Advanced Statistical Methods HW6

2021-21116 Taeyoung Chang

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Exercise 7.3

3. In Table 7.1, calculate the JS column based on (7.20).

First, we create a vector of observed batting average. Then, calculate binomial variance σ_0^2 and then derive James stein estimator $\hat{p_i^{JS}}$ according to (7.20)

```
batAvg=c(0.345, 0.333, 0.322, 0.311, 0.289, 0.289, 0.278, 0.255, 0.244, 0.233,
         0.233, 0.222, 0.222,0.222, 0.211, 0.211, 0.200, 0.145)
JSsigma<-function(x){
  sigmasq = mean(x)* (1-mean(x)) / 90   # binomial variance sigma_0 ^2
  mean(x) + (1 - (length(x)-3) * sigmasq / ((length(x)-1)*var(x)) ) * (x-mean(x))
  # James stein estimator according to (7.20)
}
pJS = JSsigma(batAvg)
as.data.frame(round(cbind(batAvg, pJS), 3))
```

```
##    batAvg   pJS
## 1   0.345 0.285
## 2   0.333 0.281
## 3   0.322 0.277
## 4   0.311 0.273
## 5   0.289 0.266
## 6   0.289 0.266
## 7   0.278 0.262
## 8   0.255 0.254
## 9   0.244 0.250
## 10  0.233 0.247
## 11  0.233 0.247
## 12  0.222 0.243
## 13  0.222 0.243
## 14  0.222 0.243
## 15  0.211 0.239
## 16  0.211 0.239
## 17  0.200 0.235
## 18  0.145 0.216
```

Exercise 7. 2

2. In Table 7.1, suppose the MLE batting averages were based on 180 at-bats for each player, rather than 90. What would the JS column look like?

We should compare two column vectors $\hat{p}^{JS}$ where the original one is generated from the data with $n = 90$

and the hypothetical another one is generated from the data with $n = 180$ where n is the number of at-bats used to measure batting average for every player.
To guess what are going to happen, look at the shape of $\hat{p}_i^{JS}$.

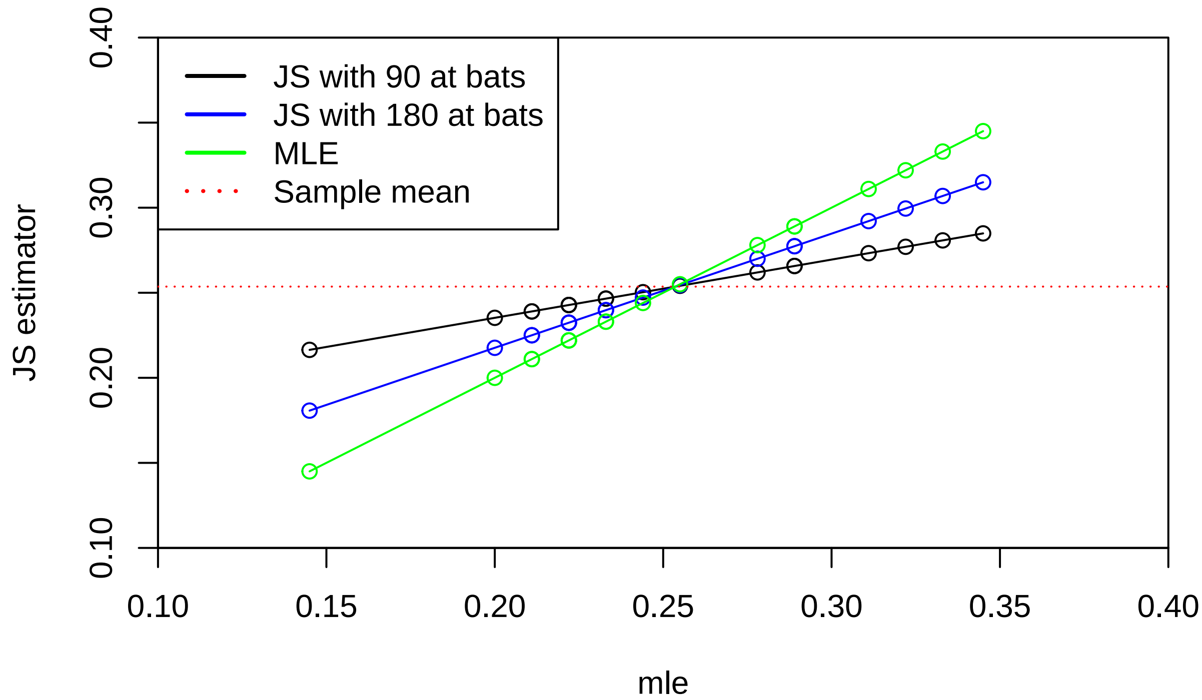
$$\hat{p}_i^{JS} = \overline{p} + \left\{ 1 - \frac{(N-3)\sigma_0^2}{\sum_i (p_i - \overline{p})^2} \right\} (p_i - \overline{p}) = \left\{ 1 - \frac{(N-3)\sigma_0^2}{\sum_i (p_i - \overline{p})^2} \right\} p_i + \frac{(N-3)\sigma_0^2}{\sum_i (p_i - \overline{p})^2} \overline{p}$$

