Part II — Classical Dynamics

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Review of Newtonian Mechanics

Some early definitions and recap [1]

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1 Review of Newtonian Mechanics

1.1 Newton's Second Law

If a particle of mass m has position vector $\mathbf{r}(t)$ and velocity $\dot{\mathbf{r}} = \frac{d\mathbf{r}}{dt}$ in an inertial frame of reference, then its acceleration $\ddot{\mathbf{r}} = \frac{d^2\mathbf{r}}{dt^2}$ is related to the force $\mathbf{F}$ acting on the particle by the *equation of motion*

$$m\ddot{\mathbf{r}} = \mathbf{F}$$

This is a system of 2nd order ODEs for the components of $\mathbf{r}$. The solution depends on the initial position $\mathbf{r}(0)$ and initial velocity $\dot{\mathbf{r}}(0)$.
Any frame of reference that moves with constant velocity w.r.t an inertial frame is also inertial.
In a non-inertial (e.g rotating) frame, the equation of motion is modified and can be defined by including 'fictitious' forces.
Newton's Second Law also applies to extended bodies if m is the total mass, $\mathbf{r}$ is the position of the centre of mass, $\mathbf{F}$ is the net force acting on the body.

The *momentum* (or *linear momentum*) of the particle is

$$\mathbf{p} = m\dot{\mathbf{r}}$$

Its *angular momentum* (about the origin) is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p} = m\mathbf{r} \times \dot{\mathbf{r}}$$

Its *kinetic energy* is $T = \frac{1}{2}m\left|\dot{\mathbf{r}}\right|^2$

Their rates of change are

$$\begin{aligned}\dot{\mathbf{p}} &= m\ddot{\mathbf{r}} = \mathbf{F} \\ \dot{\mathbf{L}} &= m\dot{\mathbf{r}} \times \dot{\mathbf{r}} + m\mathbf{r} \times \ddot{\mathbf{r}} \\ &= \mathbf{0} + \mathbf{G}\end{aligned}$$

Where $\mathbf{G}$ is the *torque* acting on the particle (or the *moment* of the force about the origin).

$$\dot{T} = m\ddot{\mathbf{r}} \cdot \dot{\mathbf{r}} = \mathbf{F} \cdot \dot{\mathbf{r}}$$

The change in KE as the particle moves along a path C is

$$\Delta T = \int_{t_1}^{t_2} \dot{T} dt = \int_{t_1}^{t_2} \mathbf{F} \cdot \dot{\mathbf{r}} dt = \int_C \mathbf{F} \cdot d\mathbf{r}$$

This line integral is the *work done* by the force along the path.

1.2 Systems of Particles

Consider N particles with masses m_i and position vectors $\mathbf{r}_i(t)$ (for $1 \leq i \leq N$)
The equation of motion of particle i is

$$m_i\ddot{\mathbf{r}}_i = \mathbf{F}_i$$

If the force $\mathbf{F}_i$ on particle i is given as a function of the positions $\mathbf{r}_j$, velocities $\dot{\mathbf{r}}_j (1 \leq j \leq N)$ and time t, then we have a system of coupled 2nd order ODEs. The solution depends on the initial positions and velocities.

Assume that $\mathbf{F}_i$ can be decomposed as

$$\mathbf{F}_i = \sum_{j=1}^{N} \mathbf{F}_{ij} + \mathbf{F}_i^{ext}$$

Where $\mathbf{F}_{ij}$ is the force on particle i due to particle j, and $\mathbf{F}_i^{ext}$ is the external force on particle i due to particles outside the system.

For an isolated system, $\mathbf{F}_i^{ext} = 0$

Newton's Third Law states that $\mathbf{F}_{ji} = -\mathbf{F}_{ij}$ and that the self-force $\mathbf{F}_{ii} = 0$ (no sum)

The equation of motion of particle i is then

$$m_i\ddot{\mathbf{r}}_i = \sum_j \mathbf{F}_{ij} + \mathbf{F}_i^{ext}$$

The *centre of mass* has position $\mathbf{R}(t)$ given by

$$M\mathbf{R} = \sum_{i=1}^{N} m_i\mathbf{r}_i$$

Where $M = \sum_{i=1}^{N} m_i$ is the total mass of the system.

