Part II — Further Complex Methods

Based on lectures by B. Groisman

Notes taken by Joseph Tedds using Dexter Chua's header and Gilles Castel's snippets.

Lent 2020

These notes are not endorsed by the lecturers, and I have modified them (often significantly) after lectures. They are nowhere near accurate representations of what was actually lectured, and in particular, all errors are almost surely mine.

Complex variable

Revision of complex variable. Analyticity of a function defined by an integral (statement and discussion only). Analytic and meromorphic continuation. Cauchy principal value of finite and infinite range improper integrals. The Hilbert transform. KramersKronig relations. Multivalued functions: definitions, branch points and cuts, integration; examples, including inverse trigonometric functions as integrals and elliptic integrals. [8]

Special functions

Gamma function: Euler integral definition; brief discussion of product formulae; Hankel representation; reflection formula; discussion of uniqueness (e.g. Wielandts theorem). Beta function: Euler integral definition; relation to the gamma function. Riemann zeta function: definition as a sum; integral representations; functional equation; *discussion of zeros and relation to p(x) and the distribution of prime numbers*. [6]

Differential equations by transform methods

Solution of differential equations by integral representation; Airy equation as an example. Solution of partial differential equations by transforms; the wave equation as an example. Causality. Nyquist stability criterion. [4]

Second order ordinary differential equations in the complex plane

Classification of singularities, exponents at a regular singular point. Nature of the solution near an isolated singularity by analytic continuation. Fuchsian differential equations. The Riemann P-function, hypergeometric functions and the hypergeometric equation, including brief discussion of monodromy. [6]

Contents

0 Introduction

Whilst the prerequisites for this course include complex analysis, this is primarily a methods course - expanding on IB complex methods.

(i) Complex variable

- Revision
- Analyticity and functions defined by integrals e.g. $\Gamma(z) = \int_0^\infty t^{z-1} e^{-t} \mathrm{d}t$ domain of z? Analytic continuation?
- Departure from analyticity (singularities at a point or at a curve) : Principal Value e.g. $PV \int_{-1}^{2} \frac{1}{x} \mathrm{d}x \stackrel{?}{=} \log 2$

(ii) Special functions Γ, β, ζ

(iii) Integral transforms of ODE and PDE

(iv) Second order ODE on $\mathbb{C}$ (1,2,3 regular singular points), hypergeometric equations

1 Complex variable

1.1 Brief revision

$z = x + iy$

Definition (Neighbourhood)**.** A *neighbourhood* of a point $z \in \mathbb{C}$ is an *open-set* containing z.

Definition (Extendend complex plane)**.** The *extended complex plane* $\mathbb{C}^*$ or $\overline{\mathbb{C}} = \mathbb{C} \cup \{\infty\}$. All directions "$\infty$" are equivalent (think of Riemann sphere).

Definition (Differentiable)**.** A function $f(z)$ is differentiable at z if

$$f'(z) = \lim_{\Delta z \to 0} \left| \frac{f(z + \Delta z) - f(z)}{\Delta z} \right|$$

exists (independent of how $\Delta z \to 0$.

Definition (Analytic)**.** We say that $f(z)$ is *analytic* (holomorphic / regular) at a point z if $\exists$ a neighbourhood of z throughout which f' exists. [Extensions to domain D]

Cauchy-Riemann Conditions If $f(z) = u(z) + iv(z)$, with $u, v \in \mathbb{R}$ is differentiable at z, then

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \frac{\partial v}{\partial x} = -\frac{\partial u}{\partial y}.$$

The converse is true for u, v differentiable at z.

Corollary. The Wirtinger derivative

$$\overline{\partial} = \frac{\partial f}{\partial \overline{z}} = 0,$$

where $\frac{\partial}{\partial \overline{z}} = \frac{1}{2}(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y})$

Theorem (Cauchy's Theorem)**.** If $f(z)$ is analytic within and on a closed bounded contour C, then

$$\oint_C f(z) \mathrm{d}z = 0.$$

Note that C bounds D - a simply connected domain and for z_0 inside C, we have Cauchy's Integral Formula

$$f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0} \mathrm{d}z.$$

With C traversed anti-clockwise.

Corollary.

$$f^{(n)}(z_0) = \frac{n!}{z_0} \oint_C \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}z,$$

and functions are differentiable infinitely many times.

Definition (Entire)**.** A function $f(z)$ is *entire* if it is analytic on $\mathbb{C}$ (not $\overline{\mathbb{C}}$).

This leads us to.

Theorem (Louiville's Theorem)**.** If f is entire and bounded on $\overline{\mathbb{C}}$, then f is constant.

Proof. Consider a circular disc or radius R centred at z_0 and we know $|f(z)| < M$. Then from our previous corollary,

$$|f^{(n)}(z_0)| \leq \frac{n!}{2\pi i} \oint \frac{|f(z)|}{|z - z_0|^{n+1}} |\mathrm{d}z| \leq \frac{n!M}{2\pi R^{n+1}} \oint |\mathrm{d}z| \leq \frac{n!M}{R^n}.$$

In particular, for $n = 1$

$$|f'(z)| \leq \frac{M}{R} \ \forall \ z \in \mathbb{C}.$$

We can choose R as large as we wish, so $|f'(z)| \underset{R\to\infty}{\to} 0$. Hence, $f'(z) = 0 \ \forall \ z$. Since $f(z) - f(0) = \int_0^z f'(\tilde{z}) \mathrm{d}\tilde{z} = 0$, we obtain $f(z) = f(0)$. □

Series Expansions

Theorem (Existence of Taylor Expansions)**.** An analytic function has a convergent expansion about any point within its domain of analyticity (proof omitted).

Laurent Series Suppose $f(z)$ has an isolated singularity at z_0 (but analytic in neighbourhood of z_0). Then

$$f(z) = \sum_{n=-\infty}^{\infty} C_n (z - z_0)^n, \quad C_n = \frac{1}{2\pi i} \oint \frac{f(z)}{(z - z_0)^{n+1}} \mathrm{d}z.$$

We can classify the singularity as follows:

$$f(z) = \sum_{n=0}^{\infty} C_n (z - z_0)^n + \sum_{n=1}^{N} \frac{C_{-n}}{(z - z_0)^n}.$$

z_0 is:

- A regular point (or zero) if all $C_{-n} = 0$
- A simple pole if $N = 1$
- A pole of order N if $N > 1$
- An essential singularity if $N \to \infty$

In the case that $0 < N < \infty$ we can write

$$f(z) = \frac{1}{(z - z_0)^N} \sum_{k=0}^{\infty} C_k (z - z_0)^k, \quad f(z) = \frac{g(z)}{(z - z_0)^N}$$

where g is analytic.

Definition (Residue)**.** The coefficient C_{-1} is called the *residue* of f at z_0, we write $\mathrm{res}(f, z_0) = C_{-1}$

For a simple pole note that $C_{-1} = \lim_{z \to z_0}(z - z_0)f(z)$. For a pole of order N,

$$C_{-1} = \frac{1}{(N-1)!}\frac{\mathrm{d}^{N-1}}{\mathrm{d}z^{N-1}}((z-z_0)^N f(z)).$$

Theorem (Residue Theorem)**.** If f is analytic in a simply connected domain except at a finite number of isolated singularities $z_1, \dots, z_0$ then

$$\oint_C f(z)\mathrm{d}z = 2\pi i \sum_{k=1}^{n} \operatorname{res}(f(z); z_k).$$

Where C is a contour traversed in the anticlockwise direction.

Lemma. (Indentation Lemma) |`https://tartarus.org/gareth/maths/Complex_Methods/rjs/indentati`
Let f have a simple pole at z_0, $\operatorname{res}(f; z_0)$. Then,

$$\lim \int_{C_\varepsilon} f(z)\mathrm{d}z = i(\beta - \alpha)\operatorname{res}(f; z_0), \quad 0 < \alpha < \beta < 2\pi.$$

Where on C_ε $z = z_0 + \varepsilon e^{i\theta}, \alpha \le \theta \le \beta$

Proof. Consider Laurent expansions of f about z_0

$$f(z) = \frac{\operatorname{res}(f, z_0)}{z - z_0} + g(z),$$

where $g(z)$ is analytic in the region $|z - z_0| < r$, where $r > 0$. By continuity of g at z_0 we can choose r small enough so g is bounded by some M. On $0 < \varepsilon < r$, we have

$$\begin{aligned}\int_{C_\varepsilon} f(z)\mathrm{d}z &= \operatorname{res}(f, z_0)\int_{C_\varepsilon}\frac{\mathrm{d}z}{z - z_0} + \int_{C_\varepsilon} g(z)\mathrm{d}z \\ &= i\operatorname{res}(f, z_0) + \int_{C_\varepsilon} g(z)\mathrm{d}z \\ &= i(\beta-\alpha)\operatorname{res}(f, z_0)\end{aligned}$$

.

Since

$$\left|\int_{C_\varepsilon} g(z)\mathrm{d}z\right| \le M \times \text{length of } C_\varepsilon = M(\beta - \alpha)\varepsilon \to 0.$$

□

1.2 Functions defined by integrals

Consider

$$F(z) = \int_C f(z, t)\mathrm{d}t,$$

where C is some path in $\mathbb{C}$ (not necessarily closed contour). For which values of z is $F(z)$ defined and analytic? Conditions on analyticity: We need to check that

(i) The integrand is continuous (jointly in t and z)

(ii) The $\int$ converges uniformly in each subset of its domain

(iii) The integrand is analytic in z for each value of t

Note: there will be no rigorous treatment of (ii).

Example.

$$F(z) = \int_{-\infty}^{\infty} e^{-zt^2}\mathrm{d}t \left(= \left(\frac{\pi}{z}\right)^{\frac{1}{2}} \right).$$

– Existence: Converges for Re$z > 0$ and diverges for Re$z < 0$. If Re$z = 0$ but $z \neq 0$ (i.e. z is imaginary) then the integrand e^{-iyt^2} oscillates increasingly rapidly. $F(z)$ is not absolutely convergent but conditionally convergent.

$$\lim_{\ell\to\infty} \int_{-\ell}^{\ell} |e^{-iyt^2}|\mathrm{d}t \to \infty \text{ but } \lim_{\ell\to\infty} \int_{-\ell}^{\ell} |e^{-iyt^2}| < \infty.$$

– Analyticity: Clearly (i), (iii) hold. It can be shown that (ii) also holds

Example. Let $F(z) = \int_0^\infty \frac{u^{z-1}}{u+1}\mathrm{d}u$

– Existence: Potential "problems" when $u = 0, \infty$. The integrand is well-behaved otherwise (except at $u = -1$, but it is outside the range of integration). No problematic values of z (consider $u^{z-1} = e^{(z-1)\log u}$). At $u = 0$, $u + 1 \approx 1$ and so we have $\int_0 \frac{u^{z-1}}{1} = \frac{u^z}{z}\Big|_0$. For $z = x + iy$,

$$|u^z| = |e^{z\log u}| = e^{x\log u}.$$

Since $\log u \underset{u\to 0}{\to} -\infty$ we must have Re $z > 0$. At $u \to \infty$, $u + 1 \approx u$ and $\int^\infty \frac{u^{z-1}}{u} = \frac{u^{z-1}}{z-1}\Big|^\infty$.

$$|u^{z-1}| = |e^{(z-1)\log u}| = e^{(x-1)\log u}.$$

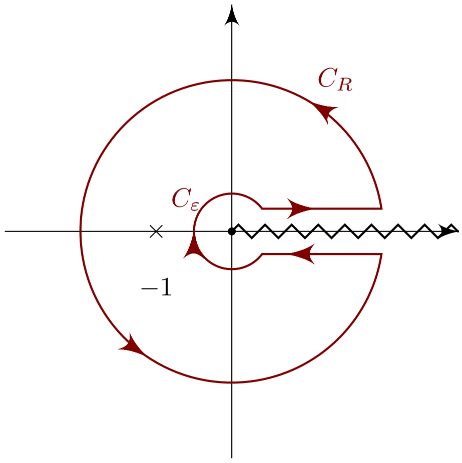
So need Re $z < 1$. Clearly for Re $z > 1$, Re $z < 0$, $F(z)$ doesn't converge. What about $z-1$, z are imaginary. (No convergence). Thus $F(z)$ is defined for $0 <$ Re $z < 1$

– Analyticity: (i), (iii) are clearly satisfied again leaving out (ii). So $F(z)$ is analytic for $0 <$ Re $z < 1$.

