Part II — Asymptotic Methods

Based on lectures by E. S. Titi

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Asymptotic expansion

Definition (Poincare) of $\phi(z) \sim \sum a_n z^{-n}$; examples; elementary properties; uniqueness; Stokes' phenomenon [4]

Asymptotics behaviour of functions defined by integrals

Integration by parts. Watsons lemma and Laplaces method. RiemannLebesgue lemma and method of stationary phase. The method of steepest descent (including derivation of higher order terms). Airy function, *and application to wave theory of a rainbow*. [7]

Asymptotic behaviour of solutions of differential equations

Asymptotic solution of second-order linear differential equations, including Liouville-Green functions (proof that they are asymptotic not required) and WKBJ with the quantum harmonic oscillator as an example. [4]

Recent developments

Further discussion of Stokes phenomenon. *Asymptotics beyond all orders*. [1]

Contents

0 Introduction

Consider $f(x) = x + 3 + \frac{1}{x^2}$ (1), from the sketch we see that $f(x) - (x+3) = \frac{1}{x^2} \underset{x\to\infty}{\to} 0$. Now, for $g(x) = x^2 + x + 3\frac{1}{x^3}$ we obtain the asymptote $y = x^2 + x + 3$ from $g(x) - (x^2 + x + 3)\frac{1}{x^3} \underset{x\to\infty}{\to} 0$. In this course, we will study

(i) Functions

(ii) Solutions to differential equations

(iii) Integrals

Important definitions:

Definition (Big $\mathcal{O}$)**.** – $f : (a, \infty) \to \mathbb{C} \setminus \mathbb{R}$, $g : (a, \infty) \to \mathbb{R}$. Assume that $g(x) > 0$ for $x \geq A \geq a$. We say that $f(x) = \mathcal{O}(g(x))$ as $x \to \infty$ if $\exists\ M > 0, B > 0$ such that

$$|f(x)| \leq Mg(x) \quad x \geq B > A.$$

(2)

– We say $f(x) = \mathcal{O}(g(x))$ as $x \to x_0$ if $\exists\ M, \delta > 0$ such that

$$\frac{|f(x)|}{g(x)} \leq M; \quad o < |x - x_0| < \delta.$$

(3) i.e.

$$\limsup_{x\to x_0} \frac{|f(x)|}{g(x)} < \infty.$$

Observation If $f(x) = \mathcal{O}(g(x))$ as $x \to x_0$, then $cf = \mathcal{O}(g(x)), c \in \mathbb{R}$

Example. Let $f(x) = \frac{1}{x^2} \sin \frac{1}{x}$ as $x \to x_0$, $f(x) = \mathcal{O}(\frac{1}{x^2})$

$$\limsup \frac{|f(x)|}{g(x)} = \limsup_{x\to 0} |\sin \frac{1}{x}| = 1 < \infty.$$

Definition (Little o)**.** We say $f(x) = o(g(x))$ as $x \to x_0$ if

$$\lim_{x\to x_0} \frac{f(x)}{g(x)} = 0.$$

More rigorously, $\forall\ \varepsilon > 0\ \exists\ \delta(\varepsilon)$ such that $0 < |x - x_0| < \delta$ then $\left|\frac{f(x)}{g(x)}\right| < \varepsilon$

Example. $f(x) = \frac{1}{x^2} \sin \frac{1}{x}$ and $g(x) = \frac{1}{x^3}$ as $x \to 0$. We claim $f(x) = o(g(x))$

$$\frac{f(x)}{g(x)} = \sin \frac{1}{x} \cdot x \underset{x\to 0}{\to} 0.$$

Example. $f(x) = 4x^2$ as $x \to \infty$

(i) $f(x) = \mathcal{O}(x^3), o(x^3)$ but not $o(x^2)$

Notation.

$$f(x) = o(g(x)) \iff f(x) \ll g(x).$$

1 Asymptotic Expansions / Series

Definition (Asymptotic sequence)**.** $\phi_n : D \subset \mathbb{C} \mapsto \mathbb{C} \setminus \mathbb{R}\, n = 0, 1, 2, \ldots$ is called an asymptotic sequence as $z \to z_0 \in D$ if

$$\phi_n(z) = o(\phi_m(z)) \text{ as } z \to z_0$$

for all

$$n > m \iff \frac{\phi_n(z)}{\phi_m(z)} \underset{z\to z_0}{\to} 0 \;\forall\; n > m.$$

Example. (i)

$$\frac{1}{x^3}, \frac{1}{x^2}, \frac{1}{x}, 1, x, x^2, \ldots, x^n, \ldots.$$

$x = 0$, $\frac{x^n}{x^m} = x^{n-m} \underset{x\to 0}{\to} 0$

(ii) $\frac{1}{(x-5)^k}, k = 0, 1, , \ldots,$ as $x \to \infty$

(iii) $\phi_n(x) = \frac{1}{x^2} \cos nx$ for $n = 0, 1, 2, \ldots$, consider

$$\frac{\phi_n(x)}{\phi_m(x)} = \frac{\cos nx}{\cos mx}.$$

But $\limsup \frac{\cos nx}{\cos mx} = 1 \neq 0$ hence is not an asymptotic sequence.

