

Probability theory II Facts

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1 Conditional Expectation

- Projection Thm for Hilbert Space

- If E is a Hilbert space and $M \subset E$ is closed and convex, then for any $y \in E$,
 \exists a unique $w \in M$ s.t. $\|y - w\| = d(y, M) := \inf\{\|y - v\| : v \in M\}$.
Denote it as $w = \text{proj}_M y$ i.e. w is a projection of y onto M .
- If E is a Hilbert space and $M \subset E$ is a closed vector subspace, then for any $y \in E$,
 - \exists a unique decomposition $y = w + v$ with $w = \text{proj}_M y \in M$ and $v \in M^\perp$
 - For $w \in M$, $w = \text{proj}_M y \Leftrightarrow \langle y - w, z \rangle = 0 \quad \forall z \in M$

* $\mathcal{L}^2 := \{ \text{Random Variable } X : E(X^2) = \int X^2 dP < \infty \}$

✓ If $X \in \mathcal{L}^2$ then $E|X| < \infty$ i.e. every element of \mathcal{L}^2 is integrable.

★ Trick : $|X| \leq X^2 + \frac{1}{4}$

✓ \mathcal{L}^2 is a vector space

★ Trick : inequality $(aX + bY)^2 \leq 2(a^2X^2 + b^2Y^2)$

- \mathcal{L}^2 is a Hilbert space with inner product $\langle X, Y \rangle = E(XY)$

★ Trick : Cauchy seq. having a subseq. converging to a point converges to the point.

- Lemma for proving \mathcal{L}^2 is a complete normed space.

- If $\{X_n\} \subset \mathcal{L}^2$ and $\|X_n - X_{n+1}\| \leq 2^{-n} \quad \forall n \in \mathbb{N}$ then $\exists X \in \mathcal{L}^2$ s.t. $X_n \rightarrow X$ a.s. and
 $\|X_n - X\| \rightarrow 0$ i.e. $X_n \rightarrow X$ in \mathcal{L}^2 .

★ Lemma : If a random variable Z satisfies $Z \geq 0$ and $E(Z) < \infty$ then $Z < \infty$ a.s.

* For $X \in \mathcal{L}^2$, $\mathcal{L}^2(X) := \{g(X) \mid g : \mathbb{R} \rightarrow \mathbb{R} \text{ is a Borel function, } E[(g(X))^2] < \infty\}$

✓ For $X \in \mathcal{L}^2$, $\mathcal{L}^2(X)$ is a vector subspace of \mathcal{L}^2 .

- For $X \in \mathcal{L}^2$, $\mathcal{L}^2(X)$ is a closed vector subspace of \mathcal{L}^2 so that $\mathcal{L}^2(X)$ is also a Hilbert space.

* Geometric definition for conditional expectation

- For $X, Y \in \mathcal{L}^2$, define $E[Y|X] = \text{Proj}_{\mathcal{L}^2(X)} Y$
- $E[Y|X] = g(X)$ a.s. for some Borel function g
- $\|Y - E[Y|X]\| = \min_{h(X) \in \mathcal{L}^2(X)} \|Y - h(X)\|$
i.e. $E[(Y - E[Y|X])^2] \leq E[(Y - h(X))^2] \quad \forall h(X) \in \mathcal{L}^2$
- For $g(X) \in \mathcal{L}^2(X)$, $g(X) = E[Y|X] \Leftrightarrow \langle Y - g(X), h(X) \rangle = 0 \quad \forall h(X) \in \mathcal{L}^2$
 $\Leftrightarrow E[(Y - g(X))h(X)] = 0 \quad \forall h(X) \in \mathcal{L}^2$

- Elementary properties of conditional expectation from geometric definition

- If $X, Y, Z \in \mathcal{L}^2$ then the followings are true.
 - $E[c|X] = c$ a.s. $\forall c \in \mathbb{R}$
 - $E[\alpha Y + \beta Z|X] = \alpha E[Y|X] + \beta E[Z|X] \quad \forall \alpha, \beta \in \mathbb{R}$
 - $E[Y|X] = E[Y]$ if X and Y are independent.

iv. $E[g(X)Y|X] = g(X)E[Y|X]$ if g satisfies $g(X) \in \mathcal{L}^2(X)$ and $\sup_x |g(x)| < \infty$

v. $E[E[Y|X]] = E[Y]$

✓ In fact, the additional assumption about boundedness of g in (iv) is not necessary. We will see later.

- Extending the definition from \mathcal{L}^2 to all integrable functions

$$E[\{Y - E[Y|X]\}I(X \in A)] = 0 \quad \forall A \in \mathcal{B}(\mathbb{R}) \quad \because I(X \in A) \in \mathcal{L}^2(X)$$

$$\int_{(X \in A)} Y dP = \int_{(X \in A)} E[Y|X] dP \quad \forall A \in \mathcal{B}(\mathbb{R})$$

$$\int_B Y dP = \int_B E[Y|X] dP \quad \forall B \in \sigma(X)$$

- $E[Y|X] \in \sigma(X)$ and $\int_B Y dP = \int_B E[Y|X] dP \quad \forall B \in \sigma(X)$. Such r.v. is unique in the sense that if any r.v. Z satisfies $Z \in \sigma(X)$ and $\int_B Y dP = \int_B Z dP \quad \forall B \in \sigma(X)$ then $Z = E[Y|X]$ a.s. provided $E|Y| < \infty$
- From the theory on \mathcal{L}^2 space, we get geometric understanding about conditional expectation. But now, from the equation above, we can guess that definition for conditional expectation may be extended to all integrable random variables.

- Proof for the uniqueness mentioned above

- (Ω, \mathcal{F}, P) : a prob. space. $Y \in \mathcal{F}$ and $E|Y| < \infty$. $\mathcal{G} \subset \mathcal{F}$ is a sub σ -field. If X is a random variable satisfying (a) $X \in \mathcal{G}$ (b) $\int_A Y dP = \int_A X dP \quad \forall A \in \mathcal{G}$ then
 - i. X is integrable
 - ii. Such X is unique in the sense that if there is another X' then $X = X'$ a.s.
 - ★ Trick : For any r.v. Z , $(Z > 0) = \bigcup_{\epsilon > 0} (Z \geq \epsilon) = \bigcup_{n \in \mathbb{N}} (Z > \frac{1}{n})$
 - ★ Lemma : For any \mathcal{F} -measurable and integrable X and Y , if $\int_A X dP = \int_A Y dP \quad \forall A \in \mathcal{F}$ then $X = Y$ a.s.

- Radon-Nikodym Thm

- If μ, ν are σ -finite measures on (Ω, \mathcal{F}) and $\nu \ll \mu$ ($\mu(A) = 0 \Rightarrow \nu(A) = 0 \quad \forall A \in \mathcal{F}$) then \exists a \mathcal{F} -measurable nonnegative function g s.t. $\nu(A) = \int_A g d\mu \quad \forall A \in \mathcal{F}$. The function g is unique in the sense that if h is another such function then $g = h$ μ -a.e.

