WROCLAW UNIVERSITY OF TECHNOLOGY
DEPARTMENT OF ELECTRONICS

Field: Electronics
Speciality: Advanced Applied Electronics

Optimization Methods: Linear programming

Author:
Jaroslaw M. Szumega

Supervisor:
Rafal Zdunek, D.Sc, K-4/W4

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Contents

Chapter 1

Solution to the given problems

Problem 1: Check the first- and second-order optimality conditions in the point: $x = [1 \quad 1]^T$ of the Rosenbrock's function: $f(\mathbf{x}) = 100(x_2 - x_1^2)^2 + (1 - x_1)^2$. Draw a contour plot of this function.

The first step will be expanding the given function, so further we can calculate the derivatives for gradient:

$$\begin{aligned} f(x) &= 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \\ &= 100(x_2^2 - 2x_2x_1^2 + x_1^4) + (1 - 2x_1 + x_1^2) \\ &= 100x_2^2 - 200x_2x_1^2 + 100x_1^4 + 1 - 2x_1 + x_1^2 \\ &= 100x_1^4 + x_1^2 - 2x_1 + 100x_2^2 - 200x_2x_1^2 + 1 \end{aligned}$$

And the point to check is:
$x = [x_1 \ x_2]^T = [1 \ 1]^T$

The First order optimality condition is:

$$\nabla f(x^*) = 0$$

$$\begin{aligned} \frac{\delta f(x)}{\delta x_2} &= 400x_1^3 + 2x_1 - 2 - 400x_2x_1 \\ &= 400 + 2 - 2 - 400 = 0 \end{aligned}$$

$$\begin{aligned} \frac{\delta f(x)}{\delta x_1} &= 200x_2 - 200x_1^2 \\ &= 200 - 200 = 0 \end{aligned}$$

In given point x = [1 1] the first–order optimality condition is fulfilled.

The Second order optimality condition is:

$$\begin{aligned}\nabla f(x^*) &= 0 \ (calculated\ in\ previous\ step)\\ \nabla^2 f(x^*) &= positive\ semi-definite\ matrix\end{aligned}$$

$$\begin{aligned}\nabla^2 f(x^*) &= \begin{bmatrix}\frac{\delta^2 f(x)}{\delta x_1^2} & \frac{\delta^2 f(x)}{\delta x_1 x_2}\\ \frac{\delta^2 f(x)}{\delta x_1 x_2} & \frac{\delta^2 f(x)}{\delta x_2^2}\end{bmatrix}\\ &= \begin{bmatrix}1200x_1^2 + 2 - 400x_2 & -400x_1\\ -400x_1 & 200\end{bmatrix}\\ for\ point\ [1\ 1] &= \begin{bmatrix}1200 + 2 - 400 & -400\\ -400 & 200\end{bmatrix}\\ H &= \begin{bmatrix}802 & -400\\ -400 & 200\end{bmatrix}\end{aligned}$$

As it can be noticed, all principal minors are positive ($H_{1,1}$ *and* $H_{2,2}$). Therefore the $H(x_*)$ is positive semi-definite.

Octave code below was written to plot contour for this task.

```
pkg load symbolic

syms x1 x2
f = @(x1,x2) 100.*(x2 - x1.^2).^2 + (1 - x1).^2;

ezcontour(f,[-3,3])
```