– Evaluating the integral: Consider

$$J = \int_C \frac{t^{z-1}}{t+1}\mathrm{d}t = \int_{C_+} + \int_{C_R} + \int_{C_-} + \int_{C_\epsilon},$$

where $R \to \infty, \epsilon \to 0$ and a simple pole at $t = -1 - e^{i\pi}$



On $C_+ : t = u, \epsilon \le u \le R$ and $\int_{C_+} = F(z)$.
On $C_- : t = ue^{2\pi i}, R \ge u \ge \epsilon$ and

$$\int_{C_-} = -(e^{2pii})^{z-1} \int_0^\infty \frac{u^{z-1}}{1+u} \mathrm{d}u = -(e^{2\pi i})^{z-1} F(z).$$

On $C_R, t = Re^{i\theta}, 0 \le \theta \le 2\pi$ and

$$\begin{aligned}\int_{C_R} &= \int_0^{2\pi} \frac{R^{z-1} e^{i\theta(z-1)}}{Re^{i\theta}+1} iRe^{i\theta} \mathrm{d}\theta \\ &= i \int_0^{2\pi} \frac{R^z e^{i\theta z}}{Re^{i\theta}+1} \mathrm{d}\theta \\ &\underset{R\to\infty}{\to} i \int_0^{2\pi} \frac{e^{i\theta(z-1)}}{R^{1-z}} \mathrm{d}\theta \\ &\underset{R\to\infty}{\to} 0.\end{aligned}$$

when Re $z < 1$ On $C_\epsilon : t = \epsilon e^{i\theta}, 2\pi \ge \theta \ge 0$ and

$$\int_{C_\epsilon} \underset{\epsilon \ll 1}{\to} i \int_{2\pi}^0 \epsilon^z e^{i\theta z} \mathrm{d}z \underset{\epsilon\to 0}{\to} 0,$$

when Re $z > 0$. So again we obtain $0 <$ Re $z < 1$. Thus

$$J = F(z)[1 - (e^{2\pi i})^{z-1}] = 2\pi \mathrm{Res}\left(\frac{t^{z-1}}{t+1}; e^{i\pi}\right) = 2\pi i (e^{i\pi})^{z-1}.$$

Hence

$$F(z) = \frac{2\pi i}{e^{i\pi z} - e^{i\pi z}} = \frac{\pi}{\sin \pi z}.$$

(Later $\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec} \pi z$).

1.3 Analytic Continuation

Let $F(z) = \int_{-\infty}^{\infty} e^{-zt^2} \mathrm{d}t$ is analytic for Re $z > 0$. Is it possible to extend its domain of analyticity? Would such an extension be unique?

Theorem (Identity Theorem). Let g_1, g_2 be analytic (holomorphic) in a connected open set $D \subseteq \mathbb{C}$ with $g_1 = g_2$ in a non-empty open subset $\tilde{D} \subseteq D$. Then $g_1 = g_2$ on D.

Proof. Expand $g_1 - g_2$ as a Taylor series about $z_0 \in \tilde{D}$. The series is 0 and convergent in D. Thus $g_1 - g_2 = 0$ on D. □

Analytic Continuation

Let D_1, D_2 be open sets with $D_1 \cap D_2 \neq \emptyset$. Let f_1 and f_2 be analytic on D_1 and D_2 respectively with $f_1 = f_2$ on $D_1 \cap D_2$. Then we say that f_2 is the AC of f_1 from D_1 to D_2. We claim that f_2 is unique.

Proof. Suppose $\exists\ \tilde{f}_2 \neq f_2$, which provides an AC (and $\tilde{f}_2 = f_1$ on $D_1 \cap D_2$). Then we can define the following function

$$g_1 = \begin{cases} f_1 & \text{on } D_1 \\ f_2 & \text{on } D_2 \end{cases} \quad g_2 = \begin{cases} f_1 & \text{on } D_1 \\ \tilde{f}_2 & \text{on } D_2 \end{cases}.$$

Then, by the Identity Theorem, $g_1 = g_2$ on $D_1 \cup D_2$. Hence $f_2 = \tilde{f}_2$ so the extension is unique. □

With three domains, in the case that we only have overlaps between consecutive domains we cannot use the Identity Theorem on D_1 and D_3. However, *AC* of f_1 to D_3 is unique if AC is possible for all domains connecting D_1 and D_3 (Monodromy Theorem). The proof is left as an exercise.

Methods of AC

(i) By Taylor expansion: Choose z_0 on the boundary of D and extend f_1 to a disk $|z - z_0| < r$, r- radius of convergence.

Example. How to obtain $f(z) = \frac{1}{1-z}$ by AC. We first note that

$$\frac{1}{1-z} = \frac{1}{1-z_0} \cdot \frac{1}{1 - \frac{z-z_0}{1-z_0}} = \frac{1}{1-z_0} \sum_{n=0}^{\infty} \left(\frac{z-z_0}{1-z_0} \right)^n,$$

which is convergent if $|z - z_0| < |1 - z_0| = r$. Now, let $f_1 = \sum_{n=0}^{\infty} z^n$. It is analytic, for $|z| < 1 = D_1$. Let $f_2 = \sum_{n=0}^{\infty} \frac{(z-\frac{i}{2})^n}{(z-\frac{i}{2})^{n+1}}$ which is analytic within the disk $|z - \frac{i}{2}| < \frac{\sqrt{5}}{2} = D_2$. So $f_1 = f_2$ on $D_1 \cap D_2$. Hence, by the Identity Theorem f_2 is the AC of f_1 into D_2. This can be continued as a chain of discs to cover the entire $\mathbb{C} \setminus \{1\}$ to obtain $\frac{1}{1-z}$, which has a simple pole at $z = 1$. A Meromorphic continuation - AC excluding singularities.

Such extensions are not always possible

Example. $f(z) = \sum_{n=0}^{\infty} z^{2^n}$, convergent in $|z| < 1$ (by ratio test). The singularities are dense (not isolated) and AC beyond $|z| < 1$ is impossible (see handout). The circle $|z| = 1$ is called a natural barrier for $f(z)$.

(ii) By contour deformation

Example. Let $F(z) = \int_{-\infty}^{\infty} \frac{e^{it}}{t-z} \mathrm{d}t$ for Im $z > 0$. We want to continue $F(z)$ to the lower half plane. Obviously, $F(z)$ is not analytic for Im $z = 0$. Why can't we just redefine F for Im $z \neq 0$? Continue F into a neighbourhood of z_1 with Im $z_1 < 0$. Define

$$F_1(z) = \int_\gamma \frac{e^{it}}{t-z} \mathrm{d}t.$$

With γ as shown. F_1 is defined and analytic $\forall\ z$ above z_1. If Im $z > 0$, then $F_1(z) = F(z)$. Indeed, the integrals agree by contour deformation as we can deform γ to the real axis without crossing any singularities. Hence F_1 is the AC of F into Im $z < 0$ (above γ). Now, let

$$G(z) = \int_{-\infty}^{\infty} \frac{e^{it}}{t-z} \mathrm{d}t, \text{Im } z \neq 0.$$

If Im $z > 0$ then $G(z) = F(z)$ by definition. If Im $z < 0$, then (considering a closed contour C) we get

$$F_1(z) + \int_{\infty}^{-\infty} \frac{e^{it}}{t-z} \mathrm{d}t = 2\pi i e^{iz}.$$

So, for Im $z > 0$, we get $F = F_1 = G(z)$, Im $z < 0, F_1 = G(z) + 2\pi i e^{iz}$. Hence $G(z)$ jumps by $2\pi i e^{iz}$ as z crosses the real axis. Thus G cannot provide an analytic continuation of F.

1.4 Cauchy Principal Value

Motivation: can we say that

$$\int_{-1}^{2} \frac{\mathrm{d}x}{x} = \log 2 - \log|-1| = \log 2?.$$

Definition (Cauchy principle value)**.** If $f(x)$ is "badly" behaved at $x = c$ and $a < c < b$, we define the *Cauchy principal value* (CPV) integral by

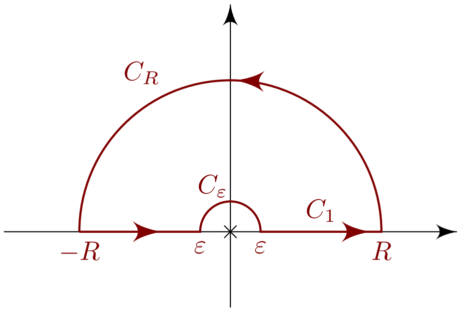
$$\mathcal{P} \int_a^b f(x) \mathrm{d}x := \lim_{\varepsilon \to 0} \left(\int_a^{c-\varepsilon} f(x) \mathrm{d}x + \int_{c+\varepsilon}^b f(x) \mathrm{d}x \right),$$

if the limit exists. For the CPV at ∞ we define

$$\mathcal{P} \int_{-\infty}^{\infty} f(x) \mathrm{d}x = \lim_{R \to \infty} \int_{-R}^{R} f(x) \mathrm{d}x,$$

if the limit exists.

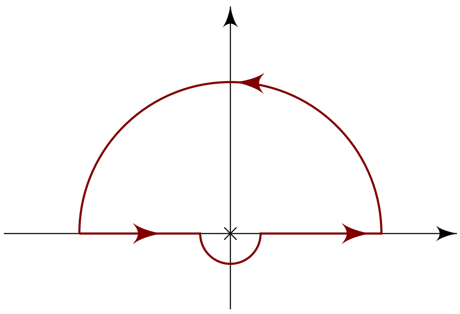
Example. Define $I = \mathcal{P} \int_{-\infty}^{\infty} \frac{f(x)}{x} \mathrm{d}x$, where $f(z)$ is analytic in the upper half-plane (including the real axis) and $|f(z)| \to 0$ as $|z| \to \infty$. This integral only makes sense as a CPV



$\oint_C \frac{f(z)}{z}\mathrm{d}z = \int_{C_1} + \int_{C_\varepsilon} + \int_{C_R} = 0$ as $f(x)$ is analytic inside. It should be clear that

$$\int_{C_1} \to I \quad \int_{C_\varepsilon} \to -i\pi f(0) \quad \int_{C_R} \to 0.$$

With the residue for C_ε coming from the indentation lemma since $z = 0$ is a simple pole unless $f(0) = 0$. So, $I = i\pi f(0)$. Alternatively, $\oint_C = I + i\pi f(0) = 2\pi i f(0)$, so $I = i\pi f(0)$ (illustrated below, uses residue of the pole at 0).



Remark. I depends on the values of $f(x)$ for real x. Hence, $f(x)$ cannot be real-valued.

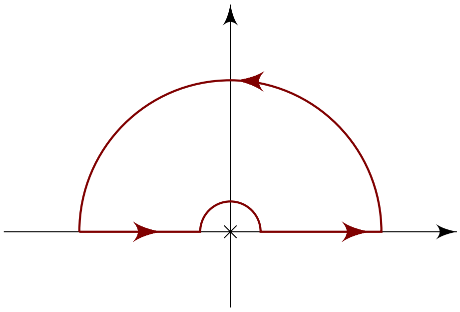
Example. Here we merely use CPV as a tool. Let $I = \int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\mathrm{d}x$. The integrand has a removable singularity at $x = 0$. The standard method is to split up the integrand

$$I = \int_{-\infty}^{\infty} \left(\frac{1}{x^2} - \frac{e^{ix}}{2x^2} - \frac{e^{-ix}}{2x^2} \right) \mathrm{d}x.$$

Of which the exponential terms do not converge at $x = 0$. So we deform the contour to exclude 0. For the second integral, we close C in the upper half plane giving zero (since there are no singularities within the closed contour). For the third integral, we close C in the lower half plane giving $-2\pi i(-i/2)$ (the minus because we are circling the pole or order two in the clockwise sense). For the first integral, we can close either in the upper half or the lower half plane, giving 0 (since in the first case the contour encloses no singularities and in the second case the residue of the enclosed singularity is 0). Therefore, I $= \pi$. We can instead use CPV, by continuity of the integral

$$I = \int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\mathrm{d}x = \mathcal{P}\int_{-\infty}^{\infty} \frac{1-\cos x}{x^2}\mathrm{d}x = \mathrm{Re}\ \mathcal{P}\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2}\mathrm{d}x.$$

Consider $J = \int_C \frac{1-e^{iz}}{z^2}\mathrm{d}z$.



