Probability theory II Assignment 3

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Exercises
Section 4.6 Uniform Integrability , Convergence in L^1
Section 4.8 Optional Stopping Theorems

4.6.6 Let $X_n \in [0,1]$ be adapted to $\mathcal{F}_n$. Let $\alpha, \beta > 0$ with $\alpha + \beta = 1$ and suppose

$$P(X_{n+1} = \alpha + \beta X_n | \mathcal{F}_n) = X_n \qquad P(X_{n+1} = \beta X_n | \mathcal{F}_n) = 1 - X_n$$

Show $P(\lim_n X_n = 0 \text{ or } 1) = 1$ and if $X_0 = \theta$, then $P(\lim_n X_n = 1) = \theta$.

Proof. $X_n \in \mathcal{F}_n$ and $0 \leq X_n \leq 1 \quad \forall n \in \mathbb{N}$. By assumption,

$$X_{n=1}|\mathcal{F}_n = \begin{cases} \alpha + \beta X_n & \text{with prob. } X_n \\ \beta X_n & \text{with prob. } 1 - X_n \end{cases}$$

Thus, we have

$$E[X_{n+1}|\mathcal{F}_n] = (\alpha + \beta X_n)X_n + \beta X_n(1 - X_n) = (\alpha + \beta)X_n = X_n$$

Therefore, X_n is a martingale. Since X_n is uniformly bounded by 0 and 1 , we can say X_n is uniformly integrable martingale. Also, $\sup_n E|X_n| \leq 1$ and by martingale convergence thm, $X_n \to X$ $a.s.$ for some integrable X . By Vitali lemma, $X_n \to X$ in L^1 and $E(X_n) \to E(X)$. Note that since X_n is a martingale. we have $\theta = E(X_0) = E(X_1) = \cdots = E(X_n) \quad \forall n \in \mathbb{N}$. Combining these, we have $\theta = E(X_0) = E(X_n) = E(X) \quad \forall n \in \mathbb{N}$
Now consider the case that $X_n = x$ where $0 < x < 1$. Then

$$X_{n+1}|X_n = \begin{cases} \alpha + \beta x & \text{with prob } x \\ \beta x & \text{with prob } 1 - x \end{cases}$$

Thus , if $0 < x < 1$ then the sequence X_n goes out of ε-ball containing x at the very next step X_{n+1} . Therefore, the limit X cannot have value $x \in (0,1)$.
On the other hand, if $X_n = 0$ then

$$X_{n+1}|X_n = \begin{cases} \alpha & \text{with prob } 0 \\ 0 & \text{with prob } 1 \end{cases} = 0 \ a.s.$$

Else, if $X_n = 1$ then

$$X_{n+1}|X_n = \begin{cases} \alpha + \beta = 1 & \text{with prob } 1 \\ \beta & \text{with prob } 0 \end{cases} = 1 \ a.s.$$

Therefore $X = 0$ or 1 $a.s.$ and $\theta = E(X) = 0 \cdot P(X = 0) + 1 \cdot P(X = 1)$ so $P(X = 1) = \theta$ □

4.6.7 Show that if $\mathcal{F}_n \uparrow \mathcal{F}_\infty$ and $Y_n \to Y$ in L^1, then $E(Y_n|\mathcal{F}_n) \to E(Y|\mathcal{F}_\infty)$ in L^1.

Proof. We want to show the L^1 convergence i.e.

$$E|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)| \to 0$$

Note that by triangle inequality, we have

$$E|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)| \leq E|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_n)| + E|E(Y|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)|$$

By Levy's thm, $E(Y|\mathcal{F}_n) \to E(Y|\mathcal{F}_\infty)$ in L^1 so that $E|E(Y|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)| \to 0$
Also,

$$E|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_n)| \le E[E[|Y_n - Y||\mathcal{F}_n]] = E|Y_n - Y| \to 0$$

by the assumption that $Y_n \to Y$ in L^1 . Therefore

$$E|E(Y_n|\mathcal{F}_n) - E(Y|\mathcal{F}_\infty)| \to 0$$

In other words, $E(Y|\mathcal{F}_n) \to E(Y|\mathcal{F}_\infty)$ in L^1 □

4.8.1 **Generalize Theorem 4.8.2 to show that if $L \le M$ are stopping times and $Y_{M\wedge n}$ is a uniformly integrable submartingale, then $EY_L \le EY_M$ and**

$$Y_L \le E(Y_M|\mathcal{F}_L)$$

Proof. We can prove this thm on four steps.