- * Definition of conditional expectation

- $(\Omega, \mathcal{F}_0, P)$: a prob. space. $\mathcal{F} \subset \mathcal{F}_0$: a sub σ -field.
 X is a random variable s.t. $X \geq 0$, $X \in \mathcal{F}_0$ and $E|X| < \infty$. Then \exists a unique r.v. Y s.t. $Y \geq 0$, $Y \in \mathcal{F}$ and $\int_A X dP = \int_A Y dP \quad \forall A \in \mathcal{F}$. Such Y is unique in the sense that if another Y' exists then $Y = Y'$ a.s.
- $Y = E[X|\mathcal{F}]$ is said to be conditional expectation of X given \mathcal{F}
 - ★ Applying Radon Nikodym thm to measures $P|_{\mathcal{F}}$ and Q on (Ω, \mathcal{F}) where Q is defined by $Q(A) = \int_A X dP \quad \forall A \in \mathcal{F}$. Note that $Q \ll P|_{\mathcal{F}}$ and Q is a finite measure.
- We can extend the definition to general integrable r.v. X
 $Y = E[X|\mathcal{F}]$ is a unique random variable s.t. $Y \in \mathcal{F}$ and $\int_A X dP = \int_A Y dP \quad \forall A \in \mathcal{F}$.
 $E[X|\mathcal{F}]$ is also integrable and the uniqueness is in the sense of a.s. equivalence relation.
 $Y = E[X|\mathcal{F}]$ can be derived by $Y = Y_1 - Y_2$ where $Y_1 = E[X^+|\mathcal{F}]$ and $Y_2 = E[X^-|\mathcal{F}]$

* Conditional expectation given a random variable

– X : integrable r.v. For a random variable Y , define $E[X|Y] := E[X|\sigma(Y)]$

✓ Y need not be integrable.

✓ Since $E[X|Y] \in \sigma(Y)$, $E[X|Y] = g(Y)$ for some Borel function g . This coincides with the definition of conditional expectation in \mathcal{L}^2 space.

* Conditional probability

– For $A \in \mathcal{F}_0$ and a sub σ -field $\mathcal{F} \subset \mathcal{F}_0$, define $P(A|\mathcal{F}) := E[I_A|\mathcal{F}]$

– For $A, B \in \mathcal{F}_0$, define $P(A|B) = P(A \cap B) / P(B)$

• Elementary properties of conditional expectation

– $(\Omega, \mathcal{F}_0, P)$: a prob. space. $\mathcal{F} \subset \mathcal{F}_0$: a sub σ -field. X, Y : integrable random variables

i. $E[c|\mathcal{F}] = c$

ii. $E[\psi(X)|\mathcal{F}] = \psi(X)$ given $E|\psi(X)| < \infty$

iii. If \mathcal{F} is a trivial σ -field i.e. $\mathcal{F} = \{\Omega, \emptyset\}$ then $E[X|\mathcal{F}] = E[X]$

iv. $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$ is a partition of Ω with $\Omega_i \in \mathcal{F}_0$ and $P(\Omega_i) > 0 \quad \forall i \in \mathbb{N}$
 $\mathcal{F} = \sigma\{\Omega_1, \Omega_2, \dots\} = \{\bigcup_{j \in \kappa} \Omega_j : \kappa \subset \mathbb{N}\} \quad (\mathcal{F} \text{ is a } \sigma\text{-field}).$ Then we have

$$E[X|\mathcal{F}] = \sum_{i=1}^{\infty} a_i I_{\Omega_i} \quad \text{with} \quad a_i = \frac{E[X I_{\Omega_i}]}{P(\Omega_i)}$$

✓ For $A \in \mathcal{F}_0$, $P(A|\mathcal{F}) = P(A|\Omega_i) I_{\Omega_i}$

★ Lemma : If $Z \in \mathcal{F}$ for such \mathcal{F} , then we can write $Z = \sum_{i=1}^{\infty} c_i I_{\Omega_i}$ where $c_i \in \mathbb{R}$

v. $E[aX + bY|\mathcal{F}] = aE[X|\mathcal{F}] + bE[Y|\mathcal{F}] \quad \forall a, b \in \mathbb{R}$

vi. $X \geq 0 \Rightarrow E[X|\mathcal{F}] \geq 0 \quad a.s.$

★ Lemma : If $Z > 0$ on A with $P(A) > 0$ then $\int_A Z dP > 0$

vii. $X \leq Y \Rightarrow E[X|\mathcal{F}] \leq E[Y|\mathcal{F}] \quad a.s.$

viii. $|E[X|\mathcal{F}]| \leq E[|X||\mathcal{F}]$

□ $|X| \leq M$ for some $M > 0 \Rightarrow |E[X|\mathcal{F}]| \leq M \quad a.s.$

ix. $E[|X||\mathcal{F}] = 0 \Rightarrow X = 0 \quad a.s.$

x. $E[E[X|\mathcal{F}]] = E[X]$

• X, Y : integrable r.v's where $X \perp\!\!\!\perp Y$. $\psi : \mathbb{R}^2 \rightarrow \mathbb{R}$ Borel measurable s.t. $E|\psi(X, Y)| < \infty$
 Define $g : \mathbb{R} \rightarrow \mathbb{R}$ by $g(x) = E[\psi(x, Y)] \quad \forall x \in \mathbb{R}$. Then $E[\psi(X, Y)|X] = g(X)$

✓ $g(x) = E[\psi(x, Y)] = \int \psi(x, Y) dP = \int_{\mathbb{R}} \psi(x, y) dPY^{-1}(y) = \int_{\mathbb{R}} \psi_x(y) d\mu_Y(y) \quad \forall x \in \mathbb{R}$
 By Fubini thm in real analysis course, it is shown that g is Borel measurable & integrable.

• Conditional expectation and convergence

– $(\Omega, \mathcal{F}_0, P)$: a probability space. $\mathcal{F} \subset \mathcal{F}_0$: a sub σ -field

i. (MCT) If $X_n \geq 0$ and $X_n \nearrow X$ a.s. with $E(X) < \infty$ then $E[X_n|\mathcal{F}] \nearrow E[X|\mathcal{F}]$ a.s.

□ If $Y_n \searrow Y$ a.s. with $E|Y_1|, E|Y| < \infty$ then $E[Y_n|\mathcal{F}] \searrow E[Y|\mathcal{F}]$ a.s.

ii. (DCT) If $|X_n| \leq Y$, $E(Y) < \infty$ and $X_n \rightarrow X$ a.s. then $E[X_n|\mathcal{F}] \rightarrow E[X|\mathcal{F}]$ a.s.

