Part II — Stochastic Financial Models

Based on lectures by J. R. Norris

Notes taken by Joseph Tedds using Dexter Chua's header and Gilles Castel's snippets.

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These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Utility and mean-variance analysis
Utility functions; risk aversion and risk neutrality. Portfolio selection with the mean-variance criterion; the efficient frontier when all assets are risky and when there is one riskless asset. The capital-asset pricing model. Reservation bid and ask prices, marginal utility pricing. Simplest ideas of equilibrium and market cleaning. State-price density. [5]

Martingales
Conditional expectation, definition and basic properties. Conditional expectation, definition and basic properties. Stopping times. Martingales, supermartingales, submartingales. Use of the optional sampling theorem. [3]

Dynamic Models
Introduction to dynamic programming; optimal stopping and exercising American puts; optimal portfolio selection. [3]

Pricing contingent conditions
Lack of arbitrage in one-period models; hedging portfolios; martingale probabilities and pricing claims in the binomial model. Extension to the multi-period binomial model. Axiomatic derivation. [4]

Brownian motion
Introduction to Brownian motion; Brownian motion as a limit of random walks. Hitting-time distributions; changes of probability. [3]

Black-Scholes model
The BlackScholes formula for the price of a European call; sensitivity of price with respect to the parameters; implied volatility; pricing other claims. Binomial approximation to BlackScholes. Use of finite-difference schemes to compute prices [6]

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0 Introduction

1 Utility and mean-variance analysis

1.1 Contingency claims and utility functions

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. A random variable X on Ω, provides a model for an investment which delivers $X(\omega)$ for consumption depending on chance $\omega \in \Omega$.

Definition (Contingent claim)**.** In this context we often use the term *contingent claim* as another name for a random variable.

Definition (Utility function)**.** By a *utility function* we mean any non-decreasing function $U : \mathbb{R} \mapsto [-\infty, \infty)$. Think of $U(x)$ as quantifying the satisfaction obtained on consuming x. Allowing $-\infty$ is a way of saying the value of x that obtains $-\infty$ is unacceptable.

We often assume the investor will act to maximise expected utility. So Y is *preferred* to X iff $\mathbb{E}[U(X)] \leq \mathbb{E}[U(Y)]$. If $\mathbb{E}[U(X)] = \mathbb{E}[U(Y)]$ the investor is said to be *indifferent* between X and Y. We say the investor is *risk averse* if they prefer $\mathbb{E}[X]$ to X for all integrable random variables X. We say *risk neutral* if indifferent between X and $\mathbb{E}[X]$.

Definition. Recall that U is a *concave* function if for all $x, y \in \mathbb{R}$, all $p \in (0,1)$

$$pU(x) + (1-p)U(y) \leq U(px + (1-p)y).$$

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Proposition. An investor with utility function U is risk averse if and only if U is concave.

Proof. Suppose risk averse. Consider the contingent claim X taking values x, y with probabilities $p, (1-p)$ respectively. Then,

$$pU(x) + (1-p)U(y) = \mathbb{E}[U(X)] \leq U(\mathbb{E}[X]) = U(px + (1-p)y).$$

Hence U is concave.
Suppose on the other hand U is concave. Let X be an integrable random variable (i.e. $\mathbb{E}[|X|] < \infty$) then by Jensen's inequality

$$\mathbb{E}[U(X)] \leq U(\mathbb{E}[X]).$$

Hence, the investor is risk averse. □

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Example. For $\gamma \in (0, \infty)$ the CARA (constant absolute relative aversion) utility function of parameter γ is given by

$$\mathrm{CARA}_\gamma(x) = -e^{-\gamma x}.$$

3 For $R \in (0,1) \cup (1, \infty)$ the CRRA (constant relative risk aversion) utility function of parameter R is given by

$$\mathrm{CRRA}_R(x) = \begin{cases} \frac{x^{1-R}}{1-R} & x > 0 \\ -\infty & x \leq 0 \end{cases}.$$

