

Part II — Statistical Modelling

Based on lectures by A. J. Coca

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Introduction to the statistical programming language R

Graphical summaries of data, e.g. histograms. Matrix computations. Writing simple functions. Simulation. [2]

Linear Models

Review of least squares and linear models. Characterisation of estimated coefficients, hypothesis tests and confidence regions. Prediction intervals. Model selection. BoxCox transformation. Leverages, residuals, qq-plots, multiple \mathbb{R}^2 and Cooks distances. [5]

Overview of basic inferential techniques

Asymptotic distribution of the maximum likelihood estimator. Approximate confidence regions. Wilks theorem. The delta method. Posterior distributions and credible intervals. [3]

Exponential dispersion families and generalised linear models (glm)

Exponential families and meanvariance relationship. Dispersion parameter and generalised linear models. Canonical link function. Iterative solution of likelihood equations. Regression for binomial data; use of logit and other link functions. Poisson regression models, and their surrogate use for multinomial data. Application to 2- and 3-way contingency tables. Hypothesis tests and model selection, including deviance and Akaike's Information Criterion. Residuals and model checking. [8]

Examples in R

Linear and generalised linear models. Interpretation of models, inference and model selection. [6]

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0 Introduction

This course is unusual in that 8 of the lectures are taken as practicals, with the following guidance.

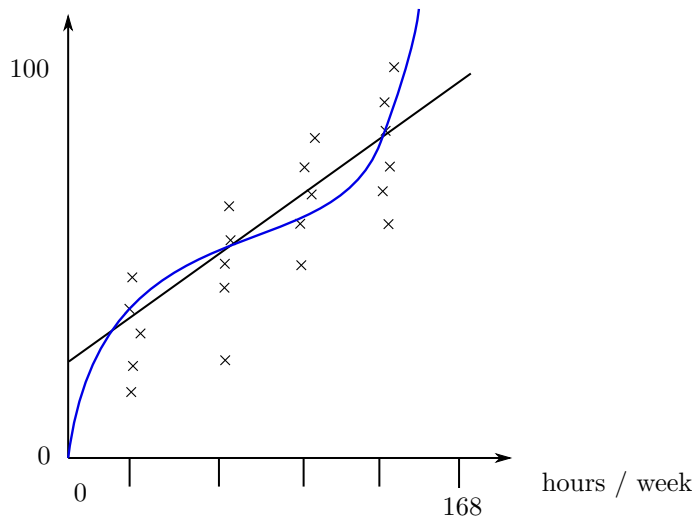
- Ideally use Linux, some things may not work on other operating systems
- Use R and R Studio

We study Data:

- $(\mathbf{x}_1, y_1), \dots, (\mathbf{x}_n, y_n), i = 1, \dots, n, n = \text{sample size.}$
- $\mathbf{x}_i = (x_{i1}, \dots, x_{ip})^T$ - predictors, covariates, independent or explanatory variables.
- y_i - targets, responses, dependent variables.

Objective: understand the functional relationship relating the y_i 's to the \mathbf{x}_i 's to develop a regression function.

Example. x_i = number of hours / week student i invests in on statistical modelling,
 y_i = final grade of student i .



In the next section we model the Y 's (note they are now upper-case) as random variables, as $Y_i = f(x_i, \theta) + \varepsilon_i$ independent.

- f is linear in θ
- $\varepsilon_i \approx$ errors / noise with potential causes as measurement errors or our limited understanding of the world.
- $\mathbb{E}[Y_i | X_i] = f(x_i, \theta) + \mathbb{E}[\varepsilon_i | x_i]$

In the sections thereafter, $\mathbb{E}[Y_i | x_i] = f_i(x_i, \theta)$, f_i is not necessarily linear in θ .

Warning. A word of caution, statistical models are not a perfect representation of the world, but they are useful approximations to make decisions.