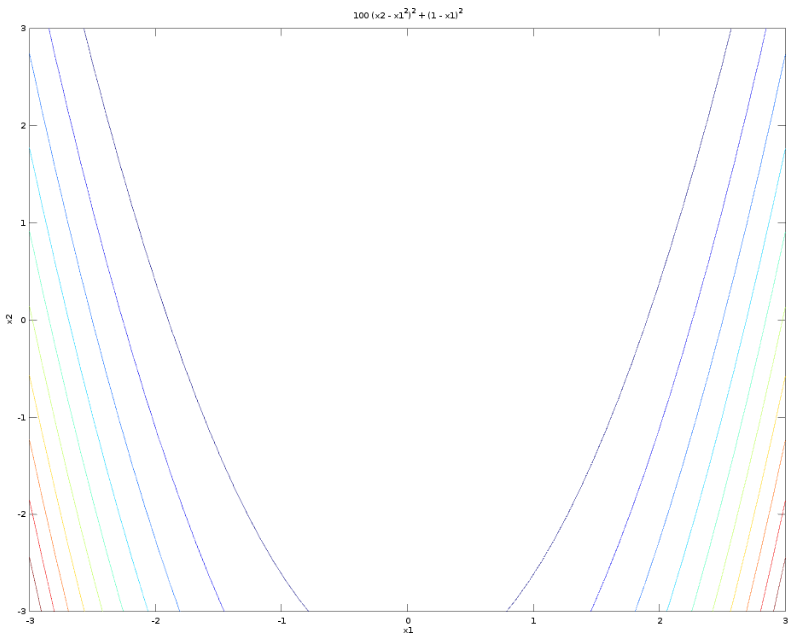

Figure 1.1 Contour plot of given function.

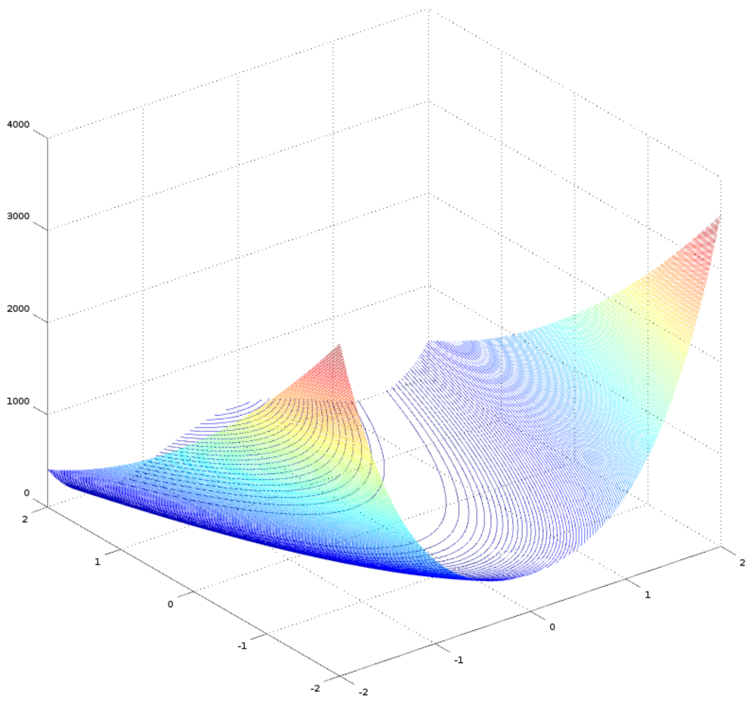

Figure 1.2 Contour plot of function in 3D version.

Problem 2: Check the first- and second-order optimality conditions for the quadratic functions:

a) $f(\mathbf{x}) = 2x_1^2 - x_1x_2 + \frac{1}{2}x_2^2 - 3x_1 + 3.5$,

b) $f(\mathbf{x}) = -\frac{3}{2}x_1^2 + x_1x_2 - \frac{1}{2}x_2^2 + 2x_1 - 1$,

c) $f(\mathbf{x}) = x_1^2 + 8x_1x_2 + \frac{1}{2}x_2^2 - 10x_1 - 9x_2 + \frac{9}{2}$.

Draw their contour plots. Are these functions convex? What kind of stationarity do they have?

Just like in the first task, we need to calculate the gradient and Hessian.// Then it can be determined which type of stationarity the functions have. **Point a)**

$$f(x) = 2x_1^2 - x_1x_2 - 3x_1 + \frac{x_2^2}{2} + 3.5$$
$$\nabla f(x^*) = 0$$

$$\frac{\delta f(x)}{\delta x_2} = 4x_1 - x_2 - 3$$
$$\frac{\delta f(x)}{\delta x_1} = -x_1 + x_2$$

