Lattice Model for Proton EDM

At first, lattice without RF is analysed to find the beta-functions, tunes, and dispersion.
Then, RF is added and beta-functions for coupled motion (in x- and z-coordinate) are found.

Data taken from proton EDM presentations is highlighted.
Input data we are not sure about is highlighted in red.

Physical Constants

Proton Rest Mass [MeV] $m_p := 938.272046$ MeV Light velocity: $c := 2.99792458 \cdot 10^{10}$

Proton g-factor $g_p := 5.5856947$

Properties of particles

Mass of particles [MeV] $M := m_p = 938.272$ MeV

Magic Momentum [MeV/c] $P := \dfrac{m_p}{\sqrt{\dfrac{g_p - 2}{2}}} = 700.74$ MeV/c

Kinetic Energy [MeV] $K := \sqrt{M^2 + P^2} - M = 232.792$ MeV

Gamma $\gamma := \dfrac{\sqrt{M^2 + P^2}}{M} = 1.248$

Velocity in units of c $\beta := \sqrt{1 - \dfrac{1}{\gamma^2}} = 0.598$

Lattice

Radius of curvature [cm] $R_0 := 4000$

Ring circumference [cm] $C := 26300$

Number of bends $N := 14$ **Bending angle [rad]** $\theta := \dfrac{2\pi}{N} = 0.449$

Length of bends [cm] $L_b := \theta \cdot R_0 = 1.795 \times 10^3$

Length of straight sections [cm] $L := \dfrac{(C - N \cdot L_b)}{N} = 83.376$

Electric Bends

The following formulas are used:

$$E_R = E_0 \left(\frac{R_0}{R}\right)^n \left[1 - \frac{n^2 - 1}{2}\frac{z^2}{R^2} + \frac{1}{24}(n^2 - 1)(n + 1)(n + 3)\frac{z^4}{R^4} + O(z^6)\right]$$

$$E_z = E_0 \left(\frac{R_0}{R}\right)^n \left[(n - 1)\frac{z}{R} - \frac{1}{6}(n^2 - 1)(n + 1)\frac{z^3}{R^3} + O(z^5)\right]$$

In linear approximation:

Radial component $E_x = -E_0\left[1 - (m + 1)\dfrac{x}{R_0}\right]$

Vertical component $E_y = -m \cdot E_0 \cdot \dfrac{y}{R}$ where $m = n - 1$

Field strength [keV/cm] $E_0 := \dfrac{P \cdot \beta \cdot 1000}{R_0} = 104.827$

$m := 0.2$ => $n := 1 + m = 1.2$

Plate separetion [cm] $\Delta x := 3$

Relative non-linear field correction at aperture: $\frac{\Delta E_R}{E_0} = \frac{n^2+1}{2\cdot R_0^2}\cdot\left(\frac{\Delta x}{2}\right)^2 = 1.716\times 10^{-7}$

Transfer Matrix without RF

Electric Bend

See H. Wollnik, Optics of charged particles for derivation

Auxiliary coefficients

Coordinate system: $\begin{pmatrix} x \\ x' \\ y \\ y' \\ z \\ \frac{\delta p}{p} \end{pmatrix}$

$$k_x := \frac{1}{R_0}\cdot\sqrt{1 - m + \frac{1}{\gamma^2}} \qquad k_y := \frac{1}{R_0}\cdot\sqrt{m}$$

$$c_x := \cos(k_x\cdot L_b) \qquad s_x := \frac{\sin(k_x\cdot L_b)}{k_x} \qquad d_x := \frac{1-\cos(k_x\cdot L_b)}{R_0\cdot k_x^2}$$

$$c_y := \cos(k_y\cdot L_b) \qquad s_y := \frac{\sin(k_y\cdot L_b)}{k_y} \qquad t_{dk} := \frac{k_x\cdot L_b - \sin(k_x\cdot L_b)}{R_0^2\cdot k_x^3} \qquad N_t := \left(1+\frac{1}{\gamma^2}\right)$$

$$R := \begin{pmatrix} c_x & s_x & 0 & 0 & 0 & d_x\cdot N_t \\ -s_x\cdot k_x^2 & c_x & 0 & 0 & 0 & \frac{s_x}{R_0}N_t \\ 0 & 0 & c_y & s_y & 0 & 0 \\ 0 & 0 & -s_y\cdot k_y^2 & c_y & 0 & 0 \\ \frac{-s_x}{R_0}N_t & -d_x N_t & 0 & 0 & 1 & -N_t^2\cdot t_{dk} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 0.858 & 1.71\times 10^3 & 0 & 0 & 0 & 645.587 \\ -1.541\times 10^{-4} & 0.858 & 0 & 0 & 0 & 0.702 \\ 0 & 0 & 0.98 & 1.783\times 10^3 & 0 & 0 \\ 0 & 0 & -2.229\times 10^{-5} & 0.98 & 0 & 0 \\ -0.702 & -645.587 & 0 & 0 & 1 & -160.129 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Drift spaces

