Part II — Further Complex Methods

Based on lectures by B. Groisman
Notes taken by Joseph Tedds using Dexter Chua's header and Gilles Castel's snippets.

Lent 2020

Complex variable
Revision of complex variable. Analyticity of a function defined by an integral (statement and discussion only). Analytic and meromorphic continuation. Cauchy principal value of finite and infinite range improper integrals. The Hilbert transform. KramersKronig relations. Multivalued functions: definitions, branch points and cuts, integration; examples, including inverse trigonometric functions as integrals and elliptic integrals. [8]

Special functions
Gamma function: Euler integral definition; brief discussion of product formulae; Hankel representation; reflection formula; discussion of uniqueness (e.g. Wielandts theorem). Beta function: Euler integral definition; relation to the gamma function. Riemann zeta function: definition as a sum; integral representations; functional equation; *discussion of zeros and relation to p(x) and the distribution of prime numbers*. [6]

Differential equations by transform methods
Solution of differential equations by integral representation; Airy equation as an example. Solution of partial differential equations by transforms; the wave equation as an example. Causality. Nyquist stability criterion. [4]

Second order ordinary differential equations in the complex plane
Classification of singularities, exponents at a regular singular point. Nature of the solution near an isolated singularity by analytic continuation. Fuchsian differential equations. The Riemann P-function, hypergeometric functions and the hypergeometric equation, including brief discussion of monodromy. [6]

Contents

0 Introduction

Whilst the prerequisites for this course include complex analysis, this is primarily a methods course - expanding on IB complex methods.

(i) Complex variable

- Revision
- Analyticity and functions defined by integrals e.g. $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t$ domain of z? Analytic continuation?
- Departure from analyticity (singularities at a point or at a curve) : Principal Value e.g. $PV\int_{-1}^2 \frac{1}{x}\mathrm{d}x \overset{?}{=} \log 2$

(ii) Special functions Γ, β, ζ

(iii) Integral transforms of ODE and PDE

(iv) Second order ODE on $\mathbb{C}$ (1,2,3 regular singular points), hypergeometric equations

1 Complex variable

1.1 Brief revision

$z = x + iy$

Definition (Neighbourhood)**.** A *neighbourhood* of a point $z \in \mathbb{C}$ is an *open-set* containing z.

Definition (Extendend complex plane)**.** The *extended complex plane* $\mathbb{C}^*$ or $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. All directions "$\infty$" are equivalent (think of Riemann sphere).

Definition (Differentiable)**.** A function $f(z)$ is differentiable at z if

$$f'(z) = \lim_{\Delta z \to 0} \left| \frac{f(z + \Delta z) - f(z)}{\Delta z} \right|$$

exists (independent of how $\Delta z \to 0$.

Definition (Analytic)**.** We say that $f(z)$ is *analytic* (holomorphic / regular) at a point z if $\exists$ a neighbourhood of z throughout which f' exists. [Extensions to domain D]

Cauchy-Riemann Conditions If $f(z) = u(z) + iv(z)$, with $u, v \in \mathbb{R}$ is differentiable at z, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

The converse is true for u, v differentiable at z.

Corollary. The Wirtinger derivative

$$\overline{\partial} = \frac{\partial f}{\partial \overline{z}} = 0,$$

where $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$

Theorem (Cauchy's Theorem)**.** If $f(z)$ is analytic within and on a closed bounded contour C, then

$$\oint_C f(z) \mathrm{d}z = 0.$$

Note that C bounds D - a simply connected domain and for z_0 inside C, we have Cauchy's Integral Formula

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \mathrm{d}z.$$

With C traversed anti-clockwise.

Corollary.

$$f^{(n)}(z_0) = \frac{n!}{z_0} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}z,$$

and functions are differentiable infinitely many times.

Definition (Entire)**.** A function $f(z)$ is *entire* if it is analytic on $\mathbb{C}$ (not $\overline{\mathbb{C}}$).

This leads us to.

Theorem (Louiville's Theorem)**.** If f is entire and bounded on $\overline{\mathbb{C}}$, then f is constant.

Proof. Consider a circular disc or radius R centred at z_0 and we know $|f(z)| < M$. Then from our previous corollary,

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi i} \oint \frac{|f(z)|}{|z - z_0|^{n+1}} |\mathrm{d}z| \leq \frac{n! M}{2\pi R^{n+1}} \oint |\mathrm{d}z| \leq \frac{n! M}{R^n}.$$

In particular, for $n = 1$

$$|f'(z)| \leq \frac{M}{R} \; \forall \; z \in \mathbb{C}.$$

We can choose R as large as we wish, so $|f'(z)| \underset{R \to \infty}{\to} 0$. Hence, $f'(z) = 0 \; \forall \; z$. Since $f(z) - f(0) = \int_0^z f'(\tilde{z}) \mathrm{d}\tilde{z} = 0$, we obtain $f(z) = f(0)$. □

Series Expansions

Theorem (Existence of Taylor Expansions)**.** An analytic function has a convergent expansion about any point within its domain of analyticity (proof omitted).

Laurent Series Suppose $f(z)$ has an isolated singularity at z_0 (but analytic in neighbourhood of z_0). Then

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n, \quad C_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}z.$$

We can classify the singularity as follows:

$$f(z) = \sum_{n=0}^{\infty} C_n (z - z_0)^n + \sum_{n=1}^{N} \frac{C_{-n}}{(z - z_0)^n}.$$

z_0 is:

– A regular point (or zero) if all $C_{-n} = 0$

– A simple pole if $N = 1$

– A pole of order N if $N > 1$

– An essential singularity if $N \to \infty$

In the case that $0 < N < \infty$ we can write

$$f(z) = \frac{1}{(z - z_0)^N} \sum_{k=0}^{\infty} C_k (z - z_0)^k, \quad f(z) = \frac{g(z)}{(z - z_0)^N}$$

where g is analytic.

Definition (Residue)**.** The coefficient C_{-1} is called the *residue* of f at z_0, we write $\text{res}(f, z_0) = C_{-1}$

For a simple pole note that $C_{-1} = \lim_{z\to z_0}(z-z_0)f(z)$. For a pole of order N,

$$C_{-1} = \frac{1}{(N-1)!}\frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}((z-z_0)^N f(z)).$$

Theorem (Residue Theorem)**.** If f is analytic in a simply connected domain except at a finite number of isolated singularities $z_1, \dots, z_0$ then

$$\oint_C f(z)\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \mathrm{res}(f(z); z_k).$$

Where C is a contour traversed in the anticlockwise direction.

