WROCLAW UNIVERSITY OF TECHNOLOGY
DEPARTMENT OF ELECTRONICS

FIELD: Electronics
SPECIALITY: Advanced Applied Electronics

Optimization Methods:
Linear programming

AUTHOR:

Jaroslaw M. Szumega

SUPERVISOR:

Rafal Zdunek, D.Sc, K-4/W4

GRADE:

WROCŁAW 15.05.2017

Contents

Chapter 1

Solution to the given problems

Problem 1: Check the first- and second-order optimality conditions in the point: $x = \begin{bmatrix} 1 & 1 \end{bmatrix}^T$ of the Rosenbrock's function: $f(\mathbf{x}) = 100\left(x_2 - x_1^2\right)^2 + \left(1 - x_1\right)^2$. Draw a contour plot of this function.

The first step will be expanding the given function, so further we can calculate the derivatives for gradient:

$$\begin{aligned} f(x) &= 100(x_2 - x_1^2)^2 + (1 - x_1)^2 \\ &= 100(x_2^2 - 2x_2 x_1^2 + x_1^4) + (1 - 2x_1 + x_1^2) \\ &= 100x_2^2 - 200x_2 x_1^2 + 100x_1^4 + 1 - 2x_1 + x_1^2 \\ &= 100x_1^4 + x_1^2 - 2x_1 + 100x_2^2 - 200x_2 x_1^2 + 1 \end{aligned}$$

And the point to check is:
$x = [x_1 \ x_2]^T = [1 \ 1]^T$

The First order optimality condition is:

$$\nabla f(x^*) = 0$$

$$\frac{\delta f(x)}{\delta x_2} = 400x_1^3 + 2x_1 - 2 - 400x_2 x_1$$

$$= 400 + 2 - 2 - 400 = 0$$

$$\frac{\delta f(x)}{\delta x_1} = 200x_2 - 200x_1^2$$

$$= 200 - 200 = 0$$

In given point x = [1 1] the first–order optimality condition is fulfilled.

The Second order optimality condition is:

$$\nabla f(x^*) = 0 \; (calculated \; in \; previous \; step)$$
$$\nabla^2 f(x^*) \; = positive \; semi - definite \; matrix$$

$$\nabla^2 f(x^*) = \begin{bmatrix} \frac{\delta^2 f(x)}{\delta x_1^2} & \frac{\delta^2 f(x)}{\delta x_1 x_2} \\ \frac{\delta^2 f(x)}{\delta x_1 x_2} & \frac{\delta^2 f(x)}{\delta x_2^2} \end{bmatrix}$$
$$= \begin{bmatrix} 1200x_1^2 + 2 - 400x_2 & -400x_1 \\ -400x_1 & 200 \end{bmatrix}$$
$$for \; point \; [1 \; 1] = \begin{bmatrix} 1200 + 2 - 400 & -400 \\ -400 & 200 \end{bmatrix}$$
$$H = \begin{bmatrix} 802 & -400 \\ -400 & 200 \end{bmatrix}$$

As it can be noticed, all principal minors are positive ($H_{1,1} \; and \; H_{2,2}$). Therefore the $H(x_*)$ is positive semi-definite.

Octave code below was written to plot contour for this task.

```
1 pkg load symbolic
2
3 syms x1 x2
4 f = @(x1,x2) 100.*(x2 - x1.^2).^2 + (1 - x1).^2;
5
6 ezcontour(f,[-3,3])
```
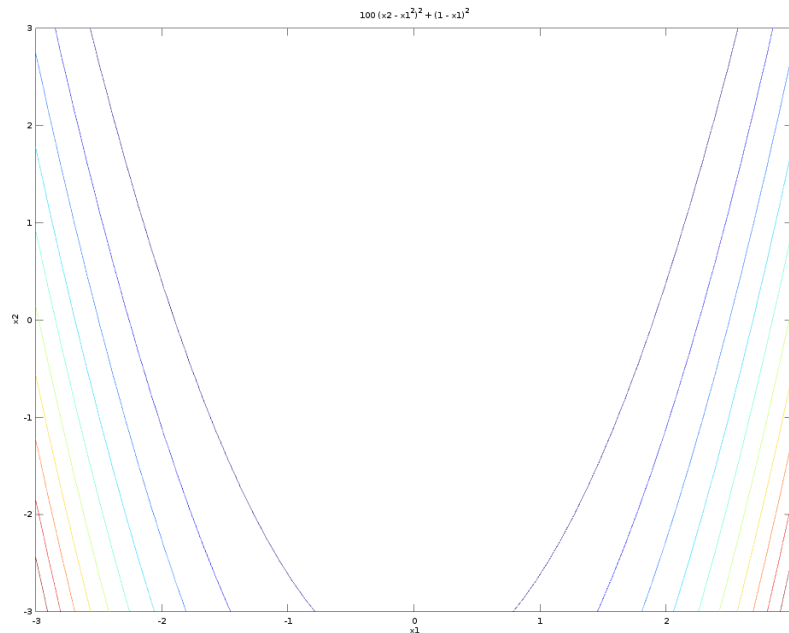

Figure 1.1 Contour plot of given function.

Problem 2: Check the first- and second-order optimality conditions for the quadratic functions:

a) $f(\mathbf{x}) = 2x_1^2 - x_1x_2 + \frac{1}{2}x_2^2 - 3x_1 + 3.5$,

b) $f(\mathbf{x}) = -\frac{3}{2}x_1^2 + x_1x_2 - \frac{1}{2}x_2^2 + 2x_1 - 1$,

c) $f(\mathbf{x}) = x_1^2 + 8x_1x_2 + \frac{1}{2}x_2^2 - 10x_1 - 9x_2 + \frac{9}{2}$.

