WROCLAW UNIVERSITY OF TECHNOLOGY
DEPARTMENT OF ELECTRONICS

FIELD: Electronics
SPECIALITY: Advanced Applied Electronics

Optimization Methods: Linear programming

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Chapter 1

Solution to the given problems

Problem 1: Check the first- and second-order optimality conditions in the point: $x = [1 \quad 1]^T$ of the Rosenbrock's function: $f(\mathbf{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$. Draw a contour plot of this function.

The first step will be expanding the given function, so further we can calculate the derivatives for gradient:

$$\begin{aligned} f(x) &= 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \\ &= 100(x_2^2 - 2x_2x_1^2 + x_1^4) + (1 - 2x_1 + x_1^2) \\ &= 100x_2^2 - 200x_2x_1^2 + 100x_1^4 + 1 - 2x_1 + x_1^2 \\ &= 100x_1^4 + x_1^2 - 2x_1 + 100x_2^2 - 200x_2x_1^2 + 1 \end{aligned}$$

And the point to check is:
$x = [x_1 \ x_2]^T = [1 \ 1]^T$

The First order optimality condition is:

$$\nabla f(x^*) = 0$$

$$\begin{aligned} \frac{\delta f(x)}{\delta x_2} &= 400x_1^3 + 2x_1 - 2 - 400x_2x_1 \\ &= 400 + 2 - 2 - 400 = 0 \end{aligned}$$

$$\begin{aligned} \frac{\delta f(x)}{\delta x_1} &= 200x_2 - 200x_1^2 \\ &= 200 - 200 = 0 \end{aligned}$$

In given point x = [1 1] the first–order optimality condition is fulfilled.

The Second order optimality condition is:

$$\begin{aligned}\nabla f(x^*) &= 0 \ (calculated\ in\ previous\ step)\\ \nabla^2 f(x^*) &= positive\ semi-definite\ matrix\end{aligned}$$

$$\begin{aligned}\nabla^2 f(x^*) &= \begin{bmatrix}\frac{\delta^2 f(x)}{\delta x_1^2} & \frac{\delta^2 f(x)}{\delta x_1 x_2}\\ \frac{\delta^2 f(x)}{\delta x_1 x_2} & \frac{\delta^2 f(x)}{\delta x_2^2}\end{bmatrix}\\ &= \begin{bmatrix}1200x_1^2 + 2 - 400x_2 & -400x_1\\ -400x_1 & 200\end{bmatrix}\\ for\ point\ [1\ 1] &= \begin{bmatrix}1200 + 2 - 400 & -400\\ -400 & 200\end{bmatrix}\\ H &= \begin{bmatrix}802 & -400\\ -400 & 200\end{bmatrix}\end{aligned}$$

As it can be noticed, all principal minors are positive ($H_{1,1}$ and $H_{2,2}$). Therefore the $H(x_*)$ is positive semi-definite.

Octave code below was written to plot contour for this task.

```
pkg load symbolic

syms x1 x2
f = @(x1,x2) 100.*(x2 - x1.^2).^2 + (1 - x1).^2;

ezcontour(f,[-3,3])
```