Lemma. (Indentation Lemma) |`https://tartarus.org/gareth/maths/Complex_Methods/rjs/indentati`
Let f have a simple pole at z_0, $\mathrm{res}(f; z_0)$. Then,

$$\lim \int_{C_\varepsilon} f(z)\mathrm{d}z = i(\beta-\alpha)\mathrm{res}(f; z_0), \quad 0 < \alpha < \beta < 2\pi.$$

Where on C_ε $z = z_0 + \varepsilon e^{i\theta}, \alpha \le \theta \le \beta$

Proof. Consider Laurent expansions of f about z_0

$$f(z) = \frac{\mathrm{res}(f, z_0)}{z-z_0} + g(z),$$

where $g(z)$ is analytic in the region $|z - z_0| < r$, where $r > 0$. By continuity of g at z_0 we can choose r small enough so g is bounded by some M. On $0 < \varepsilon < r$, we have

$$\begin{aligned}\int_{C_\varepsilon} f(z)\mathrm{d}z &= \mathrm{res}(f, z_0)\int_{C_\varepsilon}\frac{\mathrm{dz}}{z-z_0} + \int_{C_\varepsilon} g(z)\mathrm{d}z \\ &= i\mathrm{res}(f, z_0) + \int_{C_\varepsilon} g(z)\mathrm{d}z \\ &= i(\beta-\alpha)\mathrm{res}(f, z_0)\end{aligned}$$

.

Since

$$\left|\int_{C_\varepsilon} g(z)\mathrm{d}z\right| \le M \times \text{length of } C_\varepsilon = M(\beta-\alpha)\varepsilon \to 0.$$

□

1.2 Functions defined by integrals

Consider

$$F(z) = \int_C f(z,t)\mathrm{d}t,$$

where C is some path in $\mathbb{C}$ (not necessarily closed contour). For which values of z is $F(z)$ defined and analytic? Conditions on analyticity: We need to check that

(i) The integrand is continuous (jointly in t and z)

(ii) The $\int$ converges uniformly in each subset of its domain

(iii) The integrand is analytic in z for each value of t

Note: there will be no rigorous treatment of (ii).

Example.

$$F(z) = \int_{-\infty}^{\infty} e^{-zt^2} \mathrm{d}t \left(= \left(\frac{\pi}{z}\right)^{\frac{1}{2}} \right).$$

- Existence: Converges for Re$z > 0$ and diverges for Re$z < 0$. If Re$z = 0$ but $z \neq 0$ (i.e. z is imaginary) then the integrand e^{-iyt^2} oscillates increasingly rapidly. $F(z)$ is not absolutely convergent but conditionally convergent.

$$\lim_{\ell \to \infty} \int_{-\ell}^{\ell} |e^{-iyt^2}| \mathrm{d}t \to \infty \text{ but } \lim_{\ell \to \infty} \int_{-\ell}^{\ell} |e^{-iyt^2}| < \infty.$$

- Analyticity: Clearly (i), (iii) hold. It can be shown that (ii) also holds

Example. Let $F(z) = \int_0^\infty \frac{u^{z-1}}{u+1} \mathrm{d}u$

- Existence: Potential "problems" when $u = 0, \infty$. The integrand is well-behaved otherwise (except at $u = -1$, but it is outside the range of integration). No problematic values of z (consider $u^{z-1} = e^{(z-1)\log u}$). At $u = 0$, $u + 1 \approx 1$ and so we have $\int_0 \frac{u^{z-1}}{1} = \frac{u^z}{z}\Big|_0$. For $z = x + iy$,

$$|u^z| = |e^{z \log u}| = e^{x \log u}.$$

Since $\log u \underset{u \to 0}{\to} -\infty$ we must have Re $z > 0$. At $u \to \infty$, $u + 1 \approx u$ and $\int^\infty \frac{u^{z-1}}{u} = \frac{u^{z-1}}{z-1}\Big|^\infty$.

$$|u^{z-1}| = |e^{(z-1)\log u}| = e^{(x-1)\log u}.$$

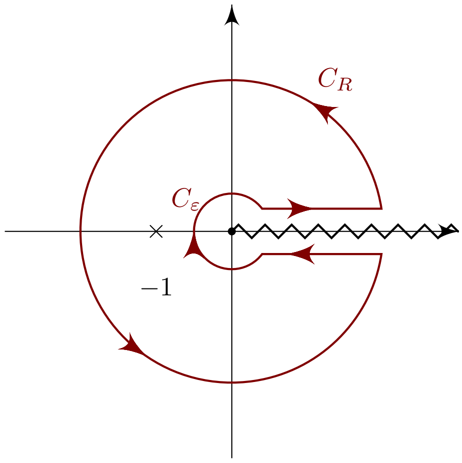
So need Re $z < 1$. Clearly for Re $z > 1$, Re $z < 0$, $F(z)$ doesn't converge. What about $z - 1$, z are imaginary. (No convergence). Thus $F(z)$ is defined for $0 <$ Re $z < 1$

- Analyticity: (i), (iii) are clearly satisfied again leaving out (ii). So $F(z)$ is analytic for $0 <$ Re $z < 1$.

- Evaluating the integral: Consider

$$J = \int_C \frac{t^{z-1}}{t+1} \mathrm{d}t = \int_{C_+} + \int_{C_R} + \int_{C_-} + \int_{C_\epsilon},$$

where $R \to \infty, \epsilon \to 0$ and a simple pole at $t = -1 - e^{i\pi}$



On $C_+ : t = u, \epsilon \le u \le R$ and $\int_{C_+} = F(z)$.
On $C_- : t = ue^{2\pi i}, R \ge u \ge \epsilon$ and

$$\int_{C_-} = -(e^{2pii})^{z-1} \int_0^\infty \frac{u^{z-1}}{1+u} \mathrm{d}u = -(e^{2\pi i})^{z-1} F(z).$$