i. $E(Y_0) \le E(Y_N) \le E(Y_\infty)$ for any stopping time N
For uniformly integrable submartingale Y_n, we have $Y_n \to Y_\infty$ $a.s.$ for some integrable r.v. Y_∞ . Also, for a stopping time N, $X_n = Y_{N\wedge n}$ is also a uniformly integrable submartingale so that $X_n \to X_\infty$ $a.s.$ for some integrable r.v. X_∞ . Note that $X_\infty = Y_N$ $a.s.$ (This is derived when we consider two cases where $N = \infty$ and $N < \infty$) . Applying Vitali lemma to both $Y_n \to Y_\infty$ and $Y_{N\wedge n} \to Y_N$, we have $E(Y_n) \to E(Y_\infty)$ and $E(Y_{N\wedge n}) \to E(Y_N)$. Using bounded optional stopping thm, we have

$$E(Y_0) \le E(Y_{N\wedge n}) \le E(Y_n)$$

Taking $n \to \infty$, we have

$$E(Y_0) \le E(Y_N) \le E(Y_\infty)$$

ii. $E(Y_L) \le E(Y_M)$ given $L \le M$
For uniformly integrable submartingale Y_n, we have $Y_n \to Y_\infty$ $a.s.$ for some integrable r.v. Y_∞ . $X_n = Y_{M\wedge n}$ is also a uniformly integrable submartingale so that $X_n \to X_\infty$ $a.s.$ for some integrable r.v. X_∞ . Note that $X_\infty = Y_M$ $a.s.$ By the result of the first step, we have

$$E(X_0) \le E(X_L) \le E(X_\infty)$$

Since $X_L = Y_{M\wedge L} = Y_L$ and $X_\infty = Y_M$ $a.s.$ we have

$$E(Y_L) \le E(Y_M)$$

iii. If $A \in \mathcal{F}_L$ then $N = L \cdot I_A + M \cdot I_{A^C}$ is a stopping time.
This is illustrated in exercise 4.4.3 of homework 2

iv. Show $E[Y_L I_A] \le E[Y_M I_A] \quad \forall\, A \in \mathcal{F}_L$
This is illustrated in exercise 4.4.4 of homework 2

□

4.8.2 **If $X_n \geq 0$ is a supermartingale, then $P(\sup X_n > \lambda) \leq EX_0/\lambda$.**

Proof. We have learned in class that for nonnegative supermartingale X_n , if N is a stopping time then $E(X_0) \geq E(X_N)$. Note that $N = \inf\{n : X_n > \lambda\}$ is a stopping time. On $(N < \infty)$, we have $X_N > \lambda$ so

$$X_N I(N < \infty) \geq \lambda I(N < \infty) \qquad E[X_N I(N < \infty)] \geq \lambda P(N < \infty)$$

Since $X_n \geq 0$, we get $E(X_0) \geq E(X_N) \geq E[X_N I(N < \infty)] \geq \lambda P(N < \infty)$. Hence

$$P(N < \infty) \leq \frac{E[X_0]}{\lambda}$$

Note that

$$(N < \infty) = (\{n : X_n > \lambda\} \neq \phi) = (X_n > \lambda \text{ for some } n \in \mathbb{N}) = (\sup_n X_n > \lambda)$$

Therefore, we can conclude that

$$P(\sup_n X_n > \lambda) \leq \frac{E[X_0]}{\lambda}$$

□

4.8.4 ***Wald's second equation.*** **Let $S_n = \xi_1 + \cdots + \xi_n$, where the ξ_i are independent with $E\xi_i = 0$ and $\text{var}(\xi_i) = \sigma^2$. Use the martingale from the previous problem to show that if T is a stopping time with $ET < \infty$, then $ES_T^2 = \sigma^2 ET$.**

Proof. As we can see in the example 4.2.2 in the textbook, $M_n = S_n^2 - n\sigma^2$ is a martingale . Since T is a stopping time , $M_{T\wedge n}$ is also a martingale . Let $P_n = M_{T\wedge n}$. Then $E(P_1) = E(P_n)$ by the property of martingale.

$$E(P_1) = E(M_1) = E(S_1^2 - \sigma^2) = E(\xi_1^2 - \sigma^2) = 0$$

$$E(P_n) = E(M_{T\wedge n}) = E[S_{n\wedge T}^2 - (n \wedge T)\sigma^2]$$

Thus we have $E[S_{T\wedge n}^2] = \sigma^2 E[T \wedge n] \quad \forall n \in \mathbb{N}$. Note that $E(T) < \infty$ assumption implies that $T < \infty$ *a.s.* Thus $S_{T\wedge n} \to S_T$ *a.s.* Also, since $T \wedge n \nearrow T$ and $E[T \wedge n] \nearrow E[T]$ by MCT, we get

$$\sup_n E[S_{T\wedge n}^2] = \sigma^2 \sup_n E[T \wedge n] = \sigma^2 E[T] < \infty$$