Then

$$J = \mathcal{P}\int_{-\infty}^{\infty} \frac{1-e^{iz}}{z^2}\mathrm{d}z = -i\pi(-i).$$

Where we calculate the residue either by the indentation lemma or directly using L'Hopital's rule. Thus $\int_{-\infty}^{\infty} \frac{1-e^{ix}}{x^2}\mathrm{d}x = \pi$ and we obtain two results

$$I = \pi, \mathcal{P}\int_{-\infty}^{\infty} \frac{-\sin x}{x^2}\mathrm{d}x = 0.$$

Definition (Hilbert transform)**.** The *Hilbert transform* of $f(x)$ is defined as

$$\mathcal{H}(f)(y) := \frac{1}{\pi}\mathcal{P}\int_{-\infty}^{\infty} \frac{f(x)\mathrm{d}x}{x-y},$$

and it is a function of y.

Observe that $\mathcal{H}$ is a linear operator, which takes $f(x)$ into a function of y. Assume $x, y \in \mathbb{R}$.
Assumption: Let's assume that f has a Fourier decomposition, so only need to consider HT of $e^{i\omega x}$ and use linearity. Let's show that

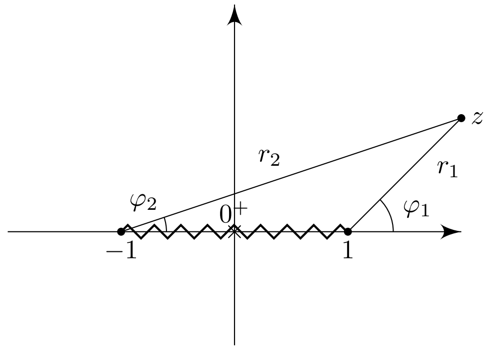
$$\mathcal{H}(e^{i\omega x})(y) = \begin{cases} ie^{i\omega x}, & w > 0 \\ -ie^{i\omega x}, & w < 0 \end{cases} = \operatorname{sgn}(\omega)ie^{i\omega x} \quad (\omega \neq 0).$$

1.5 Multi-valued Functions

For a Harmonic function, we must have

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 4\frac{\partial^2 u}{\partial \bar{z}\partial z} = 0.$$

Example. $f(z) = (1-z^2)^{\frac{1}{2}} = (1-z)^{\frac{1}{2}}(1+z)^{\frac{1}{2}}$, which has *branch points* at $z = \pm 1$. One possible parametrization is local polar coordinates by letting $z = 1 + r_1 e^{i\varphi_1} = -1 + r_2 e^{i\varphi_2}$



$$f(z) = \pm i(r_1 r_2)^{\frac{1}{2}} e^{i(\varphi_1+\varphi_2)/2} = |1-z^2|^{\frac{1}{2}} e^{i(\varphi_1+\varphi_2\pm\pi)/2}.$$

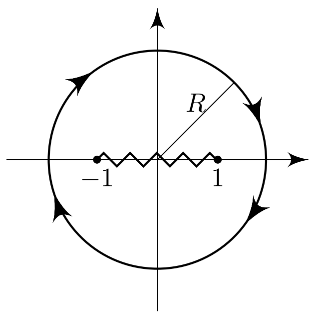
Choose the branch by choosing a branch cut. The natural choice is $[-1,1]$. Also, we specify the value of the function at $z = 0^+$ such that $f(0^+) = 1$. From our equation above, 0^+ corresponds to $\varphi_1 = \pi, \varphi_2 = 0$ i.e. $f(0^+) = e^{i(\pi+0\pm\pi)/2} = \mp 1$. So in the above equation, we take $-$ so

$$f(z) = |1-z|^{\frac{1}{2}} e^{i(\varphi_1+\varphi_2-\pi)/2}.$$

We evaluate $f(z)$ at other points by examining how φ_1, φ_2 vary along paths connecting 0^+ with z without crossing the branch cut.

Integration using a branch cut

Our aim is to evaluate $I = \int_{-1}^{1}(1-x^2)^{\frac{1}{2}}\mathrm{d}x$ using contour integration. We choose the same branch as above and let $J(z) = \int_C f(z)\mathrm{d}z$ where C is the circle $|Z| = R > 1$ traversed clockwise.



$$f(z) = -iz(1-\frac{1}{z^2})^{\frac{1}{2}} = -iz(1-\frac{1}{2}z^{-2}-\frac{1}{8}z^{-4}+\cdots,$$

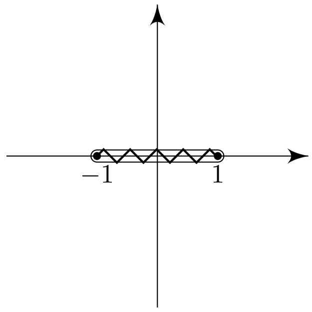
i.e. the Laurent series for $|z| > 1$. Note that the $-$ at the front of our expansion comes from our choice of branch. Now, at $z = R$, we have $\varphi_1 = \varphi_2 = 0$ i.e.

$$f(R) = \sqrt{R^2-1}e^{i(0+0-\pi)/2} = e^{-i\pi/2}\sqrt{R^2-1} = -i\sqrt{R^2-1}.$$

So

$$\begin{aligned} J &= \int_C (-iz)(1-\frac{1}{2}z^{-\frac{1}{2}}-\frac{1}{8}z^{-4}+\cdots)\mathrm{d}z \\ &= \int_0^{-2\pi}(-iRe^{i\theta})(1-\frac{e^{-2i\theta}}{2R^2}+\cdots)iRe^{i\theta}\mathrm{d}\theta = \pi. \end{aligned}$$

What about I? Let's collapse the contour onto the branch cut.



$$J = \int_{-1}^{1} (1-x^2)^{\frac{1}{2}} \mathrm{d}x + \int_{1}^{-1} -(1-x^2)^{\frac{1}{2}} \mathrm{d}x = 2I = \pi.$$

Hence, $I = \frac{\pi}{2}$
<u>The arcsin function defined as an integral</u>
Introduction: Let $z = \sin w = \frac{e^{iw}-e^{-iw}}{2i}$, and let $v = e^{iw}$. Then $v - \frac{1}{v} = 2iz, v^2 - 2izv - 1 = 0$, so $v = iz + \sqrt{1-z^2}$. Hence we can write $iw = \log v$ and

$$w = \arcsin z = -i \log(iz + \sqrt{1-z^2}).$$

Hence,

$$\frac{\mathrm{d} \arcsin z}{\mathrm{d}z} = \frac{1}{\sqrt{1-z^2}},$$

and therefore

$$\arcsin z = \int_0^z \frac{\mathrm{d}t}{(1-t^2)^{\frac{1}{2}}}.$$

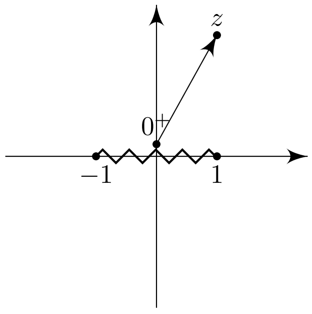
Our aim is to construct the multivariate function arcsin from a single valued function arcsin by analytic continuation. Let

$$\arcsin z_{[0,2\pi)} = \int_0^z \frac{\mathrm{d}t}{(1-t^2)^{\frac{1}{2}}},$$

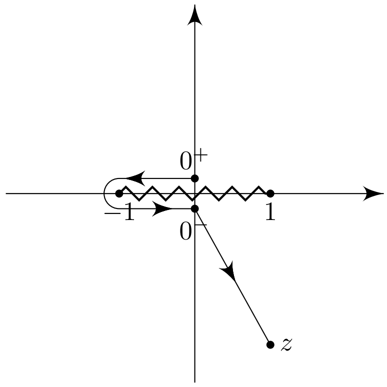
where

(i) The branch of $\sqrt{1-t^2}$ is defined by a branch cut along the real axis connecting $t=-1$ and $t=1$, with $\sqrt{1-t^2} = +1$ at the origin just above the cut (at $t = 0^+$).

(ii) The path is defined as follows:



$0 \leq \arg z < \pi.$



$\pi < \arg z < 2\pi.$

In what domain of z is $\arcsin_{[0,2\pi)} z$ analytic? Notice that

$$\frac{\mathrm{d} \arcsin_{[0,2\pi)} z}{\mathrm{d}z} = \frac{1}{\sqrt{1-z^2}},$$

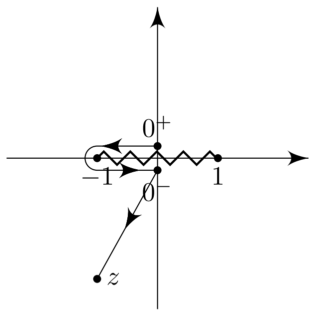
which has a branch as defined in (i). thus $\arcsin_{[0,2\pi)} z$ is not analytic on $[-1,\infty)$. It can be show, that it is discontinuous on $[-1,\infty)$ Notice $\int \frac{\mathrm{d}t}{(1-t^2)^{\frac{1}{2}}} = -2\pi$ integrating around our cut.

$$\int_{-1}^{1} \frac{\mathrm{d}t}{(1-t^2)^{\frac{1}{2}}} = -2\int_0^{\pi} \mathrm{d}\theta = -2\pi,$$

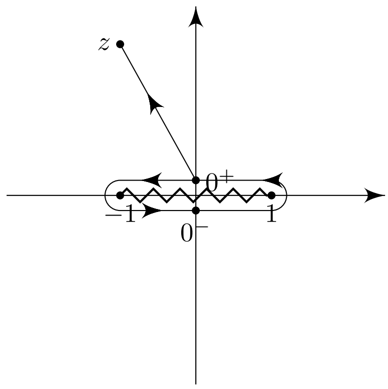
where we substituted $t = \cos\theta$. Now we obtain the analytic continuation into $\arg z > 2\pi$. Define

$$\arcsin_{(\pi,3\pi)} z = \int_0^z \frac{\mathrm{d}t}{(1-t^2)^{\frac{1}{2}}},$$

where



$\pi \leq \arg z < 2\pi.$



$2\pi < \arg z < 3\pi.$

Now

– $\arcsin_{[0,2\pi)} z = \arcsin_{(\pi,3\pi)}$ in $\pi < \arg z < 2\pi$

– The latter is analytic in $2\pi < \arg z < 3\pi$

Hence we obtain the analytic continuation of $\arcsin_{[0,2\pi)}$ into $2\pi < \arg z < 2\pi$. This process can be repeated to obtain the multivalued function $\arcsin z$. Observe that for $0 \leq \arg z \leq 2\pi$,

$$\arcsin(e^{2\pi i}z) = \arcsin_{[0,2\pi)} z + \int_C = \arcsin_{[0,2\pi)} z - 2\pi.$$

We can show that possible values of $\arcsin z = \arcsin_{[0,2\pi)} z + 2\pi n, n \in \mathbb{N}$. Also, (again for $0 \leq \arg z \leq 2\pi$)

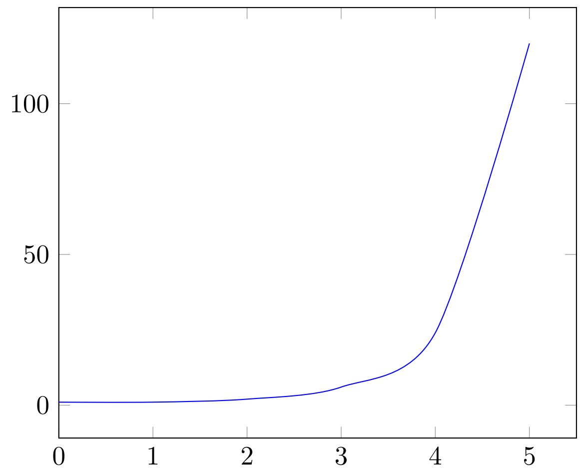
$$\sin\left(\arcsin e^{2\pi i}z\right) = \sin\left(\arcsin_{[0,2\pi)} z - 2\pi\right).$$

And $\sin\left(\arcsin e^{2\pi i}z\right) = e^{2\pi i}z = z$. Thus setting $w = \arcsin_{[0,2\pi)} z$ and using $\arcsin z = w$, we obtain $\sin w = \sin(w - 2\pi)$. Thus $\sin w$, obtained this way is 2π periodic.

2 Special Functions

2.1 The Gamma function

Motivation: find solution to the interpolation problem, finding a smooth curve $y = f(x)$ that connects the points given by $y = x!, x \in \mathbb{N}$. We want to generalise $n!$ from integers into $\mathbb{C}$.



