Advanced Statistical Methods HW5

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Exercise 6.1

1. Suppose that instead of the Poisson model (6.1), we assume a binomial model

$$\Pr\{x_k = x\} = \binom{n}{x}\theta_k^x(1-\theta_k)^{n-x},$$

n some fixed and known integer such as $n = 10$. What is the equivalent of Robbins' formula (6.5)?

$$E[\theta|x] = \frac{\int_0^\infty \theta p_\theta(x)g(\theta)\,d\theta}{\int_0^\infty p_\theta(x)g(\theta)\,d\theta}$$
$$f(x) = \int_0^\infty p_\theta(x)g(\theta)\,d\theta$$
$$\text{Robbin's Formula} \quad E[\theta|x] = \frac{(x+1)f(x+1)}{f(x)}$$

Robbin's Formula comes out from the model $X|\theta \sim Poi(\theta)$
Now, assume a new model $X|\theta \sim B(n,\theta)$ with $n = 10$

$$E[\theta|x] = \frac{\int_0^1 \theta p_\theta(x)g(\theta)\,d\theta}{\int_0^1 p_\theta(x)g(\theta)\,d\theta}$$

$$f(x) = \int_0^1 p_\theta(x)g(\theta)\,d\theta = \int_0^1 \binom{10}{x}\theta^x(1-\theta)^{10-x}g(\theta)\,d\theta$$

$$\begin{aligned}
\int_0^1 \theta p_\theta(x)g(\theta)\,d\theta &= \int_0^1 \binom{10}{x}\theta^{x+1}(1-\theta)^{10-x}g(\theta)\,d\theta \\
&= \int_0^1 \binom{10}{x}\theta^{x+1}(1-\theta)^{10-(x+1)}(1-\theta)g(\theta)\,d\theta \\
&= \int_0^1 \binom{10}{x}\theta^{x+1}(1-\theta)^{10-(x+1)}g(\theta)\,d\theta - \int_0^1 \binom{10}{x}\theta^{x+2}(1-\theta)^{10-(x+1)}g(\theta)\,d\theta \\
&= \frac{\binom{10}{x}}{\binom{10}{x+1}}f(x+1) - \int_0^1 \binom{10}{x}\theta^{x+2}(1-\theta)^{10-(x+2)}(1-\theta)g(\theta)\,d\theta \\
&= \frac{\binom{10}{x}}{\binom{10}{x+1}}f(x+1) - \frac{\binom{10}{x}}{\binom{10}{x+2}}f(x+2) + \int_0^1 \binom{10}{x}\theta^{x+3}(1-\theta)^{10-(x+2)}g(\theta)\,d\theta \\
&= \cdots
\end{aligned}$$

The last term of above will be given by

$$\pm\int_0^1 \binom{10}{x}\theta^{10}(1-\theta)^1 g(\theta)\,d\theta = \pm\frac{\binom{10}{x}}{\binom{10}{10}}\int_0^1 \binom{10}{10}\theta^{10}g(\theta)\,d\theta \mp \binom{10}{x}\int_0^1 \theta^{11}g(\theta)\,d\theta$$

Here, we can say that

$$\binom{10}{x}\int_0^1 \theta^{11} g(\theta)\,d\theta = \frac{\binom{10}{x}}{\binom{11}{11}}\int_0^1 \binom{11}{11}\theta^{11} g(\theta)\,d\theta \approx \binom{10}{x} f(11) = 0$$

Since $x \in \{1, \cdots, n = 10\}$, we can say $f(11) = 0$
Therefore,

$$\begin{aligned}
\int_0^1 \theta p_\theta(x) g(\theta)\,d\theta &= \frac{\binom{10}{x}}{\binom{10}{x+1}} f(x+1) - \frac{\binom{10}{x}}{\binom{10}{x+2}} f(x+2) + \cdots \pm \binom{10}{x} f(10) \\
&= \sum_{k=1}^{10-x} (-1)^{k-1} \frac{\binom{10}{x}}{\binom{10}{x+k}} f(x+k)
\end{aligned}$$

$$\begin{aligned}
E[\theta|x] &= \frac{\int_0^1 \theta p_\theta(x) g(\theta)\,d\theta}{\int_0^1 p_\theta(x) g(\theta)\,d\theta} \\
&= \frac{\sum_{k=1}^{10-x} (-1)^{k-1} \frac{\binom{10}{x}}{\binom{10}{x+k}} f(x+k)}{f(x)}
\end{aligned}$$

This is the equivalent of Robbin's formula for binomial model.

