MATH0109 Proving Theorems in Lean - Sheet 5

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The questions in this sheet are on vector spaces and linear maps. All vector spaces mentioned below will be vector spaces over $\mathbb{R}$. In most of the questions, the actual lean code is fairly simple, although the concepts involved can be difficult to understand. You may well find it easier to prove the theorems than to understand what they are saying.

1 Vector spaces of functions

Let X be a non-empty set (in the lean file, this set is taken to be `Fin` 100, which is $\{0, 1, \ldots, 99\}$, but the actual choice of set makes no difference). Lean knows that the set of functions $f : X \to \mathbb{R}$ is a vector space over the real numbers. Addition and scalar multiplication in this vector space are defined as

$$(f + g)(x) = f(x) + g(x), \qquad (\lambda \cdot f)(x) = \lambda \cdot f(x), \qquad (x \in X).$$

It's not necessary to prove the axioms of a vector space, because this is already in Mathlib. This vector space of functions is written "$X \to \mathbb{R}$" in lean. More generally, if V is any vector space then the set of functions $X \to V$ is also a vector space.

2 The constant linear map (Exercise 1)

Let V be a vector space. Given any vector $v \in V$, we shall write "`const` v" for the constant function $X \to V$ with value v. This means that `const` is a function from the vector space V to the vector space $X \to V$. Question 1 is to prove that the map `const` $: V \to (X \to V)$ is a linear map. This involves proving two axioms:

- `map_add'` : $\forall v, w \in V, \texttt{const}\ (v + w) = \texttt{const}\ (v) + \texttt{const}\ (w)$,
- `map_smul'` : $\forall v \in V, \lambda \in \mathbb{R}, \texttt{const}\ (\lambda \cdot v) = \lambda \cdot \texttt{const}\ (v)$.

Note that for vector spaces V and W over $\mathbb{R}$, the notation in lean for the set of linear maps from V to W is "$V \to_1 [\mathbb{R}]\ W$".

3 The lift of a linear map (Exercises 2-4)

Suppose now that we have two vector spaces V and W, and a linear map $\phi : V \to_1 [\mathbb{R}]\ W$. We can define another linear map "`lift` ϕ" from $(X \to V)$ to

$(X \to W)$. This is defined as follows: given a function $f : X \to V$, $(\texttt{lift}\ \phi)(f)$ is the composition $\phi \circ f : X \to W$. In other words,

$$\Big((\texttt{lift}\ \phi)(f)\Big)(x) = \phi(f(x)).$$

In question 2, you first need to prove that for any fixed $\phi : V \to W$, the function $\texttt{lift}\ \phi$ is a linear map from $X \to V$ to $X \to W$, i.e.

$$(\texttt{lift}\ \phi)(f + \lambda \cdot g) = (\texttt{lift}\ \phi)(f) + \lambda \cdot (\texttt{lift}\ \phi)(g), \qquad (f, g : X \to V,\ \lambda \in \mathbb{R}).$$

Once this is done, you can think of `lift` as a function from the vector space of linear maps $V \ \to_1 [\mathbb{R}]\ W$ to the vector space of linear maps $(X \to V) \ \to_1 [\mathbb{R}]\ (X \to W)$, and you must also prove that the function `lift` is a linear map. In other words

$$\texttt{lift}\ (\phi + \lambda \cdot \psi) = \texttt{lift}\ \phi + \lambda \cdot \texttt{lift}\ \psi.$$

Questions 3 and 4 are some very simple lemmas about `lift` .

Lemma 1. *If $\phi : U \to V$ and $\psi : V \to W$ are two linear maps between vector spaces then* $\texttt{lift}\ (\psi \circ \phi) = (\texttt{lift}\ \psi) \circ (\texttt{lift}\ \phi)$.

Proof. Let $f : X \to U$. Recall that $(\texttt{lift}\ \phi)(f)$ is defined to be the function $\phi \circ f$, so we have

$$\begin{aligned}(\texttt{lift}\ (\psi \circ \phi))(f) &= (\psi \circ \phi) \circ f \\ &= \psi \circ (\phi \circ f) \\ &= (\texttt{lift}\ \psi)((\texttt{lift}\ \phi)(f)).\end{aligned}$$

(the actual proof in lean is a lot simpler than this). □

Lemma 2. *If ϕ is a linear map from V to W then*

$$(\texttt{lift}\ \phi) \circ \texttt{const} \ = \texttt{const}\ \circ \phi$$

(This is an equation of linear maps from V to $X \to W$.)