Also,

$$\mathrm{CRRA}_1(x) = \begin{cases} \log x & x > 0 \\ -\infty & \text{otherwise} \end{cases}.$$

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Non-rigorous discussion Let U be concave (note that U is non-decreasing). Consider a small continent claim X. We ask whether we prefer $w + X$ to w for a given constant w. By Taylor's theorem

$$U(w+X) \approx U(w) + X \underbrace{U'(w)}_{>0} + \frac{1}{2}X^2 \underbrace{U''(w)}_{<0}.$$

$$\mathbb{E}\left[U(w+X)\right] \approx U(w) + \mathbb{E}\left[X\right] U'(w) + \frac{1}{2}\mathbb{E}\left[X^2\right] U''(w),$$

so we prefer $w + X$ if

$$2\frac{\mathbb{E}\left[X\right]}{\mathbb{E}\left[X^2\right]} > -\frac{U''(w)}{U'(w)}$$

the Arrow-Pratt coefficient of absolute risk aversion. For CARA_γ this constant is equal to γ.
Similarly, do we prefer $w(1+X)$ to w? Yes if

$$2\frac{\mathbb{E}\left[X\right]}{\mathbb{E}\left[X^2\right]} > -\frac{wU''(w)}{U'(w)}$$

the Arrow-Pratt coefficient of relative risk aversion. For CRRA_R this constant is equal to R.

1.2 Reservation prices and marginal prices

Consider an investor with concave utility function. Suppose they have available a set $\mathcal{A}$ of contingent claims, and suppose $\mathbb{E}\left[U(X)\right]$ is maximised over $\mathcal{A}$ at $X^* \in \mathcal{A}$. Let Y be another contingent claim. The investor would buy Y for price π if there exists $X \in \mathcal{A}$ such that

$$\mathbb{E}\left[U(X+Y-\pi)\right] > \mathbb{E}\left[U(X^*)\right].$$

The supremum of all such prices $\pi_b(Y)$ is the *(reservation) bid price)* for Y.

The investor would sell Y for price π if there exists $X \in \mathcal{A}$ such that

$$\mathbb{E}\left[U(X-Y+\pi)\right] > \mathbb{E}\left[U(X^*)\right].$$

The infimum of all such prices $\pi_a(Y)$ is the *(reservation) ask price* for Y.

Proposition. (Ask above, bid below) Assume $\mathcal{A}$ is convex. Then $\pi_b(Y) \leq \pi_a(Y)$

Proof. It suffices to show there is no price π at which the investor will both buy and sell. Suppose for a contradiction that there exist X_a, X_b such that

$$\mathbb{E}\left[U(X_a-Y+\pi)\right] > \mathbb{E}\left[U(X^*)\right].$$

$$\mathbb{E}\left[U(X_b+Y-\pi)\right] > \mathbb{E}\left[U(X^*)\right].$$

Now $X = \frac{X_a+X_b}{2} \in \mathcal{A}$ since $\mathcal{A}$ is convex and $U(X) \geq \frac{U(X_a-Y+\pi)+U(X_b+Y-\pi)}{2}$ since U is concave. Then we obtain the following contradiction.

$$\mathbb{E}\left[U(X^*)\right] < \frac{\mathbb{E}\left[U(X_a - Y + \pi)\right] + \mathbb{E}\left[U(X_b + Y - \pi)\right]}{2} \leq \mathbb{E}\left[U(X)\right] \leq \mathbb{E}\left[U(X^*)\right].$$

Hence there is no such π. $\square$

Recall U is concave and non-decreasing. An investor has available a set of of contingent claims $\mathcal{A}$, and seeks to maximise $\mathbb{E}\left[U(X)\right], X \in \mathcal{A}$. Assume $X^* \in \mathcal{A}$ is a maximiser. Suppose Y is another contingent claim. Assume that $\mathcal{A}$ is an affine space and that U is a differentiable and strictly concave.