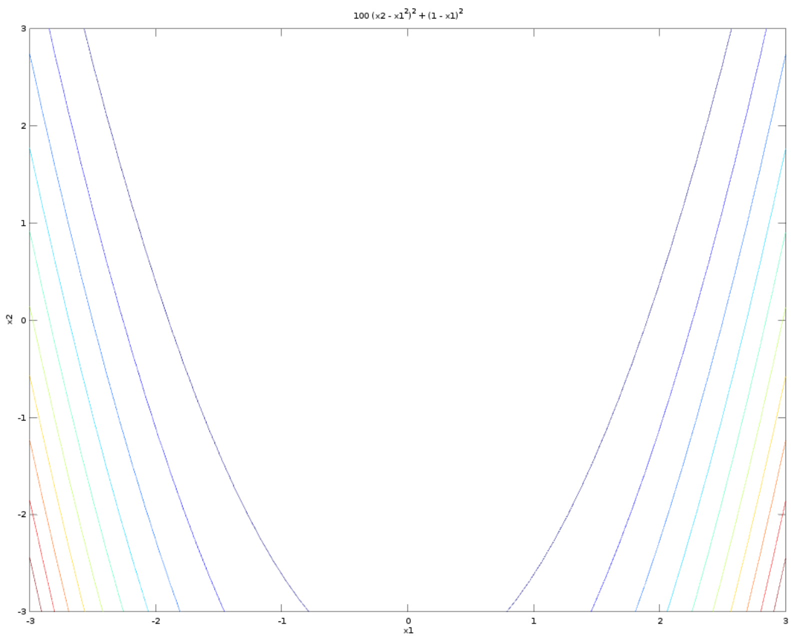

Figure 1.1 Contour plot of given function.

Problem 2: Check the first- and second-order optimality conditions for the quadratic functions:

a) $f(\mathbf{x}) = 2x_1^2 - x_1x_2 + \frac{1}{2}x_2^2 - 3x_1 + 3.5$,

b) $f(\mathbf{x}) = -\frac{3}{2}x_1^2 + x_1x_2 - \frac{1}{2}x_2^2 + 2x_1 - 1$,

c) $f(\mathbf{x}) = x_1^2 + 8x_1x_2 + \frac{1}{2}x_2^2 - 10x_1 - 9x_2 + \frac{9}{2}$.

Draw their contour plots. Are these functions convex? What kind of stationarity do they have?

Just like in the first task, we need to calculate the gradient and Hessian.// Then it can be determined which type of stationarity the functions have. **Point a)**

$$f(x) = 2x_1^2 - x_1x_2 - 3x_1 + \frac{x_2^2}{2} + 3.5$$
$$\nabla f(x^*) = 0$$

$$\frac{\delta f(x)}{\delta x_2} = 4x_1 - x_2 - 3$$
$$\frac{\delta f(x)}{\delta x_1} = -x_1 + x_2$$

To ensure the equality to zero, we can calculate that stationary point shall be:

$$x_1 = 1$$
$$x_2 = 1$$

Now we will calculate the Hessian:

$$\nabla f(x^*) = 0 \ (\textit{calculated in previous step})$$

$$\nabla^2 f(x^*) = \begin{bmatrix} \frac{\delta^2 f(x)}{\delta x_1^2} & \frac{\delta^2 f(x)}{\delta x_1 x_2} \\ \frac{\delta^2 f(x)}{\delta x_1 x_2} & \frac{\delta^2 f(x)}{\delta x_2^2} \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$$
$$H = \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$$

To establish the type of stationarity we need to check Hessian determinant and the values of principal minors:

$$det(H) = 4 + 1 = 5 > 0$$
$$H_{1,1} = 1 > 0$$
$$H_{2,2} = 4 > 0$$

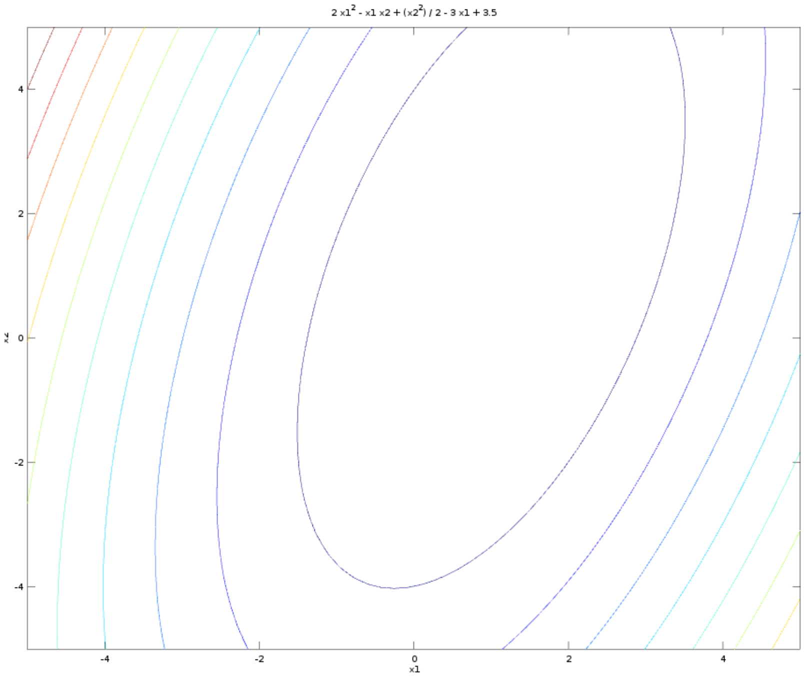

Figure 1.2 Contour plot of point a) function.

The Hessian is strictly positive definite, so we have the minimizer at calculated point.

Point b)

$$
\begin{aligned}
f(x) &= -\frac{3x_1^2}{2} + x_1x_2 + 2x_1 - \frac{x_2^2}{2} - 1 \\
\nabla f(x^*) &= 0 \\
\\
\frac{\delta f(x)}{\delta x_2} &= -3x_1 + x_2 + 2 \\
\frac{\delta f(x)}{\delta x_1} &= x_1 - x_2
\end{aligned}
$$

The calculated values for fulfilling the condition of $\nabla f(x^*) = 0$:

$$
\begin{aligned}
x_1 &= 1 \\
x_2 &= 1
\end{aligned}
$$

Now we will calculate the Hessian:

$$\nabla f(x^*) = 0 \ (\textit{calculated in previous step})$$

$$\begin{aligned}
\nabla^2 f(x^*) &= \begin{bmatrix} \frac{\delta^2 f(x)}{\delta x_1^2} & \frac{\delta^2 f(x)}{\delta x_1 x_2} \\ \frac{\delta^2 f(x)}{\delta x_1 x_2} & \frac{\delta^2 f(x)}{\delta x_2^2} \end{bmatrix} \\
&= \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} \\
H &= \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix}
\end{aligned}$$

To establish the type of stationarity we need to check Hessian determinant and the values of principal minors:

$$\begin{aligned}
det(H) \ = \ 3 \ - \ 1 \ &= \ 2 \ > \ 0 \\
H_{1,1} \ &= \ -1 \ < \ 0 \\
H_{2,2} \ &= \ -3 \ < \ 0
\end{aligned}$$

The Hessian is strictly negative definite, The maximizer is present.

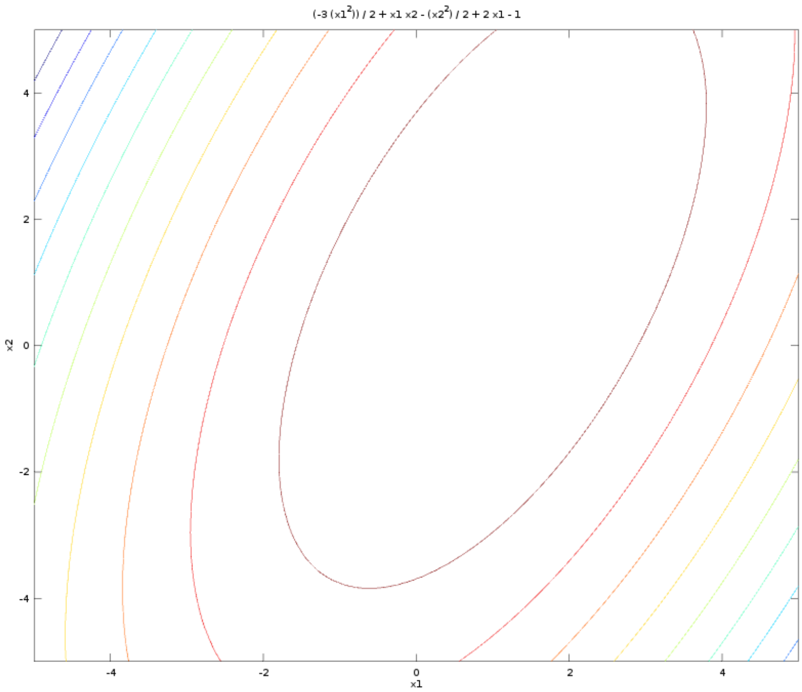

Figure 1.3 Contour plot of point b) function.

