General Topology Facts

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Contents

1 Elementary facts about Set theory

- Elementary facts about preimage and image
  - let $f : X \to Y$ and $A \subset X, B \subset Y$.
    1. $A \subset f^{-1}(f(A))$ If f is injection, equality holds.
    2. $f(f^{-1}(B)) \subset B$ If f is surjection, equality holds.
    3. Taking preimage preserves inclusion, union, intersection and difference.
    4. Taking image preserves inclusion and union. If the mapping is injective, then taking image also preserves intersection and difference.
    5. If f is invertible map, then $(f^{-1})^{-1}(A) = f(A)$ and $f^{-1}(B) = f^{-1}(B)$ where LHS is preimage of B under f while RHS is image of B under f^{-1}

* Equivalence relation and Equivalence class
  - An equivalence relation $\sim$ on a set A is a relation having the following properties
    1. (Reflexivity) $x \sim x$ for every $x \in A$
    2. (Symmetry) $x \sim y \Rightarrow y \sim x$
    3. (Transitivity) $x \sim y$ and $y \sim z \Rightarrow x \sim z$
  - An equivalence class on A determined by $x \in A$ is given as $E = \{a \mid a \sim x\}$

- Two equivalence classes are either disjoint or equal

* Partition
  - A partition of a set A is a collection of disjoint nonempty subsets of A whose union is A

- Equivalence classes forms a partition

- A partition is derived from a unique equivalence relation.

* (Simple) Order relation
  - A (simple) order relation $<$ on a set A is a relation having the following properties
    1. (Comparability) $x \neq y \Rightarrow$ either $x < y$ or $y < x$
    2. (Nonreflexivity) There is no x s.t. $x < x$
    3. (Transitivity) $x < y$ and $y < z \Rightarrow x < z$

- Well-ordering property
  - Every nonempty subset of $\mathbb{N}$ has a smallest element

- Strong induction principle
  - A : a set of positive integers. Suppose that for each $n \in \mathbb{N}$, $\{1, 2, \cdots, n-1\} \subset A$ implies $n \in A$. Then $A = \mathbb{N}$

* ω-tuple and sequence
  - ω-tuple of elements of a set X is a function $\mathbf{x} : \mathbb{N} \to X$, which is also called as a sequence in X

- X^ω is a set of all ω-tuples of elements of X

- Elementary facts about countability

  i. A subset of countable set is countable
  ii. $\mathbb{N} \times \mathbb{N}$ is countably infinite
  iii. A countable union of countable sets is countable.
  iv. A finite product of countable sets is countable
  v. A countable products of countable sets need not be countable
     - $\{0, 1\}^\omega$ is uncountable

- For a set A, there is no surjection from A to $\mathcal{P}(A)$

* Well orderedness

  - A set A with an order relation $<$ is said to be well-ordered if every nonempty subset of A has a smallest element.

- Well-ordering principle

  - For a set A, $\exists$ an order relation on A that is well-ordering.

* Strict partial order

  - A strict partial order $\prec$ on a set A is a relation having the following properties
    i. (Nonreflexivity) $a \prec a$ never holds
    ii. (Transitivity) $a \prec b$ and $b \prec c \Rightarrow a \prec c$

□ A typical example of this relation is given by "is a proper subset of".

- The maximum principle

  - If A is a set equipped with a strict partial order $\prec$,
    then $\exists$ a maximal simply ordered subset B of A

* Upper bound and maximal element

  - A : a set equipped with a strict partial order $\prec$. If $B \subset A$ then an uppber bound on B is $c \in A$ s.t. $\forall b \in B$, either $b = c$ or $b \prec c$.
  - A maximal element of A is $m \in A$ s.t. $m \prec a$ does not hold $\forall\, a \in A$

- Zorn's Lemma

  - A : a set equipped with a strict partial order $\prec$. If every simply ordered subset of A has an upper bound in A, then A has a maximal element.

- Taking advantage of Zorn's lemma, we can prove that every vector space has a basis.

2 Topological spaces & Continuous functions

2.1 Topological Spaces

* Topology $\mathcal{T}$
  - A topology $\mathcal{T}$ on a set X is a collection of subsets of X having the following properties
    i. $\mathcal{T}$ contains ϕ and X.
    ii. $\mathcal{T}$ is closed under taking arbitrary union.
    iii. $\mathcal{T}$ is closed under taking finite intersection.

* Finer, coarser and comparable topologies
  - For two topologies $\mathcal{T}$ and $\mathcal{S}$ on a given set X,
    $\mathcal{T}$ is said to be finer than $\mathcal{S}$ if $\mathcal{S} \subset \mathcal{T}$, and coarser if $\mathcal{T} \subset \mathcal{S}$.
  - $\mathcal{T}$ and $\mathcal{S}$ are said to be comparable if either $\mathcal{T} \subset \mathcal{S}$ or $\mathcal{S} \subset \mathcal{T}$

2.2 Basis for a Topology

* A basis for a topology
  - X : a set. A basis $\mathcal{B}$ for a topology on X is a collection of subsets of X s.t.
    i. For each $x \in X$, $\exists\, B \in \mathcal{B}$ s.t.$x \in B$
    ii. If $x \in B_1 \cap B_2$ where $B_1, B_2 \in \mathcal{B}$, then $\exists\, B_3 \in \mathcal{B}$ s.t. $x \in B_3 \subset B_1 \cap B_2$

* A topology generated by basis
  - X : a set. $\mathcal{B}$: A basis on X. The topology $\mathcal{T}$ generated by the basis $\mathcal{B}$ is given as :
    Declare $\mathcal{U} \subset X$ to be open if $\forall x \in \mathcal{U}$, $\exists\, B \in \mathcal{B}$ s.t. $x \in B \subset \mathcal{U}$
  - It can be denoted as $\mathcal{B} \rightsquigarrow \mathcal{T}$

□ All basis elements generating a topology are open in the topology.

• Any open set can be represented as union of basis elements

• How to obtain the basis from given topology
  - $(X, \mathcal{T})$: a topological space. $\mathcal{C}$: a collection of open sets. If for each $\mathcal{U} \underset{open}{\subset} X$ and $x \in \mathcal{U}$, $\exists\, C \in \mathcal{C}$ s.t. $x \in C \subset \mathcal{U}$, then $\mathcal{C}$ is a basis generating $\mathcal{T}$

• How to compare two topologies using bases
  - $\mathcal{T}, \mathcal{S}$: two topologies on X. $\mathcal{B} \rightsquigarrow \mathcal{T}$ and $\mathcal{C} \rightsquigarrow \mathcal{S}$. Then $\mathcal{S}$ is finer than $\mathcal{T}$
    $\Leftrightarrow$ $\forall x \in X$ and $\forall B \in \mathcal{B}$ containing x, $\exists\, C \in \mathcal{C}$ s.t. $x \in C \subset B$

□ Larger basis generates finer topology. i.e. If $\mathcal{B} \subset \mathcal{C}$ and $\mathcal{B} \rightsquigarrow \mathcal{T}, \mathcal{C} \rightsquigarrow \mathcal{S}$ then $\mathcal{T} \subset \mathcal{S}$

(Ex) The topology on $\mathbb{R}$ generated by a basis $\mathcal{B} = \{$All bounded open intervals$\}$ is called a standard topology on $\mathbb{R}$

* A subbasis for a topology

- X : a set. A subbasis $\mathcal{S}$ for a topology on X is a collection of subsets of X whose union covers X
- The topology generated by subbasis $\mathcal{S}$ is given as below :
  Declare $\mathcal{U}$ to be open if $\mathcal{U}$ is union of finite intersection of subbasis elements.

- For a topology generated by a subbasis, basis generating the topology is given by a collection of all finite intersections of subbasis elements.

☐ All subbasis elements generating a topology are open in the topology.

- If a basis $\mathcal{B}$ generates a topology $\mathcal{T}$ then $\mathcal{T}$ is a smallest topology containing every element of $\mathcal{B}$ as open sets. # 13.5

- If a subbasis $\mathcal{S}$ generates a topology $\mathcal{T}$ then $\mathcal{T}$ is a smallest topology containing every element of $\mathcal{S}$ as open sets. # 13.5

- A basis $\mathcal{B}_{\mathbb{Q}} = \{(a,b) : a < b, a, b \in \mathbb{Q}\}$ generates the standard topology on $\mathbb{R}$, which means a standard topology on $\mathbb{R}$ can be generated by a countable basis. # 13.8

2.3 The Order Topology

* Order topology

- X : a simply ordered set. An order topology on X is generated by a basis $\mathcal{B}$ given as the following: $\mathcal{B} = \{(a,b) : a,b \in X\} \cup \{[a_0, b) : b \in X\} \cup \{(a, b_0] : a \in X\}$
  where a_0, b_0 are the smallest / largest element of X resp. if exist.

(Ex) The order topology on $\mathbb{R}$ is the standard topology.

(Ex) The order topology on $\mathbb{N}$ is the discrete topology.

- The collection of open rays forms a subbasis generating order topology.

2.4 Product Topology

* Product topology

- X, Y : topological spaces. The product topology on $X \times Y$ is generated by a basis $\mathcal{B} = \{\mathcal{U} \times \mathcal{V} : \mathcal{U} \underset{open}{\subset} X,\ \mathcal{V} \underset{open}{\subset} Y\}$

$\surd$ Note that $\mathcal{B}$ itself is not a topology since it is not closed under taking union.

- If $\mathcal{B} \rightsquigarrow \mathcal{T}_X$ and $\mathcal{C} \rightsquigarrow \mathcal{T}_Y$ then $\mathcal{D} = \{B \times C : B \in \mathcal{B},\ C \in \mathcal{C}\}$ is a basis for a product topology $\mathcal{T}_{X \times Y}$ on $X \times Y$

* Projection map

- $\pi_1 : X \times Y \to X$ given by $(x,y) \mapsto x$ and $\pi_2 : X \times Y \to Y$ given by $(x,y) \mapsto y$
  π_1 and π_2 are called as projections.

* Open map and closed map
    - A map $f : X \to Y$ is said to be an open map if $\mathcal{U} \underset{open}{\subset} X \Rightarrow f(\mathcal{U}) \underset{open}{\subset} Y$
    - A map $g : X \to Y$ is said to be a closed map if $\mathcal{F} \underset{closed}{\subset} X \Rightarrow g(\mathcal{F}) \underset{closed}{\subset} Y$
- Projection maps π_1 and π_2 are open maps # 16.4
- $\mathcal{S} = \{\pi_1^{-1}(\mathcal{U}) : \mathcal{U} \underset{open}{\subset} X\} \cup \{\pi_2^{-1}(\mathcal{V}) : \mathcal{V} \underset{open}{\subset} Y\}$ is a subbasis generating a product topology on $X \times Y$

2.5 Subspace Topology

* Subspace topology
    - $(X, \mathcal{T})$, $Y \subset X$. The subspace topology on Y inherited from X is given as $\mathcal{T}_Y = \{\mathcal{U} \cap Y : \mathcal{U} \underset{open}{\subset} X\}$. It is often denoted as $Y \underset{subsp}{\subset} X$
- $(X, \mathcal{T})$, $Y \subset X$. If a basis $\mathcal{B} \rightsquigarrow \mathcal{T}$ then a basis $\mathcal{B}_Y = \{B \cap Y : B \in \mathcal{B}\}$ generates a subspace topology on Y
- $(X, \mathcal{T})$, $Y \subset X$. If a subbasis $\mathcal{S} \rightsquigarrow \mathcal{T}$ then a subbasis $\mathcal{S}_Y = \{S \cap Y : S \in \mathcal{S}\}$ generates a subspace topology on Y
- $Y \underset{subsp}{\subset} X$. If $\mathcal{U} \underset{open}{\subset} Y$ and $Y \underset{open}{\subset} X$ then $\mathcal{U} \underset{open}{\subset} X$
- Compatibility of subspace topology and product topology
    - X, Y : topological spaces. $A \subset X$, $B \subset Y$. Consider two topologies on $A \times B$
        i. Equip A, B with subspace topology and then take their product topology.
        ii. Take subspace topology inherited from the product topology on $X \times Y$

      Then those two topologies are the same.
* Convex subset of ordered set
    - X : an ordered set. $A \subset X$. A is said to be convex if $\forall\, a, b \in A$ s.t. $a < b$, $(a, b) \subset A$
- Compatibility of subspace topology and order topology
    - X : an ordered set. $A \subset X$ convex. Then the order topology on A is same as the subspace topology on A inherited from X
- $Y \underset{subsp}{\subset} X$ and $A \subset Y$. Then the topology on A inherited as a subspace of Y is the same as the topology inherited as a subspace of X # 16.1

2.6 Closed Sets and Limit points & Hausdorff space

* Closed sets : $A \subset X$ where X is topological space. A is closed if $X \setminus A$ is open

- Closedness is dual of openness
  - For a topological space X
    i. ϕ and X are closed.
    ii. Arbitrary intersection of closed sets is closed.
    iii. Finite union of closed sets is closed.

