Probability theory II Assignment 2

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Exercises
Section 4.2 Martinagles, Almost Sure Convergence
Section 4.3 Examples
Section 4.4 Doob's Inequality, Convergence in L^p

4.2.1 **Suppose X_n is a martingale w.r.t. $\mathcal{G}_n$ and let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$. Then $\mathcal{G}_n \supset \mathcal{F}_n$ and X_n is a martingale w.r.t. $\mathcal{F}_n$.**

Proof. Since X_n is a martingale w.r.t $\mathcal{G}_n$, X_n is integrable. Also, $X_n \in \mathcal{G}_n \quad \forall n \in \mathbb{N}$. Define a sequence of collections $\{\mathcal{A}_n\}_n$ as

$$\mathcal{A}_n = \{(X_j \in B) : j = 1, \cdots, n \text{ and } B \in \mathcal{B}(\mathbb{R})\}$$

Then we have $\mathcal{F}_n = \sigma(\mathcal{A}_n) \quad \forall n \in \mathbb{N}$. Take any $B \in \mathcal{B}(\mathbb{R})$ and $j \in \{1, \cdots, n\}$. Since $X_n \in \mathcal{G}_n$, we have $(X_j \in B) \in \mathcal{G}_j \subset \mathcal{G}_n$. It implies that $\mathcal{A}_n \subset \mathcal{G}_n \quad \forall n \in \mathbb{N}$. Furthermore , since $\mathcal{G}_n$ is a σ-field, we get $\sigma(\mathcal{A}_n) \subset \mathcal{G}_n \quad \forall n \in \mathbb{N}$. Therefore $\mathcal{F}_n \subset \mathcal{G}_n \quad \forall n \in \mathbb{N}$.
Now we should show that X_n is a martingale w.r.t $\mathcal{F}_n$. As we said before, X_n is integrable. Also, $X_n \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$ by the definition of $\mathcal{F}_n$. Finally, $E[X_{n+1}|\mathcal{F}_n] = X_n \quad \forall n \in \mathbb{N}$ since

$$\begin{aligned} E[X_{n+1}|\mathcal{F}_n] &= E[E[X_{n+1}|\mathcal{G}_n]\big|\mathcal{F}_n] \quad \because\ \mathcal{F}_n \subset \mathcal{G}_n \quad \text{smoothing property} \\ &= E[X_n|\mathcal{F}_n] \quad \because\ X_n \text{ is martingale w.r.t } \mathcal{G}_n \\ &= X_n \quad \because\ X_n \in \mathcal{F}_n \end{aligned}$$

Thus, X_n is a martingale w.r.t $\mathcal{F}_n$ □

Remark. *If X_n is a martingale then $\mathcal{F}_n$ defined by $\mathcal{F}_n = \sigma(X_1, \cdots, X_n)$ is the smallest filtration that makes X_n a martingale.*

4.2.4 **Let X_n, $n \geq 0$, be a submartingale with $\sup X_n < \infty$. Let $\xi_n = X_n - X_{n-1}$ and suppose $E(\sup \xi_n^+) < \infty$. Show that X_n converges a.s.**

Proof. Take $M \in \mathbb{N}$. Define $N = \inf\{n \in \mathbb{N} \cup \{0\} : X_n > M\}$. Then N is a stopping time.

$$\because (N = n) = (X_0 \leq M) \cap \cdots \cap (X_{n-1} \leq M) \cap (X_n > M) \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$$

Consider $X_{N \wedge n}$. Since X_n is a martingale and N is a stopping time, $X_{N \wedge n}$ is also a submartingale.

i. If $n < N$
   $X_{N \wedge n} = X_n$ and $X_n \leq M$. Also $X_n^+ \leq M$

ii. Else, if $n \geq N$
   $X_{N \wedge n} = X_N > M$. Note that $X_{N-1} \leq M$ and $\xi_N = X_N - X_{N-1}$. Hence we have
   $X_N = X_{N-1} + \xi_N \leq M + \xi_N^+ \leq M + \sup_n \xi_n^+ \quad \forall n \in \mathbb{N}$ and also $X_N^+ \leq M + \sup_n \xi_n^+ \quad \forall n \in \mathbb{N}$

Combining two cases , we have $X_{N \wedge n}^+ \leq M + \sup_n \xi_n^+ \quad \forall n \in \mathbb{N}$.
Taking expectation, we have $E[X_{N \wedge n}^+] \leq M + E[\sup_n \xi_n^+] \quad \forall n \in \mathbb{N}$.
By taking supremum, we get $\sup_n E[X_{N \wedge n}^+] \leq M + E[\sup_n \xi_n^+]$
Since $E[\sup_n \xi_n^+] < \infty$ by assumption, we have $\sup_n E[X_{N \wedge n}^+] < \infty$. By the submartingale convergence theorem, $X_{N \wedge n}$ converges $a.s.$
Note that if $N = \infty$ then $X_{N \wedge n} = X_n \quad \forall n \in \mathbb{N}$. Hence X_n converges $a.s.$ on $(N = \infty)$

We have taken arbitrary $M \in \mathbb{N}$ for defining a stopping time N. Hence we can write it as N_M to emphasize that it depends on the value of M . Then for each $M \in \mathbb{N}$, we have

$$(N_M = \infty) = (X_n \leq M \quad \forall n \in \mathbb{N}) = \Big(\sup_n X_n \leq M\Big)$$

Therefore $P\Big(\sup_n X_n \leq M\Big) = P(N_M = \infty) \quad \forall\ M \in \mathbb{N}$. Observe that $\Big(\sup_n X_n \leq M\Big)$ is increasing sequence of events w.r.t M . i.e. $\Big(\sup_n X_n \leq M\Big) \subset \Big(\sup_n X_n \leq M+1\Big) \quad \forall\ M \in \mathbb{N}$. Thus $(N_M = \infty)$ is also increasing sequence of events w.r.t M
By assumption $\sup_n X_n < \infty\ a.s.$,

