

Unfamiliar trajectories for a relativistic particle in a Kepler or Coulomb potential

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Relativistic particles in the Kepler and Coulomb potentials may have trajectories that are qualitatively different from the trajectories found in nonrelativistic mechanics. Spiral scattering trajectories were pointed out by C. G. Darwin in 1913 in connection with the relativistic Rutherford scattering of classical charged particles. Relativistic trajectories are of current interest in connection with Cole and Zou's computer simulation of the hydrogen ground state in classical physics. © 2004

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I. INTRODUCTION

The mechanics of a relativistic particle in a potential is usually treated as an afterthought in classical mechanics textbooks. These textbooks usually begin with an extensive treatment of nonrelativistic mechanics, the energy and momentum of relativistic particles are then noted, and the corrections to the orbits of nonrelativistic particles are sometimes mentioned.¹ However, relativistic particles in the familiar Kepler or Coulomb potential $V(r) = -\alpha/r$ can have trajectories that are qualitatively different from the trajectories found from nonrelativistic mechanics, and these unfamiliar trajectories are not mentioned in the textbooks.² For example, a relativistic particle in a $1/r$ potential can spiral into the force center (while conserving mechanical energy and angular momentum). This behavior occurs because a small increase in the velocity near the speed of light c can lead to a large increase in the mass $m/(1-v^2/c^2)^{1/2}$ so that an increase in the kinetic energy will compensate the decrease in the potential energy as the radius decreases, thus keeping the total energy constant. In addition, the increase in the linear momentum will compensate the decrease in the radius to keep the angular momentum constant. The existence of such trajectories has played a role in recent research.^{3,4}

The potential $V(r) = -\alpha/r$ ($\alpha > 0$) appears in the Kepler problem of gravitational physics and in the attraction of point charges in electrostatics. The theory of gravitation finds its relativistic form in general relativity and electrostatics has a natural extension into electrodynamics. However, we will not discuss these extended theories. The problem we consider has no curved spacetime and no radiation; it is the relativistic mechanics of a particle in a $1/r$ potential.

As a first example of the unfamiliar nature of some relativistic particle trajectories, we show that if the angular momentum is too small, there are no circular orbits for a relativistic particle in a $1/r$ potential. In contrast, in nonrelativistic mechanics, such a situation never occurs; there are always nonrelativistic circular orbits unless the angular momentum vanishes.

We follow the traditional procedure for the classification of trajectories for a particle, noting the orbits that undergo a qualitative transformation in the nonrelativistic limit and those that do not. The equations of the trajectories are then obtained. We find the familiar scattering trajectories and the familiar rosette shapes that appear in texts on old quantum

theory,⁵ and also C. G. Darwin's spiraling trajectories,² which do not appear in textbooks. Finally, we mention the research context in which these spiral trajectories have recently become of interest.

II. THE MECHANICAL PROBLEM

We compare the nonrelativistic and relativistic behavior of a particle in the potential $V(r) = -\alpha/r$. A particle in such a potential experiences a force $\mathbf{F} = -\nabla V(r) = -\hat{r}\alpha/r^2$. If we use the nonrelativistic momentum $m\mathbf{v}$ for a nonrelativistic particle, Newton's second law becomes

$$\frac{d}{dt}(m\mathbf{v}) = -\frac{\alpha}{r^2}\hat{r}. \quad (1)$$

For a relativistic particle, the momentum is $m\mathbf{v}/\sqrt{1-v^2/c^2}$, which gives

$$\frac{d}{dt}\left(\frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}}\right) = -\frac{\alpha}{r^2}\hat{r}. \quad (2)$$

If we take the dot product of Eqs. (1) and (2) with the velocity \mathbf{v} and integrate with respect to time, we obtain conservation of energy. The nonrelativistic energy \mathcal{E}_{nr} is expressed as

$$\mathcal{E}_{nr} = \frac{1}{2}mv^2 - \frac{\alpha}{r}, \quad (3)$$

while the relativistic energy E is

$$E = \mathcal{E} + mc^2 = \frac{mc^2}{\sqrt{1-v^2/c^2}} - \frac{\alpha}{r}. \quad (4)$$

Here, we have written $E = \mathcal{E} + mc^2$ so that \mathcal{E} is the energy difference from the particle rest energy. In addition, because the $1/r$ potential gives a central force, the angular momentum \mathbf{L} is conserved, giving in the nonrelativistic case

$$\mathbf{L}_{nr} = \mathbf{r} \times (m\mathbf{v}), \quad (5)$$

and in the relativistic case

$$\mathbf{L} = \mathbf{r} \times \left(\frac{m\mathbf{v}}{\sqrt{1-v^2/c^2}}\right). \quad (6)$$

We will use the conservation laws to determine the trajectories. As an introduction to the nature of the differences that

appear in relativistic mechanics, we first consider the case of circular orbits.

