Advanced Statistical Methods HW3

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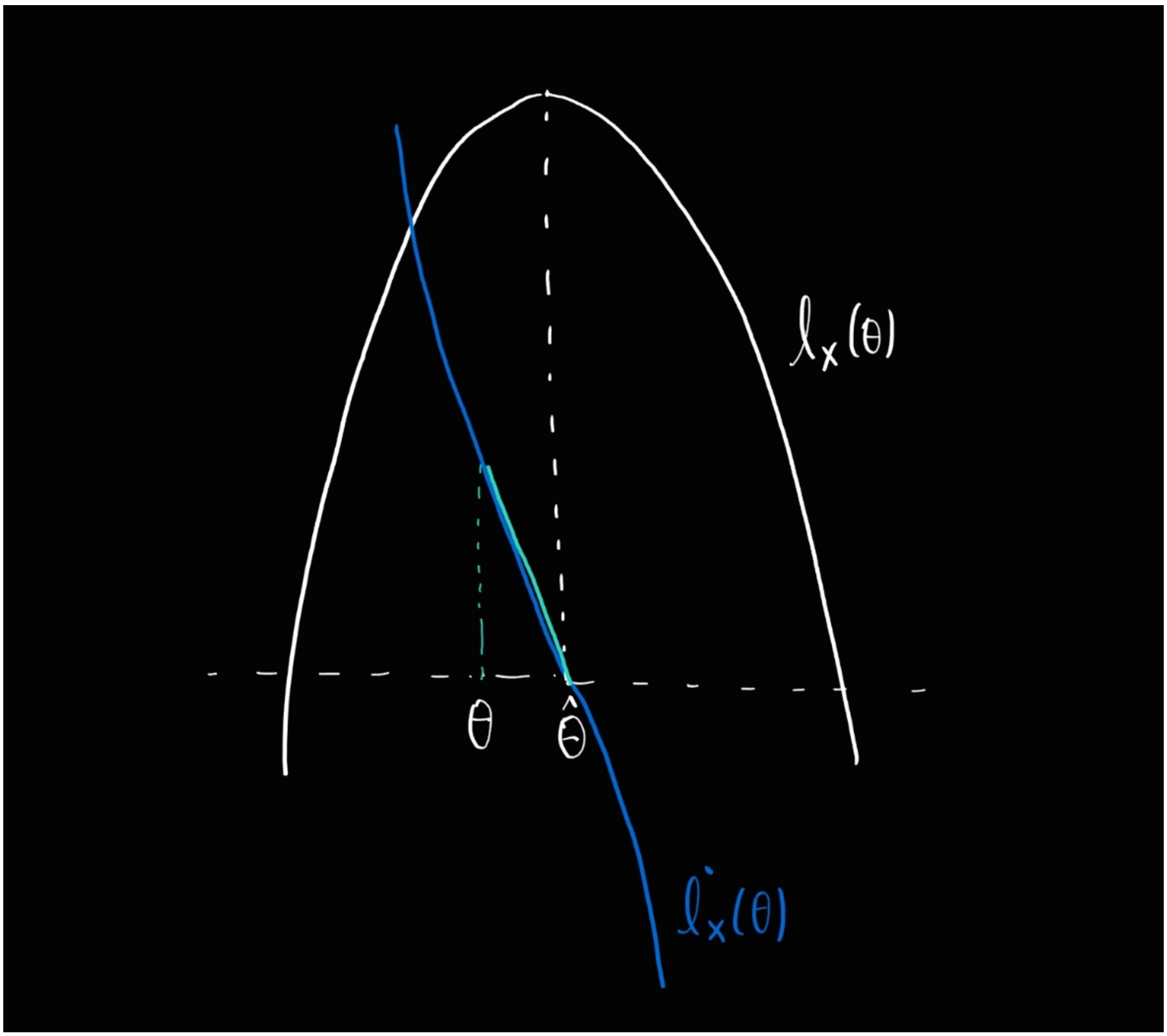
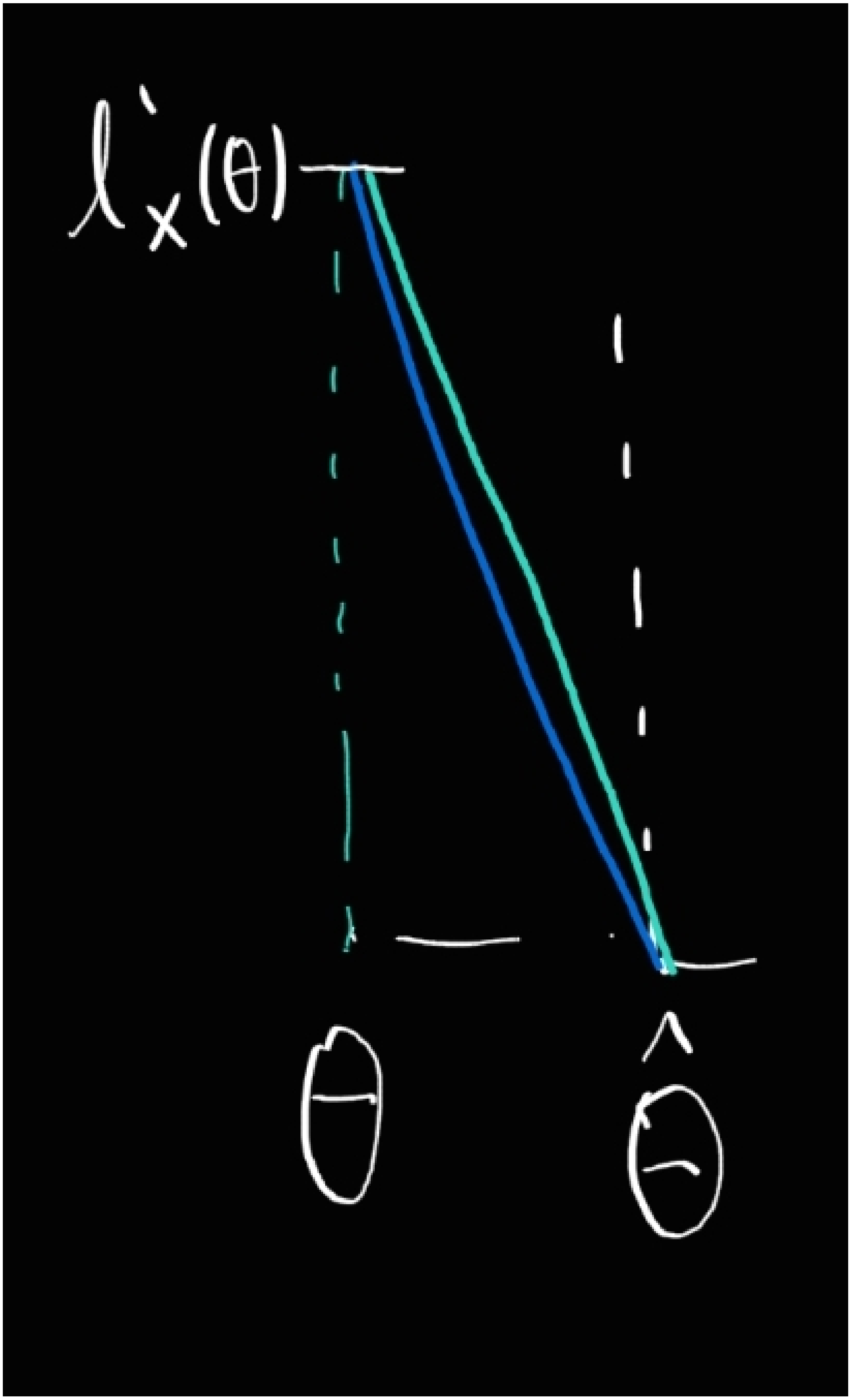
Exercise 4.2