On $C_R, t = Re^{i\theta}, 0 \le \theta \le 2\pi$ and

$$\begin{aligned} \int_{C_R} &= \int_0^{2\pi} \frac{R^{z-1}e^{i\theta(z-1)}}{Re^{i\theta}+1} iRe^{i\theta} \mathrm{d}\theta \\ &= i\int_0^{2\pi} \frac{R^z e^{i\theta z}}{Re^{i\theta}+1} \mathrm{d}\theta \\ &\underset{R\to\infty}{\to} i\int_0^{2\pi} \frac{e^{i\theta(z-1)}}{R^{1-z}} \mathrm{d}\theta \\ &\underset{R\to\infty}{\to} 0. \end{aligned}$$

when Re $z < 1$ On $C_\epsilon : t = \epsilon e^{i\theta}, 2\pi \ge \theta \ge 0$ and

$$\int_{C_\epsilon} \underset{\epsilon \ll 1}{\to} i \int_{2\pi}^0 \epsilon^z e^{i\theta z} \mathrm{d}z \underset{\epsilon\to 0}{\to} 0,$$

when Re $z > 0$. So again we obtain $0 <$ Re $z < 1$. Thus

$$J = F(z)[1 - (e^{2\pi i})^{z-1}] = 2\pi \mathrm{Res}\left(\frac{t^{z-1}}{t+1}; e^{i\pi}\right) = 2\pi i (e^{i\pi})^{z-1}.$$

Hence

$$F(z) = \frac{2\pi i}{e^{i\pi z} - e^{i\pi z}} = \frac{\pi}{\sin \pi z}.$$

(Later $\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec} \pi z$).

1.3 Analytic Continuation

Let $F(z) = \int_{-\infty}^{\infty} e^{-zt^2} \mathrm{d}t$ is analytic for Re $z > 0$. Is it possible to extend its domain of analyticity? Would such an extension be unique?

Theorem (Identity Theorem)**.** Let g_1, g_2 be analytic (holomorphic) in a connected open set $D \subseteq \mathbb{C}$ with $g_1 = g_2$ in a non-empty open subset $\tilde{D} \subseteq D$. Then $g_1 = g_2$ on D.

Proof. Expand $g_1 - g_2$ as a Taylor series about $z_0 \in \tilde{D}$. The series is 0 and convergent in D. Thus $g_1 - g_2 = 0$ on D. □

Analytic Continuation

Let D_1, D_2 be open sets with $D_1 \cap D_2 \neq \emptyset$. Let f_1 and f_2 be analytic on D_1 and D_2 respectively with $f_1 = f_2$ on $D_1 \cap D_2$. Then we say that f_2 is the AC of f_1 from D_1 to D_2. We claim that f_2 is unique.

Proof. Suppose $\exists\ \tilde{f}_2 \neq f_2$, which provides an AC (and $\tilde{f}_2 = f_1$ on $D_1 \cap D_2$). Then we can define the following function

$$g_1 = \begin{cases} f_1 & \text{on } D_1 \\ f_2 & \text{on } D_2 \end{cases} \quad g_2 = \begin{cases} f_1 & \text{on } D_1 \\ \tilde{f}_2 & \text{on } D_2 \end{cases}.$$

Then, by the Identity Theorem, $g_1 = g_2$ on $D_1 \cup D_2$. Hence $f_2 = \tilde{f}_2$ so the extension is unique. □

With three domains, in the case that we only have overlaps between consecutive domains we cannot use the Identity Theorem on D_1 and D_3. However, *AC* of f_1 to D_3 is unique if AC is possible for all domains connecting D_1 and D_3 (Monodromy Theorem). The proof is left as an exercise.

Methods of AC

(i) By Taylor expansion: Choose z_0 on the boundary of D and extend f_1 to a disk $|z - z_0| < r$, r- radius of convergence.

Example. How to obtain $f(z) = \frac{1}{1-z}$ by AC. We first note that

$$\frac{1}{1-z} = \frac{1}{1-z_0} \cdot \frac{1}{1 - \frac{z-z_0}{1-z_0}} = \frac{1}{1-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{1-z_0} \right)^n,$$

which is convergent if $|z - z_0| < |1 - z_0| = r$. Now, let $f_1 = \sum_{n=0}^{\infty} z^n$. It is analytic, for $|z| < 1 = D_1$. Let $f_2 = \sum_{n=0}^{\infty} \frac{(z-\frac{i}{2})^n}{(z-\frac{i}{2})^{n+1}}$ which is analytic within the disk $|z - \frac{i}{2}| < \frac{\sqrt{5}}{2} = D_2$. So $f_1 = f_2$ on $D_1 \cap D_2$. Hence, by the Identity Theorem f_2 is the AC of f_1 into D_2. This can be continued as a chain of discs to cover the entire $\mathbb{C} \setminus \{1\}$ to obtain $\frac{1}{1-z}$, which has a simple pole at $z = 1$. A Meromorphic continuation - AC excluding singularities.

Such extensions are not always possible

Example. $f(z) = \sum_{n=0}^{\infty} z^{2^n}$, convergent in $|z| < 1$ (by ratio test). The singularities are dense (not isolated) and AC beyond $|z| < 1$ is impossible (see handout). The circle $|z| = 1$ is called a natural barrier for $f(z)$.

(ii) By contour deformation

Example. Let $F(z) = \int_{-\infty}^{\infty} \frac{e^{it}}{t-z}\mathrm{d}t$ for Im $z > 0$. We want to continue $F(z)$ to the lower half plane. Obviously, $F(z)$ is not analytic for Im $z = 0$. Why can't we just redefine F for Im $z \neq 0$? Continue F into a neighbourhood of z_1 with Im $z_1 < 0$. Define

$$F_1(z) = \int_\gamma \frac{e^{it}}{t-z}\mathrm{d}t.$$

With γ as shown. F_1 is defined and analytic $\forall\ z$ above z_1. If Im $z > 0$, then $F_1(z) = F(z)$. Indeed, the integrals agree by contour deformation as we can deform γ to the real axis without crossing any singularities. Hence F_1 is the AC of F into Im $z < 0$ (above γ). Now, let

$$G(z) = \int_{-\infty}^{\infty} \frac{e^{it}}{t-z}\mathrm{d}t, \text{Im } z \neq 0.$$

If Im $z > 0$ then $G(z) = F(z)$ by definition. If Im $z < 0$, then (considering a closed contour C) we get

$$F_1(z) + \int_{\infty}^{-\infty} \frac{e^{it}}{t-z}\mathrm{d}t = 2\pi i e^{iz}.$$

So, for Im $z > 0$, we get $F = F_1 = G(z)$, Im $z < 0, F_1 = G(z) + 2\pi i e^{iz}$. Hence $G(z)$ jumps by $2\pi i e^{iz}$ as z crosses the real axis. Thus G cannot provide an analytic continuation of F.