- iii. (Fatou's lemma) If $X_n \geq 0$ and $X_n \rightarrow X$ a.s. with $E(X_n) < \infty$, $E(X) < \infty$ then $E[X|\mathcal{F}] \leq \liminf E[X_n|\mathcal{F}]$
- iv. (Continuity from below) $\{B_n\} \subset_{seq} \mathcal{F}_0$ s.t. $B_n \subset B_{n+1} \quad \forall n \in \mathbb{N}$. $B := \bigcup_n B_n$
Then $P(B_n|\mathcal{F}) \nearrow P(B|\mathcal{F})$
- v. (Countable additivity) If $\{C_n\} \subset_{seq} \mathcal{F}_0$ is mutually disjoint then $P(\bigcup_n C_n|\mathcal{F}) = \sum_n P(C_n|\mathcal{F})$

- Essential inequalities

- i. (Markov) $P(|X| \geq c|\mathcal{F}) \leq \frac{1}{c}E[|X||\mathcal{F}] \quad \forall c > 0$
- ii. (Jensen) If $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is convex then $\phi(E[X|\mathcal{F}]) \leq E[\phi(X)|\mathcal{F}]$ a.s.
★ Trick : For each $x \in \mathbb{R}$ and convex function $\phi : \mathbb{R} \rightarrow \mathbb{R}$, we have $\phi(x) = \sup\{ax + b : (a, b) \in S\}$ where $S = \{(a, b) \in \mathbb{R}^2 : ax + b \leq \phi(x) \quad \forall x \in \mathbb{R}\}$
- iii. (Cauchy-Schwarz) For $X, Y \in \mathcal{L}^2$, we have $E^2[XY|\mathcal{F}] \leq E[X^2|\mathcal{F}]E[Y^2|\mathcal{F}]$ a.s.

- Smoothing property of conditional expectation

- i. If $X \in \mathcal{F}$, $E|Y| < \infty$, and $E|XY| < \infty$ then $E[XY|\mathcal{F}] = XE[Y|\mathcal{F}]$ a.s.
 \checkmark $E|X| < \infty$ assumption is not required.
- If $X \in \mathcal{F}$ and $E|X| < \infty$ then $E[X|\mathcal{F}] = X$ a.s.
- ii. If $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_0$ are sub σ -fields and $E|X| < \infty$ then
 - (a) $E[E(X|\mathcal{F}_1)|\mathcal{F}_2] = E[X|\mathcal{F}_1]$
 - (b) $E[E(X|\mathcal{F}_2)|\mathcal{F}_1] = E[X|\mathcal{F}_1]$
- ★ Lemma : If $\mathcal{F}_1 \subset \mathcal{F}_2$ then $Y \in \mathcal{F}_1 \Rightarrow Y \in \mathcal{F}_2$
 \checkmark In short, “the smaller wins”. In view of information, it is similar to projection onto vector subspaces $S_1 \subset S_2 \subset S$ where $Proj_{S_1} Proj_{S_2} = Proj_{S_2} Proj_{S_1} = Proj_{S_1}$

- Def. of conditional expectation by Radon-Nikodym derivative agrees with def. in \mathcal{L}^2 space.

- If $E(X^2) < \infty$ then for $\mathcal{C} = \{Y : Y \in \mathcal{F}, E(Y^2) < \infty\}$,
 $E[\{X - E[X|\mathcal{F}]\}^2] = \inf_{Y \in \mathcal{C}} E[\{X - Y\}^2]$ and $E[X|\mathcal{F}] = \arg \min_{Y \in \mathcal{C}} E[\{X - Y\}^2]$
- ★ Lemma : If $X \in \mathcal{L}^2$ then $E[X|\mathcal{F}] \in \mathcal{L}^2$

- * Independence of a random variable and a σ -field

- A random variable X and a σ -field \mathcal{F} are said to be independent if $\sigma(X)$ and \mathcal{F} are independent

- If an integrable random variable X and a σ -field \mathcal{F} are independent then $E[X|\mathcal{F}] = E[X]$

- Two extreme cases of conditional expectations w.r.t information

- Perfect information : If $X \in \mathcal{F}$ then $E[X|\mathcal{F}] = X$
- No information : If $X \perp \mathcal{F}$ then $E[X|\mathcal{F}] = E[X]$

- * Conditional variance

$$Var(X|\mathcal{F}) := E[\{X - E[X|\mathcal{F}]\}^2|\mathcal{F}] = E[X^2|\mathcal{F}] - E^2[X|\mathcal{F}]$$

Conditional variance is defined for $X \in \mathcal{L}^2$

2 Martingales

* Definition needed for martingales

- Given a probability space (Ω, \mathcal{F}, P) , increasing sequence of sub σ -fields $\{\mathcal{F}_n\}_{n=0}^\infty$ is called a filtration.
- A random sequence $\{X_n\}_{n=0}^\infty$ is said to be adapted to $\{\mathcal{F}_n\}$ if $X_n \in \mathcal{F}_n \quad \forall n \in \mathbb{N} \cup \{0\}$

* Definition of martingale and their cousins

- $\{X_n\}_{n=0}^\infty$: a random sequence. $\{\mathcal{F}_n\}_{n=0}^\infty$: a filtration. Assume $E|X_n| < \infty \quad \forall n \in \mathbb{N} \cup \{0\}$ and $\{X_n\}$ is adapted to $\{\mathcal{F}_n\}$. Then $\{X_n\}$ is said to be a martingale (w.r.t $\{\mathcal{F}_n\}$) if $E[X_{n+1}|\mathcal{F}_n] = X_n \quad \forall n \in \mathbb{N} \cup \{0\}$
- $\{X_n\}$ is said to be a submartingale (w.r.t $\{\mathcal{F}_n\}$) if $E[X_{n+1}|\mathcal{F}_n] \geq X_n \quad \forall n \in \mathbb{N} \cup \{0\}$
- $\{X_n\}$ is said to be a supermartingale (w.r.t $\{\mathcal{F}_n\}$) if $E[X_{n+1}|\mathcal{F}_n] \leq X_n \quad \forall n \in \mathbb{N} \cup \{0\}$
- ✓ These are abbreviated to ‘mtg’, ‘submtg’, ‘supermtg’ respectively.

• Examples of martingales

- i. $\{\xi_n\}_n$ i.i.d with $E(\xi_1) = 0$. $X_0 = 0$. $X_n = \xi_1 + \dots + \xi_n$ and $\mathcal{F}_0 = \{\phi, \Omega\}$.
 $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$. Then $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$
 ★ Trick : $E[Z]$ is finite $\Leftrightarrow Z$ is integrable. (\because the definition of expectation)
- ii. Adding assumption $Var(\xi_1) = \sigma^2 < \infty$ to i. above.
 Then $\{X_n - n\sigma^2\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$
- iii. $\{\varepsilon_n\}_n$ i.i.d $\sim (0, 1)$. $X_0 = 0$. $X_{n+1} = X_n + h(X_n)\varepsilon_{n+1}$ with $h : \mathbb{R} \rightarrow \mathbb{R}$ Borel function s.t.
 $E|h(X_n)| < \infty \quad \forall n \in \mathbb{N} \cup \{0\}$ and $\mathcal{F}_0 = \{\phi, \Omega\}$. $\mathcal{F}_n = \sigma(\varepsilon_1, \dots, \varepsilon_n)$
 Then $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$
- iv. $\{\varepsilon_n\}_n$ i.i.d $\sim (0, 1)$. $Y_0 = 0$. $Y_{n+1} = \phi(Y_n)\varepsilon_{n+1}$ with $\phi(y) = w + \alpha y^2$ ($w > 0, 0 \leq \alpha < 1$)
 and $E[\phi(Y_n)] < \infty \quad \forall n \in \mathbb{N}$. and $\mathcal{F}_0 = \{\phi, \Omega\}$. $\mathcal{F}_n = \sigma(\varepsilon_1, \dots, \varepsilon_n)$.
 Let $X_0 = 0$. $X_n = Y_1 + \dots + Y_n$. Then $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$
 ✓ Such $\{Y_n\}$ is called as ARCH (autoregressive conditional heteroskedasticity) process