- Closedness and subspace topology
  - For a topological space X and $Y \underset{subsp}{\subset} X$
    i. $A \underset{closed}{\subset} Y$ iff $A = C \cap Y$ for some $C \underset{closed}{\subset} X$
    ii. $A \underset{closed}{\subset} Y$ and $Y \underset{closed}{\subset} X \Rightarrow A \underset{closed}{\subset} X$

- Closedness and product topology # 17.3
  - For topological spaces X, Y, if $A \underset{closed}{\subset} X$ and $B \underset{closed}{\subset} Y$ then $A \times B \underset{closed}{\subset} X \times Y$

* Closure and interior
  - X: a topological space. $A \subset X$. Closure $\overline{A}$ is the smallest closed set containing A. Interior A^0 is the largest open set contained in A

- $Y \underset{subsp}{\subset} X$ and $A \subset Y$. Then $\overline{A}^Y = \overline{A}^X \cap Y$. i.e. Taking closure in subspace is equivalent with taking closure in ambient space and then taking intersection with subspace.

- Pointwise description of the closure
  - $(X, \mathcal{T})$, $A \subset X$, $\mathcal{B} \rightsquigarrow \mathcal{T}$. Then $x \in \overline{A} \Leftrightarrow$ every neighborhood of x intersects A $\Leftrightarrow$ every basis element containing x intersects A

* Limit point
  - X: a topological space. $A \subset X$. x is said to be a limit point of A if any neighborhood of x intersects A in some points other than x. Often A' denotes a sets of all limit points of A

- X: a topological space. $A \subset X$. $\Rightarrow$ $\overline{A} = A \cup A'$

- X: a topological space. $A \subset X$. Then $A \underset{closed}{\subset} X \Leftrightarrow A' \subset A$

- Elementary properties of the closure # 17.6, 17.8, 17.9
  i. $A \subset B \Rightarrow \overline{A} \subset \overline{B}$
  ii. $\overline{A \cup B} = \overline{A} \cup \overline{B}$
  iii. $\bigcup_{\alpha \in I} \overline{A_\alpha} \subset \overline{\bigcup_{\alpha \in I} A_\alpha}$ where I is an arbitary index set.
  iv. $\overline{A \cap B} \subset \overline{A} \cap \overline{B}$
  v. $\overline{\bigcap_{\alpha \in I} A_\alpha} \subset \bigcap_{\alpha \in I} \overline{A_\alpha}$ where I is an arbitary index set.

vi. $\overline{A} \setminus \overline{B} \subset \overline{A \setminus B}$

vii. $\overline{A \times B} = \overline{A} \times \overline{B}$

- Relationship between closure, interior and boundary # 17.19
  - For a topological space X and $A \subset X$, the boundary of A is defined as $\partial A = \overline{A} \cap \overline{X \setminus A}$

    i. $A^0 \cap \partial A = \phi$ and $\overline{A} = A^0 \cup \partial A$
    i.e. closure is a disjoint union of interior and boundary.

    ii. $\overline{X \setminus A} = X \setminus A^0$

    iii. $\partial A = \phi$ iff A is open and closed.

    iv. A is open iff $\partial A = \overline{A} \setminus A$

* T_1 space

  - A topological space satisfying "every one point set is closed" or equivalently "every finite set is closed" is T_1 space.

- Equivalent condition for T_1 axiom # 17.15
  - A topological space X is T_1 space iff $\forall\, x, y \in X$ s.t. $x \neq y$, $\exists$ neighborhood $\mathcal{U}, \mathcal{V}$ of x, y s.t. $x \notin \mathcal{V}$ and $y \notin \mathcal{U}$ i.e. any two distinct points have neighborhoods not containing the other.

* Convergence of sequence

  - X: a topological space. $\{x_n\} \underset{seq}{\subset} X$ and $x \in X$. It is said that $\{x_n\}$ converges to x if $\forall$ neighborhood $\mathcal{U}$ of x, $\exists\, N \in \mathbb{N}$ s.t. $x_n \in \mathcal{U}\ \ \forall n \geq N$

- Closure of set and convergence of sequence
  - X : a topological space. $A \subset X$. If $\exists\, \{x_n\} \underset{seq}{\subset} A$ s.t. $x_n \to x$ then $x \in \overline{A}$

* Hausdorff space (T_2 space)

  - A topological space X is called as Hausdorff space(or T_2 space) if $\forall\, x, y \in X$ s.t. $x \neq y$, $\exists$ disjoint neighborhood $\mathcal{U}, \mathcal{V}$ of x, y
    i.e. any two distinct points are separated out by disjoint neighborhoods.

- T_2 condition is stronger than T_1 condition.
- X : T_1 space. $A \subset X$. Then $x \in A' \Leftrightarrow$ any neighborhood of x contains infinitely many points of A.
- X: Hausdorff space. Then a sequence in X converges to at most one point.
- Elementary properties of Hausdorff space

  i. Every ordered set equipped with order topology is Hausdorff

  ii. The product of two Hausdorff spaces is Hausdorff

  iii. Every subspace of a Hausdorff space is Hausdorff

2.7 Continuous Functions & Homeomorphism

* X, Y : topological spaces. For a map $f : X \to Y$, f is said to be continuous if $f^{-1}(\mathcal{V}) \underset{open}{\subset} X$ whenever $V \underset{open}{\subset} Y$

□ To check whether $f : X \to Y$ is continuous, it suffices to show one of the followings.

i. Preimage of every basis element of topology on Y is open in X
ii. Preimage of every subbasis element of topology on Y is open in X

- Equivalent conditions with continuity
  - X, Y : topological spaces. For a map $f : X \to Y$, the followings are equivalent
    i. $f^{-1}(\mathcal{V}) \underset{open}{\subset} X$ whenever $V \underset{open}{\subset} Y$ i.e. f is continuous
    ii. $f^{-1}(\mathcal{B}) \underset{closed}{\subset} X$ whenever $B \underset{closed}{\subset} Y$
    iii. $f(\overline{A}) \subset \overline{f(A)} \quad \forall A \subset X$
    iv. For each $x \in X$ and a neighborhood $\mathcal{V}$ of $f(x) \in Y$, $\exists$ a neighborhood $\mathcal{U}$ of x s.t. $f(\mathcal{U}) \subset \mathcal{V}$

√ $f : X \to Y$ continuous implies that $x \in \overline{A} \Rightarrow f(x) \in \overline{f(A)}$ on the other hand, $f : X \to Y$ continuous does not guarantee that $x \in A' \Rightarrow f(x) \in \{f(A)\}'$ # 18.2

- Continuity and convergent sequence
  - $f : X \to Y$. If f is continuous then $x_n \to x \;\Rightarrow\; f(x_n) \to f(x) \quad \forall \{x_n\} \underset{seq}{\subset} X$
- Rules for constructing continuous functions
  i. Constant function is continuous.
  ii. If $A \underset{subsp}{\subset} X$, then the inclusion function $i : A \to X$ is continuous. Indeed, the definition of subspace topology is designed to make inclusion function continuous.
  iii. Projection mapping $\pi_1 : X \times Y \to X$ or $\pi_2 : X \times Y \to Y$ is continuous. Indeed, the definition of product topology is designed to make projection map continuous.
  iv. Composites of two continuous functions are continuous.
  v. Restricting the domain preserves continuity i.e. if $f : X \to Y$ continuous and $A \underset{subsp}{\subset} X$ then restricted function $f|_A : A \to Y$ is still continuous.
  vi. Restricting or expanding the target space preserves continuity. i.e. if $f : X \to Y$ continuous then restricted $f : X \to f(X)$ is also continuous and if $Y \underset{subsp}{\subset} Z$ then expanded $f : X \to Z$ is also continuous.
- Local formulation of continuity
  - If $X = \bigcup_{\alpha \in I} \mathcal{U}_\alpha$, $\mathcal{U}_\alpha \underset{open}{\subset} X \; \forall \alpha \in I$, and each $f_\alpha : \mathcal{U}_\alpha \to Y$ is continuous then $f : X \to Y$ defined by $f(x) = f_\alpha(x) I(x \in \mathcal{U}_\alpha)$ is continuous provided $f_\alpha = f_\beta$ on $\mathcal{U}_\alpha \cap \mathcal{U}_\beta \quad \forall \alpha, \beta \in I$

- The pasting lemma
  - If $X = C_1 \cup C_2$ where $C_1, C_2 \underset{closed}{\subset} X$, and $f_1 : C_1 \to Y$ & $f_2 : C_2 \to Y$ are both continuous then $f : X \to Y$ defined by $f(x) = f_i(x)I(x \in C_i) \quad \forall i = 1, 2$ is continuous, provided $f_1 = f_2$ on $C_1 \cap C_2$

(Ex) $f, g : X \to Y$ both continuous. Y is equipped with order topology.
Then $\{x \in X : f(x) \leq g(x)\}, \{x \in X : f(x) \geq g(x)\} \underset{closed}{\subset} X$ and $\max(f, g)$ and $\min(f, g)$ are also continuous. # 18.8

(Ex) $f : A \to Y$ continuous. Y is Hausdorff. If f may be extended to continuous function $g : \overline{A} \to Y$ then such g is uniquely determined by f

* Homeomorphism

  - X, Y : topological spaces. $f : X \to Y$ is called as a homeomorphism if f is bijection and both f and f^{-1} are continuous. In this case, we say X and Y are homeomorphic and denote it as $X \underset{Homeo}{\simeq} Y$

☐ For a bijection $f : X \to Y$, f is homeomorphism $\Leftrightarrow \mathcal{U} \underset{open}{\subset} X$ iff $f(\mathcal{U}) \underset{open}{\subset} Y$.

☐ For a bijection $f : X \to Y$ where $\mathcal{B} \rightsquigarrow \mathcal{T}_X$ and $\mathcal{C} \rightsquigarrow \mathcal{T}_Y$, if $f(B) \underset{open}{\subset} Y \quad \forall B \in \mathcal{B}$ and $f^{-1}(C) \underset{open}{\subset} X \quad \forall C \in \mathcal{C}$ then f is homeomorphism.

(Ex) $(0, 1) \underset{Homeo}{\simeq} \mathbb{R}$ with homeomorphism $f(x) = \frac{1}{1+e^{-x}}$ which is a standard logistic function
$(0, 1) \underset{Homeo}{\simeq} (a, b)$ for any $a < b \in \mathbb{R}$ with homeomorphism of linear transform # 18.5
so that $(a, b) \underset{Homeo}{\simeq} \mathbb{R}$ for any $a < b \in \mathbb{R}$

√ $f : X \to Y$ bijection $\Rightarrow X, Y$ are essentially the same set.
$f : X \to Y$ isomorphism $\Rightarrow X, Y$ are essentially the same vector space.
$f : X \to Y$ homeomorphism $\Rightarrow X, Y$ are essentially the same topological space.