1.4 Cauchy Principal Value

Motivation: can we say that

$$\int_{-1}^{2} \frac{\mathrm{d}x}{x} = \log 2 - \log|-1| = \log 2?.$$

Definition (Cauchy principle value)**.** If $f(x)$ is "badly" behaved at $x = c$ and $a < c < b$, we define the *Cauchy principal value* (CPV) integral by

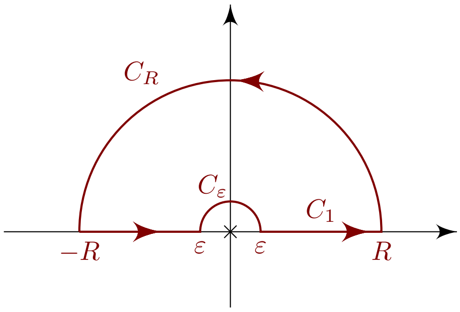
$$\mathcal{P}\int_a^b f(x)\mathrm{d}x := \lim_{\varepsilon\to 0}\left(\int_a^{c-\varepsilon} f(x)\mathrm{d}x + \int_{c+\varepsilon}^{b} f(x)\mathrm{d}x\right),$$

if the limit exists. For the CPV at ∞ we define

$$\mathcal{P}\int_{-\infty}^{\infty} f(x)\mathrm{d}x = \lim_{R\to\infty}\int_{-R}^{R} f(x)\mathrm{d}x,$$

if the limit exists.

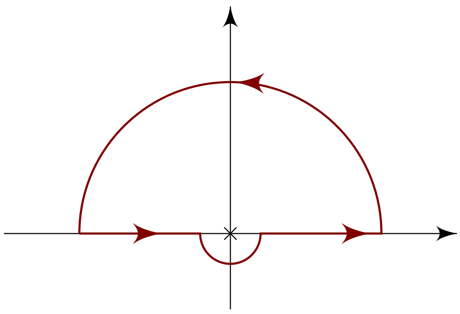
Example. Define $I = \mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)}{x}\mathrm{d}x$, where $f(z)$ is analytic in the upper half-plane (including the real axis) and $|f(z)| \to 0$ as $|z| \to \infty$. This integral only makes sense as a CPV



$\oint_C \frac{f(z)}{z}\mathrm{d}z = \int_{C_1} + \int_{C_\varepsilon} + \int_{C_R} = 0$ as $f(x)$ is analytic inside. It should be clear that

$$\int_{C_1} \to I \qquad \int_{C_\varepsilon} \to -i\pi f(0) \qquad \int_{C_R} \to 0.$$

With the residue for C_ε coming from the indentation lemma since $z = 0$ is a simple pole unless $f(0) = 0$. So, $I = i\pi f(0)$. Alternatively, $\oint_C = I + i\pi f(0) = 2\pi i f(0)$, so $I = i\pi f(0)$ (illustrated below, uses residue of the pole at 0).



Remark. I depends on the values of $f(x)$ for real x. Hence, $f(x)$ cannot be real-valued.

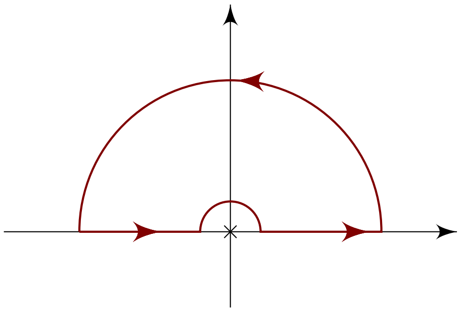
Example. Here we merely use CPV as a tool. Let $I = \int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\mathrm{d}x$. The integrand has a removable singularity at $x = 0$. The standard method is to split up the integrand

$$I = \int_{-\infty}^{\infty} \left(\frac{1}{x^2} - \frac{e^{ix}}{2x^2} - \frac{e^{-ix}}{2x^2} \right) \mathrm{d}x.$$

Of which the exponential terms do not converge at $x = 0$. So we deform the contour to exclude 0. For the second integral, we close C in the upper half plane giving zero (since there are no singularities within the closed contour). For the third integral, we close C in the lower half plane giving $-2\pi i(-i/2)$ (the minus because we are circling the pole or order two in the clockwise sense). For the first integral, we can close either in the upper half or the lower half plane, giving 0 (since in the first case the contour encloses no singularities and in the second case the residue of the enclosed singularity is 0). Therefore, I $= \pi$. We can instead use CPV, by continuity of the integral

$$I = \int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\mathrm{d}x = \mathcal{P}\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\mathrm{d}x = \mathrm{Re}\ \mathcal{P}\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2}\mathrm{d}x.$$

Consider $J = \int_C \frac{1-e^{iz}}{z^2}\mathrm{d}z$.



Then

$$J = \mathcal{P}\int_{-\infty}^{\infty} \frac{1-e^{iz}}{z^2}\mathrm{d}z = -i\pi(-i).$$

Where we calculate the residue either by the indentation lemma or directly using L'Hopital's rule. Thus $\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2}\mathrm{d}x = \pi$ and we obtain two results

$$I = \pi, \mathcal{P}\int_{-\infty}^{\infty} \frac{-\sin x}{x^2}\mathrm{d}x = 0.$$

Definition (Hilbert transform)**.** The *Hilbert transform* of $f(x)$ is defined as

$$\mathcal{H}(f)(y) := \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)\mathrm{d}x}{x-y},$$

and it is a function of y.

