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A Hamilton-like vector for the special-relativistic Coulomb problem

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Abstract
A relativistic point charge moving in a Coulomb potential does not admit a conserved Hamilton vector. Despite this fact, a Hamilton-like vector may be developed that proves useful in the derivation and analysis of the particle's orbit.

1. Introduction

As is well known, the Laplace–Runge–Lenz [1–4] (or Hermann–Bernoulli–Laplace [5]) vector is a conserved quantity in the non-relativistic Coulomb/Kepler problem. It reflects a special property—often referred to as a dynamical symmetry—of a radial inverse-square-law force within the context of non-relativistic mechanics and allows for a rather straightforward derivation of the orbit of a particle moving under the influence of such a force. An alternative [6–8] is to use the Hamilton vector, a quantity related in a simple manner to the Laplace–Runge–Lenz vector whose conservation was first discovered by W R Hamilton in 1845 [9].

Given the remarkable efficiency introduced by the Hamilton vector into the derivation of the orbit of the particle [6–8], it is only natural to ask whether the relativistic problem might not also benefit from a similar development. At first sight, the answer seems rather trivially no, since special-relativistic effects break the degeneracy that ensures conservation of the Hamilton and Laplace–Runge–Lenz vectors. The aim of this paper is to show that, despite the non-existence of a *conserved* Hamilton vector, a generalized version of this quantity may be defined that is similarly useful in the derivation and analysis of the particle's orbit. This new method should be accessible to undergraduates; it is also expected to provide additional insights to a rather general audience.

A few general comments are in order before we begin the discussion. In the relativistic case it becomes necessary to distinguish between the Coulomb (a charged particle bound by an electrostatic potential proportional to r^{-1}) and Kepler (a particle bound by gravitational attraction) problems. This is true even if we agree—as we shall do here—to neglect the dynamical aspects of the fields as a matter of definition of what we mean by the 'Coulomb problem' and the 'Kepler problem'. Non-relativistically, these two situations are formally identical (hence the use of Coulomb/Kepler in [8]), but relativistic effects require a separate treatment. Indeed, it is fairly common knowledge that a consistent, exact formulation of the Kepler problem within the framework of special relativity is not possible if we insist on describing gravity by means of a velocity-independent force law. Approximate formulations are possible but rather ill-motivated, with the simplest prescriptions leading to conceptual inconsistencies and incorrect experimental predictions [10, 11]. Stump [12] has obtained an unambiguous force law from general relativity. Unfortunately, the velocity dependence of his result abolishes every advantage the Hamilton or Laplace–Runge–Lenz vector methods have over the more traditional integration methods. For these reasons, we consider from here on the Coulomb problem only.

Finally, we wish to emphasize that the purpose of this paper is to focus on the non-conserved Hamilton vector as a new method of solution for the relativistic Coulomb problem. We include for completeness a few remarks on the classification and nature of the solutions, and some figures to illustrate the differences with the non-relativistic case. The reader should also consult a recent article by Boyer [13] for an alternative approach.

2. A Hamilton vector and the equation of the orbit

The relativistic equation of motion for a particle of mass m in a Coulomb force field $\mathbf{F} = -k\hat{\mathbf{r}}/r^2$ (we assume $k > 0$; the repulsive case may be treated along the same lines) is formally identical to the non-relativistic equation of motion

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = -k\frac{\hat{\mathbf{r}}}{r^2}. \tag{1}$$

All the distinguishing features of the relativistic behaviour arise at this level from the definition of the relativistic momentum, $\mathbf{p} = \gamma m\mathbf{v} \equiv m\mathbf{u}$.

From equation (1) one easily obtains a conservation law for the energy $E = \gamma mc^2 - k/r$, and another for the angular momentum $\mathbf{L} = \mathbf{r} \times \mathbf{p}$. We shall consider $L \neq 0$ only, since $L = 0$ characterizes a straight-line trajectory ($\mathbf{p}$ parallel to $\mathbf{r}$). Proceeding in analogy with the non-relativistic analysis in [8], we choose polar coordinates in the plane of the orbit so that $L = \gamma mr^2\dot{\theta}$. Then

$$\frac{\mathrm{d}\mathbf{p}}{\mathrm{d}t} = m\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\theta}\dot{\theta} = \frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\theta}\frac{L}{\gamma r^2}. \tag{2}$$

Substituting (2) into (1) and using $\mathrm{d}\hat{\boldsymbol{\theta}}/\mathrm{d}\theta = -\hat{\mathbf{r}}$ on the rhs we obtain

$$\frac{\mathrm{d}\mathbf{u}}{\mathrm{d}\theta} = \frac{k}{L}\gamma\frac{\mathrm{d}\hat{\boldsymbol{\theta}}}{\mathrm{d}\theta}. \tag{3}$$

Equation (3) differs from its non-relativistic counterpart by the term proportional to the gamma factor on the rhs.