Let

$$I(z) = \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t,$$

which converges and is analytic for Re $z > 0$. Now

$$I(z+1) = \int_0^\infty t^z e^{-t}\mathrm{d}t = [-t^z e^{-t}]_0^\infty + \int_0^\infty zt^{z-1}e^{-t}\mathrm{d}t = zI(z).$$

Hence we obtain the recurrence relation $I(z+1) = zI(z)$. Also, $I(1) = 1$ so $I(n+1) = n!$ for $n \in \mathbb{N}$. Idea, define

$$\Gamma(z) = \begin{cases} I(z), & \text{Re } z > 0 \\ \text{by Ac in Re} z < 0, & \text{whenever possible} \end{cases}.$$

Now, $I(z) = \frac{I(z+1)}{z}$ which is analytic for Re $z + 1 > 0$ i.e. Re $z > -1$ and $z \neq 0$. Similarly,

$$\frac{I(z+n+1)}{z(z+1)\cdots(z+n-1)(z+n)},$$

is analytic for Re $z > -(n+1)$ and $z \neq 0, -1, \dots, -n$. Hence define

$$\Gamma(z) = \begin{cases} I(z), & \text{for Re } z > 0 \\ I(z+1)/z, & \text{for Re } z > -1 \\ \vdots, & \vdots \\ \frac{I(z+n+1)}{z(z+1)\cdots(z+n)}, & \text{for Re } z > -(n+1) \end{cases}.$$

Which has simple poles at $z = 0, -1, \ldots$ as its only singularities.

$$\text{Res }(\Gamma(z), -n) = \lim_{z \to -n} (z+n)\Gamma(z) = \frac{(-1)^n}{n!}.$$

Some alternative definitions and formulae

First we look at the Euler Product Formula

Claim.

$$\Gamma(z) = \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)} \qquad \forall\, z \in \mathbb{C} \setminus \{0, -1, \ldots\}.$$

Proof. Firstly, show this for Re $z > 0$. Note $e^{-t} = \lim_{n\to\infty}\left(1 - \frac{t}{n}\right)^n$. Then,

$$\Gamma(z) = \lim_{n\to\infty} \int_0^n \left(1 - \frac{t}{n}\right)^n t^{z-1}\mathrm{d}t.$$

Letting $\tau = \frac{t}{n}$,

$$\begin{aligned}\Gamma(z) &= \lim_{n\to\infty} n^z \left[\frac{(1-\tau)^n \tau^z}{z}\right]_0^1 - \frac{n^z}{z}(-n)\int_0^1 (1-\tau)^{n-1}\tau^z \mathrm{d}\tau \\ &= \lim_{n\to\infty} (-1)^n n^z (-1)^n n! \int_0^1 \frac{\tau^{z+n-1}\mathrm{d}\tau}{z(z+1)\cdots(z+n-1)} \\ &= \lim_{n\to\infty} \frac{n! n^z}{z(z+1)\cdots(z+n)}.\end{aligned}$$

$\Gamma(z)$ is analytic in $z \in \mathbb{C}\setminus\{0,-1,\ldots\}$, and so is the RHS. They are equal in Re $z > 0$. Hence we obtain an analytic continuation by the identity theorem. $\square$

Next we consider the Gauss product formula.

Claim.

$$\Gamma(z) = \frac{1}{z}\prod_{n=1}^\infty \frac{(1+\frac{1}{n})^z}{1+\frac{z}{n}}.$$

Proof. Follows from Euler's product formula

$$\begin{aligned}\Gamma(z) &= \frac{1}{z}\lim_{n\to\infty} \frac{n^z}{(z+1)\cdots(\frac{z}{n-1)}+1)(\frac{z}{n}+1)} \\ &= \frac{1}{z}\lim_{n\to\infty}\left(\frac{\left(\frac{n+1}{n}\right)^z \left(\frac{n}{n-1}\right)^z \cdots \left(\frac{2}{1}\right)^z \left(\frac{n}{n+1}\right)^z}{(z+1)\cdots(\frac{z}{n-1)}+1)(\frac{z}{n}+1)}\right).\end{aligned}$$

As $\left(\frac{n}{n+1}\right)^z \underset{n\to\infty}{\to} 1$, we obtain the required expression. $\square$

Now the Weierstrass canonical product.

Claim.

$$\frac{1}{\Gamma(z)} = ze^{\gamma z}\prod_{k=1}^\infty \left(1+\frac{z}{k}\right)e^{-\frac{z}{k}}, \quad ,$$

where

$$\gamma = \lim_{n\to\infty}\left(1 + \frac{1}{2} + \cdots + \frac{1}{n} - \log n\right) \approx 0.577.$$

We call γ the Euler-Mascheroni constant.

Proof. By Euler's product formula

$$\frac{1}{\Gamma(z)} = z \lim_{n\to\infty} \frac{(1+z)(2+z)\cdots(n+z)}{n!n^z}.$$

Divide each term by $n, n-1, \ldots$ and use $n^{-z} = e^{\log n^{-z}}$.

$$\begin{aligned}
\frac{1}{\Gamma(z)} &= z \lim_{n\to\infty} e^{-z\log n}(1+z)\cdots(1+\frac{z}{n}) \\
&= z \lim_{n\to\infty} e^{-z(\log n-(1+\frac{1}{2}+\cdots+\frac{1}{n})}e^{-z(1+\frac{1}{2}+\cdots+\frac{1}{n})}(1+z)\cdots(1+\frac{z}{n}) \\
&= ze^{\gamma z}\prod_{k=1}^{\infty}(1+\frac{z}{k})e^{-\frac{z}{k}}.
\end{aligned}$$

□

Reflection formula

$$\Gamma(z)\Gamma(1-z) = \pi \operatorname{cosec}(\pi z), z \notin \mathbb{Z}.$$

Proof. Begin by considering Re $z \in (0,1)$, so $\Gamma(z), \Gamma(1-z)$ can be written as integrals. Thus,

$$\Gamma(z)\Gamma(1-z) = \int_0^\infty e^{-t}t^{z-1}\mathrm{d}t \int_0^\infty e^{-s}s^{-z}\mathrm{d}s.$$

By using the substitution, $t = r\sin^2\theta, s = r\cos^2\theta$ we obtain

$$\Gamma(z)\Gamma(1-z) = 2\int_0^\infty (\tan\theta)^{2z-1}\mathrm{d}\theta = \int_0^\infty \frac{u^{z+1}}{u+1}.$$

Where we substituted $\tan\theta = u^{\frac{1}{2}}$. This is an integral we have already calculated, which takes the value $\frac{\pi}{\sin \pi z}$. Both $\Gamma(z)\Gamma(1-z)$ and $\pi \operatorname{cosec} \pi z$ are analytic except at $z \in \mathbb{Z}$. They are equal in Re $z \in (0,1)$, so the result follows by analytic continuation (meromorphic continuation). □

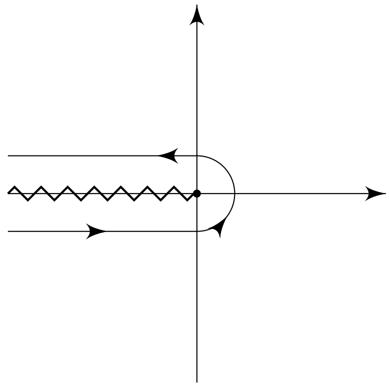
Observe $\Gamma(\frac{1}{2}) = \sqrt{\pi}$. Some key properties:

- $\Gamma(n+1) = n!, n \in \mathbb{N}$
- Res $(\Gamma(z), -n) = \frac{(-1)^n}{n!}$
- $\Gamma(\frac{1}{2}) = \sqrt{\pi}, \Gamma(1) = \Gamma(2) = 1$
- $\Gamma(z) \neq 0 \ \forall \ z$ (from recurrence relation)

The Hankel representation of $\Gamma(z)$

$$\Gamma(z) = \frac{1}{2i\sin\pi z}\int_{-\infty}^{0^+} e^t t^{z-1}\mathrm{d}t, z \neq 0, -1, \ldots,$$

where $-\pi \leq \arg t \leq \pi$ and the path is the Hankel contour



This integral represents an analytic function both in z and t. Let's prove that the formula holds for Re $z > 0$ (i.e. equality for $I(z)$). We collapse the Hankel contour onto the branch cut. Let

$$J(z) = \int_{-\infty}^{0^+} e^t t^{z-1} \mathrm{d}t, \mathrm{Re}\ z > 0.$$

Thus $J(z) = \int_{\gamma_1}, + \int_{\gamma_\varepsilon} + \int_{\gamma_2}$ with the obvious choice of contours:

- $\gamma_1 : t = xe^{-i\pi}, \infty > x > \varepsilon$
- $\gamma_\varepsilon : t = \varepsilon e^{i\theta}, -\pi < 0 < \pi$
- $\gamma_2 : t = xe^{i\pi}, \varepsilon \leq x < \infty$

$$\int_{\gamma_1} \underset{\varepsilon \to 0}{\to} \left(e^{-i\pi}\right)^z \int_\infty^0 e^{-x} x^{z-1} \mathrm{d}x.$$

$$\int_\varepsilon \underset{\varepsilon \to 0}{\to} 0, \text{ as Re } z > 0.$$

$$\int_{\gamma_2} \underset{\varepsilon \to 0}{\to} \left(e^{i\pi}\right)^z \int_0^\infty e^{-x} x^{z-1} \mathrm{d}x.$$

Thus $J(z) = 2i \sin \pi z I(z)$ for Re $z > 0$. So we complete by analytic continuation. Note that $z = 1, 2, 3, \ldots$ the zeros of $\sin \pi z$ cancel by zeros of the integral in RHS of our formula. Indeed, for $z \in \mathbb{Z}^+, t = 0$ is not a branch point, there is no branch cut. So $\exists$ no singularities within the Hankel contour so $J(z) = 0$ which cancels the zeros of $\sin \pi z$.

Computing residues of $\Gamma(z)$

For $z = -m$ $\exists$ no branch cut so the Hankel contour is a circle around the origin.

$$J(-m) = \int_{|t|=1} e^t t^{-m-1} \mathrm{d}t = 2\pi i \mathrm{Res}\ (e^t t^{-m-1}, t = 0).$$

Now,

$$e^t t^{-m-1} = \sum_{n=0}^{\infty} \frac{t^{n-m-1}}{n!},$$

by Taylor expansion of e^t. C_{-1} corresponds to $n - m - 1 = -1$, so $n = m$. Thus, $J(-m) = 2\pi i / m!$

2.2 Beta Function

$$B(p,q) = \begin{cases} \int_0^1 t^{p-1}(1-t)^{q-1}\mathrm{d}t, & \text{Re } p,\ \text{Re } q > 0 \\ \text{by AC in p for q and vice versa,} & \text{o/w} \end{cases}.$$

Setting $t = \sin^2\theta$ gives $B(p,q) = \int_0^{\frac{\pi}{2}} \sin^{2p-1}\theta\cos^{2q-1}\theta\mathrm{d}\theta$.
Properties

(i) $B(p,q) = B(q,p)$

(ii) $B(1,q) = \frac{1}{q}$

(iii) $B(p,z+1) = \frac{z}{p+z}B(p,z)$

(iv) $B(p,q) = \frac{\Gamma(p)\Gamma(q)}{\Gamma(p+q)}$. Note that if $m,n \in \mathbb{N}$ we have $B(n,m) = \frac{(n-1)!(m-1)!}{(n+m-1)!}$

We give a proof of (iii) here

Proof.

$$\begin{aligned} B(p,z+1) = \int_0^1 t^{p-1}(1-t)^z\mathrm{d}t \\ &= \int_0^1 t^{p-1}(1-t)^{z-1}\mathrm{d}t - \int_0^1 t^p(1-t)^{z-1}\mathrm{d}t \\ &= B(p,z) - \frac{p}{z}B(p,z+1). \end{aligned}$$

□

This gives the analytic continuation of $B(p,z)$ to $-1 \leq$ Re $z \leq 0$ (just as we did with Γ) The singularities are simple poles at $z = 0,-1,-2,\ldots$ for fixed p. Note that (iv) can also provide an analytic continuation. We can prove (iv) in a similar way to the proof of the reflection formula for Γ

Proof.

$$\Gamma(p)\Gamma(q) = \left(\int_0^\infty e^{-s}s^{p-1}\mathrm{d}s\right)\left(\int_0^\infty e^{-t}t^{q-1}\mathrm{d}t\right) = \Gamma(p+q)\cdot B(p,q).$$

Where we made use of the substitution $s = r\cos^2\theta, t = r\sin^2\theta$. □

2.3 The Riemann Zeta function

$$\zeta(z) := \begin{cases} \sum_{n=1}^\infty \frac{1}{n^z}, & \text{for Re } z > 1 \\ \text{and by analytic continuation,} & \text{wherever possible} \end{cases}.$$