Observe that $\mathcal{H}$ is a linear operator, which takes $f(x)$ into a function of y. Assume $x, y \in \mathbb{R}$.
Assumption: Let's assume that f has a Fourier decomposition, so only need to consider HT of $e^{i\omega x}$ and use linearity. Let's show that

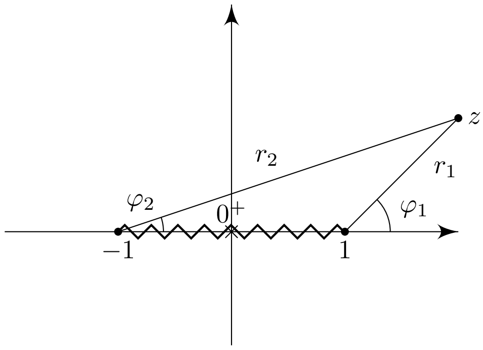
$$\mathcal{H}(e^{i\omega x})(y) = \begin{cases} ie^{i\omega x}, & w > 0 \\ -ie^{i\omega x}, & w < 0 \end{cases} = \operatorname{sgn}(\omega) ie^{i\omega x} \quad (\omega \neq 0).$$

1.5 Multi-valued Functions

For a Harmonic function, we must have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial^2 u}{\partial \bar{z}\partial z} = 0.$$

Example. $f(z) = (1-z^2)^{\frac{1}{2}} = (1-z)^{\frac{1}{2}}(1+z)^{\frac{1}{2}}$, which has *branch points* at $z = \pm 1$. One possible parametrization is local polar coordinates by letting $z = 1 + r_1 e^{i\varphi_1} = -1 + r_2 e^{i\varphi_2}$



$$f(z) = \pm i(r_1 r_2)^{\frac{1}{2}} e^{i(\varphi_1+\varphi_2)/2} = |1-z^2|^{\frac{1}{2}} e^{i(\varphi_1+\varphi_2\pm\pi)/2}.$$

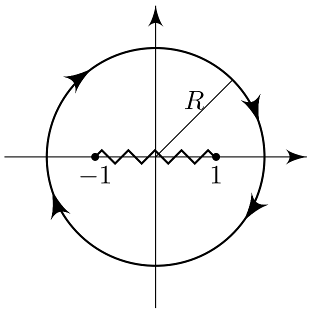
Choose the branch by choosing a branch cut. The natural choice is $[-1, 1]$. Also, we specify the value of the function at $z = 0^+$ such that $f(0^+) = 1$. From our equation above, 0^+ corresponds to $\varphi_1 = \pi, \varphi_2 = 0$ i.e. $f(0^+) = e^{i(\pi+0\pm\pi)/2} = \mp 1$. So in the above equation, we take $-$ so

$$f(z) = |1-z|^{\frac{1}{2}} e^{i(\varphi_1+\varphi_2-\pi)/2}.$$

We evaluate $f(z)$ at other points by examining how φ_1, φ_2 vary along paths connecting 0^+ with z without crossing the branch cut.

Integration using a branch cut

Our aim is to evaluate $I = \int_{-1}^{1}(1-x^2)^{\frac{1}{2}}\mathrm{d}x$ using contour integration. We choose the same branch as above and let $J(z) = \int_C f(z)\mathrm{d}z$ where C is the circle $|Z| = R > 1$ traversed clockwise.



$$f(z) = -iz(1-\frac{1}{z^2})^{\frac{1}{2}} = -iz(1-\frac{1}{2}z^{-2} - \frac{1}{8}z^{-4} + \cdots,$$

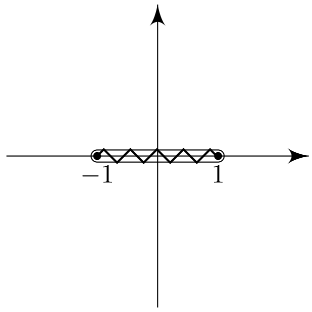
i.e. the Laurent series for $|z| > 1$. Note that the $-$ at the front of our expansion comes from our choice of branch. Now, at $z = R$, we have $\varphi_1 = \varphi_2 = 0$ i.e.

$$f(R) = \sqrt{R^2-1}e^{i(0+0-\pi)/2} = e^{-i\pi/2}\sqrt{R^2-1} = -i\sqrt{R^2-1}.$$

So

$$\begin{aligned} J &= \int_C (-iz)(1-\frac{1}{2}z^{-\frac{1}{2}} - \frac{1}{8}z^{-4} + \cdots)\mathrm{d}z \\ &= \int_0^{-2\pi} (-iRe^{i\theta})(1-\frac{e^{-2i\theta}}{2R^2} + \cdots)iRe^{i\theta}\mathrm{d}\theta = \pi. \end{aligned}$$

What about I? Let's collapse the contour onto the branch cut.



$$J = \int_{-1}^{1} (1-x^2)^{\frac{1}{2}}\,\mathrm{d}x + \int_{1}^{-1} -(1-x^2)^{\frac{1}{2}}\,\mathrm{d}x = 2I = \pi.$$

Hence, $I = \frac{\pi}{2}$

The arcsin function defined as an integral

Introduction: Let $z = \sin w = \frac{e^{iw}-e^{-iw}}{2i}$, and let $v = e^{iw}$. Then $v - \frac{1}{v} = 2iz, v^2 - 2izv - 1 = 0$, so $v = iz + \sqrt{1-z^2}$. Hence we can write $iw = \log v$ and

$$w = \arcsin z = -i\log(iz + \sqrt{1-z^2}).$$

Hence,

$$\frac{\mathrm{d}\arcsin z}{\mathrm{d}z} = \frac{1}{\sqrt{1-z^2}},$$

and therefore

$$\arcsin z = \int_0^z \frac{\mathrm{d}t}{(1-t^2)^{\frac{1}{2}}}.$$

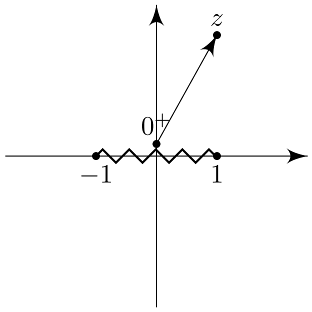
Our aim is to construct the multivariate function arcsin from a single valued function arcsin by analytic continuation. Let

$$\arcsin z_{[0,2\pi)} = \int_0^z \frac{\mathrm{d}t}{(1-t^2)^{\frac{1}{2}}},$$

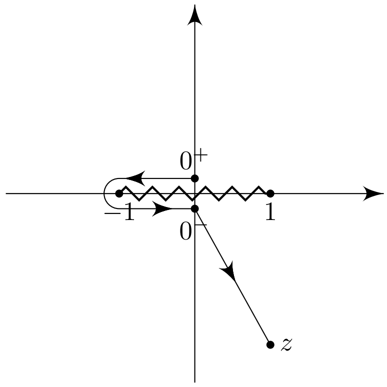
where

(i) The branch of $\sqrt{1-t^2}$ is defined by a branch cut along the real axis connecting $t=-1$ and $t=1$, with $\sqrt{1-t^2} = +1$ at the origin just above the cut (at $t = 0^+$).