To ensure the equality to zero, we can calculate that stationary point shall be:

$$x_1 = 1$$
$$x_2 = 1$$

Now we will calculate the Hessian:

$$\nabla f(x^*) = 0 \; (\textit{calculated in previous step})$$

$$\nabla^2 f(x^*) = \begin{bmatrix} \frac{\delta^2 f(x)}{\delta x_1^2} & \frac{\delta^2 f(x)}{\delta x_1 x_2} \\ \frac{\delta^2 f(x)}{\delta x_1 x_2} & \frac{\delta^2 f(x)}{\delta x_2^2} \end{bmatrix}$$
$$= \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$$
$$H = \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$$

To establish the type of stationarity we need to check Hessian determinant and the values of principal minors:

$$det(H) = 4 + 1 = 5 > 0$$
$$H_{1,1} = 1 > 0$$
$$H_{2,2} = 4 > 0$$

The Hessian is strictly positive definite, so we have the minimizer at calculated point.

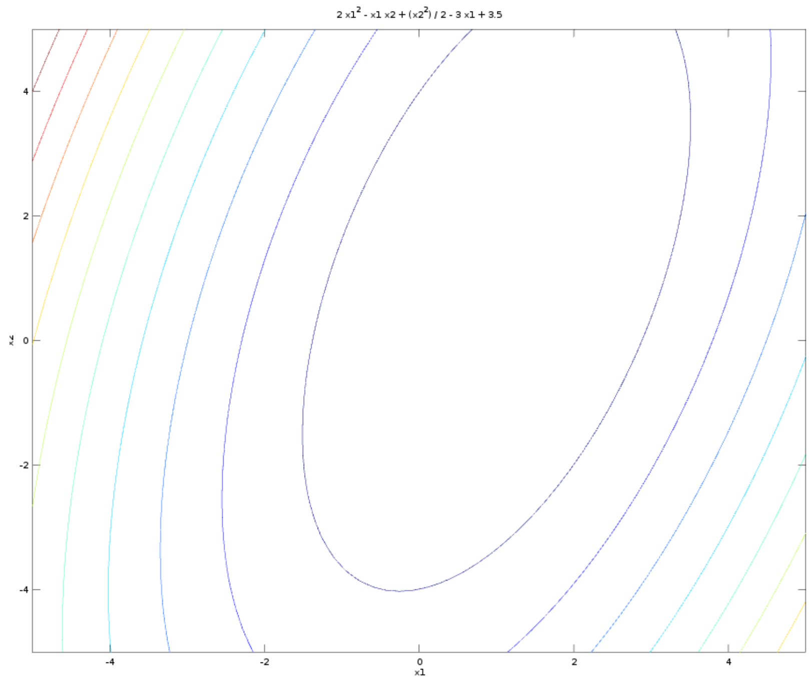

Figure 1.3 Contour plot of point a) function.

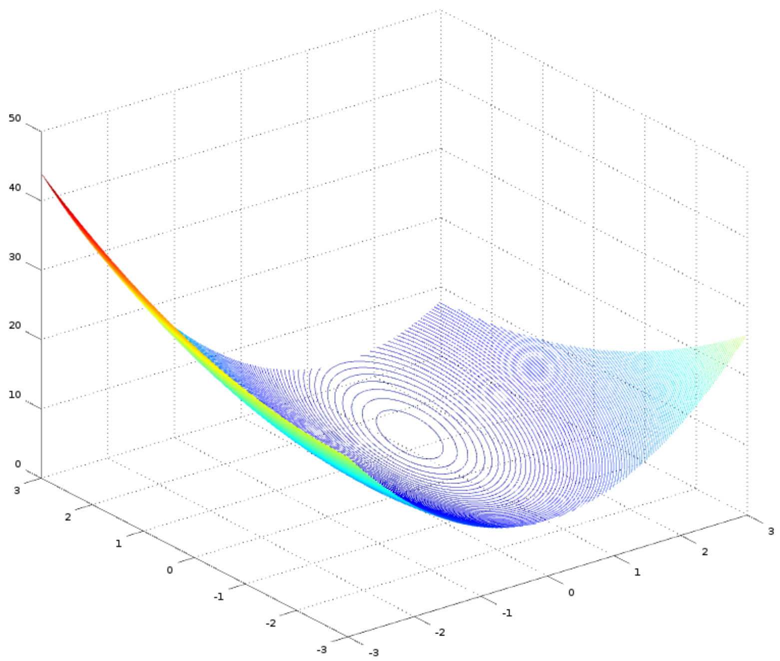

Figure 1.4 And its 3D version. We can notice the minimum.

