Part II — Stochastic Financial Models

Based on lectures by J. R. Norris

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Utility and mean-variance analysis

Utility functions; risk aversion and risk neutrality. Portfolio selection with the mean-variance criterion; the efficient frontier when all assets are risky and when there is one riskless asset. The capital-asset pricing model. Reservation bid and ask prices, marginal utility pricing. Simplest ideas of equilibrium and market cleaning. State-price density. [5]

Martingales

Conditional expectation, definition and basic properties. Conditional expectation, definition and basic properties. Stopping times. Martingales, supermartingales, submartingales. Use of the optional sampling theorem. [3]

Dynamic Models

Introduction to dynamic programming; optimal stopping and exercising American puts; optimal portfolio selection. [3]

Pricing contingent conditions

Lack of arbitrage in one-period models; hedging portfolios; martingale probabilities and pricing claims in the binomial model. Extension to the multi-period binomial model. Axiomatic derivation. [4]

Brownian motion

Introduction to Brownian motion; Brownian motion as a limit of random walks. Hitting-time distributions; changes of probability. [3]

Black-Scholes model

The BlackScholes formula for the price of a European call; sensitivity of price with respect to the parameters; implied volatility; pricing other claims. Binomial approximation to BlackScholes. Use of finite-difference schemes to compute prices [6]

Contents

0 Introduction

1 Utility and mean-variance analysis

1.1 Contingency claims and utility functions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable X on Ω, provides a model for an investment which delivers $X(\omega)$ for consumption depending on chance $\omega \in \Omega$.

Definition (Contingent claim)**.** In this context we often use the term *contingent claim* as another name for a random variable.

Definition (Utility function)**.** By a *utility function* we mean any non-decreasing function $U : \mathbb{R} \mapsto [-\infty, \infty)$. Think of $U(x)$ as quantifying the satisfaction obtained on consuming x. Allowing $-\infty$ is a way of saying the value of x that obtains $-\infty$ is unacceptable.

We often assume the investor will act to maximise expected utility. So Y is *preferred* to X iff $\mathbb{E}[U(X)] \leq \mathbb{E}[U(Y)]$. If $\mathbb{E}[U(X)] = \mathbb{E}[U(Y)]$ the investor is said to be *indifferent* between X and Y. We say the investor is *risk averse* if they prefer $\mathbb{E}[X]$ to X for all integrable random variables X. We say *risk neutral* if indifferent between X and $\mathbb{E}[X]$.

Definition. Recall that U is a *concave* function if for all $x, y \in \mathbb{R}$, all $p \in (0, 1)$

$$pU(x) + (1-p)U(y) \leq U(px + (1-p)y).$$

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Proposition. An investor with utility function U is risk averse if and only if U is concave.

Proof. Suppose risk averse. Consider the contingent claim X taking values x, y with probabilities $p, (1-p)$ respectively. Then,

$$pU(x) + (1-p)U(y) = \mathbb{E}[U(X)] \leq U(\mathbb{E}[X]) = U(px + (1-p)y).$$

Hence U is concave.
Suppose on the other hand U is concave. Let X be an integrable random variable (i.e. $\mathbb{E}[|X|] < \infty$) then by Jensen's inequality

$$\mathbb{E}[U(X)] \leq U(\mathbb{E}[X]).$$

Hence, the investor is risk averse. □

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Example. For $\gamma \in (0, \infty)$ the CARA (constant absolute relative aversion) utility function of parameter γ is given by

$$\text{CARA}_\gamma(x) = -e^{-\gamma x}.$$

3 For $R \in (0, 1) \cup (1, \infty)$ the CRRA (constant relative risk aversion) utility function of parameter R is given by

$$\text{CRRA}_R(x) = \begin{cases} \frac{x^{1-R}}{1-R} & x > 0 \\ -\infty & x \leq 0 \end{cases}.$$

Also,

$$\mathrm{CRRA}_1(x) = \begin{cases} \log x & x > 0 \\ -\infty & \text{otherwise} \end{cases}.$$

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Non-rigorous discussion Let U be concave (note that U is non-decreasing). Consider a small continent claim X. We ask whether we prefer $w + X$ to w for a given constant w. By Taylor's theorem

$$U(w+X) \approx U(w) + X \underbrace{U'(w)}_{>0} + \frac{1}{2}X^2 \underbrace{U''(w)}_{<0}.$$

$$\mathbb{E}\left[U(w+X)\right] \approx U(w) + \mathbb{E}\left[X\right] U'(w) + \frac{1}{2}\mathbb{E}\left[X^2\right] U''(w),$$

so we prefer $w + X$ if

$$2\frac{\mathbb{E}\left[X\right]}{\mathbb{E}\left[X^2\right]} > -\frac{U''(w)}{U'(w)}$$

the Arrow-Pratt coefficient of absolute risk aversion. For CARA_γ this constant is equal to γ.

Similarly, do we prefer $w(1 + X)$ to w? Yes if

$$2\frac{\mathbb{E}\left[X\right]}{\mathbb{E}\left[X^2\right]} > -\frac{wU''(w)}{U'(w)}$$

the Arrow-Pratt coefficient of relative risk aversion. For CRRA_R this constant is equal to R.

1.2 Reservation prices and marginal prices

Consider an investor with concave utility function. Suppose they have available a set $\mathcal{A}$ of contingent claims, and suppose $\mathbb{E}\left[U(X)\right]$ is maximised over $\mathcal{A}$ at $X^* \in \mathcal{A}$. Let Y be another contingent claim. The investor would buy Y for price π if there exists $X \in \mathcal{A}$ such that

$$\mathbb{E}\left[U(X+Y-\pi)\right] > \mathbb{E}\left[U(X^*)\right].$$

The supremum of all such prices $\pi_b(Y)$ is the *(reservation) bid price)* for Y.

The investor would sell Y for price π if there exists $X \in \mathcal{A}$ such that

$$\mathbb{E}\left[U(X-Y+\pi)\right] > \mathbb{E}\left[U(X^*)\right].$$

The infimum of all such prices $\pi_a(Y)$ is the *(reservation) ask price* for Y.

Proposition. (Ask above, bid below) Assume $\mathcal{A}$ is convex. Then $\pi_b(Y) \leq \pi_a(Y)$

Proof. It suffices to show there is no price π at which the investor will both buy and sell. Suppose for a contradiction that there exist X_a, X_b such that

$$\mathbb{E}\left[U(X_a - Y + \pi)\right] > \mathbb{E}\left[U(X^*)\right].$$

$$\mathbb{E}\left[U(X_b + Y - \pi)\right] > \mathbb{E}\left[U(X^*)\right].$$

Now $X = \frac{X_a + X_b}{2} \in \mathcal{A}$ since $\mathcal{A}$ is convex and $U(X) \geq \frac{U(X_a - Y + \pi) + U(X_b + Y - \pi)}{2}$ since U is concave. Then we obtain the following contradiction.

$$\mathbb{E}\left[U(X^*)\right] < \frac{\mathbb{E}\left[U(X_a - Y + \pi)\right] + \mathbb{E}\left[U(X_b + Y - \pi)\right]}{2} \leq \mathbb{E}\left[U(X)\right] \leq \mathbb{E}\left[U(X^*)\right].$$

Hence there is no such π. $\square$

Recall U is concave and non-decreasing. An investor has available a set of of contingent claims $\mathcal{A}$, and seeks to maximise $\mathbb{E}\left[U(X)\right], X \in \mathcal{A}$. Assume $X^* \in \mathcal{A}$ is a maximiser. Suppose Y is another contingent claim. Assume that $\mathcal{A}$ is an affine space and that U is a differentiable and strictly concave.

Definition (Affine space)**.** S is *affine* if $S - S$ is a vector space. This can be visualised as a vector space away from the origin.

Then X^* is unique (or $\frac{X_1^* + X_2^{ast}}{2}$ is better.

Definition (Marginal price)**.** We define the *marginal price* of Y as

$$\pi_m(Y) = \mathbb{E}\left[U'(X^*)Y\right] / \mathbb{E}\left[U'(X^*)\right].$$

Non-rigorous discussion to explain Note that for $\Xi \in \mathcal{A} - \mathcal{A}$ the map $t \mapsto \mathbb{E}\left[U(X^* + t\Xi\right]$ on $\mathbb{R}$ achieves its minimum at $t = 0$. So

$$0 = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_0 \mathbb{E}\left[U(X^* + t\Xi)\right] = \mathbb{E}\left[U'(X^*)\right].$$

It is plausible that there is a differentiable map $t \mapsto X^*(t) : \mathbb{R} \leftarrow \mathcal{A}$ such that for all t

$$\mathbb{E}\left[U(X^*(t) - tY + \pi_b(tY))\right] = \mathbb{E}\left[U(X^*)\right].$$

Then $X^*(0) = X^*$. Define $\Xi \in \frac{\mathrm{d}}{\mathrm{dt}}\Big|_0 X^*(t), \pi = \frac{\mathrm{d}}{\mathrm{dt}}\Big|_0 \pi_b(tY)$. It is plausible that $\Xi \in \mathcal{A} - \mathcal{A}$. So

$$0 = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_0 \mathbb{E}\left[U(X^*(t) - tY + \pi_b(tY))\right] = \mathbb{E}\left[U'(X^*)(\Xi - Y + \pi)\right].$$

So we see

$$\pi_m(Y) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_0 \pi_b(tY) = \frac{\mathrm{d}}{\mathrm{dt}}\bigg|_0 \pi_a(tY).$$

So marginal price is the price to buy (or sell) a small amount of Y.

1.3 Single period asset price model

Definition (Single period asset price model)**.** By a *single period asset price model*, we mean a pair of random variables (S_0, S_1) in $\mathbb{R}^d$. We write $S_n = (S_n^1, \ldots, S_n^d$ with S_n^i the price of asset i at time n.

Definition (Numeraire)**.** By a *numeraire* we mean a pair of random variables $(S_0^0, S_1^0$ in $(0, \infty)$.

Notation. We write

$$\overline{S}_n = (S_n^0, S_n) = (S_n^0, S_n^1, \ldots, S_n^{d)}.$$

Call $(\overline{S}_0, \overline{S}_1)$ an *asset price model with numeraire*

Often we take $S_0^0 = 1, S_1^0 = 1 + r$ some constant $r \in (-1, \infty)$, Then S^0 is called a *riskless bond* and r is the *interest rate*. We assume $\overline{S}_0$ is non-random as the default.

In the case without numeraire, an investor with initial wealth w_0 chooses $\theta \in \mathbb{R}^d$ subject to

$$\theta.S_0 = \sum_{i=1}^{d} \theta^i S_0^i = w_0.$$

Then the investor has wealth $\theta.S_1$ at time 1. We call θ the *portfolio*. With numeraire, investor chooses $\overline{\theta} = (\theta^0, \theta$ such that $\overline{\theta}.\overline{S}_0 = w_0$. The wealth at time 1 is $\overline{\theta}.\overline{S}_0$.

It may be that there exists a random variable $\rho \geq 0$ such that $\mathbb{E}\left[\rho S_1^i\right] = S_0^i$ for all i. Then we call ρ a *state price density*

1.4 Portfolio selection using the mean-variance criterion

Let (S_0, S_1) be an asset price model on $\mathbb{R}^d$ with S_0 non-random, S_1 has mean μ, variance V. We assume that V is invertible and S_0, μ are linearly independent. Suppose we are given w_0, w_1. The investor wishes to

$$\begin{aligned} \text{minimise} \quad & \operatorname{var}(\theta.S_1) \\ \text{subject to} \quad & \theta.S_0 = w_0, \\ & \mathbb{E}\left[\theta.S_1\right] = w_1 \end{aligned}$$

Note $\mathbb{E}\left[\theta.S_1\right] = \theta.\mu$, $\operatorname{var}(\theta.S_1) = \theta.(V\theta)$ So our problem is to

$$\begin{aligned} \text{minimise} \quad & \theta.(V\theta) \\ \text{subject to} \quad & \theta.S_0 = w_0, \\ & \theta.\mu = w_1. \end{aligned}$$

Consider $L(\theta, \lambda) = \frac{1}{2}\theta.(V\theta) - \lambda_0 \theta.S_0 - \lambda_1 \theta.\mu$ At minimising θ^*.

$$\begin{aligned} 0 &= \frac{\partial}{\partial \theta^i} L(\theta, \lambda) \\ &= (V\theta)^i - \lambda_0 S_0^i - \lambda_1 \mu^i. \end{aligned}$$

So $\theta^* = \lambda_0 A S_0 + \lambda_1 A \mu, A = V^{-1}$. Now fit the constants

$$\begin{aligned} w_0 &= \theta^*.S_0 = \lambda_0 a + \lambda_1 b \\ w_1 &= \theta^*.\mu = \lambda_0 b + \lambda_1 c \end{aligned}$$

$a = S_0.(AS_0), b = \mu(AS_0) = S_0(A\mu), c = \mu(A\mu)$. Note that $\Delta = ac - b^2 \neq 0$ by linear independence

$$\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = M \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix}, M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

$$M^{-1} = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -b & a \end{pmatrix}.$$

$$\begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = M^{-1} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}.$$

So

$$\theta^* = \frac{cw_0 - bw_1}{\Delta} AS_0 + \frac{aw_1 - bw_0}{\Delta} A\mu$$

The minimising variance is

$$\begin{aligned}\theta^*_{.}(V\theta^*) &= (\lambda_0 AS_1 + \lambda_1 A\mu)_{.}(\lambda_0 S_0 + \lambda_1 \mu) \\ &= (\lambda_0 \lambda_1) M \begin{pmatrix}\lambda_0 \\ \lambda_1\end{pmatrix} \\ &= (w_0 w_1) M^{-1} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix} \\ &= \frac{cw_0^2 - 2bw_0w_1 + aw_1^2}{\Delta} = q(w_1)\end{aligned}$$

We minimise this over w_1

$$w_1^* = \frac{b}{a} w_0, \theta^*_{\text{min}} = \frac{w_0}{a} AS_0.$$

Putting w_1^* back into q, we obtain

$$q(w_1^*) = \frac{acw_0^2 - 2b^2w_0^2}{a\Delta} + \frac{b^2}{a\Delta} w_0^2 = \frac{w_0^2}{a}$$

Suppose we seek to

$$\begin{aligned} \text{minimise} \quad & \operatorname{var}(\theta.S_1) \\ \text{subject to} \quad & \theta.S_0 = w_0, \end{aligned}$$

Consider $L(\theta, \lambda) = \frac{1}{2}\theta.(V\theta) - \lambda\theta.S_0$. At minimiser,

$$0 = \frac{\partial}{\partial \theta^i} L(\theta, \lambda) = (V\theta)^i - \lambda S_0^i.$$

So

$$V\theta^* = \lambda S_0, \quad \theta^* = \lambda AS_0, A = V^{-1}.$$

Use the constraint to find $\lambda : w_0 = \theta^*_{.} S_0 = \lambda \underbrace{a}_{S_0.}(AS_0)$. Hence $\theta^* = \frac{w_0}{a} AS_0 = \theta^*_{\text{min}}$.