Exercise 6.2

2. Define $V\{\theta \mid x\}$ as the variance of θ given x. In the Poisson situation (6.1), show that

$$V\{\theta \mid x\} = E\{\theta \mid x\} \cdot \left(E\{\theta \mid x+1\} - E\{\theta \mid x\}\right),$$

where $E\{\theta \mid x\}$ is as given in (6.5).

By the definition of conditional variance,

$$V(\theta|x) = E[\theta^2|x] - E[\theta|x]^2$$

By Robbin's formula,

$$E[\theta|x] = \frac{(x+1)f(x+1)}{f(x)}$$

Similarly,

$$E[\theta^2|x] = \frac{\int_0^\infty \theta^2 p_\theta(x) g(\theta)\,d\theta}{\int_0^\infty p_\theta(x) g(\theta)\,d\theta}$$

$$\int_0^\infty \theta^2 p_\theta(x) g(\theta)\,d\theta = \int_0^\infty \frac{e^{-\theta}\theta^{x+2}}{x!} g(\theta)\,d\theta = (x+1)(x+2)f(x+2)$$

$$\Rightarrow\ E[\theta^2|x] = \frac{(x+1)(x+2)f(x+2)}{f(x)}$$

$$E[\theta|x] \cdot E[\theta|x+1] = \frac{(x+1)f(x+1)}{f(x)} \frac{(x+2)f(x+2)}{f(x+1)} = \frac{(x+1)(x+2)f(x+2)}{f(x)} = E[\theta^2|x]$$

Therefore,

$$V(\theta|x) = E[\theta^2|x] - E[\theta|x]^2 = E[\theta|x] \cdot E[\theta|x+1] - E[\theta|x]^2 = E[\theta|x] \cdot (E[\theta|x+1] - E[\theta|x])$$

Exercise 6.3

3. Instead of (6.8), assume $g(\theta) = (1/\sigma)e^{-\theta/\sigma}$ for $\theta > 0$.

   (a) Numerically find the maximum likelihood estimate $\hat{\sigma}$ for the Poisson model (6.1) fit to the count data in Table 6.1.

   (b) Calculate the estimates of $\widehat{E}\{\theta \mid x\}$, as in the third row of Table 6.1.

6.3-(a) Assume the model below :

$$\theta \sim Exp(\sigma) \qquad x|\theta \sim Poi(\theta)$$

We will derive the marginal distribution of x given σ

$$\begin{aligned}
g_\sigma(\theta) &= \frac{1}{\sigma}\exp(-\frac{\theta}{\sigma}) \\
p(x|\theta) &= \frac{\exp(-\theta)\theta^x}{x!} \\
g_\sigma(\theta)p(x|\theta) &= \frac{1}{\sigma}\frac{1}{x!}\theta^x \exp(-\frac{\theta}{\gamma}) \quad \text{where} \quad \gamma = \frac{\sigma}{\sigma+1} \quad \text{and} \quad \sigma = \frac{\gamma}{1-\gamma} \\
f_\sigma(x) &= \int_0^\infty g_\sigma(\theta)p(x|\theta) = \frac{1}{\sigma}\frac{1}{x!}\int_0^\infty \theta^x \exp(-\frac{\theta}{\gamma})\,d\theta \\
&= \frac{1}{\sigma}\frac{1}{x!}\Gamma(x+1)\gamma^{x+1} = \frac{1}{\sigma}\gamma^{x+1} = \frac{1-\gamma}{\gamma}\gamma^{x+1}
\end{aligned}$$