where $\sigma_0^2 = \frac{\overline{p}(1-\overline{p})}{n}$. Thus we can view $\hat{p}_i^{JS}$ as a weighted sum of p_i and $\overline{p}$. Since we consider two cases where $n = 90$ and $n = 180$, plug in those values. Then we can see that weight for p_i gets larger and weight for $\overline{p}$ gets smaller as n is changed from 90 to 180. This means that $\hat{p}_i^{JS}$ comes closer to p_i and moves farther from $\overline{p}$. Of course this happens for every $i = 1, 2, \cdots, N$ where $N = 180$ in this problem.

We can explain this in heuristic way. James-Stein estimator inherits property of Bayes estimator under squared loss, which is a posterior mean. In typical cases, posterior mean can be yielded as a weighted mean of data mean and prior mean where the weight is determined by the amount of information that prior and data have respectively. If the amount of information data have grows then posterior mean tends to data mean rather than the prior mean. In James-Stein estimator, we can view the observed value p_i as data and $\overline{p}$ as an estimate of prior mean. If n was not 90 but 180, then the amount of information from data becomes higher so that James-Stein estimator comes closer to p_i rather than $\overline{p}$.

Now, it is time to check visually whether our claim is true.

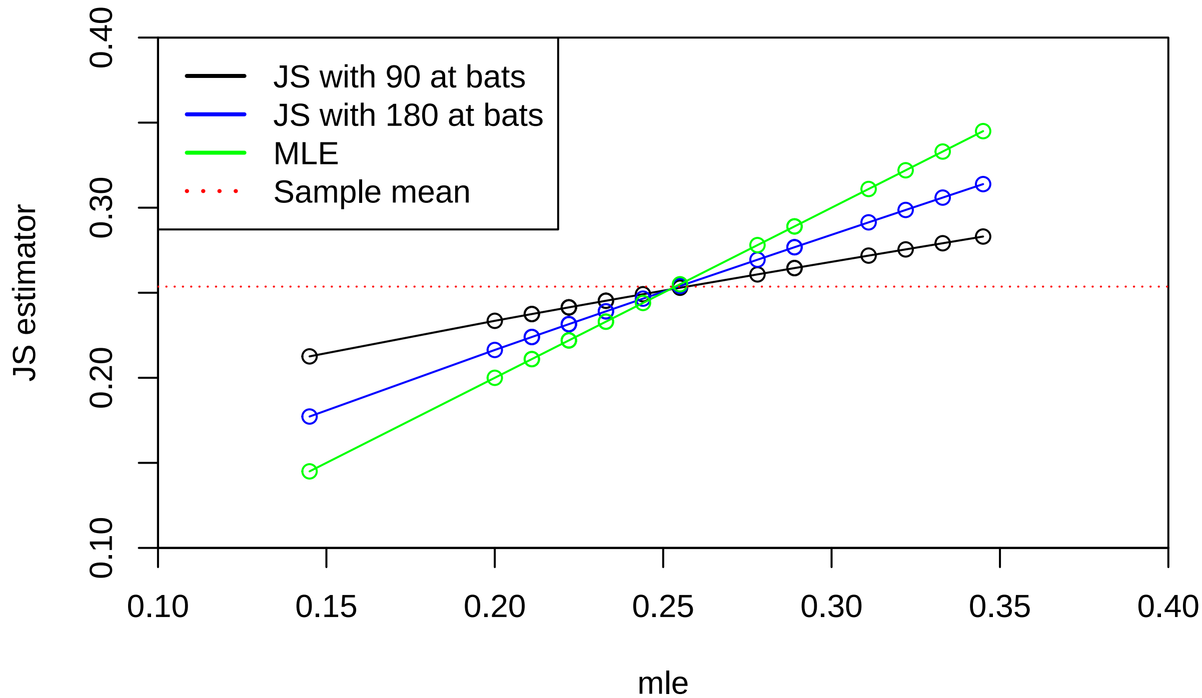
```r
JSsigma180<-function(x){
  sigmasq = mean(x)* (1-mean(x)) / 180   # binomial variance sigma_0 ^2 with n=180
  mean(x) + (1 - (length(x)-3) * sigmasq / ((length(x)-1)*var(x)) ) * (x-mean(x))
  # James stein estimator according to (7.20)
}
pJS180 = JSsigma180(batAvg)

plot(batAvg, pJS , type='n' , xlab= "mle", ylab='JS estimator', xaxs='i', yaxs='i',
     xlim=c(0.1 ,0.4), ylim=c(0.1, 0.4) )
lines(batAvg, pJS, type= 'o')
lines(batAvg, pJS180, col='blue' , type='o')
lines(batAvg, batAvg, col= 'green', type = 'o')
abline(h=mean(batAvg), col='red', lty= 'dotted')
legend('topleft', c("JS with 90 at bats" ,"JS with 180 at bats", "MLE", "Sample mean") ,
       col=c('black', 'blue', 'green', 'red'), lwd=2, lty=c('solid', 'solid', 'solid', 'dotted'))
