Part II — Stochastic Financial Models

Based on lectures by J. R. Norris

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Utility and mean-variance analysis
Utility functions; risk aversion and risk neutrality. Portfolio selection with the mean-variance criterion; the efficient frontier when all assets are risky and when there is one riskless asset. The capital-asset pricing model. Reservation bid and ask prices, marginal utility pricing. Simplest ideas of equilibrium and market cleaning. State-price density. [5]

Martingales
Conditional expectation, definition and basic properties. Conditional expectation, definition and basic properties. Stopping times. Martingales, supermartingales, submartingales. Use of the optional sampling theorem. [3]

Dynamic Models
Introduction to dynamic programming; optimal stopping and exercising American puts; optimal portfolio selection. [3]

Pricing contingent conditions
Lack of arbitrage in one-period models; hedging portfolios; martingale probabilities and pricing claims in the binomial model. Extension to the multi-period binomial model. Axiomatic derivation. [4]

Brownian motion
Introduction to Brownian motion; Brownian motion as a limit of random walks. Hitting-time distributions; changes of probability. [3]

Black-Scholes model
The BlackScholes formula for the price of a European call; sensitivity of price with respect to the parameters; implied volatility; pricing other claims. Binomial approximation to BlackScholes. Use of finite-difference schemes to compute prices [6]

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1 Utility and mean-variance analysis

1.1 Contingency claims and utility functions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable X on Ω, provides a model for an investment which delivers $X(\omega)$ for consumption depending on chance $\omega \in \Omega$.

Definition (Contingent claim)**.** In this context we often use the term *contingent claim* as another name for a random variable.

Definition (Utility function)**.** By a *utility function* we mean any non-decreasing function $U : \mathbb{R} \mapsto [-\infty, \infty)$. Think of $U(x)$ as quantifying the satisfaction obtained on consuming x. Allowing $-\infty$ is a way of saying the value of x that obtains $-\infty$ is unacceptable.

We often assume the investor will act to maximise expected utility. So Y is *preferred* to X iff $\mathbb{E}[U(X)] \leq \mathbb{E}[U(Y)]$. If $\mathbb{E}[U(X)] = \mathbb{E}[U(Y)]$ the investor is said to be *indifferent* between X and Y. We say the investor is *risk averse* if they prefer $\mathbb{E}[X]$ to X for all integrable random variables X. We say *risk neutral* if indifferent between X and $\mathbb{E}[X]$.

Definition. Recall that U is a *concave* function if for all $x, y \in \mathbb{R}$, all $p \in (0, 1)$

$$pU(x) + (1-p)U(y) \leq U(px + (1-p)y).$$

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Proposition. An investor with utility function U is risk averse if and only if U is concave.

Proof. Suppose risk averse. Consider the contingent claim X taking values x, y with probabilities $p, (1-p)$ respectively. Then,

$$pU(x) + (1-p)U(y) = \mathbb{E}[U(X)] \leq U(\mathbb{E}[X]) = U(px + (1-p)y).$$

Hence U is concave.
Suppose on the other hand U is concave. Let X be an integrable random variable (i.e. $\mathbb{E}[|X|] < \infty$) then by Jensen's inequality

$$\mathbb{E}[U(X)] \leq U(\mathbb{E}[X]).$$

Hence, the investor is risk averse. □

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Example. For $\gamma \in (0, \infty)$ the CARA (constant absolute relative aversion) utility function of parameter γ is given by

$$\mathrm{CARA}_\gamma(x) = -e^{-\gamma x}.$$

3 For $R \in (0,1) \cup (1, \infty)$ the CRRA (constant relative risk aversion) utility function of parameter R is given by

$$\mathrm{CRRA}_R(x) = \begin{cases} \frac{x^{1-R}}{1-R} & x > 0 \\ -\infty & x \leq 0 \end{cases}.$$

Also,

$$\text{CRRA}_1(x) = \begin{cases} \log x & x > 0 \\ -\infty & \text{otherwise} \end{cases}.$$

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Non-rigorous discussion Let U be concave (note that U is non-decreasing). Consider a small continent claim X. We ask whether we prefer $w + X$ to w for a given constant w. By Taylor's theorem

$$U(w+X) \approx U(w) + X \underbrace{U'(w)}_{>0} + \frac{1}{2} X^2 \underbrace{U''(w)}_{<0}.$$

$$\mathbb{E}\left[U(w+X)\right] \approx U(w) + \mathbb{E}\left[X\right] U'(w) + \frac{1}{2}\mathbb{E}\left[X^2\right] U''(w),$$

so we prefer $w + X$ if

$$2\frac{\mathbb{E}\left[X\right]}{\mathbb{E}\left[X^2\right]} > -\frac{U''(w)}{U'(w)}$$

the Arrow-Pratt coefficient of absolute risk aversion. For CARA_γ this constant is equal to γ.
Similarly, do we prefer $w(1 + X)$ to w? Yes if

$$2\frac{\mathbb{E}\left[X\right]}{\mathbb{E}\left[X^2\right]} > -\frac{wU''(w)}{U'(w)}$$

the Arrow-Pratt coefficient of relative risk aversion. For CRRA_R this constant is equal to R.