Add a riskless bond / bank account.

$$S^0 = 1, S_1^0 = 1 + r > 0.$$

Suppose we seek to

$$\begin{aligned} \text{minimise} \quad & \operatorname{var}(\overline{\theta}.\overline{S}_1) \\ \text{subject to} \quad & \overline{\theta}.\overline{S}_0 = w0 \\ & \mathbb{E}\left[\overline{\theta}.\overline{S}_1\right] = w_1 \end{aligned}$$

Recalling that $\overline{\theta} = (\theta^0, \theta), \overline{S}_n = (S_n^0, S_n)$. Now $\text{var}(\overline{\theta}.\overline{S}_1 = \theta.(V\theta)$. $\mathbb{E}\left[\overline{\theta}.\overline{S}_1\right] = \theta^0(1+r) + \theta.\mu$. So our problem is to

$$\begin{aligned} \text{minimise} \quad & \theta.(V\theta),\ V \text{ invertible} \\ \text{subject to} \quad & \theta^0 + \theta.S_0 = w_0 \quad (1) \\ & \theta^0(1+r) + \theta.\mu = w_1 \quad (2) \end{aligned}$$

Use (1) to eliminate θ^0 in (2).

$$(w_0 - \theta.S_0)(1+r) + \theta.\mu = w_1.$$

i.e.

$$\theta.(\mu - (1+r)S_0) = w_1 - (1+r)w_0.$$

Set $L(\theta, \lambda) = \frac{1}{\theta.(V\theta) - \lambda\theta.(\mu - (1+r)S_0)}$. At θ^*,

$$0 = \frac{\partial}{\partial \theta^i} L(\theta, \lambda) = (V\theta)^i - \lambda(\mu^i - (1+r)S_0^i).$$

So

$$\theta^* = \lambda \underbrace{(A\mu - (1+r)S_0)}_{\theta^*_m = \theta^*_{\text{market}}}, \quad A = V^{-1}.$$

Find λ using the remaining constraint

$$\lambda \underbrace{(c - 2b(1+r) + (1+r)^2 a)}_{>0 \text{ by Cauchy Schwarz}} = w_1 - (1+r)w_0,$$

where

$$a = S_0.(AS_0), b = \mu(AS_0) = S_0(A\mu), c = \mu(A\mu)$$

as before. So

$$\lambda = \frac{w_1 - (1+r)w_0}{(1+r)^2 a - 2b(1+r) + c}.$$

1.5 Portfolio selection using CARA utility

Take as utility function

$$U(x) = \text{CARA}_\gamma(x) = -e^{-\gamma x} \quad \gamma \in (0, \infty).$$

The investor has available the following set of contingent claims.

$$\mathcal{A} = \{\theta.S_1 : \theta.S_0 = w_0\}.$$

Suppose we seek to

$$\begin{aligned} \text{maximise} \quad & \mathbb{E}\left[U(\theta.S_1)\right] \\ \text{subject to} \quad & \theta.S_0 = w0 \end{aligned}$$

Here, S_1 has mean μ, variance V (invertible) and S_1 is Gaussian.

<u>aside</u>

$$\mathbb{E}\left[\theta.S_1\right] = \theta\mu.$$

$$\mathrm{var}(\theta.S_1) = \theta.(V\theta).$$

$\theta.S_1$ is also Gaussian. $Z \sim N(0,1), \mathbb{E}\left[e^{\lambda z}\right] = e^{-\frac{\lambda^2}{2}}$

$$\begin{aligned} \mathbb{E}\left[e^{\lambda z}\right] &= \int_{\mathbb{R}} e^{\lambda z} \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}} dz \\ &= e^{\frac{\lambda^2}{2}} \int_R \frac{1}{\sqrt{2\pi}} e^{-\frac{(z-\lambda)^2}{2}} \end{aligned}$$

.

Note

$$\begin{aligned} \mathbb{E}\left[U(\theta.S_1)\right] &= -\mathbb{E}\left[e^{-\gamma\theta.S_1}\right] \\ &= -e^{-\gamma\theta.+\frac{1}{2}\gamma^2\theta.(Vg\theta)} \end{aligned}$$

.

So our problem is to Suppose we seek to

$$\begin{array}{ll} \text{maximise} & \mathbb{E}\left[U(\theta.S_1)\right] \\ \text{subject to} & \theta.S_0 = w_0 \end{array}$$

Consider $L(\theta,\lambda) = \theta.\mu - \frac{1}{2}\gamma\theta.(V\theta) - \lambda\theta.S_0$ At maximiser θ^*

$$0 = \frac{\partial}{\partial\theta^i} L(\theta,\lambda) = \mu^i - \gamma(V\theta)^i - \lambda S_0^i.$$

So

$$\theta^* = \gamma^{-1}(A\mu - \lambda A S_0).$$

Find λ by

$$w_0 = \theta^*_. S_0 = \gamma^{-1}(b - \lambda a).$$

So $\lambda w_0 = b - \lambda a$. So $\lambda = \frac{b-\gamma w_0}{a}$. So $\theta^* = \underbrace{\frac{w_0}{a} A S_0}_{\theta^*_{\min}} + \gamma^{-1}(A\mu - \frac{b}{a} A S_0)$.

Add riskless bond $S_0^0 = 1, S_1^0 = 1 + r > 0$

$$\overline{\theta}.\overline{S}_0 = \theta^0 + \theta.S_0, \overline{\theta}.\overline{S}_1 = \theta^0(1+r) + \theta.S_1.$$

So

$$\mathbb{E}\left[U(\overline{\theta}.\overline{S}_1\right] = -e^{-\gamma(\theta\mu+\theta^0(1+r))+\frac{1}{2}\gamma^2\theta.(V\theta)}.$$

$$\begin{array}{rl} \text{with constraint} & \theta.S_0 = w_0 - \theta^0 \\ \text{maximise} & \theta.\mu + \theta^0(1+r) - \dfrac{1}{2}\gamma\theta.(V\theta). \end{array}$$

Using our constraint to eliminate θ^0

$$\theta.\mu + (w_0 - \theta.S_0)(1+r) - \frac{1}{2}\gamma\theta.(V\theta).$$

Maximising θ^* satisfies

$$\mu - (1+r)S_0 = \gamma V\theta^*.$$

So

$$\theta^* = \gamma^{-1} \underbrace{(A\mu - (1+r)AS_0)}_{\theta_{\mathrm{m}}^{ast} = \theta^*_{\mathrm{market}}}.$$

$\gamma >> 1$ means we are highly risk averse.

Critique

- Easy to estimate V, but it is hard to estimate μ 5
- Why do we assume the stock prices are Gaussian? We use Centra Limit Theorem as we can consider them as the sum of random variables, but this relies on variance conditions.
- We've allowed negative asset values, consider $S_1 \sim N(\mu, V)$. More realistically,

  $$S_0 = e^{s_0}, S_1 = e^{s_0 + \varepsilon Z} = S_0 e^{\varepsilon Z} \approx S_0(1 + \varepsilon Z).$$

  $Z \sim N(\mu, V), \varepsilon$ small.

1.6 Capital-asset pricing model

We have seen $\theta^*_{\mathrm{m}} = A\mu + (1+r)AS_0$ appear twice. Suppose we assume that the market optimises itself. Then, we should be able to observe θ^*_{m}

$$\theta^{*i}_{\mathrm{m}} = \text{ the number of shares of asset } i.$$

$$\theta^{*i}_{\mathrm{m}} S^i_n = \text{ capitalization of asset } i.$$

Notation. Set $S^m_n = \theta^*_{\mathrm{m}}.S_n, n = 0, 1,\ \mu^{\mathrm{m}} = \theta^*_{\mathrm{m}}.\mu$ Define

$$\beta^i = \frac{\mathrm{cov}(S^i_1, S^{\mathrm{m}}_1)}{\mathrm{var}\, S^{\mu}_1}$$

the *beta* or *sensitivity* something we can estimate.

Proposition. For $i = 1, \ldots, d$

$$\mu^i = (1+r)S^i_o = \beta^i(\mu^m - (1+r)S^m_0).$$

Proof. For $\theta = A\mu - (1+r)AS_0$, then

$$\mu^m - (1+r)S^m_0 = \theta.(\mu - (1+r)S_0) = \theta.(V\theta) = \mathrm{var}(\theta.S_1) = \mathrm{var}\, S^{\mu}_1.$$

So

$$\begin{aligned}
\mu^i - (1+r)S^i_0 &= e_i.(\mu - (1+r)S_0) \\
&= e_i.(V\theta) \\
&= \mathrm{cov}(S^i_1, S^{\mathrm{m}}_1 \\
&= \beta^i(\mu^{\mathrm{m}} - (1+r)S^{\mathrm{m}}_0)
\end{aligned}$$

.

□

This appears to identify μ^i from the market. Often this pricing formula is written in terms of the returns. Define R^i, R^{m} by $S_1^0 = (1+r)S_0^0$, $S_1^i = (1+R^i)S_0^i, S_1^{\mathrm{m}} = (1+R^m)S_0^m$ Then

$$\mu^i = (1 + \mathbb{E}\left[R^i\right])S_0^i.$$

$$\mu^{\mathrm{m}} = (1 + \mathbb{E}\left[R^{\mathrm{m}}\right])S_0^{\mathrm{m}}.$$

$$\operatorname{var} S_1^{\mathrm{m}} = (S_0^{\mathrm{m}})^2 \operatorname{var}(R^{\mathrm{m}}).$$

$$\operatorname{cov}(S_1^i, S_1^{\mathrm{m}}) = S_0^i S_0^{\mathrm{m}} \operatorname{cov}(R^i, R^{\mathrm{m}}) = \frac{S_0^i S_0^m \operatorname{cov}(R^i, R^{\mathrm{m}})}{(S_0^{\mathrm{m}})^2 \operatorname{var}(R^{\mathrm{m}})}((1+\mathbb{E}\left[R^{\mathrm{m}}\right])S_0^m - (1+r)S_0^{\mathrm{m}}).$$

So

$$\mathbb{E}\left[R^i\right] - r = \hat{\beta}^i(\mathbb{E}\left[R^{\mathrm{m}} - r\right].$$

2 Martingales

2.1 Conditional probabilities and expectations

$(\Omega, \mathcal{F}, \mathbb{P})$, is a probability space. Recall for an event $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$ we define $\mathbb{P}(\cdot \mid B)$

$$\mathbb{P}(A \mid B) = \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}, A \in \mathcal{F}.$$

Then $\mathbb{P}(\cdot \mid B)$ has associated expectation written $\mathbb{E}[\cdot \mid B]$. This satisfies for X a random variable

$$\mathbb{E}[X \mid B] = \frac{\mathbb{E}[X\not{\Vdash}_B]}{\mathbb{P}(B)}.$$

We will need a more general notions of conditional probabilities and expectations associated not with a single event B, but with a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$.

Definition (σ-algebra)**.** We say $\mathcal{G}$ is a *σ-algebra* if

(i) $\emptyset \in \mathcal{G}$

(ii) $A \in \mathcal{G} \implies A^c \in \mathcal{G}$

(iii) $(A_n : n \in \mathbb{N}) \in \mathcal{G} \implies \bigcup_n A \in \mathcal{G}$

Definition (Integrable)**.** We say a random variable X is *integrable* if

$$\mathbb{E}[|X|] < \infty.$$

Definition ($\mathcal{F}$-measurable)**.** X is *$\mathcal{F}$-measurable* if $\{\omega \in \Omega : X(\omega) \leq x\} \in \mathcal{F}$

Theorem. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and let X be an integrable / non-negative random variable. Then there exists an integrable / non-negative random variable Y satisfying

(i) Y is $\mathcal{G}$-measurable

(ii) $\mathbb{E}[Y1_A] = \mathbb{E}[X1_A]\,,\ \forall\ A \in \mathcal{G}$.

Moreover, if Y' is any integrable / non-negative random variable satisfying (i) and (ii) then $Y' = Y$ almost surely. We call Y (a version of) the *conditional expectation* of X given $\mathcal{G}$ and write

$$Y = \mathbb{E}[X|\mathcal{G}] \text{ a.s. .}$$

If $\mathcal{G} = \sigma(Z)$ for some random variable Z, we write $Y = \mathbb{E}[X|Z]$ a.s. If $X = 1_A$ we write $Y = \mathbb{P}(A|\mathcal{G})$ a.s.