```



From the plot above, we can see that in both cases where $n = 90$ and $n = 180$, $\hat{p}_i^{JS}$ lies between $p_i = \hat{p}_i^{MLE}$ and $\overline{p}$ because they are weighted means of those two. However, the line drawn when $n = 180$ is closer to p_i and farther from $\overline{p}$ as we explained above.
The same thing happens when we calculate the JS estimator based on (7.23) taking advantage of variance stabilizing transformation.

```
library(purrr)

# arcsin transformation p --> x
g1<-function(p){
  n=90
  2 * sqrt(n + 0.5) * asin(sqrt((n*p + 0.375)/(n + 0.75)))
}

# x vector transformed from p
x1=
  batAvg %>%
  map_dbl(g1)

# JS estimator function
JS<-function(x){
  mean(x) + (1 - (length(x)-3) / ((length(x)-1)*var(x)) ) * (x-mean(x))
}

# mu_JS
muJS1 = JS(x1)

# inverse transformation x --> p
h1<-function(mu){
  n=90
  ((n+0.75)*(sin( mu/(2*sqrt(n+0.5)) ))^2 - 0.375) / n
}
```

```r
# p_JS
pJS1 =
  muJS1 %>%
  map_dbl(h1)


#repeat same process but with n=180 this time

g2<-function(p){
  n=180
  2 * sqrt(n + 0.5) * asin(sqrt((n*p + 0.375)/(n + 0.75)))
}

x2=
  batAvg %>%
  map_dbl(g2)

muJS2 = JS(x2)

h2<-function(mu){
  n=180
  ((n+0.75)*(sin( mu/(2*sqrt(n+0.5)) ))^2 - 0.375) / n
}

pJS2 =
  muJS2 %>%
  map_dbl(h2)

# Plot similar process as above

plot(batAvg, pJS1 , type='n' , xlab= "mle", ylab='JS estimator', xaxs='i', yaxs='i',
     xlim=c(0.1 ,0.4), ylim=c(0.1, 0.4)
     )
lines(batAvg, pJS1, type= 'o')
lines(batAvg, pJS2, col='blue' , type='o')
lines(batAvg, batAvg, col= 'green', type = 'o')
abline(h=mean(batAvg), col='red', lty= 'dotted')
legend('topleft', c("JS with 90 at bats" ,"JS with 180 at bats", "MLE", "Sample mean") ,
       col=c('black', 'blue', 'green', 'red'), lwd=2, lty=c('solid', 'solid', 'solid', 'dotted'))
```



Exercise 7.5

5. Your brother-in-law's favorite player, number 4 in Table 7.1, is batting .311 after 90 at-bats, but JS predicts only .272. He says that this is due to the lousy 17 other players, who didn't have anything to do with number 4's results and are averaging only .250. How would you answer him?