We can label $f_\sigma(x)$ by $f_\gamma(x)$ since $\sigma \mapsto \gamma$ is one to one transform. To get MLE of σ, we shall yield MLE of γ and then take inverse transform.

$$\begin{aligned}
\text{loglikelihood} &= \sum_{i=1}^{N} \log f_\gamma(x) = \sum_{x=0}^{x_{max}} y_x \log f_\gamma(x) \\
\hat{\gamma} &= \text{argmax}_\gamma \sum_{x=0}^{x_{max}} y_x \log f_\gamma(x) \\
\log f_\gamma(x) &= \log\frac{1-\gamma}{\gamma} + (x+1)\log\gamma \\
\frac{\partial}{\partial\gamma}\log f_\gamma(x) &= -\frac{1}{\gamma(1-\gamma)} + \frac{x+1}{\gamma} \\
\frac{\partial^2}{\partial\gamma^2}\log f_\gamma(x) &= \frac{1-2\gamma}{(1-\gamma)^2\gamma^2} - \frac{x+1}{\gamma^2} = \frac{1}{\gamma^2} - \frac{1}{(1-\gamma)^2} - \frac{x}{\gamma^2} - \frac{1}{\gamma^2} \\
&= -\frac{1}{(1-\gamma)^2} - \frac{x}{\gamma^2} < 0
\end{aligned}$$

Note that by the last inequality, loglikelihood function is strictly concave, which makes it easy to find MLE.

```
x=c(0,1,2,3,4,5,6,7)
y=c(7840, 1317, 239, 42, 14, 4, 4, 1)
claim=as.data.frame(cbind(x,y))
claim
```

```
##   x    y
## 1 0 7840
## 2 1 1317
## 3 2  239
## 4 3   42
```

```
## 5 4   14
## 6 5    4
## 7 6    4
## 8 7    1
```

Here we reproduce the table 6.1 in the textbook. Now we shall make a R function calculating loglikelihood and derivative of loglikelihood.

```
# loglikelihood as a function of gamma
logf<-function(gamma){
  z=c(0)
  for(i in 1:length(x)){
    z[i]=log((1-gamma)/gamma)+(x[i]+1)*log(gamma)
  }
  return(sum(z*y))
}

# derivative of loglikelihood as a function of gamma
logfp<-function(gamma){
  z=c(0)
  for(i in 1:length(x)){
    z[i]=-1/(gamma*(1-gamma))+(x[i]+1)/gamma
  }
  return(sum(z*y))
}
```