The integral representation of $\zeta(z)$ is given by

$$\zeta(z) = \frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t-1}\mathrm{d}t,\ \text{Re } z > 1.$$

Proof. Recall that $\Gamma(z) = \int_0^\infty t^{z-1}e^{-t}\mathrm{d}t$ for Re $z > 0$. Let $t = ns$ with $n \in \mathbb{N}, s \in \mathbb{R}$. Then $\Gamma(z) = n^z \int_0^\infty s^{z-1}e^{-ns}\mathrm{d}s \;\forall\; n$. Hence,

$$\zeta(z)\Gamma(z) = \sum_{n=1}^{\infty} \int_0^\infty s^{z-1}e^{-ns}\mathrm{d}s, \text{ for Re } z > 1.$$

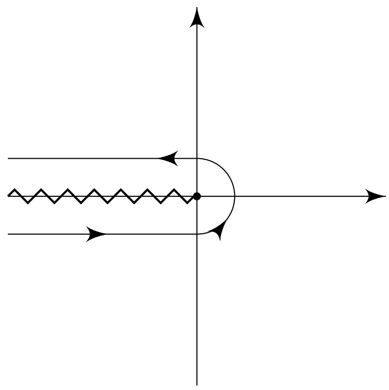
$$\begin{aligned}\zeta(z)\Gamma(z) &= \int_0^\infty t^{z-1}\left(\sum_{n=1}^{\infty} e^{-nt}\right)\mathrm{d}t \\ &= \int_0^\infty t^{z-1}\frac{e^{-t}}{1-e^{-t}}\mathrm{d}t \\ &= \int_0^\infty \frac{t^{z-1}}{e^t - 1}.\end{aligned}$$

□

Proposition.

$$\zeta(z) = \frac{\Gamma(1-z)}{2\pi i}\int_{-\infty}^{0+}\frac{t^{z-1}}{e^{-t}-1}\mathrm{d}t.$$

Note that the integrand has simple poles at $2\pi i n, n \in \mathbb{Z}$



Proof. We show that

$$\frac{\Gamma(1-z)}{2\pi i}\int_{-\infty}^{0+}\frac{t^{z-1}}{e^{-t}-1}\mathrm{d}t = \frac{1}{\Gamma(z)}\int_0^\infty \frac{t^{z-1}}{e^t-1}\mathrm{d}t, \text{for Re } z > 1.$$

Then, we prove that the LHS gives the analytic continuation of the RHS into Re $z < 1$. See our previous proof using the Hankel contour and make use of the reflection formula.

$$\int_{-\infty}^{0^+} = \int_{\gamma_1} + \int_{\gamma_\varepsilon} + \int_{\gamma_2} \underset{\varepsilon\to 0}{\to} e^{i\pi z}\int_0^\infty \frac{x^{z-1}}{e^x - 1}\mathrm{d}x + 0 - e^{i\pi z}\int_0^\infty \frac{x^{z-1}}{e^{x-1}} = 2i\sin\pi z\Gamma(z)\zeta(z).$$

Thus, by the reflection formula we are done. The integral of the LHS is entire in z and smooth in t, hence LHS provides the analytic continuation of $\zeta(z)$ into Re $z < 1$. □

Proposition. The ζ function extends to a meromorphic continuation on $\mathbb{C}$ with the only singular point being a simple pole at $z = 1$ with residue 1.

Proof. Notice that $\Gamma(1-z)$ has simple poles at $z = 1, 2, \dots$ but $\zeta(z)$ is analytic for Re $z > 1$ (from series definition). Hence, $z = 1$ is the only singularity.

$$\text{Res }(\zeta(z), 1) = \lim_{z\to 1}(z-1)\frac{\Gamma(1-z)}{2\pi i}\int_{-\infty}^{0^+}\frac{t^{z-1}}{e^{-t}-1}\mathrm{d}t.$$

For $z = 1$, there is no branch cut, so the integral is given by a circle enclosing the origin. Note for $z = -n, n = 0, 1, 2, \dots$, then

$$\Gamma(z) = \frac{(-1)^n}{n!}\frac{1}{z+n} + \text{ analytic function (in } z = -n).$$

So

$$\lim_{z\to 1}(z-1)\Gamma(1-z) \overset{n=0}{=} \lim_{\substack{z\to 1\\(n\to 0)}}(z-1)\left(\frac{(-1)^0}{0!}\frac{1}{1-z} + \text{ an } f\right) = -1.$$

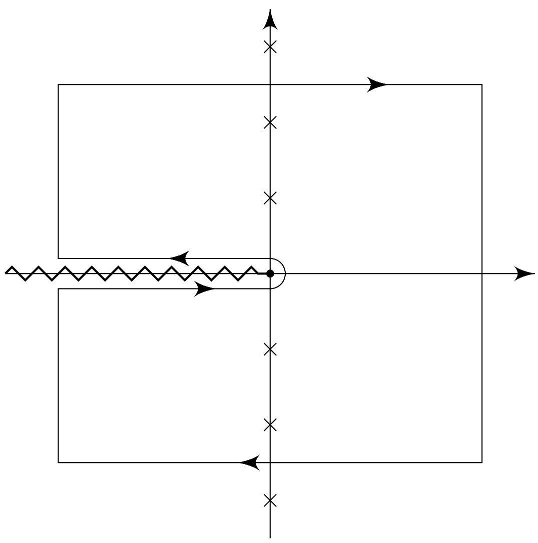
Also, $\oint_{|t|=\frac{1}{2}} = 2\pi i(-1)$, therefore Res $(\zeta, z = 1) = 1$ as required. □

Functional Equation for $\zeta(z)$

Claim.

$$\zeta(z) = 2^z\pi^{z-1}\sin\left(\frac{\pi z}{2}\right)\Gamma(1-z)\zeta(1-z)\ \forall\ z.$$

Proof. We derive this for Re $z < 0$, and then use analytic continuation. This is done by modifying the Hankel contour as follows



We close the contour as a rectangle with vertices at $z = \pm R \pm (2N-1)\pi i$. Let

$$J(z) = \oint_C \frac{t^{z-1}}{e^{-t}-1}\mathrm{d}t.$$

The integral has a branch cut with branch point at $z = 0$ and poles at $2\pi in, n = \pm 1, \pm 2, \ldots$.

$$\operatorname{Res}\left(\frac{t^{z-1}}{e^{-t}-1}, t = 2\pi in\right) = -\frac{1}{(2\pi in)^{1-z}}.$$

Hence, by the Hankel representation,

$$\begin{aligned}\frac{2\pi i}{\Gamma(1-z)}\zeta(z) &= -2\pi i \sum \text{residues} \\ &= 2\pi i \sum_{n=1}^{N}\left((2\pi in)^{z-1} + (-2\pi in)^{z-1}\right) \\ &= 2\pi i (2\pi)^{z-1}\left(\frac{i^z}{i} + \frac{(-i)^z}{-i}\right)\sum_{n=1}^{N}\frac{1}{n^{1-z}} \\ &\to 2\pi i 2^z \pi^{z-1}\sin\frac{\pi z}{2}\zeta(1-z).\end{aligned}$$

It can be shown that the contributions from the sides of the rectangle are negligible as $R, N \to \infty$ but will not be done so here. As usual we extend this to the entire complex plane by analytic continuation □

Zeros of $\zeta(z)$ From the functional equation, it follows that

- $\zeta(-2n) = 0$, where $n = 1, 2, 3, \ldots$ (at negative integers)
- $\zeta(2n) \neq 0$ (at positive integers) as the zeros of $\sin\left(\frac{\pi z}{2}\right)$ cancel with the poles of $\Gamma(1-z)$
- $\zeta(1+2n) \sim \Gamma(-2n)\zeta(-2n) \neq 0$
- $\zeta(0) \neq 0$

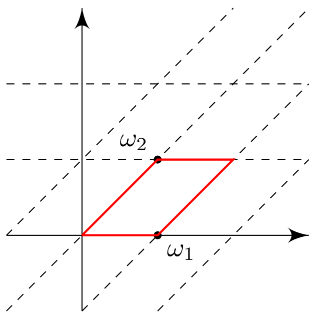
The Riemann Hypothesis All non-trivial zeros of the ζ function lie on the critical line $\operatorname{Re} z = \frac{1}{2}$.

2.4 Elliptic Functions

A periodic function f obeys $f(z+\alpha) = f(z)$ for some α, $\forall\, z$. Doubly-periodic functions have two periods ω_1, ω_2 such that $\frac{\omega_1}{\omega_2} \notin \mathbb{R}$ (i.e. $\arg\omega_1 \neq \arg\omega_2$ such that $f(z+\omega_1) = f(z+\omega_2) = f(z)$, and more generally, $f(z+m\omega_1+n\omega_2) = f(z)$, for $m, n \in \mathbb{Z}$.

Definition (Elliptic functions)**.** A doubly periodic function which is meromorphic on $\mathbb{C}$ is called *elliptic*.

Assume wlog that $\omega_1 \in \mathbb{R}$



All zeros and poles of an elliptic function correspond to zeros and poles in one cell.

Properties

(i) The number of zeros and poles in one cell is finite

(ii) An elliptic function with no poles in a cell are constant

The Weierstrass $\mathcal{P}$ function Set $\omega_{m,n} := m\omega_1 + n\omega_2$ (fixed lattices points), and define

$$\mathcal{P}(z) = \frac{1}{z^2} + \sum_{m,n} \left[\frac{1}{(z-\omega_{m,n})^2} - \frac{1}{\omega_{m,n}^2} \right], (m,n) \in \mathbb{Z}^2 \setminus \{(0,0)\}.$$

It is meromorphic with ∞-many poles at lattice points. To show double-periodicity, consider $\mathcal{P}(z+\omega_{k,l})$, for given k,l and shift the summation: $k-m \to m, l-n \to n$.

Proposition. The $\mathcal{P}$ function satisfies

$$(\mathcal{P}')^2 = 4(\mathcal{P})^3 - g_2\mathcal{P} - g_3,$$

where $g_2 = 60\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{\omega_{m,n}^4}$, $g_3 = 140\sum_{(m,n)\in\mathbb{Z}^2\setminus(0,0)} \frac{1}{\omega_{m,n}^6}$

Proof. See example sheet 3. Hint: Consider $Q(z) = \mathcal{P}(z) - \frac{1}{z^2}$, where $Q(z)$ is analytic and even (make the case by symmetry) in the neighbourhood of $z=0$. Then Taylor expand it. □

Rearrange our $\mathcal{P}$ equation and integrate

$$z = \int_{w(z)}^{\infty} \frac{\mathrm{d}s}{4s^3 - g_2 s - g_3} = \mathcal{P}^{-1}(w) - \alpha,$$

where $w = \mathcal{P}(z+\alpha)$. Thus, $\mathcal{P}$ can be given as an inverse of an elliptic integral (we call this integral and elliptic integral of the first kind). Note: compare with

$$\arcsin z = \int_0^z \frac{\mathrm{d}s}{\sqrt{1-s^2}}.$$

Example. (General elliptic ODE) Consider $w' = (w-a)(w-b)(w-c)(w-d)$. This is solved by an elliptic integral

$$z + \alpha = \int_w^{\infty} \frac{\mathrm{d}s}{[(s-a)(s-b)(s-c)(s-d)]^{\frac{1}{2}}}.$$

We can reduce it to our more useful form by moving one root to ∞ and shifting the remaining roots as follows. Consider

$$\frac{\mathrm{d}u}{[(u-a)(u-b)(u-c)(u-d)]^{\frac{1}{2}}},$$

shifting $\tilde{u} = u - d$ yields

$$\frac{\mathrm{d}\tilde{u}}{\left[(\tilde{u}-\tilde{a})(\tilde{u}-\tilde{b})(\tilde{u}-\tilde{c})\tilde{u}\right]^{\frac{1}{2}}}.$$

Then $t = \frac{1}{\tilde{u}}$ gives

$$\frac{-\mathrm{d}t}{t^2\left[(\frac{1}{t}-\tilde{a})(\frac{1}{t}-\tilde{b})(\frac{1}{t}-\tilde{c})\frac{1}{t}\right]} = \frac{A\mathrm{d}t}{[(t-t_1)(t-t_2)(t-t_3)]^{\frac{1}{2}}},$$

where A is constant. Thus, the quartic integrand has been reduced to a cubic with one root moved to ∞ (the original d is not $t = \infty$). Finally, set $t = s + \gamma$, with γ chosen so that $t_1 + t_2 + t_3$ will move to 0, then rescale s such that the leading term has coefficient 4. Then we arrive at our previous equation.