Hence, we have $\sup_n E[S_{T\wedge n}^2] < \infty$. Note that S_n is also a martingale so that $S_{T\wedge n}$ is a martingale too. By martingale L^p convergence thm, $S_{T\wedge n} \to S_T$ *a.s.* and in L^2 . This L^2 convergence gives us the fact that $E[S_{T\wedge n}^2] \to E[S_T^2]$. Therefore, taking $n \to \infty$ for $E[S_{T\wedge n}^2] = \sigma^2 E[T \wedge n] \quad \forall n \in \mathbb{N}$, we finally get

$$E[S_T^2] = \sigma^2 E[T]$$

□

4.8.3 Let $S_n = \xi_1 + \cdots + \xi_n$ where the ξ_i are independent with $E\xi_i = 0$ and $\text{var}(X_i) = \sigma^2$. $S_n^2 - n\sigma^2$ is a martingale. Let $T = \min\{n : |S_n| > a\}$. Use Theorem 4.8.2 to show that $ET \geq a^2/\sigma^2$.

Proof. Notice that T is a stopping time. If $E(T) = \infty$ then the inequality trivially holds so assume that $E(T) < \infty$. Then, by the Wald's second equation we proved in the previous exercise, we have $E[S_T^2] = \sigma^2 E[T]$. Observe that on $(T < \infty)$, we get $S_T^2 > a^2$. Hence we have

$$E[S_T^2] \geq E[S_T^2 I(T < \infty)] \geq a^2 E[I(T < \infty)] = a^2 P(T < \infty)$$

The assumption $E[T] < \infty$ implies that $T < \infty$ $a.s.$ thus $P(T < \infty) = 1$. Therefore,

$$E[T] = E[S_T^2]/\sigma^2 \geq a^2/\sigma^2$$

□

4.8.5 *Variance of the time of gambler's ruin.* Let $\xi_1, \xi_2, \ldots$ be independent with $P(\xi_i = 1) = p$ and $P(\xi_i = -1) = q = 1 - p$, where $p < 1/2$. Let $S_n = S_0 + \xi_1 + \cdots + \xi_n$ and let $V_0 = \min\{n \geq 0 : S_n = 0\}$. Theorem 4.8.9 tells us that $E_x V_0 = x/(1-2p)$. The aim of this problem is to compute the variance of V_0. If we let $Y_i = \xi_i - (p - q)$ and note that $EY_i = 0$ and

$$\text{var}(Y_i) = \text{var}(X_i) = EX_i u^2 - (EX_i)^2$$

then it follows that $(S_n - (p-q)n)^2 - n(1-(p-q)^2)$ is a martingale. (a) Use this to conclude that when $S_0 = x$ the variance of V_0 is

$$x \cdot \frac{1 - (p-q)^2}{(q-p)^3}$$

(b) Why must the answer in (a) be of the form cx?

Proof. Here $x > 0$ is assumed. (Otherwise, variance becomes negative which is nonsense)
Note that $E[Y_1] = 0$ and $Var[Y_1] = 1 - (p-q)^2$. $S_n - n(p-q) = S_0 + Y_1 + \cdots + Y_n$. We have observed that $S_n^2 - n\sigma^2$ is martingale provided $S_n = \xi_1 + \cdots + \xi_n$ and ξ_i i.i.d. with $E[\xi_1] = 0$ and $Var[\xi_1]\sigma^2$ Here, Y_i plays a role of ξ_i in the sense that it has zero mean. Thus,

$$(S_n - n(p-q))^2 - n(1-(p-q))^2$$

is a martingale. In the lecture, we have learned that $T_a < \infty$ $a.s.$ for $a < 0$. Here, it is replaced by $V_0 < \infty$ $a.s.$ for $0 < x$. (The former has starting point 0 and target point a, while the latter has starting point x and target point 0) . Since $E[V_0] = \frac{x}{1-2p} < \infty$, we can apply Wald second identity. Let $\tilde{S}_n = Y_1 + \cdots + Y_n = S_n - n(p-q) - S_0$. Applying Wald second identity, we have

$$E[\tilde{S}_{V_0}^2] = (1-(p-q)^2)E[V_0]$$

. Plugging in $\tilde{S}_n = S_n - n(p-q) - S_0$ and $E[V_0] = \frac{x}{q-p}$, we get