The definition

$$\mathbf{h} = \mathbf{u} - \frac{k}{L}\gamma\hat{\boldsymbol{\theta}} \tag{4}$$

allows us to recast equation (3) in the form

$$\frac{\mathrm{d}\mathbf{h}}{\mathrm{d}\theta} = -\frac{k}{L}\frac{\mathrm{d}\gamma}{\mathrm{d}\theta}\hat{\boldsymbol{\theta}}, \tag{5}$$

which makes it clear that, in the low-velocity limit $\gamma \approx 1$, $\mathbf{h}$ is approximately conserved and becomes the non-relativistic Hamilton vector $\mathbf{v} - \frac{k}{L}\hat{\boldsymbol{\theta}}$; it is therefore natural to refer to $\mathbf{h}$ as the relativistic Hamilton vector.

Equation (5) gives a precise way to quantify the breakdown of the dynamical symmetry of the Coulomb problem. Since $\mathbf{h}$ is no longer conserved, one might be tempted to conclude that its usefulness in the determination of the orbits does not carry over to the relativistic regime. Fortunately, this is not the case, as the following shows.

Let us write $\mathbf{h} = h_r\hat{\mathbf{r}} + h_\theta\hat{\boldsymbol{\theta}}$. Equation (5) then yields

$$h_r' - h_\theta = 0 \tag{6}$$

$$h_\theta' + h_r = -\frac{k}{L}\gamma' \tag{7}$$

with $' = \mathrm{d}/\mathrm{d}\theta$.

The rhs of equation (7) may be written in terms of h_θ' by means of the expressions for energy and angular momentum. From $E = \gamma mc^2 - k/r$,

$$\gamma = \frac{E + k/r}{mc^2}, \tag{8}$$

and, from $L = mru_\theta$ and (4), we have

$$\gamma = \frac{E}{mc^2} + \frac{k}{Lc^2}u_\theta = \frac{E}{mc^2} + \frac{k}{Lc^2}\left(h_\theta + \frac{k}{L}\gamma\right). \tag{9}$$

Solving for γ we find

$$\gamma = \frac{EL + mkh_\theta}{Lmc^2\kappa^2} \tag{10}$$

with

$$\kappa^2 = 1 - (k/Lc)^2. \tag{11}$$

Hence

$$\gamma' = \frac{k}{Lc^2\kappa^2}h_\theta', \tag{12}$$

a result that leads to a very simple form for equations (6) and (7):

$$h_r' = h_\theta \tag{13}$$

$$h_\theta' = -\kappa^2 h_r. \tag{14}$$

Equations (13) and (14) will be used together with the following formula giving r in terms of h_θ. Starting with

$$h_\theta = u_\theta - \frac{k}{L}\gamma = \frac{L}{mr} - \frac{k}{L}\frac{E + k/r}{mc^2} \tag{15}$$

and using the definition (11) we have

$$h_\theta = \frac{\kappa^2 L}{mr} - \frac{kE}{Lmc^2}. \tag{16}$$

Solving for r we get the desired equation of the orbit,

$$r = \frac{\lambda}{1 + \frac{Lmc^2}{kE}h_\theta}, \tag{17}$$

where

$$\lambda = \frac{\kappa^2 L^2 c^2}{kE}. \tag{18}$$

3. Orbits for the critical angular momentum $L = k/c$

Clearly, the $\kappa = 0\,(L = k/c)$ case must be considered separately: when $\kappa = 0$, (16) tells us that h_θ is a constant,

$$h_\theta = -\frac{E}{mc} \tag{19}$$

and we loose the connection (17) between r and h_θ. However, from (13),

$$h_r = h_{r0} - \frac{E}{mc}(\theta - \theta_0) = \frac{E}{mc}(\phi - \theta) \tag{20}$$

with $h_{r0} = h_r(\theta_0)$ and $\phi = \theta_0 + mch_{r0}/E$. Also, from (4),

$$\mathbf{u}^2 = \mathbf{h}^2 + \frac{2k}{L}\gamma h_\theta + \left(\frac{k\gamma}{L}\right)^2. \tag{21}$$