Full length:

$$O := \begin{pmatrix} 1 & L & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & L & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 83.376 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 83.376 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

1/2 length:

$$O_{0.5} := \begin{pmatrix} 1 & \frac{L}{2} & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & \frac{L}{2} & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 41.688 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 41.688 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

1 Period

An electric bend, surrounded by two 1/2 straight sections.

$$M := O_{0.5}\cdot R\cdot O_{0.5} = \begin{pmatrix} 0.852 & 1.781\times 10^3 & 0 & 0 & 0 & 674.841 \\ -1.541\times 10^{-4} & 0.852 & 0 & 0 & 0 & 0.702 \\ 0 & 0 & 0.979 & 1.865\times 10^3 & 0 & 0 \\ 0 & 0 & -2.229\times 10^{-5} & 0.979 & 0 & 0 \\ -0.702 & -674.841 & 0 & 0 & 1 & -160.129 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Betatron Tunes

Horizontal tune $\mu_x := \operatorname{acos}(M_{0,0}) = 0.551$ $\qquad \Delta Q_x := \frac{\mu_x}{2\cdot\pi} = 0.088$ $\qquad Q_x := N\cdot\Delta Q_x = 1.228$

Vertical tune $\mu_y := \operatorname{acos}(M_{2,2}) = 0.205$ $\qquad \Delta Q_y := \frac{\mu_y}{2\cdot\pi} = 0.033$ $\qquad Q_y := N\cdot\Delta Q_y = 0.457$

Courant-Snyder parameters at the period boundary

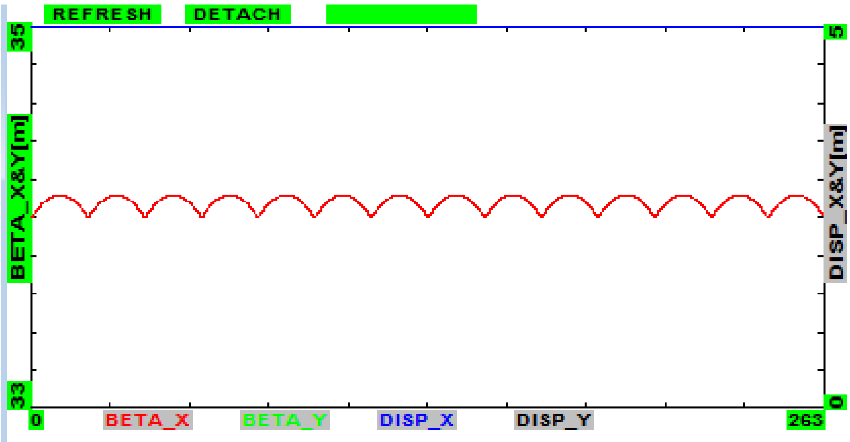
$$\boxed{\beta_x := \frac{M_{0,1}}{\sin(\mu_x)} = 3399.833}\ \text{[cm]} \qquad \alpha_x := 0$$

$$\boxed{\beta_y := \frac{M_{2,3}}{\sin(\mu_y)} = 9146.788}\ \text{[cm]} \qquad \alpha_y := 0$$

$$\gamma_x := \frac{-M_{1,0}}{\sin(\mu_x)} = 2.941 \times 10^{-4} \quad [\text{cm}^{-1}]$$

$$\gamma_y := \frac{-M_{3,2}}{\sin(\mu_y)} = 1.093 \times 10^{-4} \quad [\text{cm}^{-1}]$$

Below are the graphs of beta-functions, obtained by OptiM simulation program, horizontal one at the top and vertical one at the bottom. As one can see vgariations over period are quite small.



