Applications of the work of Munoz and Pavic [1] to accelerators are made.

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Abstract

The exact solutions in [1] for the special relativistic Coulomb problem are very attractive for accelerator simulation/tracking. The Hamilton (like) vector is zero on circular (often design) orbits. Horizontal and vertical motion is handled concurrently. It corresponds to a spherical electrode bending configuration. Different, actual, electrode cofigurations are handled with "kicks". This is pursued, with an emphasis on a proton electric dipole moment accelerator/experiment (EDM) in the planning stage.

Contents

1 Introduction

I'll take design orbits, at least arcs, as circular, and having R_0 as their radius. I'll take the arcs as having a bending electric field El_0. E tends to be reserved for energy. An ambiguity here is the choice of zero for potential energy. When the potential energy is set to zero on a design orbit, I'll designate energy as $\mathcal{E}$. Of course there are many other potential deviations/complications.

Taking a general form for the electric field as E(r) $\sim \frac{1}{r^{1+m}}$, spherical electrodes have m=1, cylindrical electrodes have m=0, and saddle shaped electrodes have m<0.

The current plans for the EDM experiment have R_0 = 40m, El_0 = 11MV/m, in a saddle shaped electrode configuration with m=-1.2.

Appendix A has information on exact values and their coding.

2 Symbols and their approximate values

Take m_p as beam particle rest mass (.94 GeV), e as beam particle charge (1.60x10^{-19} C), p_0 as beam ideal/design momentum (.70 GeV/c in the EDM), v_0 as beam ideal/design velocity (.60 in the EDM), γ_0 as beam ideal/design Lorentz factor (1.25 in the EDM). The speed of light, c, tends to be 1, but I try to keep it in the formulas.

Munoz and Pavic (and Boyer [2]) use

$$\frac{d\vec{p}}{dt} = -k\frac{\hat{r}}{r^2} \tag{1}$$

with

$$V(r) = -\frac{k}{r} \tag{2}$$

which is not zero on R_0. Thus their conserved particle/probe energy is

$$E = \gamma m_p c^2 - \frac{k}{r} \tag{3}$$

with a conserved angular momentum $\vec{L} = \vec{r} \times \vec{p}$. This can be seen from $\frac{d\vec{L}}{dt} = \frac{d(\vec{r}\times\vec{p})}{dt} = \frac{d\vec{r}}{dt} \times \vec{p} + \vec{r} \times \frac{d\vec{p}}{dt} = \frac{\vec{p}}{\gamma m_p} \times \vec{p} + \vec{r} \times (-\frac{k}{r^3}\vec{r}) = \vec{0} + \vec{0} = \vec{0}$. The orbit takes place in the plane orthogonal to $\vec{L}$ ($\vec{r} \cdot \vec{L} = \vec{r} \cdot (\vec{r} \times \vec{p}) = \vec{p} \cdot (\vec{r} \times \vec{r})$ = 0 or just from ($\vec{r}$ × anything) $\perp \vec{r}$). The same result holds for $\vec{p}$ ($\vec{p} \perp \vec{L}$). Taking polar coordinates in this plane, $\vec{r}$ = r $\hat{r}$, and $\vec{p} = \gamma m_p (\dot{r}\hat{r} + r\dot{\theta}\hat{\theta})$. Thus L = $\gamma m_p r^2 \dot{\theta}$ with its vector given by the normal to the plane. Malitsky et al [3] (page 34, Figure 5) call the direction of L, $\hat{\mathbf{n}}$, so that $\overrightarrow{L} = \gamma m_p r^2 \dot{\theta} \hat{\mathbf{n}}$. Of interest later, $\frac{d\hat{\theta}}{d\theta} = -\dot{r}$. Also V(r) $= -\frac{k}{r} + \frac{k}{R_0}$, so that

$$E = \mathcal{E} - \frac{k}{R_0} \tag{4}$$

(this is equation (61) in [3], where E $\equiv \mathcal{E}_{\mathcal{M}}$).

This orbit is not necessarily circular, but if it is, it can be viewed as corresponding to a design value

$$E_0 = \gamma_0 \, m_p \, c^2 - \frac{k}{R_0}. \tag{5}$$

I'll take

$$\frac{d\overrightarrow{p}}{dt} = -eEl_0 R_0^2 \frac{\hat{r}}{r^2} \tag{6}$$

so that

$$k = eEl_0 R_0^2 = p_0 v_0 R_0 = L_0 v_0. \tag{7}$$

The form used varies.

Equating the centripetal acceleration force to the electric force on the design arc gives

$$\frac{p_0 v_0}{R_0} = eEl_0. \tag{8}$$

Thus the design energy (5) can be written

$$E_0 = \gamma_0 \, m_p \, c^2 - p_0 v_0 \tag{9}$$

or

$$E_0 = \gamma_0 m_p c^2 - \gamma_0 m_p v_0^2 = \gamma_0 \, m_p c^2 (1 - \beta_0^2) = \frac{m_p c^2}{\gamma_0}. \tag{10}$$

For the EDM experiment,

$$E_0 = \frac{.94}{1.25} = .75 \, GeV. \tag{11}$$

Design $L_0 = R_0 p_0 = 28 \ \frac{GeV m}{c}$ so that k=28(.6)=16.8.