III. LIMITING ANGULAR MOMENTUM FOR RELATIVISTIC CIRCULAR ORBITS

For circular orbits, the particle displacement \mathbf{r} from the center of the potential is perpendicular to the velocity \mathbf{v} , and the angular momentum \mathbf{L} is perpendicular to the plane of the orbit. The magnitude of the angular momentum in the non-relativistic case is

$$L_{nr} = mrv, \quad (7)$$

and in the relativistic case

$$L = \frac{mrv}{\sqrt{1-v^2/c^2}}. \quad (8)$$

For circular orbits in the nonrelativistic case, we have

$$m \frac{v^2}{r} = \frac{\alpha}{r^2}, \quad (9)$$

and in the relativistic case

$$\frac{m}{\sqrt{1-v^2/c^2}} \frac{v^2}{r} = \frac{\alpha}{r^2}. \quad (10)$$

If we now use the angular momentum to remove either r or v , we find for the nonrelativistic case

$$v = \frac{\alpha}{L_{nr}}, \quad \text{or} \quad r = \frac{L_{nr}^2}{m\alpha}. \quad (11)$$

Because both v and r can take all values between zero and infinity in nonrelativistic mechanics, it follows that any value of the angular momentum L_{nr} can lead to a circular orbit.

The same procedure in the relativistic case leads to

$$v = \frac{\alpha}{L}, \quad (12a)$$

or

$$r = \frac{L^2}{m\alpha} \left[1 - \left(\frac{\alpha}{Lc} \right)^2 \right]^{1/2}. \quad (12b)$$

In the relativistic case, we have an upper limit for the speed $v=c$ and, accordingly, a lower limit on the magnitude of the angular momentum for a circular orbit. As seen in Eqs. (12), we must have⁶

$$L > \frac{\alpha}{c}. \quad (13)$$

Thus, there is a qualitative distinction for circular orbits between nonrelativistic and relativistic mechanics for the potential $V(r) = -\alpha/r$, and not only for this potential.⁷ The limiting value of the angular momentum in the relativistic case is $L_\alpha = \alpha/c$. In contrast, in nonrelativistic mechanics there is no limiting speed and thus no lower limit on the angular momentum for circular orbits.

The actual value of the limiting angular momentum L_α for the $1/r$ potential depends on the magnitude of α . For an electron of charge e in the field of a nucleus of atomic number Z , $\alpha = Ze^2$, and the limiting angular momentum can be written as $L_\alpha = Zm(e^2/mc^2)c = Z(e^2/\hbar c)\hbar$. Thus, for an

electron in the Coulomb potential of hydrogen, the limiting angular momentum $L_\alpha = (e^2/\hbar c)\hbar \approx (1/137)\hbar$. However, the reader should not be distracted by an angular momentum that is close to that given by Planck's constant. Our discussion involves only relativistic classical mechanics, not quantum mechanics.

IV. CLASSIFICATION OF TRAJECTORIES

The classification of orbits for the potential $V(r) = -\alpha/r$ reflects the fact that some relativistic trajectories are qualitatively different from those found in nonrelativistic mechanics. The classification can be made following the usual procedures of classical mechanics.¹ The motion is confined to a plane so that the velocity can be written in terms of polar coordinates as $\mathbf{r} = \hat{r}\dot{r} + \hat{\theta}r\dot{\theta}$. We then use the energy expression (4) and remove the $\dot{\theta}$ dependence in the velocity in favor of the angular momentum L to obtain a first-order differential equation in the radial variable r as a function of time. We will carry out the procedure for the relativistic case and then take the nonrelativistic limit to connect with the familiar Kepler orbits.

In terms of polar coordinates, the relativistic angular momentum is

$$L = \frac{mr^2\dot{\theta}}{\sqrt{1-(\dot{r}^2 + r^2\dot{\theta}^2)/c^2}}, \quad (14)$$

where we choose the orientation so that L and $\dot{\theta}$ are positive. If we solve Eq. (14) for $\dot{\theta}$ and substitute the result into Eq. (4), we find

$$E = \mathcal{E} + mc^2 = \frac{mc^2}{\sqrt{1-(\dot{r}^2/c^2) - L^2(1-\dot{r}^2/c^2)/(m^2r^2c^2 + L^2)}} - \frac{\alpha}{r}. \quad (15)$$

The solution of Eq. (15) for \dot{r}^2 is

$$\dot{r}^2 = c^2 \left[1 - \left(1 + \frac{L^2}{m^2r^2c^2} \right) \left(\frac{mc^2}{E + (\alpha/r)} \right)^2 \right]. \quad (16)$$

