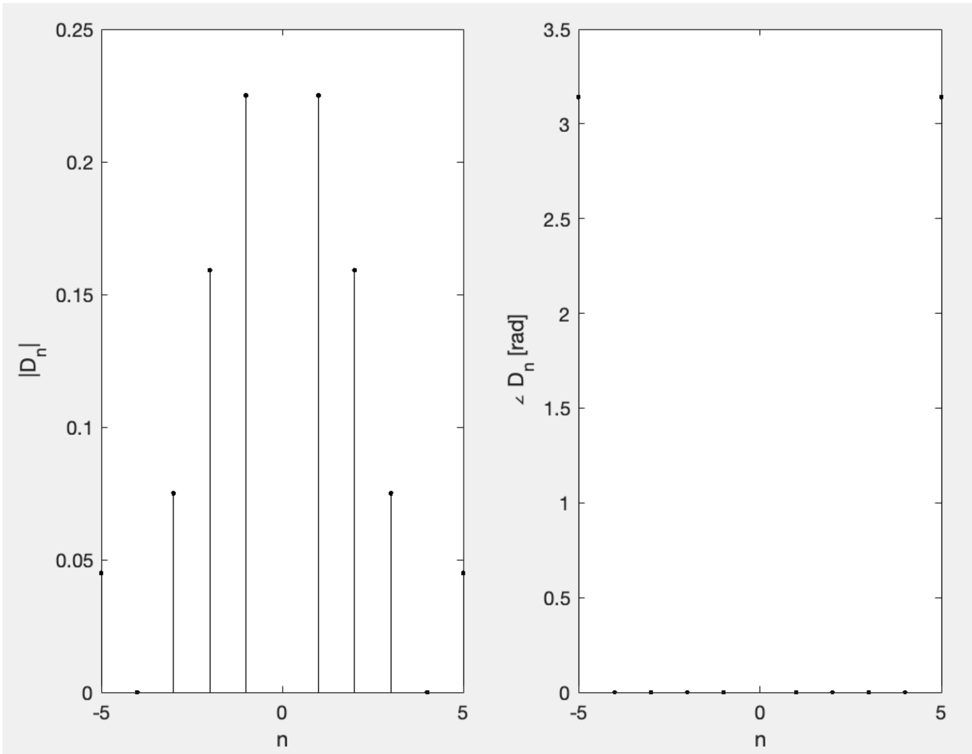
Plot: **x_2(t)**



Code: **x_3(t)**

```
clf;
n = (-5:5);
D_n = (1./(n.*pi).*sin((n.*pi)./4));
subplot(1,2,1); stem(n,abs(D_n),'.k');
xlabel('n'); ylabel('|D_n|');
subplot(1,2,2); stem(n,angle(D_n),'.k');
xlabel('n'); ylabel('\angle D_n [rad]');
```

Plot: **$x_3(t)$**

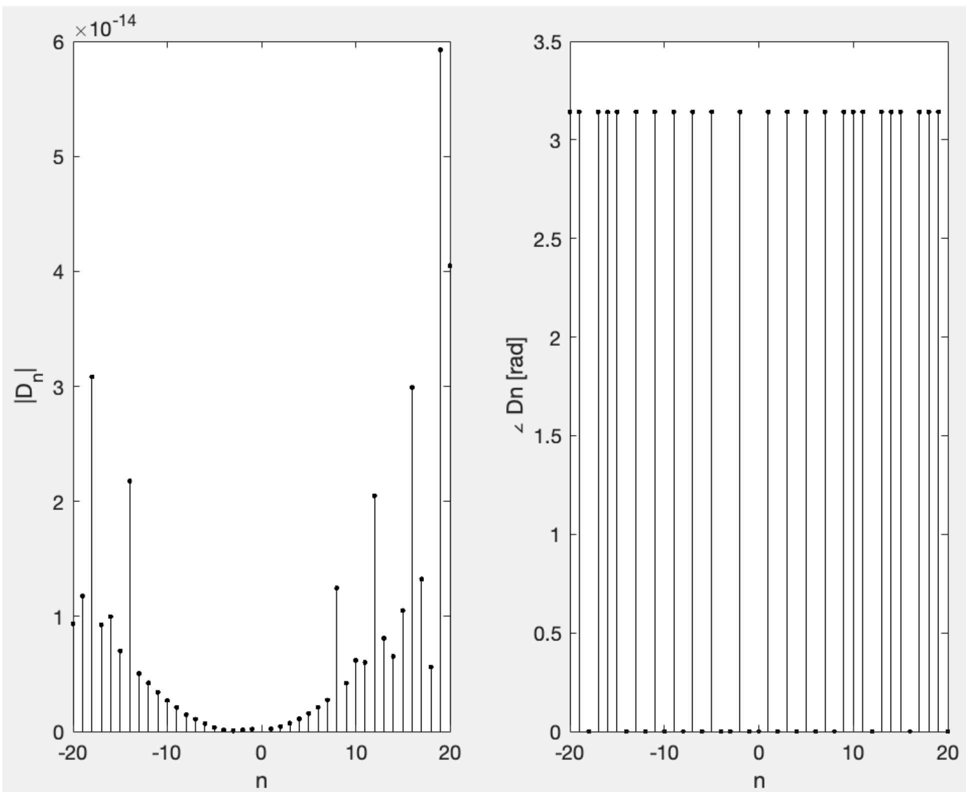


(b)