A given probe will deviate from design. For energy, and angular momentum, this can be written

$$E = E_0(1 + \delta_E) \tag{12}$$

$$L = L_0(1 + \delta_L) \tag{13}$$

An important factor in [1] is $\frac{k}{Lc}$, which can be written

$$\frac{k}{Lc} = \frac{L_0 v_0}{L_0(1 + \delta_L)c} = \frac{\beta_0}{1 + \delta_L} \tag{14}$$

This enters eq. (11) of [1] as

$$\kappa^2 \;=\; 1-(\frac{k}{Lc})^2 \;=\; 1-\frac{\beta_0^2}{(1+\delta_L)^2} \;\approx\; 1-\beta_0^2(1-2\delta_L) \;=\; 1-\beta_0^2+2\beta_0^2\delta_L \;=\; \frac{1}{\gamma_0^2}+2\beta_0^2\delta_L \tag{15}$$

Another important factor in [1] is $\frac{m_pc^2}{E}$, which can be written

$$\frac{m_pc^2}{E} \;=\; \frac{m_pc^2}{E_0(1+\delta_E)} \;=\; \frac{\gamma_0}{1+\delta_E} \tag{16}$$

it gets squared and multiplied by κ^2 to yield approximately

$$\frac{m_pc^2}{E}\frac{m_pc^2}{E}\kappa^2 \;\approx\; \frac{\gamma_0^2}{(1+\delta_E)^2}\frac{1+2\beta_0^2\gamma_0^2\delta_L}{\gamma_0^2} \;\approx\; (1-2\delta_E)(1+2\gamma_0^2\beta_0^2\delta_L) \;\approx\; 1+2\gamma_0^2\beta_0^2\delta_L-2\delta_E-4\gamma_0^2\beta_0^2\delta_L\delta_E \tag{17}$$

The cross term is needed because the first order terms approximately cancel. Eq. (35) of [1] for the eccentricity becomes

$$\epsilon \;=\; \frac{1+\delta_L}{\beta_0}\sqrt{1-(\frac{m_pc^2}{E})^2\kappa^2} \;\approx\; \frac{1+\delta_L}{\beta_0}\sqrt{-2\gamma_0^2\beta_0^2\delta_L+2\delta_E+4\gamma_0^2\beta_0^2\delta_L\delta_E} \tag{18}$$

which is stated to be real. The 2 digit approximations for the best values break down here. Appendix B has some accurate calculations for this.

It is shown in eq. (23) in Boyer[2] that for circular orbits

$$E \;=\; m_pc^2\sqrt{1-(\frac{k}{Lc})^2} \;=\; m_pc^2\sqrt{1-(\frac{L_0v_0}{Lc})^2} \tag{19}$$

so that

$$\frac{dE}{dL} \;=\; m_pc^2\frac{1}{2}(1-(\frac{L_0v_0}{Lc})^2)^{-1/2}(-1)(\frac{L_0v_0}{c})^2(-2)L^{-3} \;=\; m_p\frac{m_pc^2}{E}L_0^2v_0^2L^{-3}. \tag{20}$$

Evaluated at a given circular design orbit this gives

$$\left.\frac{dE}{dL}\right|_0 \;=\; \frac{m_p^2c^2v_0^2}{E_0L_0} \;=\; \frac{\gamma_0m_pv_0^2}{R_0\gamma_0m_pv_0} \;=\; \frac{v_0}{R_0} \tag{21}$$

to stay in a circular orbit.

Malitsky, et al [3] set potential energy on a design orbit as zero so that

$$V(r) \;=\; -eEl_0R_0^2(\frac{1}{r}-\frac{1}{R_0}) \tag{22}$$

and

$$\mathcal{E}_0 \;=\; E_0 \;+\; eEl_0R_0. \tag{23}$$

(19) can be solved for L as

$$L^2 \;=\; \frac{k^2 m_p^2 c^2}{m_p^2 c^4 - E^2}. \tag{24}$$

Taking the orientation that L and $\dot\theta$ are both positive ([2] eq. 14),

$$L \;=\; \frac{k m_p c}{\sqrt{m_p^2 c^4 - E^2}} \;=\; \frac{L_0 v_0 m_p c}{\sqrt{.94^2 - E^2}} \;=\; \frac{L_0 \beta_0 m_p c^2}{\sqrt{.88 - E^2}} \;=\; \frac{28(.6)(.94)}{\sqrt{.88 - E^2}} \;=\; \frac{15.8}{\sqrt{.88 - E^2}} \tag{25}$$

and this is plotted in figure 1.

From

$$E_0 \;=\; \gamma_0 m_p c^2 - \frac{k}{R_0} \tag{26}$$

$$\frac{k}{R_0} \;=\; \gamma_0 m_p c^2 - E_0 \;=\; \frac{m_p^2}{E_0} - E_0 \;=\; \frac{m_p^2 - E_0^2}{E_0} \tag{27}$$

$$R_0 \;=\; \frac{k E_0}{m_p^2 - E_0^2} \tag{28}$$

which is a nice form.

(18) shows that ϵ is real for the $\delta_E = 0$ vertical change ($\delta_L < 0$) illustrated for (E_0, L_0) in figure 1. [1] states that this is always true (eccentricity, ϵ, is real). Their proof is available.

Of course for circular orbits, ϵ=0, and for the probe orbits envisioned here, ϵ is very close to 0. Appendix B shows that they are rarely, if ever greater than 1E-03 for the physical situation emphasized here.