In the following, it will turn out that the limiting angular momentum, $L_\alpha = \alpha/c$, that appears in the analysis of circular orbits, represents a crucial limiting value. It is useful to have certain constraints on the energy which are associated with this limiting angular momentum. Because the particle velocity is less than c , we see from Eqs. (14) and (15) that

$$L = \frac{mr^2\dot{\theta}}{\sqrt{1-v^2/c^2}} < \frac{rmc}{\sqrt{1-v^2/c^2}} = \frac{r}{c} \left(E + \frac{\alpha}{r} \right). \quad (17)$$

Equation (17) requires that

$$L < \frac{r}{c} \left(E + \frac{\alpha}{r} \right) \quad \text{or} \quad L - \frac{\alpha}{c} < \frac{Er}{c}. \quad (18)$$

Because r is positive, it follows that if $L \geq \alpha/c$, then $E = \mathcal{E} + mc^2 > 0$. Thus orbits with angular momentum larger than L_α must have positive total (relativistic) energy. In particular, bound orbits with $L > L_\alpha$ (those that do not plunge into the potential center) cannot have arbitrarily small values of ki-

netic plus potential energy; rather the kinetic plus potential energy \mathcal{E} must be larger than $-mc^2$.

The classification of the relativistic orbits can be made by noting that the function on the right-hand side of Eq. (16) must satisfy the requirement $0 \leq \dot{r}^2 < c^2$. It is straightforward to transform this condition to

$$-L^2 c^2 < 0 \leq (E^2 - m^2 c^4) r^2 + 2E\alpha r + (\alpha^2 - L^2 c^2). \quad (19)$$

Because all quantities are real, the first inequality in Eq. (19) holds automatically and is not important. The second inequality can be treated by plotting the parabolic function $Y(r) = (E^2 - m^2 c^4) r^2 + 2E\alpha r + (\alpha^2 - L^2 c^2)$ versus r . The allowed orbits correspond to the regions of positive r where $Y(r) > 0$ and the turning points occur when $Y(r) = 0$:

$$r_{\text{turning-point}} = \frac{E\alpha \pm \sqrt{E^2 \alpha^2 + (m^2 c^4 - E^2)(\alpha^2 - L^2 c^2)}}{m^2 c^4 - E^2}. \quad (20)$$

In the nonrelativistic limit $c \rightarrow \infty$ with $E - mc^2 = \mathcal{E} \rightarrow \mathcal{E}_{nr}$, the inequality (19) becomes

$$-L_{nr}^2/m < 0 \leq \mathcal{E}_{nr} r^2 + \alpha r - L_{nr}^2/2m, \quad (21)$$

where the first inequality holds for any real L_{nr} . The parabolic function $y(r) = \mathcal{E}_{nr} r^2 + \alpha r - L_{nr}^2/2m$ can be plotted versus r with the allowed orbits corresponding to regions of $r > 0$ and $y(r) > 0$; the turning points occur when $y(r) = 0$:

$$r_{\text{turning-point}} = \frac{-\alpha}{2\mathcal{E}_{nr}} \left(1 \pm \sqrt{1 + \frac{2L_{nr}^2 \mathcal{E}_{nr}}{m\alpha^2}} \right). \quad (22)$$

A relativistic circular orbit corresponds to the square root in Eq. (20) equal to zero so that both the inner and outer turning points are at the same radius r . The vanishing square root gives a connection between E and L for a circular orbit, given by

$$E = mc^2 \sqrt{1 - \left(\frac{\alpha}{Lc} \right)^2}. \quad (23)$$

If we substitute Eq. (23) into Eq. (20), we obtain a radius r , which is the same result obtained in Eq. (12b). In addition, if we take the nonrelativistic limit of Eq. (23), we obtain

$$\begin{aligned} \mathcal{E} = E - mc^2 &= mc^2 \sqrt{1 - \left(\frac{\alpha}{Lc} \right)^2} - mc^2 \\ &\cong -\frac{1}{2} \frac{m\alpha^2}{L_{nr}^2} = \mathcal{E}_{nr}, \end{aligned} \quad (24)$$

which is the nonrelativistic expression for a circular orbit. We note that Eq. (23) gives a limit $E \geq 0$ for a circular orbit, which agrees with the result given just below Eq. (18). However, there is no lower bound for the energy in the nonrelativistic approximation appearing in Eq. (24) when we take the nonrelativistic limit $L_{nr} \rightarrow 0$ (and find $\mathcal{E}_{nr} \rightarrow -\infty$).