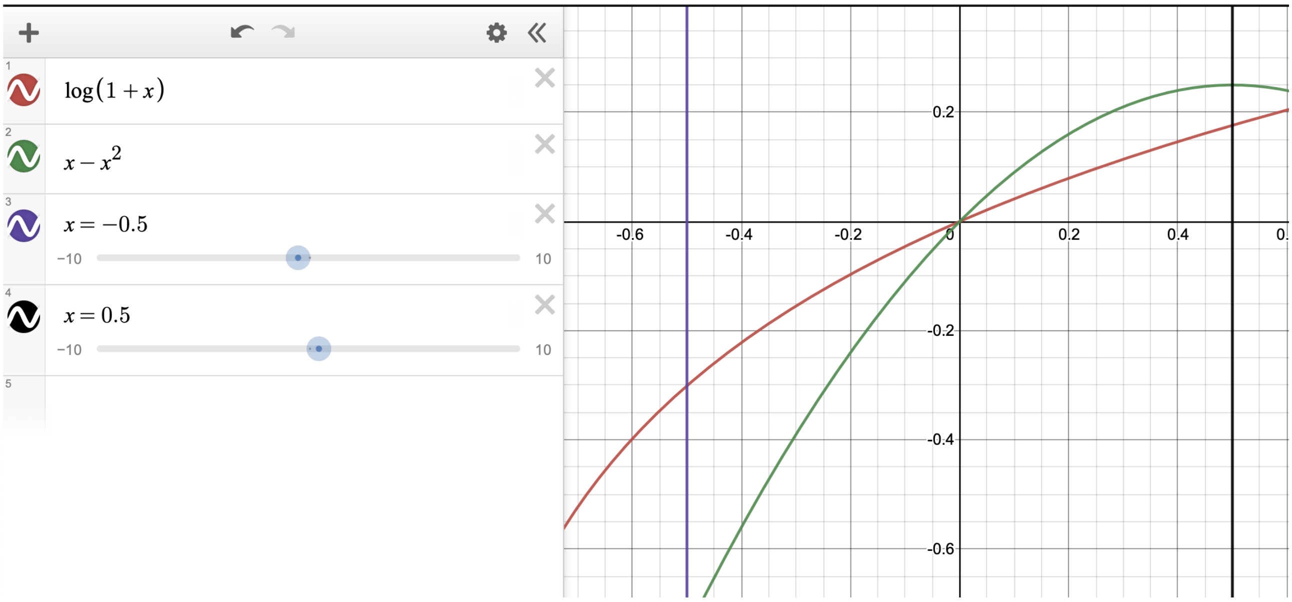
$$1 = P\Big(\sup_n X_n < \infty\Big) = P\Big(\bigcup_{M \in \mathbb{N}} \big(\sup_n X_n \leq M\big)\Big) = \lim_{M \to \infty} P\Big(\sup_n X_n \leq M\Big)$$

due to continuity from below of probability measure. Combining the results above, we have

$$1 = \lim_{M \to \infty} P(N_M = \infty) = P\Big(\bigcup_{M \in \mathbb{N}} (N_M = \infty)\Big)$$

Let $C = \Big(\bigcup_{M \in \mathbb{N}} (N_M = \infty)\Big)$. Since X_n converges $a.s.$ on $(N_M = \infty)$ for every $M \in \mathbb{N}$, we have X_n converges $a.s.$ on C with $P(C) = 1$. Therefore X_n converges $a.s.$ $\square$

4.2.7 Suppose $y_n > -1$ for all n and $\sum |y_n| < \infty$. Show that $\prod_{m=1}^{\infty}(1 + y_m)$ exists.



Proof. We shall use the result $|\log(1+x)| \leq |x - x^2| \quad \forall\ |x| \leq \frac{1}{2} \quad \cdots (*)$ by the visual result of graphing device (Desmos) .
From the assumption $\sum_n |y_n| < \infty$, we have $y_n \to 0$ as $n \to \infty$. Thus $\exists\ M \in \mathbb{N}$ s.t.

$|y_n| \leq \frac{1}{2} \quad \forall\ n \geq M$ which implies $y_n^2 \leq \frac{1}{4} \quad \forall\ n \geq M$. Since $\sum_n \left(\frac{1}{4}\right)^n$ converges , $\sum_n y_n^2$ also converges by the comparison test. Now take $N \in \mathbb{N}$ s.t. $N \geq M$ and observe that

$$\Big|\sum_{n>N} \log(1+y_n)\Big| \leq \sum_{n>N} |\log(1+y_n)| \leq \sum_{n>N} |y_n - y_n^2| \leq \sum_{n>N} |y_n| + \sum_{n>N} y_n^2 \longrightarrow 0 \quad \text{as} \quad N \to \infty$$

The second inequality comes from the two fact : $|y_n| \leq \frac{1}{2} \quad \forall\ n \geq M$ and the inequality $(*)$. The convergence to zero is derived from the fact that $\sum_n |y_n|$ and $\sum_n y_n^2$ are both convergent series and the tail part of the convergent series tends to zero.
Therefore $\sum_n \log(1+y_n)$ converges to a finite number and $\prod_n (1+y_n) = \exp(\sum_n \log(1+y_n))$ also converges to a finite number □

4.2.8 **Let X_n and Y_n be positive integrable and adapted to $\mathcal{F}_n$. Suppose**

$$E(X_{n+1}|\mathcal{F}_n) \leq (1+Y_n)X_n$$

with $\sum Y_n < \infty$ a.s. Prove that X_n converges a.s. to a finite limit by finding a closely related supermartingale to which Theorem 4.2.12 can be applied.