2. Draw a schematic graph of $\dot{l}_x(\theta)$ versus θ. Use it to justify (4.25).




$l'_x(\theta)$
θ
$\hat{\theta}$

Since loglikelihood $l_x(\theta)$ is typically concave, we can draw a schematic graph of $l_x(\theta)$ as a concave function of θ which has a maximum value at $\theta = \hat{\theta}^{MLE}$. Near $\hat{\theta}^{MLE}$, $\dot{l}_x(\theta)$ has positive value at $\theta < \hat{\theta}^{MLE}$, negaive value at $\theta > \hat{\theta}^{MLE}$, and zero value at $\theta = \hat{\theta}^{MLE}$ as drawn in the first figure. Now, from the second figure, we can justify (4.25)

$$\hat{\theta}^{MLE} \doteq \theta + \frac{\dot{l}_x(\theta)}{-\ddot{l}_x(\theta)}$$

The slope of $\dot{l}_x(\theta)$ is $\ddot{l}_x(\theta)$, which can be approximated by

$$\ddot{l}_x(\theta) \doteq \frac{\dot{l}_x(\theta)}{\theta - \hat{\theta}^{MLE}} \quad \because \dot{l}_x(\hat{\theta}^{MLE}) = 0$$

By multiplying $\theta - \hat{\theta}^{MLE}$ and dividing by $\ddot{l}_x(\theta)$ on both sides, we get the desired result.