Draw their contour plots. Are these functions convex? What kind of stationarity do they have?

Just like in the first task, we need to calculate the gradient and Hessian.// Then it can be determined which type of stationarity the functions have. **Point a)**

$$f(x) = 2x_1^2 - x_1x_2 - 3x_1 + \frac{x_2^2}{2} + 3.5$$
$$\nabla f(x^*) = 0$$

$$\frac{\delta f(x)}{\delta x_2} = 4x_1 - x_2 - 3$$
$$\frac{\delta f(x)}{\delta x_1} = -x_1 + x_2$$

To ensure the equality to zero, we can calculate that stationary point shall be:

$$x_1 = 1$$
$$x_2 = 1$$

Now we will calculate the Hessian:

$$\nabla f(x^*) = 0 \ (calculated \ in \ previous \ step)$$

$$\nabla^2 f(x^*) = \begin{bmatrix} \frac{\delta^2 f(x)}{\delta x_1^2} & \frac{\delta^2 f(x)}{\delta x_1 x_2} \\ \frac{\delta^2 f(x)}{\delta x_1 x_2} & \frac{\delta^2 f(x)}{\delta x_2^2} \end{bmatrix}$$

$$= \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$$

$$H = \begin{bmatrix} 4 & -1 \\ -1 & 1 \end{bmatrix}$$

To establish the type of stationarity we need to check Hessian determinant and the values of principal minors:

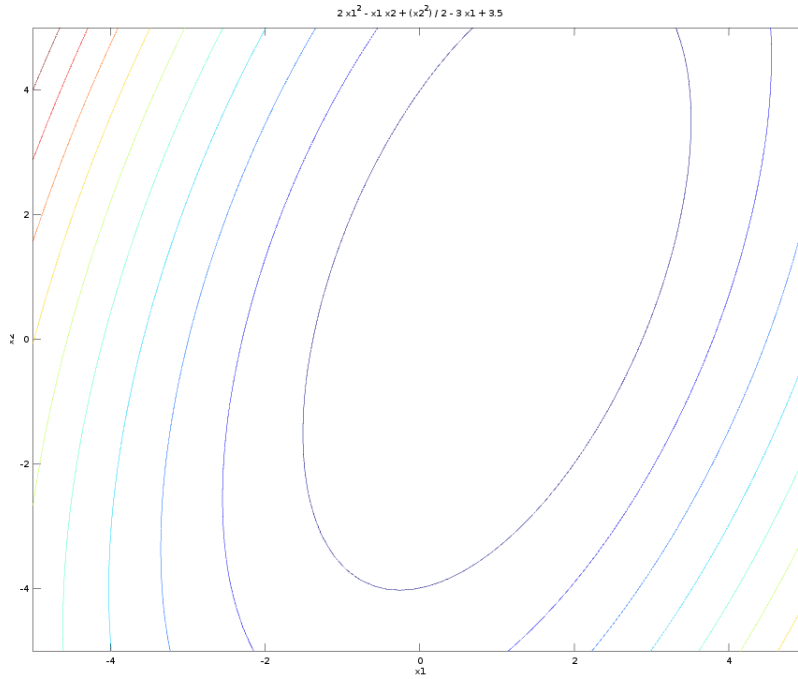$$det(H) = 4 + 1 = 5 > 0$$
$$H_{1,1} = 1 > 0$$
$$H_{2,2} = 4 > 0$$

Figure 1.2 Contour plot of point a) function.

The Hessian is strictly positive definite, so we have the minimizer at calculated point.

Point b)

$$f(x) = -\frac{3x_1^2}{2} + x_1x_2 + 2x_1 - \frac{x_2^2}{2} - 1$$

$$\nabla f(x^*) = 0$$

$$\frac{\delta f(x)}{\delta x_2} = -3x_1 + x_2 + 2$$

$$\frac{\delta f(x)}{\delta x_1} = x_1 - x_2$$

The calculated values for fulfilling the condition of $\nabla f(x^*) = 0$:

$$x_1 = 1$$
$$x_2 = 1$$

Now we will calculate the Hessian:

$$\nabla f(x^*) = 0 \; (calculated \; in \; previous \; step)$$

$$\nabla^2 f(x^*) = \begin{bmatrix} \frac{\delta^2 f(x)}{\delta x_1^2} & \frac{\delta^2 f(x)}{\delta x_1 x_2} \\ \frac{\delta^2 f(x)}{\delta x_1 x_2} & \frac{\delta^2 f(x)}{\delta x_2^2} \end{bmatrix}$$

$$= \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H = \begin{bmatrix} -3 & 1 \\ 1 & -1 \end{bmatrix}$$

To establish the type of stationarity we need to check Hessian determinant and the values of principal minors:

$$det(H) = 3 - 1 = 2 > 0$$
$$H_{1,1} = -1 < 0$$
$$H_{2,2} = -3 < 0$$

The Hessian is strictly negative definite, The maximizer is present.

Chapter 2

Listings of algorithms

2.1 Coded selected algorithms

Algorithm 1 - Simplex algorithm

Algorithm 2 - Revised Simplex algorithm

Bibliography

[1] Luenberger, D. G., & Ye, Y. (2015). Linear and nonlinear programming (Vol. 228). Springer.

[2] Zdunek R., Optimization Methods - lecture slides.