Code: **x_1(t)**

```
clf;
n = (-20:20);
D_n = 1./2.*((1./(pi.*n)).*sin((3-n).*pi )) + (1./pi.*n).*sin((3+n).*pi)
    + (1./(2.*n.*pi).*sin((1+n).*pi)) + (1./(2.*n.*pi).*sin((1-n).*pi)) ;
subplot(1,2,1); stem(n,abs(D_n),'.k');
xlabel('n'); ylabel('|D_n|');
subplot(1,2,2); stem(n,angle(D_n),'.k');
xlabel('n'); ylabel('\angle Dn [rad]');
```

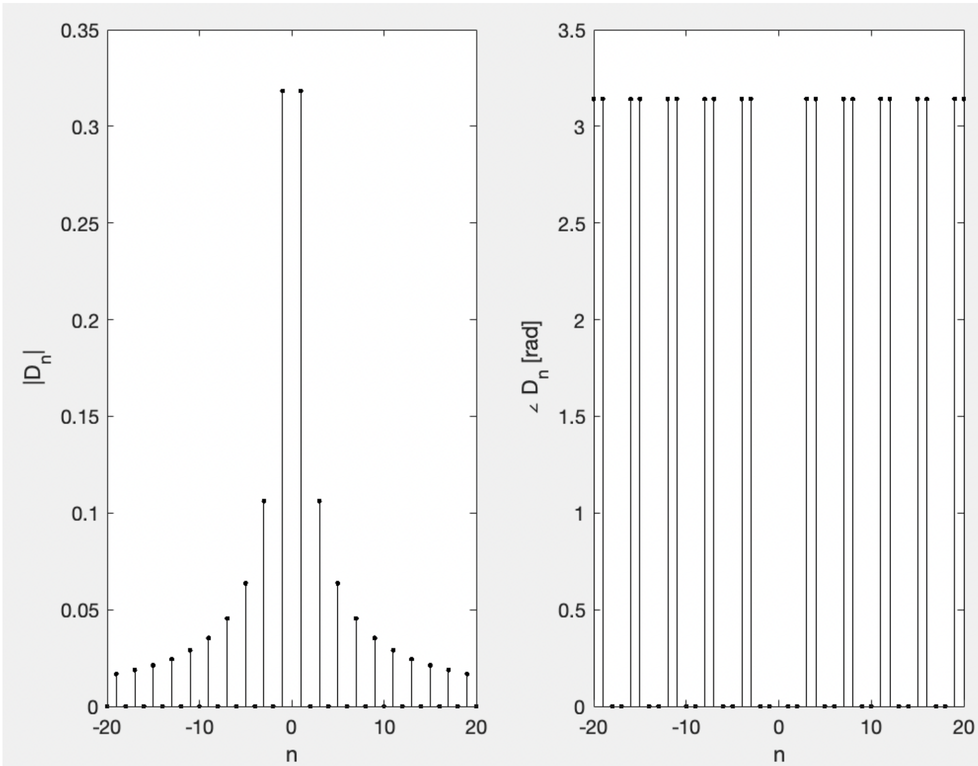
Plot: **x_1(t)**



Code: **x_2(t)**

```
clf;
n = (-20:20);
D_n = (1./(n.*pi).*sin((n.*pi)./2));
subplot(1,2,1); stem(n,abs(D_n),'.k');
xlabel('n'); ylabel('|D_n|');
subplot(1,2,2); stem(n,angle(D_n),'.k');
xlabel('n'); ylabel('\angle D_n [rad]');
```