Point b)

$$
\begin{aligned}
f(x) &= -\frac{3x_1^2}{2} + x_1x_2 + 2x_1 - \frac{x_2^2}{2} - 1 \\
\nabla f(x^*) &= 0
\end{aligned}
$$

$$
\begin{aligned}
\frac{\delta f(x)}{\delta x_2} &= -3x_1 + x_2 + 2 \\
\frac{\delta f(x)}{\delta x_1} &= x_1 - x_2
\end{aligned}
$$

The calculated values for fulfilling the condition of $\nabla f(x^*) = 0$:

$$
\begin{aligned}
x_1 &= 1 \\
x_2 &= 1
\end{aligned}
$$

Now we will calculate the Hessian:

$$\nabla f(x^*) = 0 \ (\textit{calculated in previous step})$$

$$
\begin{aligned}
\nabla^2 f(x^*) &= \begin{bmatrix} \frac{\delta^2 f(x)}{\delta x_1^2} & \frac{\delta^2 f(x)}{\delta x_1 x_2} \\ \frac{\delta^2 f(x)}{\delta x_1 x_2} & \frac{\delta^2 f(x)}{\delta x_2^2} \end{bmatrix} \\
&= \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix} \\
H &= \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix}
\end{aligned}
$$

To establish the type of stationarity we need to check Hessian determinant and the values of principal minors:

$$
\begin{aligned}
det(H) &= 3 - 1 = 2 > 0 \\
H_{1,1} &= -1 < 0 \\
H_{2,2} &= -3 < 0
\end{aligned}
$$

The Hessian is strictly negative definite, The maximizer is present.

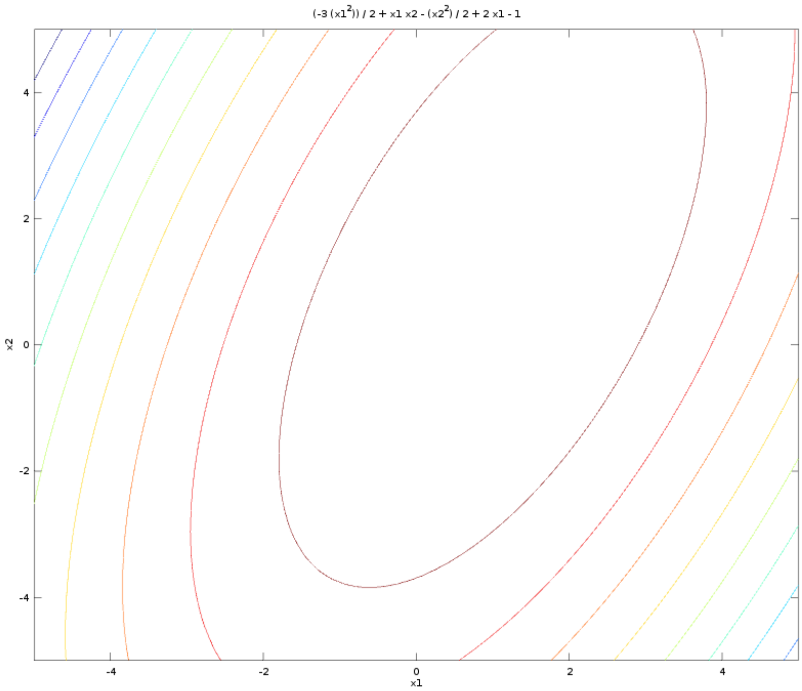

Figure 1.5 Contour plot of point b) function.