Proof. Monotonicity and uniqueness argument.
Let X' be another integrable random variable with $X \leq X'$ (pointwise greater) and suppose that Y' is an integrable random variable which satisfies (i) and (ii) with respect to X'. Set $A = \{Y \geq Y'\}$ and consider the non-negative random variable

$$Z = (Y - Y')1_A.$$

Since $A \in \mathcal{G}$

$$\mathbb{E}[Y1_A] = \mathbb{E}[X1_A] \leq \mathbb{E}[X'1_A] = \mathbb{E}[Y'1_A]\,.$$

So $\mathbb{E}[Z] \leq 0$ and so $Z = 0$ a.s. Hence, $Y \leq Y'$ a.s.
In the case that $X = X'$ a.s. we would also have $Y' \leq Y$ a.s. so $Y = Y'$ a.s.
We will omit the existence proof. ☐

Existence for $\mathcal{G} = \{\bigcup_{n \in I} B_n : I \subseteq \mathbb{N}\}$ with $(B_n : n \in \mathbb{N})$ a partition of Ω by events. Given an integrable random variable X set

$$Y = \sum_n \mathbb{E}\left[X | B_n\right] 1_{B_n},$$

where we set $\mathbb{E}\left[X|B_n\right] = 0$ if $\mathbb{P}\left(B_n\right) = 0$. Since the B_n are disjoint, only one term is non-zero so we need not worry about convergence. Note

$$|Y| = \sum_n |\mathbb{E}\left[X|B_n\right]| 1_{B_n}$$

so by monotone convergence,

$$\begin{aligned}
\mathbb{E}\left[|Y|\right] &\overset{m}{=} \sum_n \mathbb{E}\left[|\mathbb{E}\left[X|B_n\right]| 1_{B_n}\right] \\
&= \sum_n |\mathbb{E}\left[X|B_n\right]| \mathbb{P}\left(B_n\right) \\
&\geq \sum_n \mathbb{E}\left[|X||B_n\right] \mathbb{P}\left(B_n\right) \overset{m}{=} \mathbb{E}\left[|X|\right] < \infty.
\end{aligned}$$

Hence Y is integrable and $\mathcal{G}$-measurable.

Theorem (Monotone convergence theorem)**.** Let $(X_n : n \in N)$ be a sequence of non-negative random variables. Then

$$\mathbb{E}\left[\sum_n X_n\right] = \sum_n \mathbb{E}\left[X_n\right].$$

Also for $I \subseteq \mathbb{N}$, by the dominated convergence theorem,

$$\begin{aligned}
\mathbb{E}\left[Y 1_{\bigcup_{n \in I} B_n}\right] &= \mathbb{E}\left[\sum_{n \in I} Y 1_{B_n}\right] \\
&\overset{D}{\underset{Y}{=}} \sum_{n \in I} \mathbb{E}\left[Y 1_{B_n}\right] \\
&= \sum_{n \in I} \mathbb{E}\left[\mathbb{E}\left[X|B_n\right] 1_{B_n}\right] \\
&= \sum_{n \in I} \mathbb{E}\left[X|B_n\right] \mathbb{P}\left(B_n\right) \\
&= \sum_{n \in I} \mathbb{E}\left[X 1_{B_n}\right] \\
&\overset{D}{\underset{X}{=}} \mathbb{E}\left[X 1_{\bigcup_{n \in I} B_n}\right].
\end{aligned}$$

So Y also satisfies (ii) so $Y = \mathbb{E}\left[X|\mathcal{G}\right]$

Theorem (Dominated convergence thorem)**.** Let $(X_n : n \in \mathbb{N})$ be a sequence of random variables. Suppose $\sum_n |X_n| \leq Z$ for some integrable random variable Z. Then $\sum_n X_n$ is integrable

$$\mathbb{E}\left[\sum_n X_n\right] = \sum_n \mathbb{E}\left[X_n\right].$$

Proposition. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and let X and W be integrable random variables. Then

(i) $\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{G}\right]\right] = \mathbb{E}\left[X\right]$

(ii) If X is $\mathcal{G}$-measurable then $\mathbb{E}\left[X|\mathcal{G}\right] = X$ a.s.

(iii) If X is independent of $\mathcal{G}$ then $\mathbb{E}\left[X|\mathcal{G}\right] = \mathbb{E}\left[X\right]$ a.s.

(iv) If $X \geq 0$ a.s. , then $\mathbb{E}\left[X|\mathcal{G}\right] \geq 0$ a.s

(v) $\mathbb{E}\left[\alpha X + \beta W|\mathcal{G}\right] = \alpha\mathbb{E}\left[X|\mathcal{G}\right] + \beta\mathbb{E}\left[W|\mathcal{G}\right]$

Proof. (i) Take $A = \Omega$ in (ii)

(ii) Check $Y = X$ works

(iii) (Exercise) $\mathbb{E}\left[X1_A\right] = \mathbb{E}\left[X\right]\mathbb{P}\left(A\right)$, so $Y = \mathbb{E}\left[X\right]$ works.

(iv)

(v) Let Y_1 be a version of $\mathbb{E}\left[X|\mathcal{G}\right]$ and let Y_2 be a version of $\mathbb{E}\left[W|\mathcal{G}\right]$. Set $Y = \alpha Y_1 + \beta Y_2$. Then Y is integrable and $\mathcal{G}$-measurable, and for all $A \in \mathcal{G}$

$$\mathbb{E}\left[Y1_A\right] = \alpha\mathbb{E}\left[Y_1 1_A\right] + \beta\mathbb{E}\left[Y_2 1_A\right] = \alpha\mathbb{E}\left[X1_A\right] + \beta\mathbb{E}\left[W1_A\right] = \mathbb{E}\left[(\alpha X + \beta W)1_A\right].$$

Hence $Y = \mathbb{E}\left[\alpha X + \beta W|\mathcal{G}\right]$ a.s.

□

Proposition. (Tower property)
Let $\mathcal{G}$ and $\mathcal{H}$ be sub-σ-algebras of $\mathcal{F}$ with $\mathcal{H} \subseteq \mathcal{G}$ and let X be an integrable random variable. Then,

$$\mathbb{E}\left[\mathbb{E}\left[X|\mathcal{G}\right]|\mathcal{H}\right] = \mathbb{E}\left[X|\mathcal{H}\right] \text{ a.s. }.$$

This can be visualised as the orthogonal projection onto a subspace $G \subseteq H$. If $X \subset L^2(\mathbb{P}), (\mathbb{E}\left[|X|^2\right] < \infty)$ then $\mathbb{E}\left[X|\mathcal{G}\right] \in L^2(\mathbb{P})$. $X \mapsto \mathbb{E}\left[X|\mathcal{G}\right]$ is an orthogonal projection $L^2(\mathcal{F}, \mathbb{P}) \to L^2(\mathcal{G}, \mathbb{P})$

Proof. Choose a version of $Y = \mathbb{E}\left[\mathbb{E}\left[X|\mathcal{G}\right]\mathcal{H}\right]$. Then Y is integrable, $\mathcal{H}$-measurable and for all $A \in \mathcal{H}$

$$\mathbb{E}\left[Y1_A\right] \underset{A\in\mathcal{H}}{=} \mathbb{E}\left[\mathbb{E}\left[X|\mathcal{G}\right]1_A\right] \underset{A\in\mathcal{G}}{=} \mathbb{E}\left[X1_A\right].$$

Hence $Y = \mathbb{E}\left[X|\mathcal{H}\right]$ a.s. □

Proposition. (Taking out what is known)
Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$ and let X be an integrable random variable. Suppose that Z is $\mathcal{G}$-measurable and ZX is integrable. Then

$$\mathbb{E}\left[ZX|\mathcal{G}\right] = Z\mathbb{E}\left[X|\mathcal{G}\right].$$

Proof. Suppose for now $X \geq 0$. Choose a version of $Y \geq 0$ of $\mathbb{E}[X|\mathcal{G}]$. Consider first the case $Z = 1_B$ for some $B \in \mathcal{G}$. Then for all $A \in \mathcal{G}$

$$\mathbb{E}[ZY1_A] = \mathbb{E}[Y1_{A\cap B}] = \mathbb{E}[X1_{A\cap B}] = \mathbb{E}[ZX1_A].$$

This identity

$$\mathbb{E}[ZY1_A] = \mathbb{E}[ZX1_A]$$

extends to a simple $\mathcal{G}$-measurable Z ($Z = \sum_{i=1}^n a_i 1_{B_i}, B_i \in \mathcal{G}$) by linearity. Now for $Z \geq 0$ consider the $\mathcal{G}$-measurable sets $Z_n = (2^{-n} \lfloor 2^n Z \rfloor) \wedge \wedge$. Then Z_n is simple and Z_n monotonically converges Z as $n \to \infty$. Have

$$\mathbb{E}[Z_nY1_A] = \mathbb{E}[Z_nX1_A] \;\; \forall \; A \in \mathcal{G},$$

so by monotonic convergence we get $\mathbb{E}[ZY1_A] = \mathbb{E}[ZX1_A]$. For Z integrable, set $Z^{\pm} = (\pm Z) \wedge 0$. Then $Z = Z^+ - Z^-$ and for all $A \in \mathcal{G}$

$$\mathbb{E}\left[Z^{\pm}Y1_A\right] = \mathbb{E}\left[Z^{\pm}X1_A\right].$$

Subtract to see $\mathbb{E}[ZY1_A] = \mathbb{E}[ZX1_A]$. Hence $ZY = \mathbb{E}[ZX|\mathcal{G}]$ a.s. □

Proposition. Let X_1, X_2 be random variables in $(E_1, \mathcal{E}_1), (E_2, \mathcal{E}_2)$ respectively. Let $\mathcal{G}$ be a sub-σ-algebra of $\mathcal{F}$. Suppose X_1 is $\mathcal{G}$-meausrable, X_2 is independent of $\mathcal{G}$. Let F be a non-negative measurable function on $E_1 \times E_2$. We can define a non-negative measurable function f on E_1 by

$$f(x) = \mathbb{E}[F(x, X_2)]$$

and then we have

$$\mathbb{E}[F(X_1, X_2)|\mathcal{G}] = f(X_1) \text{ a.s. }.$$

Proof. For $F = 1_{B_1 \times B_2}, B_k \in \mathcal{E}_k$ check $F = 1_B$ $B \in \mathcal{E}_1 \otimes \mathcal{E}_2$ by Dynkin's lemma F is simple so $F \geq 0$ measurable monotonic convergence □

2.2 Definitions

Let $(\Omega, \mathcal{F})$ be a measurable space.

Definition (Filtration)**.** We say that $(\mathcal{F}_n)_{n\geq 0}$ is a *filtration* if $\mathcal{F}_n$ is a σ-algebra on Ω and $\mathcal{F}_n \subseteq \mathcal{F}_{n+1} \subseteq \mathcal{F}$ for all n.

Definition (Random process)**.** We say that $(X_n)_{n\geq 0}$ is a *random process* if X_n is a random variable for all n.

Definition (Adapted)**.** n. We say $(X_n)_{n\geq 0}$ is *adapted* (to $(\mathcal{F}_n)_{n\geq 0}$) if X_n is $(\mathcal{F})_n$-measurable for all n.

Definition (Natural filtration)**.** Given a process $(X_n)_{n\geq 0}$ define $(F_n^X)_{n\geq 0}$ by $\mathcal{F}_n^X = \sigma(X_k : 0 \leq k \leq n)$. We call $(F_n^X)_{n\geq 0}$ the *natural filtration* of $(X_n)_{n\geq 0}$. Filtration gives us some history, so the natural filtration of X gives us the history of X.

Definition (Martingale)**.** Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space equipped with a filtration $(\mathcal{F}_n)_{n\geq 0}$. We say that a random process $(X_n)_{n\geq 0}$ is a *martingale* if

(i) X_n is $\mathcal{F})_n$-measurable for all n.

(ii) $\mathbb{E}[|X_n|] < \infty$ for all n.

(iii) $\mathbb{E}[X_{n+1}|\mathcal{F}_n] = X_n$ a.s. for all n

If (ii) holds, we say $(X_n)_{n\geq 0}$ is integrable. Condition (iii) is called the *martingale property.* So $(X_n)_{n\geq 0}$ is a *martingale* if it is adapted, integrable and satisfies the martingale property.

Remark. – The martingale property is equivalent to

$$\mathbb{E}[(M_{n+1} - M_n)1_A] = 0 \;\forall\; A \in \mathcal{F}_n.$$

– If we take expectations of the martingale property, we get that $\mathbb{E}[M_{n+1}] = \mathbb{E}[M_n]$ so the expected value is constant.