Definition (Affine space)**.** S is *affine* if $S - S$ is a vector space. This can be visualised as a vector space away from the origin.

Then X^* is unique (or $\frac{X_1^*+X_2^{ast}}{2}$ is better.

Definition (Marginal price)**.** We define the *marginal price* of Y as

$$\pi_m(Y) = \mathbb{E}\left[U'(X^*)Y\right] / \mathbb{E}\left[U'(X^*)\right].$$

Non-rigorous discussion to explain Note that for $\Xi \in \mathcal{A} - \mathcal{A}$ the map $t \mapsto \mathbb{E}\left[U(X^* + t\Xi\right]$ on $\mathbb{R}$ achieves its minimum at $t = 0$. So

$$0 = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_0 \mathbb{E}\left[U(X^* + t\Xi)\right] = \mathbb{E}\left[U'(X^*)\right].$$

It is plausible that there is a differentiable map $t \mapsto X^*(t) : \mathbb{R} \leftarrow \mathcal{A}$ such that for all t

$$\mathbb{E}\left[U(X^*(t) - tY + \pi_b(tY))\right] = \mathbb{E}\left[U(X^*)\right].$$

Then $X^*(0) = X^*$. Define $\Xi \in \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_0 X^*(t), \pi = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_0 \pi_b(tY)$. It is plausible that $\Xi \in \mathcal{A} - \mathcal{A}$. So

$$0 = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_0 \mathbb{E}\left[U(X^*(t) - tY + \pi_b(tY))\right] = \mathbb{E}\left[U'(X^*)(\Xi - Y + \pi)\right].$$

So we see

$$\pi_m(Y) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_0 \pi_b(tY) = \left.\frac{\mathrm{d}}{\mathrm{dt}}\right|_0 \pi_a(tY).$$

So marginal price is the price to buy (or sell) a small amount of Y.

1.3 Single period asset price model

Definition (Single period asset price model)**.** By a *single period asset price model*, we mean a pair of random variables (S_0, S_1) in $\mathbb{R}^d$. We write $S_n = (S_n^1, \dots, S_n^d$ with S_n^i the price of asset i at time n.

Definition (Numeraire)**.** By a *numeraire* we mean a pair of random variables $(S_0^0, S_1^0$ in $(0, \infty)$.

Notation. We write

$$\overline{S}_n = (S_n^0, S_n) = (S_n^0, S_n^1, \dots, S_n^{d)}.$$

Call $(\overline{S}_0, \overline{S}_1)$ an *asset price model with numeraire*

Often we take $S_0^0 = 1, S_1^0 = 1 + r$ some constant $r \in (-1, \infty)$, Then S^0 is called a *riskless bond* and r is the *interest rate*. We assume $\overline{S}_0$ is non-random as the default.

In the case without numeraire, an investor with initial wealth w_0 chooses $\theta \in \mathbb{R}^d$ subject to

$$\theta . S_0 = \sum_{i=1}^{d} \theta^i S_0^i = w_0.$$

Then the investor has wealth $\theta . S_1$ at time 1. We call θ the *portfolio*. With numeraire, investor chooses $\overline{\theta} = (\theta^0, \theta$ such that $\overline{\theta} . \overline{S}_0 = w_0$. The wealth at time 1 is $\overline{\theta} . \overline{S}_0$.

It may be that there exists a random variable $\rho \geq 0$ such that $\mathbb{E}\left[\rho S_1^i\right] = S_0^i$ for all i. Then we call ρ a *state price density*

1.4 Portfolio selection using the mean-variance criterion

Let (S_0, S_1) be an asset price model on $\mathbb{R}^d$ with S_0 non-random, S_1 has mean μ, variance V. We assume that V is invertible and S_0, μ are linearly independent. Suppose we are given w_0, w_1. The investor wishes to

$$\begin{aligned} \text{minimise} \quad & \operatorname{var}(\theta . S_1) \\ \text{subject to} \quad & \theta . S_0 = w_0, \\ & \mathbb{E}\left[\theta . S_1\right] = w_1 \end{aligned}$$