3 Hamilton Like Vector: Theory

The equation of motion (6) is valid relativistically ($\vec{p} \;=\; \gamma m_p \vec{v}$) and can be recast. Letting $\vec{u} \;=\; \gamma \vec{v}$ gives

$$\frac{k}{r^2}\frac{d\hat\theta}{d\theta} \;=\; -\frac{k}{r^2}\hat r \;=\; \frac{d\vec{p}}{dt} \;=\; \frac{d(m_p \vec{u})}{dt} \;=\; m_p \frac{d\vec{u}}{dt} \;=\; m_p \frac{d\vec{u}}{d\theta}\dot\theta \;=\; \frac{d\vec{u}}{d\theta}\frac{L}{\gamma r^2} \tag{29}$$

and

$$\frac{d\vec{u}}{d\theta} \;=\; \frac{k\gamma}{L}\frac{d\hat\theta}{d\theta}. \tag{30}$$

[1] defines a "Hamilton-like vector"

$$h_r \hat r \;+\; h_\theta \hat\theta \;=\; \vec{h} \;=\; \vec{u} \;-\; \frac{k\gamma}{L}\hat\theta \tag{31}$$

which has explicit functional dependence

$$h_r(\theta)\hat{r}(\theta) \; + \; h_\theta(\theta)\hat{\theta}(\theta) \; = \; \overrightarrow{h}(\theta) \; = \; \overrightarrow{u}(\theta) \; - \; \frac{k\gamma(\theta)}{L}\hat{\theta}(\theta). \tag{32}$$

This leads to

$$\frac{d\overrightarrow{h}}{d\theta} \; = \; -\frac{k}{L}\frac{d\gamma}{d\theta}\hat{\theta} \tag{33}$$

and the observation that for non relativistic phenomena (γ approximately 1), $\overrightarrow{h}$ is conserved.

Writing out the components of (33) explicitly gives

$$h_r' \; - \; h_\theta \; = \; 0 \tag{34}$$

and

$$h_r \; + \; h_\theta' \; = \; -\frac{k}{L}\frac{d\gamma}{d\theta}. \tag{35}$$

Using $L = \gamma m_p r^2 \dot{\theta} = m_p r u_\theta$ and (3) gives

$$\gamma \; = \; \frac{E + \frac{k}{r}}{m_p c^2} \; = \; \frac{E}{m_p c^2} \; + \; \frac{k u_\theta}{L c^2} \; = \; \frac{E}{m_p c^2} \; + \; \frac{k}{L c^2}(h_\theta + \frac{k\gamma}{L}), \tag{36}$$

$$\gamma(1 - (\frac{k}{Lc})^2) \; = \; \frac{E}{m_p c^2} \; + \; \frac{k}{L c^2}h_\theta \; = \; \frac{EL + k m_p h_\theta}{L m_p c^2}, \tag{37}$$

$$\gamma = \; \frac{EL + k m_p h_\theta}{L m_p c^2 \kappa^2}, \tag{38}$$

and

$$\gamma' \; = \; \frac{k m_p h_\theta'}{L m_p c^2 \kappa^2} \; = \; \frac{k h_\theta'}{L c^2 \kappa^2}. \tag{39}$$

This can be substituted into (35) to give

$$h_r \; + h_\theta' \; = \; -\frac{k}{L}\frac{k h_\theta'}{L c^2 \kappa^2}. \tag{40}$$

ie

$$h_\theta'(1 + \frac{k^2}{L^2 c^2 \kappa^2}) \; = \; -h_r \tag{41}$$

or

$$h_\theta'(\frac{\kappa^2 + 1 - \kappa^2}{\kappa^2}) \; = \; -h_r \tag{42}$$

so that

$$h_\theta' \; = \; -\kappa^2 h_r \tag{43}$$

The scalar θ part can be written (using L $= \gamma m_p r^2 \hat{\theta}$)

$$h_\theta(\theta) \; = \; \gamma v_\theta(\theta) \; - \; \frac{k\gamma}{L} \; = \; \gamma r\dot{\theta} \; - \; \frac{k}{L}\frac{E + k/r}{m_p c^2} \; = \; \frac{L}{m_p r} \; - \; \frac{kE}{m_p c^2 L} - \frac{k^2}{m_p c^2 L r} \tag{44}$$

or

$$\frac{1}{r}(\frac{L}{m_p}-\frac{k^2}{m_pc^2L}) = \frac{1}{r}\frac{L}{m_p}(1-(\frac{k}{Lc})^2) \equiv \frac{1}{r}\frac{L}{m_p}(\kappa^2) = h_\theta+\frac{kE}{m_pc^2L} = \frac{kE}{m_pc^2L}(1+\frac{m_pc^2L}{kE}h_\theta). \tag{45}$$

Thus

$$\lambda = \frac{L^2\kappa^2c^2}{kE}. \tag{46}$$

gives

$$r = \frac{\lambda}{1+\frac{m_pc^2L}{kE}h_\theta} \tag{47}$$

Possibly this can be called the canonical prcessing ellipse equation. At any rate, it's a nice form. The ubiquity of the $\kappa^2 = 1-(\frac{k}{Lc})^2$ factor seems worth observing.

4 Hamilton Like Vector: Accelerator Tracking

Equation (44) evaluated at $\theta = 0$ (bend entrance) is of particular interest for accelerator tracking (see [3] page 31). Thus

$$h_0 \equiv h_\theta(0) = \frac{L}{m_pr_0} - \frac{kE}{Lm_pc^2}-\frac{k^2}{Lm_pc^2r_0} = \frac{L}{m_pr_0} - \frac{k(\mathcal{E}-\frac{k}{R_0})}{Lm_pc^2}-\frac{k^2}{Lm_pc^2r_0} \tag{48}$$

$$= \frac{L}{m_pr_0} - \frac{k\mathcal{E}}{Lm_pc^2}+\frac{k^2}{Lm_pc^2R_0}-\frac{k^2}{Lm_pc^2r_0} \tag{49}$$

using (4). Also, from (43)

$$h_0' \equiv h_\theta'(0) = -\kappa^2h_r = -\kappa^2u_r = -\kappa^2\gamma v_r = -\kappa^2\gamma\dot{r} = -\kappa^2\gamma\frac{dr}{d\theta}\frac{d\theta}{dt} \tag{50}$$

$$= -\kappa^2\gamma\frac{dr}{d\theta}\frac{L}{\gamma m_pr^2} = -\kappa^2\frac{L}{m_pr^2}R_0\frac{dr}{R_0d\theta} = -\kappa^2\frac{L}{m_pr^2}R_0\frac{dr}{ds} \tag{51}$$

Accelerator tracking can be viewed as a succession of initial value problems. Here, the initial values are $\theta = 0$ and $r = r_0$. r_0 has been tracked to this point, and is known. It is continuous across any boundaries. Equations to model the probe values (Appendix B) at the next "step" (design beam arc length delta, ds) are developed based on these initial/input values. These values then become the initial values at that point ($\theta = 0$ and $r =$ the new tracked r).