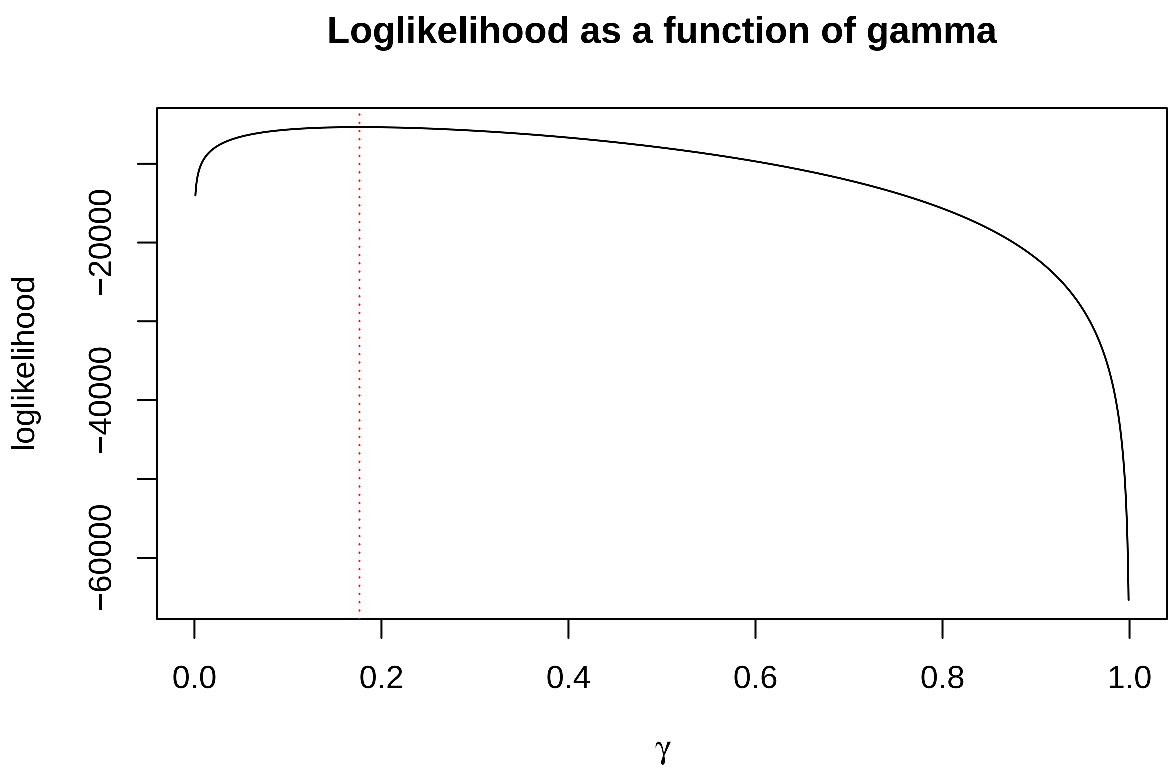
To derive a MLE $\hat{\gamma}$, we need an initial value to start numerical algorithm.
Since $\sigma > 0$ and $\gamma = \frac{\sigma}{\sigma+1}$, we have $0 < \gamma < 1$.
Also, since $E[x|\theta] = \theta$ and $E[x] = E[E[x|\theta]] = E[\theta] = \sigma$, we can use $\overline{X}$ as an estimate of σ. Then, plugin estimator $\frac{\overline{X}}{\overline{X}+1}$ would be an initial value to start numerical algorithm to find $\hat{\gamma}$.

```
N=sum(y)
samplemean=sum(x*y)/N
initialvalue=samplemean/(1+samplemean)
logfp(initialvalue)
```

```
## [1] 5.449863e-12
```

Surprisingly, if we plug in $\gamma = \frac{\overline{X}}{\overline{X}+1}$ on $\frac{\partial}{\partial\gamma}$loglikelihood, we have nearly zero value (magnitude is an order of 10^{-12}). Drawing loglikelihood as a function of gamma gives us a clear sight that this value is indeed an MLE of γ.

```
gammas=seq(from=0, to=1, by=0.001)
loglikelihood=0
for(i in 1:length(gammas)){
  loglikelihood[i]=logf(gammas[i])
}
plot(gammas, loglikelihood, type='l', xlab=expression(gamma))
title(main="Loglikelihood as a function of gamma")
abline(v=initialvalue, col='red', lty='dotted')
```



The red dotted line indicates the γ value equal to $\frac{\overline{X}}{\overline{X}+1}$. Therefore, $\hat{\gamma} = \frac{\overline{X}}{\overline{X}+1}$ and $\hat{\sigma} = \overline{X}$

```
(sigma_hat=samplemean)
```

```
## [1] 0.2143537
```

As a result, we derive $\hat{\sigma} = 0.2144$

6.3-(b) Using Robbin's formula, we will now calculate the estimates $\widehat{E}[\theta|x]$ as in the third row of table 6.1