$$E\big[(S_{V_0} - V_0(p-q) - x)^2\big] = (1-(p-q)^2)\frac{x}{q-p}$$

Note that $S_{V_0} = 0$ $a.s.$ since $V_0 < \infty$ $a.s.$. Thus , we can derive that

$$\begin{aligned} E\big[(S_{V_0} - V_0(p-q) - x)^2\big] &= E\big[(-V_0(p-q) - x)^2\big] = E\big[(V_0(p-q) + x)^2\big] \\ &= E\Big[\big\{(p-q)\big(V_0 + \frac{x}{p-q}\big)\big\}^2\Big] = (q-p)^2 E\Big[\big(V_0 - \frac{x}{q-p}\big)^2\Big] \\ &= (q-p)^2 E[\{V_0 - E(V_0)\}^2] = (q-p)^2 Var(V_0) \end{aligned}$$

Therefore, we have

$$Var(V_0) = \frac{x(1-(p-q)^2)}{(q-p)^3}$$

□

4.8.7 **Let S_n be a symmetric simple random walk starting at 0, and let $T = \inf\{n : S_n \notin (-a,a)\}$, where a is an integer. Find constants b and c so that $Y_n = S_n^4 - 6nS_n^2 + bn^2 + cn$ is a martingale, and use this to compute ET^2.**

Proof. $S_n = \xi_1 + \cdots + \xi_n$ with ξ_i i.i.d. where $P(\xi_1 = 1) = P(\xi_1 = -1) = 0.5$. $E(\xi_1) = 0$ and $Var(\xi_1) = 1$. Note that $S_n^2 - n$ is a martingale. Since T is a stopping time, $S_{T\wedge n}^2 - (T \wedge n)$ is a martingale .

$$E[S_{T\wedge n}^2 - (T\wedge n)] = E[S_1^2 - 1] = E[\xi_1^2] - 1 = 0$$

Hence $E[S_{T\wedge n}^2] = E[T\wedge n]$ $\forall n \in \mathbb{N}$. Here, we shall claim that $T < \infty$ $a.s.$
Note that wherever S_n lies between $(-a,a)$, if we have $2a$ consecutive steps of size $+1$ then we will exit the interval $(-a,a)$. It can be written as

$$(T > m \cdot 2a) \subset (m \text{ times fail of "}2a\text{ consecutive steps of size }+1'')$$

$$P(T > 2ma) \le \left(1 - (\frac{1}{2})^{2a}\right)^m$$

Taking $m \to \infty$, we have $P(T = \infty) = 0$. Hence, $T < \infty$ $a.s.$ Thus $S_{T\wedge n} \to S_T$ $a.s.$ and $S_{T\wedge n}^2 \to S_T^2$ $a.s.$ Observe that $S_T^2 = a^2$ $a.s.$ and $|S_{T\wedge n}^2| \le a^2$ so that applying BCT , we get

$$E[S_{T\wedge n}^2] \to E[S_T^2] = E[a^2] = a^2$$

Also, by MCT , $E[T\wedge n] \nearrow E[T]$. From the equality $E[S_{T\wedge n}^2] = E[T\wedge n]$ $\forall n \in \mathbb{N}$, we can conclude that $E[T] = a^2$ Now, we shall calculate $E[Y_{n+1}|\mathcal{F}_n] = Y_n$ to make Y_n be a martingale.

$$\begin{aligned} E[Y_{n+1}|\mathcal{F}_n] &= E[(S_n + \xi_{n+1})^4 - 6(n+1)(S_n + \xi_{n+1})^2 + b(n+1)^2 + c(n+1)|\mathcal{F}_n] \\ &= S_n^4 + 6S_n^2 - 6(n+1)S_n^2 - 6(n+1) + bn^2 + b(2n+1) + cn + c \\ &= S_n^5 - 6nS_n^2 + bn^2 + cn + (2b-6)n + (b+c-5) = Y_n \end{aligned}$$

To satisfy the last equation, $b = 3$ and $c = 2$ is the right choice. Since $Y_{T\wedge n}$ is a martingale , $E[Y_1] = E[Y_{T\wedge n}]$ and we have

$$E[S_{T\wedge n}^4] - 6E[(T\wedge n)S_{T\wedge n}^2] + 3E[(T\wedge n)^2] + 2E[T\wedge n] = 0$$

Taking $n \to$, by DCT and MCT , we get

$$a^4 - 6a^2E[T] + 3E[T^2] + 2E[T] = 0$$

Using $E[T] = a^2$, we get $E[T^2] = (5a^4 - 2a^2)/3$ □