In the UAL/ETEAPOT code, r_0 appears (rInput) in

$UAL/examples/Spherical/newDipoleAlgorithm.icc (52)

on line 90

double rInput = p[0]+R0; (53)

This approximation, r = x + R_0, allows (51) to become

$$h_0' = -\kappa^2 \frac{L}{m_p r^2} R_0 \frac{dr}{ds} \;=\; -\kappa^2 \frac{L}{m_p r^2} R_0 \frac{dx}{ds} \;=\; -\kappa^2 \frac{L}{m_p r^2} R_0 p[1] \qquad (54)$$

$\frac{dx}{ds}$ = p[1] has also been tracked to this point.

Code for this is in

$UAL/examples/Spherical/newDipoleAlgorithm.hh (55)

```
double h0(double r0){
    double value = L/mass/r0-k*gamma/L;
    return value;
}

double ht(double r){                       // h theta
    double fac   = k/L/mass/c/c;
    double value = L/mass/r-fac*(E+k/r);
    return value;
}

double htp(const Coordinates p,double Rsxf,double r){                    /
    double drdtheta = Rsxf*p[1];
    double value = -(L/mass/r/r)*drdtheta+(k*k/L/mass/c/c/r/r)*drdtheta;
    return value;
}

double htp2(const Coordinates p,double Rsxf,double r){                   /
    double drdtheta = Rsxf*p[1];
    double value = -kappa*kappa*(L/mass/r/r)*drdtheta;
    return value;
}
```

5 Electric Field Varying as $1/r^2$

5.1 Solution of Equation of Motion

Throughout much of this section formulas of Muñoz and Pavic[1] will be transcribed unchanged, except for bringing symbols into consistency with the rest of this report. Their formulation, though consistent with various other formalisms describing relativistic Coulomb orbits, is especially appropriate for our relativistic application. They show that the "generalized"-Hamilton vector

$$\mathbf{h} = h_r\,\hat{\mathbf{r}} + h_\theta\,\hat{\boldsymbol{\theta}} \tag{56}$$

is especially powerful in describing 2D, relativistic Kepler orbits. (Our 3D application can be formulated in such a way as to use only 2D orbits.) In the non-relativistic regime $\mathbf{h}$ is related to the (nonrelativistically conserved) Laplace-Runge-Lenz vector. As generalized by Muñoz and Pavic, though not quite conserved in our specialized relativistic regime, $\mathbf{h}$ satisfies a very simple equation of motion, whose exact solutions are sinusoids. In the accelerator context h_0 is linearly related to the relativistic factor γ^I. (As elsewhere, the superscript "I" serves as a reminder that it refers to "inside" bend elements.) This permits other orbit quantities to be expressed analytically. For example, the orbit period $T_{\rm rev.}$, so important for spin coherence analysis, can be expressed in terms of integrals that can be evaluated in closed form, when expressed in terms of h_θ.

In the code,

$UAL/examples/Spherical/newDipoleAlgorithm.icc,

the symbol for the current probe gamma immediately after entering a bend is

97 double gafti = gbefi-eVafti/m0;

This, and the other, various quantities then become inputs themselves for the (split) propagation. Thus the code symbol for the current probe gamma immediately after entering a bend is

$UAL/examples/Spherical/reference.inline

22 gamma = gafti;

The code symbols for the fixed split bend entrance values tend to be suffixed with In

25 double rIn = get_rFromProbe(x,y,z);

It turns out that the probe orbits of interest here are precessing ellipses, and there is also a

$UAL/examples/Spherical/newDipoleAlgorithm.hh

152 double get_rFromEllipse(double theta)

method. These appear to agree on rIn to 13 digits. The code symbols for the explicitly calculated/propagated split bend exit values tend to be ("near") suffixed with Out

$UAL/examples/Spherical/rotate.insert

4 double rOutFromEllipse = get_rFromEllipse(splitTheta);

...

68 double xOut = rOut*rOutHatx;

Though **h** is not conserved in general, Muñoz and Pavic show that **h** *is conserved if and only if the orbit is circular.* For the proton EDM lattice the design orbit is circular within bends. Also, neglecting the effects of the very weak, vertically focusing quads, off-momentum closed orbits are also circles. Thus $\overrightarrow{h}$ is very close to conserved for characteristic/realistic probes.

The Lorentz force equation is

$$\frac{d\mathbf{p}}{dt} = -k\,\frac{\hat{\mathbf{r}}}{r^2}, \tag{57}$$

where k is the same as the customary MKS notation for $1/(4\pi\epsilon_0)$ except for implicitly containing also the charge factor which, in our case, is e. The angular momentum is

$$\mathbf{L} = \mathbf{r} \times \mathbf{p}. \tag{58}$$

One shows easily that both the total energy

$$\mathcal{E}_{\mathcal{M}} = \gamma^I mc^2 - \frac{k}{r} = \mathcal{E} - \frac{k}{r_0} \tag{59}$$

and

$$L = \gamma^I m_p r^2 \dot{\theta}, \tag{60}$$

are constants of the motion. It has been necessary to distinguish between the Muñoz potential energy V_M and our potential energy eV, because our potential vanishes on the design orbit, while his vanishes at infinity.