Proof. We shall define a nonnegative supermartingale to utilize supermartingale convergence thm.

$$W_n := \frac{X_n}{\prod_{m=1}^{n-1}(1+Y_m)}$$

Since X_n and Y_n are positive , W_n is also positive. Also , $X_n \in \mathcal{F}_n$ and $\prod_{m=1}^{n-1}(1+Y_m) \in \mathcal{F}_{n-1} \subset \mathcal{F}_n$, we have $W_n \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$. Note that $(1+Y_n) > 1$ so that $\prod_{m=1}^{n}(1+Y_m) > 1 \quad \forall n \in \mathbb{N}$

$$E(W_n) = E\Big[\frac{X_n}{\prod_{m=1}^{n-1}(1+Y_m)}\Big] \leq E[X_n] < \infty \quad \forall n \in \mathbb{N}$$

The last inequality $E(X_n) < \infty$ comes from the assumption that X_n is integrable. Thus W_n is integrable. To show W_n is a supermartingale, it suffices to show that $E[W_{n+1}|\mathcal{F}_n] \leq W_n$

$$\begin{aligned} E[W_{n+1}|\mathcal{F}_n] &= E\Big[\frac{X_{n+1}}{\prod_{m=1}^{n}(1+Y_m)}|\mathcal{F}_n\Big] \\ &= \frac{1}{\prod_{m=1}^{n}(1+Y_m)} E[X_{n+1}|\mathcal{F}_n] \quad \because \ \frac{1}{\prod_{m=1}^{n}(1+Y_m)} \in \mathcal{F}_n \\ &\leq \frac{1}{\prod_{m=1}^{n}(1+Y_m)}(1+Y_n)X_n \quad \because \text{By assumption} \quad E[X_{n+1}|\mathcal{F}_n] \leq (1+Y_n)X_n \\ &= \frac{1}{\prod_{m=1}^{n-1}(1+Y_m)} X_n = W_n \end{aligned}$$

Therefore, W_n is a nonnegative supermartingale. By supermartingale convergence thm ,
$W_n \to W$ $a.s.$ for some integrable r.v. W. Note that

$$X_n = W_n \prod_{m=1}^{n-1}(1+Y_m)$$

Since $Y_n > 0$ and $\sum_n Y_n < \infty$ $a.s.$ by the assumption, we can apply the result of the previous exercise so that $\prod_n (1+Y_n)$ converges $a.s.$. In other words, $\prod_n (1+Y_n) \to Z$ $a.s.$ for some r.v. Z . Then we can conclude that $X_n \to WZ$ $a.s.$. Note that Z is finite by the result of the previous exercise and W is finite $a.s.$ since it is nonnegative and integrable. Hence WZ is finite $a.s.$ □

4.3.1 Give an example of a martingale X_n with $\sup_n |X_n| < \infty$ and $P(X_n = a \text{ i.o.}) = 1$ for $a = -1, 0, 1$. This example shows that it is not enough to have $\sup |X_{n+1} - X_n| < \infty$ in Theorem 4.3.1.

Proof. Let $\{U_n\}_n$ be i.i.d random sequence with $U_1 \sim Unif(0,1)$ and $\mathcal{F}_n := \sigma(U_1, \cdots, U_n)$
Let $X_0 = 0$ and define X_n for each $n \in \mathbb{N}$ as below :

$$X_{n+1} = \begin{cases} \begin{cases} 1 & \text{if } U_{n+1} \geq \frac{1}{2} \\ -1 & \text{if } U_{n+1} < \frac{1}{2} \end{cases} & \text{if } X_n = 0 \\ \begin{cases} 0 & \text{if } U_{n+1} \geq \frac{1}{n^2} \\ n^2 X_n & \text{if } U_{n+1} < \frac{1}{n^2} \end{cases} & \text{if } X_n \neq 0 \end{cases}$$

Note that $P(U_n < \frac{1}{n^2}) = \frac{1}{n^2} \quad \forall n \in \mathbb{N}$ and $\sum_n P(U_n < \frac{1}{n^2}) = \sum_n \frac{1}{n^2} < \infty$. By Borel Cantelli lemma, this implies that $P(U_{n+1} < \frac{1}{n^2} \ i.o.) = 0$ In other words,

$$P(U_{n+1} \geq \frac{1}{n^2} \text{ all but finitely many n}) = 1$$

Let $B = (U_{n+1} \geq \frac{1}{n^2}$ all but finitely many n$)$. Then $P(B) = 1$ and for each $\omega \in B$, there is large enough N s.t. sequence $X_N, X_{N+1}, X_{N+2}, \cdots$ is given by $0, \pm 1, 0, \pm 1, \cdots$.
Hence $\omega \in (X_n = 1 \ i.o.) \cap (X_n = -1 \ i.o.) \cap (X_n = 0 \ i.o.)$ whenever $\omega \in B$. Since $P(B) = 1$, we have $P(X_n = a \ i.o.) = 1$ for $a = -1, 1, 0$
Also, since $|X_n| \leq (n!)^2 \quad \forall n \in \mathbb{N}$ and $|X_n| \leq 1$ for large enough n $a.s.$ we have $\sup_n |X_n| < \infty$. By this , X_n is integrable. Also $X_n \in \mathcal{F}_n$. To show X_n is a martingale , it suffices to show that $E[X_{n+1}|\mathcal{F}_n] = X_n$

$$\begin{aligned} E[X_{n+1}|\mathcal{F}_n] &= E[X_{n+1} I(X_n = 0) + X_{n+1} I(X_n \neq 0) | \mathcal{F}_n] \\ &= E[X_{n+1} I(X_n = 0)|\mathcal{F}_n] + E[X_{n+1} I(X_n \neq 0)|\mathcal{F}_n] \\ &= I(X_n = 0) E[X_{n+1}|\mathcal{F}_n] + I(X_n \neq 0) E[X_{n+1}|\mathcal{F}_n] \\ &= I(X_n = 0)\{\frac{1}{2} \cdot 1 + \frac{1}{2} \cdot (-1)\} + I(X_n \neq 0)\Big\{(1 - \frac{1}{n^2}) \cdot 0 + \frac{1}{n^2} \cdot n^2 X_n\Big\} \\ &= 0 \cdot I(X_n = 0) + X_n \cdot I(X_n \neq 0) = X_n \end{aligned}$$