The general character of the turning points in the nonrelativistic equation (22) is controlled by the sign of the single parameter \mathcal{E}_{nr} , with $L_{nr} = 0$ a unique special case. In the full relativistic treatment, the character of the turning points in Eq. (20) depends on the signs of both $E\alpha/(m^2 c^4 - E^2)$ and $(m^2 c^4 - E^2)(\alpha^2 - L^2 c^2)$, where there is a sign change in the second term depending on the magnitude of L . When $L > L_\alpha$ so that $(\alpha^2 - L^2 c^2) < 0$, the turning point analysis is

qualitatively the same for both the relativistic and nonrelativistic cases. However, if the angular momentum is small but non-zero, $0 < L \leq \alpha/c$, so that $(\alpha^2 - L^2 c^2)$ is positive, the turning point analysis allows new possibilities that do not appear in the nonrelativistic case. We will see these new trajectories when we obtain the orbit equations.

V. ORBIT EQUATIONS

The orbit equations $r(\theta)$ can be found using the traditional analysis. If we write the relativistic particle momentum in the form

$$\mathbf{p} = \hat{r} p_r + \hat{\theta} p_\theta = \frac{m(\hat{r}\dot{r} + \hat{\theta}r\dot{\theta})}{\sqrt{1 - v^2/c^2}} = \frac{m\hat{r}\dot{r}}{\sqrt{1 - v^2/c^2}} + \hat{\theta} \frac{L}{r}, \quad (25)$$

the relativistic particle energy E can be written as

$$E = \sqrt{p^2 c^2 + m^2 c^4} - \frac{\alpha}{r}, \quad (26)$$

or

$$\left(E + \frac{\alpha}{r} \right)^2 = p^2 c^2 + m^2 c^4 = p_r^2 c^2 + \frac{L^2 c^2}{r^2} + m^2 c^4. \quad (27)$$

Now, from Eq. (25),

$$\frac{p_r}{p_\theta} = \frac{\dot{r}}{r\dot{\theta}} = \frac{1}{r} \frac{dr}{d\theta}, \quad (28)$$

and

$$p_r = \frac{L}{r^2} \frac{dr}{d\theta}. \quad (29)$$

Thus, Eq. (27) becomes

$$\left(E + \frac{\alpha}{r} \right)^2 = \left(\frac{L}{r^2} \frac{dr}{d\theta} \right)^2 c^2 + \frac{L^2 c^2}{r^2} + m^2 c^4. \quad (30)$$

If we introduce the usual r -inverse variable, $s = 1/r$, so that

$$\frac{ds}{d\theta} = -\frac{1}{r^2} \frac{dr}{d\theta}, \quad (31)$$

Eq. (30) becomes

$$(E + \alpha s)^2 = L^2 c^2 \left(\frac{ds}{d\theta} \right)^2 + s^2 L^2 c^2 + m^2 c^4. \quad (32)$$

Next we differentiate Eq. (32) with respect to θ and divide through by $ds/d\theta$ to obtain

$$\frac{d^2 s}{d\theta^2} + \left[1 - \left(\frac{\alpha}{Lc} \right)^2 \right] s - \frac{E\alpha}{L^2 c^2} = 0. \quad (33)$$

Equation (33) is a second-order linear differential equation and can be easily solved. We then substitute the solution into the first-order differential equation (32) to determine some of the integration constants.

The connection between position and time can be obtained from Eq. (16) for $\dot{r} = dr/dt$, as

$$t = \int_{r_0}^r \frac{dr}{c} \left(1 - \left[1 + \frac{L^2}{m^2 r^2 c^2} \right] \left[\frac{mc^2}{E + (\alpha/r)} \right]^2 \right)^{-1/2}. \quad (34)$$

Alternatively, the angular momentum equation can be combined with the orbital equation. Thus, Eqs. (14) and (15) can be combined to give

$$\frac{r^2 \dot{\theta}}{c^2} = \frac{L}{E + \alpha/r}, \quad (35)$$

so that from $\dot{\theta} = d\theta/dt$, we have

$$t = \int_{\theta_0}^{\theta} \frac{r^2 d\theta (E + \alpha/r)}{Lc^2}, \quad (36)$$

where $r(\theta)$ is to be evaluated from the orbital equations.

VI. SOLUTION OF THE ORBITAL EQUATIONS

We obtain different solutions of Eqs. (32) and (33) depending on the value of the angular momentum L . If $L > \alpha/c$, we have

$$s = \frac{1}{r} = \sqrt{\frac{E^2 L^2 c^2 - m^2 c^4 (L^2 c^2 - \alpha^2)}{(L^2 c^2 - \alpha^2)^2}} \times \cos \left[\sqrt{1 - \left(\frac{\alpha}{Lc} \right)^2} (\theta - \theta_0) \right] + \frac{E\alpha}{L^2 c^2 - \alpha^2}. \quad (37)$$

If $L = \alpha/c$, we have

$$s = \frac{1}{r} = \frac{1}{2} \left(\frac{E}{\alpha} \right) (\theta - \theta_0)^2 + \frac{m^2 c^4 - E^2}{2E\alpha}. \quad (38)$$

For $L < \alpha/c$, we have

$$s = \frac{1}{r} = \sqrt{\frac{m^2 c^4 (\alpha^2 - L^2 c^2) + E^2 L^2 c^2}{(\alpha^2 - L^2 c^2)^2}} \times \cosh \left[\sqrt{\left(\frac{\alpha}{Lc} \right)^2 - 1} (\theta - \theta_0) \right] - \frac{E\alpha}{\alpha^2 - L^2 c^2}. \quad (39)$$

Finally, if $L = 0$, the orbit is a straight line $\theta = \theta_0$.