In the code, $\mathcal{E}_M$ appears as E in ie

$UAL/examples/Spherical/newDipoleAlgorithm.hh

93 double E;

,

$UAL/examples/Spherical/reference.inline

26 E = gamma*m0*c*c-k/rIn;

...

Equation (60) is also a vector conservation statement. The orbit through the split bend exit takes place in the fixed plane normal to **L_In** = **rIn** $\times$ **pIn**.

Using the relations

$$\hat{\theta}' = -\hat{\mathbf{r}}, \quad \text{and} \quad \hat{\mathbf{r}}' = \hat{\theta}, \tag{61}$$

where primes stand for $d/d\theta$, the Lorentz equation can be re-expressed as

$$\frac{d\mathbf{p}}{dt} = \frac{k}{r^2}\frac{d\hat{\theta}}{d\theta}. \tag{62}$$

Defining (covariant) velocity $\mathbf{u} = \gamma^I \mathbf{v}$ (the spatial part of the relativistic 4-velocity), its equation of motion is

$$\frac{d\mathbf{u}}{dt} = \frac{k\gamma^I}{L}\frac{d\hat{\theta}}{d\theta}. \tag{63}$$

Muñoz and Pavic then introduce the generalized Hamilton vector

$$\mathbf{h} = \mathbf{u} - \frac{k}{L\gamma^I}\,\hat{\theta}, \tag{64}$$

specially defined so that the differential equation for $\mathbf{h}$ reduces to

$$\frac{d\mathbf{h}}{d\theta} = -\frac{k}{L}\frac{d\gamma^I}{d\theta}\,\hat{\theta}. \tag{65}$$

Using

$$u_\theta = \frac{L}{m_p r}, \quad \text{and} \quad h_\theta = \frac{L}{m_p r} - \frac{k}{L}\frac{\mathcal{E}_M + k/r}{m_p c^2}, \tag{66}$$

and introducing the parameter

$$\kappa^2 = 1 - \left(\frac{k}{Lc}\right)^2, \tag{67}$$

the orbit equation can be expressed in terms of h_θ;

$$r = \frac{\frac{L}{m_p} - \frac{k^2}{Lm_pc^2}}{h_\theta + \frac{k\mathcal{E}_M}{Lm_pc^2}} \equiv \frac{\lambda}{1 + \bar{\epsilon} h_\theta(\theta)}, \tag{68}$$

where

$$\lambda = \frac{L^2c^2\kappa^2}{k\mathcal{E}_M} \quad \text{and} \quad \bar{\epsilon} = \frac{Lm_pc^2}{k\mathcal{E}_M}. \tag{69}$$

This form is possibly reminiscent of ellipse equations, $r = \frac{\lambda}{1+\epsilon cos(\kappa(\theta-\theta_0))}$, (precessing and/or otherwise). This is the basis of the method

$UAL/examples/Spherical/newDipoleAlgorithm.hh

152 double get_rFromEllipse(double theta)

The actual code is

```
        double get_rFromEllipse(double theta){
        double fac = L*mass*c*c/k/E;
        return lambda/( 1+fac*C*cos( kappa*(theta-theta0) ) );
//      return lambda/(1+epsilon*cos(kappa*theta));
    }
```

showing that the probe orbits are skew, precessing ellipses. In component form Eq. (65) is

$$h_r' - h_\theta = 0, \quad h_\theta' + h_r = -\frac{k}{L}\,\gamma^{I\prime}. \tag{70}$$

After some algebra, one can express γ^I in terms of h_0;

$$\gamma^I = \frac{\mathcal{E}_M}{\kappa^2 m_p c^2} + \frac{k}{\kappa^2 L c^2}\,h_\theta. \tag{71}$$

Needed for Eqs. (70),

$$\gamma^{I\prime} = \frac{k}{\kappa^2 L c^2}\,h_\theta'. \tag{72}$$

(After several lines of algebra) Eqs. (70) reduce to

$$\boxed{\begin{aligned} h_r' &= h_\theta, \\ h_\theta' &= -\kappa^2\,h_r. \end{aligned}} \tag{73}$$

These are the equations that justify having introduced the generalized Hamilton vector. Their general solution can be written as

$$\begin{aligned} h_\theta &= \mathcal{C}\,\cos\kappa(\theta-\theta_0) \\ h_r &= \frac{\mathcal{C}}{\kappa}\,\sin\kappa(\theta-\theta_0). \end{aligned} \tag{74}$$

where $\mathcal{C}$ and θ_0 are to be chosen to match the known initial conditions. When this done, the components of **h** are available as

$UAL/examples/Spherical/newDipoleAlgorithm.hh

127 double hr(double theta)

132 double _ht(double theta)

These are the functions to be used for smoothly fitting the true, particle by particle, orbits over short intervals. (As explained elsewhere, the effects of the small deviations of the actual electric field from the Coulomb field are to be corrected for by delta function kicks between adjacent bends.) Concentrating on h_θ, and setting $\theta = 0$ at the entrance to a bend element, we have

$$h_0 = \mathcal{C}\cos\kappa\theta_0, \quad h_0' = \mathcal{C}\kappa\sin\kappa\theta_0. \tag{75}$$

Our radial coordinate $x = r - r_0$ is given by

$$x = \frac{\lambda}{1 + \bar{\epsilon}\, h_\theta} - r_0. \tag{76}$$

This form is particularly inconvenient for computational purposes—it vanishes on the design orbit in spite of being the difference of large numbers. To avoid this complication one can define $\Delta\mathcal{E}$ and $\Delta\kappa$ by

$$\mathcal{E}_M = \mathcal{E}_{M,0} + \Delta\mathcal{E}, \quad \kappa = \kappa_0 + \Delta\kappa, \tag{77}$$

and substitute into Eq.(76). While still being exact, the leading terms then cancel explicitly.