$$\widehat{E}[\theta|x] = \frac{(x+1)f_{\hat{\sigma}}(x+1)}{f_{\hat{\sigma}}(x)} f_{\sigma}(x) = \frac{1}{\sigma}\gamma^{x+1} = \frac{1}{\sigma}(\frac{\sigma}{\sigma+1})^{x+1}$$

```
f.sigma_hat<-function(x){
  (sigma_hat/(1+sigma_hat))^(x+1) /sigma_hat
}
# yield estimate of E[theta|x] using sigma_hat and Robbin's formula
Robbin<-function(x){
  (x+1)*f.sigma_hat(x+1)/f.sigma_hat(x)
}

# Reproduce table 6.1 with Gamma MLE replaced by Exponential MLE
Exp_MLE=rep(NaN, length(x))
for(i in 1:length(x)){
  Exp_MLE[i]=Robbin(x[i])
}
claim=cbind(claim, Exp_MLE)
claim
```

```
##   x    y   Exp_MLE
## 1 0 7840 0.1765167
## 2 1 1317 0.3530333
```

```
## 3 2  239 0.5295500
## 4 3   42 0.7060667
## 5 4   14 0.8825833
## 6 5    4 1.0591000
## 7 6    4 1.2356167
## 8 7    1 1.4121333
```

Exercise 6.7

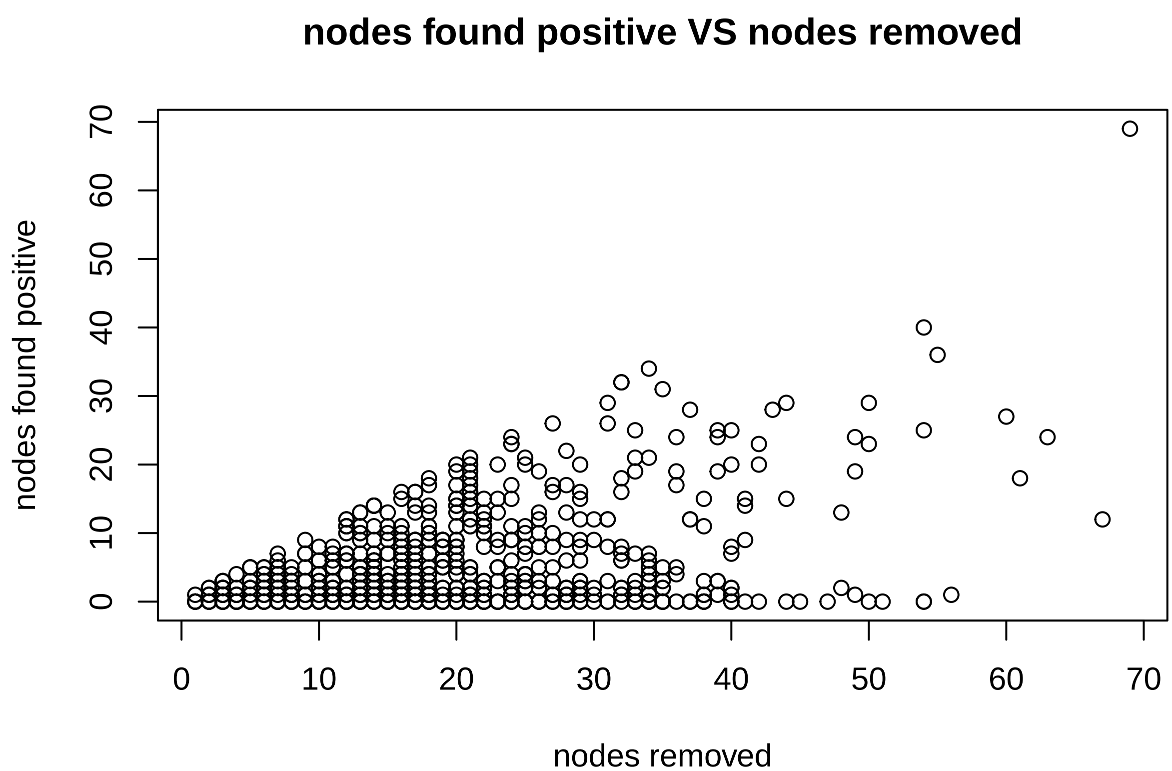
7. The nodes data of Section 6.3 consists of 844 pairs (n_i, x_i).

   (a) Plot x_i versus n_i.