Therefore X_n is a martingale w.r.t $\mathcal{F}_n$ □

4.3.3 Let X_n and Y_n be positive integrable and adapted to $\mathcal{F}_n$. Suppose $E(X_{n+1}|\mathcal{F}_n) \leq X_n + Y_n$, with $\sum Y_n < \infty$ a.s. Prove that X_n converges a.s. to a finite limit. Hint: Let $N = \inf_k \sum_{m=1}^{k} Y_m > M$, and stop your supermartingale at time N.

Proof. We want to define a supermartingale. Define W_n as below :

$$W_n := X_n - \sum_{m=1}^{n-1} Y_m \qquad \forall n \in \mathbb{N}$$

We shall show that W_n is a supermartingale w.r.t $\mathcal{F}_n$.
First, since $X_n \in \mathcal{F}_n$ and $\sum_{m=1}^{n-1} Y_m \in \mathcal{F}_{n-1} \subset \mathcal{F}_n$, we have $W_n \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$

Second, $E|W_n| \le E|X_n| + \sum_{m=1}^{n-1} E|Y_m|$ and X_n , Y_n are all integrable so that $E|W_n| < \infty$
For showing W_n is a supermartingale , it suffices to show that $E[W_{n+1}|\mathcal{F}_n] \le W_n$

$$
\begin{aligned}
E[W_{n+1}|\mathcal{F}_n] = E\Big[X_{n+1} - \sum_{m=1}^{n} Y_m \big| \mathcal{F}_n\Big] &= E[X_{n+1}|\mathcal{F}_n] - \sum_{m=1}^{n} Y_m \\
&\le X_n + Y_n - \sum_{m=1}^{n} Y_m \quad \because\ E[X_{n+1}|\mathcal{F}_n] \le X_n + Y_n \quad \text{by assumption} \\
&= X_n - \sum_{m=1}^{n-1} Y_m = W_n
\end{aligned}
$$

Therefore, W_n is a supermartingale.
Take $M \in \mathbb{N}$ and define $N = \inf\{n \in \mathbb{N} : \sum_{m=1}^{n} Y_m > M\}$. Then N is a stopping time.

$$
\because (N = n) = (\sum_{m=1}^{n-1} Y_m \le M) \cap (\sum_{m=1}^{n} Y_m > M) \in \mathcal{F}_n \quad \forall n \in \mathbb{N}
$$

Since W_n is a supermartingale and N is a stopping time, $W_{N\wedge n}$ is a supermartingale.

$$
\begin{aligned}
W_{N\wedge n} = X_{N\wedge n} - &\sum_{m=1}^{(N\wedge n)-1} Y_m \\
&\ge X_{N\wedge n} - M \quad \because \sum_{m=1}^{(N\wedge n)-1} Y_m \le \sum_{m=1}^{N-1} Y_m \le M \\
W_{N\wedge n} + M \ge X_{N\wedge n} > 0 &
\end{aligned}
$$

We have used the assumption that X_n and Y_n are positive.
Thus $W_{N\wedge n} + M$ is nonnegative supermartingale. By supermartingale convergence thm, $W_{N\wedge n} + M$ converges $a.s.$ to an integrable random variable. It implies that $W_{N\wedge n} \to W$ $a.s.$ for some integrable r.v. W . Note that if $N = \infty$ then $W_{N\wedge n} = W_n \quad \forall n \in \mathbb{N}$ so that $W_n \to W$ $a.s.$ on $(N = \infty)$
We have taken arbitrary $M \in \mathbb{N}$ for defining a stopping time N. Hence we can write it as N_M to emphasize that it depends on the value of M . Then for each $M \in \mathbb{N}$, we have

$$
(N_M = \infty) = (\sum_{m=1}^{n} Y_m \le M \quad \forall n \in \mathbb{N}) = \Big(\sum_{n} Y_n \le M\Big)
$$

Therefore $P\Big(\sum_n Y_n \le M\Big) = P(N_M = \infty) \quad \forall\, M \in \mathbb{N}$. Observe that $\Big(\sum_n Y_n \le M\Big)$ is increasing sequence of events w.r.t M . i.e. $\Big(\sum_n Y_n \le M\Big) \subset \Big(\sum_n Y_n \le M+1\Big) \quad \forall\, M \in \mathbb{N}$. Thus $(N_M = \infty)$ is also increasing sequence of events w.r.t M
By assumption $\sum_n Y_n < \infty$ $a.s.$,

$$
1 = P\Big(\sum_n Y_n < \infty\Big) = P\Big(\bigcup_{M\in\mathbb{N}} (\sum_n Y_n \le M)\Big) = \lim_{M\to\infty} P\Big(\sum_n Y_n \le M\Big)
$$

due to continuity from below of probability measure. Combining the results above, we have

$$
1 = \lim_{M\to\infty} P(N_M = \infty) = P\Big(\bigcup_{M\in\mathbb{N}} (N_M = \infty)\Big)
$$