The initial conditions of h_θ need to be determined from the known proton coordinates just as it enters the bend region. From its defining Eq. (64) (and remembering that $\mathcal{E}$ differs from $\mathcal{E}_M$ by an additive constant),

$$h_0 = u_0 - \frac{k\gamma^I}{L} \tag{78}$$

This method appears in the code as

```
double _h0(double r0){
    double value = L/mass/r0-k*gamma/L;
    return value;
}
```

which contains only known factors, all of which are continuous across the boundary. In fact r is the only factor which is not a constant of the motion. We therefore have

$$h'_0 = \frac{dh}{d\theta}\Big|_{\theta=0} = -\kappa^2 h_r(\theta=0) = -\kappa^2 u_r(\theta=0) = -\kappa^2\gamma\dot{r}$$
$$= -\kappa^2\gamma\frac{dr}{d\theta}\frac{d\theta}{dt} = -\kappa^2\frac{L}{m_p r^2}\,\frac{dr}{d\theta}\Big|_{\theta=0} = -\kappa^2\frac{L}{m_p r^2}\,r_0\,\frac{dx}{ds}\Big|_{\theta=0}. \tag{79}$$

The 3rd step has used (73). This method appears in the code as

```
double htp2(const Coordinates p,double Rsxf,double r){                          /
    double drdtheta = Rsxf*p[1];
    double value    = -kappa*kappa*(L/mass/r/r)*drdtheta;
    return value;
}
```

since $\frac{dx}{ds}\Big|_{\theta=0} = \mathrm{p}[1]$. htp can be be translated as "h theta prime". htp2 is 1 of 2 equivalent methods available for checking purposes. For normal incidence $x' \equiv dx/ds$ is continuous across the boundary. As a consistency check, the ellipse eccentricity parameter ϵ in $r = \frac{\lambda}{1+\epsilon cos(\kappa\theta)}$ in [1] (Eq. 33), should be related to h_0 by

$$\epsilon = \frac{m_p c^2}{\mathcal{E}_M}\,\frac{L}{k}\,h_0. \tag{80}$$

Also $\mathcal{C}$ and θ_0 in Eqs. (74) can be obtained by matching initial conditions.

$$\theta_0 = \frac{1}{\kappa}\,\tan^{-1}\left(\frac{1}{\kappa}\,\frac{h_0'}{h_0}\right),$$
$$\mathcal{C}^2 = h_0^2 + \frac{{h_0'}^2}{\kappa^2}. \tag{81}$$

These equations are available in the code.

In the context of accelerator physics one can ask for the similar evolution equations for vertical coordinate y. But this question is inappropriate; the Kepler orbit lies, by definition, in a single plane and we always choose the propagation plane to be the plane containing the incident velocity vector. Instead of keeping track of y we have to keep track of the orbit plane or, equivalently, the normal to the orbit plane. This vector will change discontinuously (but only by a tiny angle) in passing through thin multipole elements.

More convenient than $\overrightarrow{L}$ for representing the fixed plane of orbit is $\hat{\mathbf{n}}$

It appears in the code as

$UAL/examples/Spherical/rotate.insert

```
double nx = -Lx/L;      // normal vector for rotation
double ny = -Ly/L;      //
double nz = -Lz/L;      // basically aligned with y axes
```

This vector can be used to apply equation (4.5.27) on page 159 of [5] ($\hat{\mathbf{I}} -> \hat{\mathbf{n}}$) to obtain a unit vector, $\widehat{\boldsymbol{rOut}}$, in the direction of $\overrightarrow{\mathbf{rOut}}$.

$$\widehat{\boldsymbol{rOut}} = \cos(\phi)\widehat{\boldsymbol{rIn}} + \sin(\phi)\widehat{\boldsymbol{\theta In}} \tag{82}$$

Here, $\widehat{\boldsymbol{\theta In}} = \widehat{\boldsymbol{rIn}} \times \hat{\mathbf{n}}$. This appears in the code as

```
double rInCross_nx = rInHaty*nz-rInHatz*ny;
double rInCross_ny = rInHatz*nx-rInHatx*nz;
double rInCross_nz = rInHatx*ny-rInHaty*nx;
```

leading to

```
double co = cos(phi);  // cos(th);
double si = sin(phi);  // sin(th);
double rOutHatx = co*rInHatx+si*rInCross_nx;
double rOutHaty = co*rInHaty+si*rInCross_ny;
double rOutHatz = co*rInHatz+si*rInCross_nz;
```

The particular angle, ϕ, for each individual probe, is a little different than the design/ideal orbit angle, θ. The code for it is

```
double phi  = acos(xInHat*a*cos(th)+yInHat*yOutHat);
```

Designating $\widehat{\boldsymbol{dIn}}$ as the design orbit split bend entrance position unit vector,

$$\widehat{\boldsymbol{dIn}} = (1, 0, 0), \tag{83}$$

$\widehat{\boldsymbol{dOut}}$ as the design split bend exit orbit position unit vector,

$$\widehat{\boldsymbol{dOut}} = (\cos(\theta), 0, \sin(\theta)), \tag{84}$$

one has

$$\widehat{\boldsymbol{rOut}} = (a\cos(\theta), yOutHat, a\sin(\theta)). \tag{85}$$

The 3 conditions $\widehat{\boldsymbol{rOut}} \cdot \widehat{\boldsymbol{rOut}} = 1$, $\widehat{\boldsymbol{rOut}} \cdot \hat{\mathbf{n}} = 0$, and $\widehat{\boldsymbol{rIn}} \cdot \widehat{\boldsymbol{rOut}} = \cos(\phi)$ lead to the equation for ϕ. $\overrightarrow{\mathbf{rOut}}$ is now simply calculated by multiplying its direction by its length

```
double xOut = rOut*rOutHatx;
double yOut = rOut*rOutHaty;
double zOut = rOut*rOutHatz;
```

The $\overrightarrow{\mathbf{pOut}}$, ouput probe momentum, logic is similar.