In the nonrelativistic limit $1/c \rightarrow 0$, only the first of these solutions ($L > \alpha/c$) does not degenerate into a straight-line orbit. The relativistic orbits in Eq. (37) are the bound rosettes and unbounded scattering orbits that usually appear in textbooks.¹ Thus, if we set $E = \mathcal{E} + mc^2$ and take the limit $1/c \rightarrow 0$ in Eq. (37), we find

$$s = \frac{1}{r} = \frac{m\alpha}{L_{nr}^2} \sqrt{1 + \frac{2L_{nr}^2 \mathcal{E}_{nr}}{m\alpha^2}} \cos(\theta - \theta_0) + \frac{m\alpha}{L_{nr}^2}, \quad (40)$$

which is the standard nonrelativistic result for the hyperbolas, parabolas, and ellipses of Kepler orbits.

The various types of orbits are sketched in Figs. 1–5, showing bound and unbound relativistic trajectories. The unbound orbits, shown in Figs. 1–3, have a total relativistic energy at least as large as the particle rest energy, $E = \mathcal{E} + mc^2 \geq mc^2$. These orbits extend to spatial infinity in at least one of their time limits. For $L > \alpha/c$, these orbits are familiar scattering orbits as shown in Fig. 1, where the orbit looks like a parabolic or hyperbolic nonrelativistic orbit. However, for $L > \alpha/c$ but close to α/c , we are reminded that relativity changes the familiar nonrelativistic scattering orbits. In Fig. 2, we see that the scattering trajectory loops around the center of the potential before receding to infinity. These loops do not occur for the parabolic or hyperbolic orbits of nonrelativistic scattering. As L decreases and be-

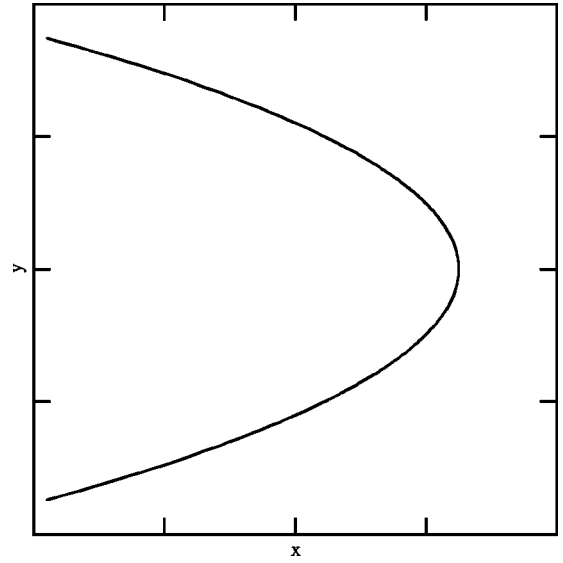


Fig. 1. Sketch of an unbound orbit of total energy $E = \mathcal{E} + mc^2 \geq mc^2$ and angular momentum $L \gg \alpha/c$ in a $1/r$ potential centered on the origin. The orbit is similar to a parabolic or hyperbolic nonrelativistic scattering orbit.

comes closer to α/c , the number of times the orbit loops around the potential center increases. When $L \leq \alpha/c$, the character of the trajectory changes. The orbital looping now continues all the way to the center of the potential; the unbound orbit extends to infinity in only one time direction and spirals into the potential center in the other (see Fig. 3). In contrast, the parabolic and hyperbolic orbits of nonrelativistic mechanics never reach the center of the potential, unless $L = 0$.

The bound orbits have total energy E smaller than the particle rest energy, $E = \mathcal{E} + mc^2 < mc^2$, corresponding to negative values of $\mathcal{E} = E - mc^2$. In nonrelativistic mechanics,

