Part II — Dynamical Systems

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Introduction
Some early definitions and recap [1]

Basic Definitions
Basic definitions for fundamentals of the course. Notation and examples of systems phase states.

Contents

0 Introduction

A *dynamical system* is a set of equations which describe how a system evolves with respect to a timelike variable. The possible states of the system define the *state space* (or *phase space*).

Example. Logistic Map

$$x_{n+1} = \mu x_n(1 - x_n) \text{ with } 0 \leq \mu \leq 4$$

Here we are working with discrete time, and our state space is [0,1]

Example. Lotka-Voltera

$$\dot{r} = r(a - br - cs)$$
$$\dot{s} = s(d - er - fs)$$
$$\text{where } a, b, ..., f > 0$$

This is an example with continuous time and state space $[0, \infty) \times [0, \infty)$

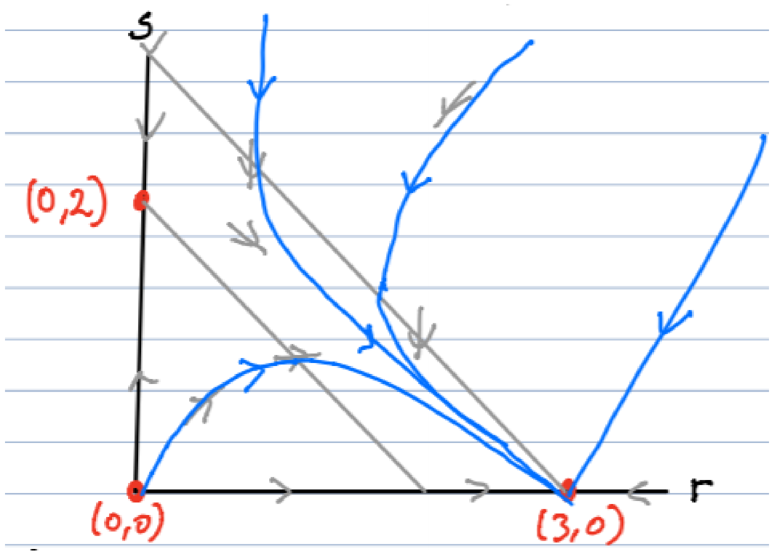
Both of these are population biology sourced models.

This course will focus on ODE's and maps in a small number of dimensions. It is usually not possible to write down a closed-form solution. Instead, we use a mix of geometric and analytic arguments, aiming to say something about general long-term behaviour... this is the 'dynamical systems approach'.

Example.

$$\dot{r} = r(3 - r - s)$$
$$\dot{s} = s(2 - r - s)$$
$$\text{for } r, s \geq 0 \text{ (the positive quadrant)}$$

First, we calculate the fixed points of the system, where $\dot{r} = \dot{s} = 0$. These can easily be worked out as $(0, 0), (0, 2)$ and $(3, 0)$. Then, we look at the sign of $\dot{r}$ and $\dot{s}$ around the fixed points, testing along suitable lines such as $r + s = 2$ or $r + s = 3$ as shown in the diagram, filling in directional arrows as we go. Finally, sketch on some solutions using the arrows as a guide, this gives us information about the long term behaviour of the system. In this example, if $r > 0$ initially, tend to the stable fixed point at $(3, 0)$



Example.

$$\dot{x} = -y + \epsilon x(\mu - x^2 - y^2)$$
$$\dot{y} = x + \epsilon y(\mu - x^2 - y^2)$$

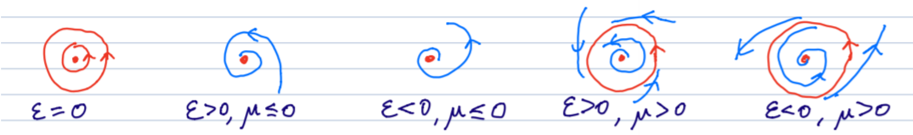
We immediately see that it would be useful to change to polar co-ordinates:

$$\dot{r} = \frac{x\dot{x} + y\dot{y}}{r}$$
$$\dot{\theta} = \frac{x\dot{y} - \dot{x}y}{r^2}$$

And therefore

$$\dot{r} = \epsilon(\mu r - r^3)$$
$$\dot{\theta} = 1$$

Here the behaviour of the system depends on the variables ϵ and μ:



There is a *periodic orbit* at $r = \mu^{\frac{1}{2}}$ for $\mu > 0$.

