Part II — Further Complex Methods

Based on lectures by B. Groisman

Notes taken by Joseph Tedds using Dexter Chua's header and Gilles Castel's snippets.

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Complex variable

Revision of complex variable. Analyticity of a function defined by an integral (statement and discussion only). Analytic and meromorphic continuation. Cauchy principal value of finite and infinite range improper integrals. The Hilbert transform. KramersKronig relations. Multivalued functions: definitions, branch points and cuts, integration; examples, including inverse trigonometric functions as integrals and elliptic integrals. [8]

Special functions

Gamma function: Euler integral definition; brief discussion of product formulae; Hankel representation; reflection formula; discussion of uniqueness (e.g. Wielandts theorem). Beta function: Euler integral definition; relation to the gamma function. Riemann zeta function: definition as a sum; integral representations; functional equation; *discussion of zeros and relation to p(x) and the distribution of prime numbers*. [6]

Differential equations by transform methods

Solution of differential equations by integral representation; Airy equation as an example. Solution of partial differential equations by transforms; the wave equation as an example. Causality. Nyquist stability criterion. [4]

Second order ordinary differential equations in the complex plane

Classification of singularities, exponents at a regular singular point. Nature of the solution near an isolated singularity by analytic continuation. Fuchsian differential equations. The Riemann P-function, hypergeometric functions and the hypergeometric equation, including brief discussion of monodromy. [6]

Contents

0 Introduction

Whilst the prerequisites for this course include complex analysis, this is primarily a methods course - expanding on IB complex methods.

(i) Complex variable

- Revision
- Analyticity and functions defined by integrals e.g. $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t$ domain of z? Analytic continuation?
- Departure from analyticity (singularities at a point or at a curve) : Principal Value e.g. $PV \int_{-1}^{2} \frac{1}{x}\mathrm{d}x \overset{?}{=} \log 2$

(ii) Special functions Γ, β, ζ

(iii) Integral transforms of ODE and PDE

(iv) Second order ODE on $\mathbb{C}$ (1,2,3 regular singular points), hypergeometric equations

1 Complex variable

1.1 Brief revision

$z = x + iy$

Definition (Neighbourhood)**.** A *neighbourhood* of a point $z \in \mathbb{C}$ is an *open-set* containing z.

Definition (Extendend complex plane)**.** The *extended complex plane* $\mathbb{C}^*$ or $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. All directions "$\infty$" are equivalent (think of Riemann sphere).

Definition (Differentiable)**.** A function $f(z)$ is differentiable at z if

$$f'(z) = \lim_{\Delta z \to 0} \left| \frac{f(z + \Delta z) - f(z)}{\Delta z} \right|$$

exists (independent of how $\Delta z \to 0$.

Definition (Anayltic)**.** We say that $f(z)$ is *analytic* (holomorphic / regular) at a point z if $\exists$ a neighbourhood of z throughout which f' exists. [Extensions to domain D]

Cauchy-Riemann Conditions If $f(z) = u(z) + iv(z)$, with $u, v \in \mathbb{R}$ is differentiable at z, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

The converse is true for u, v differentiable at z.

Corollary. The Wirtinger derivative

$$\overline{\partial} = \frac{\partial f}{\partial \overline{z}} = 0,$$

where $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$

Theorem (Cauchy's Theorem)**.** If $f(z)$ is analytic within and on a closed bounded contour C, then

$$\oint_C f(z) \mathrm{d}z = 0.$$

Note that C bounds D - a simply connected domain and for z_0 inside C, we have Cauchy's Integral Formula

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \mathrm{d}z.$$

With C traversed anti-clockwise.

Corollary.

$$f^{(n)}(z_0) = \frac{n!}{z_0} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}z,$$

and functions are differentiable infinitely many times.

Definition (Entire)**.** A function $f(z)$ is *entire* if it is analytic on $\mathbb{C}$ (not $\overline{\mathbb{C}}$).

This leads us to.

Theorem (Louiville's Theorem)**.** If f is entire and bounded on $\overline{\mathbb{C}}$, then f is constant.

Proof. Consider a circular disc or radius R centred at z_0 and we know $|f(z)| < M$. Then from our previous corollary,

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi i} \oint \frac{|f(z)|}{|z - z_0|^{n+1}} |\mathrm{d}z| \leq \frac{n!M}{2\pi R^{n+1}} \oint |\mathrm{d}z| \leq \frac{n!M}{R^n}.$$

$\square$