Formalization of finite sets in Lean

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July 17, 2023

What is a set ?

Zermelo-Fraenkel set theory:

1. Axiom of Extensionality: $\forall x, y.(\forall z.(z \in x \leftrightarrow z \in y) \rightarrow x = y)$
2. Axiom of Regularity $\forall x.(x \neq \emptyset \rightarrow \exists y.y \in x \wedge y \cap x = \emptyset)$
3. Axiom of Empty Set: $\exists y.\forall x : \neg x \in y$
4. ...

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Lean: $(s : setA) = A \rightarrow Prop$

What is a finite set ?

1. there is a list containing all elements
2. there is a tree containing all elements
3. there exists some $N \in \mathbf{N}$ s.t. every list of length at least N contains duplicates
4. there no surjection from this set by to the natural numbers
5. ...

Current state

Finite by construction

Finite by proof

Don't do it twice

Mathlib data.set

```
theorem mem_inter_iff (x : α) (a b : set α) :
    x ∈ a ∩ b ↔ (x ∈ a ∧ x ∈ b)
```

Kuratowski sets

```
lemma in_intersection_iff_in_both {A:Type u}
[decidable_eq A] (X Y: Kuratowski A) (a:A):
kuratowski_member_prop a (X ∩ Y) =
    (kuratowski_member_prop a X
    ∧ kuratowski_member_prop a Y)
```

Finite by proof