If $\epsilon \neq 0$ and μ is increased, a periodic orbit is created at $\mu = 0$, this is *bifurcation*.

At $\epsilon = 0$, the system is sensitive to small changes in ϵ, this is called *structural instability*

In 3 or more dimensions, ODEs can have much more complex behaviours, including chaos. This course will focus on chaos in maps, which is even possible in one dimension, e.g a logistic map.

1 Basic Definitions

1.1 Notation

Only consider ODEs of the form $\dot{\mathbf{x}} = \frac{d\mathbf{x}}{dt} = \mathbf{f}(\mathbf{x})$

for $\mathbf{x}$ in the state space $E \subseteq \mathbb{R}^n$ for some given n.

The n first order ODEs gives a dynamical system of order n. Since $\mathbf{f}$ does not depend explicitly on t, the system is *autonomous*.

Remark. A non-autonomous system $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$ can be made autonomous, by setting $\mathbf{y} = (\mathbf{x}, t)$ with $\dot{\mathbf{y}} = (\dot{\mathbf{x}}, 1)$

An n^{th} order ODE $\frac{d^n x}{dt^n} = g(x, \frac{dx}{dt}, ..., \frac{d^{n-1}x}{dt^{n-1}})$ can be put into the form above:
Set $\mathbf{y} = (x, \frac{dx}{dt}, ..., \frac{d^{n-1}x}{dt^{n-1}})$ with $\dot{\mathbf{y}} = (y_2, y_3, ..., y_n, g(y_1, y_2, ..., y_n))$

Similarly, only consider maps of the form $[\mathbf{x}_{n+1} = F(\mathbf{x_n})]$

1.2 Initial Value Problems

Consider the initial value problem $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x})$, $\mathbf{x}(t_0) = \mathbf{x}_0$

Provided $\mathbf{f}$ satisfies a Lipschitz condition (i.e $\exists L, a > 0$ s.t $|\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{y})| < L\,|\mathbf{x} - \mathbf{y}|$ for all $|\mathbf{x} - \mathbf{x_0}|\,, |\mathbf{y} - \mathbf{y_0}| < a$, then a unique solution $\mathbf{x}(t)$ is guaranteed to exist in a neighbourhood $(\mathbf{x_0}, \mathbf{t_0})$.

Moreover,neighbouring solutions $\mathbf{x}(t; \mathbf{x}') to \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \mathbf{x}(t_0) = \mathbf{x}'$ exist and are unique and depend continuously on $\mathbf{x}'$.

Remark. Beware! Not guaranteed existence for all time, e.g $\dot{x} = x^2, x(0) = 1$ is solved by $x(t) = (1-t)^{-1}$ but has a finite time blow-up ($|x| \to \infty$ as $t \to 1$)

Example. Non-Uniqueness ($\mathbf{f}$ not Lipschitz)

$$\begin{cases} \dot{x} = \sqrt{x} \text{ for } x \geq 0 \\ \dot{x} = 0 \text{ for } x < 0 \end{cases}$$

This is solved by

$$\begin{cases} x = 0 \text{ for } t < \tau \\ x = \frac{1}{4}(t-\tau)^2 \text{ for } t \geq \tau \end{cases}$$

Which is a family of solutions for any $\tau > 0$

For now, we assume $\mathbf{f}$ to be differentiable ($\implies$ Lipschitz).

1.3 Trajectories and Flows

Consider $\mathbf{x} = \mathbf{f}(\mathbf{x}), \boldsymbol{x} \in E$. The solution $\mathbf{x}(t)$ with $\mathbf{x}(t_0) = \mathbf{x}_0$ defines the trajectory (or integral curve, or orbit) through $\mathbf{x}_0$. Clearly position depends only on $\mathbf{x}_0$ and $t - t_0$.