1 Linear Models

1.1 Ordinary least squares (OLS)

Consider the linear regression model $Y = X\beta + \epsilon$,

$$Y = \begin{pmatrix} y_1 \\ \vdots \\ y_n \end{pmatrix} \in \mathbb{R}^n, X = \begin{pmatrix} x_1^T \\ \vdots \\ x_n^T \end{pmatrix} \in \mathbb{R}^{n \times p}, \beta = \begin{pmatrix} \beta_1 \\ \vdots \\ \beta_n \end{pmatrix} \in \mathbb{R}^p, \epsilon = \begin{pmatrix} \epsilon_1 \\ \vdots \\ \epsilon_n \end{pmatrix} \in \mathbb{R}^n.$$

where

- (i) $\mathbb{E}[\epsilon_i] = 0$ - unbiased
- (ii) $\text{var } \epsilon_i = 0$ - homoskedastic
- (iii) $\text{cov}(\epsilon_i, \epsilon_j) = 0$ - uncorrelated = linear independence \neq independence

Definition (Design matrix). The design matrix X , unless otherwise stated : $p \leq n$, and \mathbf{X} is full rank i.e. $\text{rank } X = p$.

Note, $\theta = \beta$ in the introduction. If we want intercept,

$$X = \begin{pmatrix} 1 & x_1^T \\ \vdots & \vdots \\ 1 & x_n^T \end{pmatrix}, \theta = \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \in \mathbb{R}^{p+1}$$

If we want higher order terms e.g. quadratic

$$X = \begin{pmatrix} 1 & x_1^T & x_{11}^2 & \cdots & x_{1p} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_n^T & x_{n1}^2 & \cdots & x_{np}^2 \end{pmatrix}, \theta = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \in \mathbb{R}^{2p+1}.$$

Remember, linear means linear in θ

Definition (Least squares). The *least squares estimator*, $\hat{\beta}$ is defined as

$$\hat{\beta} = \underset{\mathbf{b} \in \mathbb{R}^n}{\text{argmin}} \|Y - X\mathbf{b}\|^2.$$

On the example sheet, we will show that $\hat{\beta} = (X^T X^{-1})X^T Y$

The fitted values are given by

$$\hat{Y} = X\hat{\beta} = \overbrace{X(X^T X^{-1})X^T}^P Y = PY.$$

We call P the 'hat' matrix and it is an orthogonal projection onto the column space of X .

1.2 Orthogonal projection

Let $V \subseteq \mathbb{R}^n$ be linear. Its orthogonal complement is

$$V^\perp = \{\omega \in \mathbb{R}^n : \omega^T \cdot \mathbf{v} = 0 \ \forall \ \mathbf{v} \in V\}.$$

Fact. (i) $\mathbb{R} \cong V \oplus V^\perp$, so $\forall \ \mathbf{u} \in \mathbb{R}^n \ \exists \ \mathbf{v} \in V, \omega \in V^\perp$ such that $\mathbf{u} = \mathbf{v} + \omega$

(ii) $(V^\perp)^\perp = V$

Definition (Orthogonal projection). $\pi \in \mathbb{R}^{n \times n}$ is an *orthogonal projection* onto V if $\pi \mathbf{u} = \mathbf{v}$ whenever $\mathbf{u} = \mathbf{v} + \omega, \mathbf{v} \in V, \omega \in V^\perp$. π is an orthogonal projection if it is an orthogonal projection onto its column space.

$$Y = \begin{pmatrix} Y_{11} \\ \vdots \\ Y_{1n_1} \\ Y_{21} \\ \vdots \\ Y_{2n_2} \\ \vdots \\ Y_{J1} \\ \vdots \\ Y_{Jn_J} \end{pmatrix}, X = \begin{pmatrix} 1 & 0 & \cdots & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \vdots \\ 1 & 0 & \cdots & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 1 & 0 & \cdots & 0 \\ & & \vdots & & \\ 0 & \cdots & \cdots & 0 & 1 \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0 & \cdots & \cdots & 0 & 1 \end{pmatrix} \beta = \begin{pmatrix} \mu_1 \\ \mu_2 \\ \vdots \\ \mu_J \end{pmatrix}$$

2 Exponential families and generalised linear models

3 Specific regression problems