6 Appendix A: Best values and coding info

UAL will stand for an environmental variable set in the typical UAL build and run process.

tcsh
svn co https://ual.googlecode.com/svn/trunk ual1
cd ual1
setenv UAL ‘pwd‘
source setup-linux-ual

It ($UAL) stands for the base directory of a checked out UAL source tree.
File
$UAL/codes/UAL/src/UAL/Common/Def.hh
has some of the values approximated above.
UAL::pmass is the code representation for the best value, 0.938272013 GeV, of the proton mass. Of course the header file
#include ”UAL/Common/Def.hh”
and other software details must be included.

Other best/exact values used above approximately are

m_pc^2	0.93827231 GeV
G	1.7928474
c	2.99792458e8 m/s
g	2G + 2 = 5.5856948
γ_0	1.248107349
$\mathcal{E}_0$	$\gamma_0 m_p c^2$ 1.171064565 GeV
K_0	$\mathcal{E}_0$ - $m_p c^2$ 0.232792255 GeV
p_0c	0.7007405278 GeV
β_0	0.5983790721
E_0	.751756097
L_0	28.02962111
k	16.77233867

Internal to the code, the class
$UAL/codes/PAC/src/PAC/Beam/Position.hh
has the following correspondence with a given probe

p[0]	dx
p[1]	dpx
p[2]	dy
p[3]	dpy
p[4]	cdt
p[5]	$\frac{d\mathcal{E}}{p_0} = 0$

These are all deviations from the design orbit.

7 Appendix B: Specific Probe Values

For the particular probe configuration with just an x deviation, and no other deviation other than the corresponding change in potential energy:

p[0]=	p[1]=	p[2]=	p[3]=	p[4]=	p[5]=
dx=	dpx=	dy	dpy	cdt	$\frac{d\mathcal{E}}{p_0}$
dx	0	0	0	0	$\frac{d\mathcal{E}}{p_0}$

$$E = \gamma_0 m_p c^2 - \frac{k}{r} = 1.171064565 - \frac{16.77233867}{40.01} = 7.518608991E-01 \quad (86)$$

$$\frac{m_p c^2}{E} = 0.93827231/7.518608991E-01 = 1.247933376 \quad (87)$$

$$L = 40.01 * 0.7007405278 = 28.03662852 \quad (88)$$

$$\frac{k}{Lc} = 16.77233867/28.03662852 = .598229514 \quad (89)$$

$$\kappa^2 = 1 - (\frac{k}{Lc})^2 = 1 - .598229514 * .598229514 = .642121448 \quad (90)$$

$$\epsilon = \frac{1}{.598229514}\sqrt{1 - 1.247933376 * 1.247933376 * .642121448} \quad (91)$$

$$= \frac{1}{.598229514}\sqrt{1 - .9999999453} = 3.9E-04 \quad (92)$$

for a 1 cm offset in x.

Writing these quantities in terms of δs,

$$E \;=\; \gamma_0 m_p c^2 - \frac{k}{r} \;=\; \gamma_0 m_p c^2 - \frac{k}{R_0 + dx} \;=\; \gamma_0 m_p c^2 - \frac{k}{R_0(1+\frac{dx}{R_0})} \;\approx\; \gamma_0 m_p c^2 - \frac{k}{R_0}(1 - \frac{dx}{R_0}) \tag{93}$$

$$= \gamma_0 m_p c^2 - \frac{k}{R_0} + \frac{kdx}{R_0^2} \;=\; \frac{m_p c^2}{\gamma_0}(1 + \frac{\gamma_0}{m_p c^2}\frac{kdx}{R_0^2}) \;=\; E_0(1 + \frac{kdx}{E_0 R_0^2}) \tag{94}$$

so that

$$\delta_E \;\approx\; \frac{kdx}{E_0 R_0^2} \;=\; \frac{16.77233867 * .01}{.751756097 * 40^2} \;=\; 1.39442988E - 4 \tag{95}$$

and

$$\delta_L \;=\; \frac{dx}{R_0} \;=\; \frac{.01}{40} \;=\; 2.5E - 4 \tag{96}$$

Plugging these into (18) gives

$$\epsilon \;=\; \frac{1+\delta_L}{\beta_0}\sqrt{1 - (\frac{m_p c^2}{E})^2 \kappa^2} \approx \tag{97}$$

$$\approx\; \frac{1+\delta_L}{\beta_0}\sqrt{-2\gamma_0^2\beta_0^2\delta_L + 2\delta_E + 4\gamma_0^2\beta_0^2\delta_L\delta_E} \tag{98}$$

$$\frac{1.00025}{0.5983790721}(-2 * 1.248107349^2 0.5983790721^2 (2.5E-4) + 2 * (1.39442988E-4) + 4 * 1.248107349^2 * 0.5983790721^2 * (2.5E \tag{99}$$

$$= 1.67159924(-2.78885977E-04 + 2.78885977E-04 + 7.77768367E-08)^{.5} \tag{100}$$

$$= 1.67(7.77768367E-08)^{.5} \tag{101}$$

$$= 1.67(2.79E-04) \;=\; 4.7E-04 \tag{102}$$

a 21% error.