The *flow*, written as a function

$$\boldsymbol{\phi_t}(\mathbf{x}) : E \times \mathbb{R} \to E$$

Is such that

$$\frac{\partial}{\partial t}\boldsymbol{\phi_t}(\mathbf{x}) = \mathbf{f}(\boldsymbol{\phi_t}(\mathbf{x})), \boldsymbol{\phi_0}(\mathbf{x}) = \mathbf{x}$$

So $\boldsymbol{\phi_t}(\mathbf{x_0})$ is the position if we start at $\mathbf{x_0}$ and time t elapses.

The solution to

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \mathbf{x}(t_0) = \mathbf{x_0}$$

is

$$\mathbf{x}(t) = \boldsymbol{\phi_{t-t_0}}(\mathbf{x_0})$$

Clearly we also have that

$$\boldsymbol{\phi_s}(\boldsymbol{\phi_t}(\mathbf{x})) = \boldsymbol{\phi_{s+t}}(\mathbf{x}) = \boldsymbol{\phi_t}(\boldsymbol{\phi_s}(\mathbf{x}))$$

1.4 Orbits, Invariant Sets and Limit Sets

Definition. Using the idea of flow, we make the following definitions:

(i) The *orbit/trajectory* of $\boldsymbol{\phi_t}$ through $\mathbf{x_0}$ is the set

$$\Theta(\mathbf{x_0}) = \{\boldsymbol{\phi_t}(\mathbf{x_0}) : -\infty < t < \infty\}$$

(ii) The *forward orbit* of $\boldsymbol{\phi_t}$ through $\mathbf{x_0}$ is the set

$$\Theta^+(\mathbf{x_0}) = \{\boldsymbol{\phi_t}(\mathbf{x_0}) : t \geq 0\}$$

and similarly the *backwards orbit*

$$\Theta^-(\mathbf{x_0}) = \{\boldsymbol{\phi_t}(\mathbf{x_0}) : t \leq 0\}$$

(iii) A set of points $\Lambda \subset E$ is an *invariant set* under $\mathbf{f}$ if

$$\mathbf{x} \in \Lambda \implies \Theta(\mathbf{x}) \subset \Lambda$$

Clearly any orbit $\Theta(\mathbf{x}_0)$ is invariant, as is any union of orbits.

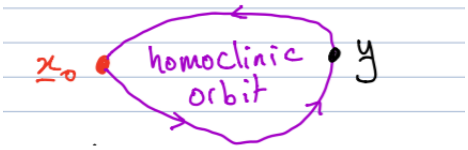
(iv) $\mathbf{x_0}$ is a *fixed point* (equilibrium, stationary/critical point) if $\mathbf{f}(\mathbf{x_0}) = \mathbf{0}$. Then $\mathbf{x} = \mathbf{x_0}$ for all time $\Theta(\mathbf{x_0}) = \mathbf{x_0}$

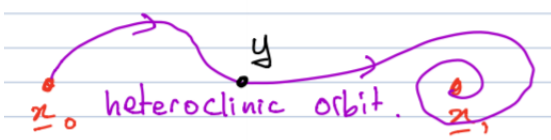
(v) $\mathbf{x_0}$ is a *periodic point* with period T if $\boldsymbol{\phi_T}(\mathbf{x_0}) = \mathbf{x_0}$ for some $T > 0$ but $\boldsymbol{\phi_t}(\mathbf{x_0}) \neq \mathbf{x_0}$ for $0 < t < T$. The set $\{\boldsymbol{\phi_t}(\mathbf{x_0}) : 0 \leq t < T\}$ is the *periodic orbit* through $\mathbf{x_0}$

(vi) A *limit cycle* is an isolated periodic orbit, i.e there are no other periodic orbits in some small neighbourhood

(vii) If $\mathbf{x_0}$ is a fixed point and $\exists \mathbf{y} \neq \mathbf{x_0}$ s.t $\boldsymbol{\phi_t}(\mathbf{y}) \to \mathbf{x_0}$ as both $t \to \infty$ and $t \to -\infty$ then $\Theta(\mathbf{y})$ is a *homoclinic orbit*

