Part II —

Based on lectures by

Notes taken by Joseph Tedds using Dexter Chua's header and Gilles Castel's snippets.

Michaelmas 2019

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Measure spaces, σ-algebras, π-systems and uniqueness of extension, statement $\star$ and proof $\star$ of Carathodorys extension theorem. Construction of Lebesgue measure on $\mathbb{R}$. The Borel σ-algebra of $\mathbb{R}$. Existence of non-measurable subsets of $\mathbb{R}$. Lebesgue-Stieltjes measures and probability distribution functions. Independence of events, independence of σ-algebras. The Borel-Cantelli lemmas. Kolmogorovs zero-one law. [6]

Measurable functions, random variables, independence of random variables. Construction of the integral, expectation. Convergence in measure and convergence almost everywhere. Fatous lemma, monotone and dominated convergence, differentiation under the integral sign. Discussion of product measure and statement of Fubinis theorem. [6]

Chebyshevs inequality, tail estimates. Jensens inequality. Completeness of L^p for $1 \leq p \leq \infty$ The Hölder and Minkowski inequalities, uniform integrability. [4]

L^2 as a Hilbert space. Orthogonal projection, relation with elementary conditional probability. Variance and covariance. Gaussian random variables, the multivariate normal distribution. [2]

The strong law of large numbers, proof for independent random variables with bounded fourth moments. Measure preserving transformations, Bernoulli shifts. Statements $\star$ and proofs $\star$ of maximal ergodic theorem and Birkhoffs almost everywhere ergodic theorem, proof of the strong law. [4]

The Fourier transform of a finite measure, characteristic functions, uniqueness and inversion. Weak convergence, statement of Lvys convergence theorem for characteristic functions. The central limit theorem. [2]

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0 Introduction

– Week 1 - Lebesgue measure

– Week 2 - Abstract measure theory

– Week 3 - Integration

– Week 4 - Measure theoretic foundations of probability theory

– Week 5 - Random variables, modes of convergence

– Week 6 - Hilbert spaces, C^p spaces

– Week 7 - Fourier transform, central limit theorem

– Week 8 - Ergodic theory

1 Boolean algebras and Finitely additive measures

Definition (Boolean Algebra)**.** A family of subset of X is called a *Boolean algebra* if it is stable under complementation and finite unions and contains the empty set. In other words

- $\emptyset \in \mathcal{B}$
- $\forall\ A, B \in \mathcal{B}, A^c \in \mathcal{B}$ and $A \cup B \in \mathcal{B}$

Remark. Clearly $\mathcal{B}$ is also stable under finite intersection and difference and symmetric difference i.e.

$$\begin{aligned} A, B \in \mathcal{B} \implies & A \cap B \in \mathcal{B} \\ & A \setminus B \in \mathcal{B} \\ & A \triangle B \in \mathcal{B}. \end{aligned}$$

Example.
- The *trivial* Boolean algebra $\mathcal{B} = \{\emptyset, X\}$
- The *discrete* Boolean algebra $\mathcal{B} = 2^X = \mathcal{P}(X)$, the family of subsets of X
- X topological space, the Boolean algebra of *constructible sets* is the family of all finite unions of *locally closed sets* (locally closes $= U \cap F$, for U open, F, closed).

Definition (Finitely additive measure)**.** A *finitely additive measure* on $(X, \mathcal{B})$ is a function $m : \mathcal{B} \mapsto [0, \infty]$ such that

(i) $m(\emptyset) = 0$

(ii) $m(E \cup F) = m(E) + m(F)$ whenever $E, F \subseteq \mathcal{B}$ are disjoint.

Remark. A finitely additive measure on (X, B) is also

(i) *monotone* - $E \subseteq F$ are in $\mathcal{B}$ then $m(E) \leq m(F)$

(ii) *subadditive* - $\forall\ E, F \in \mathcal{B}, M(E \cup F) \leq m(E) + m(F)$

Example.
(i) $\mathcal{B} = 2^X, m(E) :=$ number of E, is called the *counting measure on X*

(ii) $\mathcal{B} = 2^X$ if $f : X \mapsto [0, \infty]$ a function, $m_f(E) = \sum_{e \in E} f(e)$ is a finitely additive measure on X

(iii) $X = \bigcup_i^n X_i \quad X_i$ pairwise disjoint, let $\mathcal{B}$ be the Boolean algebra generated by this partition. If you assign some weight say $a_i \geq 0$ to each X_i, you can define a finitely additive measure on $\mathcal{B}$

$$m(E) = \sum_{i, X_i \subseteq E} a_i.$$

2 Jordan measure on $\mathbb{R}$

Definition (elementary)**.** A subset $E \subseteq \mathbb{R}^d$ is called *elementary* if it is a finite union of *boxes*. A *box* is a product of finite intervals

$$B = \prod_{i=1}^{d} I_i, \quad I_i = \text{ an interval } \subseteq \mathbb{R}.$$

For instance $(a, b), [a, b], (a, b], [a, b)$.