√ A homeomorphism is simultaneously an open map and a closed map.

√ If two spaces X, Y are homeomorphic then X and Y have same topological properties, which are described in terms of open sets.

* Embedding

  - X, Y : topological spaces. $f : X \to Y$ is called as an embedding if f is injective and $X \underset{Homeo}{\simeq} f(X)$

(Ex) $f : X \to X \times Y$ defined by $x \mapsto (x, y_0)$ for some $y_0 \in Y$ is an embedding so that $X \underset{Homeo}{\simeq} X \times \{y_0\}$ # 18.4

2.8 The Product Topology : Infinite product

* J-tuple for arbitrary index set J / Infinite Cartesian product
  - J-tuple of elements of a set X is a function $\mathbf{x} : J \to X$, also denoted as $\mathbf{x} = (x_\alpha)_{\alpha \in J}$
  - X^J is a set of all J-tuples of elements of X
  - √ Given $J = \mathbb{R}$, X^J is { all functions shaped as $f : \mathbb{R} \to X$}
  - $\prod_{\alpha \in J} X_\alpha$ is defined as $\{(x_\alpha)_{\alpha \in J} \in X^J \ \ where\ X = \bigcup_{\alpha \in J} X_\alpha : x_\alpha \in X_\alpha \ \ \forall \alpha \in J\}$

* Box topology and Product topology
  - For a family of topological spaces $\{X_\alpha\}_{\alpha \in J}$, there are two types of topology we can impose on the product $\prod_{\alpha \in J} X_\alpha$
    i. The box topology on $\prod_{\alpha \in J} X_\alpha$ is generated by a basis $\mathcal{B} = \{\prod_\alpha \mathcal{U}_\alpha : \mathcal{U}_\alpha \underset{open}{\subset} X_\alpha\}$
    ii. The product topology on $\prod_{\alpha \in J} X_\alpha$ is generated by
    a subbasis $\mathcal{S} = \{\pi_\beta^{-1}(\mathcal{U}_B) : \mathcal{U}_\beta \underset{open}{\subset} X_\beta\}$ where $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ is projection

- Comparison of the box topology and the product topology
  - On $\prod_{\alpha \in J} X_\alpha$, the box topology is generated by $\{\prod_\alpha \mathcal{U}_\alpha : \mathcal{U}_\alpha \underset{open}{\subset} X_\alpha\}$ while the product topology is generated by $\{\prod_\alpha \mathcal{U}_\alpha : \mathcal{U}_\alpha \underset{open}{\subset} X_\alpha$ and $\mathcal{U}_\alpha = X_\alpha$ except for finitely many values of $\alpha\}$
  - For finite product space $\prod_{i=1}^n X_i$, box topology and product topology are the same.
  - In general, the box topology is finer than the product topology

√ Projection map $\pi_\beta : \prod_{\alpha \in J} X_\alpha \to X_\beta$ is continuous by definition of product topology. In fact, projection map is also continuous when box topology is given. But the product topology is the smallest topology which makes projection mapping continuous.
In this sense of optimality, it is suggested that the box topology may declare too many open sets.

√ Whenever we consider the product $\prod_\alpha X_\alpha$, we shall assume it is equipped with the product topology unless we specifically state otherwise.

- let $\{\mathcal{B}_\alpha\}$ be a basis for the topology on X_α for each $\alpha \in J$
  i. $\{\prod_\alpha B_\alpha : B_\alpha \in \mathcal{B}_\alpha\}$ is a basis for box topology on $\prod_{\alpha \in J} X_\alpha$
  ii. $\{\prod_\alpha B_\alpha : B_\alpha \in \mathcal{B}_\alpha$ for fin. many $\alpha's$ and $B_\alpha = X_\alpha$ for all remaining $\alpha's\}$ is a basis for product topology on $\prod_{\alpha \in J} X_\alpha$

(Ex) Standard topology on $\mathbb{R}^n$ generated by $\{\prod_{i=1}^n (a_i, b_i) : \forall\, a_i, b_i \in \mathbb{R}\}$

- Properties of product space no matter which topology we use
  i. Product topology and subspace topology are compatible
  ii. Product of Hausdorff spaces is Hausdorff space.
  iii. Product of closures is same as the closure of the product i.e. $\prod \overline{A_\alpha} = \overline{\prod A_\alpha}$

- Componentwise continuity equivalent to continuity given product topology
  - $f : X \to \prod Y_\alpha$ is given by $x \mapsto (f_\alpha(x))_{\alpha\in J}$ where $f_\alpha : X \to Y_\alpha$ for each α
    i. If f is continuous then each component f_α is continuous given box topology or product topology on $\prod Y_\alpha$.
    ii. Given product topology on $\prod Y_\alpha$, if each component f_α is continuous then f is continuous.

(Ex) The switching map $s : X \times Y \to Y \times X \quad (x, y) \mapsto (y, x)$ is continuous # Final Test

- Compoenentwise convergence equivalent to convergence given product topology # 19.6
  - $\{\mathbf{x}_n\} \underset{seq}{\subset} \prod_\alpha X_\alpha$ and $\mathbf{x} \in \prod_\alpha X_\alpha$
    i. If $\mathbf{x}_n \to \mathbf{x}$ then $\pi_\alpha(\mathbf{x}_n) \to \pi_\alpha(\mathbf{x})$ for each α, given box topology or product topology on $\prod_\alpha X_\alpha$
    ii. Given product topology on $\prod_\alpha X_\alpha$, if each component $\pi_\alpha(\mathbf{x}_n) \to \pi_\alpha(\mathbf{x})$ then $\mathbf{x}_n \to \mathbf{x}$

- Componentwise linear transform is homeomorphism # 19.8
  - $\{a_n\}, \{b_n\} \underset{seq}{\subset} \mathbb{R}$ where $a_i > 0\ \forall i \in \mathbb{N}$. $f : \mathbb{R}^\omega \to \mathbb{R}^\omega$ is defined as
    $\{x_n\} \mapsto \{a_n x_n + b_n\}$ $\forall$ real sequence $\{x_n\}$. f is homeomorphism of $\mathbb{R}^\omega$ with itself. This holds no matter $\mathbb{R}^\omega$ is equipped with product topology or box topology.

2.9 The Metric Topology

* A metric d on a set X is a function $d : X \times X \to \mathbb{R}$ satisfying

  i. (Positive definite) $d(x, y) \to 0\ \forall x, y \in X$ and $d(x, y) = 0 \Leftrightarrow x = y$
  ii. (Symmetric) $d(x, y) = d(y, x)\ \forall x, y \in X$
  iii. (Triangle Inequality) $d(x, y) + d(y, z) \geq d(x, z)\ \forall x, y, z \in X$

* ϵ-ball centered at x ; $B_d(x, \epsilon)$

  - $B_d(x, \epsilon) = \{y \mid d(x, y) < \epsilon\}$

- For any point inside the ball, there is a smaller ball centered at the point, contained in the given ball. i.e. $\forall y \in B_d(x, \epsilon), \exists\, \delta > 0$ s.t. $B_d(y, \delta) \subset B_d(x, \epsilon)$

* Metric topology on (X, d)

  - A metric topology on (X, d) is generated by a basis $\mathcal{B} = \{B_d(x, \epsilon) : x \in X, \epsilon > 0\}$

* Metric space and metrizability

  - (X, d) is a metric space if X is equipped with the metric topology induced by d
  - X is said to be metrizable if $\exists$ a metric d on X which induces a metric topology same with the one imposed on X. Obviously every metric space is metrizable.

√ Not every topological space is metrizable.

- A space homeomorphic to metric space is metrizable
  - If (X, d) is a metric space and $X \underset{Homeo}{\simeq} Y$ with homeomorphism $f : X \to Y$ then Y is metrizable with metric ρ defined by $\rho(y, z) = d(g(y), g(z))$ where $g = f^{-1}$

(Ex) A standard topology on $\mathbb{R}$ is same with metric topology induced by metric $d(x, y) = |x-y|$

(Ex) A discrete topology on a set X is metrizable by the metric $d(x, y) = 1 - I(x = y)$

* Boundedness and diameter
  - (X, d) : a metric space. $A \subset X$. For nonempty A, diameter of A is defined as diam(A)=sup$\{d(a, b) : a, b \in A\}$. A is said to be bounded if diam(A)$< \infty$

- Useful facts about diameter and closure
  - (X, d) : a metric space. $A \subset X$. $\Rightarrow$ diam(A)=diam($\overline{A}$)

* Standard bounded metric
  - (X, d) : a metric space. $\overline{d} : X \times X \to \mathbb{R}$ defined as $\overline{d}(x, y) = \min\{d(x, y), 1\}$ is called as a standard bounded metric corresponding to d

- Collection of small balls is sufficient to generate a metric topology.
  - For a metric space (X, d), $\mathcal{B}_1 = \{B_d(x, \epsilon) : x \in X, 0 < \epsilon < 1\}$ is a basis for the metric topology.

- Standard bounded metric $\overline{d}$ induces same metric topology with corresponding metric d

- ℓ_p-norm and metric on Euclidean space
  - For $\mathbf{x} \in \mathbb{R}^n$, $\|\mathbf{x}\|$ is ℓ_2-norm(or also called as Euclidean norm) and $\|\mathbf{x}\|_\infty$ is ℓ_∞-norm
  - Denote $d(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|$ and $\rho(\mathbf{x}, \mathbf{y}) = \|\mathbf{x} - \mathbf{y}\|_\infty$ where d is called as Euclidean metric and ρ is called as square metric.

- Relationship between Euclidean metric and square metric
  - For Euclidean metric d and square metric ρ and $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ inequality below holds ; $\rho(\mathbf{x}, \mathbf{y}) \leq d(\mathbf{x}, \mathbf{y}) \leq \sqrt{n}\, \rho(\mathbf{x}, \mathbf{y})$

- How to compare two metric topologies using bases
  - $\mathcal{T}_d, \mathcal{T}_{d'}$ are two metric topologies on X induced by d, d' respectively.
    $\mathcal{T}_d \subset \mathcal{T}_{d'} \Leftrightarrow \forall x \in X, \forall \epsilon > 0,\ \exists\, \delta > 0$ s.t. $B_{d'}(x, \delta) \subset B_d(x, \epsilon)$

- Standard topology on $\mathbb{R}^n$ is induced by Euclidean metric d or square metric ρ

☐ Denote a metric on $\mathbb{R}^n$ induced by ℓ_1-norm as d'. Then $d(\mathbf{x}, \mathbf{y}) \leq d'(\mathbf{x}, \mathbf{y}) \leq \sqrt{n}\, d(\mathbf{x}, \mathbf{y})$ holds and d' also induces standard topology on $\mathbb{R}^n$ # 20.1

* Uniform metric and uniform topology
  - J : an arbitrary index set. A uniform metric $\overline{\rho}$ on $\mathbb{R}^J$ is defined by
    $\overline{\rho}(\mathbf{x}, \mathbf{y}) = \sup\{\overline{d}(x_\alpha, y_\alpha) : \alpha \in J\}\ \forall\, \mathbf{x} = (x_\alpha)_{\alpha \in J}, \mathbf{y} = (y_\alpha)_{\alpha \in J}$
    where $\overline{d}$ is a standard bounded metric on $\mathbb{R}$

- A metric topology on $\mathbb{R}^J$ induced by uniform metric is called as uniform topology.

- On $\mathbb{R}^J$, Product topology $\subset$ Uniform topology $\subset$ Box topology.
  If J is finite, all three topologies are the same and if J is infinite then all three are different.