Exercise 4.3

3. You observe $x_1 \sim \text{Bin}(20, \theta)$ and, independently, $x_2 \sim \text{Poi}(10 \cdot \theta)$. Numerically compute the Cramér–Rao lower bound (4.33). Hint: Fisher information adds for independent observations.

let $x = (x_1, x_2)$.

$$I(\theta) = E\Big[\big(\frac{\partial}{\partial\theta}\log f(x;\theta)\big)^2\Big] = -E\Big[\frac{\partial^2}{\partial\theta^2}\log f(x;\theta)\Big] = -E\Big[\frac{\partial^2}{\partial\theta^2}\log f(x_1;\theta)\Big] - E\Big[\frac{\partial^2}{\partial\theta^2}\log f(x_2;\theta)\Big] \quad \because x_1, x_2 \quad indep.$$

We can analytically derive a closed form of $I(\theta)$ and Cramer-Rao lower bound which is $I(\theta)^{-1}$

$$\begin{aligned}
f(x_1;\theta) &= \binom{20}{x_1}\theta^{x_1}(1-\theta)^{20-x_1} \\
\log f(x_1;\theta) &= \log\binom{20}{x_1} + x_1\log\theta + (20-x_1)\log(1-\theta) \\
\frac{\partial}{\partial\theta}\log f(x_1;\theta) &= \frac{x_1}{\theta} - \frac{20-x_1}{1-\theta} \\
\frac{\partial^2}{\partial\theta^2}\log f(x_1;\theta) &= -\frac{x_1}{\theta^2} - \frac{20-x_1}{(1-\theta)^2} \\
-E\Big[\frac{\partial^2}{\partial\theta^2}\log f(x_1;\theta)\Big] &= \frac{20\theta}{\theta^2} + \frac{20(1-\theta)}{(1-\theta)^2} = \frac{20}{\theta(1-\theta)} \\
f(x_2;\theta) &= \frac{e^{-10\theta}(10\theta)^{x_2}}{x_2!} \\
\log f(x_2;\theta) &= -10\theta + x_2\log(10\theta) - \log x_2! \\
\frac{\partial}{\partial\theta}\log f(x_2;\theta) &= -10 + \frac{x_2}{\theta} \\
\frac{\partial^2}{\partial\theta^2}\log f(x_2;\theta) &= -\frac{x_2}{\theta^2} \\
-E\Big[\frac{\partial^2}{\partial\theta^2}\log f(x_1;\theta)\Big] &= \frac{10\theta}{\theta^2} = \frac{10}{\theta}
\end{aligned}$$

Hence we have

$$\begin{aligned}
I(\theta) &= \frac{20}{\theta(1-\theta)} + \frac{10}{\theta} = \frac{10(3-\theta)}{\theta(1-\theta)} \\
CRLB(\theta) &= \frac{1}{I(\theta)} = \frac{\theta(1-\theta)}{10(3-\theta)}
\end{aligned}$$

Now we have to numerically compute CRLB given observed data x_1 and x_2. We shall find $\hat{\theta}^{MLE}$ and then CRLB can be computed as $CRLB(\hat{\theta}^{MLE})$ or $1/I(x)$ where

$$I(x) = -\frac{\partial^2}{\partial\theta^2}\log f(x;\theta)\Big|_{\theta=\hat{\theta}^{MLE}} = -\Big(-\frac{x_1}{\theta^2} - \frac{20-x_1}{(1-\theta)^2} - \frac{x_2}{\theta^2}\Big)\Big|_{\theta=\hat{\theta}^{MLE}} = \frac{x_1+x_2}{\theta^2} + \frac{20-x_1}{(1-\theta)^2}\Big|_{\theta=\hat{\theta}^{MLE}}$$

is an observed Fisher information.
We shall now consider an algorithm to find $\hat{\theta}^{MLE}$ numerically.

$$\begin{aligned}
\dot{l}_x(\theta) &= \frac{x_1}{\theta} - \frac{20-x_1}{1-\theta} - 10 + \frac{x_2}{\theta} = \frac{x_1+x_2}{\theta} - \frac{20-x_1}{1-\theta} - 10 \\
\ddot{l}_x(\theta) &= -\frac{x_1+x_2}{\theta^2} - \frac{20-x_1}{(1-\theta)^2} < 0 \quad \forall\, 0<\theta<1 \\
\dot{l}_x(\theta) &= 0 \Leftrightarrow (1-\theta)(x_1+x_2) - \theta(20-x_1) - 10\theta(1-\theta) = 0 \Leftrightarrow 10\theta^2 - (30+x_2)\theta + (x_1+x_2) = 0
\end{aligned}$$

Define a function g by

$$g(\theta) = 10\theta^2 - (30+x_2)\theta + (x_1+x_2) = 10\Big(\theta - \frac{30+x_2}{20}\Big)^2 + constant$$