Substituting $\mathbf{u}^2 = (\gamma^2 - 1)c^2$ and $L = k/c$ we obtain

$$-c^2 = \mathbf{h}^2 + 2c\gamma h_\theta. \tag{22}$$

Using (8), (19) and (20) in this result yields the equation of the orbit

$$r = \frac{2k/E}{(\theta - \phi)^2 + \delta} \tag{23}$$

where

$$\delta = \left(\frac{mc^2}{E}\right)^2 - 1. \tag{24}$$

The relativistic Hamilton vector for the $\kappa = 0\,(L = k/c)$ case is

$$\mathbf{h} = -\frac{E}{mc}[(\theta - \phi)\hat{\mathbf{r}} + \hat{\boldsymbol{\theta}}]. \tag{25}$$

3.1. $E < mc^2$

If $E < mc^2$, $\delta > 0$ and r reaches a maximum $r_{\max}$ at $\theta = \phi$, with

$$r_{\max} = \frac{2k}{E\delta} = \frac{2kE}{m^2c^4 - E^2}. \tag{26}$$

Let us set $\theta_0 = 0$. Then $\phi = mch_{r0}/E = \gamma_0 mcv_{r0}/E$, and a positive radial component of the initial velocity would result in an outward motion until $\theta = \phi$ followed by an inward spiral towards $r = 0$, as illustrated in figure 1. If the radial component of the initial velocity is negative, the particle simply spirals inwards.

3.2. $E \geqslant mc^2$

If $E \geqslant mc^2$, $\delta \leqslant 0$ and the angle is bounded by

$$(\theta - \phi)^2 > |\delta|. \tag{27}$$

As $(\theta - \phi)^2 \to |\delta|$, $r \to \infty$ and $E = \gamma_\infty mc^2$, so that $|\delta| = \beta_\infty^2$. The particle may, for instance, begin at infinity with speed β_∞ and then spiral inwards, reaching $r = 0$ as $\theta \to \infty$. Figure 2 below shows a particle starting at a finite value of r with $v_{r0} > 0$.

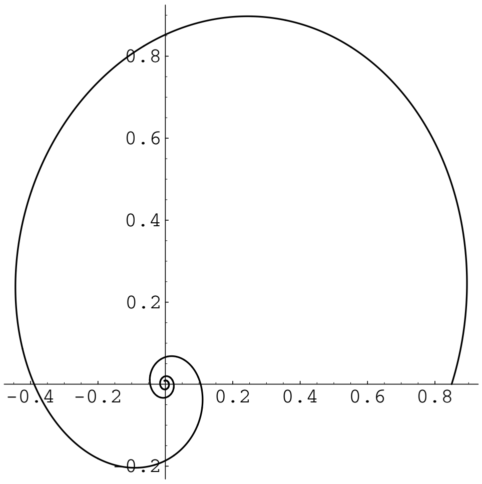

Figure 1. An unstable bound orbit for $L = k/c$, $E = 0.47mc^2$ and $v_{r0} > 0$.

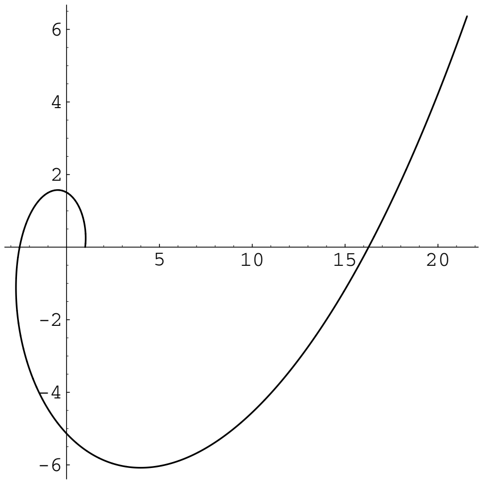

Figure 2. An unbound orbit for $L = k/c$, $E = 2.47mc^2$ and $v_{r0} > 0$.