What he said about JS estimator is not totally wrong. Only direct evidence for the true batting average P_4 of player number 4 is p_4. Other p_i's are indeed indirect evidence. JS estimator takes advantage of those indirect evidence to estimate P_4. Derived JS estimates are weighted mean of p_4 and $\overline{p}$ and for the player4, since $p_4 > \overline{p}$, we have $\hat{p}_4^{JS}$ smaller than p_4 . Maybe our brother-in law is unhappy with the result of this situation, however we can say to him that by shrinking every observed batting average toward sample mean $\overline{p}$, we are able to have lower the risk of estimator for overall 18 players' true batting average. As we can see in the table 7.1, in reality, our JS estimate is more accurate than MLE compared with true batting average for player 4.

Problem 2

We shall show that in $M = 0$ case, the Bayes risk of James-Stein estimator is $NB + 2/(A+1)$. Our model is

$$\mu \sim N(0, AI) \quad , \quad x|\mu \sim N(\mu, I)$$

$$\mu|x \sim N(Bx, BI) \quad \text{where} \quad B = \frac{A}{A+1}$$

$$x \sim N(0, (A+1)I) \quad , \quad S = x^T x \sim (A+1)\chi^2(N)$$

$$\hat{\mu}^{JS} = \hat{B}x \quad , \quad \hat{B} = 1 - \frac{N-2}{S}$$

Note that $E(S) = N(A+1)$ and $E(\frac{A+1}{S}) = \frac{1}{N-2} \cdots (*)$

$$
\begin{aligned}
E[(\hat{\mu}_i^{JS}-\mu_i)^2|x] &= E[(\hat{\mu}_i^{JS}-E(\mu_i|x)+E(\mu_i|x)-\mu_i))^2|x] \\
&= E[(E(\mu_i|x)-\mu_i)^2|x]+E[(\hat{\mu}_i^{JS}-E(\mu_i|x))^2|x] \quad \because \text{Cross Product=0} \\
&= Var(\mu_i|x)+(Bx_i-\hat{B}x_i)^2 = B+(B-\hat{B})^2x_i^2 \\
E[\|\hat{\mu}^{JS}-\mu\|^2\,|x] &= NB+(B-\hat{B})^2S \\
E[\|\hat{\mu}^{JS}-\mu\|^2] &= NB+E[(B-\hat{B})^2S]
\end{aligned}
$$

Using $(*)$ above, we have

$$
\begin{aligned}
(B-\hat{B})^2S = (B-1+\frac{N-2}{S})^2S = C^2S-2C(N-2)+(N-2)^2\frac{1}{S} \quad \text{where} \quad C=1-B \\
E[(B-\hat{B})^2S] = CN-2C(N-2)+C(N-2) = CN-C(N-2) = 2C = 2(1-B)
\end{aligned}
$$

Therefore,

$$
E[\|\hat{\mu}^{JS}-\mu\|^2] = NB+E[(B-\hat{B})^2S] = NB+2(1-B) = NB+2/(1+A)
$$

Problem 3

Let $\hat{\mu}_i$ be the i-th coordinate of the JS-estimator in the setting of p. 93 (of the textbook). Compare the risk of $\hat{\mu}_i$ with that of the MLE of μ_i.