Let $C = \left(\bigcup_{M\in\mathbb{N}} (N_M = \infty)\right)$. Since $W_n \to W$ $a.s.$ on $(N_M = \infty)$ for every $M \in \mathbb{N}$, we have $W_n \to W$ $a.s.$ on C with $P(C) = 1$. Therefore $W_n \to W$ $a.s.$ with W being integrable.
Since $X_n = W_n + \sum_{m=1}^{n-1} Y_m$, $X_n \to X$ $a.s.$ where $X = W + \sum_n Y_n$ with W and $\sum_n Y_n$ being finite $a.s.$. Therefore X_n converges $a.s.$ to a finite limit X □

4.3.4 Let $p_m \in [0, 1)$. Use the Borel-Cantelli lemmas to show that

$$\prod_{m=1}^{\infty}(1-p_m) = 0 \quad \text{if and only if} \quad \sum_{m=1}^{\infty} p_m = \infty.$$

Proof. Define a random sequence $\{X_n\}_n$ such that $X_n \overset{ind}{\sim} \text{Bern}(p_n) \quad \forall n \in \mathbb{N}$. Using this random sequence, we can write that

$$\prod_n (1-p_n) = P(X_n = 0 \quad \forall n \in \mathbb{N})$$

($\Leftarrow$) Suppose $\sum_n p_n = \infty$. Then $\sum_n P(X_n = 1) = \infty$. Since X_n's are independent, we can apply Borel Cantelli lemma so that $P(X_n = 1 \ i.o.) = 1$. Thus $P(X_n = 0 \text{ all but finitely many } n) = 0$.
Since $(X_n = 0 \quad \forall n \in \mathbb{N}) \subset (X_n = 0 \text{ all but finitely many } n)$, we have $P(X_n = 0 \quad \forall n \in \mathbb{N}) = 0$
Therefore $\prod_n (1-p_n) = 0$
($\Rightarrow$) Suppose $\sum_n p_n < \infty$. Since the tail part of the convergent series tends to zero, we have large N s.t. $\sum_{n>N} p_n < 1$ so that $1 - \sum_{n>N} p_n > 0$

$$\begin{aligned} P(X_n = 0 \quad \forall n > N) &= P\Big(\bigcap_{n>N}(X_n = 0)\Big) = 1 - P\Big(\bigcup_{n>N}(X_n = 1)\Big) \\ &\geq 1 - \sum_{n>N} P(X_n = 1) = 1 - \sum_{n>N} p_n > 0 \end{aligned}$$

$$\begin{aligned} P(X_n = 0 \quad \forall n \in \mathbb{N}) &= P(X_1 = 0)P(X_2 = 0)\cdots P(X_N = 0)P(X_n = 0 \quad \forall n > N) \\ &= (1-p_1)(1-p_2)\cdots(1-p_N)P(X_n = 0 \quad \forall n > N) > 0 \quad \because p_n < 1 \ \forall n \text{ is assumed} \end{aligned}$$

Thus $\prod_n (1-p_n) > 0$. As a result, $\sum_n p_n < \infty$ implies $\prod_n (1-p_n) > 0$. By contrapositive, we have the right direction proved. □

Remark. *Notice that for the left direction, we have not taken advantage of assumption* $p_n < 1 \quad \forall n$

4.3.5 Show $\sum_{n=2}^{\infty} P(A_n | \cap_{m=1}^{n-1} A_m^c) = \infty$ implies $P(\cap_{m=1}^{\infty} A_m^c) = 0$.

Proof. Define $p_1 = P(A_1)$ and $p_n = P(A_n | \bigcap_{m=1}^{n-1} A_m^C) \quad \forall n \in \mathbb{N}$.
By assumption, we have $\sum_n p_n = \infty$. From the previous exercise, we get $\prod_n (1-p_n) = 0 \quad \cdots (*)$
$1 - p_1 = P(A_1^C)$. What about $1 - p_n$ for each $n > 1$?

$$\begin{aligned} 1 - p_n &= 1 - P\Big(A_n | \bigcap_{m=1}^{n-1} A_m^C\Big) = 1 - E\Big[I_{A_n} | \bigcap_{m=1}^{n-1} A_m^C\Big] = P(A_n^c | \bigcap_{m=1}^{n-1} A_m^C) \\ &= E\Big[1 - I_{A_n} | \bigcap_{m=1}^{n-1} A_m^C\Big] = E\Big[I_{A_n^C} | \bigcap_{m=1}^{n-1} A_m^C\Big] = P\Big(A_n^c | \bigcap_{m=1}^{n-1} A_m^C\Big) = \frac{P\Big(\bigcap_{m=1}^{n} A_m^C\Big)}{P\Big(\bigcap_{m=1}^{n-1} A_m^C\Big)} \end{aligned}$$

Therefore, for each $n \in \mathbb{N}$, we can calculate $\sum_{m=1}^{n}(1-p_m)$ as below :

$$\prod_{m=1}^{n}(1-p_m) = P(A_1^C)\frac{P(A_1^C \cap A_2^C)}{P(A_1^C)}\frac{P(A_1^C \cap A_2^C \cap A_3^C)}{P(A_1^C \cap A_2^C)}\cdots\frac{P\big(\bigcap_{m=1}^{n} A_m^C\big)}{P\big(\bigcap_{m=1}^{n-1} A_m^C\big)} = P\big(\bigcap_{m=1}^{n} A_m^C\big)$$

Therefore, using continuity from above of probability measure , we have

$$\prod_{n}(1-p_n) = \lim_{n\to\infty}\prod_{m=1}^{n}(1-p_m) = \lim_{n\to\infty} P\big(\bigcap_{m=1}^{n} A_m^C\big) = P\big(\bigcap_{n} A_n^C\big)$$

By $(*)$ above, we have $P\big(\bigcap_n A_n^C\big) = 0$ $\square$

4.4.1 Show that if $j \le k$, then $E(X_j; N = j) \le E(X_k; N = j)$ and sum over j to get a second proof of $EX_N \le EX_k$.