4. Orbits for $L > k/c$

We may now return to the cases when $\kappa \neq 0$. For $\kappa^2 > 0$ (i.e., $L > k/c$), (13) and (14) are satisfied by

$$h_\theta = A \cos\kappa\theta + B \sin\kappa\theta \tag{28}$$

$$h_r = \frac{1}{\kappa} (A \sin\kappa\theta - B \cos\kappa\theta). \tag{29}$$

The above form of h_θ together with (17) clearly indicates that we are now dealing with orbits where r reaches a minimum value $r_{\min} \neq 0$ ($L = \infty$ is trivially ruled out, and $E = 0$ implies $L = kv_\theta/c^2 < k/c$, or $\kappa^2 < 0$) . For simplicity, let us choose $\theta = 0$ at

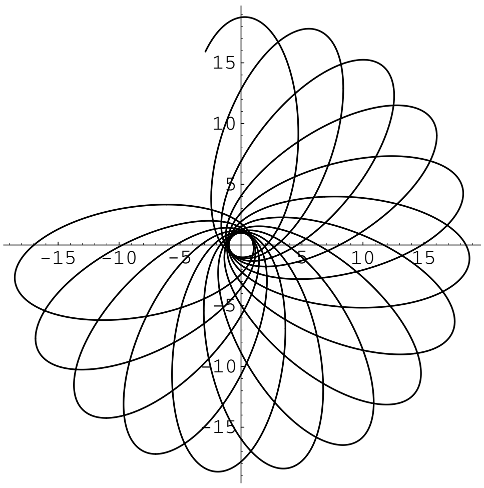

Figure 3. A precessing stable bound orbit for $L = 3.23k/c$, $E = 0.99mc^2$.

$r = r_{\text{min}}$. Then $h_r(\theta = 0) = 0$, since $v_r = 0$ at $r = r_{\text{min}}$ and $h_r = u_r = \gamma v_r$. Letting $h_\theta(\theta = 0) = \gamma_0(v_0 - k/L) \equiv h_0$, these conditions yield

$$h_\theta = h_0 \cos\kappa\theta \tag{30}$$

and, from (17),

$$r = \frac{\lambda}{1 + \frac{Lmc^2}{kE} h_0 \cos\kappa\theta} . \tag{31}$$

h_0 may be expressed in terms of more familiar quantities such as the energy and angular momentum by means of the equations $h_0 = u_0 - \gamma_0 k/L, \kappa^2 c^2 \gamma_0 = E/m + kh_0/L$ (see equation (10)) and $(u_0/c)^2 = \gamma_0^2 - 1$, with the result

$$h_0 = c\sqrt{\left(\frac{E}{mc^2}\right)^2 - \kappa^2} , \tag{32}$$

the negative sign solution being reducible to the orbit equation below by a redefinition of the angle θ. Hence

$$r = \frac{\lambda}{1 + \epsilon \cos\kappa\theta} , \tag{33}$$

with

$$\lambda = \frac{\kappa^2 L^2 c^2}{kE} = \frac{k}{E}\left[\left(\frac{Lc}{k}\right)^2 - 1\right] \tag{34}$$

and

$$\epsilon = \frac{Lc}{k}\sqrt{1 - \left(\frac{mc^2}{E}\right)^2 \kappa^2} . \tag{35}$$

We leave it to the reader to show that ϵ is always real.

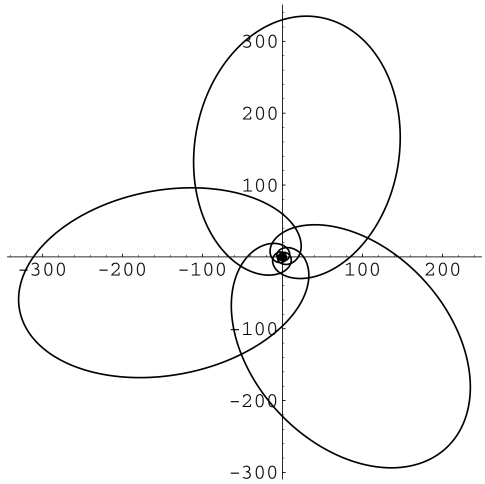

Figure 4. A precessing stable bound orbit for $L = 1.009k/c$, $E = 0.77mc^2$.

