Exercises 1.4 - 1.6

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Question 1.4

Which of the following statements are true? If the statement is true, prove it; if not, give a counterexample. Let x and y be real numbers.

a) If x is rational and y is irrational, then $x + y$ is irrational.
Proof:
Assume for the sake of contradiction that $x + y$ is rational,

$$\begin{aligned}
&\implies x + y = \frac{m}{n}, \quad m, n \in \mathbb{Z} \\
&\implies \frac{p}{q} + y = \frac{m}{n}, \quad p, q \in \mathbb{Z} \\
&\implies y = \frac{m}{n} - \frac{p}{q} = \frac{mq - np}{nq} \\
&\implies y \in \mathbb{Q} \quad \text{and} \quad y \notin \mathbb{Q}
\end{aligned}$$

This is a contradiction and thus $x + y$ must be irrational.

b) If x is rational and y is irrational, then xy is irrational.
Counterexample:
Let $x = 0$ and $y = \sqrt{2}$.

$$xy = 0 \in \mathbb{Q}$$

.

c) If x and y are irrational, then so is $x + y$
Counterexample:
Let $x = \sqrt{2}$ and $y = 2 - \sqrt{2}$,

$$x + y = \sqrt{2} + (2 - \sqrt{2}) = 2 \in \mathbb{Q}$$

d) If x and y are irrational, then so is xy.
Counterexample:
Let $x = \sqrt{2}$ and $y = \sqrt{8}$,

$$\begin{aligned}
xy &= \sqrt{2} \cdot \sqrt{8} \\
&= \sqrt{16} \\
&= 4 \in \mathbb{Q}
\end{aligned}$$

e) If x and y are irrational, then $x + y$ is rational.
Counterexample
Let $x = \sqrt{2}$, $y = \sqrt{3}$, and assume for the sake of contradiction that $x + y$ is rational.

$$\implies \sqrt{2} + \sqrt{3} = \frac{p}{q}$$

We square both sides,

$$\begin{aligned}
&\implies 5 + \sqrt{6} = \frac{p^2}{q^2} \\
&\implies \sqrt{6} = \frac{p^2}{q^2} - 5 \\
&\implies \sqrt{6} \in \mathbb{Q}
\end{aligned}$$

This is a contradiction and thus we get that $x + y$ is irrational.

f) If x and y are irrational, then xy is rational.
Counterexample:
Let $x = \sqrt{2}$ and $y = \sqrt{3}$. $xy = \sqrt{6}$, which is irrational.

Question 1.5

Let x, y be real numbers, with $x < y$. Show that, if x and y are rational, then there exists an irrational number u such that $x < u < y$.
Proof:
Let $u = x + (2 - \sqrt{2})(y - x)$.
We know a rational number plus or minus an irrational number is irrational, thus $2 - \sqrt{2}$ will be irrational. And in particular this number will be less than 1, which implies that $(2 - \sqrt{2})(y - x) < (y - x)$, and that, $x < x + (2 - \sqrt{2})(y - x) < y$.

Question 1.6

Let $x, y \in \mathbb{R}$, with $x < y$. Show that there exists a rational number q such that $x < q < y$
Proof:
Let n be some positive integer, such that

$$n > \frac{1}{y - x} \in \mathbb{Q}$$
$$\implies \frac{1}{n} \in \mathbb{Q} < y - x,$$

and we know n exists by The Axiom of Archimedes. Looking at the series defined by $\frac{1}{n}$,

$$\cdots - \frac{2}{n}, -\frac{1}{n}, \frac{0}{n}, \frac{1}{n}, \frac{2}{n}, \cdots$$

we get that there must exists some element of this series within the bounds from x to y; the differences between the elements are smaller than the difference between the bounds.