Summing the equations of motion and using Newton's Third Law, we find

$$\begin{aligned} M\ddot{\mathbf{R}} &= \sum_i \sum_j \mathbf{F}_{ij} + \sum_i \mathbf{F}_i^{ext} \\ &= \mathbf{0} + \mathbf{F}^{ext} \end{aligned}$$

Where $\mathbf{F}^{ext}$ is the *net external force.*

The total linear momentum of the system

$$\mathbf{P} = \sum_{i=1}^{N} \mathbf{p}_i = \sum_i m_i\dot{\mathbf{r}}_i = M\dot{\mathbf{R}}$$

satisfies $\dot{\mathbf{P}} = M\ddot{\mathbf{R}} = \mathbf{F}^{ext}$

The total angular momentum of the system about the origin is

$$\mathbf{L} = \sum_i m_i\mathbf{r}_i \times \dot{\mathbf{r}}_i$$

and satisfies

$$\dot{\mathbf{L}} = \sum_i m_i\mathbf{r}_i \times \ddot{\mathbf{r}}_i = \sum_i \mathbf{r}_i \times \mathbf{F}_i$$

If the *strong version* of Newton's Third Law applies, i.e that $\mathbf{F}_{ij} = -\mathbf{F}_{ji}$ and is parallel to $\mathbf{r}_i - \mathbf{r}_j$ then the internal torques cancel.

$$\begin{aligned} \sum_i \sum_j \mathbf{r}_i \times \mathbf{F}_{ij} &= \frac{1}{2}\sum_i \sum_j \mathbf{r}_i \times (\mathbf{F}_{ij} - \mathbf{F}_{ji}) \\ &= \frac{1}{2}\sum_i \sum_j (\mathbf{r}_i - \mathbf{r}_j) \times \mathbf{F}_{ij} = \mathbf{0} \end{aligned}$$

Then we have

$$\dot{\mathbf{L}} = \sum_i \mathbf{r}_i \times \mathbf{F}_i^{ext} = \sum_i \mathbf{G}_i^{ext} = \mathbf{G}^{ext} \text{ (net external torque)}$$

For an *isolated system*, both $\mathbf{P}$ and $\mathbf{L}$ are conserved.

2 Lagrange's Equations

2.1 Generalised Co-ordinates

While Newtonian mechanics uses vectors ($\mathbf{F} = m\mathbf{a}$), the Lagrangian approach is more flexible.

A system with n degrees of freedom requires n independent *generalised coordinates* $q_i(t)$, $i = 1, 2, ..., n$ to specify its configuration.

The *generalised velocities* are $\dot{q}_i = \frac{dq_i}{dt}$

Why do this?

(i) Non-Cartesian coordinates e.g polar coordinates for problems with circular or spherical symmetry

(ii) Systems with constraints e.g confining surfaces, curves, strings, rods

Lagrangian mechanics deals mostly with systems that are *conservative* (ideal, non-dissipative). Their dynamics can be derived from a scalar function called the Lagrangian

$$\mathcal{L}(\mathbf{q}, \dot{\mathbf{q}}, t) = T - V$$

Notation:

$$\mathbf{q} \text{ stands for } (q_1, q_2, ..., q_n)$$
$$\dot{\mathbf{q}} \text{ stands for } (\dot{q}_1, \dot{q}_2, ..., \dot{q}_n)$$
$$\frac{\partial \mathcal{L}}{\partial \mathbf{q}} \text{ stands for } (\frac{\partial \mathcal{L}}{\partial q_1}, ...)$$
$$\mathbf{q} \cdot \mathbf{p} = \sum_{i=1}^{n} q_i p_i$$

2.2 Hamilton's Principle

If a system evolves from an initial configuration $\mathbf{q_1} = \mathbf{q}(t_1)$ to a final configuration $\mathbf{q_2} = \mathbf{q}(t_2)$, the *action* is defined as the functional

$$S[\mathbf{q}] = \int_{t_1}^{t_2} \mathcal{L} dt$$

This is a functional of t and the path $\mathbf{q}(t)$ taken in the configuration space.