(ii) The path is defined as follows:



$0 \leq \arg z < \pi.$



$\pi < \arg z < 2\pi.$

In what domain of z is $\arcsin_{[0,2\pi)} z$ analytic? Notice that

$$\frac{\mathrm{d} \arcsin_{[0,2\pi)} z}{\mathrm{d}z} = \frac{1}{\sqrt{1-z^2}},$$

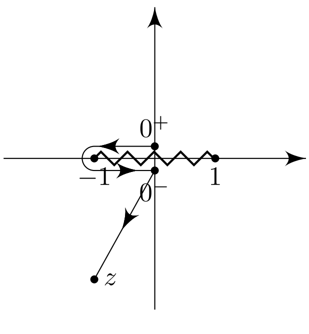
which has a branch as defined in (i). thus $\arcsin_{[0,2\pi)} z$ is not analytic on $[-1, \infty)$. It can be show, that it is discontinuous on $[-1, \infty)$ Notice $\int \frac{\mathrm{d}t}{(1-t^2)^{\frac{1}{2}}} = -2\pi$ integrating around our cut.

$$\int_{-1}^{1} \frac{\mathrm{d}t}{(1-t^2)^{\frac{1}{2}}} = -2 \int_0^{\pi} \mathrm{d}\theta = -2\pi,$$

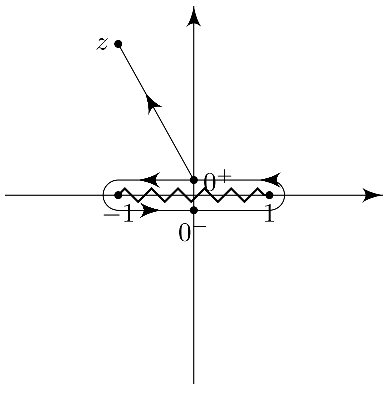
where we substituted $t = \cos\theta$. Now we obtain the analytic continuation into $\arg z > 2\pi$. Define

$$\arcsin_{(\pi,3\pi)} z = \int_0^z \frac{\mathrm{d}t}{(1-t^2)^{\frac{1}{2}}},$$

where



$\pi \leq \arg z < 2\pi.$



$2\pi < \arg z < 3\pi.$

Now

- $\arcsin_{[0,2\pi)} z = \arcsin_{(\pi,3\pi)}$ in $\pi < \arg z < 2\pi$

– The latter is analytic in $2\pi < \arg z < 3\pi$

Hence we obtain the analytic continuation of $\arcsin_{[0,2\pi)}$ into $2\pi < \arg z < 2\pi$. This process can be repeated to obtain the multivalued function $\arcsin z$. Observe that for $0 \leq \arg z \leq 2\pi$,

$$\arcsin(e^{2\pi i} z) = \arcsin_{[0,2\pi)} z + \int_C = \arcsin_{[0,2\pi)} z - 2\pi.$$

We can show that possible values of $\arcsin z = \arcsin_{[0,2\pi)} z + 2\pi n, n \in \mathbb{N}$. Also, (again for $0 \leq \arg z \leq 2\pi$)

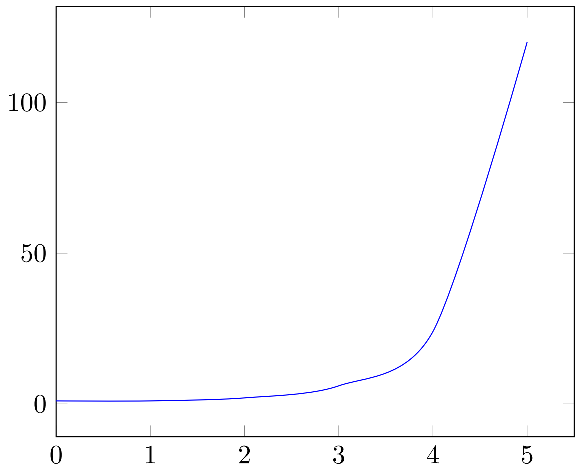
$$\sin\left(\arcsin e^{2\pi i} z\right) = \sin\left(\arcsin_{[0,2\pi)} z - 2\pi\right).$$

And $\sin\left(\arcsin e^{2\pi i} z\right) = e^{2\pi i} z = z$. Thus setting $w = \arcsin_{[0,2\pi)} z$ and using $\arcsin z = w$, we obtain $\sin w = \sin(w - 2\pi)$. Thus $\sin w$, obtained this way is 2π periodic.

2 Special Functions

2.1 The Gamma function

Motivation: find solution to the interpolation problem, finding a smooth curve $y = f(x)$ that connects the points given by $y = x!, x \in \mathbb{N}$. We want to generalise $n!$ from integers into $\mathbb{C}$.



Let

$$I(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t,$$

which converges and is analytic for Re $z > 0$. Now

$$I(z+1) = \int_0^\infty t^z e^{-t} \mathrm{d}t = [-t^z e^{-t}]_0^\infty + \int_0^\infty z t^{z-1} e^{-t} \mathrm{d}t = zI(z).$$

Hence we obtain the recurrence relation $I(z+1) = zI(z)$. Also, $I(1) = 1$ so $I(n+1) = n!$ for $n \in \mathbb{N}$. Idea, define

$$\Gamma(z) = \begin{cases} I(z), & \text{Re } z > 0 \\ \text{by Ac in Re} z < 0, & \text{whenever possible} \end{cases}.$$

Now, $I(z) = \frac{I(z+1)}{z}$ which is analytic for Re $z + 1 > 0$ i.e. Re $z > -1$ and $z \neq 0$. Similarly,

$$\frac{I(z+n+1)}{z(z+1)\cdots(z+n-1)(z+n)},$$

is analytic for Re $z > -(n+1)$ and $z \neq 0, -1, \ldots, -n$. Hence define

$$\Gamma(z) = \begin{cases} I(z), & \text{for Re } z > 0 \\ I(z+1)/z, & \text{for Re } z > -1 \\ \vdots, & \vdots \\ \frac{I(z+n+1)}{z(z+1)\cdots(z+n)}, & \text{for Re } z > -(n+1) \end{cases}.$$

Which has simple poles at $z = 0, -1, \ldots$ as its only singularities.

$$\text{Res }(\Gamma(z), -n) = \lim_{z\to -n} (z+n)\Gamma(z) = \frac{(-1)^n}{n!}.$$

Some alternative definitions and formulae
First we look at the Euler Product Formula

Claim.