☐ Representation of basis element for uniform topology on $\mathbb{R}^\omega$ #20.6

  - Define $U(\mathbf{x}, \epsilon) = \prod_{n=1}^{\infty}(x_n - \epsilon, x_n + \epsilon) \quad \forall\, \mathbf{x} = (x_n)_{n\in\mathbb{N}}$ and $\forall\ 0 < \epsilon < 1$
    Basis element for uniform topology on $\mathbb{R}^\omega$ is $B_{\overline{\rho}}(\mathbf{x}, \epsilon) = \bigcup_{\delta<\epsilon} U(\mathbf{x}, \delta)$

- $\mathbb{R}^\omega$ equipped with product topology is metrizable by metric D defined as
  $D(\mathbf{x}, \mathbf{y}) = \sup\{\frac{\overline{d}(x_n, y_n)}{n} : n \in \mathbb{N}\}\ \forall\ \mathbf{x} = (x_n)_{n\in\mathbb{N}}, \mathbf{y} = (y_n)_{n\in\mathbb{N}}$

☐ Countable product of metric spaces is metrizable.

- Subspace topology and metric topology are compatible.

  - $(X, d),\ A \subset X$. Subspace topology on A inherited from metric space X agrees with the metric topology on A induced by restricted metric $d|_A$

- Every metric space is Hausdorff space.

- On metric space (X, d), taking metric $(x, y) \mapsto d(x, y)$ is a continuous map. #20.3

- Metric topology on X induced by a metric d is the smallest topology that makes the mapping of taking metric d continuous # 20.3

- (X, d) : a metric space. Then metric d' defined as $d'(x, y) = \frac{d(x,y)}{1+d(x,y)}$ is a bounded metric which imposes same topology induced by d #20.11

2.10 Metric Topology and Continuous Functions

- $\epsilon - \delta$ definition of continuity carries over to general metric spaces.

  - $(X, d_X), (Y, d_Y)$, $f : X \to Y$, f is continuous iff $\forall x \in X, \epsilon > 0,\ \exists\, \delta > 0$ s.t.
    $y \in X$ and $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$

- The sequence lemma

  - X : a topological space. $A \subset X$

    i. If $\exists\, \{x_n\} \underset{seq}{\subset} A$ converging to $x \in X$ then $x \in \overline{A}$

    ii. Provided X is metrizable, if $x \in \overline{A}$ then $\exists\, \{x_n\} \underset{seq}{\subset} A$ converging to $x \in X$

- Convergent sequence definition of continuity carries over to general metric spaces.

  - X, Y : topological spaces. $f : X \to Y$

    i. If f is continuous then $x_n \to x \;\Rightarrow\; f(x_n) \to f(x) \quad \forall\, \{x_n\} \underset{seq}{\subset} X$

    ii. Provided X is metrizable, if $x_n \to x \;\Rightarrow\; f(x_n) \to f(x) \quad \forall\, \{x_n\} \underset{seq}{\subset} X$ then f is continuous.

- Elementary algebraic operations '+', '−', '×', and '÷' are all continuous. # 21.12
- Additional methods of constructing continuous functions
  - X : a topological space. f, g : real valued functions defined on X.
    If f, g are continuous then $f+g$, $f-g$, and $f \cdot g$ are continuous and f/g is also continuous given $g \neq 0$ on X.
- Uniform convergence
  * (Y, d) : a metric space. $\{f_n\}$ is a a function seq. $f_n : X \to Y$ and $f : X \to Y$.
    $\{f_n\}$ is said to converge uniformly to f if $\forall \epsilon > 0, \exists N \in \mathbb{N}$ s.t.
    $d(f_n(x), f(x)) < \epsilon \quad \forall n \geq N$ for any $x \in X$ i.e. N does not depend on $x \in X$.
  - Often denoted as $f_n \rightrightarrows f$
- Uniform limit theorem
  - X : a topological space. (Y, d) : a metric space. $\{f_n\}$: a function seq. $f_n : X \to Y$ and $f : X \to Y$. If each f_n is continuous and $f_n \rightrightarrows f$ then f is continuous.
    i.e. the uniform limit of sequence of continuous functions must be continuous.

□ $\mathbb{R}^X$ is a function space $\{f \mid f : X \to \mathbb{R}\}$ for a given set X. The uniform metric on $\mathbb{R}^X$ is defined as $\overline{\rho}(f, g) = \sup\{\overline{d}(f(x), g(x)) : x \in X\} \quad \forall f, g \in \mathbb{R}^X$ where $\overline{d}$ is standard bounded metric on $\mathbb{R}$.
Given uniform topology equipped on $\mathbb{R}^X$, for a sequence of real-valued functions $\{f_n\}$ and a real-valued function f defined on X, $\quad f_n \rightrightarrows f \Leftrightarrow \{f_n\}$ converges to f in $\mathbb{R}^X$. # 21.7

- $(X, d_X), (Y, d_Y)$: metric spaces. $f : X \to Y$ f is isometry i.e. $d_X(x_1, x_2) = d_Y(f(x_1), f(x_2))$
  Then f is an (isometric) embedding i.e. $X \underset{Homeo}{\simeq} f(X)$ # 21.2
- X : a topological space. (Y, d) : a metric space. $\{f_n\}$: a function seq. $f_n : X \to Y$ and $f : X \to Y$. $\{x_n\} \underset{seq}{\subset} X$ and $x \in X$. If $x_n \to x$ and $f_n \rightrightarrows f$ then $f_n(x_n) \to f(x)$ # 21.8

2.11 Quotient Topology

* Quotient map
  - X, Y : topological spaces. $p : X \to Y$ is a surjective map. p is said to be a quotient map if $\mathcal{V} \underset{open}{\subset} Y \Leftrightarrow p^{-1}(\mathcal{V}) \underset{open}{\subset} X$, or equivalently, $F \underset{closed}{\subset} Y \Leftrightarrow p^{-1}(F) \underset{closed}{\subset} X$

√ Every quotient map is continuous. Every continuous surjective open map or closed map is a quotient map. Bijective quotient map and homeomorphism are the same.

√ Composition of quotient map is quotient map.

√ Product of quotient maps need not be a quotient map.

√ A quotient map does not preserve Hausdorff condition in general.

- $p : X \to Y$ continuous. If $\exists f : Y \to X$ continuous s.t. $p \circ f = (identity)_Y$ then p is a quotient map. # 22.2

* Quotient topology

  - X : a topological space. A : a set. $p : X \to A$ is a surjective map. The quotient topology on A induced by p is the unique topology on A which makes p into a quotient map. Indeed the quotient topology on A declares $\mathcal{U} \underset{open}{\subset} A$ whenever $p^{-1}(\mathcal{U}) \underset{open}{\subset} X$

* Quotient space

  * X : a topological space. X^* : a partition of X i.e. $X^* = \{X_\alpha : X = \bigcup_{\alpha \in J} X_\alpha$ is a disjoint union$\}$. Define a surjective map $p : X \to X^*$ by $x \mapsto X_\alpha \; \forall x \in X_\alpha$.
    X^* is said to be a quotient space of X if it is equipped with quotient topology induced by p.

  √ Since a mapping $\alpha \mapsto X_\alpha$ is a bijection, $X^* \simeq J$ as sets. Also we can define equivalence relation "$\sim$" by the partition X^* given by $x \sim y \Leftrightarrow p(x) = p(y)$. By this reason, we denote $X^* = X/\sim$. If $B \underset{open}{\subset} X^*$ then B is a collection of equivalence classes whose union is opet subset of X.

- Continuous functions on the quotient space.

  - $p : X \to Y$ is a quotient map. $g : X \to Z$ is a map which is constant on $p^{-1}(\{y\})$ for each $y \in Y$. Define $f : Y \to Z$ as an induced map by g s.t. $f(y) = g(p^{-1}(y)) \; \forall y \in Y$

    i. f is continuous $\Leftrightarrow$ g is continuous
    ii. f is a quotient map $\Leftrightarrow$ g is a quotient map

  □ $g : X \to Z$ is a continuous surjection. $X^* = \{g^{-1}(\{z\}) : z \in Z\}$
    (Note that since g is surjective, $X^* \simeq Z$ as sets) Regard X^* as the quotient space and let $f : X^* \to Z$ be induced by g as above. Then all the following hold true.

    i. f is continuous bijection
    ii. f is homemorphism $\Leftrightarrow$ g is a quotient map
    iii. Z is Hausdorff space $\Rightarrow$ X^* is Hausdorff.

3 Connectedness and Compactness

3.1 Connected Spaces

* Connectedness and separation

  - X : a topological space. A separtion of X is a nonempty disjoint pair of open sets $\mathcal{U}, \mathcal{V} \subset X$ whose union is X. X is said to be connected if $\nexists$ separation of X.

  √ Connectedness is a topological property so that if $X \underset{Homeo}{\simeq} Y$ and X is connected then Y is also connected.

  √ If $(\mathcal{U}, \mathcal{V})$ is a separation of X, then $\mathcal{U}, \mathcal{V}$ are both open and closed (or clopen) in X.

- X is connected $\Leftrightarrow$ X is the only nonempty clopen subset of X.

(Ex) As proved in the next section, $\mathbb{R}^n$ is connected. So there is no proper nonempty clopen subset of Euclidean space. It means that if $A \subset \mathbb{R}^n$ is open then A is not closed and if $B \subset \mathbb{R}^n$ is closed then B is not open, provided A, B are nonempty and not $\mathbb{R}^n$ itself.

- Separation of subspace

  - $Y \underset{subsp}{\subset} X$. A separation of Y is a pair of disjoint nonempty subsets $A, B \subset Y$ whose union is Y, satisfying neither of which contains limit point(taken in X) of the other.

(Ex) $\mathbb{Q}$ is totally disconnected i.e. the only connected subspaces of $\mathbb{Q}$ are one-point sets.

- $Y \underset{subsp}{\subset} X$. If $(\mathcal{U}, \mathcal{V})$ is a separation of X and Y is connected then $Y \subset \mathcal{U}$ or $Y \subset \mathcal{V}$.

- Union of connected subspaces of X having a common point is connected.

- $A \underset{subsp}{\subset} X$. If A is connected and $A \subset C \subset \overline{A}$ then C is also connected.

□ $A \underset{subsp}{\subset} X$. A is connected $\Rightarrow$ $\overline{A}$ is connected.

- Image of a connected space under continuous map is connected.

- A finite product of connected spaces is connected.

□ An arbitrary product of connected space is also connected. #23.10

- If X are equipped with the discrete topology then X is totally disconnected # 23.5

- $A \subset X$. If $C \underset{subsp}{\subset} X$ is connected and intersects both A and $X \setminus A$ then C intersects ∂A # 23.6

3.2 Connected Subspaces of $\mathbb{R}$

- $\mathbb{R}$ is connected and every interval or every ray in $\mathbb{R}$ is connected.
- Intermediate value theorem ; IVT
  - X : a topological space. Y : an ordered space. If X is connected and $f : X \to Y$ is continuous then for $x, y \in X$ and $\alpha \in Y$ s.t. α lies between $f(x)$ and $f(y)$ then $\exists\, z \in X$ s.t. $f(z) = \alpha$

  □ $f : \mathbb{R} \to \mathbb{R}$. $x, y \in \mathbb{R}$ s.t. $x < y$. If f is continuous and γ lies between $f(x)$ and $f(y)$ then $\exists\, z \in (x, y)$ s.t. $f(z) = \gamma$

* Path Connectedness
  - $x, y \in X$. p is called as a path from x to y if $p : [a, b] \to X$ is a continuous map with $p(a) = x$, $p(b) = y$. X is said to be path connected if every pair $x, y \in X$ can be joined by a path in X.