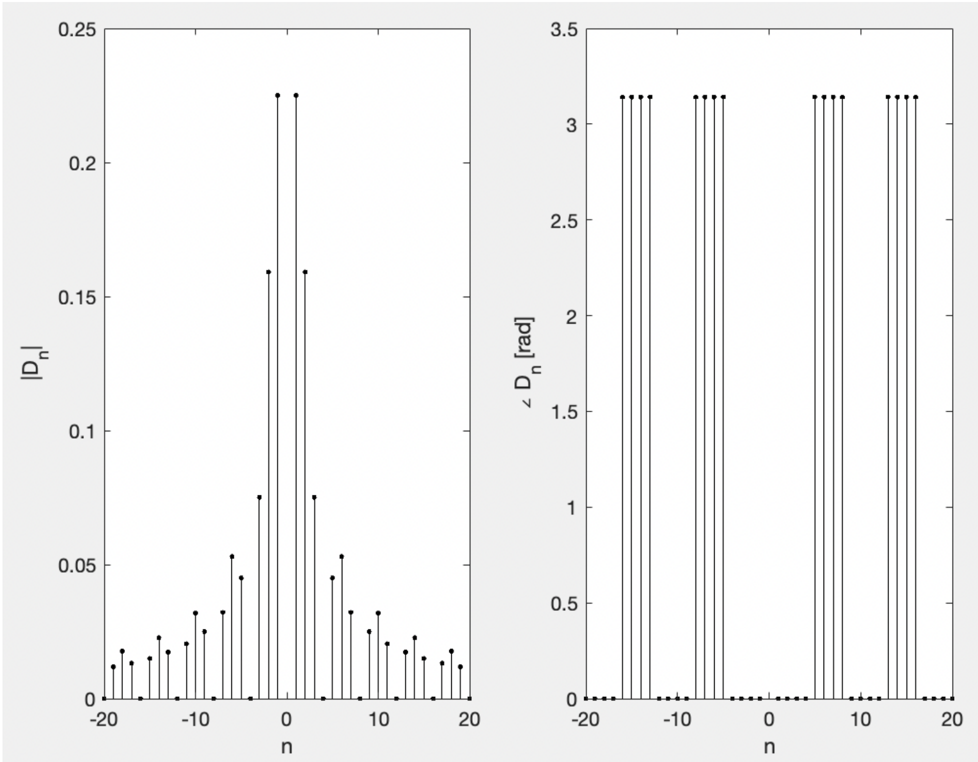
Plot: **$x_2(t)$**



Code: **$x_3(t)$**

```
clf;
n = (-20:20);
D_n = (1./(n.*pi).*sin((n.*pi)./4));
subplot(1,2,1); stem(n,abs(D_n),'.k');
xlabel('n'); ylabel('|D_n|');
subplot(1,2,2); stem(n,angle(D_n),'.k');
xlabel('n'); ylabel('\angle D_n [rad]');
```

Plot: **x_3(t)**

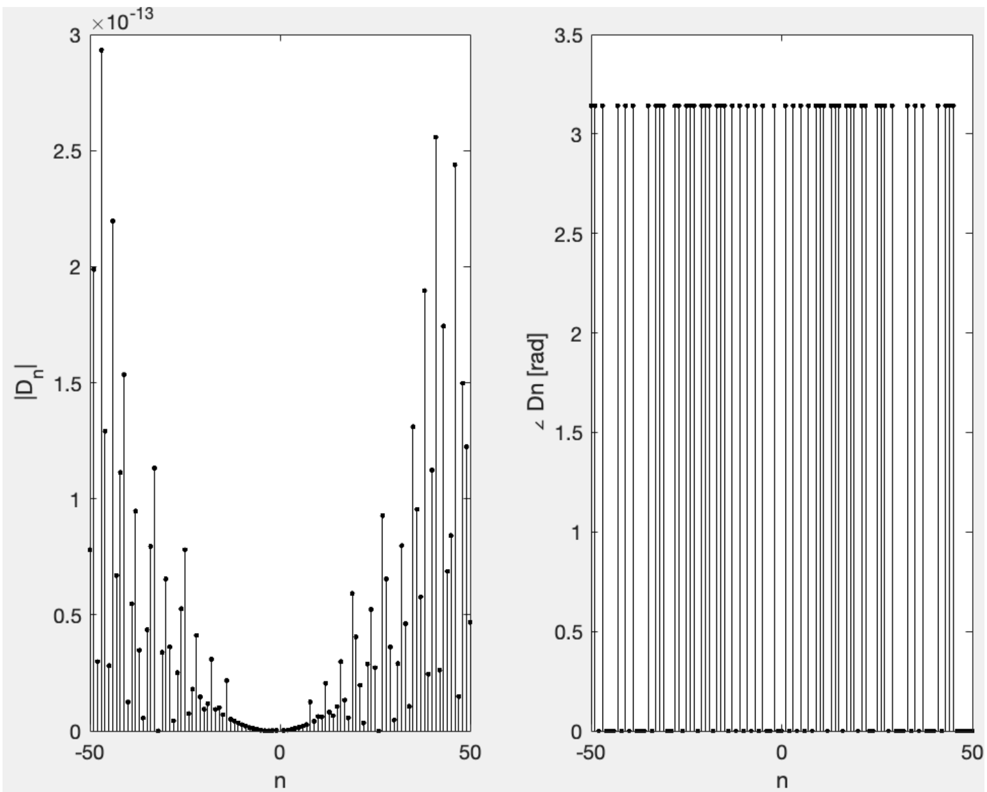

0.25
0.2
0.15
0.1
0.05
0
-20
-10
0
10
20
n
|D_n|
3.5
3
2.5
2
1.5
1
0.5
0
-20
-10
0
10
20
n
∠ D_n [rad]

(c)

Code: $\mathbf{x_1(t)}$

```
clf;
n = (-50:50);
D_n = 1./2.*((1./(pi.*n)).*sin((3-n).*pi )) + (1./pi.*n).*sin((3+n).*pi)
    + (1./(2.*n.*pi).*sin((1+n).*pi)) + (1./(2.*n.*pi).*sin((1-n).*pi)) ;
subplot(1,2,1); stem(n,abs(D_n),'.k');
xlabel('n'); ylabel('|D_n|');
subplot(1,2,2); stem(n,angle(D_n),'.k');
xlabel('n'); ylabel('\angle Dn [rad]');
```