Definition (Supermartingale)**.** If (i) and (ii) hold and also

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] \leq X_n \text{ a.s. },$$

then we say that $(X_n)_{n\geq 0}$ is a *supermartingale*

Definition (submartingale)**.** If (i) and (ii) hold and also

$$\mathbb{E}[X_{n+1}|\mathcal{F}_n] \geq X_n \text{ a.s. },$$

then we say that $(X_n)_{n\geq 0}$ is a *submartingale*

2.3 Examples

Let $(X_n)_{n\geq 1}$ be a sequence of iid random variables. Set $\mathcal{F}_0 = \{\emptyset, \Omega\}, \mathcal{F}_n = \sigma(X_1, \dots, X_n)$. Set $S_0 = 0, Z_0 = 1$ and define

$$S_n = X_1 + \cdots + X_n, \quad Z_n = \prod_{i=1}^{n} X_i \; n \geq 1.$$

In the case that X_1 is integrable and $\mathbb{E}[X_1] = 0$ the process $(S_n)_{n\geq 0}$ is a martingale - called an additive martingale. In the case that $X_1 \geq 0$ and $\mathbb{E}[X_1] = 1$ the process $(Z_n)_{n\geq 0}$ is a martingale.
Adapted: X_i is $\mathcal{F})_n$-measurable if for all $i \leq n$ so S_n and Z_n are $\mathcal{F}_n$-measurable.
Integrable: Use $|S_n| \leq |X_1| + \cdots + |X_n|$ then $\mathbb{E}[|S_n|] \leq n\mathbb{E}[|X_1|] < \infty$.

$$0 \leq Z_n = \prod_{i=1}^{n} X_i \text{ so } \mathbb{E}[Z_n] = \prod_{i=1}^{n} \mathbb{E}[X_1] = 1 < \infty.$$

Martingale property:

$$\begin{aligned}\mathbb{E}[S_{n+1}|\mathcal{F}_n] &= \mathbb{E}[X_{n+1} + S_n|\mathcal{F}_n] \\ &= \mathbb{E}[X_{n+1}|\mathcal{F}_n] + \mathbb{E}[S_n|\mathcal{F}_n] \\ &= \mathbb{E}[X_{n+1}] + S_n \\ &= S_n..\end{aligned}$$

$$\begin{aligned}\mathbb{E}\left[Z_{n+1}|\mathcal{F}_n\right] = \mathbb{E}\left[X_{n+1}Z_n|\mathcal{F}_n\right] \\ &= Z_n\mathbb{E}\left[X_{n+1}|F_n\right] \\ &= Z_n\mathbb{E}\left[X_{n+1}\right] = Z_n \text{ a.s..}\end{aligned}$$

Example. Let $(X_n)_{n\geq 0}$ be a Markov chain with countable state-space S and transition matrix P with $\lambda_x = \mathbb{P}(X_0 = x)$. Set $\mathcal{F} = \sigma(X_0, \dots, X_n)$. Define for f bounded or non-negative (on S)

$$Pf(x) = \sum_{y\in S} p_{xy}f(y) = \mathbb{E}_x\left[f(X_1)\right].$$

Fix $x_0, \dots, x_n \in S$ and set $A = \{X_0 = x_0, \dots, X_n = x_n\}$. Then,

$$\begin{aligned}\mathbb{E}\left[f(X_{n+1})\right]1_A &= \sum_{y\in S}\lambda_{x_0}p_{x_0x_1}\cdot p_{x_{n-1}x_n}p_{x_ny}f(y) \\ &= pf(x_n)\mathbb{P}(A) \\ &= \mathbb{E}\left[pf(X_n)1_A\right].\end{aligned}$$

So $\mathbb{E}\left[f(X_{n+1}1_A =\right] = \mathbb{E}\left[pf(X_n)1_A\right]$ for all $A \in \mathcal{F}_n$. We've shown that

$$\mathbb{E}\left[f(X_{n+1})|\mathcal{F}_n\right] = pf(X_n).$$

We say that f is subharmonic if $f(x) \leq Pf(x)\ \forall\ x \in S$. Suppose that f is subharmonic and set $M_n = f(X_n)$. Then,

$$\mathbb{E}\left[M_{n+1}|\mathcal{F}_n\right] = pf(X_n) \text{ a.s } \geq f(X_n) = M_n.$$

So $(M_n)_{n\geq 0}$ is a submartingale.

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Example. Continuing with our Markov chain theme, take $A, B \subseteq S$ disjoint. Set $T = \inf\{n \geq 0 : X_n \in A \cup B\}$ and define

$$u(x) = \mathbb{P}_x\left(T < \infty, X_T \in A\right).$$

Then $M_n = U(X_{T\wedge n}$ is a martingale (E_x)

2.4 Optional stopping

Definition (Stopping time)**.** We say that a random time $T : \Omega \to \mathbb{Z}^+ \cup \{\infty\}$ is a *stopping time* if $\{T \leq n\} \in \mathcal{F}_n$ for all n. Equivalently, $\{T = n\} \in \mathcal{F}_n$ for all n.

For a Markov chain, for $A \subseteq S$

$$T_a = \inf\{n \geq 0 : X_n \in A\},\ L_A = \sup\{n \geq 0 : X_n \in A\},$$

T_A is a stopping time, but in general L_A is not.

Theorem (Optional stopping time)**.** Let $(M_n)_{n\geq 0}$ be a martingale and let T be a bounded stopping time. Then $\mathbb{E}\left[M_T\right] = \mathbb{E}\left[M_0\right]$.

Proof. Choose n such that $T \leq n$. Note that

$$M_T = M_0 + (M_1 - M_0) + \cdots + (M_T - M_{T-1}) = M_0 + \sum_{k=1}^{n} 1_{\{k \leq T\}}(M_k - M_{k-1}).$$

Since T is a stopping time $\{k \leq T\} = \{T \leq k-1\}^c \in \mathcal{F}_{k-1}$. Since $(M_n)_{n\geq 0}$ is a martingale

$$\mathbb{E}\left[1_{\{k \leq T\}}(M_k - M_{k-1})\right] 0.$$

So by taking expectations of our sum, we obtain the desired result. □

Theorem. Let $(M_n)_{n\geq 0}$ be a martingale and let T be a stopping time. Suppose that there is a constant $C < \infty$ such that at least one of (i) and (ii) holds

(i) $\mathbb{P}(T < \infty) = 1$ and $|M_n| \leq C$ a.s. for all n.

(ii) $\mathbb{E}[T] < \infty$ and $|M_n - M_{n-1}| \leq C$ a.s. for all $n \leq T$.

Then, $\mathbb{E}[M_T] = \mathbb{E}[M_0]$

Proof. Since T is a stopping time, so is $T \wedge n$ for all n, so $\mathbb{E}[M_{T\wedge n}] = \mathbb{E}[M_0]$ by the previous theorem. So it suffices to show $\mathbb{E}[M_{T\wedge n}] \to \mathbb{E}[M_T]$ as $n \to \infty$. Since $T < \infty$ a.s. $M_{T\wedge n} \to M_T$ a.s. If (i) holds, then $|M_{T\wedge n}| \leq C$ for all n a.s. so $\mathbb{E}[M_{T\wedge n} \to \mathbb{E}[M_T]]$ by bounded convergence. On the other hand, if (ii) holds then $|M_{T\wedge n}| \leq |M_0| + CT$ a.s. so $\mathbb{E}[M_{T\wedge n}] \to \mathbb{E}[M_T]$ by dominated convergence using $|M_0| + CT$ as the dominating random variable. □

Example. Take $(X_n)_{n\geq 1}$ a sequence of iid random variables. Suppose $\mathbb{P}(X_1 = 1) = \mathbb{P}(X_1 = -1) = \frac{1}{2}$. Set $S_n = X_1 + \cdots + X_n$, $S_0 = 0$ and $T = \inf\{n \geq 0 : S_n = 1\}$. We know that $T < \infty$ a.s. and that $(S_n)_{n\geq 0}$ is a martingale. But$\mathbb{E}[S_T] = 1 \neq 0 = \mathbb{E}[S_0]$

Example. Suppose that $\mathbb{P}(X_1 = 0) = \mathbb{P}(X_1 = 2) = \frac{1}{2}$. Set $Z_n = \prod_{i=1}^{n} X_i, Z_0 = 1$.

$$T = \inf\{n \geq 0 : Z_n = 0\}.$$

Then $\mathbb{E}[T] = 2 < \infty$ but $\mathbb{E}[Z_T] = 0 \neq 1 = \mathbb{E}[Z_0]$.

Theorem. Let $(M_n)_{n\geq 0}$ be a martingale and let T be a stopping time. Then, $(M_{T\wedge n})_{n\geq 0}$ is also a martingale.

Proof. We have for all n

$$M_{T\wedge n} = M_0 + \sum_{k=1}^{n} 1_{\{k \leq T\}}(M_k - M_{k-1}).$$

Since $(M_n)_{n\geq 0}$ is adapted and integrable and $\{k \leq T\} = \{T \leq k-1\}^c \in \mathcal{F}_{k-1}$, $\mathbb{E}[|M_{T\wedge n}|] < \infty$ and $M_{T\wedge n}$ is $\mathcal{F}_n$ measurable for all n. Hence $(M_{T\wedge n})_{n\geq 0}$ is also adapted and integrable.
Now

$$M_{T\wedge(n+1)} - M_{T\wedge n} - 1_{n+1\leq T}(M_{n+1} - M_n).$$

So by taking out what is known

$$\mathbb{E}\left[M_{T\wedge(n+1)} - M_{T\wedge n}|\mathcal{F}_n\right] = \mathbb{E}\left[1_{\{n+1\leq T\}}(M_{n+1} - M_n|\mathcal{F}_n\right] = 1_{n+1\leq T}\mathbb{E}[M_{n+1} - M_n|F_n] = 0 \text{ a.s. }.$$

Hence $(M_{T\wedge n})_{n\geq 0}$ is a martingale. □

Definition (Previsible)**.** We say that a process $(H_n)_{n\geq 1}$ is *previsible* if H_n is $\mathcal{F}_{n-1}$-measurable.

Theorem. Let $(M_n)_{n\geq 0}$ be a martingale and let $(H_n)_{n\geq 1}$ be previsible process. Define

$$Y_n - \sum_{k=1}^{n} H_k(M_k - M_{k-1}) \text{ and } Y_0 = 0.$$

Suppose that $|H_n| \leq C$ a.s. for all n for some constant $C < \infty$. Then $(Y_n)_{n\geq 0}$ is a martingale.

Proof. We have for all n

$$Y_n = 0 + \sum_{k=1}^{n} H_k(M_k - M_{k-1}).$$

Since $(M_n)_{n\geq 0}$ is adapted and integrable and $(H_n)_{n\geq 1}$ is previsible and bounded we see $\mathbb{E}\left[|Y_n|\right] < \infty$ and Y_n is $\mathcal{F}_n$ measurable for all n. Hence $(Y_n)_{n\geq 0}$ is also adapted and integrable.
Now

$$Y_{n+1} - Y_n = H_n(M_{n+1} - M_n).$$

So by taking out what is known

$$\mathbb{E}\left[Y_{n+1} - Y_n|\mathcal{F}_n\right] = \mathbb{E}\left[H_{n+1}(M_{n+1} - M_n|\mathcal{F}_n\right] = H_{n+1}\mathbb{E}\left[M_{n+1} - M_n|F_n\right] = 0 \text{ a.s. } .$$

Hence $(Y_n)_{n\geq 0}$ is a martingale. □

Financial / gambling interpretations In a casino a martingale is a fair game - given what we know, there is no expected gain or loss. The optional stopping theorem says that

$$\mathbb{E}\left[M_T\right] = \mathbb{E}\left[M_0\right] \;\; T \leq n.$$

Suppose we hold an asset with price $(M_n)_{n\geq 0}$. Our last result says there is no way to invest boundedly in $(M_n)_{n\geq 0}$ to give positive expected reward.

3 Pricing contingent claims

3.1 Multi-period asset price model

Notation. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be equipped with a filtration $(\mathcal{F}_n)_{0\le n\le T}$ (with T a constant $\in \mathbb{N}$). Let $(\overline{S}_n)_{0\le n\le T}$ be a random process in $\mathbb{R}^{d+1}$.

$$\overline{S}_n = (S_n^0, S_n) = (S_n^0, S_n^1, \dots, S_n^d).$$

We assume that $(S_n^0)_{0\le n\le T}$ is a numeraire, that is , $S_n^0 > 0$ for all n. Often interpret $(S_n^0)_{0\le n\le T}$ as a bond or a bank account. Then write

$$S_n^0 = (1+r_n)S_{n-1}^0, r_n \in (-1,\infty)$$

and call r_n the *interest rate*. We interpret $S_n^i, i = 1,\dots,d$ as the price of the ith *risky assets* at time n. There are T time periods: 0 to 1, 1 to 2, up to $\underbrace{T-1 \text{ to } T}_{T\text{th time period}}$.

Note theta $S_n^i - S_{n-1}^i$ is the price change of the ith asses over the nth time period.
We are not mainly looking at the absolute prices, more the discounted prices

$$X_n^i = \frac{S_n^i}{S_n^0}.$$

Let

$$\overline{X}_n = (X_n^0, X_n^1, \dots, X_n^d) = (1, X_n).$$

Let $(\theta_n)_{1\le n\le T}$ be a random process in $\mathbb{R}^{d+1}$. Write

$$\overline{\theta}_n = (\theta_n^0, \theta_n) = (\theta_n^0, \theta_n^1, \dots, \theta_n^d).$$

Suppose an investor holds θ_n^i units of asset i for the nth time period. The total price of a portfolio at the start of the nth period

$$n = 1 : \sum_{i=0}^{d} \theta_1^i S_0^i = \overline{\theta}_1.\overline{S}_0$$
$$n \ge 2 : \overline{\theta}_n.\overline{S}_{n-1}.$$

The total price of the portfolio at the end of the nth period is $\overline{\theta}_n.\overline{S}_n$.

Definition (Self-financing)**.** We say that $(\overline{\theta}_n)_{1\le n\le T}$ is *self-financing* if $\overline{\theta}_n.\overline{S}_n = \overline{\theta}_{n+1}.S_n$ for $n = 1,\dots,T-1$.