- Every path connected space is connected.
- Image of path connected space under continuous map is path connected. # 24.8
- Product of path connected spaces is path connected. # 24.8
- Union of path connected subspaces of X having a common point is path connected. # 24.8
- $A \underset{subsp}{\subset} X$. "$A$ is path connected" need not imply that $\overline{A}$ is path connected. # 24.8

(Ex) No two spaces of $(0, 1)$ $(0, 1]$, $[0, 1]$ are homeomorphic. # 24.1

(Ex) $\mathbb{R}^n$ and $\mathbb{R}$ are not homeomorphic for every $n > 1$ # 24.1

3.3 Components and Local Connectedness

* (Connected) Component
  - A (connected) component of X is a maximal connected subspace of X. Formally, we can set equivalence relation $x \sim y$ if $\exists$ connected subspace of X containing x, y. Then the equivalence classes are called components of X.

- The components of X are connected disjoint subspaces of X whose union is X. Each nonempty connected subspace of X intersects only one of them.
- Every component is always a closed subspace.
- Any nonempty clopen subset of X contains whole points of component which it intersects. If any nonempty clopen subset of X is connected then it is a component of X.

* Path component
  - A path component of X is a maximal path connected subspace of X.

- The path components of X are path connected disjoint subspaces of X whose union is X. Each nonempty path connected subspace of X intersects only one of them.

□ Each path component is contained in a component

* Local Connectedness and Local path connectedness

  - X is said to be locally connected / path connected at x if for any neighborhood $\mathcal{U}$ of x, $\exists$ connected / path connected neighborhood $\mathcal{V}$ of x s.t. $x \in \mathcal{V} \subset \mathcal{U}$. If this happens for every $x \in X$ then X is called locally connected / path connected.

  √ Local connectedness means each point has arbitrary samll connected neighborhood.

- X is locally connected $\Leftrightarrow \forall\, \mathcal{U} \underset{open}{\subset} X$, each component of $\mathcal{U}$ is open in X.

- X is locally path connected $\Leftrightarrow \forall\, \mathcal{U} \underset{open}{\subset} X$, each path component of $\mathcal{U}$ is open in X.

√ If X is locally connected / path connected then every component / path component is open. Since every component is always closed, locally connected space has a partition consisting of clopen connected subspaces.

- If X is locally path connected then the components and the path components of X are the same.

□ If X is locally path connected then X is connected $\Leftrightarrow$ X is path connected.

- If X is locally connected and compact then the number of components of X is finite. #Final test.

3.4 Compact Spaces

* Compactness

  - X is said to be compact if every open covering of X contains a finite subcollection that also covers X.

□ Trivially, every finite set is compact.

- Compactness of Subspace

  - $Y \underset{subsp}{\subset} X$. Y is compact $\Leftrightarrow$ every covering of Y by open sets in X contains a finite subcollection that also covers Y.

- Every closed subspace of a compact space is compact.

- Every compact subspace of a Hausdorff space is closed.

- In Hausdorff space, a compact subspace and a point outside the subspace can be separated out by disjoint neighborhoods. i.e. if X is Hausdorff, $C \subset X$ is compact and $x \notin C$ then $\exists\, \mathcal{U}, \mathcal{V}$ s.t. $x \in \mathcal{U}, C \subset \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \phi$

☐ Two disjoint compact subspaces of Hausdorff space can be separated out by disjoint neighborhoods. i.e. if X is Hausdorff, $A, B \subset X$ are compact then $\exists\, \mathcal{U}, \mathcal{V}$ s.t. $A \subset \mathcal{U}, B \subset \mathcal{V}$ and $\mathcal{U} \cap \mathcal{V} = \phi$ #26.5

- Every compact subspace of a metric space is closed and bounded in that metric. # 26.4
- The image of compact space under a continuous map is compact.

(Ex) There is no continuous surjective map from a sphere S^2 to $\mathbb{R}^2$

- Finite union of compact spaces is compact.
- If $f : X \to Y$ is continuous, X is compact and Y is Hausdorff then f is homeomorphism
- The tube lemma
  - Suppose Y is compact. $x \in X$. If $N \underset{open}{\subset} X \times Y$ contains the slice $\{x\} \times Y$ then $\exists\, W \underset{open}{\subset} X$ neighborhood of x s.t. $\{x\} \times Y \subset W \times Y \subset N$

☐ Generalization of the tube lemma #26.9

  - $A \underset{subsp}{\subset} X$, $B \underset{subsp}{\subset} Y$. Suppose A and B are compact. If $N \underset{open}{\subset} X \times Y$ contains $A \times B$ then $\exists\, \mathcal{U} \underset{open}{\subset} X$, $\mathcal{V} \underset{open}{\subset} Y$ s.t. $A \times B \subset \mathcal{U} \times \mathcal{V} \subset N$

- The product of finitely many compact spaces is compact.
- Furthermore, using Zorn's lemma, it is proved that arbitary product of compact spaces is compact. (Tychonoff theorem)
- Finite intersection property ; F.I.P.
  - A collection $\mathcal{C}$ of subsets of X is said to have the finite intersection property (F.I.P.) if for any finite subcollection $\{C_1, \cdots, C_n\}$ of $\mathcal{C}$, the intersection $\bigcap_{i=1}^{n} C_i$ is nonempty

  √ A nested sequence of nonempty sets is a typical example of collection having F.I.P.
- Dual definition of Compactness
  - X is compact $\Leftrightarrow$ for any collection $\mathcal{C}$ of closed sets in X having F.I.P. , the intersection $\bigcap_{C \in \mathcal{C}} C$ is nonempty.
- If Y is compact then projection map $\pi_1 : X \times Y \to X$ is a closed map. #26.7
- Closed graph theorem #26.8
  - $f : X \to Y$. The graph of f is defined as $G_f = \{(x, f(x)) : x \in X\} \subset X \times Y$. Suppose Y is compact Hausdorff. Then the followings are equivalent.
    (a) f is continuous. (b) $G_f \underset{closed}{\subset} X \times Y$ (c) $X \underset{Homeo}{\simeq} G_f$
  - If Y is compact Hausdorff, then the following holds true.
    $f : X \to Y$ continuous $\Leftrightarrow$ the map $(identity)_X \times f : X \to X \times Y \quad x \mapsto (x, f(x))$ is embedding of X to a closed set of $X \times Y$, which is a graph of f.

3.5 Compact Subspaces of $\mathbb{R}$

- Every closed interval in $\mathbb{R}$ is compact.

□ $[a_1, b_1] \times \cdots \times [a_n, b_n]$ in $\mathbb{R}^n$ is compact.

- Heine-Borel theorem
  - $A \underset{subsp}{\subset} \mathbb{R}^n$. Then A is compact $\Leftrightarrow$ A is closed and bounded in the Euclidean metric d or the square metric ρ.

- Max-Min value theorem
  - X : a topological space. Y : an ordered space. If X is compact and $f : X \to Y$ is continuous then $\exists\, x_m,\, x_M$ s.t. $f(x_m) \leq f(x) \leq f(x_M) \quad \forall x \in X$

* Distance from a point to a set.
  - (X, d) : a metric space. $A \subset X$ nonempty. For each $x \in X$, the distance from x to A is defined as $d(x, A) = \inf\{d(x, a) : a \in A\}$

- Properties of distance from a point to a set. # 27.2

  i. $d(x, A) = 0 \Leftrightarrow x \in \overline{A}$

  ii. If A is compact then $d(x, A) = d(x, a)$ for some $a \in A$

  iii. Define $N(A, \epsilon) = \{x \in X : d(x, A) < \epsilon\}$. Then $N(A, \epsilon) = \bigcup_{a \in A} B_d(a, \epsilon)$

  iv. If A is compact then $\forall\, \mathcal{U} \underset{open}{\subset} X$ containing A, then $\exists\, \epsilon > 0$ s.t. $N(A, \epsilon) \subset \mathcal{U}$

- For a given nonempty $A \subset X$, mapping $x \mapsto d(x, A)$ from X to $\mathbb{R}$ is continuous.
  In fact, $|d(x, A) - d(y, A)| \leq d(x, y) \quad \forall x, y \in X$ holds true.

- The Lebesgue number lemma
  - (X, d) : a metric space. let $\mathcal{A}$ be an open covering of X. If X is compact then $\exists\, \delta > 0$ (depending on $\mathcal{A}$) satisfying $\forall\, B \subset X$ with $diam(B) < \delta$, $\exists\, A \in \mathcal{A}$ s.t. $B \subset A$. Such δ is called a Lebesgue number for the covering $\mathcal{A}$.

* Uniform continuity
  - $(X, d_X), (Y, d_Y)$: metric spaces. $f : X \to Y$ is said to be uniformly continuous if $\forall \epsilon > 0$, $\exists\, \delta > 0$ s.t. $d_X(x, y) < \delta \Rightarrow d_Y(f(x), f(y)) < \epsilon$ for any pair x, y of X

- Uniform continuity theorem
  - $(X, d_X), (Y, d_Y)$: metric spaces. If X is compact and $f : X \to Y$ is continuous then f is uniformly continuous.

* Isolated point
  - X : a topological space. $x \in X$ is said to be an isolated point of X if $\{x\} \underset{open}{\subset} X$

- Limit point and isolated point
  - X : a topological sapce. $A \subset X$. For $x \in A$,
    1. If x is a limit point of A then x is not an isolated point of A.
    2. If x is an isolated point of A then x is not a limit point of A.
- If nonempty X is compact Hausdorff and have no isolated point, then X is uncountable.

□ Any intervals in $\mathbb{R}$ is uncountable and $\mathbb{R}$ itself is uncountable.

(Ex) Cantor set C

i. We can represent $C = \bigcap_{n=1}^{\infty} I_n$ where each set I_n is a union of 2^n many disjoint closed intervals with length $1/3^n$. All endpoints of these intervals lie in C.

ii. C is totally disconnected.

iii. C is compact.

iv. Every point of C is a limit point of C so that C has no isolated point.

v. C is uncountable.

3.6 Limit point Compactness & Sequential Compactness

* Limit point compactness

- X : a topological space. X is said to be limit point compact if every infinite subset of X has a limit point. (Also called as "Bolzano-Weierstrass property")

- Compactness is stronger than limit point compactness

* Sequential compactness

- X : a topological space. X is said to be sequentially compact if every sequence in X has a convergent subsequence converging to a point in X.

* Totally Boundedness

- (X, d) : a metric space. X is said to be totally bounded if $\forall\ \epsilon > 0$, $\exists$ a finite covering of X consisting of ϵ-balls.

- (X, d) : a metric space. If X is sequentially compact then
  1. The Lebesgue number lemma also holds.
  2. X is totally bounded.
- For any metric space, compactness, limit point compactness, and sequential compactness are all equivalent.
- Fixed point of shrinking map in compact metric space # 28.7
  - (X, d) : a metric space. $f : X \to X$. f is called as a shrinking map if $d(f(x), f(y)) < d(x, y)$ whenver $x \neq y$. If X is compact and f is a shrinking map on X then f has a unique fixed point i.e. $\exists\,!\, x \in X$ s.t. $f(x) = x$

* Countable compactness

- X : a topological space. X is said to be countably compact if every countable open covering of X contains a finite subcollection that also covers X.

- Relation of countable compactness and limit point compactness # 28.4

  - Countable compactness implies limit point compactness
    With T_1 axiom, limit point compactness and countable compactness are equivalent.

- (X, d) : a metric space. If X is compact and $f : X \to X$ is a isometry then f is homeomorphism. # 28.6

(Notes) Various concepts of compactness

- $(Compactness) \Rightarrow (Countable\ Compactness) \Rightarrow (Limit\ point\ Compactness)$
- $(Compactness) \underset{Lindelof}{\Leftarrow} (Countable\ Compactness) \underset{T_1\ axiom}{\Leftarrow} (Limit\ point\ Compactness)$
- In metric space, $(Compactness) \Leftrightarrow (Limit\ point\ Compactness) \Leftrightarrow (Sequential\ Compactness)$
- In metric space, $(Compact\ subspacce) \Rightarrow (Closed\ and\ Bounded\ subspace)$
- In Eucldiean space, $(Compact\ subspacce) \Leftrightarrow (Closed\ and\ Bounded\ subspace)$

3.7 Local Compactness

* Local compactness

- X : a topological space. X is said to be locally compact at $x \in X$ if $\exists$ compact $C \subset X$ containing some neighborhood of x. X is said to be locally compact if it is locally compact at every point.