• Elementary facts about Martingales

- Every martingale is a submartingale and a supermartingale
- If $\{X_n\}$ is a submartingale then $\{-X_n\}$ is a supermartingale
 ✓ We develop theory about martingales often assuming submartingale since every martingale is submartingale and every supermartingale is negative version of submartingale
- If $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$ then $E[X_n|\mathcal{F}_m] = X_m$ whenever $n \geq m$
- If $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$ then $E[X_n|\mathcal{F}_m] \geq X_m$ whenever $n \geq m$
- If $\{X_n\}$ is a supermartingale w.r.t $\{\mathcal{F}_n\}$ then $E[X_n|\mathcal{F}_m] \leq X_m$ whenever $n \geq m$
- If $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$ then $\{E[X_n]\}$ is constant.
- If $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$ then $\{E[X_n]\}$ is increasing.
- If $\{X_n\}$ is a supermartingale w.r.t $\{\mathcal{F}_n\}$ then $\{E[X_n]\}$ is decreasing.

- Convex transformation of martingale

- If $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a convex function s.t. $E|\phi(X_n)| < \infty \quad \forall n \in \mathbb{N}$ then $\{\phi(X_n)\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$
- If $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a convex and increasing function s.t. $E|\phi(X_n)| < \infty \quad \forall n \in \mathbb{N}$ then $\{\phi(X_n)\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$
- If $\{X_n\}$ is a supermartingale w.r.t $\{\mathcal{F}_n\}$ and $\phi : \mathbb{R} \rightarrow \mathbb{R}$ is a concave and increasing function s.t. $E|\phi(X_n)| < \infty \quad \forall n \in \mathbb{N}$ then $\{\phi(X_n)\}$ is a supermartingale w.r.t $\{\mathcal{F}_n\}$
- (Ex) If $\{X_n\}$ is a martingale and $E[|X_n|^p] < \infty$ for some $p \geq 1$, then $\{|X_n|^p\}$ is a submartingale
- (Ex) If $\{X_n\}$ is a submartingale then for any $a \in \mathbb{R}$, $\{(X_n - a)^+\}$ is a submartingale
- (Ex) If $\{X_n\}$ is a supermartingale then for any $a \in \mathbb{R}$, $\{X_n \wedge a\}$ is a supermartingale
- (Ex) If $\{X_n\}$ is a submartingale then $\{X_n^+\}$ is a submartingale and $\{X_n^-\}$ is a supermartingale

- * Predicable sequence and a process using it

- For a filtration $\{\mathcal{F}_n\}_{n=0}^\infty$, a random sequence $\{H_n\}_{n=1}^\infty$ is said to be a predicable sequence (w.r.t $\{\mathcal{F}_n\}$) if $H_n \in \mathcal{F}_{n-1} \quad \forall n \in \mathbb{N}$
 \checkmark A letter H stands for a ‘height’
- Suppose $\{X_n\}$ is adapted to $\{\mathcal{F}_n\}$. For a predicable sequence $\{H_n\}$ (w.r.t $\{\mathcal{F}_n\}$), we define a process $\{(H \cdot X)_n\}$ by

$$(H \cdot X)_n = \sum_{m=1}^n H_m(X_m - X_{m-1})$$

- \checkmark Note that $\{(H \cdot X)_n\}$ is adapted to $\{\mathcal{F}_n\}$
- \checkmark The definition above can be extended from $\{(H \cdot X)_n\}_{n \in \mathbb{N}}$ to $\{(H \cdot X)_n\}_{n \in \mathbb{N} \cup \{0\}}$ with additionally defining $(H \cdot X)_0 = 0$. Obviously $(H \cdot X)_0 \in \mathcal{F}_0$. For the following theorems using this process, we can regard it as $\{(H \cdot X)_n\}_{n \in \mathbb{N} \cup \{0\}}$

- Elementary facts about martingale transform with predicable sequence

- Let $\{X_n\}_{n=0}^\infty$ and $\{H_n\}_{n=1}^\infty$ be a random sequence and $\{H_n\}$ is a predicable sequence w.r.t. a filtration $\{\mathcal{F}_n\}_{n=0}^\infty$. Assume $E|X_n H_n| < \infty$, $E|X_{n-1} H_n| < \infty \quad \forall n \in \mathbb{N}$
 - If $\{X_n\}$ is a martingale (w.r.t $\{\mathcal{F}_n\}$) then $\{(H \cdot X)_n\}$ is also a martingale
 - If $\{X_n\}$ is a submartingale (w.r.t $\{\mathcal{F}_n\}$) and $H_n \geq 0$ then $\{(H \cdot X)_n\}$ is also a submartingale
 - If $\{X_n\}$ is a supermartingale (w.r.t $\{\mathcal{F}_n\}$) and $H_n \geq 0$ then $\{(H \cdot X)_n\}$ is also a supermartingale
- \checkmark The condition “ $E|X_n H_n| < \infty$, $E|X_{n-1} H_n| < \infty \quad \forall n \in \mathbb{N}$ ” can be replaced with “For each $n \in \mathbb{N}$, H_n is bounded”.

- * Stopping time

- A (extended) random variable N taking values of $\mathbb{N} \cup \{0, \infty\}$ is said to be a stopping time (w.r.t a filtration $\{\mathcal{F}_n\}$) if an event $(N = n) \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$

$$\begin{aligned} (N \leq n) &= \bigcup_{j=0}^n (N = j) \in \mathcal{F}_n & (N > n) &= (N \leq n)^C \in \mathcal{F}_n \\ (N < n) &= \bigcup_{j=0}^{n-1} (N = j) \in \mathcal{F}_{n-1} & (N \geq n) &= (N < n)^C \in \mathcal{F}_{n-1} \end{aligned}$$

- $(N \geq n)$ is a \mathcal{F}_{n-1} -measurable event. $I(N \geq n)$ is \mathcal{F}_{n-1} -measurable random variable. Hence, $\{I(N \geq n)\}_n$ is a predictable sequence given N is a stopping time.

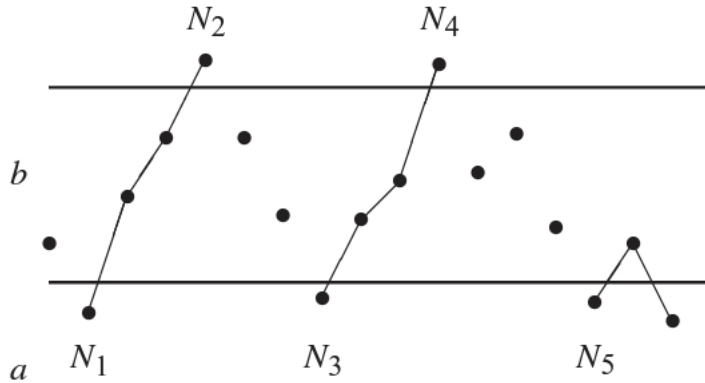
- Martingale stopped by stopping time

- Let $\{X_n\}$ be a random sequence adapted to $\{\mathcal{F}_n\}$. Let N be a stopping time w.r.t $\{\mathcal{F}_n\}$ and put $H_n = I(N \geq n) \quad \forall n \in \mathbb{N}$. Then $(H \cdot X)_n = X_{N \wedge n} - X_0$.
- The process $\{X_{N \wedge n}\}_n$ is said to be a martingale stopped by stopping time N , provided $\{X_n\}$ is a martingale.
 - ★ If $\{X_n\}$ and $\{Y_n\}$ are martingales (w.r.t. $\{\mathcal{F}_n\}$) then $\{X_n + Y_n\}$ is also a martingale. The same holds for submartingales and supermartingales too.
- If $\{X_n\}$ is a martingale and N is a stopping time then $\{X_{N \wedge n}\}$ is martingale.
- If $\{X_n\}$ is a submartingale and N is a stopping time then $\{X_{N \wedge n}\}$ is submartingale.
- If $\{X_n\}$ is a supermartingale and N is a stopping time then $\{X_{N \wedge n}\}$ is supermartingale.