Denote by $|B|$ the "volume" of a box B.

$$B = \prod_{i=1}^{d} |b_i - a_i| \text{ if } B = \prod_{i=1}^{d} I_i \text{ and } (a_i, b_i) \subseteq I_i \subseteq [a_i, b_i].$$

Proposition. Let B be a box in $\mathbb{R}^d$ and let $\mathcal{E}(B)$ be the family of elementary subsets of B

(i) $\mathcal{E}(B)$ is a Boolean algebra

(ii) Every $E \in \mathcal{E}(B)$ can be written as a finite *disjoint* union of boxes

(iii) If $E \in \mathcal{E}(B)$ is written in 2 ways $E = \bigcup_i^N B_i = \bigcup_j^{N'} B'_j$ with B_i, B'_j pairwise disjoint, then $\sum_{i=1}^{N} |B_i| = \sum_{i=1}^{N'} |B'_j|$

Proof. When $d = 1$ it is obvious

Exercise. $d > 1$

□

Proposition. We may set $m(E) = \sum_{i=1}^{N} |B_i|$ whenever E is an elementary set written as $E = \bigcup_i^N B_i$ for B_i pairwise disjoint. Then m is a finitely additive measure on $(B, \mathcal{E}(B))$.

Definition (Joran measurable set)**.** A subset $A \subseteq \mathbb{R}^d$ is called *Jordan-measurable* if $\forall\, \epsilon > 0\ \exists\, E, F$ elementary sets such that

- $E \subseteq A \subseteq F$
- $m(F \setminus E) < \epsilon$

Definition. If A is Jordan measurable, then set

$$m(A) = \inf\{m(F), A \subseteq F, F \text{ elementary}\}$$

Remark. This implies that

$$m(A) = \sup\{m(F), A \subseteq F, F \text{ elementary}\}$$

indeed,

$$\forall\, \epsilon\ \exists\, E, F\ E \subseteq A \subseteq F : m(F \setminus E) < \epsilon.$$

So $m(E) = m(F) - m(F \setminus E) \geq m(A) - \epsilon$.

Proposition. Let B be a box. The family $J(B)$ of Jordan measurable subsets of B is a Boolean algebra and m is a finitely additive measure on $(B, J(B))$.

Proof. Exercise □

Remark. $A \subseteq [0,1]$ is Jordan measurable $\iff$ 1_A is Riemann-integrable.

Example.

$$f_n(x) = \mathbb{1}_{[0,1] \cap \frac{1}{n!} \in \mathbb{Z}} \ \forall \ x, \quad f_n(x) \to \mathbb{1}_{\mathbb{Q} \cap [0,1]}(x).$$

3 Lebesgue measurable set

Definition (Outer-measure)**.** To a subset E of $\mathbb{R}^d$ we associate its *outer-measure*

$$m^*(E) = \inf\left\{\sum_{i\geq 1} m(B_i), E \subseteq \bigcup_{i\geq 1} B_i, B_i \text{ boxes}\right\}.$$

Definition (Lebesgue measurable set)**.** A subset $E \subseteq \mathbb{R}^d$ is called *Lebesgue measurable* if

$$\forall\, \varepsilon > 0\ \exists\, C = \bigcup_{i\geq 1} B_i,$$

a countable union of boxes, such that

$$m^*(C \setminus E) < \varepsilon, E \subseteq C.$$

Remark.
- $m^*(E + x) = m^*(E),\ \forall\, E,\ \forall\, x \in \mathbb{R}^d$
- We can take open boxes if we wish
- Jordan measurable sets are Lebesgue measurable

Our main proposition for this section is as follows:

Proposition.
(i) m^* extends to m on Jordan measurable sets.

(ii) The family $\mathcal{L}$ of Lebesgue measurable sets is a Boolean algebra, stable under countable unions.

(iii) m^* is a countably additive measure on $(\mathbb{R}^d, \mathcal{L})$. i.e.

$$m^*\left(\bigcup_{n\geq 1} E_n\right) = \sum_{n\geq 1} m^*(E_n) \text{ for } E_n \text{ pairwise disjoint.}$$

Remark.
- $\mathbb{Q}$ is in $\mathcal{L}$.
- m^* when restricted the family $\mathcal{L}$ is called the Lebesge measure
- Not every subset of $\mathbb{R}^d$ is in $\mathcal{L}$.
- m^* is not finitely additive on all subsets of $\mathbb{R}^d$.