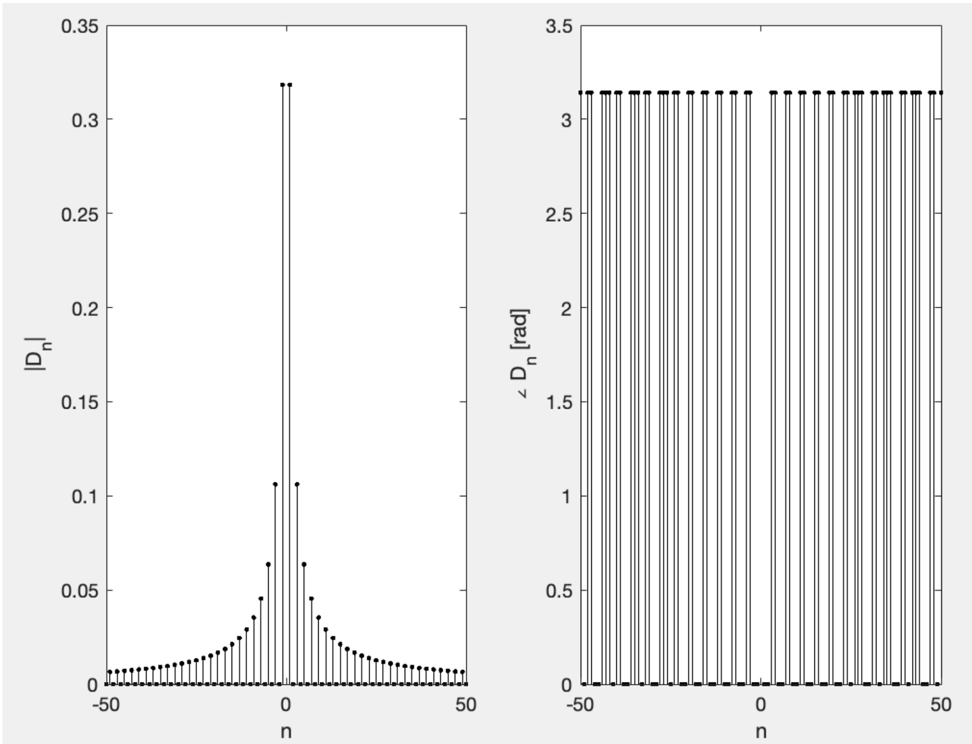
Plot: $\mathbf{x_1(t)}$



Code: $\mathbf{x_2(t)}$

```
clf;
n = (-50:50);
D_n = (1./(n.*pi).*sin((n.*pi)./2));
subplot(1,2,1); stem(n,abs(D_n),'.k');
xlabel('n'); ylabel('|D_n|');
subplot(1,2,2); stem(n,angle(D_n),'.k');
xlabel('n'); ylabel('\angle D_n [rad]');
```

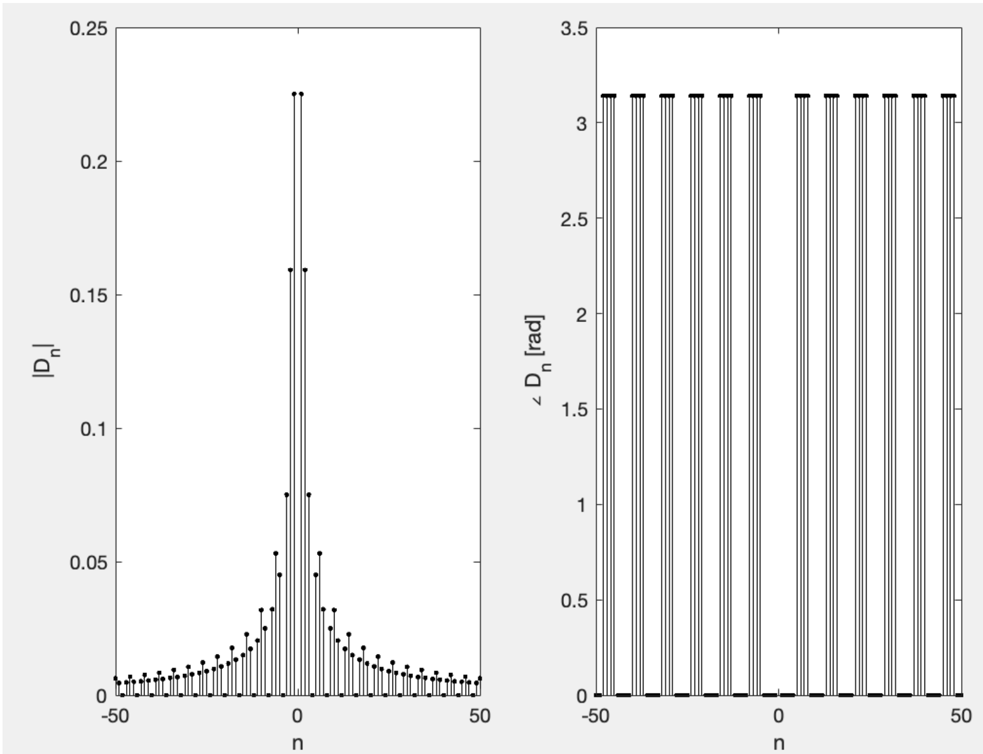
Plot: **x_2(t)**



Code: **x_3(t)**

```
clf;
n = (-50:50);
D_n = (1./(n.*pi).*sin((n.*pi)./4));
subplot(1,2,1); stem(n,abs(D_n),'.k');
xlabel('n'); ylabel('|D_n|');
subplot(1,2,2); stem(n,angle(D_n),'.k');
xlabel('n'); ylabel('\angle D_n [rad]');
```

