Independent Variable Forms for Coulomb Relativistic Radius: Targeting Time of Flight Results.

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Contents

1 Overview

Accelerator tracking tends to give $r = r(s)$, where independent variable s is arc length on design orbit.

Conservation of vector angular momentum, $\vec{L}$, in a Coulomb ($\frac{d\vec{p}}{dt} = -k\frac{\hat{\mathbf{r}}}{r^2}$) central field makes the motion planar. This in turn makes polar coordinates natural, and θ the independent variable. This is where I start. The aim is to have time as an effective independent variable to get accurate time of flight formulae.

The utility of the Hamilton-like vector, $\vec{h}$, for calculating the exact $r(\theta)$ functional dependency of a relativistic charged particle in a Coulomb field has been discussed previously [1], [2]. A completely general full period formula suitable for matching injected probe input parameters has not been given, however.

This note aims to gather/exhibit relevant results for achieving this or a numerical equivalent. It relies on a concurrent program, 2D_5Values.cc, version controlled in the UAL section of Google codes.

Symbolically, the following equations are motivated

$$r = \frac{\lambda}{1 + C(Acos(\kappa\theta) + Bsin(\kappa\theta))}, \tag{1}$$

and

$$\frac{dt}{d\theta} = \frac{r}{Lc^2}(k + r\mathcal{E}). \tag{2}$$

The first is an explicit function for r that has θ as the independent variable, the second allows examination of time, t, as the independent variable. 2D_5Values.cc has numerical values for the quantities referred to here. Some of these values are accepted physical constants, some are geared for a frozen spin proton Electric Dipole Moment experiment with design radius $r_D = 40m$. MKS units are used throughout. The symbols are presumably reasonably familiar, except for energy (often designated E, but here designated $\mathcal{E}$). A symbol for electric field is also desirable.

"Total energy", $\mathcal{E}$, is also conserved. The script $\mathcal{E}$ is to distinguish the energies corresponding to the different potentials used in conventional accelerator tracking ($V(r_D)$=0), and here ($\mathcal{V}(\infty)$=0).

The relationship between the energies is

$$E = \mathcal{E} - \frac{k}{r_D} \tag{3}$$

Possibly the most important relationship for our purposes is

$$\vec{h}_D = \vec{0} \tag{4}$$

(the design Hamilton-like vector is 0!) as it is used to define the circular design input values. This is a small change from an earlier orientation that used eccentricity, $\epsilon = 0$, to define a circle. An examination of [1] Eqs. (32) and (35) shows that they are all but equivalent.

2 Syntax

If 2D_5Values is run with other than 10 inputs (including ./2D_5Values) the message is

```
Usage: ./2D_5Values k(Design) rD(esign) gD(esign) dr th0 thD0 rD0 dg0 Ngrid
```

A sample input, geared for the Design orbit, is

```
./2D_5Values +2.6872238219e-09 40 +1.248107349 0 0 +4.484738326227404e+06 0 0 100
```

This excessive, and inconsistent, precision must be addressed, but will be adhered to for the time being.

The +2.6872238219e-09 is "design k" (equivalent to design bending electric field).

The 40 is the already mentioned design radius $r_D = 40m$.

The +1.248107349 is design/"frozen" γ_D.

The +4.484738326227404e+06 is design angular frequency. It is of particular interest, and a formula is given in Appendix : Design Angular Frequency ($\dot{\theta}|_D$). This formula is used to give this value.

100 is the grid for θ (number of θ values used in the sum approximation to the integral).

The 0s indicate design (0 deviation from design).

3 Results

Design period is

$$T = 2\pi r_D/v_D = 6.28318531 * 40/1.793895330477678e+08 = 251.327412/1.793895330477678e+08 \tag{5}$$

$$= 1.40101492e-06 seconds. \tag{6}$$

The above example run gives

$$T = 1.41502507e-06 seconds. \tag{7}$$

Obviously more investigation is needed!

Again, design $\overrightarrow{h}_D = \overrightarrow{0}$.

The actual calculated value of h_θ to start is +3.05e-08 m/s. It may be possible to improve on this, but this may be the effective zero for this numerical work.

4 Appendix: Formula Overview

The Hamilton-like vector is

$$\vec{h} = \gamma\vec{v} - \frac{k\gamma}{L}\hat{\boldsymbol{\theta}} = \gamma\dot{r}\hat{\mathbf{r}} + (\gamma r\dot{\theta} - \frac{k\gamma}{L})\hat{\boldsymbol{\theta}} = (\gamma\dot{r}, \gamma r\dot{\theta} - \frac{k\gamma}{L}) = (h_r, h_\theta) \tag{8}$$

by definition.

It is shown in the given references that

$$h_\theta'' = -\kappa^2 h_\theta \tag{9}$$

so

$$h_\theta = Acos(\kappa\theta) + Bsin(\kappa\theta). \tag{10}$$

Here κ is a constant for the given fixed k, L.

Thus

$$A = h_\theta|_0 \tag{11}$$

$$B\kappa = h_\theta'|_0. \tag{12}$$

It is also shown in the given references that

$$h_\theta' = -\kappa^2 h_r \tag{13}$$

so that

$$A = (\gamma r\dot{\theta} - \frac{k\gamma}{L})|_0 = \gamma_0 r_0 \dot{\theta}_0 - \frac{k\gamma_0}{L} \tag{14}$$

$$B = -\kappa h_r|_0 = -\kappa(\gamma\dot{r})|_0. = -\kappa\gamma_0\dot{r}_0 \tag{15}$$

All of these values are parameters for a given injected particle. "Machine constants" are designated with no subscript, or with a D (Design) subscript. Injected particle specific values are designated with a 0 (angle or time) subscript.