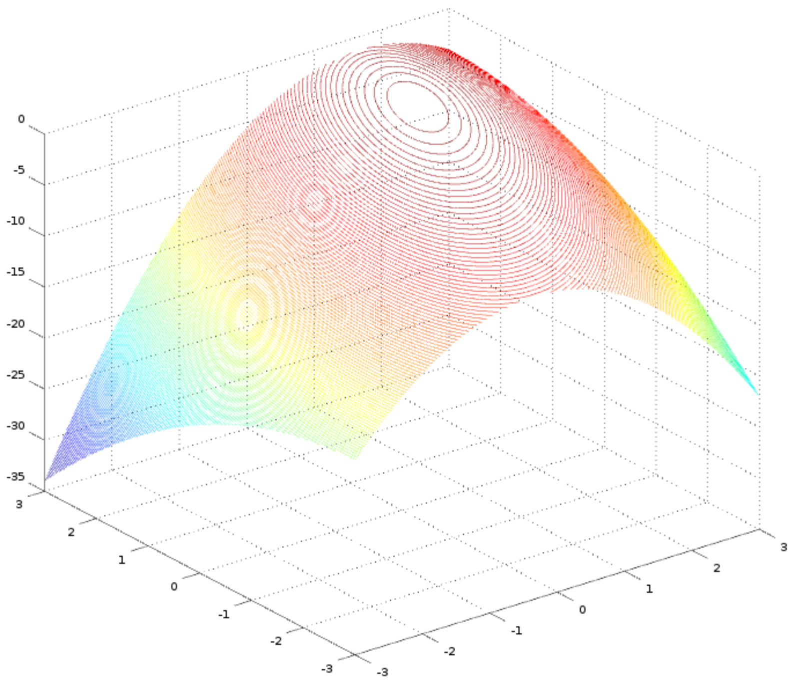

Figure 1.6 3D version with maximum observed.

Point c)

In third point, the following function needs to be analyzed:

$$f(x) = x_1^2 + 8x_1x_2 - 10x_1 + \frac{x_2^2}{2} - 9x_2 + 4$$

The task routine remains all the same as before:

$$\nabla f(x^*) = 0$$

$$\frac{\delta f(x)}{\delta x_2} = 2x_1 + 8x_2 - 10$$
$$\frac{\delta f(x)}{\delta x_1} = 8x_1 + x_2 - 9$$

To ensure the equality to zero, we can calculate that stationary point shall be:

$$x_1 = 1$$
$$x_2 = 1$$

Now we will calculate the Hessian:

$$\nabla f(x^*) = 0 \ (\textit{calculated in previous step})$$

$$\nabla^2 f(x^*) = \begin{bmatrix} \frac{\delta^2 f(x)}{\delta x_1^2} & \frac{\delta^2 f(x)}{\delta x_1 x_2} \\ \frac{\delta^2 f(x)}{\delta x_1 x_2} & \frac{\delta^2 f(x)}{\delta x_2^2} \end{bmatrix}$$
$$= \begin{bmatrix} 2 & 8 \\ 8 & 1 \end{bmatrix}$$
$$H = \begin{bmatrix} 2 & 8 \\ 8 & 1 \end{bmatrix}$$

To establish the type of stationarity we need to check Hessian determinant and the values of principal minors:

$$det(H) = 2 - 64 = -62 < 0$$

The determinant of Hessian is smaller than zero – at this point we know that the critical point of function is a saddle point.