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)} \qquad \forall\, z \in \mathbb{C} \setminus \{0, -1, \ldots\}.$$

Proof. Firstly, show this for Re $z > 0$. Note $e^{-t} = \lim_{n\to\infty} \left(1 - \frac{t}{n}\right)^n$. Then,

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1} \mathrm{d}t.$$

Letting $\tau = \frac{t}{n}$,

$$\begin{aligned}
\Gamma(z) &= \lim_{n\to\infty} n^z \left[\frac{(1-\tau)^n \tau^z}{z}\right]_0^1 - \frac{n^z}{z}(-n)\int_0^1 (1-\tau)^{n-1}\tau^z \mathrm{d}\tau \\
&= \lim_{n\to\infty} (-1)^n n^z (-1)^n n! \int_0^1 \frac{\tau^{z+n-1}\mathrm{d}\tau}{z(z+1)\cdots(z+n-1)} \\
&= \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}.
\end{aligned}$$

$\Gamma(z)$ is analytic in $z \in \mathbb{C} \setminus \{0, -1, \ldots\}$, and so is the RHS. They are equal in Re $z > 0$. Hence we obtain an analytic continuation by the identity theorem. □

Next we consider the Gauss product formula.

Claim.

$$\Gamma(z) = \frac{1}{z} \prod_{n=1}^{\infty} \frac{(1+\frac{1}{n})^z}{1+\frac{z}{n}}.$$

Proof. Follows from Euler's product formula

$$\begin{aligned}
\Gamma(z) &= \frac{1}{z} \lim_{n\to\infty} \frac{n^z}{(z+1)\cdots(\frac{z}{n-1)}+1)(\frac{z}{n}+1)} \\
&= \frac{1}{z} \lim_{n\to\infty} \left(\frac{\left(\frac{n+1}{n}\right)^z \left(\frac{n}{n-1}\right)^z \cdots \left(\frac{2}{1}\right)^z \left(\frac{n}{n+1}\right)^z}{(z+1)\cdots(\frac{z}{n-1)}+1)(\frac{z}{n}+1)} \right).
\end{aligned}$$

As $\left(\frac{n}{n+1}\right)^z \underset{n\to\infty}{\rightarrow} 1$, we obtain the required expression. □

Now the Weierstrass canonical product.

Claim.

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{k=1}^{\infty} \left(1 + \frac{z}{k}\right) e^{-\frac{z}{k}}, \quad ,$$

where

$$\gamma = \lim_{n\to\infty} \left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right) \approx 0.577.$$

We call γ the Euler-Mascheroni constant.

Proof. By Euler's product formula

$$\frac{1}{\Gamma(z)} = z \lim_{n\to\infty} \frac{(1+z)(2+z)\cdots(n+z)}{n!n^z}.$$

Divide each term by $n, n-1, \dots$ and use $n^{-z} = e^{\log n^{-z}}$.

$$\begin{aligned}\frac{1}{\Gamma(z)} &= z \lim_{n\to\infty} e^{-z\log n}(1+z)\cdots(1+\frac{z}{n})\\ &= z\lim_{n\to\infty} e^{-z(\log n-(1+\frac{1}{2}+\cdots+\frac{1}{n})}e^{-z(1+\frac{1}{2}+\cdots+\frac{1}{n})}(1+z)\cdots(1+\frac{z}{n})\\ &= ze^{\gamma z}\prod_{k=1}^{\infty}(1+\frac{z}{k})e^{-\frac{z}{k}}.\end{aligned}$$

□

Reflection formula

$$\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec}(\pi z), z \notin \mathbb{Z}.$$

Proof. Begin by considering Re $z \in (0,1)$, so $\Gamma(z), \Gamma(1-z)$ can be written as integrals. Thus,

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty e^{-t}t^{z-1}\mathrm{d}t \int_0^\infty e^{-s}s^{-z}\mathrm{d}s.$$

By using the substitution, $t = r\sin^2\theta, s = r\cos^2\theta$ we obtain

$$\Gamma(z)\Gamma(1-z) = 2\int_0^\infty (\tan\theta)^{2z-1}\mathrm{d}\theta = \int_0^\infty \frac{u^{z+1}}{u+1}.$$

Where we substituted $\tan\theta = u^{\frac{1}{2}}$. This is an integral we have already calculated, which takes the value $\frac{\pi}{\sin\pi z}$. Both $\Gamma(z)\Gamma(1-z)$ and $\pi \operatorname{cosec}\pi z$ are analytic except at $z \in \mathbb{Z}$. They are equal in Re $z \in (0,1)$, so the result follows by analytic continuation (meromorphic continuation). □

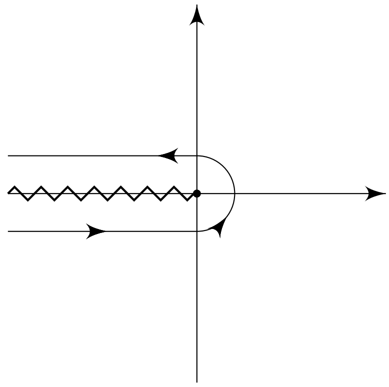
Observe $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Some key properties:

- $\Gamma(n+1) = n!, n \in \mathbb{N}$
- Res $(\Gamma(z), -n) = \frac{(-1)^n}{n!}$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}, \Gamma(1) = \Gamma(2) = 1$
- $\Gamma(z) \neq 0 \ \forall\ z$ (from recurrence relation)

The Hankel representation of $\Gamma(z)$

$$\Gamma(z) = \frac{1}{2i\sin\pi z}\int_{-\infty}^{0^+} e^t t^{z-1}\mathrm{d}t, z \neq 0, -1, \dots,$$

where $-\pi \le \arg t \le \pi$ and the path is the Hankel contour



This integral represents an analytic function both in z and t. Let's prove that the formula holds for Re $z > 0$ (i.e. equality for $I(z)$). We collapse the Hankel contour onto the branch cut. Let

$$J(z) = \int_{-\infty}^{0^+} e^t t^{z-1} \mathrm{d}t, \mathrm{Re}\ z > 0.$$

Thus $J(z) = \int_{\gamma_1}, + \int_{\gamma_\varepsilon} + \int_{\gamma_2}$ with the obvious choice of contours:

- $\gamma_1 : t = xe^{-i\pi}, \infty > x > \varepsilon$
- $\gamma_\varepsilon : t = \varepsilon e^{i\theta}, -\pi < 0 < \pi$
- $\gamma_2 : t = xe^{i\pi}, \varepsilon \le x < \infty$

$$\int_{\gamma_1} \underset{\varepsilon\to 0}{\to} \left(e^{-i\pi}\right)^z \int_\infty^0 e^{-x} x^{z-1} \mathrm{d}x.$$

$$\int_\varepsilon \underset{\varepsilon\to 0}{\to} 0, \text{ as Re } z > 0.$$

$$\int_{\gamma_2} \underset{\varepsilon\to 0}{\to} \left(e^{i\pi}\right)^z \int_0^\infty e^{-x} x^{z-1} \mathrm{d}x.$$

Thus $J(z) = 2i \sin \pi z I(z)$ for Re $z > 0$. So we complete by analytic continuation. Note that $z = 1, 2, 3, \dots$ the zeros of $\sin \pi z$ cancel by zeros of the integral in RHS of our formula. Indeed, for $z \in \mathbb{Z}^+, t = 0$ is not a branch point, there is no branch cut. So $\exists$ no singularities within the Hankel contour so $J(z) = 0$ which cancels the zeros of $\sin \pi z$.

Computing residues of $\Gamma(z)$

For $z = -m$ $\exists$ no branch cut so the Hankel contour is a circle around the origin.

$$J(-m) = \int_{|t|=1} e^t t^{-m-1} \mathrm{d}t = 2\pi i \mathrm{Res}\ (e^t t^{-m-1}, t = 0).$$

Now,

$$e^t t^{-m-1} = \sum_{n=0}^\infty \frac{t^{n-m-1}}{n!},$$

by Taylor expansion of e^t. C_{-1} corresponds to $n - m - 1 = -1$, so $n = m$. Thus, $J(-m) = 2\pi i/m!$

2.2 Beta Function

$$B(p,q) = \begin{cases} \int_0^1 t^{p-1}(1-t)^{q-1}\mathrm{d}t, & \text{Re } p,\ \text{Re } q > 0 \\ \text{by AC in p for q and vice versa}, & \text{o/w} \end{cases}.$$

Setting $t = \sin^2\theta$ gives $B(p,q) = \int_0^{\frac{\pi}{2}} \sin^{2p-1}\theta \cos^{2q-1}\theta \mathrm{d}\theta$.
Properties

(i) $B(p,q) = B(q,p)$

(ii) $B(1,q) = \frac{1}{q}$

(iii) $B(p,z+1) = \frac{z}{p+z}B(p,z)$

(iv) $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. Note that if $m,n \in \mathbb{N}$ we have $B(n,m) = \frac{(n-1)!(m-1)!}{(n+m-1)!}$

We give a proof of (iii) here

Proof.

$$B(p,z+1) = \int_0^1 t^{p-1}(1-t)^z \mathrm{d}t$$
$$= \int_0^1 t^{p-1}(1-t)^{z-1}\mathrm{d}t - \int_0^1 t^p(1-t)^{z-1}\mathrm{d}tzz \quad = B(p,z) - \frac{p}{z}B(p,z+1).$$

□

This gives the analytic continuation of $B(p,z)$ to $-1 \le$ Re $z \le 0$ (just as we did with Γ) The singularities are simple poles at $z = 0, -1, -2, \ldots$ for fixed p. Note that (iv) can also provide an analytic continuation. We can prove (iv) in a similar way to the proof of the reflection formula for Γ

Proof.

$$\Gamma(p)\Gamma(q) = \left(\int_0^\infty e^{-s}s^{p-1}\mathrm{d}s\right)\left(\int_0^\infty e^{-t}t^{q-1}\mathrm{d}t\right) = \Gamma(p+q)\cdot B(p,q).$$

Where we made use of the substitution $s = r\cos^2\theta, t = r\sin^2\theta$. □

2.3 The zeta function

$$\zeta(z) := \begin{cases} \sum_{n=1}^\infty \frac{1}{n^z}, & \text{for Re } z > 1 \\ \text{and by analytic continuation}, & \text{wherever possible} \end{cases}.$$

The integral representation of $\zeta(z)$ is given by

$$\zeta(z) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t - 1}\mathrm{d}t,\ \text{Re } z > 1.$$

Proof. Recall that $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t$ for Re $z > 0$. Let $t = ns$ with $n \in \mathbb{N}, s \in \mathbb{R}$. Then $\Gamma(z) = n^z\int_0^\infty s^{z-1}e^{-ns}\mathrm{d}s\ \forall\ n$. Hence,

$$\zeta(z)\Gamma(z) = \sum_{n=1}^\infty \int_0^\infty s^{z-1}e^{-ns}\mathrm{d}s,\ \text{for Re } z > 1.$$

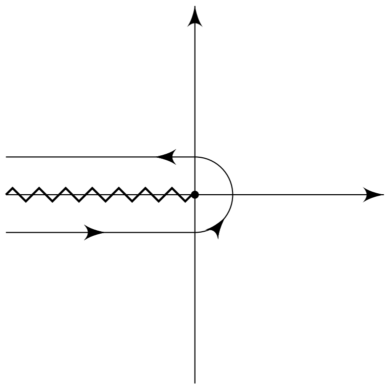
$$\begin{aligned}\zeta(z)\Gamma(z) &= \int_0^\infty t^{z-1}\left(\sum_{n=1}^{\infty} e^{-nt}\right)\mathrm{d}t \\ &= \int_0^\infty t^{z-1}\frac{e^{-t}}{1-e^{-t}}\mathrm{d}t \\ &= \int_0^\infty \frac{t^{z-1}}{e^t-1}.\end{aligned}$$

□

Proposition.

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i}\int_{-\infty}^{0+}\frac{t^{z-1}}{e^{-t}-1}\mathrm{d}t.$$

Note that the integrand has simple poles at $2\pi i n, n \in \mathbb{Z}$



Proof. We show that

$$\frac{\Gamma(1-z)}{2\pi i}\int_{-\infty}^{0+}\frac{t^{z-1}}{e^{-t}-1}\mathrm{d}t = \frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t-1}\mathrm{d}t, \text{for Re } z > 1.$$

Then, we prove that the LHS gives the analytic continuation of the RHS into Re $z < 1$. □