- Stopping time and Upcrossing

- Suppose $\{X_n\}_{n=0}^\infty$ is a submartingale w.r.t $\{\mathcal{F}_n\}$. Let $a < b$. Define N_j 's as below :

$$\begin{aligned}
 N_1 &= \inf\{m \geq 0 : X_m \leq a\} & N_2 &= \inf\{m > N_1 : X_m \geq b\} \\
 N_3 &= \inf\{m > N_2 : X_m \leq a\} & N_4 &= \inf\{m > N_3 : X_m \geq b\} \\
 &\vdots & &\vdots \\
 N_{2k-1} &= \inf\{m > N_{2k-2} : X_m \leq a\} & N_{2k} &= \inf\{m > N_{2k-1} : X_m \geq b\} \\
 &\vdots & &\vdots
 \end{aligned}$$



- Every N_j for $j \in \mathbb{N}$ is stopping time w.r.t $\{\mathcal{F}_n\}$. $N_1 < N_2 < N_3 \dots$ provided all N_j 's are finite. (It is possible that $N_j = \infty$ provided it has a form of $\inf(\emptyset)$)
- ‘Upcrossing’ is a case where the submartingale $\{X_n\}$ crosses from below a to above b .
- $\mathcal{U}_n := \sup\{k : N_{2k} \leq n\}$ is the number of upcrossings completed by time n

- Upcrossing inequality

- Suppose $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$. If stopping time N_j and the number of upcrossings \mathcal{U}_n are defined as above then

$$(b - a)E[\mathcal{U}_n] \leq E[(X_n - a)^+] - E[(X_0 - a)^+]$$

- Submartingale convergence theorem

- If $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$ with $\sup_n E(X_n^+) < \infty$ then $X_n \rightarrow X$ a.s. for some integrable random variable X
 - ★ Trick : If $X_n \rightarrow X$ a.s. then $X_n^+ \rightarrow X^+$ a.s. and $X_n^- \rightarrow X^-$ a.s.
 - ★ Lemma : If the number of upcrossings of $[a, b]$ by submartingale $\{X_n\}$ is finite for any $a, b \in \mathbb{Q}$, then $\lim_n X_n$ exists. i.e. X_n converges to some r.v. almost surely.
- If $\{X_n\}$ is a nonnegative supermartingale w.r.t $\{\mathcal{F}_n\}$ then $X_n \rightarrow X$ a.s. for some integrable random variable X s.t. $E(X) \leq E(X_0)$

- Example of martingale which converges almost surely but not in L^1

- $\{\xi_n\}_n$ i.i.d with $P(\xi_1 = 1) = P(\xi_1 = -1) = 1/2$. Let $S_0 = 1$, $S_n = S_{n-1} + \xi_n$ and $\mathcal{F}_0 = \{\phi, \Omega\}$, $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n)$. Then $\{S_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$
Let $N = \inf\{n \in \mathbb{N} : S_n = 0\}$. Then N is a stopping time.
 $X_n := S_{N \wedge n}$ so that $X_n = S_n$ if $n < N$ and $X_n = 0$ if $n \geq N$. $\{X_n\}$ is a nonnegative integer valued martingale w.r.t $\{\mathcal{F}_n\}$. $X_n \rightarrow 0$ a.s. but $X_n \not\rightarrow 0$ in \mathcal{L}^1 .

- If $\{X_n\}_{n \in \mathbb{N} \cup \{0\}}$ is a negative submartingale w.r.t $\{\mathcal{F}_n\}_{n \in \mathbb{N} \cup \{0\}}$ then so is $\{X_n\}_{n \in \mathbb{N} \cup \{0, \infty\}}$ w.r.t $\{\mathcal{F}_n\}_{n \in \mathbb{N} \cup \{0, \infty\}}$ where $X_\infty = \lim_n X_n$ and $\mathcal{F}_\infty = \sigma(\bigcup_{n=0}^\infty \mathcal{F}_n)$

- If $\{X_n\}_{n \in \mathbb{N}}$ is a martingale w.r.t $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ and $X_n \rightarrow X_\infty$ a.s. then $X_\infty \in \mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F}_n)$

- Doob's decomposition

- Any submartingale $\{X_n\}$ can be written as $X_n = M_n + A_n$ where $\{M_n\}$ is a martingale and $\{A_n\}$ is a predictable increasing sequence with $A_0 = 0$. Also, this expression is unique in the sense that if $X_n = M'_n + A'_n$ is another expression then $M_n = M'_n$ and $A_n = A'_n$ a.s.
- ✓ The exact form of M_n, A_n for given X_n is $A_n = A_{n-1} + E[X_n | \mathcal{F}_{n-1}] - X_{n-1} \quad \forall n \in \mathbb{N}$ and $M_n = X_n - A_n \quad \forall n \in \mathbb{N} \cup \{0\}$ (Since $A_0 = 0$, $A_n = \sum_{k=1}^n (E[X_k | \mathcal{F}_{k-1}] - X_{k-1})$)

- * Almost sure convergence on a restricted event

- Define “ $X_n \rightarrow X$ a.s. on B ” for measurable set B as
 $P((X_n \rightarrow X) \cap B) = P(B)$ or equivalently, $P((X_n \not\rightarrow X) \cap B) = 0$
- ★ Lemma : $X_n \rightarrow X$ a.s. on $B \Rightarrow X_n \rightarrow X$ a.s. on A whenever $A \subset B$
- ★ Lemma : $X_n \rightarrow X$ a.s. on $A_k \quad \forall k \in \mathbb{N} \Rightarrow X_n \rightarrow X$ a.s. on $\bigcup_{k \in \mathbb{N}} A_k$

- Martingales with bounded increments either converge or oscillate between ∞ and $-\infty$

- Let $\{X_n\}$ be a martingale with $|X_n - X_{n-1}| \leq M < \infty \quad \forall n \in \mathbb{N}$ for some $M > 0$. Define disjoint subsets $C, D \subset \Omega$ by

$$C = (\lim_n X_n \text{ exists and } -\infty < \lim_n X_n < \infty)$$

$$D = (\limsup X_n = \infty \text{ and } \liminf X_n = -\infty)$$

Then $P(C \cup D) = 1$

- Conditional Borel-Cantelli second lemma

- Let $\{\mathcal{F}_n\}_{n \in \mathbb{N} \cup \{0\}}$ be a filtration with $\mathcal{F}_0 = \{\phi, \Omega\}$. If $A_n \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$ then

$$(A_n \text{ i.o.}) = \left(\sum_{n=1}^{\infty} P(A_n | \mathcal{F}_{n-1}) = \infty \right) \text{ a.s.}$$