1.2 Reservation prices and marginal prices

Consider an investor with concave utility function. Suppose they have available a set $\mathcal{A}$ of contingent claims, and suppose $\mathbb{E}\left[U(X)\right]$ is maximised over $\mathcal{A}$ at $X^* \in \mathcal{A}$. Let Y be another contingent claim. The investor would buy Y for price π if there exists $X \in \mathcal{A}$ such that

$$\mathbb{E}\left[U(X + Y - \pi)\right] > \mathbb{E}\left[U(X^{*)}\right].$$

The supremum of all such prices $\pi_b(Y)$ is the *(reservation) bid price)* for Y.

The investor would sell Y for price π if there exists $X \in \mathcal{A}$ such that

$$\mathbb{E}\left[U(X - Y + \pi)\right] > \mathbb{E}\left[U(X^*)\right].$$

The infimum of all such prices $\pi_a(Y)$ is the *(reservation) ask price* for Y.

Proposition. (Ask above, bid below) Assume $\mathcal{A}$ is convex. Then $\pi_b(Y) \leq \pi_a(Y)$

Proof. It suffices to show there is no price π at which the investor will both buy and sell. Suppose for a contradiction that there exist X_a, X_b such that

$$\mathbb{E}\left[U(X_a - Y + \pi)\right] > \mathbb{E}\left[U(X^*)\right].$$

$$\mathbb{E}\left[U(X_b + Y - \pi)\right] > \mathbb{E}\left[U(X^*)\right].$$

Now $X = \frac{X_a + X_b}{2} \in \mathcal{A}$ since $\mathcal{A}$ is convex and $U(X) \geq \frac{U(X_a - Y + \pi) + U(X_b + Y - \pi)}{2}$ since U is concave. Then we obtain the following contradiction.

$$\mathbb{E}\left[U(X^*)\right] < \frac{\mathbb{E}\left[U(X_a - Y + \pi)\right] + \mathbb{E}\left[U(X_b + Y - \pi)\right]}{2} \leq \mathbb{E}\left[U(X)\right] \leq \mathbb{E}\left[U(X^*)\right].$$

Hence there is no such π. □

Recall U is concave and non-decreasing. An investor has available a set of of contingent claims $\mathcal{A}$, and seeks to maximise $\mathbb{E}\left[U(X)\right], X \in \mathcal{A}$. Assume $X^* \in \mathcal{A}$ is a maximiser. Suppose Y is another contingent claim. Assume that $\mathcal{A}$ is an affine space and that U is a differentiable and strictly concave.

Definition (Affine space)**.** S is *affine* if $S - S$ is a vector space. This can be visualised as a vector space away from the origin.

Then X^* is unique (or $\frac{X_1^* + X_2^{ast}}{2}$ is better.

Definition (Marginal price)**.** We define the *marginal price* of Y as

$$\pi_m(Y) = \mathbb{E}\left[U'(X^*)Y\right] / \mathbb{E}\left[U'(X^*)\right].$$

Non-rigorous discussion to explain Note that for $\Xi \in \mathcal{A} - \mathcal{A}$ the map $t \mapsto \mathbb{E}\left[U(X^* + t\Xi\right]$ on $\mathbb{R}$ achieves its minimum at $t = 0$. So

$$0 = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_0 \mathbb{E}\left[U(X^* + t\Xi)\right] = \mathbb{E}\left[U'(X^*)\right].$$

It is plausible that there is a differentiable map $t \mapsto X^*(t) : \mathbb{R} \leftarrow \mathcal{A}$ such that for all t

$$\mathbb{E}\left[U(X^*(t) - tY + \pi_b(tY))\right] = \mathbb{E}\left[U(X^*)\right].$$

Then $X^*(0) = X^*$. Define $\Xi \in \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_0 X^*(t), \pi = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_0 \pi_b(tY)$. It is plausible that $\Xi \in \mathcal{A} - \mathcal{A}$. So

$$0 = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_0 \mathbb{E}\left[U(X^*(t) - tY + \pi_b(tY))\right] = \mathbb{E}\left[U'(X^*)(\Xi - Y + \pi)\right].$$

So we see

$$\pi_m(Y) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_0 \pi_b(tY) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_0 \pi_a(tY).$$

So marginal price is the price to buy (or sell) a small amount of Y.

1.3 Single period asset price model

Definition (Single period asset price model)**.** By a *single period asset price model*, we mean a pair of random variables (S_0, S_1) in $\mathbb{R}^d$. We write $S_n = (S_n^1, \ldots, S_n^d$ with S_n^i the price of asset i at time n.