Note $\mathbb{E}\left[\theta . S_1\right] = \theta . \mu$, $\operatorname{var}(\theta . S_1) = \theta . (V\theta)$ So our problem is to

$$\begin{aligned} \text{minimise} \quad & \theta . (V\theta) \\ \text{subject to} \quad & \theta . S_0 = w_0, \\ & \theta . \mu = w_1. \end{aligned}$$

Consider $L(\theta, \lambda) = \frac{1}{2}\theta . (V\theta) - \lambda_0 \theta . S_0 - \lambda_1 \theta . \mu$ At minimising θ^*.

$$\begin{aligned} 0 &= \frac{\partial}{\partial \theta^i} L(\theta, \lambda) \\ &= (V\theta)^i - \lambda_0 S_0^i - \lambda_1 \mu^i. \end{aligned}$$

So $\theta^* = \lambda_0 A S_0 + \lambda_1 A \mu, A = V^{-1}$. Now fit the constants

$$\begin{aligned} w_0 &= \theta^* . S_0 = \lambda_0 a + \lambda_1 b \\ w_1 &= \theta^* . \mu = \lambda_0 b + \lambda_1 c \end{aligned}$$

$a = S_0 . (AS_0), b = \mu(AS_0) = S_0(A\mu), c = \mu(A\mu)$. Note that $\Delta = ac - b^2 \neq 0$ by linear independence

$$\begin{pmatrix} w_0 \\ w_1 \end{pmatrix} = M \begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix}, M = \begin{pmatrix} a & b \\ b & c \end{pmatrix}.$$

$$M^{-1} = \frac{1}{\Delta} \begin{pmatrix} d & -b \\ -b & a \end{pmatrix}.$$

$$\begin{pmatrix} \lambda_0 \\ \lambda_1 \end{pmatrix} = M^{-1} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}.$$

So
$$\theta^* = \frac{cw_0 - bw_1}{\Delta} AS_0 + \frac{aw_1 - bw_0}{\Delta} A\mu$$
The minimising variance is
$$\begin{aligned}\theta^*_{.}(V\theta^*) &= (\lambda_0 AS_1 + \lambda_1 A\mu)_{.}(\lambda_0 S_0 + \lambda_1 \mu)\\ &= (\lambda_0 \lambda_1) M \begin{pmatrix}\lambda_0\\ \lambda_1\end{pmatrix}\\ &= (w_0 w_1) M^{-1} \begin{pmatrix} w_0 \\ w_1 \end{pmatrix}\\ &= \frac{cw_0^2 - 2bw_0 w_1 + aw_1^2}{\Delta} = q(w_1)\end{aligned}$$
We minimise this over w_1
$$w_1^* = \frac{b}{a} w_0, \theta^*_{\min} = \frac{w_0}{a} AS_0.$$
Putting w_1^* back into q, we obtain
$$q(w_1^*) = \frac{acw_0^2 - 2b^2 w_0^2}{a\Delta} + \frac{b^2}{a\Delta} w_0^2 = \frac{w_0^2}{a}$$

Suppose we seek to
$$\begin{array}{rl}\text{minimise} & \operatorname{var}(\theta.S_1)\\ \text{subject to} & \theta.S_0 = w_0,\end{array}$$

Consider $L(\theta, \lambda) = \frac{1}{2}\theta.(V\theta) - \lambda\theta.S_0$. At minimiser,
$$0 = \frac{\partial}{\partial\theta^i} L(\theta, \lambda) = (V\theta)^i - \lambda S_0^i.$$
So
$$V\theta^* = \lambda S_0, \quad \theta^* = \lambda AS_0, A = V^{-1}.$$
Use the constraint to find $\lambda : w_0 = \theta^*_{.} S_0 = \lambda \underbrace{a}_{S_0.}(AS_0)$. Hence $\theta^* = \frac{w_0}{a} AS_0 = \theta^*_{\min}$.
Add a riskless bond / bank account.
$$S^0 = 1, S_1^0 = 1 + r > 0.$$
Suppose we seek to
$$\begin{array}{rl}\text{minimise} & \operatorname{var}(\overline{\theta}.\overline{S}_1)\\ \text{subject to} & \overline{\theta}.\overline{S}_0 = w0\\ & \mathbb{E}\left[\overline{\theta}.\overline{S}_1\right] = w_1\end{array}$$

