Proof that the eccentricity in Section 4 is real

From the definition (11) of κ^2 we see that $\kappa^2 \leq 1$, and also that the assumption $\kappa^2 > 0$ of Section 4 implies

$$(k/Lc)^2 < 1 \tag{1}$$

Now, if $E \geq mc^2$,

$$(mc^2/E)^2\kappa^2 \leq 1 \tag{2}$$

so

$$1 - (mc^2/E)^2\kappa^2 \geq 0 \tag{3}$$

and (35) tells us that ϵ is real.

The case $E < mc^2$ is slightly more involved. It may be dealt with by starting from the angular momentum and energy evaluated at $r = r_{\min}$,

$$L = mc\gamma_0\beta_0 r_{\min} \tag{4}$$

$$E = \gamma_0 mc^2 - \frac{k}{r_{\min}} \tag{5}$$

Combining these equations,

$$\frac{E}{mc^2} = \gamma_0\left(1 - \frac{k\beta_0}{Lc}\right) \tag{6}$$

from which it follows that

$$\left(\frac{E}{mc^2}\right)^2 = \gamma_0^2\left(1 - \frac{2k\beta_0}{Lc}\right) + \left(\frac{k}{Lc}\right)^2(\gamma_0^2 - 1) \tag{7}$$

Putting this result in the form

$$\left(\frac{k}{Lc}\right)^2 + \left(\frac{E}{mc^2}\right)^2 = \gamma_0^2\left(\frac{k}{Lc} - \beta_0\right)^2 + 1 \tag{8}$$

shows that

$$\left(\frac{k}{Lc}\right)^2 + \left(\frac{E}{mc^2}\right)^2 > 1 \tag{9}$$

Therefore

$$\left(\frac{E}{mc^2}\right)^2 > 1 - \left(\frac{k}{Lc}\right)^2 \equiv \kappa^2 \tag{10}$$

or

$$1 - \left(\frac{mc^2}{E}\right)^2\kappa^2 > 0 \tag{11}$$

and ϵ is real again.