Using [1] Eq. (17), and Appendix: $r = r(\theta)$, this gives (1) where

$$\lambda = \frac{\kappa^2 L^2 c^2}{k\mathcal{E}} \tag{16}$$

and

$$C = \frac{Lmc^2}{k\mathcal{E}}. \tag{17}$$

A simple numerical integration of (2) is then possible. The goal is to be able to investigate this, and more sophisticated approaches.

5 Appendix: $r = r(\theta)$

$$h_\theta = \gamma r\dot{\theta} - k\gamma/L = \frac{L}{mr} - \frac{k}{L}\frac{\mathcal{E} + k/r}{mc^2} = \kappa^2\frac{L}{mr} - \frac{k\mathcal{E}}{Lmc^2} \tag{18}$$

so

$$r = \frac{\lambda}{1 + \frac{Lmc^2}{k\mathcal{E}}h_\theta} \tag{19}$$

where λ is as above. κ^2 is defined as

$$\kappa^2 = 1 - (\frac{k}{Lc})^2. \tag{20}$$

6 Appendix: $dt/d\theta$

Following [1] Eq. (2)

$$L = \gamma m r^2 \dot{\theta} \tag{21}$$

so

$$\frac{1}{\dot{\theta}} = \frac{\gamma m r^2}{L}. \tag{22}$$

Thus

$$h_\theta = \gamma r \dot{\theta} - \frac{k\gamma}{L} = \frac{L}{mr} - \frac{k\gamma}{L} \tag{23}$$

$$\gamma = \frac{L^2}{kmr} - \frac{L h_\theta}{k} \tag{24}$$

and

$$\frac{1}{\dot{\theta}} = \frac{mr^2}{L}(\frac{L^2}{kmr} - \frac{L h_\theta}{k}) = \frac{Lr}{k} - \frac{mr^2 h_\theta}{k} = \frac{r}{k}(L - mrh_\theta). \tag{25}$$

Using [1] Eq. (16), the factor

$$L - mrh_\theta \tag{26}$$

can be written

$$L - mr(\frac{\kappa^2 L}{mr} - \frac{k\mathcal{E}}{Lmc^2}) = L(1-\kappa^2) + \frac{rk\mathcal{E}}{Lc^2} \tag{27}$$

Using (20), this is

$$L\frac{k^2}{L^2c^2} + \frac{rk\mathcal{E}}{Lc^2} = \frac{k}{Lc^2}(k + r\mathcal{E}). \tag{28}$$

Finally

$$\frac{dt}{d\theta} = \frac{r}{Lc^2}(k + r\mathcal{E}) \tag{29}$$

which is (2)

7 Appendix: Design Angular Frequency ($\dot{\theta}_D$)

On a design (circular) orbit γ_D is given by

$$\gamma_D = \frac{\mathcal{E}_D + k_D/r_D}{mc^2} \tag{30}$$

so is constant. Also $\dot{r}_D = 0$ so $(\overrightarrow{h_D})_r = 0$ so $(\overrightarrow{h_D})_\theta = 0$. Thus $\overrightarrow{h}_D = \overrightarrow{0}$. Plugging this into (14) at $\theta = 0$ gives

$$0 = \gamma_D(r_D\dot{\theta}_D - \frac{k_D}{L_D}) \tag{31}$$

$$r_D\dot{\theta}_D = \frac{k_D}{L_D} \tag{32}$$

$$L_D\dot{\theta}_D = \frac{k_D}{r_D} \tag{33}$$

$$\gamma_D m {r_D}^2 \dot{\theta}_D^2 = \frac{k_D}{r_D} \tag{34}$$

$$\gamma_D\dot{\theta}_D^2 = \frac{k_D}{mr_D^3} \tag{35}$$

$$\dot{\theta}_D^2 = \frac{k_D}{\gamma_D m {r_D}^3} \tag{36}$$

$$\dot{\theta}_D^4 = (\frac{k_D}{m{r_D}^3})^2 (1 - \frac{r_D^2\dot{\theta}_D^2}{c}) \tag{37}$$

$$\dot{\theta}_D^4 + (\frac{k_D}{m{r_D}^3})^2 \frac{r_D^2}{c}\dot{\theta}_D^2 - (\frac{k_D}{m{r_D}^3})^2 = 0 \tag{38}$$

$$\dot{\theta}_D^4 + (\frac{k_D}{m{r_D}^2 c})^2 \dot{\theta}_D^2 - (\frac{k_D}{m{r_D}^3})^2 = 0 \tag{39}$$

$$2\dot{\theta}_D^2 = -(\frac{k_D}{mr_D^2 c})^2 + \sqrt{(\frac{k_D}{mr_D^2 c})^4 + 4(\frac{k_D}{mr_D^3})^2} \tag{40}$$

8 References

References

[1] Munoz, G., Pavic, I. A Hamilton-like vector for the special-relativistic Coulomb problem European Journal of Physics 27(2006) 1007-1018

[2] Malitsky, N., Talman, J., Talman, R. Appendix UALcode: Development of the UAL/ETEAPOT Code for the Proton EDM Experiment 2013

[3] Malitsky, N., Talman, J., Talman, R. Development of the UAL/ETEAPOT Code for the Proton EDM Experiment March 26, 2011