Example. (Euler's Top) Consider

$$\begin{aligned} w_1' &= w_2 w_3 \\ w_2' &= w_1 w_3 \\ w_3' &= w_1 w_2. \end{aligned}$$

which gives

$$\begin{aligned} w_2' w_2 - w_3' w_3 = 0, \text{ so } w_3^2 &= w_2^2 + B \\ w_1' w_1 - w_3' w_3 = 0, \text{ so } w_3^2 &= w_1^2 + C, \end{aligned}$$

for constants B, C and

$$w_3'^2 = (w_3^2 - B)(w_3^2 - C)$$

which is a special case of the elliptic ODE.

3 Transform Methods

3.1 Solution of ODEs by integral representation

Idea: Look at general solutions as integrals in $\mathbb{C}$.

Example. (Airy equation)

$$\omega''(x) + x\omega(x) = 0.$$

In this course, the convention is that we take $+$, often you will see the Airy equation written with a $-$. Let us first attempt to solve via the Fourier transform.

$$\hat{\omega}(k) = \int_{-\infty}^{\infty} e^{-ikx}\omega(x)\mathrm{d}x,$$

so

$$-k^2\hat{\omega}(k) + i\hat{\omega}'(k) = 0.$$

Solving and inverting give us

$$\omega(x) = \frac{A}{2\pi}\int_{-\infty}^{\infty} \exp\left(-\frac{ik^3}{3} + ikx\right)\mathrm{d}k.$$

There exists a second, linearly independent solution which is not bounded for large x and hence doesn't admit a Fourier transform.
Instead, we consider the equation in $\mathbb{C}$, with $\omega(z)$. Let

$$\omega(z) = \int_\gamma e^{zt} f(t)\mathrm{d}t,$$

where we will determine $f(t)$ uniquely and then establish suitable contours γ.
Assume $\frac{\partial f}{\partial z} = 0$, then

$$\omega''(z) = \int_\gamma t^2 e^{zt} f(t)\mathrm{d}t.$$

Our equation now becomes

$$\int_\gamma (t^2 + z) f(t) e^{zt}\mathrm{d}t = 0.$$

After integrating $\int_\gamma z f(t) e^{zt}\mathrm{d}t$ by parts, we obtain

$$\left[e^{zt} f(t)\right]_\gamma + \int_\gamma e^{zt}\left(t^2 f(t) - \frac{\mathrm{d}f}{\mathrm{d}t}\right)\mathrm{d}t.$$

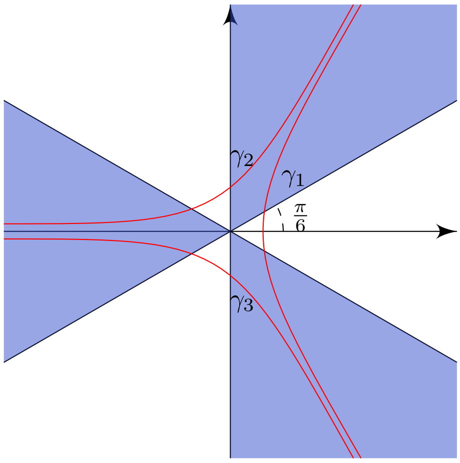
Note $[e^{zt}f(t)]_\gamma = 0$ if $f(t)$ is entire and γ is closed. We demand that $t^2 f(t) - \frac{\mathrm{d}f}{\mathrm{d}t} = 0$ hence

$$f(t) = Ae^{\frac{t^3}{3}}.$$

Having found f, we can choose γ such that $\left[e^{zt}e^{\frac{t^3}{3}}\right]_\gamma = 0$. We choose γ, which starts / ends at ∞ (a closed contour will give a trivial solution). Set $t = |t|e^{i\theta}$. For large $|t|$, we can write

$$e^{zt+\frac{t^3}{3}} \sim e^{\frac{t^3}{3}} = e^{\frac{1}{3}|t|^3\cos 3\theta} e^{i\frac{1}{3}|t|^3 \sin 3\theta}.$$

Which tends to 0 as $|t| \to \infty$ if $\cos\theta < 0$. $\oint_{\gamma_1+\gamma_2+\gamma_3} = 0$ by Cauchy.



So only 2 γ 's give linearly independent solutions (no proof given here for their linear independence). So we choose any pair of contours.
γ_1 : Deform γ_1 to Im axis i.e. $z = iy$

$$w_1(z) = Ai\int_{-\infty}^{\infty} e^{-izy-i\frac{y^3}{3}}\mathrm{d}y = A\left[i\int_{-\infty}^{\infty}\cos(zy-\frac{y^3}{3})\mathrm{d}y - \int_{-\infty}^{\infty}\sin(zy-\frac{y^3}{3})\right],$$

(the solution obtained by Fourier transform) which we note both integrals are real for real z.

$$Ai = \frac{1}{\pi}\int_0^{\infty}\cos(zy-\frac{y^3}{3})\mathrm{d}y.$$

Airy's function of the 1st kind. Note that this is a different A, and the previous one was just a constant.
$\gamma_2 : t = iy, y \in (\infty, 0), t = -x, x \in [0, \infty)$

$$w_2(z) = A\left[i\int_{\infty}^{0} e^{izy-i\frac{y^3}{3}}\mathrm{d}y - \int_0^{\infty} e^{-zx-\frac{x^3}{3}}\mathrm{d}x\right].$$

We take the real part of the first integral, and the second integral is real for real z (but unbounded as $z \to \infty$) . Taking the real part,

$$w_2(z) = -A\int_0^{\infty}\left[\sin(\frac{x^3}{3}-zx) - \exp\left(-zx-\frac{x^3}{3}\right)\right]\mathrm{d}x.$$

Set $A = -\frac{1}{\pi}$ to obtain Bi (z), Airy's function of the second kind.

Example. For equations of the form

$$aw'' + bw' + cw = 0,$$

where $w = w(z)$ and a, b, c are low-order polynomials in z (we require the latter, so we could use integration by parts), set

$$w(z) = \int_{\gamma} K(z,t)f(t)\mathrm{d}t,$$

substitute this back in and integrate by parts to eliminate the terms with z^n. The factor $K(z,t)$ is called a *kernel*. Three following kernels are commonly used:

$$\begin{aligned} e^{zt} &\quad \text{(Laplace kernel)} \\ (z-t)^{\gamma} &\quad \text{(Euler kernel)} \\ t^{z} &\quad \text{(Mellin kernel).} \end{aligned}$$

An example of Euler kernel:

$$w(z) = \int_{\gamma} t^{a-c}(1-t)^{c-b-1}(t-z)^{-a}\mathrm{d}t,$$

which satisfies the *hypergeometric equation*

$$z(1-z)w'' + [c-(a+b-1)z]w' - abw = 0,$$

with constant a, b, c provided that

$$[t^{a-c+1}(1-t)^{c-b}(t-z)^{1-a}]_{\gamma} = 0.$$

The integrand might have branch points, possibly at $t = 0, 1, z$, depending on the values of a, b, c and on whether the exponents are integers. If there are any branch cuts, then γ must not cross them, of course.

3.2 Solving PDEs by integral transform

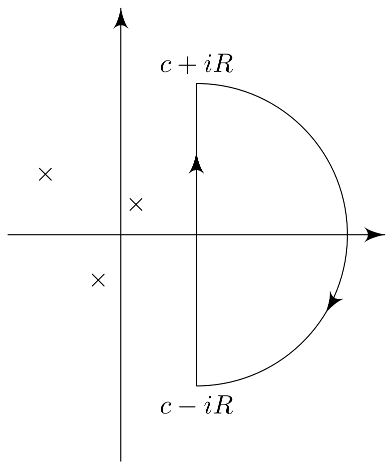
Laplace Transform : for functions which have support on semi-intervals e.g. $f(x) = 0$, for $x < 0$

$$\hat{f}(p) = \int_0^{\infty} e^{-px} f(x)\mathrm{d}x,$$

the Laplace transform, and

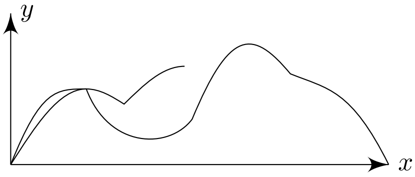
$$f(x) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{px}\hat{f}(p)\mathrm{d}p,$$

the inverse Laplace transform, where the (Bromwich) contour is chosen so it passes to the right of all singularities.



Closing the contour to the right will yield $f(x) = 0$ for $x < 0$. Then closing to the left obtain $f(x)$ for $x \geq 0$.

Example. (Waves on a finite string)



Small transverse oscillations with displacement $y(x,t)$, which satisfies

$$\frac{\partial^2 y}{\partial t^2} = c^2 \frac{\partial^2 y}{\partial x^2},$$

$y = 0$ for $t < t_0$. The string is fixed at $x = 0$ and movable at $x = \ell$. At $t = 0$, it is given a discontinuous instantaneous "nudge" and held. This is an initial value problem. $y(0,t) = 0, y(\ell,t) = y_0(t > 0), y(x,0) = \frac{\partial y}{\partial t}\Big|_x = 0(0 \leq x \leq \ell)$. Note: alternative physical models exist e.g. a capacitor. What we need to consider, which transform (FT or LT) and with respect to which variable? IVP suggests a LT with respect to t

$$y(x,t) = \frac{1}{2\pi i} \int_{\tilde{C}-i\infty}^{\tilde{C}+i\infty} e^{pt} \hat{y}(x,p) \mathrm{d}p.$$

Taking the LT with respect to t yields an ODE in x :

$$p^2 \hat{y}(x,p) = c^2 \frac{\partial^2 \hat{y}(x,p)}{\partial x^2}.$$

This has the solution $\hat{y}(x,p) = A(p) \sinh \frac{px}{x} + B(p) \cosh \frac{px}{c}$. Take the LT of the boundary conditions.

$$y(0,t) = 0 \implies \hat{y}(0,p) = 0 \implies B(p) = 0.$$

$$y(\ell,t) = y_0 \implies \hat{y}(\ell,p) = \int_0^\infty y_0 e^{-pt} \mathrm{d}t = \frac{y_0}{p}.$$

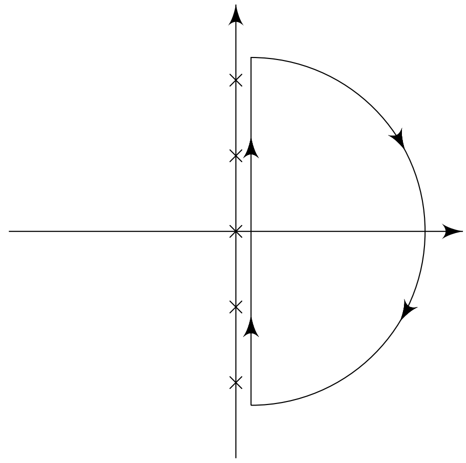
So $A(p) \sinh \frac{\ell p}{c} = \frac{y_0}{p}$. Hence,

$$\hat{y}(x,p) = \frac{y_0}{p} \frac{\sinh\left(\frac{xp}{c}\right)}{\sinh\left(\frac{\ell p}{c}\right)} \implies y(x,t) = \frac{y_0}{2\pi i} \int_{\tilde{C}-i\infty}^{\tilde{C}+i\infty} \frac{\sinh\left(\frac{xp}{c} e^{pt}\right)}{\sinh\left(\frac{\ell p}{c}\right) p} \mathrm{d}p.$$

We perform inversion by contour integration. The integrand has simple poles at $p_m := \frac{m\pi c i}{\ell}$ where m is an integer. For $|p| \to \infty$ with $\operatorname{Re} p > 0$

$$F(x,t,p) \to \frac{e^{xp/c}}{p e^{\ell p/c}} e^{pt} = \frac{1}{p} e^{(x-\ell+ct)p/c}.$$

If $x - \ell + ct < 0$,



So $y(x,t) = 0$ for $t < \frac{\ell - x}{c}$. The solution at x "switches on" at $t = \frac{\ell - x}{c}$. If $|p| \to \infty$ with Re $p < 0$ then

$$F(x,t,p) \to \frac{1}{p}\frac{e^{-xp/c}}{e^{-\ell p/c}}e^{pt} = \frac{1}{p}e^{(-x+\ell+ct)p/c}.$$

So, if $-x+\ell+ct > 0$, we close the path to the left.