- ★ Define “ $A = B$ a.s.” for measurable sets A and B by $P(A \Delta B) = 0$ where $A \Delta B$ denotes the symmetric difference of two sets.
- ★ $\sum_{k=1}^n I_{A_k}$ is a submartingale whose martingale component of Doob’s decomposition is

$$\sum_{k=1}^n I_{A_k} - \sum_{k=1}^n \left(E \left[\sum_{j=1}^k I_{A_j} | \mathcal{F}_{k-1} \right] - \sum_{j=1}^{k-1} I_{A_j} \right) = \sum_{k=1}^n I_{A_k} - \sum_{k=1}^n P(A_k | \mathcal{F}_{k-1})$$

and this is the martingale we exploit in the proof of conditional B-C 2nd lemma

- ★ Trick : $(A_n \text{ i.o.}) = \left(\sum_{n=1}^{\infty} I_{A_n} = \infty \right)$
- ✓ Given $\{A_n\}$ is independent, by setting $\mathcal{F}_n = \sigma(A_1, \dots, A_n)$, conditional Borel-Cantelli second lemma implies original Borel-Cantelli second lemma which is given by

$$\sum_n P(A_n) = \infty \Rightarrow P(A_n \text{ i.o.}) = 1$$

* Branching process (Galton-Watson process)

- Let $\{\xi_i^n\}_{i \in \mathbb{N}, n \in \mathbb{N}}$ be i.i.d nonnegative integer-valued random variables.
Define a Galton-Watson process $\{Z_n\}_{n \in \mathbb{N} \cup \{0\}}$ as below :

$$\begin{aligned} Z_0 &= 1 \\ Z_{n+1} &= \begin{cases} \xi_1^{n+1} + \dots + \xi_{Z_n}^{n+1} = \sum_{j=1}^{Z_n} \xi_j^{n+1} & \text{if } Z_n > 0 \\ 0 & \text{if } Z_n = 0 \end{cases} \end{aligned}$$

- ✓ The idea behind the definitions is that Z_n is the population in the n -th generation and each member of the n -th generation gives birth independently to an identically distributed number of offspring.
- $P(\xi_1^1 = k) \quad \forall k \in \mathbb{N} \cup \{0\}$ is called the offspring distribution. $\mu = E(\xi_1^1)$ is the expected number of offspring per individual.

• Properties of the branching process

- Let $\mathcal{F}_n = \sigma(\{\xi_i^m : i \in \mathbb{N}, 1 \leq m \leq n\}) \quad \forall n \in \mathbb{N}$, $\mathcal{F}_0 = \{\phi, \Omega\}$. If $\mu = E(\xi_1^1) \in (0, \infty)$ then $\{Z_n / \mu^n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$ and $E(Z_n) = \mu^n \quad \forall n \in \mathbb{N}$
- If $\mu = E(\xi_1^1) \in (0, 1)$ then $Z_n = 0$ for large enough n ’s a.s. i.e. the species goes extinct.

• Inequality for bounded stopping time

- If $\{X_n\}$ is a submartingale and N is a stopping time with $P(N \leq K) = 1$ for some $K \in \mathbb{N}$ then

$$E(X_0) \leq E(X_N) \leq E(X_K)$$

- ✓ Since $\{X_n\}$ is a submartingale, $E(X_0) \leq E(X_j) \leq E(X_K)$ whenever $0 \leq j \leq K$. This thm tells us that similar inequality still holds true when the index is random.

- If $\{X_n\}$ is a martingale and N is a stopping time with $P(N \leq K) = 1$ for some $K \in \mathbb{N}$ then

$$E(X_0) = E(X_N) = E(X_K)$$

• Doob's inequality

- Let $\{X_n\}_{n \in \mathbb{N} \cup \{0\}}$ be a submartingale. Take $n \in \mathbb{N}$ and define $\bar{X}_n = \max_{0 \leq m \leq n} X_m$. Let $\lambda > 0$ and define an event $A = (\bar{X}_n \geq \lambda)$. Then the inequality below holds true.

$$\lambda P(A) \leq E[X_n I_A] \leq E[X_n^+ I_A] \leq E[X_n^+]$$

- Let $\{X_n\}_{n \in \mathbb{N} \cup \{0\}}$ be a supermartingale. Take $n \in \mathbb{N}$ and define $\bar{X}_n = \max_{0 \leq m \leq n} X_m$. Let $\lambda > 0$ and define an event $A = (\bar{X}_n \geq \lambda)$. Then the inequality below holds true.

$$\lambda P(A) \leq E[X_0] - E[X_n I_{A^c}] \leq E[X_0] + E[X_n^-]$$

✓ Note that $P(A)$ involves $\max_{0 \leq m \leq n}$ term while $E[X_n^+]$ or $E[X_n^-]$ only depends on n

• Doob's L^p maximal inequality

- If $\{X_n\}_{n \in \mathbb{N} \cup \{0\}}$ is a nonnegative submartingale, then for $1 < p < \infty$ and $\bar{X}_n = \max_{0 \leq m \leq n} X_m$, the inequality below holds true.

$$E(\bar{X}_n^p) \leq \left(\frac{p}{p-1}\right)^p E[X_n^p]$$

- If $\{X_n\}_{n \in \mathbb{N} \cup \{0\}}$ is a martingale then for $1 < p < \infty$ and $|\bar{X}_n| = \max_{0 \leq m \leq n} |X_m|$, the inequality below holds true.

$$E|\bar{X}_n|^p \leq \left(\frac{p}{p-1}\right)^p E|X_n|^p$$

★ Lemma : If $X \geq 0$ then $E(X) = \int_0^\infty P(X > t) dt$

• L^p convergence thm

- If $\{X_n\}$ is a martingale with $\sup_n E|X_n|^p < \infty$ for some $p > 1$ then $X_n \rightarrow X$ a.s. and $X_n \rightarrow X$ in L^p for some integrable r.v. X

✓ For a martingale convergence thm, the condition was $\sup_n E(X_n^+) < \infty$

★ Trick : $a, b \in \mathbb{R}$ and $p \geq 1 \Rightarrow |a + b|^p \leq 2^p(|a|^p + |b|^p)$

* σ -field generated by a stopping time

- Let τ be a stopping time w.r.t. a filtration $\{\mathcal{F}_n\}$. Then we define \mathcal{F}_τ as the following :

$$\mathcal{F}_\tau = \{A \in \mathcal{F} : A \cap (\tau = n) \in \mathcal{F}_n \quad \forall n \in \mathbb{N}\}$$

✓ Note that \mathcal{F}_τ is indeed a σ -field.