It is natural to assume that $(\overline{\theta}_n)_{1\le n\le T}$ is previsible. The investor choose $\overline{\theta}_n$ given what is known at time $n-1$. We have a value process given by

$$V_0 = \overline{\theta}_1.\overline{X}_0, V_n = \overline{\theta}_n.\overline{X}_n \overset{\text{(self financing)}}{=} \overline{\theta}_{n+1}\overline{X}_n \; n = 1,\dots,T.$$

Proposition. Let $(\theta_n)_{1\le n\le T}$ be previsible process in $\mathbb{R}^d$ and let $V_0 \in \mathbb{R}$. There exists a unique previsible process $(\theta_n^0)_{1\le n\le T} \in \mathbb{R}$ such that (for $\overline{\theta}_n = (\theta_n^0, \theta_n)$) $(\overline{\theta}_n)_{1\le n\le T}$ is a self-financing portfolio, with initial value V_0. Moreover, the associated value process is given by

$$V_n = v_0 + \sum_{k=1}^{n} \theta_k (X_k - X_{k+1}).$$

Proof. The equations $\overline{\theta}_1.\overline{X}_0 = V_0$ and $\overline{\theta}_n.\overline{X}_n = \overline{\theta}_{n+1}.\overline{X}_n$ for $n = 1, \ldots, T$ which express that $(\overline{\theta}_n)_{1 \le n \le T}$ has initial value V_0 and is self-financing. This can be written as

$$\theta_1^0 + \theta_1.X_0 = V_0, \theta_n^0 + \theta_n.Xn = \theta_{n+1}^0 + \theta_{n+1}.X_n.$$

These equations can be solved uniquely for $(\theta_n^0)_{1 \le n \le T}$ which is then previsible. Since $X_n^0 = 1$ for all n,

$$V_n - V_{n-1} = \theta_n.(X_n - X_{n-1}).$$

So by induction, we are done. □

3.2 Examples of contingent claims

Context: asset price model $(\overline{S_n})_{1 \le n \le T}$. Take $\mathcal{F}_T = \sigma(\overline{S}_0, \overline{S}_T)$. By a contingent claim of *maturity* T we mean any $\mathcal{F}_T$-measurable random variable C. Interpret this as a contract which pays C to the investor at time T.

Notation. We write

$$x^+ = \max\{x, 0\} = x \vee 0, \ x^- = \max\{-x, 0\} = (-x) \vee 0.$$

Example. (d = 1)

(i) $(S_T - K)^+$ (European) *call* of *strike price* K. This confers the right but not the obligation (the *option*) to buy one unit of the stock / asset at time T for price K.

(ii) $(S_T - K)^- = (K - S_T)^+$ *put* of K. This is the option to sell one unit at time T for price K.

(iii) Note $(S_T - K)^+ - (S_T - K)^- = S_T - K$ this is a *forward contract* it obliges you to buy a unit of stock at time T at price K.

There are more exotic options which depend not just on the final value S_T, such as *barrier options* which are *knocked in* or *knocked out* when the price crosses a given level.

Example. The *up-and-out call* C given by

$$C = \begin{cases} (S_T - K)^+ & \text{if } \max_{0 \le n \le T} S_n < B \\ 0 & \text{otherwise} \end{cases}.$$

Example. The *down-and-out call* C given by

$$C = \begin{cases} (S_T - K)^- & \text{if } \min_{0 \le n \le T} S_n \le B' \\ 0 & \text{otherwise} \end{cases}.$$

3.3 Arbitrage and completeness

(By an equivalent probability measure $\tilde{\mathbb{P}}$ on $(\Omega, \mathcal{F})$ we mean a probability measure $\tilde{\mathbb{P}}$ such that for some random variable $\rho \ge 0$

(i) $\mathbb{P}(\rho > 0) = 1$

(ii) $\tilde{\mathbb{P}}(A) = \mathbb{E}[\rho 1_A]$ $A \in \mathcal{F}$

Then we write $\tilde{\mathbb{P}} \sim \mathbb{P}$ and $\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} = \rho$ a.s. and call ρ (a version of) the density of $\tilde{\mathbb{P}}$ with respect to $\mathbb{P}$. Let $(\overline{S}_n)_{0 \le n \le T}$ be an asset price model in $\mathbb{R}^{d+1}$.

Definition (Arbitrage)**.** By an *arbitrage* for $(\overline{S}_n)_{1 \le n \le T}$ we mean a previsible self-financing portfolio $(\overline{\theta}_n)_{1 \le n \le T}$ with initial value $V_0 = 0$ such that $V_T \ge 0$ a.s. and $V_T > 0$ with positive probability (w.p.p.)

It is often considered reasonable to use an assumption that there is no such arbitrage in constraining models used. Recall $V_t = V_0 \sum_{k=1}^{n} \theta_k . (X_k - X_{k-1})$. We sometimes call $(\theta_n)_{1 \le n \le T}$ an arbitrage for $(X_n)_{0 \le n \le T}$.

Proposition. Suppose that $(X_n)_{1 \le n \le T}$ is a martingale (each component is one). Then $(X_n)_{0 \le n \le T}$ has no arbitrage.

Proof. Let $(\theta_n)_{1 \le n \le T}$ be a previsible process in $\mathbb{R}^d$ and set $V_n = \sum_{k=1}^{n} \theta_k (X_k - X_{k-1})$. Suppose $V_T \ge 0$ a.s. Note $\mathbb{E}[V_T | \mathcal{F}_T] = V_T$ a.s. Suppose in a reverse induction that for $n \le T$ $\mathbb{E}[V_T | \mathcal{F}_N] = V_n$ a.s. Them $V_N \ge 0$ a.s. Now $V_n = V_{n-1} + \theta_n (X_n - X_{n-1})$. Fix $R < \infty$ and set $A = \{|\theta_n| \le R, |V_{n-1}| \le R\}$. Then $A \in \mathcal{F}_{n-1}$ and

$$1_A V_n = 1_A V_{n-1} + (1_A \theta_n)(X_n - X_{n-1}).$$

Take $\mathbb{E}[\cdot | \mathcal{F}_{n-1}]$ taking out what is known to see

$$1_A \mathbb{E}[V_n | \mathcal{F}_{n-1}] = 1_A V_{n-1} + 1_A V_{n-1} + 1_A \theta_n \underbrace{\mathbb{E}[X_n - X_{n-1} | \mathcal{F}_{n-1}]}_{=0}.$$

But R was arbitrary so induction proceeds. We see

$$\mathbb{E}[V_T | \mathcal{F}_0] = V_0 = 0,$$

so $\mathbb{E}[V_T] = 0$ so $V_T = 0$ a.s. Hence there can be no arbitrage. □

3.4 Characterisation of a single-period model with no arbitrage

Proposition. Let Y be a random variable in $\mathbb{R}^d$ the following are equivalent

(i) There is no $\theta \in \mathbb{R}^d$ such that $\theta . Y \ge 0$ a.s. and $\theta . Y > 0$ w.p.p.

(ii) There is an equivalent probability measure $\tilde{\mathbb{P}}$ under which $\tilde{\mathbb{E}}[|Y|] < \infty$ with $\tilde{\mathbb{E}}[Y] = 0$

Suppose (i) holds. Then for all $\theta \in \mathbb{R}^d$ such that $\theta . Y \ne 0$ w.p.p. then $\theta . Y > 0$ w.p.p. It will be sufficient to consider the case where

$$\phi(\theta) = \mathbb{E}\left[e^{\theta . Y}\right] < \infty \ \forall \ \theta \in \mathbb{R}^d.$$

In general we could switch to the equivalent probability measure $\tilde{\mathbb{P}}$ given by $\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}} \propto e^{-|Y|^2}$. Then under $\tilde{\mathbb{P}}$,

$$\tilde{\mathbb{E}}\left[e^{\theta . Y}\right] \propto \mathbb{E}\left[e^{\theta . Y - |Y|^2}\right] \le e^{\frac{|\theta|^2}{4}},$$

where the inequality follows from noting that $e^{-|Y|^2+\theta.Y-\frac{|\theta|^2}{4}}e^{\frac{|\theta|^2}{4}}=e^{-|Y-\frac{\theta}{2}|^2}e^{\frac{|\theta|^2}{4}}$. Assume this has been done and drop the tildes. Since $\phi(\theta)<\infty$ for all θ, ϕ is differentiable on $\mathbb{R}^d$ with $\phi'(\theta)=\mathbb{E}\left[Ye^{\theta.Y}\right]$. We will show that ϕ achieves a minimum at some $\theta^*\in\mathbb{R}^d$. We can define an equivalent probability measure $\tilde{\mathbb{P}}$ by $\frac{\mathrm{d}\tilde{\mathbb{P}}}{\mathrm{d}\mathbb{P}}=e^{\theta^*_.Y/\phi(\theta^*)}$. Then

$$\tilde{\mathbb{E}}\left[|Y|\right]=\mathbb{E}\left[|Y|e^{\theta.Y}\right]/\phi(\theta^*)<\infty$$

and

$$\tilde{\mathbb{E}}\left[Y\right]=\mathbb{E}\left[Ye^{\theta^*Y}\right]/\phi(\theta^*)=0.$$

Set $E_0=\{\theta\in\mathbb{R}^d:\theta.Y=0\text{ a.s. }\}$ a vector space $\subseteq\mathbb{R}^d$ and set $E_1=E_0^{\perp}$. For $\theta=\theta_0+\theta_1,\theta_i\in E_i$ have $\phi(\theta)=\phi(\theta_1)$. So it will suffice to show ϕ achieves a minimum on E_1. Since $\phi(0)=1$ it will suffice to show that $\phi(\theta)\geq 1$ for all $\theta\in E_1$ with $|\theta|$ is sufficiently large. Set $\psi(t)=0\vee t\wedge 1$, then $|\psi(t)-\psi(t')|\leq|t-t'|$ for all t,t'. Define

$$f(\theta)=\mathbb{E}\left[\psi(\theta.Y)\right].$$

Then f is continuous by the bounded convergence theorem and $f(\theta)>0$ for all $\theta\in S_1=\{\theta\in E_1:|\theta|=1\}$. But S is compactso

$$\varepsilon=\frac{1}{2}\inf_{\theta\in S}f(\theta)>0.$$

Now for $\theta\in S$

$$\mathbb{P}\left(\theta_{cdot}Y\geq\varepsilon\right)=\mathbb{P}\left(\theta.Y-\varepsilon\geq 0\right)\geq\mathbb{E}\left[\psi(\theta.Y-\varepsilon)\right]\geq\mathbb{E}\left[\psi(\theta.Y\right]-\varepsilon\geq\varepsilon.$$

So for $t\geq 0$, $\phi(t\theta)=\mathbb{E}\left[e^{t\theta.Y}\right]\geq\varepsilon e^{t\varepsilon}\geq 1$ whenever $t\geq\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}$ as required.

We assume here that $\mathcal{F}_0=\{\emptyset,\Omega\}$ and consider the asset price model $(\overline{S}_0,\overline{S}_1)$. Note, since $\mathcal{F}_0$ is trivial, all $\mathcal{F}_0$ measurable random variables are constants including $X_0=S_n/S_n^0$ for any previsible process θ_1. So for the associated portfolio with $V_0=0$ we have $V_1=\underbrace{\theta_1.}_{\theta}\underbrace{(X_1-X_0)}_{Y}$. We showed (there exists no $\theta\in\mathbb{R}^d$ such that $\theta.Y\geq 0$ a.s. and $\theta.Y>0$ w.p.p) $\iff$ (there exists an equivalent probability measure $\tilde{\mathbb{P}}$ such that $\tilde{\mathbb{E}}\left[|Y|\right]<\infty$ and $\tilde{\mathbb{E}}\left[Y\right]=0$). This says (there is no arbitrage for $(\overline{S}_0,\overline{S}_1)$) $\iff$ (there exists and equivalent probability measure $\tilde{\mathbb{E}}\left[X_1\right]<\infty$ and $\tilde{\mathbb{E}}\left[X_1\right]=X_0$) i.e. (X_0,X_1) is a $\tilde{\mathbb{P}}$-martingale.

3.5 Fundamental theorem of asset prices

Definition (Equivalent martingale measure)**.** We say that $\tilde{\mathbb{P}}$ is an *equivalent martingale measure* if $\tilde{\mathbb{P}}\sim\mathbb{P}$ and, under $\tilde{\mathbb{P}}$, $(X_n)_{0\leq n\leq T}$ is a martingale. ($X_n=S_n/S_n^0$) the discounted asset price and $\tilde{\mathbb{E}}\left[X_{n+1}|\mathcal{F}_n\right]=\tilde{X}_n$ a.s.). We also use the name *risk-neutral measure*.

Theorem. Let $(\overline{S}_n)_{0\leq n\leq T}$ be an asset price model with numeraire. Then the following statements are equivalent:

(i) $(\overline{S}_n)_{0\leq n\leq T}$ has no arbitrage.

(ii) There exists an equivalent martingale measure

Proof. The proof is given in James Norris's official course notes appendix. We've shown that (ii) $\implies$ (i) in an earlier proposition and that (i) $\implies$ (ii) when $T = 1$ and $\mathcal{F}_0$ is trivial. □

3.6 Completeness

Definition (Attainable)**.** We say that a time-T contingent claim C (i.e. an $\mathcal{F}_T$-measurable random variable is *attainable* or *replicable* if there is a previsible self-financing portfolio $(\overline{\theta}_n)_{1\leq n\leq T}$ such that $C = \overline{\theta}_T.\overline{S}_T$. This is equivalent to the condition that there exists an $\mathcal{F}_0$-measurable random variable V_0 and a previsible process $(\theta_n)_{1\leq n\leq T}$ in $\mathbb{R}^d$ such that for $X_n = S_n/S_n^0$

$$D := \frac{C}{S_T^0} = V_0 + \sum_{n=1}^{T} \theta_n (X_n - X_{n-1}) = V_T.$$

We call V_0 the *fair price* or *risk-neutral price* for C. Suppose you are offered C at time 0 for a price $U_0 < V_0$. You could buy borrowing U_0 from the bank. You could also trade to deliver at time T the amount $-C + V_0 S_T^0$. At time T you have $-C + V_0 S_T^0 + C - U_0 S_T^0$. You are left with $(V_0 - U_0)S_T^0 > 0$.

Definition (Complete)**.** If all time-T contingent claims are attainable, then we say $(\overline{S}_n)_{0\leq n\leq T}$ is *complete*.