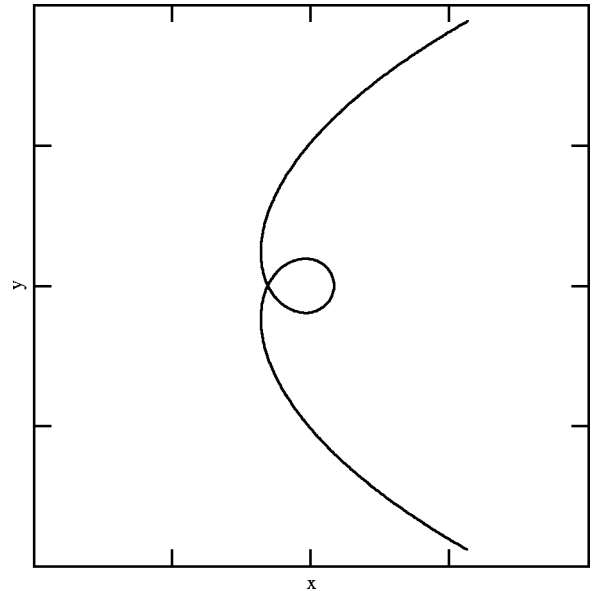


Fig. 2. Sketch of an unbound orbit of energy $E \geq mc^2$ and angular momentum L slightly larger than α/c in a $1/r$ potential. Because L is only slightly larger than α/c , the orbit makes loops around the scattering center. Such loops do not occur for an unbounded nonrelativistic orbit, which is always a parabola or hyperbola.

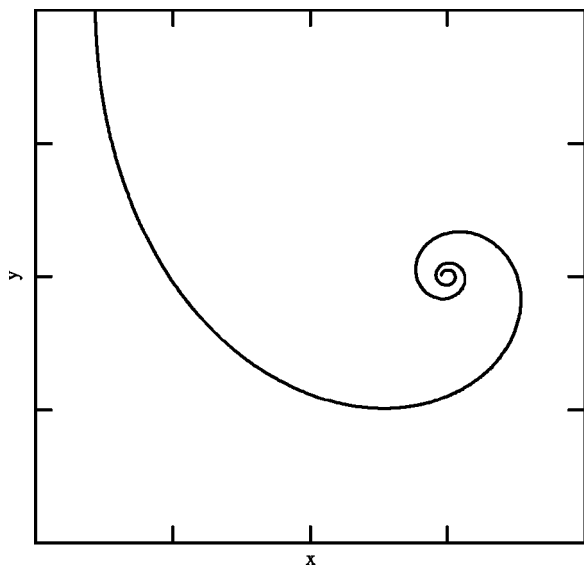


Fig. 3. Sketch of an unbound orbit of energy $E \geq mc^2$ and angular momentum $L \leq \alpha/c$. One end of the orbit is at infinity and the other corresponds to a spiral into or out of the center of the potential. The nonrelativistic limit corresponds to a straight line orbit with $L=0$.

all the bound orbits are ellipses unless $L_{nr}=0$. For $L > \alpha/c$, the relativistic orbits take the rosette shape of precessing ellipses shown in Fig. 4. These orbits have two turning points. The orbits are familiar as the Sommerfeld relativistic orbits of old quantum theory,⁵ and reduce to the familiar ellipses in the nonrelativistic limit. For $L \leq \alpha/c$, the relativistic orbits spiral out from the center of the $1/r$ potential and back into the center at early and late times, as shown in Fig. 5. In the nonrelativistic limit, these orbits become straight-line trajectories for $L_{nr}=0$.

From Eqs. (34) and (36), we find that all the trajectories involve finite time changes for finite changes in r or θ . For the trajectories with $L \leq \alpha/c$ (which spiral into the potential center), the particles arrive at the potential center in a finite

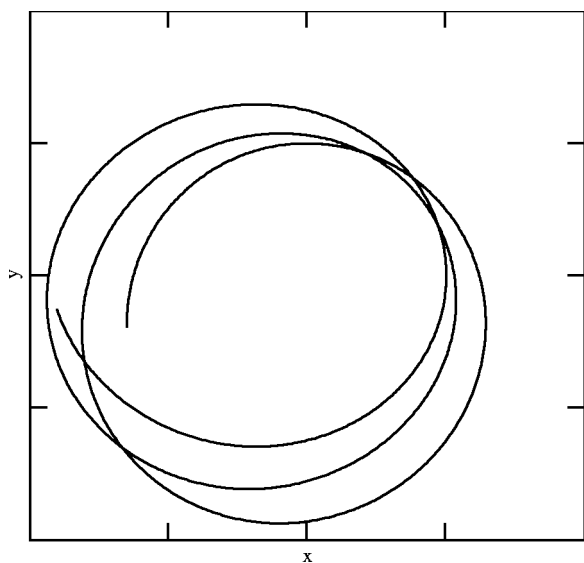


Fig. 4. Sketch of a bound orbit of energy $0 < E < mc^2$ and angular momentum $L > \alpha/c$. The orbit has two turning points and, in the nonrelativistic limit, reduces to the familiar elliptical orbit of nonrelativistic mechanics.