Plot: **$x_3(t)$**

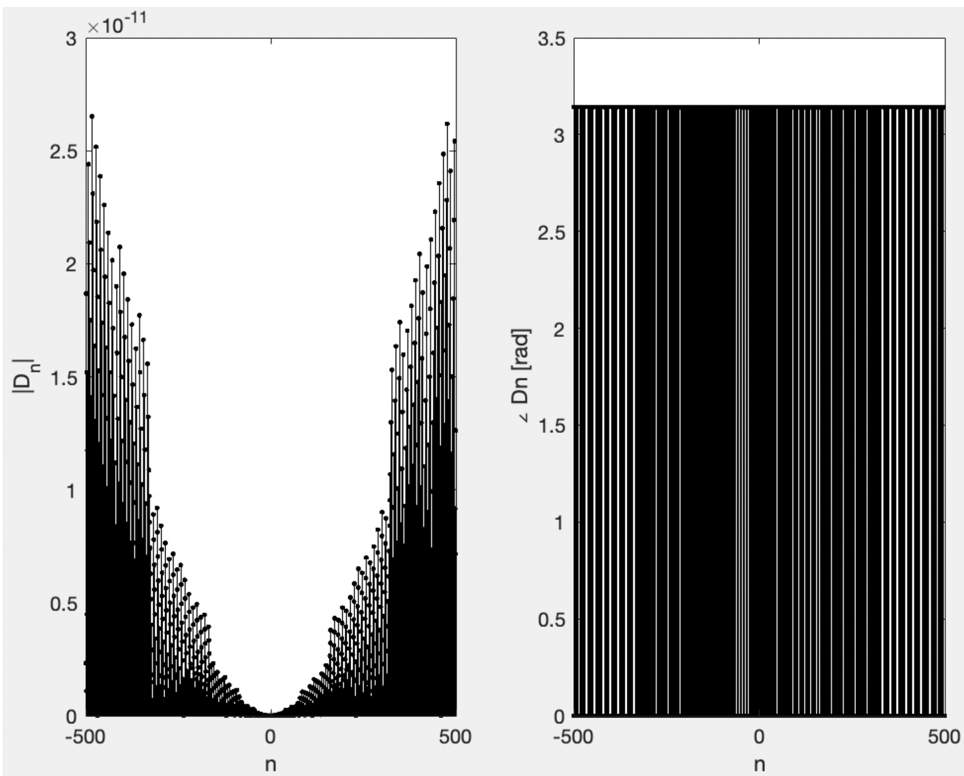

0.25
0.2
0.15
0.1
0.05
0
$|D_n|$
-50
0
50
n
3.5
3
2.5
2
1.5
1
0.5
0
$\angle D_n$ [rad]
-50
0
50
n

(d)

Code: **$x_1(t)$**

```
clf;
n = (-500:500);
D_n = 1./2.*((1./(pi.*n)).*sin((3-n).*pi )) + (1./pi.*n).*sin((3+n).*pi)
    + (1./(2.*n.*pi).*sin((1+n).*pi)) + (1./(2.*n.*pi).*sin((1-n).*pi)) ;
subplot(1,2,1); stem(n,abs(D_n),'.k');
xlabel('n'); ylabel('|D_n|');
subplot(1,2,2); stem(n,angle(D_n),'.k');
xlabel('n'); ylabel('\angle Dn [rad]');
```

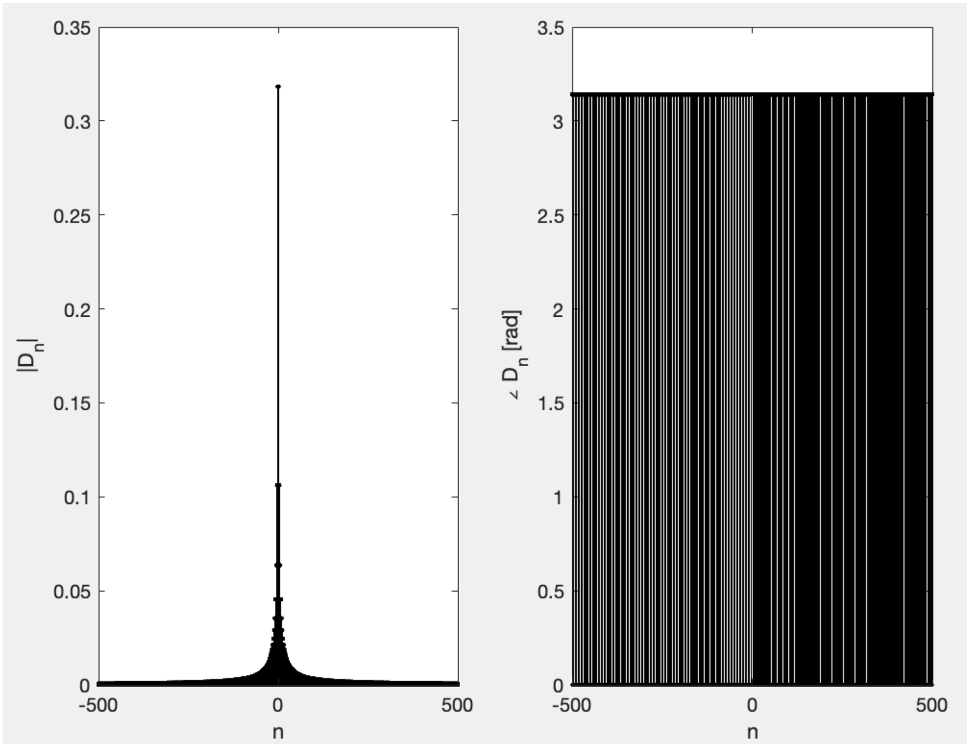
Plot: **$x_1(t)$**



Code: **$x_2(t)$**

```
clf;
n = (-500:500);
D_n = (1./(n.*pi).*sin((n.*pi)./2));
subplot(1,2,1); stem(n,abs(D_n),'.k');
xlabel('n'); ylabel('|D_n|');
subplot(1,2,2); stem(n,angle(D_n),'.k');
xlabel('n'); ylabel('\angle D_n [rad]');
```