Point c)

In third point, the following function needs to be analyzed:

$$f(x) = x_1^2 + 8x_1x_2 - 10x_1 + \frac{x_2^2}{2} - 9x_2 + 4$$

The task routine remains all the same as before:

$$\nabla f(x^*) = 0$$

$$\frac{\delta f(x)}{\delta x_2} = 2x_1 + 8x_2 - 10$$
$$\frac{\delta f(x)}{\delta x_1} = 8x_1 + x_2 - 9$$

To ensure the equality to zero, we can calculate that stationary point shall be:

$$x_1 = 1$$
$$x_2 = 1$$

Now we will calculate the Hessian:

$$\nabla f(x^*) = 0 \ (\mathit{calculated\ in\ previous\ step})$$

$$\begin{aligned}\nabla^2 f(x^*) &= \begin{bmatrix} \frac{\delta^2 f(x)}{\delta x_1^2} & \frac{\delta^2 f(x)}{\delta x_1 x_2} \\ \frac{\delta^2 f(x)}{\delta x_1 x_2} & \frac{\delta^2 f(x)}{\delta x_2^2} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 8 \\ 8 & 1 \end{bmatrix} \\ H &= \begin{bmatrix} 2 & 8 \\ 8 & 1 \end{bmatrix}\end{aligned}$$

To establish the type of stationarity we need to check Hessian determinant and the values of principal minors:

$$det(H) = 2 - 64 = -62 < 0$$

The determinant of Hessian is smaller than zero – at this point we know that the critical point of function is a saddle point.

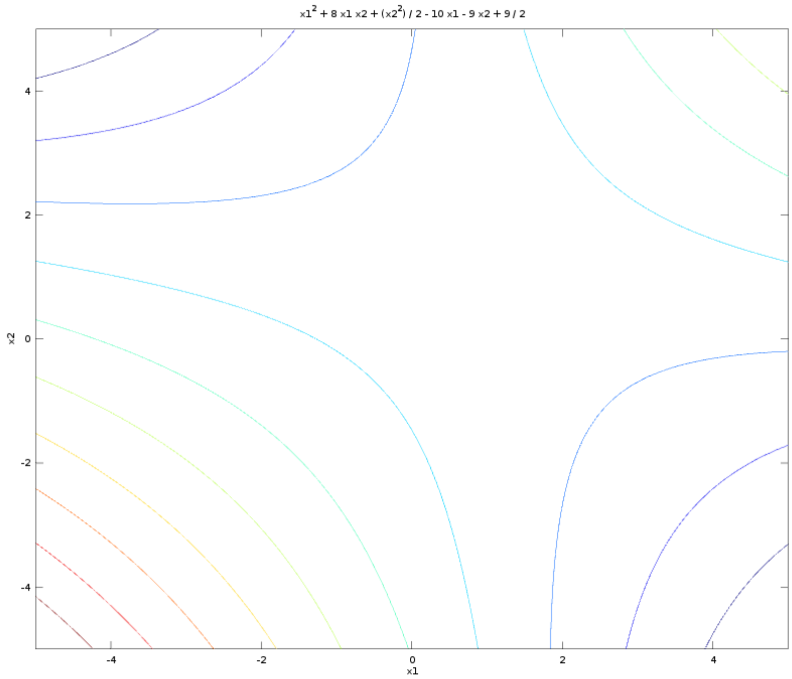

Figure 1.4 The contour plot of point c) function.

Chapter 2

Listings of algorithms

2.1 Coded selected algorithms

Algorithm 1 - Simplex algorithm

Algorithm 2 - Revised Simplex algorithm

Bibliography

[1] Luenberger, D. G., & Ye, Y. (2015). Linear and nonlinear programming (Vol. 228). Springer.

[2] Zdunek R., Optimization Methods - lecture slides.