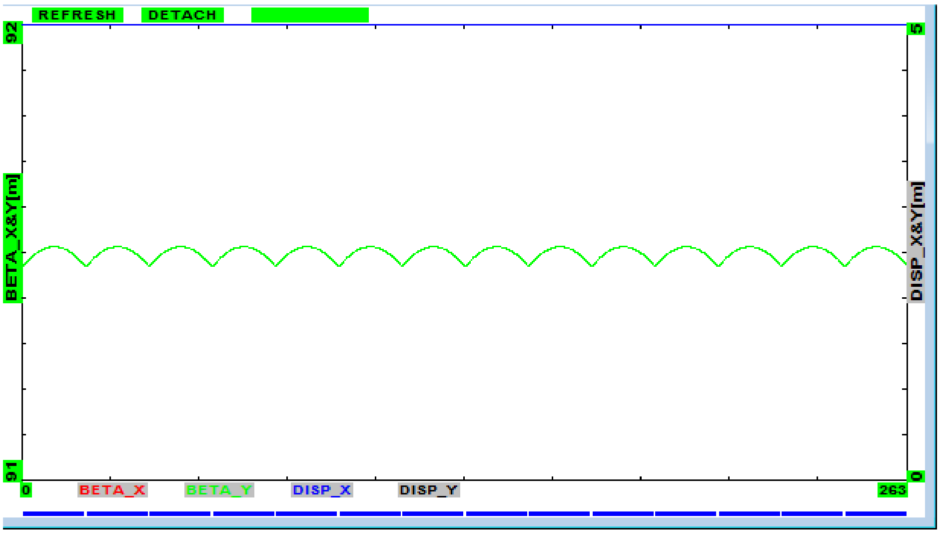
Dispersion

$$\boxed{D := \frac{M_{0,5}}{1 - M_{0,0}} = 4554.807}\ \text{[cm]}$$

D is calculated at the period boundary. Simular to the beta-functions variations of the dispersion over period are quite small.

This dispersion coinsides with the disspersion for the case of zero length of the straight lines

$$\frac{N_t}{R_0 \cdot k_x^2} = 4554.807$$



1-turn transfer matrix

$$M_{2\pi} := M^N = \begin{pmatrix} 0.135 & 3.369 \times 10^3 & 0 & 0 & 0 & 3.939 \times 10^3 \\ -2.914 \times 10^{-4} & 0.135 & 0 & 0 & 0 & 1.327 \\ 0 & 0 & -0.965 & 2.415 \times 10^3 & 0 & 0 \\ 0 & 0 & -2.886 \times 10^{-5} & -0.965 & 0 & 0 \\ -1.327 & -3.939 \times 10^3 & 0 & 0 & 1 & -4.094 \times 10^4 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

Momentum compaction $\alpha := -\frac{{M_{2\pi}}_{4,0} \cdot D + {M_{2\pi}}_{4,5}}{C} = 1.787$ - there are no the transition energy for this ring

Slip-factor: $\eta := \alpha - \frac{1}{\gamma^2} = 1.145$ $\qquad K_s := \frac{\alpha \cdot \gamma^2 - 1}{\gamma^2 - 1} = 3.197$

RF parameters

Voltage [MeV] $V_0 := 0.002$

Harmonic number $h := 1$

Synchrotron tune: $Q_s := \sqrt{\frac{V_0 \cdot h \cdot K_s}{2 \cdot \pi \cdot m_p \cdot \gamma}} = 9.322 \times 10^{-4}$

Bucket height $\Delta p\_p_{max} := \frac{2 \cdot Q_s}{h \cdot \eta} = 1.629 \times 10^{-3}$

RF frequency: $f_{RF} := \frac{c \cdot \beta}{C} = 6.821 \times 10^5$

Beam emittances, rms beam sizes and angular spreads

Ratio of horizontal aperture to the rms beam size $n_\sigma := 2.96$

Rms hor. emittances $\varepsilon_x := \left(\frac{\Delta x}{2\cdot n_\sigma}\right)^2 \cdot \frac{1}{\beta_x}$ $\quad \varepsilon_{xn} := \beta\cdot\gamma\cdot\varepsilon_x$ $\quad \varepsilon_x\cdot 10^4 = 0.755$ mm mrad

$\varepsilon_{xn}\cdot 10^4 = 0.564$ mm mrad

Rms hor. angular spread $\sigma_{\theta x} := \sqrt{\frac{\varepsilon_x}{\beta_x}}$ $\quad \sigma_{\theta x} = 1.491\times 10^{-4}$

Thermodynamic equilibrium (minimum IBS) is acheved when $\frac{\varepsilon_x}{\beta_x} = \frac{\varepsilon_y}{\beta_y} = \frac{\beta_x\cdot\varepsilon_x\cdot\sigma_p^{\,2}}{\beta_x\cdot\varepsilon_x + \sigma_p^{\,2}\cdot D^2}$