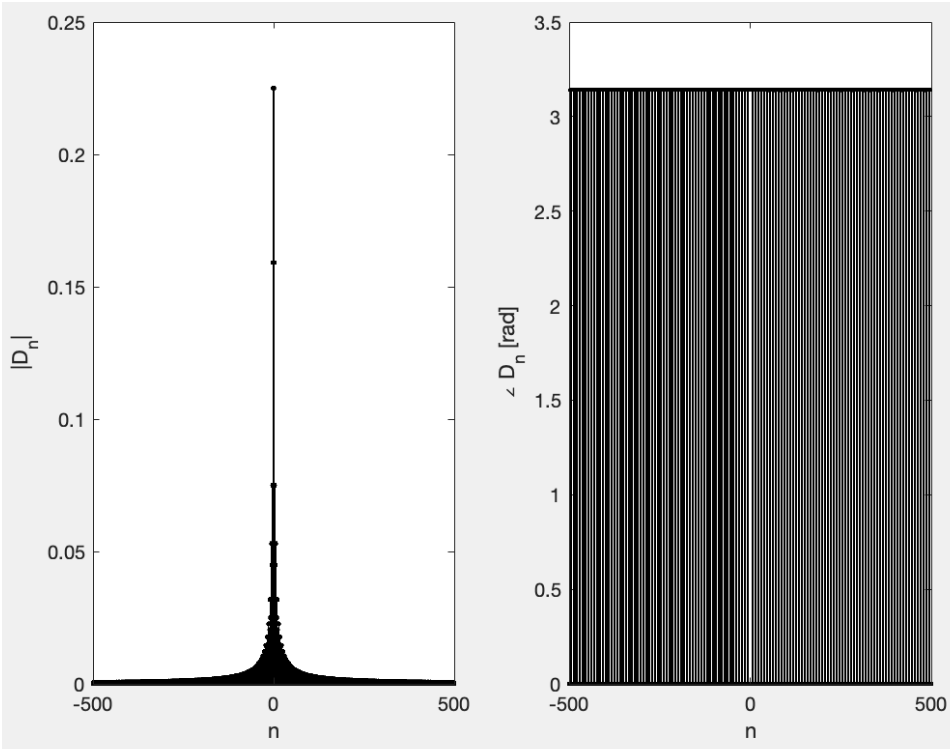
Plot: **$x_2(t)$**



Code: **$x_3(t)$**

```
clf;
n = (-500:500);
D_n = (1./(n.*pi).*sin((n.*pi)./4));
subplot(1,2,1); stem(n,abs(D_n),'.k');
xlabel('n'); ylabel('|D_n|');
subplot(1,2,2); stem(n,angle(D_n),'.k');
xlabel('n'); ylabel('\angle D_n [rad]');
```

Plot: **$x_3(t)$**

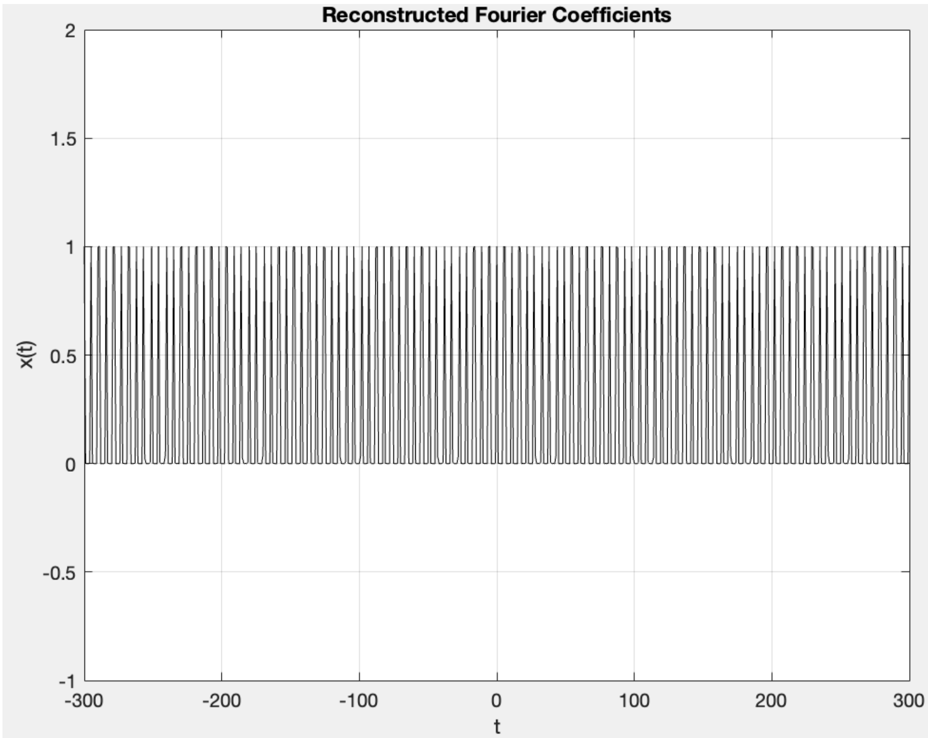

0.25
0.2
0.15
0.1
0.05
0
$|D_n|$
-500
0
500
n
3.5
3
2.5
2
1.5
1
0.5
0
$\angle D_n$ [rad]
-500
0
500
n

Problem A.5: Write a MATLAB function that takes a MATLAB generate D_n set and reconstructs the original time-domain signal from which the Fourier coefficients had been derived. For example given the set of truncated Fourier coefficients $\{D_n, n = 0, +/- 1, \ldots, +/- 20\}$, your code should reconstruct the time-domain signal from this set using Equation (1). Note: Use the time variable t defined with the MATLAB command t = [-300:1:300].