Lemma. m^* is

(i) Monotone i.d. $E \subseteq F m^*(E) \leq m^*(F)$

(ii) Countably subadditive $\forall\, E_n \subseteq \mathbb{R}^d$

$$m^*\left(\bigcup_{n\geq 1} E_n\right) \leq \sum_{n\geq 1} m(E_n).$$

Proof. (i) Clear

(ii) By definition of m^*, $\forall\, \varepsilon > 0\ \exists\ C_n = \bigcup_{i\geq 1} B_{n,i}$ a countable union of boxes such that $E_n \subseteq C_n$ and

$$m^*(E_n) + \frac{\varepsilon}{2^n} \geq \sum_{i\geq 1} m(B_{n,i)}$$
$$\implies \left(\sum_{n\geq 1} m^*(E_n)\right) + \varepsilon \geq \sum_{n,i} m(B_{n,i)}$$
$$\bigcup E_n \subseteq \bigcup_{n,i} B_{n,i}$$
$$\implies \sum m^*(E_n) \geq m^*\left(\bigcup_{n\geq 1} E_n\right)$$

□

Remark. It is easy to check (see the example sheet) that a finitely additive measure on a Boolean algebra is countably additive iff it has the "continuity property".

Definition (Continuity property)**.** Let X be a set, $\mathcal{B}$ a Boolean algebra of subsets of X. Let m be a finitely additive measure on X such that $m(X) < \infty$. We say that $(X, \mathcal{B}, m)$ have the *continuity property* if

$$\forall\ E_n \in \mathcal{B}, E_{n+1} \subseteq E_n \text{ and } \bigcap_n E_n = \emptyset \implies \lim_{n\to\infty} m(E_n) = 0.$$

Proposition. The Jordan measure has the continuity property on elementary sets

Proof. Suppose not. We get $E_{n+1} \subseteq E_n, \bigcap_n E_n = \emptyset$ and $m(E_n) \not\to 0, E_n$ elementary. $\exists\ F_n \subseteq E_n$ elementary sets $m(F_n) \geq m(E_n) - \frac{\varepsilon}{2^n}$ and F_n closed. By Heine-Borel, since

$$\bigcap_n F_n = \emptyset \implies \exists\ N < \infty \bigcap_{n=1}^{N} F_n = \emptyset$$

(The F_i are closed and bounded and hence compact; in particular, F_1 is compact. Since the intersection of all the F_i is $\emptyset$ then the open sets $F_1 \setminus F_n \subseteq F_1$ form an open cover of F_1. Since F_1 is compact, there is a finite subcover and in particular $\exists N$ such that $\bigcup_{n=1}^{N} F_1 \setminus F_n = F_1$)

Then,

$$\begin{aligned} m(E_n \setminus (F_1\cap, \dots, \cap F_n)) &= m\left(\bigcup_{i=1}^{n} E_n \setminus F_n\right) \\ &\leq \sum_{i=1}^{n} m(E_n \setminus F_i) \\ &\leq \sum_{i=1}^{n} m(E_i \setminus F_i) \\ &\leq \sum_{i=1}^{n} \frac{\varepsilon}{2^i} \leq \varepsilon. \end{aligned}$$

them $m(F_1 \cap \ldots, \cap F_n) \geq m(E_n) - \varepsilon \geq 2\varepsilon - \varepsilon \geq \varepsilon > 0$. For $n = N$ this gives a contradiction □

We can now begin our proof of the main proposition

Proof. (i) To show $m^* = m$ on Jordan measurable sets

- It is clear $m^*(A) \leq m(A)$ by definition
- We need to show converse inequality
- First suppose A is elementary, Pick $\varepsilon > 0, A \subseteq \bigcup_{n\geq 1} B_n, m^*(A) + \varepsilon \geq \sum_{n\geq 1} m(B_n)$. Let $E_n = A \setminus (B_1 \cup, \ldots, \cup B_n)$ an elementary set. $E_{n+1} \subseteq E_n, \bigcap_n E_n = \emptyset \implies m(E_n) \underset{n\to\infty}{\to} 0$ but

$$\begin{aligned} m(A) &\leq m(A \setminus B_1 \cup, \ldots, \cup B_n) + m(B_1 \cup, \ldots, \cup B_n) \\ &\leq \underset{\to 0}{m(E_n)} + \sum_{i=1}^{n} m(B_i). \end{aligned}$$

  So $m(A) \leq m^*(A) + \varepsilon$, ε arbitrary $\implies m(A) \leq m^*(A)$
- In general, if A is Jordan measurable, $\forall\, \varepsilon > 0\ \exists\, E$ elementary

$$E \subseteq A, m(A) \leq m(E) + \varepsilon.$$

□