=> $\varepsilon_y := \varepsilon_x\cdot\frac{\beta_y}{\beta_x}$ $\quad \varepsilon_y\cdot 10^4 = 2.032$ mm mrad

$\sigma_{pe} := \sqrt{\frac{\varepsilon_x\cdot\beta_x}{\beta_x^{\,2} - D^2}} = 1.672i\times 10^{-4}$ - i.e. the beam cannot be in equilibrium due to operation above transition

In this casem it looks reasonable to choose the momentum spread using aperture limitation as the only problem

$\sigma_p := \frac{\Delta x}{2\cdot n_\sigma\cdot D} = 1.113\times 10^{-4}$

Bunch length $\quad \sigma_s := \frac{C\cdot\eta\cdot\sigma_p}{2\cdot\pi\cdot Q_s} = 571.879$ cm

Space charge tune shifts

$N_p := 2\cdot 10^{10}$ $\quad r_p := 1.535\cdot 10^{-16}$ cm $\quad \sigma_x := \sqrt{\varepsilon_x\cdot\beta_x + (\sigma_p\cdot D)^2}$ $\quad \sigma_y := \sqrt{\varepsilon_y\cdot\beta_y}$

$\sqrt{\varepsilon_x\cdot\beta_x} = 0.507$ cm

$\sigma_p\cdot D = 0.507$ cm

$\sigma_x = 0.717$ cm

$\sigma_y = 1.363$ cm

$$\delta Q_{xSC} := \frac{r_p\cdot N_p\cdot C}{(2\cdot\pi)^{1.5}\cdot\beta^2\cdot\gamma^3\cdot\sigma_s}\cdot\frac{\beta_x}{(\sigma_x+\sigma_y)\cdot\sigma_x}$$

$$\delta Q_{ySC} := \frac{r_p\cdot N_p\cdot C}{(2\cdot\pi)^{1.5}\cdot\beta^2\cdot\gamma^3\cdot\sigma_s}\cdot\frac{\beta_y}{(\sigma_x+\sigma_y)\cdot\sigma_y}$$

$\delta Q_{xSC} = 0.029$

$\delta Q_{ySC} = 0.042$

Intrabeam scattering

$\theta_x := \sqrt{\frac{\varepsilon_x}{\beta_x}}$ $\quad \theta_y := \sqrt{\frac{\varepsilon_y}{\beta_y}}$

$\theta_p := \sigma_p\cdot\frac{\sqrt{\beta_x\cdot\varepsilon_x}}{\gamma\cdot\sigma_x}$ $\quad r_{min} := \frac{2\cdot r_p}{\beta^2\cdot\gamma^2\cdot\left(\theta_x^{\,2}+\theta_y^{\,2}+\theta_p^{\,2}\right)}$

$L_c := \ln\left(\frac{\sqrt{\sigma_x\cdot\sigma_y}}{r_{min}}\right)$ $\quad r_{max\omega} := \sqrt{\frac{2\cdot\pi\cdot\sigma_x\cdot\sigma_y\cdot\sigma_s\cdot\beta^2\cdot\gamma^2\cdot\left(\theta_x^{\,2}+\theta_y^{\,2}+\theta_p^{\,2}\right)}{2\cdot N_p\cdot r_p}}$

$\theta_p = 6.303\times 10^{-5}$

$r_{min} = 1.137\times 10^{-8}$

$L_c = 18.281$

$r_{max\omega} = 3.929$ cm

Program for evaluation of the below integral is hidden in the region below

$$R_D(x,y,z) = \frac{3}{2}\cdot\int_0^\infty \frac{1}{\sqrt{(t+x)\cdot(t+y)\cdot(t+z)^3}}\,dt$$

$$\psi_{ibs}(a,b,c) := \left(-2\cdot a^2\cdot R_D\left(c^2,b^2,a^2\right) + b^2\cdot R_D\left(c^2,a^2,b^2\right) + c^2\cdot R_D\left(a^2,b^2,c^2\right)\right)\cdot\frac{\sqrt{2}\cdot\sqrt{a^2+b^2+c^2}}{\pi\cdot 3}$$

$$\begin{pmatrix} d\varepsilon x\_dt \\ d\varepsilon y\_dt \\ d\sigma p2\_dt \end{pmatrix} := \frac{1}{4\cdot\sqrt{2}}\cdot\frac{r_p^{\,2}\cdot c\cdot N_p\cdot L_c}{\sigma_x\cdot\sigma_y\cdot\sigma_s\cdot\beta^3\cdot\gamma^5\sqrt{\theta_x^{\,2}+\theta_y^{\,2}+\theta_p^{\,2}}}\cdot\begin{pmatrix} \beta_x\cdot\psi_{ibs}(\theta_x,\theta_y,\theta_p) + \gamma^2\cdot\frac{D^2}{\beta_x}\cdot\psi_{ibs}(\theta_p,\theta_x,\theta_y) \\ \beta_y\cdot\psi_{ibs}(\theta_y,\theta_p,\theta_x) \\ \gamma^2\cdot\psi_{ibs}(\theta_p,\theta_x,\theta_y) \end{pmatrix}$$