Proof. Let $v \in V$ and $x \in X$. After unravelling the definitions, you have

$$\begin{aligned}((\texttt{lift}\ \phi)(\texttt{const}\ v))(x) &= \phi((\texttt{const}\ v)(x)) \\ &= \phi(v) \\ &= \texttt{const}\ (\phi(v))(x).\end{aligned}$$

Therefore $(\texttt{lift}\ \phi)(\texttt{const}\ (v)) = \texttt{const}\ (\phi(v))$, so $\texttt{lift}\ \phi \circ \texttt{const}\ = \texttt{const}\ \circ \phi$. (Again, this is a lot easier to prove in lean than it is to think about). □

4 A sequence of vector spaces

As we said above, the set of functions $X \to \mathbb{R}$ is a vector space. Iterating this, we can form a sequence V_n of vector spaces, beginning with the 1-dimensional vector space $\mathbb{R}$:

$$\begin{aligned}V_0 &= \mathbb{R}, \\ V_1 &= (X \to \mathbb{R}), \\ V_2 &= (X \to (X \to \mathbb{R})), \\ V_3 &= (X \to (X \to (X \to \mathbb{R}))), \ldots\end{aligned}$$

In the lean file, we call the sequence V_n of vector spaces "`MultiFun` n". These are defined recursively by stating that `MultiFun` $0 = \mathbb{R}$ and `MultiFun` $(n+1) = (X \to$ `MultiFun` $n)$. There is a simple proof by induction that each of these is a vector space. Notice that we can think of an element $f \in V_2$ as a function of two variables in X with values in $\mathbb{R}$, or alternatively as a function from X to V_1. This means that $f(x, y)$ is a real number and $f(x)$ is an element of V_1, i.e. an function from X to $\mathbb{R}$. Similarly, an element f of V_n is a real valued function of n variables in X, and $f(x)$ is a function of $n-1$ variables in X.

5 A sequence of linear maps (Exercise 5)

We can also define a sequence of linear maps $\partial_n : V_n \ \to_1 [\mathbb{R}]\ V_{n+1}$. These are defined recursively as follows. We begin by defining $\partial_0 : \mathbb{R} \ \to_1 [\mathbb{R}]\ (X \to \mathbb{R})$ to be the linear map `const` :

$$\partial_0 = \texttt{const} \ . \tag{1}$$

This means that for any $f \in V_0$ and any $x \in X$ we have

$$\texttt{d\_zero\_apply}: \quad (\partial_0 f)(x) = f. \tag{2}$$

Assume now that we have already defined the linear map $\partial_n : V_n \ \to_1 [\mathbb{R}]\ V_{n+1}$, and we would like to define $\partial_{n+1} : V_{n+1} \ \to_1 [\mathbb{R}]\ V_{n+2}$. We have two linear maps

$$\texttt{const} \ : V_{n+1} \ \to_1 [\mathbb{R}]\ (X \to V_{n+1}), \qquad \texttt{lift}\ \partial_n : (X \to V_n) \ \to_1 [\mathbb{R}]\ (X \to V_{n+1}).$$

Notice that since $V_{n+1} = (X \to V_n)$ and $V_{n+2} = (X \to V_{n+1})$, both of the linear maps above may be interpreted as linear maps from V_{n+1} to V_{n+2}. We define $\partial_{n+1} : V_{n+1} \to V_{n+2}$ by

$$\partial_{n+1} = \texttt{const} \ - \texttt{lift}\ \partial_n. \tag{3}$$

Note that this means for any $f \in V_{n+1}$ and $x \in X$,

$$\texttt{d\_succ\_apply}: \quad (\partial_{n+1} f)(x) = f - \partial_n(f(x)). \tag{4}$$

Exercise 5 is to rove the equation above. In the equation above, we are thinking of f as a function $X \to V_n$ and $\partial_{n+1} f$ as a function $X \to V_{n+1}$.

If we prefer instead to think of f as a function of $n+1$ variables and $\partial_{n+1} f$ as a function of $n+2$ variables, then this means

$$(\partial_{n+1} f)(x_0, \ldots, x_{n+1}) = f(x_1, \ldots, x_{n+1}) - (\partial_n(f(x_0)))(x_1, \ldots, x_{n+1}).$$

6 Exercises 6-8

In these exercises we check what these definitions mean for ∂_n with $n = 1, 2, 3$.