$$
\begin{aligned}
(\hat{\mu}_i-\mu_i)^2 &= (\hat{\mu}_i-x_i+x_i-\mu_i)^2 = (\hat{\mu}_i-x_i)^2+(x_i-\mu_i)^2+2(\hat{\mu}_i-x_i)(x_i-\mu_i) \\
&= (\hat{\mu}_i-x_i)^2+(x_i-\mu_i)^2+2(\hat{\mu}_i-\mu_i)(x_i-\mu_i)-2(x_i-\mu_i)^2 \\
&= (\hat{\mu}_i-x_i)^2-(x_i-\mu_i)^2+2(\hat{\mu}_i-\mu_i)(x_i-\mu_i)
\end{aligned}
$$

$$
E[(\hat{\mu}_i-\mu_i)(x_i-\mu_i)] = E[\hat{\mu}_i(x_i-\mu_i)] = E[g(x_i)(x_i-\mu_i)]
$$

where g is given by

$$
g(x_i) = \frac{1}{N}\sum_{j=1}^N x_j + \Big(1-\frac{N-3}{\sum_{j=1}^N(x_j-\overline{x})^2}\Big)(x_i-\frac{1}{N}\sum_{j=1}^N x_j)
$$

Here, every x_j with $j \neq i$ is treated as fixed. By stein's identity, we have

$$
\begin{aligned}
E[(\hat{\mu}_i-\mu_i)(x_i-\mu_i)] = E[g(x_i)(x_i-\mu_i)] &= E\Big[\frac{\partial}{\partial x_i}g(x_i)\Big] \\
&= E\Big[\frac{1}{N}+(1-\frac{1}{N})-(N-3)\frac{\partial}{\partial x_i}\frac{x_i-\frac{1}{N}\sum_{j=1}^N x_j}{\sum_{j=1}^N(x_j-\overline{x})^2}\Big] \\
&= 1-(N-3)E\Big[\frac{(1-\frac{1}{N})S-2(x_i-\overline{x})^2}{S^2}\Big] \quad \text{where} \quad S=\sum_{j=1}^N(x_j-\overline{x})^2
\end{aligned}
$$

$$
\begin{aligned}
E[(\hat{\mu}_i-x_i)^2] &= E\Big[\big(\overline{x}+(1-\frac{N-3}{S})(x_i-\overline{x})-x_i\big)^2\Big] = E\Big[\big(\frac{N-3}{S}(x_i-\overline{x})\big)^2\Big] \\
&= (N-3)E\Big[\frac{(N-3)(x_i-\overline{x})^2}{S^2}\Big] \\
E[(x_i-\mu_i)^2] &= E[(\hat{\mu}_i^{MLE}-\mu_i)^2] = 1
\end{aligned}
$$

Combining above three results and plugging in on the first equation, we have

$$
\begin{aligned}
E[(\hat{\mu}_i-\mu_i)^2] &= E[(\hat{\mu}_i-x_i)^2]-E[(x_i-\mu_i)^2]+2E[(\hat{\mu}_i-\mu_i)(x_i-\mu_i)] \\
&= (N-3)E\Big[\frac{(N-3)(x_i-\overline{x})^2}{S^2}\Big]-1+2\Big\{1-(N-3)E\Big[\frac{(1-\frac{1}{N})S-2(x_i-\overline{x})^2}{S^2}\Big]\Big\} \\
&= 1+(N-3)E\Big[\frac{(N+1)(x_i-\overline{x})^2/S-2(1-\frac{1}{N})}{S}\Big] \\
&= E[(\hat{\mu}_i^{MLE}-\mu_i)^2]+(N-3)E\left[\frac{(N+1)(x_i-\overline{x})^2/S-2(1-\frac{1}{N})}{S}\right]
\end{aligned}
$$

Hence,

$$
E[(\hat{\mu}_i-\mu_i)^2]-E[(\hat{\mu}_i^{MLE}-\mu_i)^2]=(N-3)E\left[\frac{(N+1)(x_i-\overline{x})^2/S-2(1-\frac{1}{N})}{S}\right]
$$

We cannot guarantee that this expectation is positive or negative for every $i \in \{1,\cdots,N\}$. For some i, this expectation may be strictly positive so that $E[(\hat{\mu}_i-\mu_i)^2] > E[(\hat{\mu}_i^{MLE}-\mu_i)^2]$. This means that not every $\hat{\mu}_i^{JS}$ improves $\hat{\mu}_i^{MLE}$. Especially, it can happen when μ_i is genuinely outstanding (whether in a positive or negative sense) compared to others.