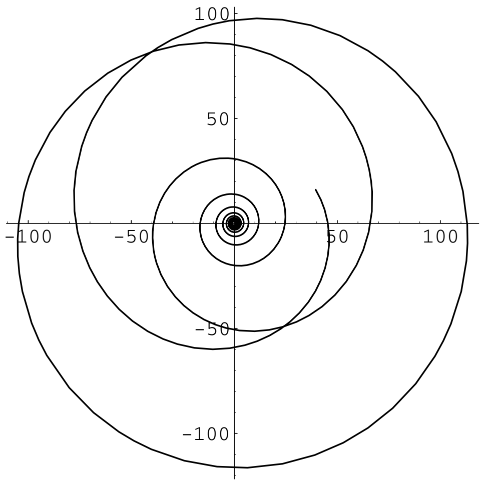

Figure 5. A precessing stable bound orbit for $L = 1.0005k/c$, $E = 0.17mc^2$.

4.1. $E < mc^2$

If $\epsilon < 1$ (equivalently, $E < mc^2$), the solution (33) represents a bound orbit where r varies between $r_{\min} = \lambda/(1+\epsilon)$ and $r_{\max} = \lambda/(1-\epsilon)$. The well-known fact that these orbits are, in general, not closed but precess by the amount $\Delta\theta = 2\pi(1/\kappa - 1)$ should be apparent from (33) (note that the orbits will close for those values of the angular momentum that make κ a rational number). The familiar rosettes found in some textbooks (see figure 3) arise for $\kappa \approx 1 (L \gg k/c)$, whereas for small $\kappa\,(L \approx k/c)$ the shape of the orbit will differ considerably form this simple picture (see figures 4 and 5).

The relativistic Hamilton vector for the $\kappa^2 > 0\,(L > k/c)$ case is

$$\mathbf{h} = h_0 \left(\frac{1}{\kappa} \sin\kappa\theta\, \hat{\mathbf{r}} + \cos\kappa\theta\, \hat{\boldsymbol{\theta}} \right) \tag{36}$$

and while it is not possible to find a rotating system where this vector is constant, (13) and (14) show that the quantity

$$\mathbf{j} = \kappa h_r\, \hat{\mathbf{r}} + h_\theta \hat{\boldsymbol{\theta}} \tag{37}$$

satisfies

$$\frac{\mathrm{d}\mathbf{j}}{\mathrm{d}t} = \boldsymbol{\omega} \times \mathbf{j} \tag{38}$$

with

$$\boldsymbol{\omega} = (1-\kappa)\dot{\theta}\, \hat{\mathbf{z}} = \frac{(1-\kappa)L}{\gamma m r^2}\, \hat{\mathbf{z}}. \tag{39}$$

Hence, from the general relationship for time derivatives between fixed and rotating coordinate systems,

$$\left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right)_{\text{fixed}} = \left(\frac{\mathrm{d}\mathbf{A}}{\mathrm{d}t}\right)_{\text{rot}} + \boldsymbol{\omega} \times \mathbf{A}, \tag{40}$$

we see that $\mathbf{j}$ is constant in a system rotating with the angular velocity $\boldsymbol{\omega}$. This justifies viewing the precessing bound orbits as a combination of a closed elliptical motion plus a rotation with the angular velocity (39). Note, however, that $\boldsymbol{\omega}$ is, in general, neither a constant nor a simple 2π-periodic function of θ (the explicit functional form may be obtained from equations (8), (33) and (39)).

4.2. $E \geqslant mc^2$

If $\epsilon \geqslant 1$ (i.e., $E \geqslant mc^2$), the solution (33) represents an unbound orbit where $r \to \infty$ as $\theta \to \theta_\infty$, with θ_∞ the solution of $\cos \kappa\theta = -1/\epsilon$. Since this restricts θ_∞ to values lying in intervals between

$$-\frac{\pi}{\kappa} \leqslant \theta_\infty \leqslant \frac{\pi}{\kappa} \quad (\epsilon = 1) \tag{41}$$

and

$$-\frac{\pi}{2\kappa} \leqslant \theta_\infty \leqslant \frac{\pi}{2\kappa} \quad (\epsilon \to \infty), \tag{42}$$

it follows that $\kappa \approx 1$ will give rise to a typical scattering trajectory where the particle comes in from infinity, reaches closest approach at $r_{\text{min}} = \lambda/(1+\epsilon)$, and then recedes back to infinity, as in figure 6.