Given

$$y''(x) + \frac{3}{x + 10^{-9}} \cdot y'(x) + y(x) = 0$$

$y(0) = 1 \qquad y'(0) = 0 \qquad y := \mathrm{Odesolve}(x, 4)$

$$\tau_x := \frac{\varepsilon_x}{d\varepsilon x\_dt} = 48.53 \quad \mathrm{s}$$

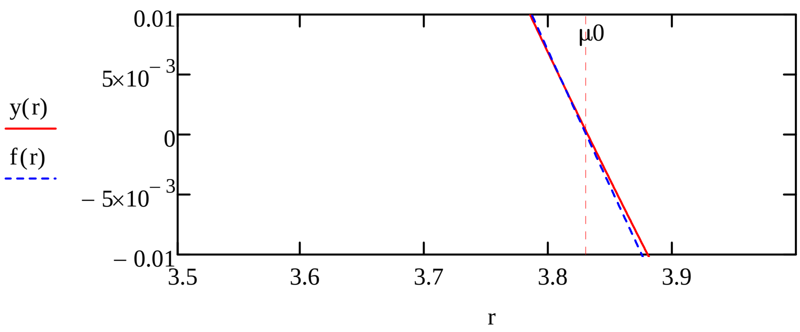
$$\tau_y := \frac{\varepsilon_y}{d\varepsilon y\_dt} = -222.843 \quad \mathrm{s}$$

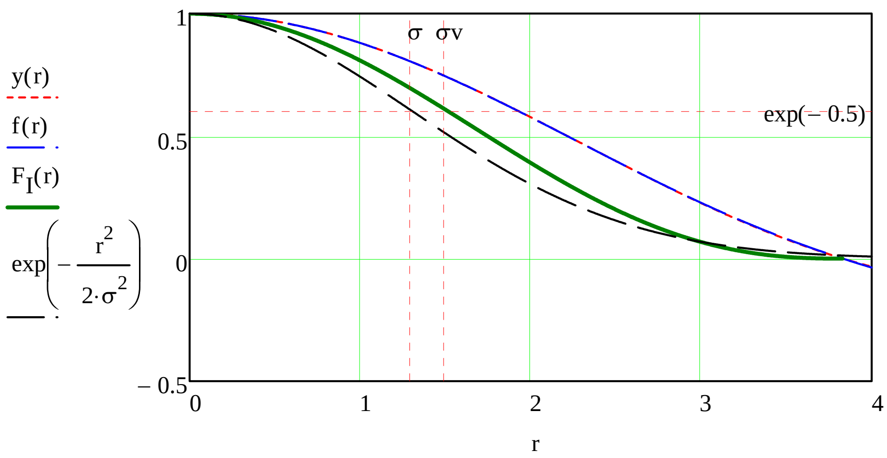
$$f(r) := 1 - \frac{r^2}{8} + \frac{r^4}{196} - \frac{r^6}{11950} \qquad \mu 0 := 3.8317 \qquad y(\mu 0) = -1.061 \times 10^{-5}$$

$$\tau_s := \frac{\sigma_p^{\,2}}{d\sigma p2\_dt} = 39.851 \quad \mathrm{s}$$

$$F_I(r) := \frac{1}{5.287} \cdot \int_0^{\sqrt{\mu 0^2 - r^2}} f\left(\sqrt{r^2 + x^2}\right) \cdot x^2 \, dx$$

$$\sigma := \sqrt{\frac{\int_0^{\mu 0} F_I(r) \cdot r^2 \, dr}{\int_0^{\mu 0} F_I(r) \, dr}} \qquad \sigma v := 1.5$$





$$\frac{\mu 0}{\sigma} = 2.96$$

$$\frac{\mu 0}{\sigma v} = 2.554$$

$$K \cdot \theta_x^{\,2} = 5.172 \times 10^{-6}$$