Note that $\dot{l}_x(\theta) = 0 \Leftrightarrow g(\theta) = 0$ on $\theta \in (0,1)$. Hence, if we can find a solution $\theta \in (0,1)$ of $g(\theta) = 0$, then it is a solution of likelihood equation so that it is $\hat{\theta}^{MLE}$. Note that since $\frac{30+x_2}{20} > 1.5$, by the shape of g, $g(\theta) = 0$ must have a root bigger than 1.5 , which implies that we should guide the algorithm to do not find a solution outside of desired area $(0,1)$.
Here we can show an algorithm to find $\hat{\theta}^{MLE}$. To generate a sample x_1, x_2, we set a true value of θ as 0.7. To find a solution θ in $(0,1)$, we shall use Newton's method with an initial value

$$\hat{\theta}^{MME} = \frac{x_1+2x_2}{40} \qquad \because\ \theta = E\big[\frac{X_1}{20}\big],\ \theta = E\big[\frac{X_2}{10}\big] \Rightarrow 2\theta = E\big[\frac{X_1}{20} + \frac{X_2}{10}\big]$$

```
theta=0.7

set.seed(100)
x1=rbinom(1, size=20, prob=theta)
x2=rpois(1, lambda=10*theta)
print(c(x1, x2))
```

```
## [1] 15  5
```

```
# error tolerance value for terminating the alogirthm
tol=1e-8

# function g defined above
g=function(theta, x1, x2){
  10*theta^2-(30+x2)*theta+(x1+x2)
}
# derivative of g with respect to  theta
gprime=function(theta, x1, x2){
  20*theta-(30+x2)
}

# find mle given data and error tolerance
# 'mix' stands for
binompois_mle<-function(x1,x2, tol){
  # initial value is theta.mme
  theta.current=(x1+2*x2)/40
```

```r
  while(TRUE){
    # update theta value by newton's method
    theta.new = theta.current - g(theta.current, x1,x2) / gprime(theta.current, x1,x2)

    # break the iteration if |g(theta)| < error tolerance
    if(abs(g(theta.new,x1,x2))<tol ) break

    # If theta deviates from (0,1) then end the iteration and print the message
    if(theta.new > 1 | theta.new < 0) {
      print("out of parameter sapce (0,1)")
      break
    }
    theta.current = theta.new
  }
  return(theta.new)
}

theta.hat=binompois_mle(x1, x2, tol)
theta.hat
```

```
## [1] 0.7192236
```

Given $\theta = 0.7$, the sample $x_1 = 15$, $x_2 = 5$ is generated. From this sample, we calculate $\hat{\theta}^{MLE} = 0.71922$ which is quite close to the true value of θ.

```r
CRLB.true=theta*(1-theta)/(10*(3-theta))
CRLB.plugin=theta.hat*(1-theta.hat)/(10*(3-theta.hat))
obsinfo=(x1+x2)/theta.hat^2+(20-x1)/(1-theta.hat)^2
CRLB.obsinfo=1/obsinfo
print(cbind(CRLB.true, CRLB.plugin, CRLB.obsinfo))
```

```
##        CRLB.true CRLB.plugin CRLB.obsinfo
## [1,] 0.009130435 0.008854047  0.009795578
```

`CRLB.plugin` equals to $CRLB(\hat{\theta}^{MLE})$ and `CRLB.obsinfo` equals to $1/I(x)$ We can check that `RCLB.plugin` and `RCLB.obsinfo`, which are the resulted computation for RCLB by our numerical method, are pretty close to the true value of Cramer Rao lower bound.

Problem 2

2. Recreate Figure 4.2 with n=20 and n=100.

We shall generate random n samples from $Cauchy(0, 1)$ distribution for 10000 times. Each time we will find MLE of θ which is a location parameter of cauchy distribution. (Note that the true value is set as 0). True value of CRLB is $1/I(\theta)$ where

$$\begin{aligned}
f(x_1;\theta) &= \frac{1}{\pi}\frac{1}{1+(x_1-\theta)^2} \\
\log f(x_1;\theta) &= -\log\pi - \log\{1+(x_1-\theta)^2\} \\
\frac{\partial}{\partial\theta}\log f(x_1;\theta) &= \frac{2(x_1-\theta)}{1+(x_1-\theta)^2} \\
\frac{\partial^2}{\partial\theta^2}\log f(x_1;\theta) &= \frac{2\{(x_1-\theta)^2-1\}}{\{1+(x_1-\theta)^2\}^2} \\
I_1(\theta) &= -E\Big[\frac{\partial^2}{\partial\theta^2}\log f(x_1;\theta)\Big] = \frac{1}{2} \\
I(\theta) &= -E\Big[\frac{\partial^2}{\partial\theta^2}\log f(x;\theta)\Big] = \sum_{i=1}^{n} -E\Big[\frac{\partial^2}{\partial\theta^2}\log f(x_i;\theta)\Big] = n\cdot\frac{1}{2}
\end{aligned}$$