However, small values of κ extend the range of θ_∞, thereby leading to rather atypical trajectories where the particle may loop several times around the scattering centre before heading off to infinity. As a simple illustration, consider the case $E = mc^2 (\epsilon = 1)$ and $L = 3k/\sqrt{5}c$. Then $\kappa = 2/3$, and the particle will move down along the positive y axis, loop once around the scattering centre, and leave along the negative y axis. Figure 7 shows an incoming free particle looping several times before receding back to infinity.

5. Orbits for $L < k/c$

Our last possibility is $\kappa^2 < 0$ (i.e., $L < k/c$). Following steps that closely parallel the $\kappa^2 > 0$ case we find

$$r = \frac{\tilde{\lambda}}{\tilde{\epsilon}\cosh \tilde{\kappa}(\theta + \varphi) - 1}, \tag{43}$$

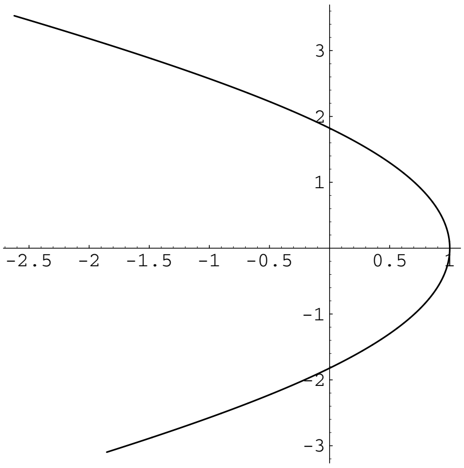

Figure 6. An unbound orbit for $L = 1.8k/c$, $E = 1.77mc^2$.

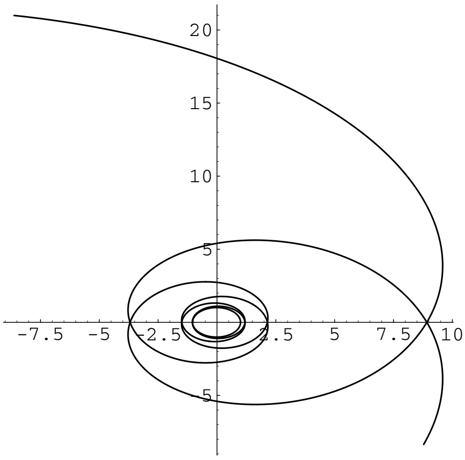

Figure 7. A looping unbound orbit for $L = 1.009k/c$, $E = 1.77mc^2$.

with

$$\tilde{\kappa} = \sqrt{(k/Lc)^2 - 1}, \tag{44}$$

$$\tilde{\lambda} = \frac{\tilde{\kappa}^2 L^2 c^2}{kE} = \frac{k}{E}\left[1 - \left(\frac{Lc}{k}\right)^2\right], \tag{45}$$

$$\tilde{\epsilon} = \frac{Lc}{k}\sqrt{1 + \left(\frac{mc^2}{E}\right)^2 \tilde{\kappa}^2}, \tag{46}$$

and φ may be found from $\tanh \tilde{\kappa}\varphi = \tilde{\kappa} h_{r0}/h_{\theta 0}$. Note that the sign of φ is determined by the sign of v_{r0}, since h_θ is always negative when $L < k/c$,

$$h_\theta = \gamma\left(v_\theta - \frac{k}{L}\right) = -\gamma c\left(\frac{k}{Lc} - \beta_\theta\right), \tag{47}$$

and the sign of $h_{r0} = \gamma_0 v_{r0}$ is determined by the sign of v_{r0}.

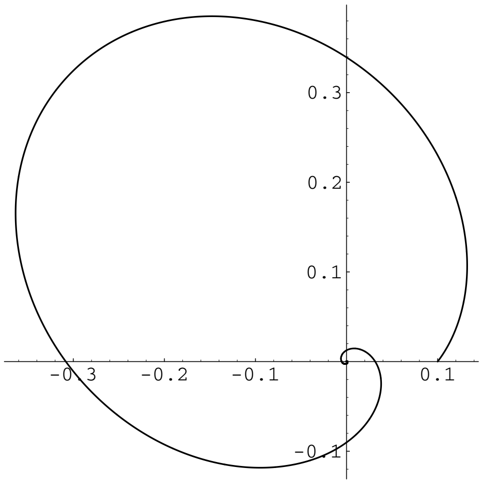

Figure 8. An unstable bound orbit for $L = 0.89k/c$, $E = 0.49mc^2$ and $v_{r0} > 0$.