✓ τ is \mathcal{F}_τ -measurable

✓ If $\{X_n\}$ is adapted to $\{\mathcal{F}_n\}$ then X_τ is \mathcal{F}_τ -measurable

• Bounded optional stopping thm

- Let $\{X_n\}$ be a submartingale. Let σ and τ be two bounded stopping times s.t. $\sigma \leq \tau \leq B$ a.s. for some $B \in \mathbb{N}$. Then $E[X_\tau | \mathcal{F}_\sigma] \geq X_\sigma$ a.s.

✓ $X_\tau = \sum_{n=0}^B X_n I(\tau = n)$ is well-defined and integrable.

✓ By defining property of submartingale, $E[X_m | \mathcal{F}_n] \geq X_n \quad \forall m \geq n$. The thm tells us that this property is preserved even when indices are stopping times if they are bounded.

★ Trick : For a random variable X and a σ -field \mathcal{F} ,

i. $(X \leq a) \in \mathcal{F} \quad \forall a \in \mathbb{R} \Rightarrow (X \in A) \in \mathcal{F} \quad \forall A \in \mathcal{B}(\mathbb{R})$

ii. For $S \in \mathcal{F}$, $(X \leq a) \cap S \in \mathcal{F} \quad \forall a \in \mathbb{R} \Rightarrow (X \in A) \cap S \in \mathcal{F} \quad \forall A \in \mathcal{B}(\mathbb{R})$

★ Lemma : For any \mathcal{F} -measurable and integrable X and Y ,

i. If $\int_A X dP = \int_A Y dP \quad \forall A \in \mathcal{F}$ then $X = Y$ a.s.

ii. If $\int_A X dP \leq \int_A Y dP \quad \forall A \in \mathcal{F}$ then $X \leq Y$ a.s.

★ Lemma : $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\} \Rightarrow \int_A X_n dP \leq \int_A X_{n+1} dP \quad \forall A \in \mathcal{F}_n$

* Uniform integrability

– A collection of r.v.'s $\{X_t : t \in T\}$ is said to be uniformly integrable if

$$\lim_{a \rightarrow \infty} \sup_{t \in T} \int_{|X_t| \geq a} |X_t| dP = \lim_{a \rightarrow \infty} \sup_{t \in T} E|X_t| I(|X_t| \geq a) = 0$$

✓ Denote it as $\{X_t\}_{t \in T}$ u.i.

✓ A uniformly integrable family is well-controlled in the sense that if $\{X_t\}_{t \in T}$ u.i. then $\exists M > 0$ s.t. $\sup_{t \in T} E|X_t| \leq M + 1 < \infty$

✓ If $\{X_t\}_{t \in T}$ is uniformly integrable then each X_t is integrable.

• If $\{X_t\}_{t \in T}$ is dominated by a nonnegative integrable r.v. X i.e. $|X_t| \leq X$ a.s. $\forall t \in T$ then $\{X_t\}_{t \in T}$ is uniformly integrable.

★ Lemma: If X is integrable then $\int_{|X| \geq a} |X| dP = E|X| I(|X| \geq a) \rightarrow 0$ as $a \rightarrow \infty$

• Equivalent condition for uniform integrability

– $\{X_t\}_{t \in T}$ is uniformly integrable iff both of two conditions below are satisfied.

i. $\sup_t E|X_t| < \infty$

ii. $\forall \varepsilon > 0, \exists \delta > 0$ s.t. $\sup_t \int_A |X_t| dP < \varepsilon$ whenever $A \in \mathcal{F}$ and $P(A) < \delta$

• Elementary properties of uniform integrable family

– If $\{X_n\}_{n \in \mathbb{N}}$ and $\{Y_n\}_{n \in \mathbb{N}}$ are both uniformly integrable then $\{X_n + Y_n\}_{n \in \mathbb{N}}$ u.i.

– If $|X_n| \leq |Y_n| \quad \forall n \in \mathbb{N}$ and $\{Y_n\}_{n \in \mathbb{N}}$ is uniformly integrable then $\{X_n\}_{n \in \mathbb{N}}$ u.i.

• Vitali's lemma

– For $p \geq 1$, if $\{X_n\} \subset L^p$ and $X_n \xrightarrow[seq]{P} X$ then the followings are equivalent.

i. $\{X_n^p\}_{n \in \mathbb{N}}$ is uniformly integrable.

ii. $X \in L^p$ and $X_n \rightarrow X$ in L^p

iii. $E|X_n|^p \rightarrow E|X|^p < \infty$

★ Lemma : For a r.v. Z , continuity set $\{z \in \mathbb{R} : P(Z = z) = 0\}$ is dense in \mathbb{R}

• If $\{X_n\}_{n \in \mathbb{N}}$ is uniformly integrable and $X_n \xrightarrow{D} X$ then $E|X_n| \rightarrow E|X|$ and $E(X_n) \rightarrow E(X)$

★ Lemma : If $Y_n \rightarrow Y$ in L^1 then $E|Y_n| \rightarrow E|Y|$ and $E(Y_n) \rightarrow E(Y)$

* Regular martingale and closable martingale

– Let $\{X_n\}_{n \in \mathbb{N} \cup \{0\}}$ be a martingale.

- i. $\{X_n\}$ is said to be regular if $\exists X \in L^1$ s.t. $X_n = E[X|\mathcal{F}_n]$ a.s. $\forall n \in \mathbb{N}$
 - ii. $\{X_n\}$ is said to be closable if $\exists X_\infty \in L^1$ s.t. $X_n \rightarrow X_\infty$ a.s. , $X_\infty \in \mathcal{F}_\infty$ where $\mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F}_n)$ and $E[X_\infty|\mathcal{F}_n] = X_n \quad \forall n \in \mathbb{N}$ so that $\{X_n\}_{n \in \mathbb{N} \cup \{0, \infty\}}$ is a martingale w.r.t $\{\mathcal{F}_n\}_{n \in \mathbb{N} \cup \{0, \infty\}}$
- ✓ Every closable martingale is regular.

• For a martingale $\{X_n\}_{n \in \mathbb{N}}$, the followings are equivalent.

- i. $\{X_n\}$ is regular.
- ii. $\{X_n\}$ is uniformly integrable and converges a.s.
- iii. $\{X_n\}$ converges in L^1
- iv. $\{X_n\}$ is closable.

□ For a martingale $\{X_n\}_{n \in \mathbb{N}}$ w.r.t $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$

- If $X_n \rightarrow X$ in L^1 then $X_n \rightarrow X$ a.s. and $X_n = E[X|\mathcal{F}_n] \quad \forall n \in \mathbb{N}$
- If $\{X_n\}$ is uniformly integrable then $X_n \rightarrow X$ a.s. for some integrable r.v. X and $X_n = E[X|\mathcal{F}_n] \quad \forall n \in \mathbb{N}$
- If $X_n = E[X|\mathcal{F}_n]$ for some integrable r.v. X then $\{X_n\}$ is uniformly integrable and \exists integrable r.v. $X_\infty \in \mathcal{F}_\infty$ s.t. $E[X_\infty|\mathcal{F}_n] = X_n \quad \forall n \in \mathbb{N}$ and $X_n \rightarrow X_\infty$ a.s. and in L^1 .

• Levy's thm

- If $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ is a filtration and $\mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F}_n)$ then for an integrable r.v. X , $E[X|\mathcal{F}_n] \rightarrow E[X|\mathcal{F}_\infty]$ a.s. and in L^1 .