√ Clearly compactness is stronger than local compactness.

(Ex) Euclidean space is locally compact.

- X : a topological space. X is locally compact Hausdorff $\Leftrightarrow \exists$ a topological space Y s.t. (a) Y is compact Hausdorff. (b) $Y \setminus X$ is one point set. (c) X is an open subspace of Y

  - $Y = X \cup \{\infty\}$ equipped with topoogy $\{\mathcal{U} : \mathcal{U} \underset{open}{\subset} X\} \cup \{Y \setminus C : C \subset X \text{ compact }\}$
  - Such space Y is unique up to homeomorphism.

√ If X is compact Hausdorff then obtained $Y = X \cup \{\infty\}$ is not interesting since in this case ∞ is an isolated point of Y. If X is non-compact but locally compact Hausdorff, then ∞ in obtained $Y = X \cup \{\infty\}$ is limit point of X in Y, so that $\overline{X} = Y$.

* Compactification

- Y : compact Hausdorff space. If X : a proper subspace of Y s.t. $\overline{X} = Y$ then Y is said to be a compactification of X. If $Y \setminus X$ is one point set then Y is said to be the one point compactification of X.

- ☐ X is locally compact Hausdorff but non-compact $\Leftrightarrow$ X has the one point compactification.

- $X_1 \underset{Homeo}{\simeq} X_2$ are locally compact Hausdorff spaces with homeomorphism f. Then f can be extended to homeomorphism of their one point compactifications. # 29.5

(Ex) One point compactification of $\mathbb{R}$ is homeomorphic to the circle S^1 #29.6

(Ex) One point compactification of $\mathbb{N}$ is homeomorphic to $\{1/n : n \in \mathbb{N}\} \cup \{0\}$ #29.8

- Natural local property of local compactness
  - X : Hausdorff space. X is locally compact $\Leftrightarrow$ $\forall\, x \in X$ and $\forall$ neighborhood $\mathcal{U}$ of x, $\exists$ a neighborhood $\mathcal{V}$ of x s.t. $x \in \overline{\mathcal{V}} \subset \mathcal{U}$ and $\overline{V}$ is compact.

- Every open subspace of locally compact Hausdorff space is locally compact Hausdorff. Every closed subspace of locally compact Hausdorff space is locally compact Hausdorff.

- X is homeomorphic to an open subspace of a compact Hausdorff space
$\Leftrightarrow$ X is locally compact Hausdorff.

- √ Our favorite well-behaved spaces are metrizable spaces and compact Hausdorff spaces. If given space is not one of those spaces then the next best thing is that it is a subspace of one of those spaces. Notice that subspace of metrizable space is also metrizable so nothing new happens. However, subspace of compact Hausdorff space can be modeled as a locally compact Hausdorff space.

4 Countability and Separation Axioms

4.1 The Countability Axioms

* First countablility

  – X : a topological space. $x \in X$. X is said to have a countable basis at x if $\exists$ a countable collection $\mathcal{B}$ of neighborhoods of x s.t. each neighborhood $\mathcal{U}$ of x contains at least one of the elements of $\mathcal{B}$. X is said to be first countable if X has a countable basis at every point.

☐ Every metric space is first countable.

• Revisiting sequence lemma and convergent sequence definition of continuity

  – Provided X is first countable, if $x \in \overline{A}$ then $\exists \{x_n\} \underset{seq}{\subset} A$ converging to $x \in X$

  – Provided X is first countable, if $x_n \to x \;\Rightarrow\; f(x_n) \to f(x) \quad \forall \{x_n\} \underset{seq}{\subset} X$ then f is continuous.

* Second countability

  – X is said to be second countable if X has a countable basis for the topology which X is equipped with.

☐ Obviously, second countability is stronger than first countability.

(Ex) $\mathbb{R}$ is second countable with a countable basis $\mathcal{B}_{\mathbb{Q}} = \{(a,b) : a, b \in \mathbb{Q}\}$

• The set of all finite subsets of $\mathbb{N}$ is countable.

(Ex) $\mathbb{R}^\omega$ equipped with product topology is second countable.

√ Not every metric space is second countable.

• First and Second countability are preserved under taking subspace or countable product.

* Lindelof space

  – X : a topological space. X is said to be a Lindelof space if every open covering of X contains a countable subcollection which also covers X.

* Separable space

  – X : a topological space. X is called as separable if $\exists$ a countable dense subset of X

• Lindelof space, Separable space, and second countability

  i. Second countable space is Lindelof space and separable.
  ii. For metric space, (Separable) $\Leftrightarrow$ (Second countable) $\Leftrightarrow$ (Lindelof) # 30.5

(Ex) $I = [0,1]$. Impose the uniform metric on $\mathbb{R}^I$. let $\mathcal{C}(I,\mathbb{R}) \underset{subsp}{\subset} \mathbb{R}^I$ be a space of continuous real valued functions on $[0,1]$. $\mathcal{C}(I,\mathbb{R})$ has a countable dense subset $Q(I,\mathbb{R})$ which is a set of all poloynomials on I with rational coefficients. Hence, $\mathcal{C}(I,\mathbb{R})$ has a countable basis.

- X is Lindelof space $\Leftrightarrow$ For every collection $\mathcal{A}$ of subsets of X having countable intersection property, $\bigcap_{A \in \mathcal{A}} \overline{A}$ is nonempty. # 37.2
- Every compact metric space has a countable basis. # 30.4
- G_δ set in X is a set that equals a countable intersection of open sets in X. If X is first countable T_1 space then every one point set in X is a G_δ set. #30.1

4.2 The Separation Axioms

* Regular space(T_3 space) and Normal space(T_4 space)

  i. A topological space X is said to be regular (or T_3 space) if X is T_1 space and each pair of a closed set and a point outside the set are separated out by disjoint neighborhoods. i.e. $\forall B \underset{closed}{\subset} X$ and $x \notin B$, $\exists$ disjoint $\mathcal{U}, \mathcal{V} \underset{open}{\subset} X$ s.t. $x \in \mathcal{U}, B \subset \mathcal{V}$

  ii. A topological space X is said to be normal (or T_4 space) if X is T_1 space and each pair of disjoint closed sets are separated out by disjoint neighborhoods. i.e. $\forall$ disjoint $A, B \underset{closed}{\subset} X$, $\exists$ disjoint $\mathcal{U}, \mathcal{V} \underset{open}{\subset} X$ s.t. $A \subset \mathcal{U}, B \subset \mathcal{V}$

- Local ways to formulate regularity and normality

  i. Provided X is a T_1 space, X is regular $\Leftrightarrow \forall x \in X$ and $\forall$ neighborhood $\mathcal{U}$ of x, $\exists$ a neighborhood $\mathcal{V}$ of x s.t. $\overline{V} \subset \mathcal{U}$

  ii. Provided X is a T_1 space, X is normal $\Leftrightarrow \forall A \underset{closed}{\subset} X$ and $\forall\, \mathcal{U} \underset{open}{\subset} X$ containing A, $\exists\, \mathcal{V} \underset{open}{\subset} X$ containing A s.t. $\overline{V} \subset \mathcal{U}$

- Regularity is preserved under taking subspace or taking product.
- In regular space, every pair of distinct points have neighborhoods whose closures are disjoint. # 31.1
- Every locally compact Hausdorff space is regular. # 32.3
- Every regular space with a countable basis is normal.

☐ Every regular Lindelof space is normal. # 32.4

- Every metric space is normal.
- Every compact Hausdorff space is normal.
- Every well-ordered set given order topology is normal.

* Separated Sets

  – $A, B \subset X$ is said to be separated if $\overline{A} \cap B = \phi$ and $A \cap \overline{B} = \phi$

  ✓ (Disjoint closed sets)$\Rightarrow$(Separated sets)$\Rightarrow$(Disjoint sets)

* Completely normal space(T_5 space)

  – A topological space X is said to be completely normal if every subspace of X is normal.

- Given X is T_1 space, X is completely normal iff every pair of separated sets in X are separated out by disjoint neighborhoods. # 32.6

(Notes) Summary of facts about separation axioms T_1, T_2, T_3, T_4, T_5

  – (Weaker) $T_1 \to T_2 \to T_3 \to T_4 \to T_5$ (Stronger)
  – Two distinct points in X are separated by ...
    i. neighborhoods that does not contain the other point if X is T_1 space
    ii. disjoint neighborhoods if X is Hausdorff space.
    iii. neighborhoods whose closures are disjoint if X is regular space.
  – These are separated out by disjoint neighborhoods in X...
    i. X is Hausdorff $\to$ A compact set and a point outside the set (furthermore, two disjoint compact sets)
    ii. X is regular $\to$ A closed set and a point outside the set
    iii. X is normal $\to$ Two disjoint closed sets.
    iv. X is completely normal $\to$ Two separated sets.
  – $T_1 \underset{Finite\ set}{\Rightarrow} T_2 \underset{Locally\ Compact}{\Rightarrow} T_3 \underset{Lindelof}{\Rightarrow} T_4$ and $T_2 \underset{Compact}{\Rightarrow} T_4$
  – Hausdorff condition and regularity are preserved under taking subspace or taking product. But normality is not preserved under taking subspace or taking product. Counterexample : $\mathbb{R}^J$ is not normal if J is uncountable index set. # 32.9

4.3 The Urysohn Lemma / The Urysohn metrization Theorem / Tietze Extension Theorem

* Separated by continuous function

  – X : a topological space. $A, B \subset X$. If $\exists$ a continuous function $f : X \to [0,1]$ s.t. $f(A) = \{0\}$ and $f(B) = \{1\}$ then A and B are said to be separated by a continuous function.

- Urysohn lemma

  – If X is normal then every pair of disjoint closed sets in X can be separated by a continuous function.

  √ Indeed, "if and only if " holds for this lemma.

* Completely regurlar space($T_{3.5}$ space)

  – A topological space X is said to be completely regular (or $T_{3.5}$ space) if X is T_1 space and each pair of a closed set and a point outside the set are separated out by continuous function.

$\surd$ (Weaker) $T_3 \to T_{3.5} \to T_4$ (Stronger)

- Complete regularity is preserved under taking subspace or taking product.
- Every pair of disjoint closed sets in completely regular space are separated by a continuous function if one of those sets is compact. #33.8

(Ex) Given a topological space X, let $\mathcal{C}(X)$ be the set of all continuous real-valued functions defined on X. Note that $\mathcal{C}(X)$ is a vector space.
If X is normal then "X is a infinite set $\Rightarrow \mathcal{C}(X)$ is an infinite dimensional vector space"

- Given continuous function $f : \mathbb{R}^2 \to \mathbb{R}$, define $\mathcal{U}_f$ by $\mathcal{U}_f = \{x \in \mathbb{R}^2 : f(x) \neq 0\}$. Then $\mathcal{B} = \{\mathcal{U}_f \mid f : \mathbb{R}^2 \to \mathbb{R} \text{ continuous }\}$ generates the standard topology on $\mathbb{R}^2$
i.e. the collection of support of continuous real-valued functions defined on $\mathbb{R}^2$ coincides with the standard topology on $\mathbb{R}^2$. # Final Test
- Urysohn metrization theorem
  - Every regular space with a countable basis is metrizable.
- If X is compact Hausdorff then X is metrizable $\Leftrightarrow$ X is second countable. # 34.3
- Tietze Extension theorem
  - X is normal space and $A \underset{closed}{\subset} X$. Then any continuous map $f : A \to [a, b]$ can be extended to a continuous map $g : X \to [a, b]$. Also any continuous map $f : A \to \mathbb{R}$ can be extended to a continuous map $g : X \to \mathbb{R}$.
Furthermore, $f : A \to \mathbb{R}^J$ can be extended to a continuous map $g : X \to \mathbb{R}^J$ for any index set J. (The last one # 44.2)

(Ex) $S = \{(x, sin(1/x)) : 0 < x \leq 1\}$. $\overline{S}$ is called as "topologist's sine curve" Since $\overline{S}$ is closed subset of a normal space $\mathbb{R}^2$, if there is a continuous function $f : \overline{S} \to \mathbb{R}$ then there is a continuous extension $g : \mathbb{R}^2 \to \mathbb{R}$ for a given f

5 Paracompactness

5.1 Local Finiteness

* Locally Finite

  - X : a topological space. A collection $\mathcal{A}$ of subsets of X is called locally finite if every point $x \in X$ has a neighborhood intersecting only finitely many elements of $\mathcal{A}$

• If $\mathcal{A}$ is locally finite collection of subsets of X then

  i. Any subcollection of $\mathcal{A}$ is locally finite
  ii. The collection $A' = \{\overline{A} : A \in \mathcal{A}\}$ is locally finite
  iii. $\bigcup_{A\in\mathcal{A}} \overline{A} = \overline{\bigcup_{A\in\mathcal{A}} A}$ (Without any assumption, $\subset$ direction holds true in general.)