Proposition. Assume that $\mathcal{F}_0 = \{\emptyset, \Omega\}$ and $\mathcal{F} = \mathcal{F}_T = \sigma(\overline{S}_1, \ldots, \overline{S}_T)$

(i) Let C be a *non-negative* time-T attainable contingent claim. Suppose that $\tilde{\mathbb{P}}$ is an equivalent martingale measure. Then the fair price V_0 is given by

$$V_0 = \tilde{\mathbb{E}}\left[D\right], D = \frac{C}{S_T^0}.$$

(ii) If $(\overline{S}_n)_{0\leq n\leq T}$ is complete and $(S_n^0)_{0\leq n\leq T}$ is non-random, then there is at most one equivalent martingale measure.

Proof. (i) There exists a previsible process $(\theta_n)_{1\leq n\leq T}$ such that $D = V_0 + \sum_{n=1}^{T} \theta_n (X_n - X_{n-1})$. By the argument from the proof of our earlier proposition (if $(X_n)_{0\leq n\leq T}$ is a martingale then there is no arbitrage, via reverse induction) this implies

$$\tilde{\mathbb{E}}\left[D|\mathcal{F}_0\right] = V_0 \text{ a.s .}$$

Since $\mathcal{F}_0$ is trivial we are done.

(ii) If $(S_n^0)_{0\leq n\leq T}$ is non-random then $\tilde{\mathbb{E}}\left[C\right] = V_0 S_T^0$ so $\tilde{\mathbb{E}}\left[C\right]$ does not depend on the choice of $\tilde{\mathbb{P}}$. If $(\overline{S}_n)_{0\leq n\leq T}$ is complete, this is true for $C = 1_A$ for all $A \in \mathcal{F}_T$ so $\tilde{\mathbb{P}}$ is uniquely determined.

□

3.7 Binomial model

Fix parameters $r, a, b \in (-1, \infty)$, $p \in (0,1)$ and $S_0 \in (0, \infty)$. We say that $(S_n^0, S_n)_{0 \le n \le T}$ is a *binomial model* with *interest rate* r and parameters $a < b$ and p if

$$S_n^0 = (1+r)^n, S_n = S_0 \prod_{k=1}^{n} (1 + R_k),$$

where $R_1, \ldots, R_T$ are i.i.d. random variables with

$$\mathbb{P}(R_1 = a) = 1 - p, \mathbb{P}(R_1 = b) = p.$$

It is also called the Cox-Ross-Rubinstein model. We'll take $\mathcal{F}_n = \sigma(R_1, \ldots, R_n)$.

Proposition. Let $(S_n^0, S_n)_{0 \le n \le T}$ be a binomial model with interest rate r and parameters $a < b$ and p. Then $(S_n^0, S_n)_{0 \le n \le T}$ has an arbitrage unless $r \in (a, b)$

Proof. Consider the self-financing portfolio $(\overline{\theta}_n)_{0 \le n \le T}$ with $\theta_1 = 1, \theta_n = 0$ for $n \ge 2$, with $V_0 = 0$. Then

$$V_T = \theta_1 (X_1 - X_0) = \frac{S_0(1 + R_1)}{1 + r} - S_0 = \frac{S_0(R_1 - r)}{1 + r}.$$

If $r \le a$, then $V_t \ge 0$ a.s. and $V_T = S_0(b - r)/(1 + r) > 0$ with probability $p > 0$. If $r \ge b$, then $V_T \le 0$ a.s. and $V_T = S_0(a - r)/(1 + r) < 0$ with probability $1 - p > 0$. So in both cases there is an arbitrage $(\overline{\theta}_n)_{0 \le n \le T}$ if $r \le a$, $(-\overline{\theta}_n)_{0 \le n \le T}$ if $r \ge b$. $\square$

Given a different $p^* \in (0,1)$ we can define an equivalent probability measure $\mathbb{P}^*$ by

$$\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} = \left(\frac{1 - p^*}{1 - p}\right)^{D_T} \left(\frac{p^*}{p}\right)^{U_T},$$

where $U_T = T - D_T$ which is the number of times $1 \le t \le T$ such that $R_t = b$. In fact, under $\mathbb{P}^*, (S_n^0, S_n)_{0 \le n \le T}$ is a binomial model with p changed to p^*. For

$$\begin{aligned}
\mathbb{P}^*(R_1 = r_1, \ldots, R_T = r_T) &= \left(\frac{1 - p^*}{1 - p}\right)^{d_T} \left(\frac{p^*}{p}\right)^{u_T} \mathbb{P}(R_1 = r_1, \ldots, R_t = r_T) \\
&= (1 - p^*)^{d_T} p^{*u_T}.
\end{aligned}$$

So under $\mathbb{P}^*$, $R_1, \ldots, R_T$ are i.i.d. random variables with

$$\mathbb{P}^*(R_1 = a) = 1 - p^*, \mathbb{P}^*(R_1 = b) = p^*.$$

In the case $r \in (a, b)$ we will take $p^* = \frac{r-a}{b-a}$.

Proposition. Let $(S_n^0, S_n)_{0 \le n \le T}$ be a binomial model with interest rate r and parameters $a < b$ and p. Suppose $r \in (a, b)$. Then under $\mathbb{P}^*$ the discounted asset price $(X_n)_{0 \le n \le T}$ is a martingale. Thus $\mathbb{P}^*$ is an equivalent martingale measure. In particular this implies that there is no arbitrage.

Proof. We have
$$X_n = \frac{S_n}{S_n^0} = S_0 \prod_{k=1}^n \left(\frac{1+R_k}{1+r}\right).$$
Note $r = (1-p^*) + p^* b$. So
$$\mathbb{E}^*\left[\frac{1+R_n}{1+r}\right] = \frac{1+(1-p^*)a + p^* b}{1+r} = 1.$$
Hence $(X_n)_{0\le n\le T}$ is a multiplicative martingale. □

In fact, (we'll show) for the binomial model, every contingent claim C is attainable, i.e. the model is complete. Hence, C has fair price
$$V_0 = \frac{\mathbb{E}^*[C]}{(1+r)^T}, \left(V_T = \frac{C}{S_T^0} = V_0 + \sum_{n=1}^T \theta_n (X_n - X_{n-1})\right).$$
In the case $C = f(S_T)$ then
$$V_0 = (1+r)^{-T} \sum_{k=0}^T \binom{T}{k} (1-p^*)^{T-k} p^{*k} f(S_0(1+a)^{T-k}(1+b)^k).$$
More generally, if $C = f(S_0, \ldots, S_T)$
$$V_0 = (1+r)^{-T} \sum \mathbb{P}^*(S_0 = s_0, \ldots, S_T = s_t) f(s_0, \ldots, s_T),$$
where we sum over all possible paths $(s_n)_{0\le n\le T}$ starting from S_0. We can compute this efficiently as follows. Set $f_T(s_0, \ldots, s_T) = f(s_0, \ldots, s_T)$ and define for $n \le T-1$ recursively
$$f_n(s_0, \ldots, s_n) = (1-p^*) f_{n+1}(s_0, \ldots, s_n, (1+a)s_n) + p^* f_{n+1}(s_0, \ldots, s_n, (1+b)s_n).$$
Proposition. We have
$$\mathbb{E}^*[C|\mathcal{F}_n] = f_n(S_0, \ldots, S_n).$$
In particular, $\mathbb{E}^*[C] = \mathbb{E}^*[C|\mathcal{F}_0] = f_0(S_0)$

Proof. Check the notes. □

Recall $C = f(S_0, \ldots, S_T), D = C/(1+r)^T, f_T = f$
$$f_n(S_0, \ldots, S_n) = p^* f_{n+1}(S_0, \ldots, S_n, (1+b)S_n) + (1-p^*) f_{n+1}(S_0, \ldots, S_n, (1+a)S_n),$$
$\mathbb{E}^*[C|\mathcal{F}_n] = f_n(S_0, \ldots, S_n)$ a.s. Define
$$\Delta_n(S_0, \ldots, S_n) = \frac{f(S_0, \ldots, S_{n-1}, (1+b)S_{n-1}) - f_n(S_0, \ldots, S_{n-1}, (1+a)S_{n-1})}{(1+r)^{T-n}(b-a)S_{n-1}}.$$
Proposition. Define $\theta_n = \Delta_n(S_0, \ldots, S_{n-1})$. Then $(\theta_n)_{1\le n\le T}$ is a replicating portfolio for C.

Proof. Define $V_n = (1_r)^{-T}\mathbb{E}[C|\mathcal{F}_n]$. Fix a path $S_0, \dots, S_{n-1}$ starting from S_0, set $\phi(x) = f_n(S_0, \dots, S_{n-1}, (1+x)S_n)$ and define $\Omega_0 = \{S_1 = s_1, \dots, S_{n-1} = s_{n-1}\}, \Omega_x = \Omega_0 \cap \{R_n = x\}$. Then $\Omega_U = \Omega_a \cup \Omega_r$. Then on Ω_a

$$\begin{aligned} V_n V_{n-1} &= (1+r)^{-T}(f_n(S_0, \dots, S_{n-1}, (1+a)S_{n-1}) - f_{n-1}(S_0, \dots, S_{n-1})) \\ &= (1+r)^{-T}(\phi(a) - f_{n-1}(S_0, \dots, S_{n-1}) \\ &= p^* \frac{(\phi(a) - \phi(b)}{(1+r)^T} \end{aligned}$$

.

and

$$X_n - X_{n-1} = \frac{1+a}{1+r}X_{n-1} - X_{n-1} = \frac{a-r}{(1+r)^n}S_{n-1}.$$

So

$$\theta_n(X_n - X_{n-1}) = \frac{\phi(b) - \phi(a)}{(1+r)^{T-n}(b-a)S_{n-1}} \frac{(a-r)S_{n-1}}{(1+r)^n} = V_n - V_{n-1}.$$

A similar calculation shows

$$\theta_n = (X_n - X_{n-1}) = V_n - V_{n-1} \text{ on } \Omega_b.$$

hence everywhere since $S_1, \dots, S_{n-1}$ were arbitrary. Now sum to see

$$D = V_T = V_0 + \sum_{n=1}^{T} \theta_n (X_n - X_{n-1}).$$

So, $(\theta_n)_{1 \le n \le T}$ is a replicating portfolio for C, hence the binomial model is complete. □

3.8 Joint distribution of a random walk and its maximum

Proposition. Let $(W_n)_{0 \le n \le T}$ be a simple random walk on $\mathbb{Z}$ starting from 0 with $\mathbb{P}(W_1 = 1) = p = 1 - \mathbb{P}(W_1 = -1)$. Set $M_T = \max_{0 \le n \le T} W_n$. Then for all $k, m \ge 0$ with $2k - T \le m \le k$

$$\mathbb{P}(M_t = m \cap W_T = 2k - T) = \left(\binom{T}{k-m} - \binom{T}{k-m-1}\right) p^k (1-p)^{T-k}.$$

Note that k is the number of up steps.

Proof. Set $E = \{(W_n)_{0 \le n \le T} : W_0 = 0, W_n = W_{n-1} \pm 1\}$ and consider the map $\phi : E \to E$ obtained by reflecting the path at level m on its first time there. Since $\phi(\phi(w)) = w$ for all w, ϕ is a bijection. Consider,

$$A = \{w \in E : W_T = 2k - T, \max_{0 \le n \le T} w_n \ge m\}.$$

Note that $\phi(A) = \{w \in E : W_T = 2m - 2k + T\}$ but the probability of any given path $w \in A$ is $p^k(1-p)^{T-k}$. Hence

$$\mathbb{P}(M_T \ge m \cap W_T = 2k - T) = |\phi(A)| p^k (1-p)^{T-k} = \binom{T}{k-m} p^k (1-p)^{T-k}.$$

This implies the result by subtracting the corresponding formula for $m + 1$. □

Suppose a, b satisfy $(1+a)(1+b) = 1$. Set

$$S_n^0 = (1+r)^n, S_n = S_0(1+b)^{W_n}.$$

Then $(S_n^0, S_n)_{0 \leq n \leq T}$ is a binomial model. Consider any contingent claim C of the form

$$C = F(S_T, \max_{0 \leq n \leq T} S_n),$$

e.g. any barrier option. Then

$$V_0 = \frac{\mathbb{E}^*\left[C\right]}{(1+r)^T} = (1+r)^{-T} \sum_{m,k \geq 0, 2k-T \leq m \leq k} \left(\binom{T}{k-m} - \binom{T}{k-m-1} \right) p^{*k}(1-p^*)^{T-k} f(2k-T, m),$$

where $f(x, m) = F(S_0(1+b)^x, S_0(1+b)^m)$.

