Dedekind Cuts

The rational numbers $\mathbb{Q}$ form an ordered field, but fail to be complete: there exist nonempty subsets of $\mathbb{Q}$ that are bounded above but have no least upper bound, e.g. $\{x \in \mathbb{Q} \mid x^2 < 2\}$.

The real numbers were introduced to fix this shortcoming: they form a complete ordered field containing $\mathbb{Q}$. But how can we construct the real numbers?

Dedekind cuts provide an explicit construction of the real numbers as a complete ordered field. In this sheet we will define Dedekind cuts and establish some of their properties.

A *Dedekind cut* is a partition of the rationals $D = (A, A^c)$ satisfying four conditions:

- A is nonempty;
- A^c is nonempty
- A is a *down-set*, i.e. if $x \in A$ and $y < x$ then $y \in A$
- A has no maximum element.

Note that any Dedekind cut $D = (A, A^c)$ is uniquely determined by the set A, so we will often refer to a set $A \subset \mathbb{Q}$ as a Dedekind cut (where in fact we really mean (A, A^c)).

In Lean we introduce the following definition.

```
structure Dedekind where
  A          : Set ℚ
  nonempty   : A.Nonempty
  nonempty'  : Aᶜ.Nonempty
  down       : ∀ ⦃x y⦄, x < y → y ∈ A → x ∈ A
  no_max     : ∀ ⦃x⦄, x ∈ A → ∃ y ∈ A, x < y
```

Since we wish to *construct* the real numbers as Dedekind cuts, we should think of a Dedekind cut *as* a real number.

For any rational $q \in \mathbb{Q}$ we can define the associated Dedekind cut

$$\texttt{rat}(q) = \{x \in \mathbb{Q} \mid x < q\}.$$

You will need to prove that this does indeed define a cut.

There is a natural ordering on Dedekind cuts given by

$$D \leq E \iff D.A \subseteq E.A.$$

With this definition we will prove that we have a preorder and that the function $\texttt{rat} : \mathbb{Q} \to \texttt{Dedekind}$ is an order embedding of the rationals in the reals, i.e. `rat` is injective and satisfies $p \leq q \iff \texttt{rat}(p) \leq \texttt{rat}(q)$.

Examples of Dedekind cuts that don't correspond to a rational include

$$D_{\sqrt{2}} = \{x \in \mathbb{Q} \mid x^2 < 2 \vee x < 0\}.$$

It is possible to define addition and multiplication on Dedekind cuts and then prove that under these operations they form an ordered field however we won't do this here. (If we did, we could then prove that the square of $D_{\sqrt{2}}$ is 2.)

One question on the sheet (`root_n_add_two`) involves proving that for each $n \in \mathbb{N}$ there exists a Dedekind cut corresponding to $\sqrt{n+2}$.

$$D_{\sqrt{n+2}} = \{x \in \mathbb{Q} \mid x^2 < n+2 \vee x < 0\}.$$

The proof is straightforward but the fact that $D_{\sqrt{n+2}}$ has no maximum element requires a little thought: if $x \in D_{\sqrt{n+2}}$ then either $x < 0$ and so we can take $y = 0$ to be a larger element of $D_{\sqrt{n+2}}$. Otherwise $0 \leq x$ and $x^2 < n+2$, in which case you can check that

$$y = x + \frac{((n+2) - x^2)}{(n+2)(2x+1)}$$

satisfies $x < y$ and $y \in D_{\sqrt{n+2}}$. We recommend that you work out this proof on paper first.

The last fact that you will prove about Dedekind cuts is completeness: if S is a set of Dedekind cuts that is nonempty and bounded above then it has a supremum i.e. there exists a least upper bound for S.

The supremum of S is simply the union of the cuts in S:

$$\sup(S) = \bigcup_{D \in S} D.$$