Theorem. Hamilton's Principle
The physical path taken is such that the action has a stationary value. i.e the first variation

$$\delta S = \delta \int \mathcal{L} dt = 0$$

subject to $\mathbf{q}(t_1)$ and $\mathbf{q}(t_2)$ being fixed.
According to the calculus of variations, this means that the functional derivative vanishes.

$$\frac{\delta S}{\delta q_i} = \frac{\partial \mathcal{L}}{\partial q_i} - \frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = 0$$

for each $i = 1, ..., n$
This is the Euler-Lagrange equation for q_i.

In analytical mechanics we call these *Lagrange's equations* and interpret them as the equations of motion of the system:

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{q}_i} = \frac{\partial \mathcal{L}}{\partial q_i}$$

or

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{q}}} = \frac{\partial \mathcal{L}}{\partial \mathbf{q}}$$

Since $\mathcal{L}$ depends on $\mathbf{q}$ and $\dot{\mathbf{q}}$ these are n 2nd order ODEs for $q_i(t)$ which are usually nonlinear.
The LHS of Lagrange's equations can be interpeted as $\frac{dp_i}{dt}$ where $p_i = \frac{\partial \mathcal{L}}{\partial \dot{q}_i}$ is the *generalised momentum* or *conjugate momentum* to coordinate q_i.

Remark. Proof of E-L equations can be found in IB Variational Principles, so is omitted.

Is the Lagrangian unique?
No. If we add to $\mathcal{L}$ the expression

$$\frac{d}{dt}f(\mathbf{q}, t) = \frac{\partial f}{\partial t} + \frac{\partial f}{\partial \mathbf{q}} \cdot \dot{\mathbf{q}}$$

then we add to $S = \int \mathcal{L} dt$ the quantity

$$f(\mathbf{q}_2, t_2) - f(\mathbf{q}_1, t_1)$$

Which is independent of the path.
Indeed, the extra terms in Lagrange's equations cancel out. (exercise)

Example. N particles with potential forces
The standard form of $\mathcal{L}$ is the *difference* between KE and PE.

$$\begin{aligned}\mathcal{L} &= T - V \\ &= \sum_{i=1}^{N} \frac{1}{2} m_i \left|\dot{\mathbf{r}}_\mathbf{i}\right|^2 - V(\mathbf{r}_1, ..., \mathbf{r}_n)\end{aligned}$$

We identify $\mathbf{q}$ with $(\mathbf{r_1}, ..., \mathbf{r}_n)$:

$$\frac{\partial \mathcal{L}}{\partial \dot{\mathbf{r}}_i} = m_i \dot{\mathbf{r}}_\mathbf{i} \frac{\partial \mathcal{L}}{\partial \mathbf{r}_i} = -\nabla_i V$$

Lagrange's equations give

$$m_i \ddot{\mathbf{r}}_i = -\nabla_i V$$

are then exactly equivalent to Newton's Second Law.

Example. Particle in a rotating frame
If $\mathbf{r}(t)$ is the position vector measured in a frame rotating with angular velocity $\boldsymbol{\omega}(t)$ about an axis through the origin, then

$$\mathcal{L} = \frac{1}{2} m \left|\dot{\mathbf{r}} + \boldsymbol{\omega} \times \mathbf{r}\right|^2 - V$$

where the first term is the *absolute velocity*, including that due to rotation of the frame

$$= \frac{1}{2} m \dot{x}_i \dot{x}_i + m \epsilon_{ijk} \dot{x}_i \omega_j x_k + \frac{1}{2} m \epsilon_{ijk} \epsilon_{ilm} \omega_j x_k \omega_l x_m - V$$

Lagrange's equations are

$$\frac{d}{dt}\frac{\partial \mathcal{L}}{\partial \dot{x}_i} = \frac{\partial \mathcal{L}}{\partial x_i}$$

$$\frac{d}{dt}(m\dot{x}_i + m\epsilon_{ijk}\omega_j x_k) = m\epsilon_{kji}\dot{x}_k\omega_j + m\epsilon_{kji}\epsilon_{klm}\omega_j\omega_l x_m - \frac{\partial V}{\partial x_i}$$

After some manipulation, leads to

$$m(\ddot{\mathbf{r}} + 2\boldsymbol{\omega} \times \dot{\mathbf{r}} + \dot{\boldsymbol{\omega}} \times \mathbf{r} + \omega \times (\omega \times \mathbf{r})) = -\nabla V$$

Which is the equation of motion for a rotating frame, obtained without vectors.