   (b) Perform a cubic regression of x_i versus n_i and add it to the plot.

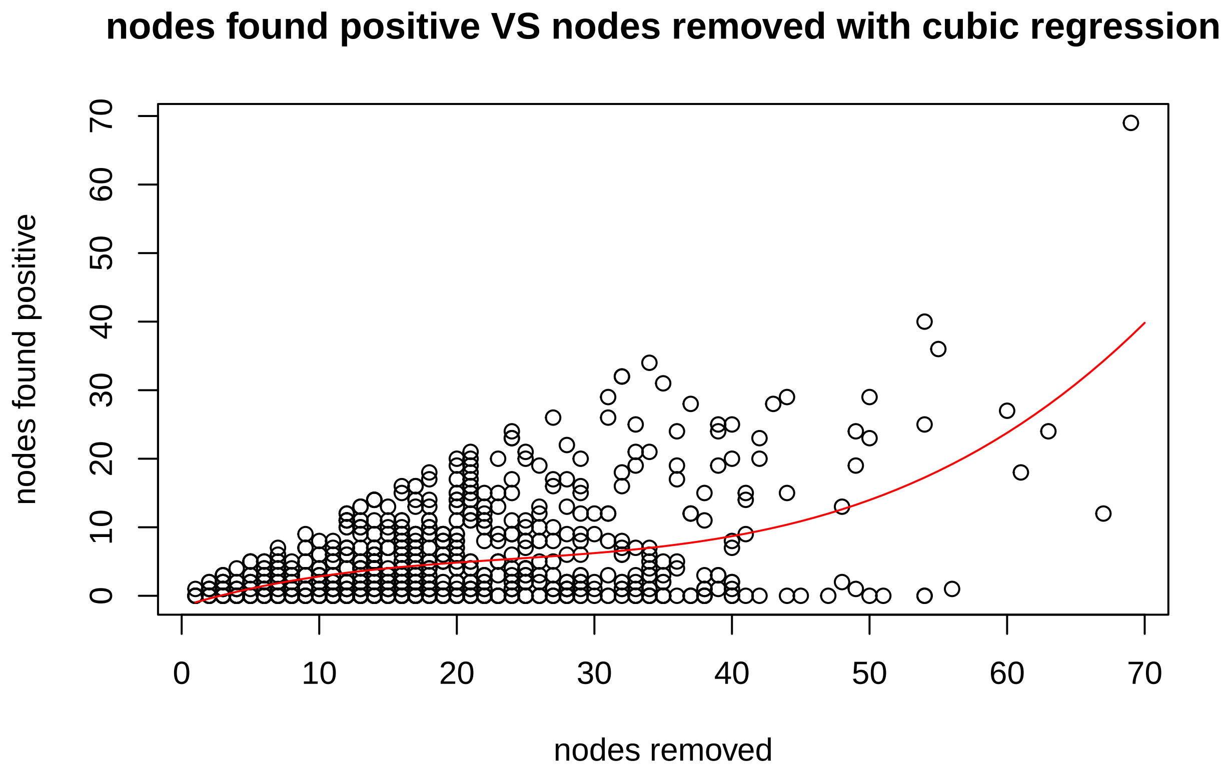
   (c) What would you expect the plot to look like if the values of n_i were assigned randomly before surgery?

```
nodes=read.table('https://web.stanford.edu/~hastie/CASI_files/DATA/nodes.txt', header=T)
plot(nodes$n,nodes$x, xlab='nodes removed', ylab='nodes found positive')
title(main="nodes found positive VS nodes removed")
```




```
# Formula for cubic linear regression model
form= as.formula(paste("x ~ ",paste("I( n^", seq(from = 1, to = 3, by = 1), ")",collapse = "+")))
# Calculate beta coefficients for cubic linear regression model
cubic=lm(form, data=nodes)
# Fitted value for n_i's
fit=0
for(i in 1:70){
  fit[i]=sum(c(1, i, i^2, i^3)*cubic$coef)
```

```
}
plot(nodes$n,nodes$x, xlab='nodes removed', ylab='nodes found positive')
# Red line added to represent cubic regression
lines(fit, col='red')
title(main="nodes found positive VS nodes removed with cubic regression")
```



6.7-(c) Notice that in the above plot, n_i values are concentrated on small values of n_i's, which are on $[0, 30]$. If the values of n_i were assigned randomly before surgery then we can expect that n_i values on the plot may be more scattered than the above. But it may not affect the shape of regression fit of x_i versus n_i.