$$\oint = y(x,t) + \frac{y_0}{2\pi i}\int_{C_{R_1}} \frac{e^{(-x+\ell+ct)p/c}}{p}\mathrm{d}p = y_0 \sum_{m=-\infty}^{\infty} \text{Res } (F(x,t;p);p_m).$$

For $\frac{\pi}{2} < \theta < \frac{3\pi}{2}, p = \varepsilon + Re^{i\theta}$ etc, we have that $e^{(-x+\ell+ct)R\cos\theta} \to 0$ so the integral tends to 0.The contribution from the section parallel to the real axis is negligible. In the case of $m = 0$ we have that

$$\lim_{p\to 0} \frac{\frac{x}{c}\cosh\left(\frac{xp}{c}\right)e^{pt} + p\sinh\left(\frac{xp}{c}\right)e^{pt}}{\frac{\ell}{c}\cosh\left(\frac{\ell p}{c}\right)} = \frac{x}{\ell}.$$

In the case that $m \neq 0$, the residue is given by L'Hopital once more as

$$\frac{\sinh\left(i\frac{m\pi x}{c}\right)\exp\left(i\frac{m\pi c}{\ell}t\right)}{im\pi\cosh(im\pi)} = \frac{i\sin\left(\frac{m\pi x}{\ell}\right)e^{i\frac{m\pi c}{\ell}t}}{im\pi\cos(m\pi)}.$$

Thus,

$$\begin{aligned} y(x,t) &= \frac{x}{l}y_0 + y_0\sum_{m\neq 0}(-1)^m\frac{\sin\left(\frac{m\pi x}{l}\right)e^{i\frac{m\pi c}{\ell}t}}{m\pi} \\ &= y_0\left(\frac{x}{\ell} + 2\sum_{m=1}^{\infty}(-1)^m\frac{\sin\left(\frac{m\pi x}{\ell}\right)\cos\left(\frac{m\pi c}{\ell}t\right)}{m\pi}\right). \end{aligned}$$

This is a Fourier series. Notice that ranges for which this equation holds ($ct > x - \ell$) and the range for which $y = 0$ holds $ct < \ell - x$ overlap i.e. in $-\ell + x < ct < \ell - x$. Note: the Fourier series must sum up to 0 within the overlap. The solution only "switches on" at x at time $\frac{\ell-x}{c}$. For an alternative inversion method see the e-handout (non-examinable). This will give

$$y(x,t) = y_0 \sum_{n=0}^{\infty} \left[H(xt + x - (2n+1)\ell) - H(ct - x - (2n+1)\ell \right] .$$

This represents a linear combination of terms caused by reflection at $x = 0$ and $x = \ell$

4 Second-order ODE in the complex plane

Consider

$$w'' + p(z)w' + q(z)q = 0,$$

where $p(z), q(z)$ and $w(z)$ are meromorphic on $\mathbb{C}$.

4.1 Classification of singular points

(i) The point $z = z_0$ is an *ordinary point* (OP) of the above equation if p and q are both analytic at z_0. Otherwise, z_0 is a *singular point* (SP).

(ii) If z_0 is a SP, but $(z - z_0)p(z)$ and $(z - z_0)^2 q(z)$ are analytic at z_0, then z_0 is a *regular singular point* (RSP). Otherwise, z_0 is an *irregular singular point.* There exist two linearly independent solutions around RSPs.

For linear ODEs the singularities of the solutions are independent of the ICs, they are fully determined by p and q. This does not hold for non-linear ODEs. For example, E

$$w'' + w^2 = 0 \implies \frac{\mathrm{d}w}{w^2} = -\mathrm{d}z \implies w(z) = (z - z_0)^{-1},$$

where the singularity at z_0 is movable, it depends on constants of integration. <u>What about $z = \infty$?</u> We can extend the definitions (i) and (ii) by setting $z = \frac{1}{t}$ and considering $t = 0$ as follows. By chain rule with $w(z) = w\left(\frac{1}{t}\right)$ we obtain

$$\frac{\mathrm{d}^2 w}{\mathrm{d}t^2} + \left(\frac{2}{t} - \frac{p(\frac{1}{t})}{t^2}\right)\frac{\mathrm{d}w}{\mathrm{d}t} + \frac{q(\frac{1}{t})}{t^4} w = 0.$$

- $t = 0 (z = \infty)$ is an OP if $\frac{2}{t} - \frac{p(\frac{1}{t})}{t^2} = -f(t)$ is analytic at $t = 0$. So, $p(z) = \frac{2}{z} + \frac{f(\frac{1}{z})}{z^2}$ and (similarly) $q(z) = \frac{1}{z^4} g(\frac{1}{z})$, where $f(\frac{1}{z}), g(\frac{1}{z})$ are both analytic at $z = \infty$.
- Similarly, $t = 0 (z = \infty)$ is a RSP if $t\left(\frac{2}{t} - \frac{p(\frac{1}{t})}{t^2}\right)$ and $\frac{q(\frac{1}{t})}{t^2}$ are analytic at $t = 0$. Thus $p(z) = 2 + \frac{1}{z} f(\frac{1}{z})$ and $q(z) = \frac{1}{z^2} g(\frac{1}{z})$, for f and g analytic at $z = \infty$. Note: $p(z)$ has at most a simple pole at ∞ and $q(z)$ has at most a double pole at ∞.

Example. (Legendre equation)

$$w'' - \frac{2z}{1 - z^2} w' + \frac{n(n+1)}{1 - z^2} w = 0,$$

which has RSPs at $z = \pm 1, \infty$. Check that $z = \infty$ is a RSP.

$$q(z) = \frac{n(n+1)}{1 - z^2} = \frac{1}{z^4} g(\frac{1}{z}), g(\frac{1}{z}) = \frac{z^4}{1 - z^2} n(n+1).$$

$g(\frac{1}{z})$ is not analytic at $z = \infty$ so is not an OP.

$$p(z) = \frac{-2z}{1 - z^2} = 2 + \frac{f(\frac{1}{z})}{z}, f(\frac{1}{z}) = -\frac{2}{1 - z^2},$$

which is analytic at ∞. Our $g(\frac{1}{z})$ is found by dividing the previous by z^2, which turns out to be analytic at $z = \infty$. Hence, $z = \infty$ is a RSP. Note, in many cases the ODE can be approximated at ∞

$$w'' + \frac{2}{z}w' - \frac{n(n+1)}{z^2}w \sim 0.$$

This is clearly in the form required for an RSP.

4.2 The indicial equation

Consider a RSP at $z = 0$, and write $W(z) = z^\sigma \sum_{n=0}^{\infty} a_n z^n$. (Around it) by definition of RSP

$$p(z) = \sum_{m=-1}^{\infty} p_m z^m, q(z) = \sum_{m=-2}^{\infty} q_m z^m.$$

If we plug this back into our standard equation for second-order ODEs

$$\sum_{n=0}^{\infty}\left[a_n(n+\sigma)(n+\sigma-1)z^{n+\sigma-2} + \sum_{m=-1}^{\infty} p_m a_n(n+\sigma)z^{n+m+\sigma-1} + \sum_{m=-2}^{\infty} q_m a_n z^{n+m-\sigma}\right] = 0.$$

Compare coefficients of lowest order in m (assuming $a_0 \neq 0$), we obtain for $n = 0$

$$\sigma(\sigma-1) + p_{-1}\sigma + q_{-2} = 0, \sigma^2 + p_{-1} + q_{-2} = 0.$$

We call this the indicial equation. The roots σ_1, σ_2 are *exponents* of our equation at $z = 0$. So,

$$w_1(z) = z^{\sigma_1}(a_0 + a_1 z + a_2 z^2 + \cdots), w_2(z) = z^{\sigma_2}(b_0 + b_1 z + \cdots).$$

($z^{\sigma_1}, z^{\sigma_2}$ are the leading terms near $z = 0$.
Now consider a RSP at ∞. Letting $t = \frac{1}{z}$ we get

$$w\left(\frac{1}{t}\right) t^\sigma \sum_{n=0}^{\infty} a_n t^n \implies w(z) = z^{-\sigma} \sum_{n=0}^{\infty} a_n z^{-n}.$$

Now

$$p(z) = \frac{2}{z} + \frac{1}{z} f(\frac{1}{z}) = \frac{2}{z} + \frac{1}{z}\sum_{m=0}^{\infty} b_m z^{-m} = \sum_{m=1}^{\infty} p_m z^{-m}.$$

$$q(z) = \frac{1}{z^2} g(z) = \sum_{m=2}^{\infty} q_m z^{-m}.$$

Substitute back into our second order ODE to obtain

$$\sum_{n=0}^{\infty}\left[a_n(-n-\sigma)(-n-\sigma-1)z^{-n-\sigma-2} + \sum_{m=1}^{\infty} p_m a_n(-n-\sigma)z^{-n-m-\sigma-1} + \sum_{m=2} q_m a_n z^{-n--m-\sigma}\right] = 0.$$

For lowest order term, taking $n = 0$ we get the indicial equation for a RSP at ∞

$$\sigma^2 + (1-p_1)\sigma + q_2 = 0.$$

4.3 Solutions near ordinary and regular singular points

Example.

$$w'' + \frac{4}{z-1}w' + \frac{2}{z-1}w = 0,$$

with p, q analytic in $|z| < 1$. Note that $z = 0$ is an OP and $z = 1, \infty$ are RSP. AT $z = 0$, $w_1(z) = \sum_{n=0}^{\infty} z^n, w_2(z) = \sum_{n=1}^{\infty} nz^n$. Near $z = 1$, $w(z) = (z-1)^\sigma \sum_{n=0}^{\infty} a_n(z-1)^n$ with the indicial equation giving us $\sigma^2 + 3\sigma + 2 = 0$ so $\sigma_{1,2} = -1, -2$. Hence $w_1(z) = \frac{1}{1-z}$ and $w_2(z) = \frac{1}{(1-z)^2}$ (near $z - 1$). At $z = \infty$, $z - 1 \approx z$, so

$$w'' + \frac{4}{z}w' + \frac{2}{z^2}w = 0.$$

Now, the indicial equation is given by $\sigma^2 - 3\sigma + 2 = 0$ ($\sigma_1 = 1, \sigma_2 = 2$. Thus,

$$w_1(z) = z^{-1}\sum_{n=0}^{\infty} z^{-n} = \frac{1}{z-1}, w_2(z) = z^{-2}\sum_{n=0}^{\infty}(n+1)z^{-n}.$$

We consider equations of the form

$$w'' + p(z)w' + q(z)q = 0,$$

where $p(z), q(z)$ and $w(z)$ are meromorphic on $\mathbb{C}$.
Solutions near an ordinary point (OP)

Theorem. If $p(z)$ and $q(z)$ are analytic in the disk $|z| < R(z = 0$ is an OP), then there exist two linearly independent solutions to the equation above w_1, w_2 such that

(i) w_1, w_2 are analytic in $|z| < R$ (and possibly in a larger disk)

(ii) $w_1(0) \neq 0, w_2(0) = 0$, but $w_2'(0) \neq 0$ (i.e. the roots of the indicial equation are 0 and 1.

Proof. (Outline - non-examinable) By substituting $p(z) = \sum_{n=0}^{\infty} p_n z^n, q(z) = \sum_{n=0}^{\infty} q_n z^n$ and $w(z) = \sum_{n=0}^{\infty} a_n z^n$ into our 2nd order equation, equating coefficients of z_n and checking the radius of convergence. (Do not attempt it. There are better ways e.g. Picard iteration) □

Remark. – Note that (for example) z and z^2 cannot both satisfy an equation of the above form, for which $z = 0$ is an OP.

– An example of an equation with solutions that are analytic in a disk larger than the disk in which $p(z)$and $q(z)$ are analytic:

$$w'' - \frac{2}{z-1}w' + \frac{2}{(z-1)^2}w = 0,$$

which has solution $w(z) = A(z-1) + B(z-1)^2$ an entire function despite the singular point of the equation at $z = 1$.

Solutions near a regular singular point Suppose $zp(z)$ and $z^2q(z)$ are analytic in the disk $|z| < R$. Let

$$p(z) = \sum_{n=-1}^{\infty} p_n z^n, \text{ and } q(z) = \sum_{n=-2}^{\infty} q_n z^n.$$

The indicial equation is

$$\sigma^2 + (p_{-1} - 1)\sigma + q_{-2} = 0.$$

This equation can be though in connection with the $|z| \ll 1$ approximation of the 2nd order ODE

$$w'' + p_{-1}z^{-1}w' + q_{-2}z^{-2}w \sim 0,$$

which has solutions

$$z^{\sigma_1}, z^{\sigma_2} \text{ if } \sigma_1 \neq \sigma_2,$$
$$z^{\sigma_1}, z^{\sigma_1} \log z \text{ if } \sigma_1 = \sigma_2,$$

where σ_1, σ_2 are the roots of the indicial equation.