Thus, we have the true value of CRLB (asymptotic variance of MLE) given by

$$CRLB(\theta) = \frac{1}{n\cdot\frac{1}{2}} = \frac{2}{n}$$

Note that $CRLB(\hat{\theta}^{MLE}) = CRLB(\theta)\Big|_{\theta=\hat{\theta}^{MLE}}$ is independent of observed data x in this case because $CRLB(\theta)$ is constant. Fisher suggested the observed information bound $1/I(x)$ as an estimate for asymptotic variance of MLE, which is given as

$$I(x) = \sum_{i=1}^{n} -\frac{\partial^2}{\partial\theta^2}\log f(x_i;\theta)\Bigg|_{\theta=\hat{\theta}^{MLE}} = \sum_{i=1}^{n}\frac{2\{1-(x_i-\theta)^2\}}{\{1+(x_i-\theta)^2\}^2}\Bigg|_{\theta=\hat{\theta}^{MLE}}$$

Unlike plug in estimate of CRLB, the observed information bound $1/I(x)$ depends on the observed data x . Now, we shall generate a list of MLE's and the observed information bounds corresponding to 10000 times of randomly generated data samples of size n.

```
# l'(theta) where l(theta) is loglikelihood function
cauchy_lprime<-function(theta, x){
  n=length(x)
  sum=0
  for(i in 1:n){
    numerator=2*(x[i]-theta)
    denominator=1+(x[i]-theta)^2
    sum = sum + numerator/denominator
  }
  return(sum)
}

# l''(theta) where l(theta) is loglikelihood function
cauchy_ldprime<-function(theta,x){
  n=length(x)
  sum=0
  for(i in 1:n){
    numerator = 2*((x[i]-theta)^2-1)
    denominator = (1+(x[i]-theta)^2)^2
    sum = sum + numerator/denominator
  }
  return(sum)
}
```

```r
# Function to find mle using Newton's method
cauchy_mle<-function(x, tol){
  # Use initial value as median since location parameter of cauchy distribution is population median
  theta.current=median(x)
  iter=0
  while(TRUE){
    theta.new = theta.current - cauchy_lprime(theta.current, x) / cauchy_ldprime(theta.current, x)
    if(abs(cauchy_lprime(theta.new,x))<tol ) break
    theta.current = theta.new
    iter=iter+1
    # Print a message if the algorithm converges too slow
    if(iter>1e+5) {
      print('too slow for iteration to converge')
      break
    }
  }
  return(theta.new)
}
```

To find MLE, we need to find a solution of likelihood equation. First, we have tried to use Newton's method.

```r
# We will generate 10000 datasets.
N=10000

# List of MLEs and observed Information bounds will be stored in X
X=matrix(0, ncol=2, nrow=N)
colnames(X)<-c('MLE', 'Obs.Info.Bound')

# sample size
n=20

#error tolerance
tol=1e-6

set.seed(100)

for(i in 1:N){
  # Generate sample of size n from cauchy(0,1)
  x=rcauchy(n, location=0, scale=1)
  mle = cauchy_mle(x, tol)
  # Too large value of theta may be a wrong answer
  if(abs(mle)>1e+03) {
    print("Abnormal value of mle is yielded")
    # Draw another sample and find mle again
    while(abs(mle)>1e+03){
      x=rcauchy(n, location=0, scale=1)
      mle=cauchy_mle(x, tol)
    }
  }
  # Calculate observed information bound
  obs_info_bound = -1/ cauchy_ldprime(mle, x)
  X[i,]=c(mle, obs_info_bound)
}
```

```
## [1] "Abnormal value of mle is yielded"
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## [1] "Abnormal value of mle is yielded"
```

```
X=as.data.frame(X)
# Sorting the list by the order statistics of observed information bound
X=X[order(X$Obs.Info.Bound),]
head(X, 10)
```

```
##               MLE Obs.Info.Bound
## 6155 -31.97291766    -3.33141318
## 5206  15.33740838    -1.81613613
## 8575  -6.82551482    -0.96347451
## 2856  -0.23648680     0.03724461
## 6964  -0.08596687     0.04185940
## 6361   0.38495939     0.04203593
## 5730  -0.22412033     0.04216019
## 3055   0.20981631     0.04228430
## 5542  -0.03992384     0.04234671
## 8977   0.24383388     0.04259172
```