Problem2

2. Show that the marginal distribution of "x" (in the missing species problem is negative binomial.

In the missing species problem, we assumed that

$$x|\theta \sim Poi(\theta) \qquad \theta \sim \Gamma(\nu, \sigma) \quad \text{where} \quad \nu, \sigma > 0$$

We can derive marginal density of x as below :

$$
\begin{aligned}
g_{\nu,\sigma}(\theta) &= \frac{1}{\Gamma(\nu)\sigma^{\nu}}\theta^{\nu-1}\exp(-\frac{\theta}{\sigma}) \\
p(x|\theta) &= \frac{\exp(-\theta)\theta^{x}}{x!} \\
g_{\nu,\sigma}(\theta)p(x|\theta) &= \frac{1}{\Gamma(\nu)\sigma^{\nu}}\frac{1}{x!}\theta^{x+\nu-1}\exp(-\frac{\theta}{\gamma}) \quad \text{where} \quad \gamma = \frac{\sigma}{\sigma+1} \quad \text{and} \quad \sigma = \frac{\gamma}{1-\gamma} \\
f_{\nu,\sigma}(x) &= \int_0^{\infty} g_{\nu,\sigma}(\theta)p(x|\theta) = \frac{1}{\Gamma(\nu)\sigma^{\nu}}\frac{1}{x!}\int_0^{\infty}\theta^{x+\nu-1}\exp(-\frac{\theta}{\gamma})\,d\theta \\
&= \frac{1}{\Gamma(\nu)\sigma^{\nu}}\frac{1}{x!}\Gamma(x+\nu)\gamma^{x+\nu} = \frac{1}{\Gamma(\nu)}\frac{1}{x!}\Gamma(x+\nu)(\frac{1-\gamma}{\gamma})^{\nu}\gamma^{x+\nu} \\
&= \frac{\Gamma(x+\nu)}{\Gamma(\nu)x!}(1-\gamma)^{\nu}\gamma^{x}
\end{aligned}
$$

Note that $0 < \gamma < 1$. Assuming ν is positive integer, we have

$$
f_{\nu,\sigma}(x) = \frac{\Gamma(x+\nu)}{\Gamma(\nu)x!}(1-\gamma)^{\nu}\gamma^{x} = \frac{(x+\nu-1)!}{(\nu-1)!x!}(1-\gamma)^{\nu}\gamma^{x} = \binom{x+\nu-1}{\nu-1}(1-\gamma)^{\nu}\gamma^{x} \qquad x = 0,1,\cdots
$$

which is the probability mass function of $NegBin(\nu, 1-\gamma)$ where ν is the number of successes, x is the number of failures, and $1-\gamma$ is the probability of success.

Problem 3

The problem is to show that

$$
E(t) = e_1\frac{1-(1+\gamma t)^{-\nu}}{\gamma\nu} \quad \text{where} \quad \gamma = (1/\sigma+1)^{-1}
$$

Our model is given as

$$
x_k \sim Poi(\theta_k) \quad x_k(t) \sim Poi(\theta_k t) \quad \forall\, k = 1,\cdots,S
$$

where x_k and $x_k(t)$ are independent. To use parametric Bayes, we assume gamma prior for $\theta \sim \Gamma(\nu,\sigma)$ $E(t)$ is the expected number of new species after time t and $e_x = E[y_x]$ where y_x represents the number of species which is observed exactly x times.