The relativistic Hamilton vector for the $\kappa^2 < 0\,(L < k/c)$ case is

$$\mathbf{h} = A\left[\frac{1}{\tilde{\kappa}}\sinh\tilde{\kappa}(\theta+\varphi)\,\hat{\mathbf{r}} + \cosh\tilde{\kappa}(\theta+\varphi)\hat{\boldsymbol{\theta}}\right] \tag{48}$$

where

$$A = -\sqrt{h_{\theta 0}^2 - (\tilde{\kappa}h_{r0})^2} = -c\sqrt{\tilde{\kappa}^2 + \gamma_0^2\left(\frac{k\beta_{\theta 0}}{Lc} - 1\right)^2}. \tag{49}$$

5.1. $E < mc^2$

If $\tilde{\epsilon} > 1 (E < mc^2)$, the particle will always end up at $r = 0$ irrespective of the initial conditions. Figure 8 gives an example for $v_{r0} > 0$.

5.2. $E \geqslant mc^2$

If $\tilde{\epsilon} \leqslant 1 (E \geqslant mc^2)$, a particle launched from r_0 at $\theta = 0$ will either end up at infinity when $v_{r0} > 0$ (figure 9), or fall toward $r = 0$ when $v_{r0} < 0$ (figure 10). Whether or not these trajectories involve a looping motion around the central body depends on the initial conditions.

It follows from the present results and those in section 3 that there are no stable bound orbits for $L \leqslant k/c$.

The $E \geqslant mc^2$ trajectories were apparently obtained for the first time by Darwin in 1913 [14]. Some of the most significant differences with the non-relativistic problem—spiral trajectories and the role of the critical angular momentum $L = k/c$—were clearly recognized by him. In his paper, Darwin also mentions the repulsive case and considers the effects of radiation. However, because he concentrated on scattering trajectories, there is no mention of bound orbits or the stability condition $L > k/c$. Sommerfeld [15],[1] on the other hand, treated the bound case only. Interestingly, Sommerfeld does not mention Darwin's paper, nor does he pay attention to the $L < k/c$ case. The full spectrum of relativistic orbits was recently

[1] In reference [15], on page 45, Sommerfeld states that his results do not differ from those of Wacker in his dissertation 'Über Gravitation und Elektromagnetismus' (Tübingen, 1909), which deals with the planetary (Kepler) problem.

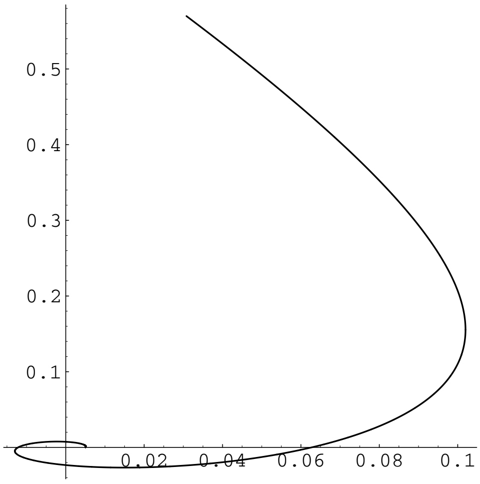

Figure 9. An unbound orbit for $L = 0.999k/c$, $E = 1.49mc^2$ and $v_{r0} > 0$.

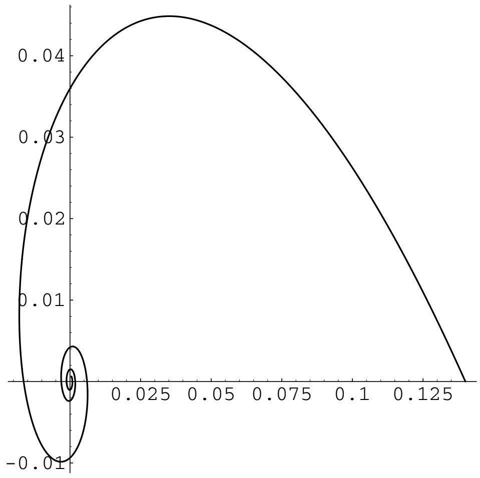

Figure 10. An unbound orbit for $L = 0.999k/c$, $E = 1.49mc^2$ and $v_{r0} < 0$.

discussed in detail by Boyer [13] following a standard integration procedure. Kern [16] has derived the same results based on a relativistic generalization of the Runge–Lenz vector. A treatment by Yoshida [17] using a rotating Runge–Lenz vector gives a different perspective on the precessing orbits (it also clarifies the confusion created by the use of non-relativistic expressions for momentum and angular momentum in section 3 of his earlier paper [18].) While there is some literature available on the Runge–Lenz vector for this particular problem, and even on post-Newtonian extensions for more general problems [19], the present approach based on a relativistic Hamilton vector is new.