We can see that performance of Newton's method for finding MLE is not quite good in this case.
- Since we know the true value of θ, we can say that too large $\hat{\theta}$ value is abnormal to estimate θ. Probably what the algorithm found is local maximizer of $\dot{l}_x(\theta)$ other than $\hat{\theta}^{MLE}$.
- For some sample data, the algorithm does not converge even after 100000 iterations. It is too slow.
As an alternative, we shall use Dekker's method which is a hybrid method combining a bisection method with a secant method.

```
# To use a function `swap`
library(seqinr)

cauchy_mle<-function(x, tol){
  # Start an algorithm with initial interval [a,b]
  # Here [a,b] is chosen as [-k, k] where k=|med(X_i)|+3
  # For bisection method to begin, l'(a)l'(b) < 0 should be satisfied.
  a = -(abs(median(x))+3)
  theta.past = a
  theta.current=abs(median(x))+3
  iter=0
  while(cauchy_lprime(theta.past, x)*cauchy_lprime(theta.current, x ) > 0){
    # If bisection l'(a)l'(b) < 0 is not satisfied then
```

```
    # give small fluctuation to a and b to attain the condition
    warning("the initial values does not saitsfy starting condition
    of bisection method. Shifting it a little...")
    theta.past = theta.past + rnorm(1)
    theta.current =  theta.current + rnorm(1)
    a=theta.past
    # If this procedure takes too much iteration, return NA
    iter=iter+1
    if(iter>1e+3) {
      print('initial value shifing is too complicated')
      return(NA)
      break
    }
  }
  iter=0
  while(TRUE){
    # Calculate the ratio which is used instead of l''(theta)
    # This is a difference between Newton's method and secant method
    ratio = (cauchy_lprime(theta.current, x)-cauchy_lprime(theta.past, x)) / (theta.current-theta.past)

    # For current bisection interval [a,b], calculate middle point m
    middle=(a + theta.current)/2

    # Propose a updated theta value , which is mainly done by a secant method
    # If two previous theta value are the same, then secant method cannot be used
    # so that middle point is proposed as new theta.
    proposal = ifelse(cauchy_lprime(theta.current, x)-cauchy_lprime(theta.past, x) !=0 ,
                      theta.current-cauchy_lprime(theta.current, x) / ratio, middle)

    # Determine an updated theta. Proposed theta becomes updated theta if it lies between m and b
    # Other wise, middle point becomes an updated theta
    if(middle<theta.current){
      theta.new=ifelse(proposal<=theta.current & proposal>=middle, proposal, middle)
    }
    else{
      theta.new=ifelse(proposal<=middle & proposal>=theta.current, proposal, middle)
    }

    # End the iteration if " |l'(theta)| < error tolerance "  is attained.
    if(abs(cauchy_lprime(theta.new,x))<tol ) break

    # Setting a bisection interval [a,b] for the next step
    if(cauchy_lprime(theta.new, x)*cauchy_lprime(a, x) > 0) a=theta.current
    if(abs(cauchy_lprime(theta.new,x)) > abs(cauchy_lprime(a, x)) ) swap(a, theta.new)

    theta.past=theta.current
    theta.current=theta.new

    # Print a message if the algorithm converges too slow
    iter=iter+1
    if(iter>1e+5) {
      print('too slow for iteration to converge')
      break
```

```r
    }

  }
  return(theta.new)
}

# We will generate 10000 datasets.
N=10000

# List of MLEs and observed Information bounds will be stored in X
X=matrix(0, ncol=2, nrow=N)
colnames(X)<-c('MLE', 'Obs.Info.Bound')

# sample size
n=20

#error tolerance
tol=1e-6

set.seed(100)

for(i in 1:N){
  # Generate sample of size n from cauchy(0,1)
  x=rcauchy(n, location=0, scale=1)
  mle = cauchy_mle(x, tol)
  # Too large value of theta may be a wrong answer
  if(abs(mle)>1e+03) {
    print("Abnormal value of mle is yielded")
    # Draw another sample and find mle again
    while(abs(mle)>1e+03){
      x=rcauchy(n, location=0, scale=1)
      mle=cauchy_mle(x, tol)
    }
  }
  # Calculate observed information bound
  obs_info_bound = -1/ cauchy_ldprime(mle, x)
  X[i,]=c(mle, obs_info_bound)
}
```

```
## Warning in cauchy_mle(x, tol): the initial values does not saitsfy starting condition
##     of bisection method. Shifting it a little...

## Warning in cauchy_mle(x, tol): the initial values does not saitsfy starting condition
##     of bisection method. Shifting it a little...

## Warning in cauchy_mle(x, tol): the initial values does not saitsfy starting condition
##     of bisection method. Shifting it a little...
```

```r
X=as.data.frame(X)
# Sorting the list by the order statistics of observed information bound
X=X[order(X$Obs.Info.Bound),]
head(X, 10)
```

```
##                MLE Obs.Info.Bound
## 8568   0.152569546     0.03937138
```

```
## 6490 -0.244318638     0.04103515
## 8520  0.006857701     0.04189009
## 3908  0.243447185     0.04359162
## 2124  0.383324723     0.04378321
## 2221  0.184737462     0.04431370
## 8465  0.199806715     0.04458801
## 3624  0.313135109     0.04483765
## 7976  0.601368498     0.04499235
## 2507  0.294786221     0.04532065
```

Thanks to Dekker's method, the problems we had when we used Newton's method are solved. We can guess that this is because we start the algorithm in restricted interval which $\hat{\theta}^{MLE}$ is most likely to lie in.
Now it is time to recreate Figure 4.2