One sees that, for this case,

$$\delta_E \;\approx\; \frac{kdx}{E_0 R_0^2} \;=\; \frac{k}{E_0 R_0}\frac{dx}{R_0} \;=\; \frac{\gamma_0}{m_p c^2}\frac{R_0 p_0 v_0}{R_0}\delta_L \;=\; \frac{\gamma_0}{m_p c^2}\gamma_0 m_p v_0^2 \delta_L \;=\; \gamma_0^2\beta_0^2\delta_L \tag{103}$$

showing that only the cross term in the expression for ϵ (18) remains

$$\epsilon \approx \frac{1+\delta_L}{\beta_0}\sqrt{4\gamma_0^2\beta_0^2\delta_L\delta_E} \;=\; \frac{1+\delta_L}{\beta_0}\sqrt{4\gamma_0^2\beta_0^2\delta_L\gamma_0^2\beta_0^2\delta_L} \tag{104}$$

$$= (1+\delta_L)(2)\gamma_0^2\beta_0\delta_L \;\approx\; 2(1.25)^2(0.60)2.5E-4 \;=\; 4.7E-04 \tag{105}$$

8 References

References

[1] Munoz, G., Pavic, I. A Hamilton-like vector for the special-relativistic Coulomb problem European Journal of Physics 27(2006) 1007-1018

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[3] Malitsky, N., Talman, J., Talman, R. Development of the UAL/ETEAPOT Code for the Proton EDM Experiment March 26, 2011

[4] Jackson, J. D., Classical Electrodynamics, Second Edition, John Wiley & Sons, 1975

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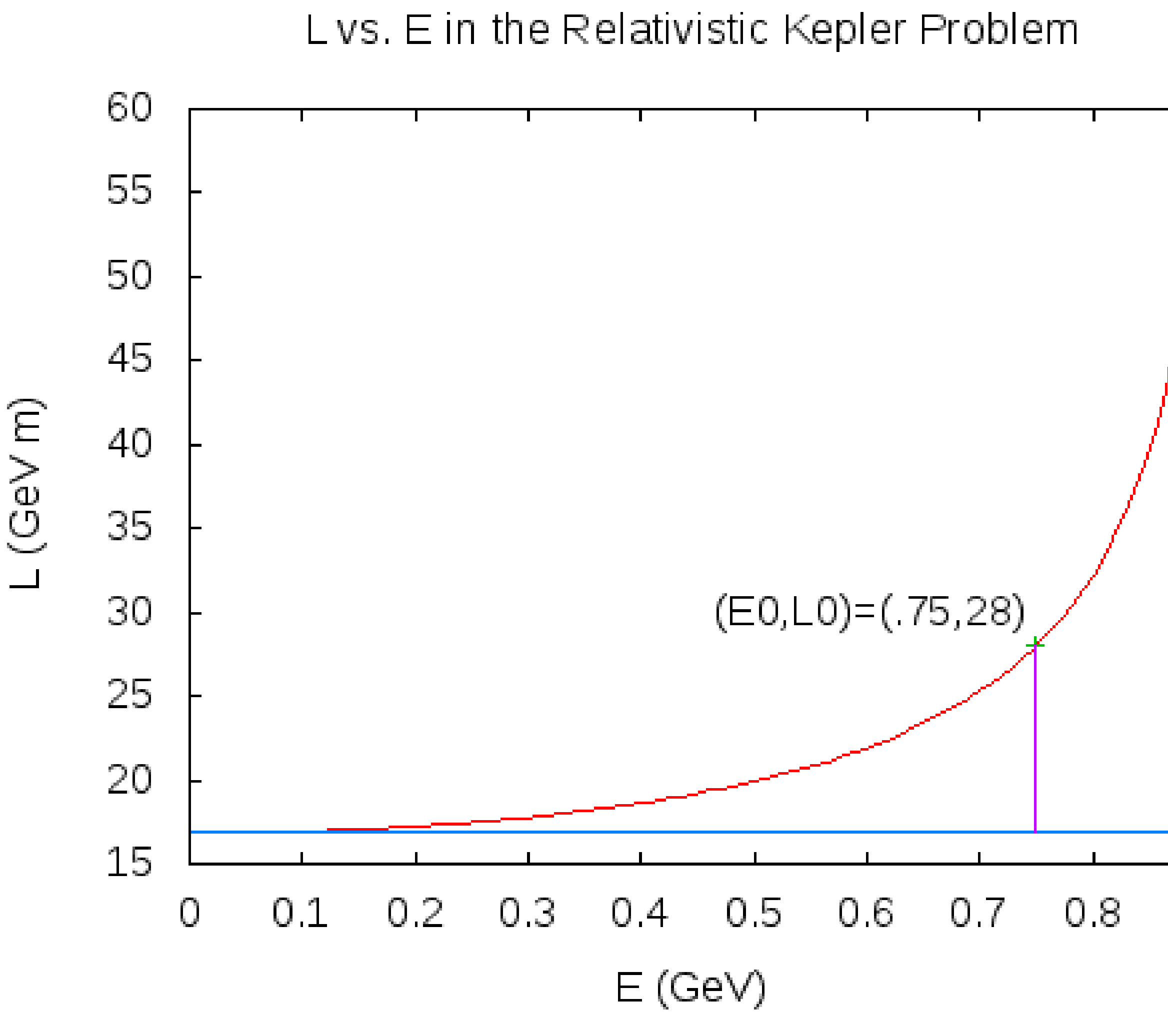

Figure 1: "Angular Momentum (L) versus Energy (E) in the Relativistic Kepler Problem"