√ If $\mathcal{A}$ is locally finite collection of subsets of X then it is guaranteed that $\bigcup_{A\in\mathcal{A}} \overline{A}$ is closed.

* Countably locally finite

  - X : a topological space. A collection $\mathcal{A}$ of subsets of X is called countably locally finite if $\mathcal{A}$ can be represented as a countable union of locally finite collections.

√ Trivially, every finite collection is locally finite and every countable collection is countably locally finite.

• Assume X is second countable. For a collection $\mathcal{A}$ of subsets of X, $\mathcal{A}$ is countably locally finite $\Leftrightarrow$ $\mathcal{A}$ is countable. # 39.5

* Refinement

  - $\mathcal{A}, \mathcal{B}$: collections of subsets of X. $\mathcal{B}$ is said to be a refinement of $\mathcal{A}$ (or $\mathcal{B}$ refines $\mathcal{A}$) if $\forall\ B \in \mathcal{B}$, $\exists\ A \in \mathcal{A}$ s.t. $B \subset A$

• If X is a metrizable space then every open covering of X has a countably locally finite open refinement that covers X. (Take advantage of Well-ordering principle)

5.2 Paracompactness

* Paracompact

  - A topological space X is said to be paracompact if every open covering of X has locally finite open refinement that covers X

√ X is compact iff every open covering of X has finite open refinement that covers X. Paracompactness is a kind of local notion that can generalize the concept of compactness.

(Ex) Euclidean space is paracompact.

• Every paracompact Hausdorff space is normal.

• Every closed subspace of paracompact space is paracompact.

- If X is regular space then the following holds true :
  "Every open covering of X has countably locally finite open refinement that covers X"
  $\Rightarrow$ "Every open covering of X has locally finite open refinement that covers X"

- Every metrizable space is paracompact.

* Support of a real-valued function

  - $\phi : X \to \mathbb{R}$ is a map. Support of ϕ is defined as $support(\phi) = \overline{\{x \in X : \phi(x) \neq 0\}}$

* Partition of unity

  - $\{\mathcal{U}_\alpha\}_{\alpha \in J}$: an open covering of X (We may assume all $\mathcal{U}_\alpha$'s are distinct to get rid of redundancy). A family of continuous functions $\{\phi_\alpha : X \to [0,1]\}_{\alpha \in J}$ is said to be a partition of unity dominated by $\{\mathcal{U}_\alpha\}_{\alpha \in J}$ if the followings are satisfied :
    i. $support(\phi_\alpha) \subset \mathcal{U}_\alpha \quad \forall\, \alpha \in J$ "ϕ_α vanishes outside $\mathcal{U}_\alpha$; ϕ_α's are local data."
    ii. $\{support(\phi_\alpha)\}_{\alpha \in J}$ is locally finite collection of closed sets.
    iii. $\sum_{\alpha \in J} \phi_\alpha(x) = 1 \quad \forall\, x \in X$

  √ By local finiteness condition in (ii), for each $x \in X$, there is a neighborhood of x intersecting only finitely many supports of ϕ_α's. Therefore the sum taken in the (iii) is in fact a finite sum for each $x \in X$. The name of a partition of 'unity' comes from the property (iii)

- Shrinking lemma

  - If X is a paracompact Hausdorff space then for an open covering $\{\mathcal{U}_\alpha\}_{\alpha \in J}$ of X, $\exists$ a locally finite open covering $\{\mathcal{V}_\alpha\}_{\alpha \in J}$ s.t. $\overline{\mathcal{V}_\alpha} \subset \mathcal{U}_\alpha \quad \forall\, \alpha \in J$

- If X is paracompact Hausdorff and $\{\mathcal{U}_\alpha\}_{\alpha \in J}$ is an open covering of X then $\exists$ a partition of unity on X dominated by $\{\mathcal{U}_\alpha\}_{\alpha \in J}$

(Ex) X is paracompact Hausdorff space. $X = \mathcal{U} \cup \mathcal{V}$ where $\mathcal{U}$, $\mathcal{V}$ are two nonempty open subsets of X. ($\mathcal{U}$ and $\mathcal{V}$ need not be disjoint). Given two continuous functions $f : \mathcal{U} \to [1, \infty)$ and $g : \mathcal{V} \to [1, \infty)$, there exists a continuous map $h : X \to [1, \infty)$ s.t. $h = f$ on $X \setminus \mathcal{V}$ and $h = g$ on $X \setminus \mathcal{U}$ # Final Test

(Ex) A smooth bump function

  - We have encountered a bump function in proving Urysohn lemmma or a partition of unity, which has a support in small local area and vanishes elsewhere.
    We can construct a smooth bump function defined on Euclidean space as below :
    i. Define $f : \mathbb{R} \to \mathbb{R}$ by $f(x) = exp(-1/x^2) \quad \forall x \neq 0$ and $f(0) = 0$
    ii. Then for some $r > 0$, define $g_r : \mathbb{R}^n \to \mathbb{R}$ by $g_r(x) = f(r^2 - |x|^2) I(|x| \leq r)$
    g_r is a smooth bump function defined on $\mathbb{R}^n$

6 Complete Metric Spaces & Function Spaces

6.1 Complete Metric Space

* Cauchy Sequence & Completeness

  – (X, d) : a metric space. $\{x_n\} \underset{seq}{\subset} X$ is said to be a Cauchy sequence if $\forall\, \epsilon > 0, \exists\, N \in \mathbb{N}$ s.t. $d(x_n, x_m) < \epsilon \quad \forall\, n, m \geq N$. If every cauchy sequence in X converges then X is said to be complete

√ Any convergent sequence is a Cauchy sequence in a metric space.

√ Every closed subspace of complete metric space is complete.

√ If (X, d) is a metric space and $\overline{d}$ is a standard bounded metric for d then for $\{x_n\} \underset{seq}{\subset} X$ and $x \in X$,

  i. $\{x_n\}$ is a Cauchy sequence under $d \Leftrightarrow \{x_n\}$ is a Cauchy sequence under $\overline{d}$

  ii. $\{x_n\}$ converges to x under $d \Leftrightarrow \{x_n\}$ converges to x under $\overline{d}$

  iii. (X, d) is complete $\Leftrightarrow$ $(X, \overline{d})$ is complete

  i.e. Cauchyness & convergence of sequence and completeness of metric space only care about small distances.

- (X, d) : a metric space. If every Cauchy sequence in X has a convergent subsequence then X is complete.
- Euclidean space is complete metric space in either Euclidean metric or square metric.
- $\mathbb{R}^\omega$ equipped with a metric D which induces product topology is a complete metric space.

(Ex) $\mathbb{Q}$, as a subspace of metric space $\mathbb{R}$, is not complete.

* Function space and uniform metric

  – (Y, d) : a metric space. J : an arbitrary index set. Then Y^J can be viewed as a function space $Y^J = \{f \mid f : J \to Y\}$. A uniform metric $\overline{\rho}$ on Y^J corresponding to d on Y is defined by $\overline{\rho}(f, g) = \sup\{\overline{d}(f(\alpha), g(\alpha)) : \alpha \in J\} \quad \forall\, f, g : J \to Y$ where $\overline{d}$ is a standard bounded metric for d

- If (Y, d) is complete metric space then given any set J, $(Y^J, \overline{\rho})$ is complete metric space. i.e. completeness of target space implies the completeness of function space induced by the uniform metric.

(Ex) $\mathbb{R}^n$ is a complete metric space under Euclidean or square metric. $\mathbb{R}^\omega$ is also a complete metric space under metric D. But $\mathbb{R}^J$ with product topology is not a metrizable space if J is uncountable since it is not normal. Instead, if we impose uniform topology on $\mathbb{R}^J$ then $\mathbb{R}^J$ is complete metric space.

* Notations for function space
  - X : a topological space. (Y,d) : a metric space. Y^X is a function space defined as $Y^X = \{f \mid f : X \to Y\}$. We can impose uniform topology on Y^X
    There are two important subspaces of Y^X
    i. $\mathcal{C}(X,Y) = \{f \mid f : X \to Y \ continuous\}$
    ii. $\mathcal{B}(X,Y) = \{f \mid f : X \to Y \ bounded\}$
  - $\surd$ Continuity of function $f : X \to Y$ depends on the topology on X so we need a topological space X as a domain. Also a function is called as bounded if the image is bounded in the metric of the target space.

- X : a topological space. (Y,d) : a metric space. A function space Y^X is equipped with the uniform topology. If Y is complete then (a) Y^X is complete (b) $\mathcal{C}(X,Y), \mathcal{B}(X,Y) \underset{closed}{\subset} Y^X$ so that both $\mathcal{C}(X,Y)$ and $\mathcal{B}(X,Y)$ are complete.

* Sup metric for function space
  - X : a topological space. (Y,d) : a metric space. We can define sup metric ρ on $\mathcal{B}(X,Y)$ defined as $\rho(f,g) = \sup\{d(f(x),g(x)) : x \in X\} \quad \forall f,g \in \mathcal{B}(X,Y)$

$\surd$ Relation b.w. sup metric and uniform metric is $\overline{\rho}(f,g) = \min\{1, \rho(f,g)\} \ \forall f,g \in \mathcal{B}(X,Y)$. Briefly speaking, uniform metric $\overline{\rho}$ is a standard bounded metric for ρ on $\mathcal{B}(X,Y)$

(Ex) By completeness of $\mathbb{R}$, given any topological space X, $\mathcal{B}(X,\mathbb{R})$ is complete metric space under sup metric. Also, if X is compact, then $\mathcal{C}(X,\mathbb{R})$ is complete under sup metric.

* Completion
  - For metric space (X,d), a completion X' of X is a complete metric space s.t. X is a dense subset of X'

- For a metric space (X,d), there is an isometric embedding of X into a complete metric space.

- X, Y : metric spaces. If $f : X \to Y$ is uniformly continuous, then $\{x_n\} \underset{seq}{\subset} X$ is a Cauchy sequence $\Rightarrow f(\{x_n\}) \underset{seq}{\subset} Y$ is also a Cauchy sequence.