(viii) If $\mathbf{x_0} \neq \mathbf{x}$ are both fixed points and $\exists \mathbf{y}$ s.t $\boldsymbol{\phi_t}(\mathbf{y}) \to \mathbf{x_0}$ as $t \to -\infty$ and $\boldsymbol{\phi_t}(\mathbf{y}) \to \mathbf{x}$ as $t \to \infty$ then $\Theta(\mathbf{y})$ is a *heteroclinic orbit*





Definition. If we are interested in the long-term behaviour of trajectories, it is not enough to think of $\lim_{t\to\infty} \boldsymbol{\phi_t}(\mathbf{x})$ since this may not exist, such as in a periodic orbit. Instead, we use the following definitions:

(i) The *ω-limit set* of $\mathbf{x}$ is

$$\omega(\mathbf{x}) = \{\mathbf{y} : \exists \text{ an infinite sequence } t_1, t_2, ... \to \infty \text{ with } \boldsymbol{\phi_{t_n}}(\mathbf{x}) \to \mathbf{y}\}$$

(ii) The *α-limit set* of $\mathbf{x}$ is

$$\alpha(\mathbf{x}) = \{\mathbf{y} : \exists \text{ an infinite sequence } t_1, t_2, ... \to -\infty \text{ with } \boldsymbol{\phi_{t_n}}(\mathbf{x}) \to \mathbf{y}\}$$

Remark. $\omega(\mathbf{x})$ is an invariant set:

$$\boldsymbol{\phi_{t_n}}(\mathbf{x}) \to \mathbf{y} \implies \boldsymbol{\phi_{t_n+T}}(\mathbf{x}) \to \boldsymbol{\phi_T}(\mathbf{y})$$

So if $\mathbf{y} \in \omega(\mathbf{x})$, so is $\boldsymbol{\phi_T}(\mathbf{y})$

If $\Theta(\mathbf{x})$ is bounded, we have that $\omega(\mathbf{x})$ is non-empty, closed, bounded and connected.

Example.

$$\dot{r} = r(1-r^2)$$
$$\dot{\theta} = 1$$
$$\text{for } 0 < |\mathbf{x}| < 1$$

Then we have

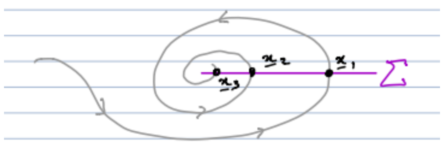
$$\omega(\mathbf{x}) = \{r = 1\}$$
$$\alpha(\mathbf{x}) = \{\mathbf{0}\}$$

For maps: fixed points solve $\mathbf{F}(\mathbf{x}) = \mathbf{x}$. Other definitions are largely similar to ODEs, but swap $\boldsymbol{\phi_t}(\mathbf{x})$ with $\mathbf{F} * (\mathbf{x})$. Orbits of period N are called *N-cycles*.

Remark. We can make maps from ODEs in two natural ways:

(i) Take snapshots of fixed time interval δt: $\mathbf{x_{n+1}} = \mathbf{F}(\mathbf{x}_n) = \boldsymbol{\phi_{\delta t}}(\mathbf{x})$

(ii) Poincare return map: successive intersections with line/surface $\sum$



1.5 Topological Equivalence and Structural Stability of Flows

Please see handout, annotated in lecture 3.

2 Fixed Points

2.1 Linearisation

If $\mathbf{f}$ is sufficiently smooth near a fixed point $\mathbf{x}_0$, Taylor expand (knowing $\mathbf{f}(x_0) = 0$)

$$\dot{\mathbf{y}} = A\mathbf{y} + O(|\mathbf{y}|^2)$$
$$\text{where } \mathbf{y} = \mathbf{x} - \mathbf{x_0}$$
$$\text{and } A_{ij} = \frac{\partial f_i}{\partial x_j}|_{\mathbf{x}_0}$$

is the Jacobian matrix of $\mathbf{f}$ at $\mathbf{x}_0$

The hope is that the flow near $\mathbf{x}_0$ of the full system is like that of $\dot{\mathbf{y}} = A\mathbf{y}$.