Proof. Assume X_n is a submartingale and N is a stopping time w.r.t a filtration $\mathcal{F}_n$. Suppose $N \le k$ $a.s.$ for some $k \in \mathbb{N}$. Then

$$X_N = \sum_{j=0}^{k} X_j I(N=j) \;\; a.s.$$

We shall claim that

$$E[X_j I(N=j)] \le E[X_k I(N=j)] \quad \forall\, j \le k$$

Choose $j \le k$. Since X_n is a submartingale , $X_j \le E[X_k|\mathcal{F}_j]$ holds true. For $A_j \in \mathcal{F}_j$,

$$\begin{aligned} E[X_j I_{A_j}] &= \int_{A_j} X_j\, dP \le \int_{A_j} E[X_k|\mathcal{F}_j]\, dP \quad \because X_j \le E[X_k|\mathcal{F}_j] \\ &= \int_{A_j} X_k\, dP \quad \because \text{ def. of conditional expectation} \\ &= E[X_k I_{A_j}] \end{aligned}$$

$(N=j) \in \mathcal{F}_j$ since N is a stopping time. Therefore, our claim is proved. Using the claim, we can show that

$$E[X_N] = \sum_{j=0}^{k} E[X_j I(N=j)] \le \sum_{j=0}^{k} E[X_k I(N=j)] = E[X_k]$$

The last equality holds since $N \le k$ $a.s.$ implies that $\{(N=0), \cdots, (N=k)\}$ is a partition of Ω in almost sure sense. i.e. $\sum_{j=0}^{k} I(N=j) = 1$ $a.s.$
Therefore, we have proved that $E[X_N] \le E[X_k]$ $\square$

4.4.2 **Generalize the proof of Theorem 4.4.1 to show that if X_n is a submartingale and $M \le N$ are stopping times with $P(N \le k) = 1$, then $EX_M \le EX_N$.**

Proof. Define $K_n = I(M < n \le N) \quad \forall n \in \mathbb{N}$. Then K_n is predictable since

$$(K_n = 1) = (N \ge n) \cap (M < n) = (N \le n-1)^C \cap (M \le n-1) \in \mathcal{F}_{n-1}$$

Then, we can define a process $(K \cdot X)_n$ as below :

$$\begin{aligned}
(K \cdot X)_n &= \sum_{j=1}^{n} K_j (X_j - X_{j-1}) = \sum_{j=1}^{n} I(M < j \le N)(X_j - X_{j-1}) \\
&= \sum_{j=1}^{n} I(M+1 \le j \le N)(X_j - X_{j-1}) \\
&= \sum_{j=(M\wedge n)+1}^{N\wedge n} X_j - X_{j-1} = X_{N\wedge n} - X_{M\wedge n} \qquad \forall n \in \mathbb{N}
\end{aligned}$$

Define $(K \cdot X)_0 = 0$. Then since X_n is a submartingale and K_n is a predictable sequence, $\{K \cdot X\}_{n\in\mathbb{N}\cup\{0\}}$ is a submartingale. It implies that $\{X_{N\wedge n} - X_{M\wedge n}\}_{n\in\mathbb{N}\cup\{0\}}$ is a submartingale. Note that if Y_n is a martingale then $E[Y_i] \le E[Y_j]$ whenever $i \le j$. Plugging in $i = 0$ and $j = k$ on $X_{N\wedge n} - X_{M\wedge n}$, we have $E[X_0 - X_0] \le E[X_N - X_M]$ since $M \le N \le k$ *a.s.* . Therefore, $0 \le E[X_N] - E[X_M]$ i.e. $E[X_N] \ge E[X_M]$ $\square$

4.4.3 **Suppose $M \le N$ are stopping times. If $A \in \mathcal{F}_M$, then**

$$L = \begin{cases} M & \text{on } A \\ N & \text{on } A^c \end{cases} \quad \text{is a stopping time.}$$

Proof. $\mathcal{F}_M = \{A \in \mathcal{F} : A \cap (M = n) \in \mathcal{F}_n \quad \forall n \in \mathbb{N}\}$. Take $A \in \mathcal{F}_M$ and define L as above. Choose arbitrary $n \in N$. We want to show that $(L = n) \in \mathcal{F}_n$. Note that

$$(L = n) = \big((M = n) \cap A\big) \bigcup \big((N = n) \cap A^C\big)$$

Since $A \in \mathcal{F}_M$, $(M = n) \cap A \in \mathcal{F}_n$ by definition of $\mathcal{F}_M$.
Observe that since $M \le N$, $(N = n) = (N = n) \cap (M \le N)$. Using this equality, we get

$$\begin{aligned}
(N = n) \cap A^C &= (N = n) \cap (M \le n) \cap A^C \\
&= (N = n) \cap \Big(\bigcup_{k=1}^{n} \{(M = k) \cap A^C\} \Big)
\end{aligned}$$

Since $A \in \mathcal{F}_M$ and $\mathcal{F}_M$ is a σ-field , we have $A^C \in \mathcal{F}_M$ so that $(M = k) \cap A^C \in \mathcal{F}_k \quad \forall\, k \in \mathbb{N}$. Therefore $\bigcup_{k=1}^{n} \{(M = k) \cap A^C\} \in \mathcal{F}_n$. Also, since N is a stopping time , $(N = n) \in \mathcal{F}_n$
Thus $(N = n) \cap A^C \in \mathcal{F}_n$ and combining with $(M = n) \cap A \in \mathcal{F}_n$, we have $(L = n) \in \mathcal{F}_n$ $\square$

4.4.4 Use the stopping times from the previous exercise to strengthen the conclusion of Exercise 4.4.2 to $X_M \le E(X_N|\mathcal{F}_M)$.