- X, Y : metric spaces. $A \subset X$. If Y is complete and $f : A \to Y$ is uniformly continuous then f can be uniquely extended to a uniformly continuous function $g : \overline{A} \to Y$ # 43.2

- X : metric space. X is complete $\Leftrightarrow$ for every nested seq. $\{A_n\}$ of nonempty closed sets in X s.t. $\text{diam}(A_n) \to 0$, the intersection $\bigcap_n A_n$ is a one point set. # 43.4

- Fixed point of contraction map in complete metric space # 43.5
  - (X,d) : a metric space. $f : X \to X$. f is called as a contraction map if $\exists\, \alpha < 1$ s.t. $d(f(x),f(y)) \le \alpha d(x,y) \quad \forall x,y \in X$. If X is complete and f is a contraction map on X then f has a unique fixed point i.e. $\exists\,!\, x \in X$ s.t. $f(x) = x$

- X, Y : topological spaces. $e : X \times \mathcal{C}(X,Y) \to Y$ defined as $(x,f) \mapsto f(x)$ is called as an evaluation map. If Y is metrizable with metric d and $\mathcal{C}(X,Y)$ is equipped with uniform topology corresponding to d then evaluation map e is continuous. # 43.8

- For a metric space (X,d), X is compact $\Leftrightarrow$ X is complete and totally bounded.

6.2 Space-filling Curve

- $I = [0, 1]$. There is a continuous surjection $f : I \to I^2$ called "Peano space-filling curve".
- There is a continuous surjection from $\mathbb{R}$ to $\mathbb{R}^n$ for any $n \in \mathbb{N}$ # 44.2

(Ex) There is a continuous surjective map from a circle S^1 to $S^1 \times S^1$ # Final Test0

6.3 Pointwise, Uniform, and Compact Convergence

- Recall that on a function space Y^X, where X is a set and Y is a metric space, we can impose a uniform topology which is a metric topology induced by uniform metric $\overline{\rho}$ or sup metric ρ. We've known that $f_n \to f$ in Y^X w.r.t. uniform topology $\Leftrightarrow f_n \rightrightarrows f$ uniformly.

* Topology of Pointwise convergence

  - X : a set. Y : a topological space. Define $S(x,\mathcal{U}) = \{f : f \in Y^X, f(x) \in \mathcal{U}\}$ for each $x \in X$ and $\mathcal{U} \underset{open}{\subset} Y$. The topology of pointwise convergence is a topology on Y^X is generated by a subbasis $\mathcal{S} = \{S(x,\mathcal{U}) : x \in X, \mathcal{U} \underset{open}{\subset} Y\}$

  √ Typical basis element of this topology containing some $f \in Y^X$ is all functions that are close to f at finitely many points.

  √ We can represent $f \in Y^X$ as $f = (f(x))_{x\in X}$ and with this notation, $S(x,\mathcal{U}) = \{f : f = (f(x))_{x\in X},\ \pi_x(f) \in \mathcal{U}\} = \pi_x^{-1}(\mathcal{U})$, which is a standard subbasis element for the product topology on Y^X. Thus the topology of pointwise convergence on Y^X is just the product topology we've already known.

- $f_n \to f$ in Y^X in Y^X w.r.t. topology of pointwise convergence $\Leftrightarrow f_n \to f$ pointwise.

√ Under the topology of pointwise convergence, the limit function of sequence of continuous functions need not be continuous.

* Topology of compact convergence

  - X : a topological space. (Y, d) : a metric space. For each $f \in Y^X$, $C \subset X$ compact and $\epsilon > 0$, define $B_C(f, \epsilon) = \{g : g \in Y^X,\ \sup\{d(f(x), g(x)) : x \in C\} < \epsilon\}$. The topology of compact convergence is a topology on Y^X is generated by a basis $\mathcal{B} = \{B_c(f, \epsilon) : f \in Y^X, C \subset X \text{ compact}, \epsilon > 0\}$

  √ Typical basis element of this topology containing some $f \in Y^X$ is all functions that are close to f uniformly on some compact set C.

- $f_n \to f$ in Y^X w.r.t. topology of compact convergence $\Leftrightarrow f_n \rightrightarrows f$ uniformly on C for every compact $C \subset X$.

* Compactly generated space.

  - X : topological space. X is said to be compactly generated if
    $A \underset{open}{\subset} X \Leftrightarrow A \cap C \underset{open}{\subset} C$ for every compact $C \subset X$ or equivalently
    $B \underset{closed}{\subset} \Leftrightarrow B \cap C \underset{closed}{\subset} C$ for every compact $C \subset X$

- Every locally compact or first countable space is compactly generated.

- If X is compactly generated then $f : X \to Y$ is continuous $\Leftrightarrow$ $f|_C : C \to Y$ is continuous for every compact $C \subset X$

- X : compactly generated space. Y : a metric space. Then $\mathcal{C}(X, Y) \underset{closed}{\subset} Y^X$ given the topology of compact convergence, so that under the topology of compact convergence, the limit function of sequence of continuous functions is continuous.

- X : a topological space. Y : a metric space. For the function space Y^X, we have the following inclusions of topologies as below :
(Top. of pointwise convergence) $\subset$ (Top. of compact convergence) $\subset$ (Uniform top.)
If X is compact then compact convergence topology and uniform topology are the same.

√ 'More open sets' makes 'convergence of sequence' much harder.

(Ex) $\{f_n : (-1, 1) \to \mathbb{R}\}$ is a seq. of functions defined as $f_n(x) = \sum_{k=1}^{n} kx^k$. f_n converges to f defined as $f(x) = \frac{x}{(1-x)^2}$ in compact convergence topology while f_n does not converge in uniform topology. # 46.5

* Compact-open topology

  - X, Y : topological spaces. Define $S(C, \mathcal{U}) = \{f : f \in \mathcal{C}(X, Y),\ f(C) \subset \mathcal{U}\}$ for each $C \subset X$ compact and $\mathcal{U} \underset{open}{\subset} Y$. The compact-open topology is a topology on $\mathcal{C}(X, Y)$ generated by a subbasis $\mathcal{S} = \{S(C, \mathcal{U}) : C \subset X \text{ compact},\ \mathcal{U} \underset{open}{\subset} Y\}$

√ Definition of uniform topology and compact convergence topology appeals to the metric on space Y. The compact-open topology is a natural topology on $\mathcal{C}(X, Y)$ which is purely topological, not involving any metric.

- X : a topological space. Y : a metric space. On $\mathcal{C}(X, Y)$, the compact-open topology and the comopact convergence topology are the same.

√ If Y is metrizable then the compact convergence topology on $\mathcal{C}(X, Y)$ does not depend on the metric we choose. For example, if Y is Euclidean space, then choosing Euclidean metric or square metric does not change the compact convergence topology on $\mathcal{C}(X, \mathbb{R}^n)$. Additionally, if X is compact and Y is metrizable then the uniform topology on $\mathcal{C}(X, Y)$ does not depend on the metric we choose.

- If X is locally compact Hausdorff and $\mathcal{C}(X, Y)$ is equipped with compact-open topology then the evaluation map $e : X \times \mathcal{C}(X, Y) \to Y$ defined by $(x, f) \mapsto f(x)$ is continuous.

√ Taking a slice, for a fixed $x \in X$, if f and g in $\mathcal{C}(X, Y)$ are closed w.r.t. compact-open topology then $f(x)$ and $g(x)$ are close in Y.

- X, Y : topoogical spaces. Given $f : X \times Z \to Y$, there is an induced map $F : Z \to \mathcal{C}(X, Y) \quad z \mapsto F(z)$ where $F(z) : X \to Y \quad x \mapsto f(x, z)$. Conversely, given $F : Z \to \mathcal{C}(X, Y)$, there is an induced function $f : X \times Z \to Y$ defined as $(x, z) \mapsto F(z)(x)$. Suppose $\mathcal{C}(X, Y)$ is equipped with compact-open topology. Then

  i. If f is continuous then induced F is continuous.

  ii. The converse also holds true provided X is locally compact Hausdorff.

- X, Y, Z : topological spaces. If $\mathcal{C}(X,Y), \mathcal{C}(Y,Z)$ and $\mathcal{C}(X,Z)$ are equipped with compact-open topology and Y is locally compact Hausdorff then composition mapping $\circ : \mathcal{C}(X,Y) \times \mathcal{C}(Y,Z) \to \mathcal{C}(X,Z) \quad (f,g) \mapsto g \circ f$ is continuous. # 46.7

- Define an oscillation map by $osc : C(I, \mathbb{R}) \to \mathbb{R} \quad f \mapsto \max_{x \in I} f(x) - \min_{x \in I} f(x)$ where I is a compact interval in $\mathbb{R}$. If $\mathcal{C}(I, \mathbb{R})$ is equipped with a compact-open topology then the oscillation map is continuous.

7 Baire Spaces

We introduce a class of topological spaces called Baire space. Many spaces we've learned in this course are Baire spaces. The defining condition of Baire space is quite unnatural, but it can be a very useful tool in Analysis and Topology.

- Recall that for $A \subset X$, $\overline{X \setminus A} = X \setminus A^0$ holds true because of optimality in definition of closure and interior. Hence A has empty interior $\Leftrightarrow X \setminus A$ is dense in X

(Ex) $\mathbb{Q}$ has empty interior because $\mathbb{R} \setminus \mathbb{Q}$ is dense in $\mathbb{R}$.
$[0,1] \times \{0\}$ is has empty interior as a subset of $\mathbb{R}^2$. $\quad \mathbb{Q} \times \mathbb{R}$ has empty interior.

* Baire Space

  - A topological space X is said to be a Baire space if
    - ∗ Given any countable collection $\{A_n\}$ of closed subsets of X where each one has empty interior, their union $\bigcup_n A_n$ has empty interior. (Closed set formulation)
    - ∗ Given any countable collection $\{\mathcal{U}_n\}$ of open subsets of X where each one is dense in X, their intersection $\bigcap_n \mathcal{U}_n$ is dense in X. (Open set formulation)

  Two conditions above are equibalent.

(Ex) $\mathbb{Q}$ is not a Baire space. $\quad \mathbb{N}$ is a Baire space. $\quad \mathbb{R} \setminus \mathbb{Q}$ is a Baire space. # 48.6

- Baire category theorem
  - If X is a compact Hausdorff space or a complete metric space then X is a Baire space.

- Every open subspace of a Baire space is also a Baire space.

(Ex) Euclidean space is a Baire space since it is complete metric space. Closed subspace of complete space is also complete so every closed subspace of Euclidean space is a Baire space. Also every open subspace of Euclidean space is a Baire space. Note that since locally compact Hausdorff space is homeomorphic to open subspace of compact Hausdorff space, every locally compact Hausdorffspace is a Baire space.

- If X is a compact Hausdorff space or a complete metric space then every G_δ set in X equipped with subspace topologoy is a Baire space. # 48.5

- How discontinuous the pointwise limit function of seq. of continuous function can be
  - X : a topological space. Y : a metric space. $\{f_n : X \to Y\}$ is a sequence of continuous functions having pointwise limit $f : X \to Y$. If X is a Baire space then the continuity set $C_f = \{x \in X : f \text{ is continuous at } x\}$ is dense in X.
- If D is a countable dense subset of $\mathbb{R}$ then there is no function $f : \mathbb{R} \to \mathbb{R}$ which is continuous precisely on D. # 48.7
- If $\{f_n : \mathbb{R} \to \mathbb{R}\}$ is a sequence of continuous functions having pointwise limit $f : \mathbb{R} \to \mathbb{R}$ then f is continuous at uncountably many points of $\mathbb{R}$. #48.8

(Ex) $f : \mathbb{R} \to \mathbb{R}$ defined as $f(x) = \begin{cases} 1/n & x \in \mathbb{Q} \cap (0,1) = \{q_n\}_{n\in\mathbb{N}},\ x = q_n \\ 0 & x \notin \mathbb{Q} \cap (0,1) \end{cases}$
which is called as "salt and pepper function". f is continuous at every irrational and not continuous at every rational, as we proved in Analysis course. On the other hand, there is no function $g : \mathbb{R} \to \mathbb{R}$ s.t. g is continuous at every rational and not continuous at every irrational.