Lemma 3. *Let $f : X \to \mathbb{R}$ (i.e. $f \in V_1$). Then for any $x, y \in X$ we have*

$$\texttt{d\_one\_apply}: \quad (\partial_1 f)(x, y) = f(y) - f(x).$$

Proof. By `d_succ_apply` we have the following equation in V_1:

$$(\partial_1 f)(x) = f - \partial_0(f(x)).$$

Hence for any $y \in X$ we have

$$(\partial_1 f)(x, y) = f(y) - \partial_0(f(x))(y).$$

The result now follows from `d_zero_apply`. □

Lemma 4. *Let $f : X \to X \to \mathbb{R}$ (i.e. $f \in V_2$). Then for any $x, y \in X$ we have*

$$\texttt{d\_two\_apply} : \quad (\partial_1 f)(x, y, z) = f(y, z) - f(x, z) + f(x, y).$$

Proof. By `d_succ_apply` we have the following equation in V_1:

$$(\partial_1 f)(x) = f - \partial_1(f(x)).$$

Hence for any $y \in X$ we have

$$(\partial_1 f)(x, y, z) = f(y, z) - \partial_1(f(x))(y, z).$$

The result now follows from `d_one_apply`. □

In question 8 you should state and prove a formula for $(\partial_3 f)(w, x, y, z)$ for $f \in V_3$.

7 Exercises 9-11

Questions 9, 10 and 11 are to prove the following facts about the sequence ∂_n of linear maps.

Lemma 5. *For any natural number n, the composition $\partial_{n+1} \circ \partial_n : V_n \to V_{n+1}$ is the zero linear map.*

Proof. The proof is by induction on n. For the case $n = 0$ we must prove that $\partial_1 \circ \partial_0 = 0$. By definition we have

$$\begin{aligned}\partial_1 \circ \partial_0 &= (\texttt{const} - \texttt{lift const}) \circ \texttt{const} \\ &= \texttt{const} \circ \texttt{const} - (\texttt{lift const}) \circ \texttt{const}.\end{aligned}$$

By lemma 2 we have $(\texttt{lift const}) \circ \texttt{const} = \texttt{const} \circ \texttt{const}$, so $\partial_1 \circ \partial_0 = 0$.

Assume now that $\partial_{n+1} \circ \partial_n = 0$. We have

$$\begin{aligned}\partial_{n+2} \circ \partial_{n+1} &= (\texttt{const} - \texttt{lift } \partial_{n+1}) \circ \partial_{n+1} \\ &= \texttt{const} \circ \partial_{n+1} - (\texttt{lift } \partial_{n+1}) \circ \partial_{n+1}.\end{aligned}$$

By lemma 2, we can rewrite this as

$$\partial_{n+2} \circ \partial_{n+1} = (\texttt{lift } \partial_{n+1}) \circ \texttt{const} - (\texttt{lift } \partial_{n+1}) \circ \partial_{n+1}.$$

Replacing the last ∂_{n+1} by $\texttt{const} - \texttt{lift } \partial_n$ and expanding out we get

$$\partial_{n+2} \circ \partial_{n+1} = \texttt{lift } \partial_{n+1} \circ \texttt{const} - \texttt{lift } \partial_{n+1} \circ \texttt{const} + \texttt{lift } \partial_{n+1} \circ \texttt{lift } \partial_n.$$

The first two terms cancel, so we are left with

$$\partial_{n+2} \circ \partial_{n+1} = \texttt{lift}\ \partial_{n+1} \circ \texttt{lift}\ \partial_n.$$

By Lemma 1 and the inductive hypothesis, we have

$$\partial_{n+2} \circ \partial_{n+1} = \texttt{lift}\ (\partial_{n+1} \circ \partial_n) = \texttt{lift}\ 0.$$

Since `lift` is linear (exercise 2 above), we must have $\texttt{lift}\ 0 = 0$. □

Lemma 6. *Let $f \in V_{n+1}$ for some natural number n (i.e. $f : X \to V_n$). Then for any $x \in X$ we have $\partial_{n+1} f = 0$ if and only if $f = \partial_n(f(x))$.*

Proof. This follows immediately from equation 4 (`d_succ_apply`). □

Lemma 7. *For any natural number n, the kernel of $\partial_{n+1} : V_{n+1} \to V_{n+2}$ is equal to the image of $\partial_n : V_n \to V_{n+1}$.*

Proof. Here (and only here) we use the fact that X is non-empty, and we choose an element $x \in X$.

Let $f \in V_{n+1}$. Suppose first that $f \in \ker \partial_{n+1}$. This means that $\partial_{n+1} f = 0$. By lemma 4, we have $f = \partial_n(f(x))$, so f is in the image of ∂_n.

Conversely, suppose that f is in the image of ∂_n. This means $f = \partial_n g$ for some $g \in V_n$. It follows that

$$\partial_{n+1} f = \partial_{n+1}(\partial_n g) = (\partial_{n+1} \circ \partial_n)(g).$$

By lemma 3, $\partial_{n+1} f = 0$ so f is in the kernel of ∂_{n+1}. □