Practice M1

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1.1 Show that $a+b$ can be less than $\min(a,b)$

Let $\left.\begin{array}{l} a = -4 \\ b = -6 \end{array}\right\} \Rightarrow \min(a,b) = -6$

$a + b = (-4)+(-6) = -10$

$\therefore a+b < \min(a,b)$

1.7 Prove the correctness of the following recursive algorithm to multiply two natural numbers, for all integer constants $c \geq 2$.

function multiply (y, z)

comment Return the product yz.

1. if $z = 0$ then return (0) else
2. return (multiply $(cy, \lfloor z/c \rfloor) + y \cdot (z \bmod c)$)

Induction proof

$\forall k \geq 0$

$\forall$ integers y, z such that $|z| = k$,

we have multiply $(y,z) = y \cdot z$

Base case:

$|z| = 0$ implies that $z = 0$

multiply $(y,z) = y \cdot z = y \cdot 0 = 0 \Rightarrow$ true

Assume that multiply (y,z) gives the correct answer where

$z \leq n$

$c \geq 2$

$y \geq 0$

Proof:

Recursive case

mult $(y-1, z) + y$ breaks down into:

$\left.\begin{array}{l} A = \text{multiply}\left(cy, \left\lfloor \frac{(n+1)}{c} \right\rfloor\right) \\ B = y\left(\lfloor n+1 \rfloor \bmod c\right) \end{array}\right\} \Rightarrow$ multiply $(y,z) = A + B$

Because $c \geq 2 \Rightarrow \left(\frac{\lfloor n+1 \rfloor}{c} \rfloor\right) < (n+1)$

$A = \text{multiply}\left(cy, \left(\frac{(n+1)}{c}\right)\right) = cy * \left\lfloor \frac{(n+1)}{c} \right\rfloor$

$A = cy * \left\lfloor \frac{(n+1)}{c} \right\rfloor = \left\lfloor y * \left(\frac{z}{c}\right) \right\rfloor * c$

$\left\lfloor \frac{z}{c} \right\rfloor \cdot c + (z \bmod c) = z$

$\Rightarrow \left\lfloor \frac{z}{c} \right\rfloor \cdot c = z - (z \bmod c)$

$A = y * \left\lfloor \frac{z}{c} \right\rfloor \cdot c = y\left(z - (z \bmod c)\right) = yz - y(z \bmod c)$

$\Rightarrow A + B = yz - y(z \bmod c) + y(z \bmod c) = yz$ which is the product returned.

$\blacksquare$ The recursive algorithm is correct.

1.10 Prove that $\sum_{i=1}^{n} i = n(n+1)/2$ for $n \geq 0$, by induction

Base case:

Show that $n = 1$ is true

$\sum_{i=1}^{1} i \stackrel{?}{=} \frac{1(1+1)}{2}$

$1 \stackrel{?}{=} \frac{1(2)}{2}$

$1 \stackrel{?}{=} 1$ ☑ yes

Inductive hypothesis:

Assume that $n = m$ is true

Assume that $\sum_{i=1}^{m} i = \frac{m(m+1)}{2}$ for some m in the set of positive integers

Inductive step:

Show that $n = m$ is true $\Rightarrow n = m+1$ is also true

$\sum_{i=1}^{m+1} i = \sum_{i=1}^{m} i + m + 1$

[because $i = 1, 2, 3, \ldots, m, m+1$

$\sum_{i=1}^{m+1} i = 1 + 2 + 3 + \ldots + m + (m+1)$]

$= \frac{m(m+1)}{2} + (m+1) \cdot \frac{2}{2}$

$= \frac{1}{2}(m+1)\left[m + 2\right]$

$= \frac{1}{2}(m+1)\left[(m+1)+1\right]$

$\sum_{i=1}^{m+1} i = \frac{1}{2}(m+1)\left[(m+1)+1\right]$

$\blacksquare$ Since the statement is true for $n = 1$ and true for $n = m$ implied that $n = m+1$ is also true ~~for~~, the statement is true for all positive integers.