$$
\begin{aligned}
E(t) &= \sum_{k=1}^{S} P(x_k = 0, x_k(t) > 0) = \sum_{k=1}^{S}\int_0^{\infty} P(x_k = 0, x_k(t) > 0 \,|\, \theta_k)g(\theta_k)\,d\theta_k \\
&= \sum_{k=1}^{S}\int_0^{\infty} e^{-\theta_k}(1-e^{-\theta_k t})g(\theta_k)\,d\theta_k = S\int_0^{\infty} e^{-\theta}(1-e^{-\theta t})g(\theta)\,d\theta \\
e_x &= E\Big[\sum_{k=1}^{S} I(x_k = x)\Big] = \sum_{k=1}^{S} P(x_k = x) = \sum_{k=1}^{S}\int_0^{\infty} P(x_k = x \,|\, \theta_k)g(\theta_k)\,d\theta_k \\
&= \sum_{k=1}^{S}\int_0^{\infty}\frac{e^{-\theta_k}\theta_k^{x}}{x!}g(\theta_k)\,d\theta_k = S\int_0^{\infty}\frac{e^{-\theta}\theta^{x}}{x!}g(\theta)\,d\theta \\
e_1 &= S\int_0^{\infty}\theta e^{-\theta}g(\theta)\,d\theta \\
\frac{E(t)}{e_1} &= \frac{S\int_0^{\infty} e^{-\theta}(1-e^{-\theta t})g(\theta)\,d\theta}{S\int_0^{\infty}\theta e^{-\theta}g(\theta)\,d\theta} = \frac{\int_0^{\infty} e^{-\theta}g(\theta)\,d\theta - \int_0^{\infty} e^{-\theta(1+t)}g(\theta)\,d\theta}{\int_0^{\infty}\theta e^{-\theta}g(\theta)\,d\theta}
\end{aligned}
$$

Now plug in

$$
g(\theta) = g_{\nu,\sigma}(\theta) = \frac{1}{\Gamma(\nu)\sigma^{\nu}}\theta^{\nu-1}\exp(-\frac{\theta}{\sigma})
$$

Then

$$
\begin{aligned}
\int_0^\infty e^{-\theta} g(\theta)\, d\theta &= \int_0^\infty \frac{1}{\Gamma(\nu)\sigma^\nu} \theta^{\nu-1} \exp(-\frac{\theta}{\gamma}) = \frac{\Gamma(\nu)\gamma^\nu}{\Gamma(\nu)\sigma^\nu} = (\frac{\gamma}{\sigma})^\nu = (1-\gamma)^\nu \\
\int_0^\infty \theta e^{-\theta} g(\theta)\, d\theta &= \int_0^\infty \frac{1}{\Gamma(\nu)\sigma^\nu} \theta^{(\nu+1)-1} \exp(-\frac{\theta}{\gamma}) = \frac{\Gamma(\nu+1)\gamma^{\nu+1}}{\Gamma(\nu)\sigma^\nu} = \nu(\frac{\gamma}{\sigma})^\nu \gamma = \gamma\nu(1-\gamma)^\nu \\
\int_0^\infty e^{-\theta(1+t)} g(\theta)\, d\theta &= \int_0^\infty \frac{1}{\Gamma(\nu)\sigma^\nu} \theta^{\nu-1} \exp(-\frac{\theta}{\gamma/(1+\gamma t)}) = \frac{\Gamma(\nu)(\frac{\gamma}{1+\gamma t})^\nu}{\Gamma(\nu)\sigma^\nu} = \Big(\frac{1}{1+\gamma t}\Big)^\nu (1-\gamma)^\nu
\end{aligned}
$$

$$
\begin{aligned}
&\frac{E(t)}{e_1} = \frac{(1-\gamma)^\nu - \Big(\frac{1}{1+\gamma t}\Big)^\nu (1-\gamma)^\nu}{\gamma\nu(1-\gamma)^\nu} = \frac{1-(1+\gamma t)^{-\nu}}{\gamma\nu} \\
&E(t) = e_1 \frac{1-(1+\gamma t)^{-\nu}}{\gamma\nu}
\end{aligned}
$$