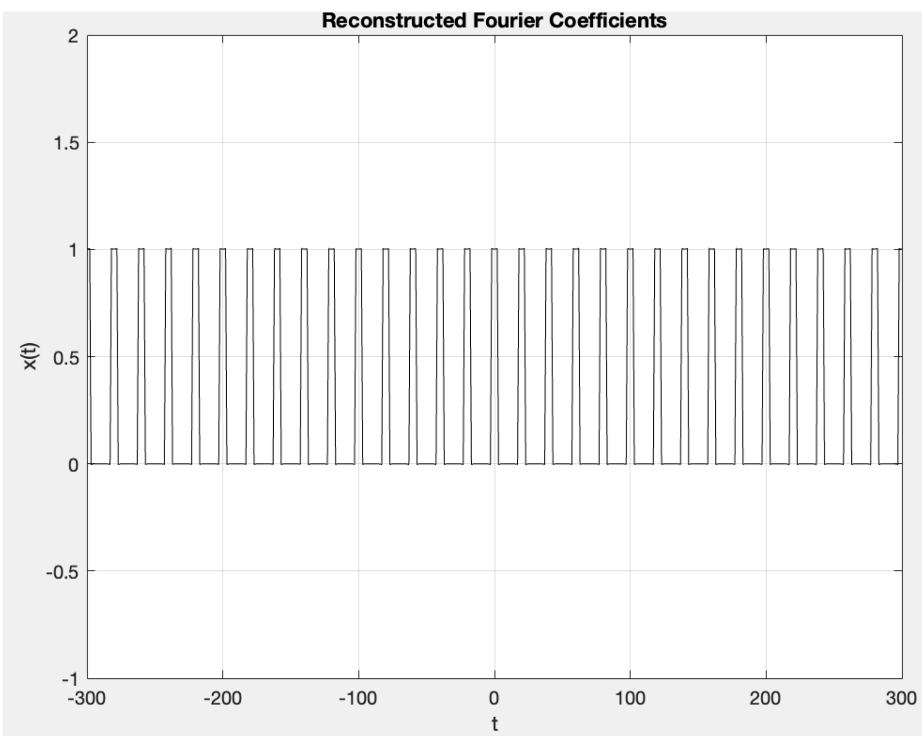
```
n=-500:500;
t=[-300:1:300];
w=pi*0.1; %change the w for x1, x2 and x3
x=zeros(size(t));
for i = 1:length(n)
 x=x+D(i)*exp(j*n(i)*w*t);
end

figure(5);
plot(t,x,'k')
xlabel('t'); ylabel('x(t)');
axis([-300 300 -1 2]);
title('Reconstructed Fourier Coefficients');
grid;
```

Problem A.6: Reconstruct the time-domain signals $x_1(t)$, $x_2(t)$, and $x_3(t)$ with the Fourier coefficient sets you generated in Problem A.4. Plot each reconstructed signal.

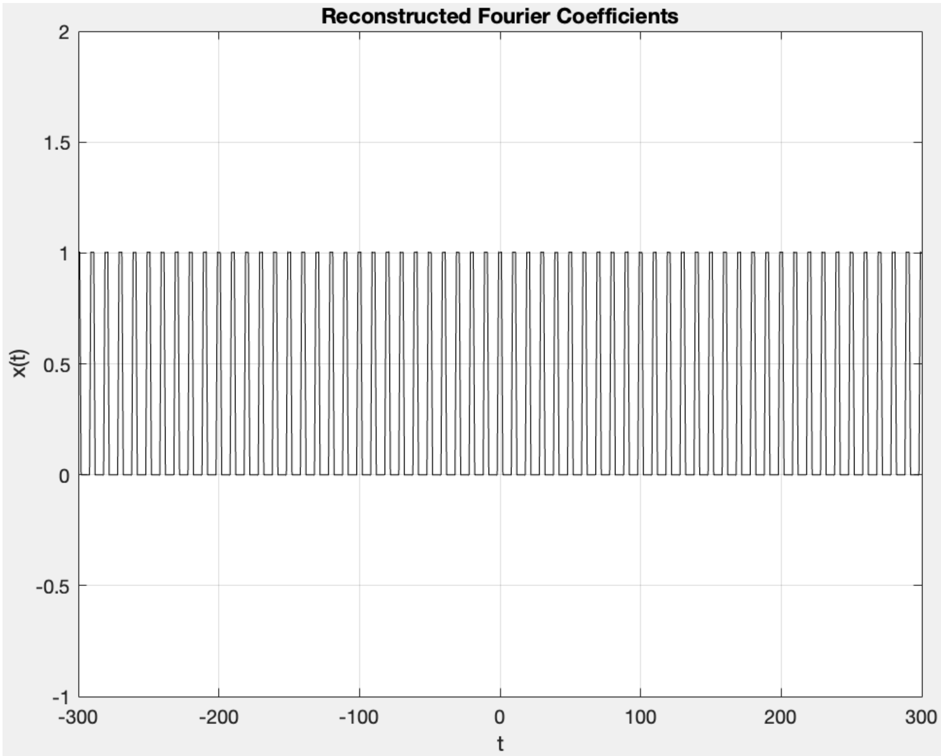
$x_1(t)$:



$x_2(t)$:

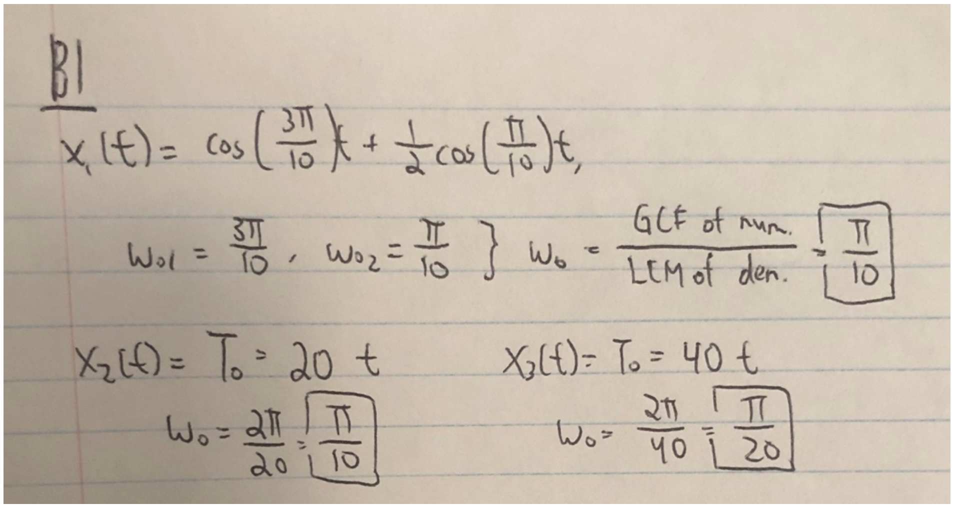


$x_3(t)$:


Reconstructed Fourier Coefficients
x(t)
t
2
1.5
1
0.5
0
-0.5
-1
-300
-200
-100
0
100
200
300

B. Discussion

Problem B.1: Determine the fundamental frequencies of $x_1(t)$, $x_2(t)$ and $x_3(t)$.


B1

$$x_1(t) = \cos\left(\frac{3\pi}{10}\right)t + \frac{1}{2}\cos\left(\frac{\pi}{10}\right)t,$$

$$\left.\omega_{01} = \frac{3\pi}{10},\ \omega_{02} = \frac{\pi}{10}\right\} \omega_0 = \frac{\text{GCF of num.}}{\text{LCM of den.}} = \boxed{\frac{\pi}{10}}$$

$$x_2(t) = T_0 = 20t \qquad x_3(t) = T_0 = 40t$$

$$\omega_0 = \frac{2\pi}{20} = \boxed{\frac{\pi}{10}} \qquad \omega_0 = \frac{2\pi}{40} = \boxed{\frac{\pi}{20}}$$

Problem B.2: What is the main difference between the Fourier coefficients of $x_1(t)$ and $x_2(t)$?

The main difference between the Fourier coefficients of $x_1(t)$ and x2(t) is that one is derived from an expression and the other one is derived from a graph with the D_n equations being significantly different.

Problem B.3: Signals $x_2(t)$ and x3(t) have the rectangular pulse shape but different periods. How are these characteristics reflected in their respective Fourier coefficients?

The difference in the periods for pulses $x_2(t)$ and $x_3(t)$ are reflected in the Fourier coefficients. The $x_3(t)$ signal has a smaller fundamental frequency compared to the $x_2(t)$ signal for its Fourier coefficients.

Problem B.4: The Fourier coefficient D_0 represents the DC value of the signal. Let $x_4(t)$ be the periodic waveform shown in Figure 2. Derive D_0 of $x_4(t)$ from D_0 of $x_2(t)$.

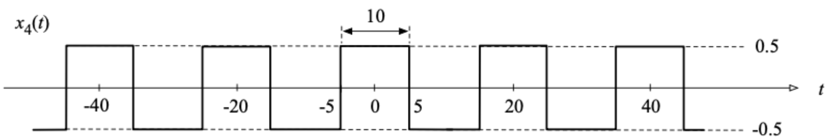


Figure 2: Periodic function $x_4(t)$.

Waveforms $x_2(t)$ and $x_4(t)$ are very similar and so the D_0 of $x_4(t)$ can be derived from x2(t) where $D_0 = 0.5$.

Problem B.5: Using the results of Problem A.6, explain how the reconstructed signal changes as you increase the number of Fourier coefficients used in reconstruction. Discuss for both $x_1(t)$ and $x_2(t)$.

$x_1(t)$ has a w value which is greater than $x_2(t)$ and as a result the reconstructed Fourier coefficients occur more often in $x_1(t)$ as shown in the graphs. Increasing the number of Fourier coefficients used in reconstruction for the graphs result in a higher accuracy.

Problem B.6: How many Fourier coefficients do you need to perfectly reconstruct the periodic waveforms discussed in this lab experiment?

To perfectly reconstruct the periodic waveforms discussed in the lab experiment, we would need an infinite number of D_n values for perfect reconstruction.

Problem B.7: Let x(t) be an arbitrary periodic signal. Instead of storing x(t) on a computer, we consider storing the corresponding Fourier coefficients. When we need to access x(t), we read the Fourier coefficients stored on the computer hard drive and reconstruct the signal. Is this a viable scenario? Explain your answer.

It is not viable because a periodic signal has an infinite number of D_n values, if there are a finite number of values, then the D_n values can be stored as it takes a certain amount of storage. However, at the same time, if the D_n values are finite, if it takes up too much storage on the computer hard drive, it would not be optimal to do so.