Part II — Principles of Quantum Mechanics

Based on lectures by D. Skinner
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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Review of Newtonian Mechanics
Some early definitions and recap [1]

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0 Introduction

1 Postulates of Quantum Mechanics

(i) Any physical system can be described by some state in a Hilbert space $|\phi\rangle \in \mathcal{H}$ Further, any complete set, orthogonal set $\{|\phi_a\rangle, |\phi_b\rangle, ...\}$ of states in $\mathcal{H}$ is in 1:1 correspondence with all possible outcomes of the measurement of some physical quantity.
If a system is in the general state $|\psi\rangle = \sum_a c_a|\phi_a\rangle$ then the probability we obtain a result corresponding to $|\phi_b\rangle$ is

$$\mathbb{P}(|\psi\rangle \text{ in } |\phi_b\rangle) = \frac{|\langle\phi_b|\psi\rangle|^2}{\langle\phi_b|\phi_b\rangle\langle\psi|\psi\rangle}$$

$$= \frac{|c_b\langle\phi_b|\psi\rangle|^2}{\langle\phi_b|\phi_b\rangle\langle\psi|\psi\rangle}$$

$$= \frac{|c_b|^2\langle\phi_b|\psi\rangle}{\langle\psi|\psi\rangle}$$

Remark. Which $\mathcal{H}$ we use (e.g $\mathbb{C}^n, L^2(\mathbb{R}^3, d^3x), ...$) depends on the system we wish to describe. Eg a single structureless particle in $\mathbb{R}^3$ may be described using $\mathcal{H} \approx L^2(\mathbb{R}^3, d^3x)$, whereas to describe a system with some internal structure, we may need a larger $\mathcal{H}$.

Remark. Quantum mechanics is inherently probabilistic. This interpretation originally from considering scattering exponents (Born rule)

Remark.

$$\langle\psi|\psi\rangle = 1$$

And likewise using *orthonormal* basis. Then

$$\mathbb{P}(|\psi\rangle \text{ in } |\phi_b\rangle) = |\langle\phi_b|\psi\rangle|^2$$

Even when using normalised states, we're still forced to rescale $|\psi\rangle \to e^{i\phi}|\psi\rangle$ for some constant $\phi \in \mathbb{R}$. Combining this with the normalisation, physical states really correspond to rays in $\mathcal{H}$, ie

$$\{|\psi\rangle \sim \lambda|\psi\rangle; \lambda \in \mathbb{C}^*\}$$

The zero vector $\mathbf{0} \in \mathcal{H}$ is the unique state with zero norm so never represents a physical system.

(ii) Observable quantities correspond to Hermitian operators.
Let Q be a Hermitian operator, and $\{|n\rangle\}$ its basis of eigenstates of $Q|n\rangle = q_n|n\rangle$.
Then the *expectation value* of Q in state $|\psi\rangle$ is

$$\langle Q\rangle_\psi = \langle\psi|Q|\psi\rangle = \sum_{n,m} c_n\bar{c}_m\langle m|Q|n\rangle = \sum_n |c_n|^2 q_n$$

Where $|\psi\rangle = \sum_n c_n|n\rangle$

Remark. Properties of measurements

(a) For Hermitian operators $O \leq \||Q|\psi\rangle\|^2 = \langle\psi|Q^+Q|\psi\rangle$

$$\implies 0 \leq \|(Q - \langle Q\rangle_\psi|\psi\rangle)\|^2 = \langle\psi|(Q - \langle Q\rangle_\psi)^2|\psi\rangle = \langle Q^2\rangle - \langle Q\rangle_\psi^2$$

Define the RMS deviation $\Delta Q_\psi = \sqrt{\langle Q^2\rangle - \langle Q\rangle_\psi^2}$ as a measure of the uncertainty of Q in state $|\psi\rangle$. So we're only certain of the outcome of an exponent if we already know our system is in an eigenstate of the corresponding operator.

(b) We're not saying anything about how to describe the physics of actually carrying out a measurement. In particular, we do *not* say this process has anything to do with applying the corresponding operator.

(iii) Our state evolves in time according to Schrodinger's equation

$$i\hbar = H|\psi\rangle$$

For some operator H, the 'Hamiltonian', that is possibly time dependent.

1.1 The Harmonic Oscillator

Inspired by the classical case, we take the Hamiltonian to be

$$H = \frac{1}{2m}P^2 + \frac{1}{2}m\omega^2 X^2$$

for $\omega \in \mathbb{R}$, $m \in \mathbb{R}$.
To analyse this, begin by introducing dimensionless operator

$$A = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega X + iP)$$

$$A^\dagger = \frac{1}{\sqrt{2m\hbar\omega}}(m\omega X - iP)$$

We have

$$\begin{aligned} A^\dagger A &= \frac{1}{\sqrt{2m\hbar\omega}}((m\omega X + iP)(m\omega X - iP)) \\ &= \frac{1}{\sqrt{2m\hbar\omega}}(P^2 + m^2\omega^2X^2 + m\omega i[X,P]) \\ &= \frac{1}{\hbar\omega}(\frac{p^2}{2m} + \frac{1}{2}m\omega^2X^2) - \frac{1}{2} = \frac{H}{\hbar\omega} - \frac{1}{2} \end{aligned}$$

so $H = \hbar\omega(N + \frac{1}{2})$ where $N = A^\dagger A$
A is the *lowering operator*
$A^\dagger$ is the *raising operator*
N is the *number operator*

First, compute

$$[A, A^\dagger] = \frac{1}{2m\hbar\omega}(m^2\omega^2[X,X] + [P,P] - im\omega[X,P] + im\omega[P,X])$$

$$= \frac{1}{2m\hbar\omega}(2m\hbar\omega) = 1$$

Whilst $[A, A] = 0 = [A^\dagger, A^\dagger]$