Proof. Assume X_n is a submartingale and $M \le N$ are stopping times with $N \le k$ $a.s.$
Note that $X_M \in \mathcal{F}_M$ by definition of $\mathcal{F}_M$ and $E[X_N|\mathcal{F}_M] \in \mathcal{F}_M$ by definition of conditional expectation . Hence, if we can show

$$\int_A X_M \, dP \le \int_A E[X_N|\mathcal{F}_M] \, dP \quad \forall\, A \in \mathcal{F}_M$$

then $X_M \le E[X_N|\mathcal{F}_M]$ is proved. Notice that for $A \in \mathcal{F}_M$, we have $\int_A E[X_N|\mathcal{F}_M] \, dP = \int_A X_N \, dP$
Thus, it suffices to show that

$$\int_A X_M \, dP \le \int_A X_N \, dP \quad \forall\, A \in \mathcal{F}_M \quad \cdots (*)$$

Take $A \in \mathcal{F}_M$. To use the result of the previous exercise, define L by

$$L = M \cdot I_A + N \cdot I_{A^C}$$

By Exer. 4.4.3 , L is a stopping time. Notice that $L \le N$ since $M \le N$. Using Exer. 4.4.2 , we have $E[X_L] \le E[X_N]$. By the definition of L, we have

$$\begin{aligned}
&X_L = X_M I_A + X_N I_{A^C} \\
&E[X_L] = E[X_M I_A] + E[X_N I_{A^C}] \\
&E[X_N] = E[X_N I_A] + E[X_N I_{A^C}] \\
&E[X_M I_A] + E[X_N I_{A^C}] \le E[X_N I_A] + E[X_N I_{A^C}] \quad \because\ E[X_L] \le E[X_N] \\
&E[X_M I_A] \le E[X_N I_A] \quad \text{i.e.} \quad \int_A X_M \, dP \le \int_A X_N \, dP
\end{aligned}$$

Hence, we have proved $(*)$ holds true. We can conclude that $X_M \le E[X_N|\mathcal{F}_M]$ □

4.4.5 Prove the following variant of the conditional variance formula. If $\mathcal{F} \subset \mathcal{G}$, then

$$E\big(E[Y|\mathcal{G}] - E[Y|\mathcal{F}]\big)^2 = E(E[Y|\mathcal{G}])^2 - E(E[Y|\mathcal{F}])^2$$

Proof. We shall take advantage of the fact that for any integrable r.v. X and a σ-field $\mathcal{F}$, we have $E[E[X|\mathcal{F}]] = E[X]$. Also, we will denote $E^2[X] := \{E(X)\}^2$

$$\begin{aligned}
&E\Big[\big(E[Y|\mathcal{G}] - E[Y|\mathcal{F}]\big)^2\Big] = E\Big[E\Big[\big(E[Y|\mathcal{G}] - E[Y|\mathcal{F}]\big)^2|\mathcal{F}\Big]\Big] = E[Z] \\
&\text{where} \quad Z := E\Big[\big(E[Y|\mathcal{G}] - E[Y|\mathcal{F}]\big)^2|\mathcal{F}\Big] = E[W|\mathcal{F}] \\
&\text{with} \quad W := \big(E[Y|\mathcal{G}] - E[Y|\mathcal{F}]\big)^2 \\
&W = E^2[Y|\mathcal{G}] + E^2[Y|\mathcal{F}] - 2E[Y|\mathcal{G}]E[Y|\mathcal{F}] \\
&\star \text{We shall take } \ E[\,\cdot\,|\mathcal{F}] \\
&E[E^2[Y|\mathcal{F}]|\mathcal{F}] = E^2[Y|\mathcal{F}] \quad \because\ E^2[Y|\mathcal{F}] \in \mathcal{F} \\
&E\big[E[Y|\mathcal{G}]E[Y|\mathcal{F}]|\mathcal{F}\big] = E[Y|\mathcal{F}]E\big[E[Y|\mathcal{G}]|\mathcal{F}\big] = E[Y|\mathcal{F}]E[Y|\mathcal{F}] \quad \because\ \mathcal{F} \subset \mathcal{G} \\
&\Rightarrow\ Z = E[W|\mathcal{F}] = E\big[E^2[Y|\mathcal{G}]|\mathcal{F}\big] + E^2[Y|\mathcal{F}] - 2E^2[Y|\mathcal{F}] = E\big[E^2[Y|\mathcal{G}]|\mathcal{F}\big] - E^2[Y|\mathcal{F}] \\
&E[Z] = E\Big[E\big[E^2[Y|\mathcal{G}]|\mathcal{F}\big]\Big] - E\big[E^2[Y|\mathcal{F}]\big] = E\big[E^2[Y|\mathcal{G}]\big] - E\big[E^2[Y|\mathcal{F}]\big]
\end{aligned}$$

Since $E\big[\big(E[Y|\mathcal{G}] - E[Y|\mathcal{F}]\big)^2\big] = E[Z]$, we have

$$E\big[\big(E[Y|\mathcal{G}] - E[Y|\mathcal{F}]\big)^2\big] = E\big[E^2[Y|\mathcal{G}]\big] - E\big[E^2[Y|\mathcal{F}]\big]$$