We consider the 2D case

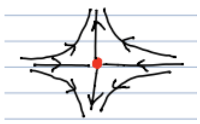
$$A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$
$$D = DetA = ad - bc$$
$$T = TrA = a + d$$

Then the eigenvalues λ satisfy

$$\lambda^2 - T\lambda + D = 0 \implies \lambda = \frac{1}{2}(T \pm \sqrt{T^2 - 4D})$$

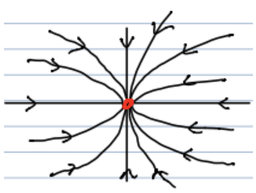
By a suitable change of basis, we can put A into *canonical form*:

(i) **Saddle point** ($D < 0$)
$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ wlog $\lambda_1 < 0 < \lambda_2$



(ii) **Stable node** ($D > 0, T < 0, T^2 > 4D$)
$A = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$ wlog $\lambda_1 < \lambda_2 < 0$

$y_2 \propto e^{\lambda_2 t}$, $y_1 \propto e^{\lambda_1 t}$, so $y_2 \propto y_1^{\frac{\lambda_2}{\lambda_1}}$



(iii) **Unstable node** ($D > 0, T > 0, T^2 > 4D$)
Here $\lambda_1 > \lambda_2 > 0$, same as (ii) but reverse time.

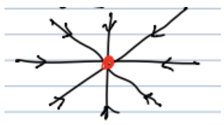
(iv) **Stable focus** $(D > 0, T < 0, T^2 < 4D)$
Complex λ, $\lambda = \rho \pm i\omega$ with $\rho < 0, \omega \neq 0$ ($\omega > 0$ wlog)
$A = \begin{pmatrix} \rho & -\omega \\ \omega & \rho \end{pmatrix} \implies \dot{r} = \rho r \qquad \dot{\theta} = \omega$

(v) **Unstable focus** $(D > 0, T > 0, T^2 < 4D)$
$\lambda = \rho \pm i\omega \qquad \rho > 0$, which is the same as (iv) except spiralling outwards.

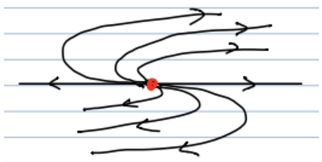
The rest are all ***degenerate*** cases:

(i) If $D > 0, T^2 = 4D$ we have a boundary between a focus and node.
There are two sub-cases:

(a) If $A = \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix}$ then we have a **stellar node**

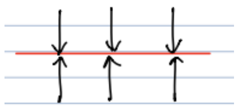


(b) If $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$ then we have an **improper node**

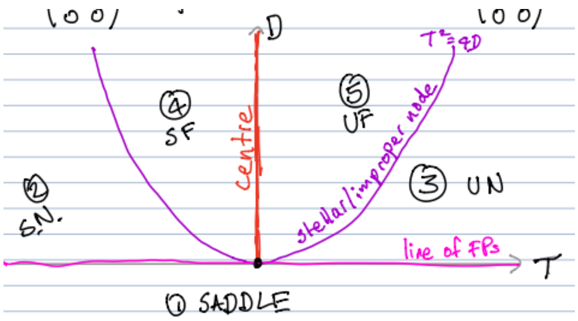


(ii) If $D > 0, T = 0$ then we have a boundary between stable/unstable foci called a **centre**
$A = \begin{pmatrix} 0 & -\omega \\ \omega & 0 \end{pmatrix}$

(iii) If $D = 0, T \neq 0$ we are between a saddle and a node, where one $\lambda = 0$, and the other $\neq 0$. $A = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}$



(iv) If $D = 0, T = 0 \qquad \lambda = 0, 0$
If $A = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$ then we have a plane of fixed points, else if $A = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$ for example then we have a line of fixed points.



The above diagram illustrates a plot of T against D, and the resulting fixed point classification for each case.