Advanced Statistical Methods HW2

2021-21116 Taeyoung Chang

Exercise 3.4

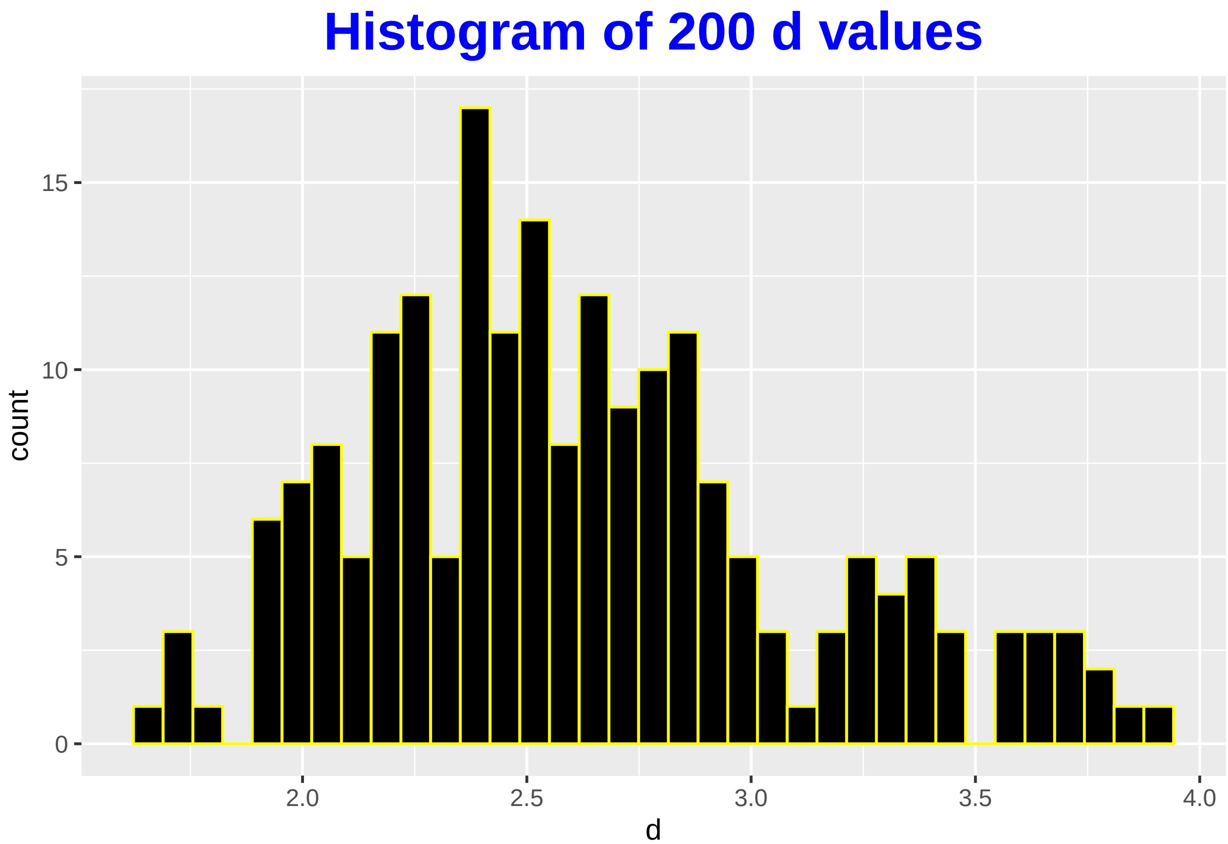
4. (a) Run the following simulation 200 times:
   - $x_i \overset{\text{ind}}{\sim} \mathcal{N}(\mu_i, 1)$ for $i = 1, 2, \ldots, 500$
   - $\mu_i = 3i/500$
   - $i_{\max}$ = index of largest x_i
   - $d = x_{i_{\max}} - \mu_{i_{\max}}$

   (b) Plot the histogram of the 200 d values.

   (c) What is the relation to Figure 3.4?

```
set.seed(123)
d=0
for(k in 1:200){  # 200 times of simulation
  x=0
  for(i in 1:500){  # the number of sample x_i's is 500
    mu=3 * i / 500  # mu_i
    x[i] = rnorm(1, mean=mu, sd=1)  # sample x_i from N( mu_i, 1 )
  }
  d[k] = x[which.max(x)] - 3 * which.max(x) /500  # x_{i_max} - mu_{i_max}
}
summary(d)
```

```
##    Min. 1st Qu.  Median    Mean 3rd Qu.    Max.
##   1.669   2.259   2.544   2.621   2.870   3.921
```



In figure 3.4 , we can see the histogram of unbiased effect-size estimates for 6033 genes. A model for the effect-size x_i for i-th gene is

$$X_i \sim N(\mu_i, 1) \qquad i = 1, 2, \cdots, N(= 6033)$$

Observed maximum value of x_i among 6033 values is $x_{610} = 5.29$. Textbook claims that $x_{610} = 5.29$ was likely to be an overestimate of an effect-size μ_{610}. Why?
It is true that x_{610} is individually unbiased for μ_{610} because $E[X_i] = \mu_i$. However, if we see this same value $x_{610} = 5.29$ as $\max_{\{i=1,\cdots,N\}} x_i$, then we can figure out why it is overestimate for μ_{610}. Since $\max_{\{i=1,\cdots,N\}} X_i > X_j \quad \forall\, j = 1, \cdots, N$,

$$E\Big[\max_{\{i=1,\cdots,N\}} X_i\Big] > E[X_j] = \mu_j \quad , \quad E\Big[\max_{\{i=1,\cdots,N\}} X_i - \mu_j\Big] > 0 \quad \forall\, j = 1, \cdots, N$$

Therefore

$$E\big[X_{i_{\max}} - \mu_{i_{\max}}\big] > 0$$

Indeed 200 d values in this exercise was simulated samples of $X_{i_{\max}} - \mu_{i_{\max}}$ assuming $\mu_j = 3j\ /\ 500 \quad \forall\, j$. From the histogram and five summary statistics of 200 d values above, we can see that the values of $x_{i_{\max}} - \mu_{i_{\max}}$ has mean value 2.6 and median value 2.5 and no value of $X_{i_{\max}} - \mu_{i_{\max}}$ is smaller than 1.5
This tells us that $x_{i_{\max}} - \mu_{i_{\max}}$ is expected to have value about 2.5 , which is stricly bigger than zero, and it indicates that using $x_{610} = 5.29$ is likely to be an overestimate of the effect-size μ_{610}