$\square$

4.4.6 Suppose in addition to the conditions introduced earlier that $|\xi_m| \le K$ and let $s_n^2 = \sum_{m\le n} \sigma_m^2$. Exercise 4.2.2 implies that $S_n^2 - s_n^2$ is a martingale. Use this and Theorem 4.4.1 to conclude

$$P\left(\max_{1\le m\le n} |S_m| \le x\right) \le (x+K)^2/\operatorname{var}(S_n)$$

Proof. Let $\{\xi_n\}_{n\in\mathbb{N}}$ be independent random seq. with $E[\xi_i] = 0$ and $E[\xi_i^2] = \sigma_i^2 < \infty \quad \forall\, i \in \mathbb{N}$
$S_n, \mathcal{F}_n$ and s_n^2 are defined by $S_n = \xi_1 + \cdots + \xi_n$, $\mathcal{F}_n = \sigma(\xi_1, \cdots, \xi_n)$ and $s_n^2 = Var(S_n) = \sum_{i=1}^n \sigma_i^2$
Especially, $S_0 = 0, s_0^2 = 0$ and $\mathcal{F}_0 = \{\phi, \Omega\}$. In addition, assume $|\xi_n| \le K \quad \forall n \in \mathbb{N}$ for some $K > 0$
Note that S_n is integrable and $S_n \in \mathcal{F}_n$. Since s_n^2 is finite constant for each $n \in \mathbb{N}$, we have $S_n^2 - s_n^2$ is integrable and $S_n^2 - s_n^2 \in \mathcal{F}_n$. To show $S_n^2 - s_n^2$ is a martingale , it suffices to show that $E[S_{n+1}^2 - s_{n+1}^2|\mathcal{F}_n] = S_n^2 - s_n^2$.

$$\begin{aligned} E[S_{n+1}^2 - s_{n+1}^2|\mathcal{F}_n] &= E[(S_n + \xi_{n+1})^2 - s_n^2 - \sigma_{n+1}^2|\mathcal{F}_n] = E[S_n^2 - s_n^2 + \xi_{n+1}^2 - \sigma_{n+1}^2 + 2S_n\xi_{n+1}|\mathcal{F}_n] \\ &= S_n^2 - s_n^2 + E[\xi_{n+1}^2 - \sigma_{n+1}^2] + 2S_nE[\xi_{n+1}] = S_n^2 - s_n^2 \end{aligned}$$

Take $x > 0$. Define $A := \big(\max_{1\le m\le n}|S_m| > x\big)$. Note that we want to find the upper bound of $P(A^C)$ in this problem. Let $N = \inf\{m \in \mathbb{N} : |S_m| > x\}$. N is a stopping time since

$$(N = n) = (|S_1| \le x) \cap \cdots \cap (|S_{n-1}| \le x) \cap (|S_n| > x) \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$$

Due to the fact that $S_n^2 - s_n^2$ is a martingale and N is a stopping time , we have $S_{N\wedge n}^2 - s_{N\wedge n}^2$ is also a martingale. Since $N \wedge n \le n$, we can apply bounded optional stopping theorem.

$$E[S_0^2 - s_0^2] = E[S_{N\wedge n}^2 - s_{N\wedge n}^2] = E[S_n^2 - s_n^2]$$

We will use the left equality, which implies that $E[S_{N\wedge n}^2 - s_{N\wedge n}^2] = 0 \quad \cdots (*)$

i. On $A = \big(\max_{1\le m\le n}|S_m| > x\big)$
$|S_m| > x$ for some $m \in \{1, \cdots, n\}$ so that $N \le n \;\Rightarrow\; N \wedge n = N$
$|S_{N\wedge n}| = |S_N| = |S_{N-1} + \xi_N| \le |S_{N-1}| + |\xi_N| \le x + K$

ii. On $A^c = \big(\max_{1\le m\le n}|S_m| \le x\big)$
$|S_m| > x$ is not attined for all $m \in \{1, \cdots, n\}$ so that $N > n \;\Rightarrow N \wedge n = n$
$|S_{N\wedge n}| = |S_n| \le x$

Using $(*)$, we have the following results.

$$\begin{aligned}
&0 = E[S_{N\wedge n}^2 - s_{N\wedge n}^2] = E[S_{N\wedge n}^2 I_A + S_{N\wedge n}^2 I_{A^C}] - E[s_{N\wedge n}^2 I_A + s_{N\wedge n}^2 I_{A^C}] \\
&E[S_{N\wedge n}^2 I_A] \le E[(x+K)^2 I_A] = (x+K)^2 P(A) \\
&E[S_{N\wedge n}^2 I_{A^C}] \le E[x^2 I_{A^C}] = x^2 P(A^C) \\
&E[s_{N\wedge n}^2 I_A] = E[s_N^2 I_A] \ge 0 \quad \surd\ s_N^2 \text{ is random ; not constant} \\
&E[s_{N\wedge n}^2 I_{A^C}] = E[s_n^2 I_{A^C}] = s_n^2 P(A^C) = Var(S_n) P(A^C) \\
&\Rightarrow\ 0 = E[S_{N\wedge n}^2 - s_{N\wedge n}^2] \le (x+K)^2 P(A) + x^2 P(A^C) - Var(S_n) P(A^C) \\
&\Rightarrow 0 \le (x+K)^2 - \{Var(S_n) + (x+K)^2 - x^2\} P(A^C) \\
&\Rightarrow (x+K)^2 \ge \{Var(S_n) + (x+K)^2 - x^2\} P(A^C) \ge Var(S_n) P(A^C) \\
&\therefore\ P(A^C) \le \frac{(x+K)^2}{Var(S_n)}
\end{aligned}$$

Since $A^C = \left(\max_{1\le m\le n} |S_m| \le x\right)$, we have proved our desired result. $\square$