Recalling that $\overline{\theta} = (\theta^0, \theta), \overline{S}_n = (S_n^0, S_n)$. Now $\mathrm{var}(\overline{\theta}.\overline{S}_1 = \theta.(V\theta)$. $\mathbb{E}\left[\overline{\theta}.\overline{S}_1\right] = \theta^0(1+r) + \theta.\mu$. So our problem is to

$$\begin{array}{ll} \text{minimise} & \theta.(V\theta),\ V \text{ invertible} \\ \text{subject to} & \theta^0 + \theta.S_0 = w_0 \quad (1) \\ & \theta^0(1+r) + \theta.\mu = w_1 \quad (2) \end{array}$$

Use (1) to eliminate θ^0 in (2).

$$(w_0 - \theta.S_0)(1+r) + \theta.\mu = w_1.$$

i.e.

$$\theta.(\mu - (1+r)S_0) = w_1 - (1+r)w_0.$$

Set $L(\theta, \lambda) = \frac{1}{\theta.(V\theta) - \lambda\theta.(\mu - (1+r)S_0)}$. At θ^*,

$$0 = \frac{\partial}{\partial \theta^i} L(\theta, \lambda) = (V\theta)^i - \lambda(\mu^i - (1+r)S_0^i).$$

So

$$\theta^* = \lambda \underbrace{(A\mu - (1+r)S_0)}_{\theta_m^* = \theta_{\text{market}}^*}, \quad A = V^{-1}.$$

Find λ using the remaining constraint

$$\lambda \underbrace{(c - 2b(1+r) + (1+r)^2 a)}_{>0 \text{ by Cauchy Schwarz}} = w_1 - (1+r)w_0,$$

where

$$a = S_0.(AS_0), b = \mu(AS_0) = S_0(A\mu), c = \mu(A\mu)$$

as before. So

$$\lambda = \frac{w_1 - (1+r)w_0}{(1+r)^2 a - 2b(1+r) + c}.$$

1.5 Portfolio selection using CARA utility

Take as utility function

$$U(x) = \mathrm{CARA}_\gamma(x) = -e^{-\gamma x} \quad \gamma \in (0, \infty).$$

The investor has available the following set of contingent claims.

$$\mathcal{A} = \{\theta.S_1 : \theta.S_0 = w_0\}.$$

Suppose we seek to

$$\begin{array}{ll} \text{maximise} & \mathbb{E}\left[U(\theta.S_1)\right] \\ \text{subject to} & \theta.S_0 = w0 \end{array}$$

Here, S_1 has mean μ, variance V (invertible) and S_1 is Gaussian.

<u>aside</u>

$$\mathbb{E}\left[\theta.S_1\right] = \theta\mu.$$

$$\mathrm{var}(\theta.S_1) = \theta.(V\theta).$$

$\theta.S_1$ is also Gaussian. $Z \sim N(0,1), \mathbb{E}\left[e^{\lambda z}\right] = e^{-\frac{\lambda^2}{2}}$

$$\begin{aligned}\mathbb{E}\left[e^{\lambda z}\right] &= \int_{\mathbb{R}} e^{\lambda z}\frac{1}{\sqrt{2\pi}}e^{-\frac{z^2}{2}}dz \\ &= e^{\frac{\lambda^2}{2}}\int_R \frac{1}{\sqrt{2\pi}}e^{-\frac{(z-\lambda)^2}{2}}\end{aligned}$$

.