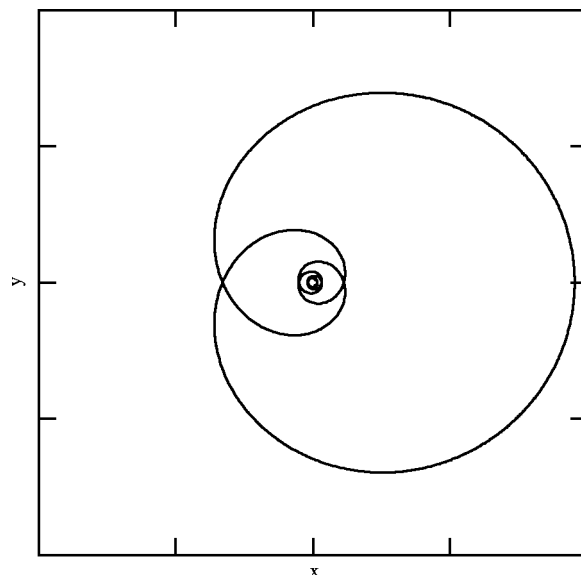


Fig. 5. Sketch of a bound orbit of energy $E < mc^2$ and angular momentum $L \leq \alpha/c$. The orbit ends in spirals to and from the potential center. The nonrelativistic limit is a straight-line orbit with $L=0$.

time starting from a finite radius. This result can be seen from Eq. (34), which connects the change in the radius r with the elapsed time. For $L = \alpha/c$, an expansion for small r gives an integral of the form

$$t = \int_0 dr \left(\frac{1}{2c^2} \sqrt{\frac{2\alpha}{Er}} + \mathcal{O}(\sqrt{r}) \right), \quad (41)$$

which is well behaved at the lower limit. If $L < \alpha/c$, an expansion for small r gives

$$t = \int_0 dr \left(\frac{1}{c^2 \sqrt{1 - L^2 c^2 / \alpha^2}} + \mathcal{O}(r) \right). \quad (42)$$

The square root is real because of the condition $L < \alpha/c$, and again the integral is finite. Thus, all the trajectories involve only finite times in the vicinity of the $1/r$ potential center.

VII. RELEVANCE OF THE TRAJECTORIES TO CURRENT RESEARCH

The existence of classical relativistic scattering trajectories that spiral into the center of the $1/r$ potential while preserving energy and angular momentum was pointed out by Darwin² in 1913. Darwin's analysis was made in connection with electron scattering based on Rutherford's model of the nuclear atom. In particular, Darwin was concerned with the spiral trajectories, and suggested that they should not occur in nature because they would cause chemical transmutation. He speculated that "There must therefore be some way by which the electron can escape from the extreme neighborhood of the nucleus."² He suggested that the presence of these unobserved spiral trajectories indicated a further failure of classical electromagnetic theory to describe experimental observation. Darwin's work was published just before Bohr proposed his stationary-state model of the atom.

Although Darwin was concerned primarily with the Rutherford scattering problem of relativistic mechanics, he estimated the radiative corrections for his scattering trajectories.

It is interesting that a relativistic treatment of the old classical problem of radiative atomic collapse gives a different result from the familiar nonrelativistic treatment. In both the nonrelativistic and relativistic analyses, the radiative loss of mechanical energy as a charged classical particle moves in a Coulomb potential is such as to make the orbit more circular. In the nonrelativistic treatment, the charged particle radiates an infinite amount of energy as it spirals into the center at ever higher speed and ever lower energy $\mathcal{E}_{nr} = -Ze^2/2r \rightarrow -\infty$. However, in the relativistic case, the radiative loss of energy ends when the particle speed in the circular orbit has reached the limit $v \rightarrow c$, the angular momentum in Eq. (13) has reached $L \rightarrow Ze^2/c$, the radius in Eq. (12) has shrunk to $r \rightarrow 0$, and the total energy in Eq. (23) has reached $E \rightarrow 0$. Thus, a charged particle of rest energy mc^2 that starts at a large radius and small velocity with total energy $E = mc^2$ will radiate away the energy mc^2 , and not the divergent radiation energy found in the nonrelativistic treatment.

Today, physicists are not concerned about Darwin's spiral trajectories nor the finite radiation energy loss given by a relativistic treatment of atomic collapse. Of course, most physicists are not aware of their existence, because quantum mechanics has changed our views of the atom and has made such awareness unnecessary. However, occasionally some researchers are interested to see just how far one can push a classical or semi-classical interpretation of atomic physics. It has been suggested⁸ that the inclusion of classical electromagnetic zero-point radiation might allow an understanding of electron motion as a sort of Brownian motion which avoids the problem of atomic collapse. Thus, an electron might indeed lose energy through radiation while accelerating in the electric field of the nucleus, but the electron might also pick up energy from the zero-point radiation. The balance between these two effects might account for the atomic structure.