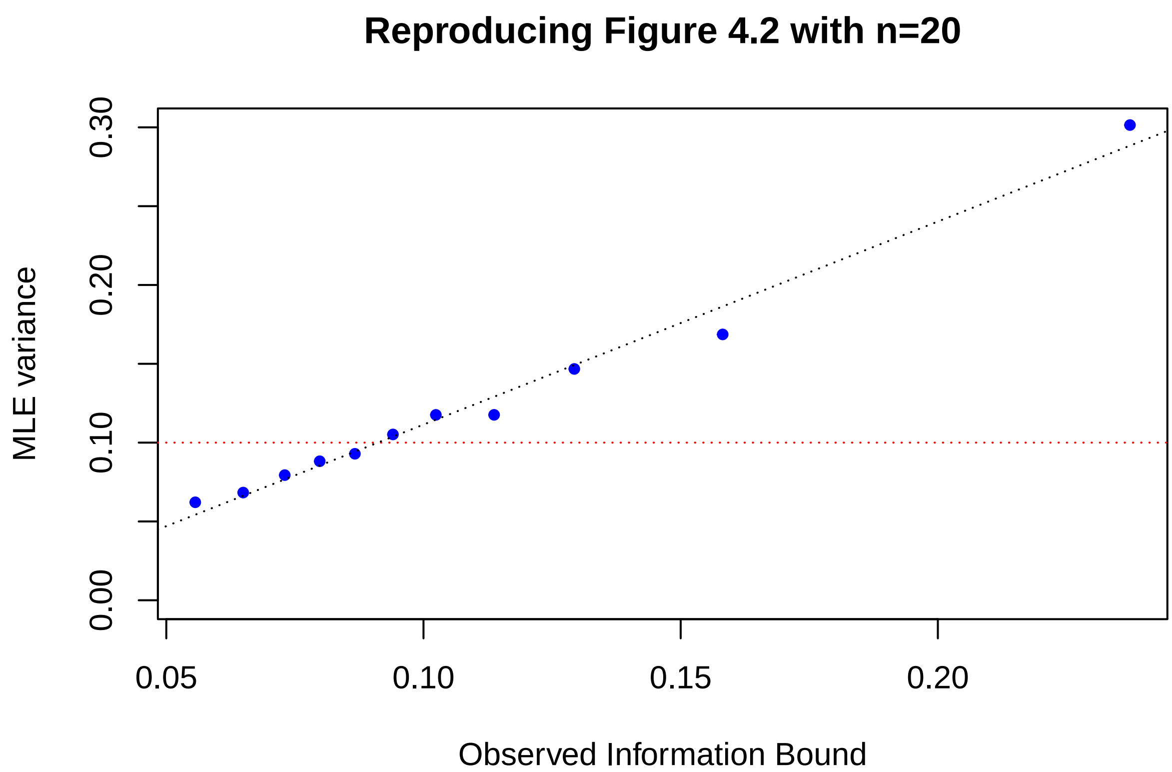
```
# Create a function which splits the list of MLE's and observed information bounds
# by the size of observation information bounds
# and then calculate variance of MLE & median of observed information bounds in each group
splitter<-function(X){
  percent=c(0,5,15,25,35,45,55,65,75,85,95,100)
  N=nrow(X)
  last.indices=N*percent/100
  var_theta=rep(0, 11)
  med_infobound=rep(0,11)
  for(i in 2:12){
    indices=(last.indices[i-1]+1):last.indices[i]
    thetas=X[indices, 1]
    infobounds=X[indices,2]
    var_theta[i-1]=var(thetas)
    med_infobound[i-1]=median(infobounds)
  }
  as.data.frame(cbind(var_theta, med_infobound))
}

S=splitter(X)

# Recreate Figure 4.2 with n = 20
plot(S$med_infobound, S$var_theta, col='blue', pch=20,
     xlab='Observed Information Bound', ylab='MLE variance', main='Reproducing Figure 4.2 with n=20',
     ylim=c(0,round(max(S$var_theta),2))
     )

# CRLB(theta) which does not depend on the data
unconditional.variance= 1/ (1/2 * n)
abline(h=unconditional.variance, lty='dotted', col='red')

# linear regression fit of Var(theta) ~ Med(Obs.Info.Bound)
fit<-lm(var_theta~med_infobound, data=S)
abline(fit$coefficients, lty='dotted')
```



Repeat this procedure for $n = 100$ case.

```
# We will generate 10000 datasets.
N=10000

# List of MLEs and observed Information bounds will be stored in X
X=matrix(0, ncol=2, nrow=N)
colnames(X)<-c('MLE', 'Obs.Info.Bound')

# sample size
n=100

#error tolerance
tol=1e-6


for(i in 1:N){
  # Generate sample of size n from cauchy(0,1)
  x=rcauchy(n, location=0, scale=1)
  mle = cauchy_mle(x, tol)
  # Too large value of theta may be a wrong answer
  if(abs(mle)>1e+03) {
    print("Abnormal value of mle is yielded")
    # Draw another sample and find mle again
    while(abs(mle)>1e+03){
      x=rcauchy(n, location=0, scale=1)
      mle=cauchy_mle(x, tol)
    }
  }
  # Calculate observed information bound
```

```r
  obs_info_bound = -1/ cauchy_ldprime(mle, x)
  X[i,]=c(mle, obs_info_bound)
}

X=as.data.frame(X)
# Sorting the list by the order statistics of observed information bound
X=X[order(X$Obs.Info.Bound),]
head(X, 10)
```