Definition (Asymptotic Exapansion)**.** Let $\phi_n : D \subset \mathbb{C} \to \mathbb{C}\backslash\mathbb{R}$ be an asymptotic sequence about $z \to z_0$. We say the sum $\sum_{n=0}^{\infty} a_n\phi_n$ is an *asymptotic expansion* of $f(z)$ as $z \to z_0$ if

$$f(z) - \sum_{n=0}^{N} a_n\phi_n(z) = o(\phi_N(z)) \; z \to z_0 \;\forall\; N = 0, 1, \ldots.$$

Or

$$\frac{f(z) - \sum_{n=0}^{N} a_n\phi_n(z)}{\phi_N(z)} \underset{z\to z_0}{\to} 0.$$

Notation. In this case, we write $f(z) \sim \sum_{n=0}^{\infty} a_n\phi_n(z)$.

Remark. We do not require $\sum a_n\phi_n(z)$ to converge. $*$

Proposition. If $f(z) \sim \sum_{n=0}^{\infty} a_n\phi_n(z)$ then

$$a_{N+1} = \lim_{z\to z_0} \frac{f(z) - \sum_{k=0}^{N} a_k\phi_k(z)}{\phi_{N+1}(z)}.$$

Consider $f(z) \sim \sum a_n\phi_n(z)$ for $n = 0, 1, 2, \ldots$ then

$$a_0 = 0 = \lim_{z\to z_0} \frac{f(z)}{\phi_0(z)}, \ldots.$$

Then

$$a_n = \lim_{z\to z_0} \frac{f(z)}{\phi_n(z)} = 0.$$

Example. (Taylor) $f : [a,b] \to \mathbb{R}, C^{\infty}$. Then $f(x) \sim \sum_{n=0}^{\infty} \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n$ $x_0 \in (a,b)$.

$$\frac{f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n}{(x-x_0)^N} \underset{x \to x_0}{\to} 0.$$

We can write

$$f(x) - \sum_{n=0}^{N} \frac{f^{(n)}(x_0)}{n!}(x-x_0)^n = R_N(x) = \int_{x_0}^{x} \frac{(x-t)^N}{N!} f^{(N+1)}(t) \mathrm{d}t.$$

Now

$$|R_n(x)| \leq \max_{a \leq t \leq b} f^{N+1}(t) \int_{x_0}^{x} \frac{|(x-t)^N|}{N!} \mathrm{d}t = \frac{|x-x_0|^{N+1}}{(N+1)!} \max |f^{(N+1)}(t).$$

So

$$\left| \frac{R_N(x)}{(x-x_0)^N} \right| \leq \frac{|x-x_0|}{(N+1)!} \max |f^{(N+1)}(t)| \underset{x \to x_0}{\to} 0.$$

Proposition. (i) Let $f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z)$ and $g(z) \sim \sum_{n=0}^{\infty} b_n \phi_n(z)$, then

$$\alpha f(z) + \beta g(z) \sim \sum_{n=0}^{\infty} (\alpha a_n + \beta b_n) \phi_n(z).$$

(ii) Let $f(z) \sim \sum_{n=0}^{\infty} a_n (z-z_0)^n$ and $g(z) \sim \sum_{n=0}^{\infty} b_n (z-z_0)^n$, then

$$f(z)g(z) \sim \sum_{n=0}^{\infty} c_n (z-z_0)^n, c_n = \sum_{k=0}^{n} a_k b_{n-k}.$$

Proposition. The asymptotic expansion is unique, that is if $f(z) \sim \sum_{n=0}^{\infty} a_n \phi_n(z)$ and $f(z) \sim \sum_{n=0}^{\infty} b_n \phi_n(z)$ then $a_n = b_n \ \forall \ n$

$$a_0 - b_0 = \frac{(a_0 - b_0)\phi_0(z)}{\phi_0(z)} = \frac{(a_0 + \phi_0(z) - f(z)) + (f(z) - b_0 \phi_0(z))}{\phi_0(z)} \underset{z \to z_0}{\to} 0.$$

The rest are done by induction.

Consider

$$H(x) = \begin{cases} e^{-\frac{1}{x^2}} & x \neq 0 \\ 0 & x = 0 \end{cases}.$$

$f^{(n)}(0) = 0$

$$\sin x = \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}.$$

But

$$\sin x + H(x) \sim \sum_{n=0}^{\infty} \frac{1}{(2n+1)!} x^{2n+1}.$$