Recently Cole and Zou³ have carried out simulations of the classical hydrogen ground state assuming classical zero-point radiation. They conclude that the electron neither falls into the nucleus nor is ionized, but rather assumes a stationary probability distribution. This ground state probability distribution approximates that given by the Schrödinger equation. This agreement is remarkable and requires confirmation and further understanding. However, in contrast to these computer simulation results are various attempts to calculate the electron Brownian motion within the basic zero-point model using nonrelativistic mechanics.⁹ These analytic calculations suggest that the electron would not fall into the nucleus due to radiation emission, but rather would be ejected from the atom (self-ionization) due to the absorption of excessive zero-point energy in the plunging orbits of very small angular momentum. However, as we have emphasized here, the nonrelativistic orbits of very small angular momenta represent the region of failure of the nonrelativistic approximation. For small angular momentum $L < \alpha/c$, the correct relativistic orbits spiral in toward the nucleus while conserving the total energy and angular momentum. This behavior will lead to a totally different interaction with zero-

point radiation from that assumed in the nonrelativistic calculations.⁹ Hence, the validity of the self-ionization conclusions in the analytic work are questionable.⁴

We see that nearly a century after Darwin's work, a knowledge of the orbits of a relativistic particle in a $1/r$ potential again seems significant. An understanding of the possibilities and limitations of a classical or semi-classical treatment of hydrogen depends upon a correct treatment of the mechanical trajectories of a relativistic particle in the Coulomb potential.

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¹See for example, H. Goldstein, C. Poole, and J. Safko, *Classical Mechanics* (Addison-Wesley, New York, 2002), 3rd ed., pp. 316–317; J. V. Jose and E. J. Saletan, *Classical Dynamics: A Contemporary Approach* (Cambridge, New York, 1998), pp. 211–212; E. Saletan and A. H. Cromer, *Theoretical Mechanics* (Wiley, New York, 1971); H. C. Corben and P. Stehle, *Classical Mechanics* (Dover, New York, 1994) (a re-publication of the 1960 edition). Goldstein, Poole, and Safko discuss the relativistic one-dimensional harmonic oscillator in Sec. 7.9. Jose and Saletan treat the relativistic Kepler problem in Sec. 5.1, but do not mention Darwin's spiral trajectories.

²Such orbits appear in the work of C. G. Darwin, "On some orbits of an electron," *Philos. Mag.* **25**, 201–210 (1913) in connection with the scattering of β particles in Rutherford's model for an atom. Darwin wrote just before the publication of Bohr's model for hydrogen.

³D. C. Cole and Y. Zou, "Quantum mechanical ground state of Hydrogen obtained from classical electrodynamics," *Phys. Lett. A* **317**, 14–20 (2003).

⁴T. H. Boyer, "Comments on Cole and Zou's calculation of the Hydrogen ground state in classical physics," *Found. Phys. Lett.* **16**, 607–611 (2003).

⁵A. E. Ruark and H. C. Urey, *Atoms, Molecules, and Quanta* (Dover, New York, 1965), p. 134.

⁶The existence of a lower limit for the angular momentum appears in the work of U. Torkelsson, "The special and general relativistic effects on orbits around point masses," *Eur. J. Phys.* **19**, 459–464 (1998). He remarks, "For relativistic motion in the Coulomb potential the centrifugal barrier disappears at a small, but still finite, specific angular momentum." He also notes that "there are no stable orbits" for small angular momentum, $L < \alpha/c$.

⁷J. M. Aguirregabiria (private communication) has pointed out that other potentials of the form $V(r) = kr^n$ (where k and n are constants) also involve interesting nonrelativistic and relativistic restrictions on angular momentum for circular orbits.

⁸See the review by L. de la Pena and A. M. Cetto, *The Quantum Dice - An Introduction to Stochastic Electrodynamics* (Kluwer Academic, Dordrecht, 1996). A brief introduction is given by T. H. Boyer, "Random electrodynamics: The theory of classical electrodynamics with classical electromagnetic zero-point radiation," *Phys. Rev. D* **11**, 790–808 (1975).

⁹T. Marshall and P. Claverie, "Stochastic electrodynamics of nonlinear systems. I. Particle in a central field of force," *J. Math. Phys.* **21**, 1918–1925 (1980); P. Claverie, L. Pesquera, and F. Soto, "Existence of a constant stationary solution for the hydrogen atom problem in stochastic electrodynamics," *Phys. Lett. A* **80**, 113–116 (1980); A. Denis, L. Pesquera, and P. Claverie, "Linear response of stochastic multiperiodic systems in stationary states with application to stochastic electrodynamics," *Physica A* **109**, 178–192 (1981); P. Claverie and F. Soto, "Nonrecurrence of the stochastic process for the hydrogen atom problem in stochastic electrodynamics," *J. Math. Phys.* **23**, 753–759 (1982).