6. Conclusions

The main results of this paper are equations (13) and (14) for the relativistic Hamilton vector (4). All the scattering trajectories as well as the bound orbits may be obtained from this basic set together with (17), provided $L \neq k/c$. The solution of the problem for $L = k/c$ is given by equation (23). For completeness, expressions for the full relativistic Hamilton vector have been given in each case (see equations (25), (36) and (48)).

From the point of view of the Hamilton vector, the breakdown of the dynamical $SO(4)$ symmetry of the non-relativistic Coulomb problem is encapsulated in the non-constant γ factor, more precisely in the term $\mathrm{d}\gamma/\mathrm{d}\theta$ on the rhs of (5). It is a simple matter to see that this equation allows for orbits with a conserved $\mathbf{h}$. These orbits have $\mathrm{d}\gamma/\mathrm{d}\theta = 0$ and equation (8) shows that they must be circular orbits. Conversely, since for a circular orbit γ must be constant from (8), we have the result that $\mathbf{h}$ is conserved if and only if the orbit is circular. Furthermore, for circular orbits $v_r = 0$, so $h_r = 0$, and (13) then requires $h_\theta = 0$. Thus the orbit is circular if and only if $\mathbf{h} = 0$. As the reader may verify, such orbits have radii and angular momenta given by $r = \kappa^2 L^2 c^2/kE$ and $L = (k/c)[1 - (E/mc^2)^2]^{-1/2} > k/c$, respectively. Amusingly, equation (5) also states that even ultra-relativistic particles can have orbits whose shapes resemble those of non-relativistic orbits provided $\mathrm{d}\gamma/\mathrm{d}\theta \approx 0$, i.e., provided the eccentricity is small.

As stated in the introduction, our purpose in this paper was to prove that, as far as the classical relativistic Coulomb problem is concerned, the Hamilton vector deserves to be regarded on a par with other useful methods of solution such as direct integration or the Laplace–Runge–Lenz vector. We also hope to have convinced the reader that the Hamilton vector provides some insights complementing the latter approaches.

References

[1] Laplace P S 1969 *Celestial Mechanics* vol 1 (New York: Chelsea Publishing Co.) p 344, equations [572]
[2] Runge C 1919 *Vektoranalysis* vol 1 (Leipzig: Hirzel) p 70
[3] Lenz W 1924 *Z. Phys.* **24** 197
[4] Goldstein H 1980 *Classical Mechanics* 2nd edn (Reading, MA: Addison-Wesley)
[5] Goldstein H 1976 *Am. J. Phys.* **44** 1123
[6] Martínez-y-Romero R P, Núñez-Yépez H N and Salas-Brito A L 1993 *Eur. J. Phys.* **14** 71
[7] Guillaumín-España E, Salas-Brito A L and Núñez-Yépez H N 2003 *Am. J. Phys.* **71** 585
[8] Muñoz G 2003 *Am. J. Phys.* **71** 1292
[9] Hamilton W R 1967 *The Mathematical Papers of Sir William Rowan Hamilton* vol 3, ed H Halberstam and R E Ingram (Cambridge: Cambridge University Press) pp 441–48
[10] Engelke R and Chandler C 1970 *Am. J. Phys.* **38** 90
[11] Sposito G 1976 *An Introduction to Celestial Dynamics* (New York: Wiley) pp 252–6
[12] Stump D R 2003 *Am. J. Phys.* **56** 1097
[13] Boyer T H 2004 *Am. J. Phys.* **72** 992
[14] Darwin C G 1913 *Phil. Mag.* **25** 201
[15] Sommerfeld A 1916 *Ann. Phys.* **51** 1
[16] Kern E 1984 *Z. Naturforsch.* A **39** 720
[17] Yoshida T 1988 *Phys. Rev.* A **38** 19
[18] Yoshida T 1987 *Am. J. Phys.* **55** 1133
[19] Argüeso F and Sanz J L 1984 *J. Math. Phys.* **25** 2935