4 Dynamic Programming

In brief,

$$X_{n+1} = F(n, X_n, u_n, \varepsilon_{n+1}), X_0 = x.$$

We call u_n the control, ε_{n+1} is an external random variable. We intend to maximise

$$\mathbb{E}\left[\sum_{n=0}^{T-1} r(n, X_n, u_n) + R(X_T)\right]$$

with r a running reward and R the final reward where these are both non-negative.
Suppose given a measurable function

$$F : \{0, 1, \dots, T-1\} \times E \times A \times [0, 1] \to E,$$

where E is a state space and A an action space (all space here are measurable), and given a sequence $(\varepsilon_n)_{1\leq n\leq T}$ of independent $U[0, 1]$ random variables. Given an initial time k, we say that $u = (u_n)_{k\leq n\leq T-1}$ is *adapted control* if u_n is $\mathcal{F}_n$-measurable A-valued random variable for all n. Here $\mathcal{F}_n = \sigma(\varepsilon_1, \dots, \varepsilon_n), \mathcal{F}_0 = \{\emptyset, \Omega\}$. Given an initial state x and an adapted control, we define recursively

$$X_k = x, X_{n+1} = F(n, X_n, u_n, \varepsilon_{n+1}), n = k, k+1, \dots, T-1.$$

We'll write $X_n = X_n(k, x)$ to make explicit the dependence on (k, x). Define

$$V^u(k, x) = \mathbb{E}\left[\sum_{n=k}^{T-1} r(n, X_n^u(k, n), u_n) + R(X_T^u(k, x))\right].$$

Define the *value function*

$$V : \{0, 1, \dots, T-1\} \times E \to [0, \infty]$$

by

$$V(k, x) = \sup_u V^u(k, x)$$

using an adapted control.
Heuristic argument :

$$V(k, x) = \sup_a \{r(k, x, a) + \mathbb{E}[V(k+1, F(k, x.a, \varepsilon_{k+1}))]\}.$$

Proposition. (Bellman equation)
Define a function $v : \{0, 1, \dots, T-1\} \times E \to [0, \infty]$ by the backward recursion relation

$$v(T, X) = R(x), v(n, x) = \sup_{a\in A}\{r(n, x, a) + Pv(n, x, a)\},$$

where $Pv(n, x, a) = \mathbb{E}[v(n+1, F(n, x, a, \varepsilon_{n+1}))]$. Suppose there is a measurable function

$$a : \{0, 1, \dots, T-1\} \times E \to A$$

such that for all n and x

$$v(n, x) = r(n, x, a(n, x)) + Pv(n, x, a(n, x)).$$

Then $V = v$. Moreover, we can define an optimal control $u^* = (u_n^*)_{k\le n\le T}$ from (k,x) by forward recursion

$$u_n^* = a(n, X_n^{u^*}(k,x))(u_k^* = a(k,x)).$$

Proof. It will suffice to consider the case $k = 0$. Fix an adapted control $u = (u_n)_{0\le n\le T-1}$. Write X_n for X^u. We have

$$\mathbb{E}\left[v(n+1, X_{n+1})|\mathcal{F}_n\right] = \mathbb{E}\left[v(n+1), F(n, X_n, u_n, \varepsilon_{n+1})|\mathcal{F}_n\right] = Pv(n, X_n, u_n) \text{ a.s. } .$$

Consider $M_n = \sum_{j=0}^{n-1} r(j, X_j, u_j) + v(n, X_n)$ i.e. the reward before n plus the expected guessed optimal reward after. Then

$$\begin{aligned}
\mathbb{E}\left[M_{n+1}|\mathcal{F}_n\right] &= \sum_{j=0}^{n-1} r(k, X_j, u_j) + r(n, X_n, u_n) + \mathbb{E}\left[v(n+1, X_{n+1})|\mathcal{F}_n\right] \\
&= \sum_{j=0}^{n-1} r(k, X_j, u_j) + r(n, X_n, u_n) + Pv(n, x, u_n) \\
&\le \sum_{j=0}^{n-1} r(j, X, u_j) + v(n, X_n) = M_n,
\end{aligned}$$

with equality if $u = u^*$. Hence $V^u(0,x) = \mathbb{E}\left[M_T\right] \le M_0 = v(0,x) = V^{u^*}(0,x)$. Since u was arbitrary, this shows $V = v$ and that u^* is optimal. □

4.1 American options

An American call (or put) is a contract of expiry T and state K under which the holder has the right but not the obligation to buy (or sell) one unit of the underlying asset $(S_n)_{0\le n\le T}$ at price K at any stopping time $\tau \le T$ chosen by the holder. Let's assume $S_n^0 = (1+r)^n$, so we think of the American call (or put) as the family of contingent claims.

$$\begin{Bmatrix} (C_\tau : \tau \le T, \tau \text{ a stopping time }) \\ (P_\tau : \tau \le T, \tau \text{ a stopping time }) \end{Bmatrix},$$

where $C_\tau = (1+r)^{T-\tau}(S_\tau - K)^+, P_\tau = (1+r)^{T-\tau}(K - S_\tau)^+$. We will show how to price these options in the case of a binomial model, parameters $a < b$. Since this model is complete, given any contingent claim C there is a unique V_0 such that, with initial wealth V_0 we can replicate C by trading in the market. Moreover,

$$V_0 = \mathbb{E}^*\left[C\right]/(1+r)^T,\ p^* = \frac{r-a}{b-a}.$$

So the investor will wish to choose τ to maximise

$$\mathbb{E}^*\left[C_\tau\right]/(1+r)^T \text{ or } \mathbb{E}^*\left[P_\tau\right]/(1+r)^T.$$

Consider first the call.
Fix a stopping time $\tau \le T$ and fix $n \in \{0, 1, \dots, T\}$. Note $\mathbb{E}^*\left[S_T|\mathcal{F}_n\right] = (1+r)^{T-n}S_n$ as

$$X_n = \frac{S_n}{(1+r)^n} \text{ is a martingale.}$$

Consider the event $A = \{S_n \geq K, \tau = n\} \in \mathcal{F}_n$. Then

$$\begin{aligned}
\mathbb{E}^*\left[C_T 1_{\{\tau=n\}}\right] &\geq \mathbb{E}^*\left[C_T 1_A\right] \\
&\geq \mathbb{E}^*\left[(S_T - K)1_A\right] \\
&= \mathbb{E}^*\left[((1+r)^{T-n}S_n - K)1_A\right] \\
&\geq \mathbb{E}^*\left[(1+r)^{T-n}(S_n - K)1_A\right] \\
&= \mathbb{E}^*\left[C_\tau 1_{\{\tau=n\}}\right].
\end{aligned}$$

Sum over n to see that $\mathbb{E}^*[C_T] \geq \mathbb{E}^*[C_\tau]$. Hence the investor should always wait until T. So

$$V_0 = \frac{\mathbb{E}^*[C_\tau]}{(1+r)^T} = (1+r)^{-T}\sum_{n=0}^{T}\binom{T}{n}(1-p^*)^{T-n}(p^*)^n(S_0(1+a)^{T-n}(1+b)^n - K)^+.$$

That is, the American and European calls are equivalent.
Turn to the American put. We seek to solve the full optional stopping time problem

$$\text{maximise } \mathbb{E}^*\left[(1+r)^{T-\tau}P_\tau\right]$$

over all stopping times $\tau \leq T$. This is a dynamic programming problem

$$A = \{\text{exercise, wait}\}\ E_n = \{S_0(1+a)^{n-k}(1+b)^k\ k = 0, 1, \ldots, n\} \cup \{\text{stop}\}.$$

The Bellman equation is

$$v(T,x) = (K-x)^+, \quad V(n,x) = \max\{(1+r)^{T-n}(K-x), (1-p^*)v(n+1, x(1+a)) + p^* v(n+1, x(1+b))\}$$

for $n = 0, 1, \ldots, T-1$ and $x \in E_n$. We could solve this by backwards recursion. Note that the maximum is always achieved (finite set). So,

$$V_0 = \sup_{\tau \leq T} \mathbb{E}^*[P_\tau] = v(0, S_0)$$

and the optimal stopping time is $\tau^* = \min\{n \geq 0 : (1+r)^{T-n}(K-x) = v(n,x)\}$.

5 Brownian motion

Let $(B_t)_{t\geq 0}$ be a real-valued random process. We say that $(B_t)_{t\geq 0}$ is *Brownian motion* if $B_0 = 0$ and

(i) for all $s, t \geq 0, B_{s+t} - B_s$ is Gaussian of mean 0 and variance t and independent of $\mathcal{F}_s = \sigma(B_r : r \leq s)$

(ii) for all $\omega \in \Omega$ the map $t \mapsto B_t(\omega) : [0, \infty) \to \mathbb{R}$ is continuous i.e. is a *continuous random process.*

Sometimes, we replace the condition $B_0 = 0$ with $B_0 = x$. Then we call $(B_t)_{t\geq 0}$ a Brownian motion starting from x. But $x = 0$ is the default.

Proposition. Let $(B_t)_{t\geq 0}$ be a continuous process with $B_0 = 0$. Then TFAE

(i) $(B_t)_{t\geq 0}$ is a Brownian motion

(ii) $(B_t)_{t\geq 0}$ is a zero mean Gaussian process ($(B_{t_1}, \dots, B_{t_n})$ is Gaussian for all $t_1, \dots, t_n,\ n$) with covariance $\mathbb{E}[B_s B_t] = s \wedge t$

Proof. Suppose (i) holds. Zero mean is clear by taking $s = 0$. To see that $(B_t)_{t\geq 0}$ is Gaussian, we note that for $0 = t_0 \leq t_1 \leq \dots \leq t_n, (B_{t_1}, \dots, B_{t_n})$ is a linear function of the independent Gaussian random variables $(B_{t_1} - B_{t_0}, \dots, B_{t_n} - B_{t_{n-1}})$. For the covariance for $s \leq t$

$$\mathbb{E}[B_s B_t] = \mathbb{E}[B_s(B_s + (B_t - B_s))] = \mathbb{E}\left[B_s^2\right] + \mathbb{E}[B_s(B_t - B_s)] = S$$

since $B_s \sim N(0, s)$ and $B_t - B_s$ is independent of B_s. Hence (ii) holds.
Suppose (ii) holds. Then for $s, t \geq 0$ $B_{s+t} - B_s$ is zero-mean Gaussian with

$$\operatorname{var}(B_{s+t} - B_t) = \mathbb{E}\left[(B_{s+t} - B_s)^2\right] = s + t + s - 2s = t.$$

To show $B_{s+t} - B_s$ is independent of $\mathcal{F}_s$ it suffices to show that B_{s+1} is independent of $(B_{r_1}, \dots, B_{r_n})$ for all n all $r_1, \dots, r_n \leq s$. By (ii)

$$\mathbb{E}[(B_{s+t} - B_s)B_{r_k}] = ((s + t) \wedge r_k) - s \wedge r_k = 0.$$

So $\operatorname{cov}(B_{t+s} - B_s, (B_{r_1}, \dots, B_{r_n})) = 0$. So this follows by a property of Gaussian random variables. □

Proposition. Let $(B_t)_{t\geq 0}$ be a Brownian motion and let $\sigma \in (0, \infty)$. Set $\tilde{B}_t = \sigma^{-1} B_{\sigma^2 t}$. Then $(\tilde{B}_t)_{t\geq 0}$ is a Brownian motion. This is called the *scaling property*

Proposition. Let $(B_t)_{t\geq 0}$ be a Brownian motion. Then $(B_t)_{t\geq 0}$ exits every bounded interval almost surely.

Proof. Fix an interval I of length L say. Consider the events $A_n = \{|B_n - B_{n-1}| > L\}$. These are independent with

$$\mathbb{P}(A_n) = \mathbb{P}(|B_1| > L) > 0,$$

hence $\mathbb{P}(\bigcup_n A_n) = 1$. But on $\bigcup_n A_n$ if $(B_t)_{t\geq 0}$ if $B_{n-1} \in I$ then $B_n \notin I$. So $(B_t)_{t\geq 0}$ exits I eventually a.s. □

Definition (Stopping time)**.** We say that a random variable $T : \Omega :\to [0, \infty]$ is a *stopping time* if $\{T \geq t\} \in \mathcal{F}_t$ for all $t \geq 0$.

Define $\mathcal{F}_T = \{A \in \mathcal{F} : A \cap \{T \leq t\} \in \mathcal{F}_t \ \forall \ t \geq 0\}$ "what we know up to T ".

Proposition. (Strong Markov property) Let $(B_t)_{t\geq 0}$ be a Brownian motion and let T be an a.s. finite stopping time. Define

$$\tilde{B}_t = \begin{cases} B_{T+t} - B_t, & \text{if } T < \infty \\ 0, & \text{otherwise} \end{cases}.$$

Then $(\tilde{B}_t)_{t\geq 0}$ is a Brownian motion and is independent of $\mathcal{F}_T$

Proposition. Let $(B_t)_{t\geq 0}$ be a Brownian motion. Definite $T_a = \inf\{t \geq 0 : B_t = a\}$. Then for all $a \in \mathbb{R}, T_a$ is an almost surely finite stopping time.

Proof. We have for $a > 0$

$$\{T_a \leq t\} = \{B_t = a\} \cup \bigcap_{r\in\mathbb{Q}, r<a} \bigcup_{s\in\mathbb{Q} s<t} \{B_s > r\} \in \mathcal{F}_t.$$

Set $T = T_1 \wedge T_{-1}$ a stopping time. Then $T < \infty$ a.s. by 5.3?. Then by symmetry $\mathbb{P}(B_T = 1) = \mathbb{P}(B_T = -1) = \frac{1}{2}$. By strong Markov, the sequence of integers hit by $(B_t)_{t\geq 0}$ ignoring immediate repeats is a simple symmetric random walk on $\mathbb{Z}$. The random walks is recurrent so it hits every integer a.s.. By continuity, $(B_t)_{t\geq 0}$ hits every real a.s.. So $\mathbb{P}(T_a < \infty) = 1$ for all $a \in \mathbb{R}$. $\square$

Theorem. Assume $(\Omega, \mathcal{F}, \mathbb{P})$ is "not discrete". Let m be a probability measure on $\mathbb{R}$, mean 0 and variance 1. There exists a process $B = (B_t)_{t\geq 0}$ and, for $k \geq 1$, there is a random process $(W_{(\mathrm{k})})_{t\geq 0}$ such that

(i) B is a Brownian motion.

(ii) $(W^{(k)}_{\frac{n}{k}})_{n\geq 0}$ is a random walk with step division m, and $(W^{(k)}_t)_{t\geq 0}$ is obtained by linear interpolation.

(iii) $W^{(k)}_t/\sqrt{k} \to B(t)$ as $k \to \infty$ uniformly on compact subsets of $\mathbb{R}$.