Theorem. Let $z = 0$ be a RSP of the standard 2nd order ODE. Then there exist two linearly independent solutions $w_1(z), w_2(z)$, such that

- If $\sigma_1 - \sigma_2 \notin \mathbb{Z}$, then

$$w_1(z) = z^{\sigma_1}u_1(z) \text{ and } w_2(z) = z^{\sigma_2}u_2(z),$$

  where u_1, u_2 are analytic for $|z| < R$ (and maybe in a larger disk), and $u_1(0) \neq 0 \neq u_2(0)$.

- If $\sigma_1 = \sigma_2$, then

$$w_1(z) = z^{\sigma_1}u_1(z) \text{ and } w_2(z) = w_1(z)\log z + z^{\sigma_1}u_2(z),$$

  where u_1, u_2 are analytic for $|z| < R$ (and maybe in a larger disk), and $u_1(0) \neq 0 \neq u_2(0)$.

- If $\sigma_1 \neq \sigma_2$ but $\sigma_1 - \sigma_2 \in \mathbb{Z}$ (say $\sigma_1 > \sigma 2$), then

$$w_1(z) - z^{\sigma_1}u_1(z) \text{ and } w_2(z) = Cw_1(z)\log z + z^{\sigma_2}u_2(z),$$

  where C is some constant, which may sometimes be zero.

Proof. (i) Substitute $w(z) = z^\sigma \sum_{n=0}^{\infty} a_n z^n$into the 2nd order ODE and (assuming $a_0 \neq 0$) equate the coefficients of z^{n+z} to zero to obtain

$$\begin{cases} a_0F(\sigma) = 0, & \text{for } n = 0, \\ a_nF(n+\sigma) = -\sum_{k=0}^{n-1} a_k\left[(k+\sigma)p_{n-k-1} + q_{n-k-2}\right], & \text{for } n > 0, \end{cases}$$

where $F(x) = x(x-1) + p_{-1}x + q_{-2} = (x-\sigma_1)(x-\sigma_2)$. Since $a_0 \neq 0$, the first equation is the indicial equation. Its exponents (i.e. roots) satisfy

$$\sigma_1 + \sigma_2 = 1 - p_{-1} \text{ and } \sigma_1\sigma_2 = q_{-2}.$$

If $\sigma_1 - \sigma_2 \notin \mathbb{Z}$, then the second equation gives the recurrence relations for a_k, which determine two linearly independent solutions which can be written in the form of (i).

□

4.4 Fuchsion Equations

$$w'' + p(z)w' + q(z)w = 0.$$

Two RSPs

We've shown: $w'' + \left(\frac{1+A}{z} + \frac{1-A}{z-1}\right)w' + \frac{Q}{z^2(z-1)^2}w = 0$, for RSPs at $z = 0, 1$, where A is a constant. Consider exponents in the vicinity $z = 0$: We have $w'' + \frac{1+A}{z}w' + \frac{Q}{z^2}w \sim 0$ The indicial equation is $\sigma^2 + A\sigma + Q = 0$. Similarly, near $z = 1$ we get: $w'' + \frac{1-A}{z-1}W + \frac{Q}{(z-1)^2} \sim 0$ so $\sigma^2 - A\sigma + Q = 0$. We can use Mobius transforms to move the RSPs. In particular, we can use $t = \frac{1}{z}$ to move $0 \to \infty, 1 \to 1$. We have

$$t^4\frac{\mathrm{d}^2}{\mathrm{d}t^2} + \left[2t^3 - t^2\left(t(1+A) + \frac{t(1-A)}{1-t}\right)\right]\frac{\mathrm{d}w}{\mathrm{d}t} + \frac{t^4Q}{(t-1)^2}w = 0,$$

and we obtain

$$\frac{\mathrm{d}^2w}{\mathrm{d}t^2} + \frac{1-A}{t-1}\frac{\mathrm{d}w}{\mathrm{d}t} + \frac{Q}{(t-1)^2}2w = 0.$$

The indicial equation at $t = 1$ is $\sigma^2 - A\sigma + Q = 0$ (same as before $z = 1$); at $t = \infty$ we get $\sigma^2 + A\sigma + Q = 0$ (same as $z = 0$). Note that solutions near $z = 0$ take the form $w = z^{\sigma_1}u_1(z) + z^{\sigma_2}u_2(z)$, where u_1, u_2 are analytic. The solutions at $t = \infty$ take the form $w(t) = t^{-\sigma_1}u_1\left(\frac{1}{t}\right) + t^{-\sigma_2}u_2\left(\frac{1}{t}\right)$.

Three RSPs : The Papperitz Equation (Riemann's P equation)

We will show that there are 8 parameters in the P-equation

- 3 from the positions of the RSPs
- 6 exponents (2 per each RSP)
- -1 from a constraint on the exponents: they must add up to 1

Suppose the (distinct) RSPs are at $z = a, b, c$. All other points are regular (including ∞) wlog, $p(z)$ must take the form

$$p(z) = \frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c} + P(z),$$

where $P(z)$ is entire. $z = \infty$ is an OP, so $p(z) = \frac{2}{z} + O(\frac{1}{z^2})$ so by Liouville's theorem $P(z) = 0$, and $zp(z) \underset{z\to\infty}{\to} 2$ so $(1-\alpha-\alpha') + (1-\beta-\beta') + (1-\gamma-\gamma') = 2$. So, $\alpha + \alpha' + \beta + \beta' + \gamma + \gamma' = 1$. Next, wlog

$$q(z) = \frac{k_ax - k'_a}{(z-a)^2} + \frac{k_bz + k'_b}{(z-b)^2} + \frac{k_cz + k'_c}{(z-c)^2} + Q_2(z) = \frac{Q_1(z)}{(z-a)^2(z-b)^2(z-c)^2} + Q_2(z),$$

with Q_1 a polynomial of degree at most 5 and Q_2 entire. $z = \infty$ is an OP implies that $z^4q(z)$ is bounded as $z \to \infty$, so by Liouville's theorem $Q_2 = 0$ and Q_1 is at most quadratic. Hence we can write,

$$q(z) = \frac{1}{(z-a)(z-b)(z-c)}\left(\frac{q_a}{z-a} + \frac{q_b}{z-b} + \frac{q_c}{z-c}\right).$$

We can specify $\alpha, \alpha', \beta, \beta', \gamma, \gamma'$ by writing

$$\begin{aligned} q_a &= \alpha\alpha'(a-b)(a-c) \\ q_b &= \beta\beta'(b-a)(b-c) \\ q_c &= \gamma\gamma'(c-a)(c-b) \end{aligned}$$

.

So, we now fixed $\alpha, \alpha' \dots$ and a, b, c under the condition $\alpha + \alpha' + \dots$. This gives the 8 degrees of freedom. Thus we obtain the P equation

$$w'' + \left(\frac{1-\alpha-\alpha'}{z-a} + \frac{1-\beta-\beta'}{z-b} + \frac{1-\gamma-\gamma'}{z-c}\right) w'$$
$$- \frac{(b-c)(c-a)(a-b)}{(z-a)(z-b)(z-c)} \left[\frac{\alpha\alpha'}{(z-a)(b-c)} + \frac{\beta\beta'}{(z-b)(c-a)} + \frac{\gamma\gamma'}{(z-c)(a-b)}\right] w = 0$$

Near $z = a$, the P-equation can be approximated as

$$w'' + \frac{1-\alpha-\alpha'}{z-a} w' + \frac{\alpha\alpha'}{(z-a)^2} w \sim 0,$$

so the indicial equation takes the form $\sigma^2 + (\alpha + \alpha')\sigma + \alpha\alpha' = 0$ hence our exponents at $z = a$ are $\sigma_1 = \alpha, \sigma_2 = \alpha$. Similarly, β, β' and γ, γ' are the exponents near b and c respectively. We write

$$P \begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}$$

for the 2-d vector space of solutions of the P equation. This is known as the Papperitz (P)-symbol. It is common to write

$$w(z) = P \begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix},$$

although this is an abuse of notation. We simply mean that w solves the corresponding P equation.

4.5 The hypergeometric equation

The Mobius transformation acts on the P-symbol as follows $(a, b, c, z) \to (a', b', c', z')$, so

$$P \begin{Bmatrix} a & b & C & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P \begin{Bmatrix} a' & b' & c' & \\ \alpha & \beta & \gamma & z' \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}.$$

We also need to transform the P-symbol by exponential shifting.
Exponential shifting

Claim.

$$\left(\frac{z-a}{z-b}\right)^\sigma \left(\frac{z-b}{z-c}\right)^\delta P \begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P \begin{Bmatrix} a & b & c & \\ \alpha+\sigma & \beta-\sigma+\delta & \gamma-\delta & z \\ \alpha'+\sigma & \beta-\sigma+\delta & \gamma-\delta & \end{Bmatrix}.$$

This means that if

$$w(z) = P \begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix},$$

then

$$w_1(z) = \left(\frac{z-a}{z-b}\right)^\sigma \left(\frac{z-b}{z-c}\right)^\delta w(z)$$

satisfies the P-equation which corresponds to the P-symbol of the RHS above.

Proof. (given for $\delta = 0$) Let us prove the equivalent:

$$\left(\frac{z-a}{z-b}\right)^\sigma P \begin{Bmatrix} a & b & c & \\ \alpha-\sigma & \beta+\sigma & \gamma & z \\ \alpha'-\sigma & \beta'+\sigma & \gamma' & \end{Bmatrix} = P \begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix}.$$

Then, w_1 is given by $w(z) = \left(\frac{z-a}{z-b}\right)^\sigma w_1(z)$. We assume that w satisfies the RHS and want to show that w_1 satisfies the LHS. Near $z = a$:

$$w(z) = (z-a)^\alpha \sum_{n=0}^{\infty} a_n (z-a)^n,$$

so

$$w_1(z) = (z-b)^\sigma \sum_{n=0}^{\infty} a_n (z-a)^{n+\alpha-\sigma} = (z-a)^{\alpha-\sigma} \sum_{n=0}^{\infty} c_n (z-a)^n.$$

This is possible since $(z-b)^\sigma$ is analytic at $z = 0$. So, we shifted to α to $\alpha - \sigma$. Hence,

$$\left(\frac{z-a}{z-b}\right)^\sigma \left(\frac{z-b}{z-c}\right)^\delta P \begin{Bmatrix} a & b & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P \begin{Bmatrix} a & b & c & \\ \alpha+\sigma & \beta-\sigma+\delta & \gamma-\delta & z \\ \alpha'+\sigma & \beta-\sigma+\delta & \gamma-\delta & \end{Bmatrix}.$$

□

Note, when $b \to \infty$, we multiply by $(-b)^\sigma$ and take $b \to \infty$ to obtain

$$(z-a)^\delta P \begin{Bmatrix} a & \infty & c & \\ \alpha & \beta & \gamma & z \\ \alpha' & \beta' & \gamma' & \end{Bmatrix} = P \begin{Bmatrix} a & \infty & c & \\ \alpha+\sigma & \beta-\sigma & \gamma & z \\ \alpha'+\sigma & \beta'-\sigma & \gamma' & \end{Bmatrix}.$$

To obtain the hypergeometric equation (HGE), we

- Move $(a, b, c) \to (0, 1, \infty)$ by suitable Mobius transform
- Shift exponents, so $\alpha = \beta = 0$.
- Rename $\gamma = A$, $\gamma' = B$, $\alpha' = 1 - C$, then $\beta' = C - A - B$

So the P-symbol becomes

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & A & z \\ 1-C & C-A-B & B & \end{Bmatrix},$$

which corresponds to the HGE

$$w'' + \left(\frac{c}{z} + \frac{1+A+B-C}{z-1}\right)w' + \frac{AB}{z(z-1)}w = 0,$$

with RSPs at $z = 0, 1, \infty$. Note on notation, $A \to a, B \to b, C \to c$ (use lower case), so we write

$$P\begin{Bmatrix} 0 & 1 & \infty & \\ 0 & 0 & a & z \\ 1-c & c-a-b & b & \end{Bmatrix}.$$

One of the exponents is 0, so $\exists$ a solution that is analytic at $z = 0$ with $w(0) = 1$. This is known as the hypergeometric function $F(a, b; c; z)$.