Note

$$\begin{aligned}\mathbb{E}\left[U(\theta.S_1)\right] &= -\mathbb{E}\left[e^{-\gamma\theta.S_1}\right] \\ &= -e^{-\gamma\theta.+\frac{1}{2}\gamma^2\theta.(Vg\theta)}\end{aligned}$$

.

So our problem is to Suppose we seek to

$$\begin{array}{ll}\text{maximise} & \mathbb{E}\left[U(\theta.S_1)\right] \\ \text{subject to} & \theta.S_0 = w_0\end{array}$$

Consider $L(\theta,\lambda) = \theta.\mu - \frac{1}{2}\gamma\theta.(V\theta) - \lambda\theta.S_0$ At maximiser θ^*

$$0 = \frac{\partial}{\partial\theta^i}L(\theta,\lambda) = \mu^i - \gamma(V\theta)^i - \lambda S_0^i.$$

So

$$\theta^* = \gamma^{-1}(A\mu - \lambda AS_0).$$

Find λ by

$$w_0 = \theta^*_. S_0 = \gamma^{-1}(b - \lambda a).$$

So $\lambda w_0 = b - \lambda a$. So $\lambda = \frac{b-\gamma w_0}{a}$. So $\theta^* = \underbrace{\frac{w_0}{a}AS_0}_{\theta^*_{\min}} + \gamma^{-1}(A\mu - \frac{b}{a}AS_0)$.

Add riskless bond $S_0^0 = 1, S_1^0 = 1 + r > 0$

$$\overline{\theta}.\overline{S}_0 = \theta^0 + \theta.S_0, \overline{\theta}.\overline{S}_1 = \theta^0(1+r) + \theta.S_1.$$

So

$$\mathbb{E}\left[U(\overline{\theta}.\overline{S}_1\right] = -e^{-\gamma(\theta\mu+\theta^0(1+r))+\frac{1}{2}\gamma^2\theta.(V\theta)}.$$

$$\begin{array}{rl}\text{with constraint} & \theta.S_0 = w_0 - \theta^0 \\ \text{maximise} & \theta.\mu + \theta^0(1+r) - \dfrac{1}{2}\gamma\theta.(V\theta).\end{array}$$

Using our constraint to eliminate θ^0

$$\theta.\mu + (w_0 - \theta.S_0)(1+r) - \frac{1}{2}\gamma\theta.(V\theta).$$

Maximising θ^* satisfies

$$\mu - (1+r)S_0 = \gamma V\theta^*.$$

So

$$\theta^* = \gamma^{-1} \underbrace{(A\mu - (1+r)AS_0)}_{\theta_{\mathrm{m}}^{ast} = \theta^*_{\mathrm{market}}}.$$

$\gamma >> 1$ means we are highly risk averse.

Critique

- Easy to estimate V, but it is hard to estimate μ 5
- Why do we assume the stock prices are Gaussian? We use Centra Limit Theorem as we can consider them as the sum of random variables, but this relies on variance conditions.
- We've allowed negative asset values, consider $S_1 \sim N(\mu, V)$. More realistically,

  $$S_0 = e^{s_0}, S_1 = e^{s_0 + \varepsilon Z} = S_0 e^{\varepsilon Z} \approx S_0(1 + \varepsilon Z).$$

  $Z \sim N(\mu, V), \varepsilon$ small.

1.6 Capital-asset pricing model

We have seen $\theta^*_{\mathrm{m}} = A\mu + (1+r)AS_0$ appear twice. Suppose we assume that the market optimises itself. Then, we should be able to observe θ^*_{m}

$$\theta^{*i}_{\mathrm{m}} = \text{ the number of shares of asset } i.$$

$$\theta^{*i}_{\mathrm{m}} S^i_n = \text{ capitalization of asset } i.$$

Notation. Set $S^m_n = \theta^*_{\mathrm{m}}.S_n, n = 0, 1,\ \mu^{\mathrm{m}} = \theta^*_{\mathrm{m}}.\mu$ Define

$$\beta^i = \frac{\mathrm{cov}(S^i_1, S^{\mathrm{m}}_1)}{\mathrm{var}\, S^\mu_1}$$

the *beta* or *sensitivity* something we can estimate.