Definition (Numeraire)**.** By a *numeraire* we mean a pair of random variables $(S_0^0, S_1^0$ in $(0, \infty)$.

Notation. We write

$$\overline{S}_n = (S_n^0, S_n) = (S_n^0, S_n^1, \ldots, S_n^{d)}.$$

Call $(\overline{S}_0, \overline{S}_1)$ an *asset price model with numeraire*

Often we take $S_0^0 = 1, S_1^0 = 1 + r$ some constant $r \in (-1, \infty)$, Then S^0 is called a *riskless bond* and r is the *interest rate*. We assume $\overline{S}_0$ is non-random as the default.

In the case without numeraire, an investor with initial wealth w_0 chooses $\theta \in \mathbb{R}^d$ subject to

$$\theta.S_0 = \sum_{i=1}^{d} \theta^i S_0^i = w_0.$$

Then the investor has wealth $\theta.S_1$ at time 1. We call θ the *portfolio*. With numeraire, investor chooses $\overline{\theta} = (\theta^0, \theta$ such that $\overline{\theta}.\overline{S}_0 = w_0$. The wealth at time 1 is $\overline{\theta}.\overline{S}_0$.

It may be that there exists a random variable $\rho \geq 0$ such that $\mathbb{E}\left[\rho S_1^i\right] = S_0^i$ for all i. Then we call ρ a *state price density*

1.4 Portfolio selection using the mean-variance criterion

Let (S_0, S_1) be an asset price model on $\mathbb{R}^d$ with S_0 non-random, S_1 has mean μ, variance V. We assume that V is invertible and S_0, μ are linearly independent. Suppose we are given w_0, w_1. The investor wishes to

$$\begin{aligned} \text{minimise} \quad & \operatorname{var}(\theta.S_1) \\ \text{subject to} \quad & \theta.S_0 = w_0, \\ & \mathbb{E}\left[\theta.S_1\right] = w_1 \end{aligned}$$

Note $\mathbb{E}\left[\theta.S_1\right] = \theta.\mu$, $\operatorname{var}(\theta.S_1) = \theta.(V\theta)$ So our problem is to

$$\begin{aligned} \text{minimise} \quad & \theta.(V\theta) \\ \text{subject to} \quad & \theta.S_0 = w_0, \\ & \theta.\mu = w_1. \end{aligned}$$

Consider $L(\theta, \lambda) = \frac{1}{2}\theta.(V\theta) - \lambda_0\theta.S_0 - \lambda_1\theta.\mu$ At minimising θ^*.

$$\begin{aligned} 0 &= \frac{\partial}{\partial \theta^i} L(\theta, \lambda) \\ &= (V\theta)^i - \lambda_0 S_0^i - \lambda_1 \mu^i. \end{aligned}$$

So $\theta^* = \lambda_0 A S_0 + \lambda_1 A\mu, A = V^{-1}$. Now fit the constants

$$\begin{aligned} w_0 &= \theta^*.S_0 = \lambda_0 a + \lambda_1 b \\ w_1 &= \theta^*.\mu = \lambda_0 b + \lambda_1 c \end{aligned}$$

$a = S_0.(AS_0), b = \mu(AS_0) = S_0(A\mu), c = \mu(A\mu)$. Note that $\Delta = ac - b^2 \neq 0$ by linear independence

$$\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = M \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix}, M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

$$M^{-1} = \frac{1}{\Delta}\begin{pmatrix} d & -b \\ -b & a \end{pmatrix}.$$

$$\begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = M^{-1}\begin{pmatrix} w_0 \\ w_1 \end{pmatrix}.$$

So

$$\theta^* = \frac{cw_0 - bw_1}{\Delta} AS_0 + \frac{aw_1 - bw_0}{\Delta} A\mu$$

The minimising variance is

$$\begin{aligned}
\theta^*_\cdot (V\theta^*) &= (\lambda_0 AS_1 + \lambda_1 A\mu)_\cdot (\lambda_0 S_0 + \lambda_1 \mu) \\
&= (\lambda_0 \lambda_1) M \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} \\
&= (w_0 w_1) M^{-1} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \\
&= \frac{cw_0^2 - 2bw_0 w_1 + aw_1^2}{\Delta} = q(w_1)
\end{aligned}$$

We minimise this over w_1 where $2bw_0 \neq 2aw_1$

$$w_1^* = \frac{b}{a} w_0, \theta^*_{\min} = \frac{w_0}{a} AS_0.$$

Putting w_1^* back into q, we obtain

$$q(w_1^*) = \frac{acw_0^2 - 2b^2 w_0^2}{a\Delta} + \frac{b^2}{a\Delta} w_0^2 = \frac{w_0^2}{a}$$