Advanced Statistical Methods HW4

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Exercise 5.6

6. If $x \sim \text{Mult}_L(n, \boldsymbol{\pi})$, use the Poisson trick (5.44) to approximate the mean and variance of x_1/x_2. (Here we are assuming that $n\pi_2$ is large enough to ignore the possibility $x_2 = 0$.) Hint: In notation (5.41),

$$\frac{S_1}{S_2} \doteq \frac{\mu_1}{\mu_2}\left(1 + \frac{S_1 - \mu_1}{\mu_1} - \frac{S_2 - \mu_2}{\mu_2}\right).$$

The Posson trick of (5.44) in textbook tells us that

$$\text{If } N \sim Poi(n) \text{ and } \mathbf{X}|N \sim Multi_L(N, \pi) \text{ then } \mathbf{X} \sim Poi(n\pi)$$

where $\mathbf{X} = (X_1, \cdots X_L)$ and $\mathbf{X} \sim Poi(n\pi)$ means $X_1, \cdots, X_L$ are independent Poisson random variables having possibly different parameters $X_j \overset{ind}{\sim} Poi(n\pi_j)\ \forall\, j = 1, \cdots, L$. By using this trick, if $\mathbf{X} \sim Multi_L(n, \pi)$ then $X_1 \dot{\sim} Poi(n\pi_1),\ X_2 \dot{\sim} Poi(n\pi_2)$ independently.

Here, we shall use the hint. THe hint is derived from linear approximation of the function $(x, y) \mapsto x/y$

$$\text{Let } g : \mathbb{R} \times (\mathbb{R} - \{0\}) \to \mathbb{R} \text{ be given by } g(x, y) = x/y$$

$$\begin{aligned} g(x, y) &\approx g(x_0, y_0) + \nabla g(x_0, y_0) \cdot (x - x_0,\ y - y_0) \\ &= \frac{x_0}{y_0} + \frac{1}{y}(x - x_0) - \frac{x_0}{y_0^2}(y - y_0) = \frac{x_0}{y_0}\{1 + \frac{x - x_0}{x_0} - \frac{y - y_0}{y_0}\} \end{aligned}$$

Therefore, using the hint for X_1 and X_2 in our problem, we have

$$\frac{X_1}{X_2} \approx \frac{n\pi_1}{n\pi_2}\{1 + \frac{X_1 - n\pi_1}{n\pi_1} - \frac{X_2 - n\pi_2}{n\pi_2} = \frac{\pi_1}{\pi_2}\{1 + \frac{X_1 - n\pi_1}{n\pi_1} - \frac{X_2 - n\pi_2}{n\pi_2}\}$$

Taking expectation, we have

$$E[\frac{X_1}{X_2}] \approx \frac{\pi_1}{\pi_2}(1 + 0 + 0) = \frac{\pi_1}{\pi_2}$$

Now, taking variance with considering independence, we have

$$Var[\frac{X_1}{X_2}] \approx (\frac{\pi_1}{\pi_2})^2\{Var(\frac{X_1 - n\pi_1}{n\pi_1}) + Var(\frac{X_2 - n\pi_2}{n\pi_2})\} = (\frac{\pi_1}{\pi_2})^2\{(\frac{1}{n\pi_1})^2 n\pi_1 + (\frac{1}{n\pi_2})^2 n\pi_2\} = \frac{1}{n}(\frac{\pi_1}{\pi_2})^2(\frac{1}{\pi_1} + \frac{1}{\pi_2})$$

Exercise 5.7

7. Show explicitly how the binomial density $\text{bi}(12, 0.3)$ is an exponential tilt of $\text{bi}(12, 0.6)$.

For given n, pdf of $B(n, p)$ is expressed as $f_p(x) = \binom{n}{x} p^x (1-p)^{n-x} = \binom{n}{x}\exp\{x \log p + (n - x)\log(1 - p)\}$
Hence, for any p and p_0 with fixed n , we have the ratio of binomial densities at two parameter values p and p_o as

$$\begin{aligned} \frac{f_p(x)}{f_{p_0}(x)} &= \frac{\exp\{x \log p + (n - x)\log(1 - p)\}}{\exp\{x \log p_0 + (n - x)\log(1 - p_0)\}} = \exp\left\{x \log \frac{p}{p_0} + (n - x)\log\frac{1 - p}{1 - p_0}\right\} \\ &= exp\{x\big(\log\frac{p}{1 - p} - \log\frac{p_0}{1 - p_0}\big) + n\log\frac{1 - p}{1 - p_0}\} \end{aligned}$$

Now, we shall plug in $n = 12$, $p = 0.3$ and $p_0 = 0.6$ on above.

$$f_p(x) = \exp\{x(\log(0.3/0.7) - \log(0.6/0.4)) + 12 \cdot \log(0.7/0.4)\} f_{p_0}(x) = c \cdot \exp(-1.253x) f_{p_0}(x)$$

where $c = (0.4/0.7)^{12}$ is normalizing constant.
Hence, binomial density $B(12, 0.3)$ is an exponential tilt of $B(12, 0.6)$ with the exponential factor $\exp(-1.253x)$