An alternative approach to Brownian motion:
Let $X = (X(t) : t \geq 0)$ be a Gaussian process with stationary independent increments i.e. if $t_1 < \cdots < t_n$, then $X(t_1) - X(0), X(t_2) - X(t_1) \ldots, X(t_n) - X(t_{n-1})$ are independent and the distribution of $X(t_i) - X(t_{i-1})$ depends on the length $t_i - t_{i-1}$ and $X(0) = 0, X(t) \sim N(0, t)$.
$X_0(s) = sX(1), s \in [0, 1]$. X_1 by linear interpolation of $(0, X(0)), (\frac{1}{2}, X(\frac{1}{2}), (1, X(1))$. Similarly, X_n is determined by values on multiples of 2^{-n}. Q = dyadic rationals = $\{\frac{k}{2^n} : n \geq 0, 0 \leq k \leq 2^n\}$. We certainly have $X_n(s) \to X(s)$ as $n \to \infty$ for $S \in Q$. We need uniform convergence on Q (it occurs a.s.). Let $B(s) = \lim_{q\to S, S\in Q} X(q)$, exists a.s.

5.1 Change of measure

$B = (B_t)_{t\geq 0}$ we say a Brownian motion has drift c if $\tilde{B}_t = B_t + ct$.

Proposition. Fix $T \geq 0, c \in \mathbb{R}$. let B be Brownian motion, and $\tilde{B}$ as above. For all events $A \subseteq C[0,T]$ such that A is measurable.

$$\mathbb{P}\left(\tilde{B} \in A\right) = \mathbb{E}\left[1_{\{B\in A\}} e^{cB_T - \frac{1}{2}c^2 T}\right].$$

Proof. See Norris's notes for more detail. Let $X \sim N(0,s)$ and $\tilde{X} = X + cs \sim N(cs, s)$

$$\begin{aligned}
\mathbb{P}\left(\tilde{X} \in I\right) &= \int_I \frac{1}{\sqrt{2\pi s}} \exp\left(-\frac{(x-cs)^2}{2s}\right) \mathrm{d}x \\
&= \int_I \frac{1}{\sqrt{2\pi s}} e^{-\frac{x^2}{2s}} e^{cx - \frac{1}{2}c^2 s} \mathrm{d}x \\
&= \mathbb{E}\left[1_{X\in I} e^{cX - c^2 s/2}\right].
\end{aligned}$$

It suffices to prove the result for events of the form: $0 = t_0 \leq t_1 \leq \cdots \leq t_n = T$

$$A = \bigcap_{k=1}^{n} \{f : C[t_{k-1}, t_k] : f(t_k) - f(t_{k-1}) \in I_k\}$$

for intervals $I_1, \ldots, I_k$. We obtain the result for such events by using the above and the independence of increments $B(t_{k+1}) - B(t_k)$ of Brownian motion. □

5.2 Reflection principle

Proposition. Let $B = (B_t)_{\geq 0}$ be a BM, let $a \geq 0$, and define

$$\tilde{B}_t = \begin{cases} B_t & \text{for } t \leq T_a \\ 2a - B_t & \text{for } t > T_a \end{cases}.$$

Then, $\tilde{B} = (\tilde{B}_t)_{t\geq 0}$

Proof. Let $Y_t = B_{t\wedge T_a}, Z_t = B_{T_a + t} - B_{T_a}$ for $t \geq 0$. Since T_a is a stopping time, Y_t, Z_t are independent processes, hence $Y_t, -Z_t$ are independent processes and $-Z$ is a BM. $(Y,Z) \stackrel{d}{=} (Y,-Z)$. (Y,Z) generates B, $(Y,-Z)$ generates $\tilde{B}$. Hence $B \stackrel{d}{=} \tilde{B}$. □

Let $M_t = \sup\{B_s :; s \leq t\}$ take $a \geq 0, x \leq a$, consider the event $\{M_t \geq a, B_t \leq x\} = \{\tilde{B}_t \geq 2a - x\}$.

$$\mathbb{P}\left(M_t \geq a, B_t \leq x\right) = \mathbb{P}\left(\tilde{B}_t \geq 2a - x\right) = \mathbb{P}\left(B_t \geq 2a - x\right).$$

To find the distribution of M_t it is enough to find

$$\begin{aligned}
\mathbb{P}\left(M_t \geq a\right) &= \mathbb{P}\left(B_t \geq a\right) + \mathbb{P}\left(B_t < a, M_t \geq a\right) \\
&= \mathbb{P}\left(B_t \geq a\right) + \mathbb{P}\left(B_t \geq a\right) \\
&= 2\mathbb{P}\left(B_t \geq a\right) = \mathbb{P}\left(|B_t| \geq a\right).
\end{aligned}$$

Hence, $M_t \stackrel{d}{=} |B_t|$. What is the mgf of M_t? By the rescaling property, we know that $M_t \stackrel{d}{=} \sqrt{t} M_1$

$$\begin{aligned}
\mathbb{E}\left[e^{uM_1}\right] &= \mathbb{E}\left[E^{u|B_1|}\right] \\
&= 2\int_0^\infty e^{ux} \frac{1}{\sqrt{2\pi}} \exp(-\frac{1}{2}x^2)\mathrm{d}x \\
&= 2e^{\frac{u^2}{2}} \int_0^\infty \frac{1}{2\pi} \exp(-\frac{1}{2}(x-u)^2)\mathrm{d}x \\
&= 2e^{\frac{u^2}{2}} \Phi(u).
\end{aligned}$$

Where Φ is taken to be the $N(0,1)$ distribution function. So, in particular, the mgf of M_t is

$$\mathbb{E}\left[e^{uM_t}\right] = \mathbb{E}\left[e^{u\sqrt{t}M_1}\right] = 2e^{\frac{tu^2}{2}} \Phi(u\sqrt{t}).$$

5.3 Hitting probabilities

Proposition. Let $(B_t)_{t\geq 0}$ be a BM, let $a > 0$. T_a has density function

$$h_a(t) = \frac{a}{\sqrt{2\pi t^3}} e^{-\frac{a^2}{2t}}, t \geq 0.$$

Proof. $\mathbb{P}\left(T_a \leq t\right) = \mathbb{P}\left(M_t \geq a\right) = 2\mathbb{P}\left(B_t \geq a\right)$.

$$\begin{aligned}
\mathbb{P}\left(B_t \geq a\right) &= \mathbb{P}\left(B_1 \geq \frac{a}{\sqrt{t}}\right) \\
&= \int_{\frac{a}{\sqrt{t}}} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{y^2}{2}\right)\mathrm{d}y
\end{aligned}$$

.

We differentiate to get the claimed result. □

5.4 Killing

Let $(B_t)_{t\geq 0}$ be a BM, $a > 0$. Imagine, B is killed when it hits a.

Proposition. Let $a > 0, t > 0, x \in \mathbb{R}$

$$\mathbb{E}_x\left[f(B_t)1_{\{T_a > t\}}\right] = \int_{-\infty}^a f(y) p_t^a(x,y)\mathrm{d}y,$$

where $p_t^a(x,y) = p_t(x,y) - p_t(x, 2a-y)$ and $p_t(x,y)$ is the pdf of B_t given $B_0 = x = N(x,t)$ pdf.

Proof. It is enough to consider the function

$$f(u) = \begin{cases} 1 & n \leq b \\ 0 & n > b \end{cases} = 1_{(-\infty,b]}(n)$$

for $b \leq a$.

$$\begin{aligned}
\mathbb{E}_x\left[f(B_t)1_{\{T_a>t\}}\right] &= \mathbb{P}_x\left(M_t < a, B_t \leq b\right)\\
&= \mathbb{P}\left(B_t \leq b\right) - \mathbb{P}\left(M_t \geq a, B_t \leq b\right)\\
&= \mathbb{P}\left(B_t \leq b\right) - \mathbb{P}\left(B_t \geq 2a-b\right)\\
&= \int_{-\infty}^{b} p_t(x,y)\mathrm{d}y - \int_{2a-b}^{\infty} p_t(x,y)\mathrm{d}y\\
&= \int_{-\infty}^{b} [p_t(x,y) - p_t(x,2a-y)]\mathrm{d}y\\
&= \int_{-\infty}^{a} f(y)p_t^a(x,y)\mathrm{d}y
\end{aligned}$$

.

□

6 Black-Scholes model

6.1 Black-Scholes pricing formula

By a *Black-Scholes model* we mean any pair of processes $(S_t^0)_{t\geq 0}$ and $(S_t)_{t\geq 0}$ of the form

$$S_t^0 = e^{rt}, \; S_t = S_0 e^{\sigma B_t + \mu t},$$

where $r, \mu \in \mathbb{R}$ and $S_0, \sigma \in (0, \infty)$ and $(B_t)_{t\geq 0}$ is a Brownian motion. We interpret $(S_t^0)_{t\geq 0}$ as a *riskless bond* of *interest rate* r. We interpret $(S_t)_{t\geq 0}$ as the price of a *risky asset*. We call μ the *drift*, σ the *volatility*. The process $(e^{\sigma B_t - \sigma^2 t/2})_{t\geq 0}$ is a martingale so for the special choice $\mu = \mu^* = r - \frac{\sigma^2}{2}$ the *discounted asset price* $(e^{-rt}S_t)_{t\geq 0}$ is a martingale.

Proposition. Let $(S_t^0, S_t)_{t\geq 0}$ be a Black-Scholes model with interest rate r, drift μ, volatility σ. Fix $T \in (0, \infty)$ Define an equivalent probability measure $\mathbb{P}^*$ by

$$\frac{\mathrm{d}\mathbb{P}^*}{\mathrm{d}\mathbb{P}} = e^{\lambda B_T - \lambda^2 \frac{T}{2}},$$

where $\sigma\lambda = \mu^* - \mu = r - \frac{\sigma^2}{2} - \mu$. Then under $\mathbb{P}^*$, the discounted asset price $(e^{-rt}S_t)_{0\leq t\leq T}$ is a martingale. Note since $\frac{\log S_t}{t} \to \mu$ a.s. we cannot have $T = \infty$ here.

Proof. Set $B_t^* = B_t - \lambda t$. Under $\mathbb{P}^*$ by Cameron-Martin formula, $(B_t)_{0\leq t\leq T}$ is a BM with drift λ, so $(B_t^*)_{0\leq t\leq T}$ is a BM. Now,

$$\sigma B_t^* + \mu^* t = \sigma B_t - \sigma\lambda t + \mu^* t = \sigma B_t + \mu t.$$

Hence

$$S_t = S_0 e^{\sigma B_t + \mu t} = S_0 e^{\sigma B_t^* + \mu^* t}.$$

So under $\mathbb{P}^*(e^{-rt}S_t)_{0\leq t\leq T}$ is a martingale. □

From now on, we will write $\mathbb{P}^*$ as a signal that we take $\mu = \mu^*$.

Definition (Contingent claim)**.** By a time T *contingent claim* we meany any $\mathcal{F}_T$-measurable random variable C considered as an amount payable to the investor at time T.

Definition (Black-Scholes price)**.** The *Black-Scholes price* for the claim C is defined by

$$V_0 = e^{-rT}\mathbb{E}^*\left[C\right].$$

We assume that the investor is free to trade in the asset and the bond.

6.2 Black-Scholes PDE

Let $(S_t^0, S_t)_{t\geq 0}$ be a Black-Scholes model as above. We'll find a PDE to determine V_0 when $C = F(S_T)$. First we find a PDE for the density of S_t on $(0, \infty)$ when $S_0 = s$ and $\mu = \mu^*$.

$$\mathbb{P}^*\left(S_t \in \mathrm{d}y\right) = \rho(t, s, y)\mathrm{d}y.$$

Set $Z = \log S_t = \log s + \sigma B_t + \mu t$, then Z has density $p(\sigma^2 t, x(t), z)$ where $x(t) = \log s + \mu t$

$$p(t, x, z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{|x-z|^2}{2t}}.$$

Now $S_t = Y = e^{Z_t}$. So $y\rho(t, s, y) = p(\sigma^2 t, x(t), z)$ since $y = e^z, \mathrm{d}y = y\mathrm{d}z$. Next $sy\rho'(t, s, y) = p'(\sigma^2 t, x(t), z)$ and

$$s(y\rho' + sy\rho'')(t, sy) = p''(\sigma^2 t, x(t), z).$$

So

$$\begin{aligned} y\dot{\rho}(t, s, y) &= \sigma^2 \dot{p}(\sigma^2 t, x(t), z) + \mu p'(\sigma^2 t, x(t), z) \\ &= (\frac{1}{2}\sigma^2 p'' + \mu p')(\sigma^2 t, x(t), z) \\ &= (\frac{1}{2}\sigma^2 s(y\rho' + sy\rho'') + \mu sy\rho')(t, s, y), \end{aligned}$$

that is

$$\dot{\rho} = \frac{1}{2}\sigma^2 s^2 \rho''(\mu + \frac{1}{2}\sigma^2)s\rho'.$$

Proposition. Assume F is continuous and at most polynomial growth on $(0, \infty)$ ($F(s) \leq C(1 + S^n)$ for some $C, n < \infty$). Write $V(t, s)$ for the time t value of $F(S_T)$ given $S_t = s$. Thus

$$V(t, s) = e^{-r(T-t)}\mathbb{E}^* \left[F(S_T)|S_t = s\right] = e^{-r(T-t)}\mathbb{E}^* \left[F(S_{T-t}|S_0 = s\right].$$

Then V is $C^{1,2}$ on $[0, T) \times (0, \infty)$ and is continuous on $[0, T] \times (0, \infty)$ and satisfies the final value problem

$$\mathcal{L}V = \dot{V} + \frac{1}{2}\sigma^2 s^2 V'' + rsV' - rV = 0, \ V(T, \cdot) = F.$$

Proof. Set $v(t, s, y) = e^{-r(T-t)}\rho(T - t, s, y)$ $(\mu = \mu^*)$ so that

$$V(t, s) = \int_0^\infty v(t, s, y)F(y)\mathrm{d}y.$$

(Our condition on F ensures we can differentiate the above under the integral to obtain

$$\mathcal{L}V(t, s) = \int_0^\infty \mathcal{L}v(t, s, y)F(y)\mathrm{d}y$$

and V is continuous on $[0, T] \times (0, \infty)$ from the continuity of BM and dominated convergence.) □