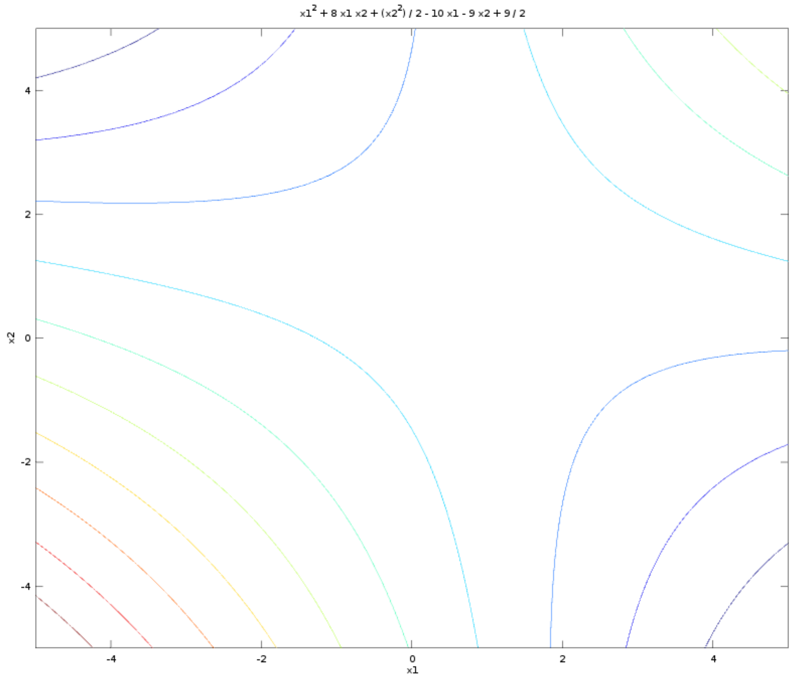

Figure 1.7 The contour plot of point c) function.

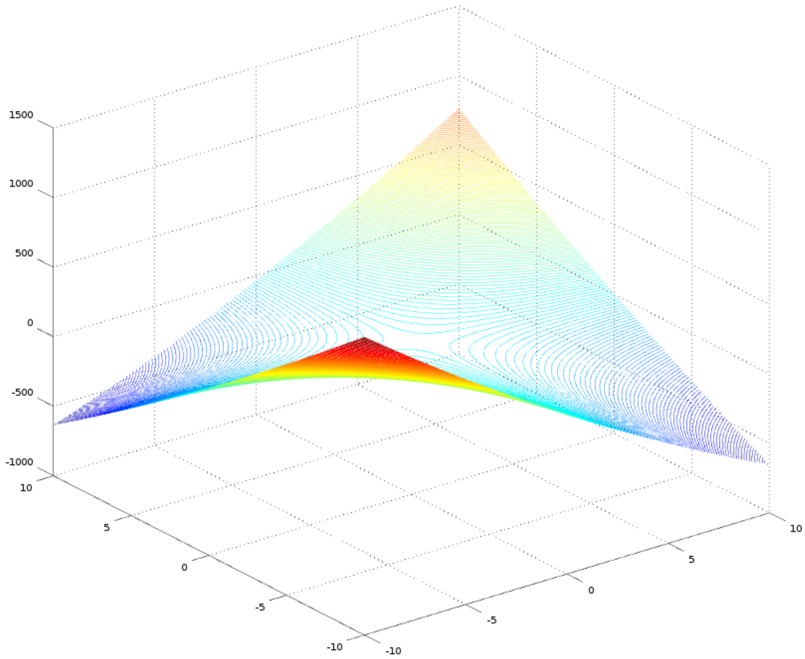

Figure 1.8 Visualization of point c) saddlepoint.

Problem 3: For the quadratic function $f : \Re^3 \rightarrow \Re$:

$$f(\mathbf{x}) = \mathbf{x}^T \mathbf{G} \mathbf{x}, \text{ where } \mathbf{G} = \begin{bmatrix} \alpha & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix},$$

determine the parameter α for which the function f is strictly convex.

To ensure the function is convex, the presented matrix: $\begin{bmatrix} \alpha & 1 & 1 \\ 3 & 2 & 1 \\ 2 & 2 & 3 \end{bmatrix}$ has to be strictly positive–definite. Following this requirements, it needs to fulfill the following conditions:

- matrix determinant ¿ 0,
- first principal minor ¿ 0.

So now we establish the determinant:

$$\begin{aligned} det(G) &= 6\alpha + 2 + 6 - 4 - 2\alpha - 9 \\ &= 4\alpha - 5 \\ 4\alpha - 5 &> 0 \\ \alpha &> 5/4 \end{aligned}$$

And the first principal minor:

$$det(M1) = 2\alpha - 3$$

$$\begin{aligned} 2\alpha - 3 &> 0 \\ \alpha &> 3/2 \end{aligned}$$

In the end, if $alpha > 1.5$, then the given function will be convex.

Chapter 2

Listings of algorithms

2.1 Coded selected algorithms

Algorithm 1 - Simplex algorithm

Algorithm 2 - Revised Simplex algorithm

Bibliography

[1] Luenberger, D. G., & Ye, Y. (2015). Linear and nonlinear programming (Vol. 228). Springer.

[2] Zdunek R., Optimization Methods - lecture slides.