Proposition. For $i = 1, \dots, d$

$$\mu^i = (1+r)S^i_o = \beta^i(\mu^m - (1+r)S^m_0).$$

Proof. For $\theta = A\mu - (1+r)AS_0$, then

$$\mu^m - (1+r)S^m_0 = \theta.(\mu - (1+r)S_0) = \theta.(V\theta) = \mathrm{var}(\theta.S_1) = \mathrm{var}\, S^\mu_1.$$

So

$$\begin{aligned} \mu^i - (1+r)S^i_0 &= e_i.(\mu - (1+r)S_0) \\ &= e_i.(V\theta) \\ &= \mathrm{cov}(S^i_1, S^{\mathrm{m}}_1 \\ &= \beta^i(\mu^{\mathrm{m}} - (1+r)S^{\mathrm{m}}_0) \end{aligned}$$

.

□

This appears to identify μ^i from the market. Often this pricing formula is written in terms of the returns. Define R^i, R^{m} by $S_1^0 = (1+r)S_0^0$, $S_1^i = (1+R^i)S_0^i, S_1^{\mathrm{m}} = (1+R^m)S_0^m$ Then

$$\mu^i = (1 + \mathbb{E}\left[R^i\right])S_0^i.$$

$$\mu^{\mathrm{m}} = (1 + \mathbb{E}\left[R^{\mathrm{m}}\right])S_0^{\mathrm{m}}.$$

$$\operatorname{var} S_1^{\mathrm{m}} = (S_0^{\mathrm{m}})^2 \operatorname{var}(R^{\mathrm{m}}).$$

$$\operatorname{cov}(S_1^i, S_1^{\mathrm{m}}) = S_0^i S_0^{\mathrm{m}} \operatorname{cov}(R^i, R^{\mathrm{m}}) = \frac{S_0^i S_0^m \operatorname{cov}(R^i, R^{\mathrm{m}})}{(S_0^{\mathrm{m}})^2 \operatorname{var}(R^{\mathrm{m}})}((1+\mathbb{E}\left[R^{\mathrm{m}}\right])S_0^m - (1+r)S_0^{\mathrm{m}}).$$

So

$$\mathbb{E}\left[R^i\right] - r = \hat{\beta}^i(\mathbb{E}\left[R^{\mathrm{m}} - r\right].$$

2 Martingales

2.1 Conditional probabilites and expectations

$(\Omega, \mathcal{F}, \mathbb{P})$, is a probability space. Recall for an event $B \in \mathcal{F}$ with $\mathbb{P}(B) > 0$ we define $\mathbb{P}(\cdot \mid B)$

$$\mathbb{P}\left(A \mid B\right) = \frac{\mathbb{P}\left(A \cap B\right)}{\mathbb{P}\left(B\right)}, A \in \mathcal{F}.$$

Then $\mathbb{P}\left(\cdot \mid B\right)$ has associated expectation written $\mathbb{E}\left[\cdot \mid B\right]$. This satisfies for X a random variable

$$\mathbb{E}\left[X \mid B\right] = \frac{\mathbb{E}\left[X \mathbb{1}_B\right]}{\mathbb{P}\left(B\right)}.$$

We will need a more general notions of conditional probabilities and expectations associated not with a single event B, but with a sub-σ-algebra $\mathcal{G} \subseteq \mathcal{F}$.

Definition (σ-algebra)**.** We say $\mathcal{G}$ is a σ-*algebra* if

(i) $\emptyset \in \mathcal{G}$

(ii) $A \in \mathcal{G} \implies A^c \in \mathcal{G}$

(iii) $(A_n : n \in \mathbb{N}) \in \mathcal{G} \implies \bigcup_n A \in \mathcal{G}$