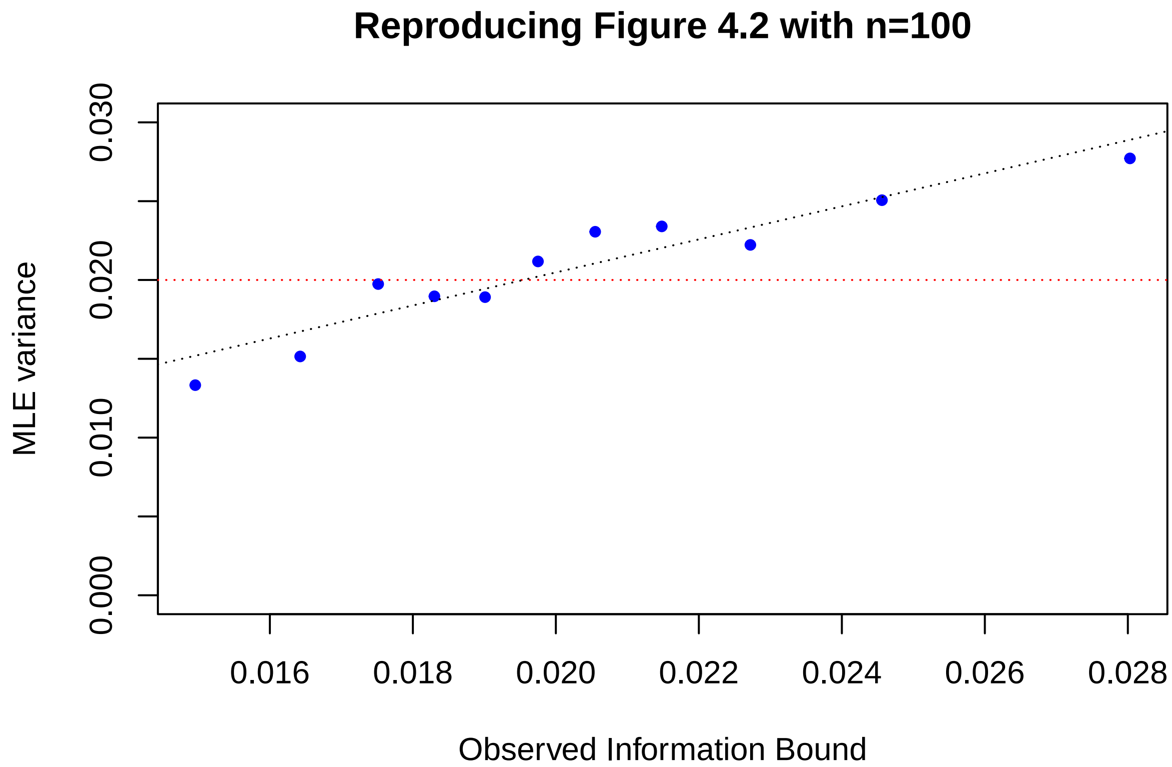
```
##                MLE Obs.Info.Bound
## 1671  0.012069513     0.01250294
## 6269  0.108151029     0.01275402
## 8788 -0.131474024     0.01285321
## 1807  0.025828327     0.01293256
## 1557 -0.112150663     0.01294998
## 2823  0.002080506     0.01304947
## 1002 -0.065360000     0.01316855
## 7944 -0.033495051     0.01317008
## 4780 -0.050738197     0.01322443
## 7132  0.237019746     0.01325483
```

```r
S=splitter(X)

# Recreate Figure 4.2 with n = 20
plot(S$med_infobound, S$var_theta, col='blue', pch=20,
     xlab='Observed Information Bound', ylab='MLE variance', main='Reproducing Figure 4.2 with n=100',
     ylim=c(0,round(max(S$var_theta),2))
     )

# CRLB(theta) which does not depend on the data
unconditional.variance= 1/ (1/2 * n)
abline(h=unconditional.variance, lty='dotted', col='red')

# linear regression fit of  Var(theta) ~ Med(Obs.Info.Bound)
fit<-lm(var_theta~med_infobound, data=S)
abline(fit$coefficients, lty='dotted')
```


Reproducing Figure 4.2 with n=100
MLE variance
0.000
0.010
0.020
0.030
0.016
0.018
0.020
0.022
0.024
0.026
0.028
Observed Information Bound