• Conditional DCT (generalized version)

- Let $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ be a filtration and $\mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F}_n)$. If $X_n \rightarrow X$ a.s. and $|X_n| \leq Z$ for some integrable r.v. Z , then $E[X_n|\mathcal{F}_n] \rightarrow E[X|\mathcal{F}_\infty]$ a.s.

* Potential

- A nonnegative supermartingale $\{X_n\}$ is said to be potential if $E(X_n) \rightarrow 0$
- ✓ If $\{X_n\}$ is potential then $\{X_n\}$ is uniformly integrable and $X_n \rightarrow 0$ a.s.

• Riesz decomposition

- Let $\{X_n\}$ be a uniformly integrable nonnegative supermartingale. Then we can express X_n as $X_n = M_n + V_n$ where $\{M_n\}$ is uniformly integrable martingale and $\{V_n\}$ is potential. Furthermore, such decomposition is unique.

• If $\{X_n\}$ is uniformly integrable submartingale, then for any stopping time N , stopped process $\{X_{N \wedge n}\}$ is also uniformly integrable submartingale.

★ Lemma : If $X_n \rightarrow X$ a.s. then $X_n^+ \rightarrow X^+$ a.s. and $X_n^- \rightarrow X^-$ a.s.

- Inequality for unbounded stopping time

- If $\{X_n\}$ is uniformly integrable submartingale then for any stopping time N , we have $E(X_0) \leq E(X_N) \leq E(X_\infty)$ where $X_n \rightarrow X_\infty$ a.s.

- If $\{X_n\}$ is uniformly integrable martingale then for any stopping time N , we have $E(X_0) = E(X_N) = E(X_\infty)$ where $X_n \rightarrow X_\infty$ a.s.

- Optional stopping thm

- If $L \leq M$ are stopping times and $\{X_n\}$ is uniformly integrable submartingale then $E[X_L] \leq E[X_M]$ and $X_L \leq E[X_M | \mathcal{F}_L]$ a.s.

- Suppose $\{X_n\}$ is a submartingale and $E[|X_{n+1} - X_n| | \mathcal{F}_n] \leq B$ a.s. $\forall n \in \mathbb{N}$. If N is a stopping time with $E(N) < \infty$ then $\{X_{N \wedge n}\}$ is uniformly integrable and $E(X_0) \leq E(X_N)$

✓ Note that $E(N) < \infty$ condition implies that N is almost surely finite.

★ Lemma : $E|X| < \infty \Leftrightarrow \sum_n P(|X| \geq n) < \infty$

- If $\{X_n\}$ is a nonnegative supermartingale and N is a stopping time then $E(X_0) \geq E(X_N)$

- Comment for X_N with stopping time N and (sub)martingale $\{X_n\}$

- $X_N = \sum_{n=0}^{\infty} X_n I(N = n)$

- Note that N can take value of $N = \infty$. Thus, for X_N to make sense, N should be almost surely bounded or X_∞ is well-defined.

- If X_∞ is well-defined such that $X_n \rightarrow X_\infty$ a.s. then $X_{N \wedge n} \rightarrow X_N$ a.s.

- How can we figure out integrability of X_N ?

- i. If N is bounded a.s.

- $N \leq K$ a.s. for some $K \in \mathbb{N}$. Hence $E|X_N| \leq \sum_{n=0}^K E|X_n| < \infty$

- ii. If $\{X_n\}$ is uniformly integrable submartingale

- $X_n \rightarrow X_\infty$ a.s. $\Rightarrow X_{N \wedge n} \rightarrow X_N$ a.s. Since $\{X_{N \wedge n}\}$ is also uniformly integrable submartingale, by Vitali lemma, $X_N \in L^1$ i.e. X_N is integrable.

- iii. If $\{X_n\}$ is nonnegative supermartingale

- $X_n \rightarrow X_\infty$ a.s. $\Rightarrow X_{N \wedge n} \rightarrow X_N$ a.s. By inequality for bounded stopping time, $E[X_{N \wedge n}] \leq E[X_0]$ and using Fatou's lemma, we have $0 \leq E[X_N] \leq E[X_0] < \infty$

- Asymmetric simple random walk

- Let $\{\xi_i\}_{i \in \mathbb{N}}$ be i.i.d. random seq. s.t. $P(\xi_1 = 1) = p$ and $P(\xi_1 = -1) = q = 1 - p$ for some $0 < p < 1$. $S_0 = 0$, $S_n = \xi_1 + \dots + \xi_n$ and $\mathcal{F}_n = \sigma(\xi_1, \dots, \xi_n) \quad \forall n \in \mathbb{N}$

- i. For $\psi : \mathbb{R} \rightarrow \mathbb{R}$ defined by $\psi(x) = \left(\frac{1-p}{p}\right)^x$, $\{\psi(S_n)\}$ is a martingale.

- ii. Define $T_m = \inf\{n \in \mathbb{N} \cup \{0\} : S_n = m\}$ for $m \in \mathbb{Z}$ where $\inf(\emptyset)$ is interpreted as ∞ . For any $a, b \in \mathbb{Z}$ s.t. $a < 0 < b$, we have

$$P(T_a < T_b) = \frac{\psi(b) - \psi(0)}{\psi(b) - \psi(a)}$$

$$p > q \Rightarrow T_b < \infty \text{ a.s.}, \quad p < q \Rightarrow T_a < \infty \text{ a.s.}, \quad p = q \Rightarrow T_a, T_b < \infty \text{ a.s.}$$

iii. If $p > q$ ($p > \frac{1}{2}$) then

$$P(\inf_n S_n \leq a) = P(T_a < \infty) = \left(\frac{1-p}{p}\right)^{-a} \quad \forall a < 0$$

$$E[T_b] = \frac{b}{2p-1} \quad \forall b > 0$$

★ Trick : For $|r| < 1$, $\sum_{n=1}^{\infty} nr^n$ converges to a finite number $r/(1-r)^2$

• Square integrable Martingales

- Suppose $\{X_n\}$ is a martingale with $X_0 = 0$ and $E[X_n^2] < \infty \quad \forall n \in \mathbb{N}$
- Since X_n^2 is a submartingale, we can find $X_n^2 = M_n + A_n$ which is a Doob's decomposition for X_n^2 . Here, A_n is called the increasing process associated with X_n

$$A_n = \sum_{k=1}^n E[X_k^2 | \mathcal{F}_{k-1}] - X_{k-1}^2 = \sum_{k=1}^n E[(X_k - X_{k-1})^2 | \mathcal{F}_{k-1}] \quad \text{with } A_0 = 0$$

- Since A_n is increasing, $A_{\infty} = \lim_n A_n = \sup_n A_n$ exists.
- A_n can be thought of as a path by path measurement of the variance at time n and A_{∞} can be viewed as the total variance in the path.
- A_n is integrable and $E[X_n^2] = E[A_n] \quad \forall n \in \mathbb{N}$
- Two results about boundedness

$$E[\sup_n X_n^2] \leq 4E[A_{\infty}] \quad \text{and} \quad E[\sup_n |X_n|] \leq 3E[\sqrt{A_{\infty}}]$$

- Two results about convergence

i. X_n converges to a finite limit *a.s.* on $(A_{\infty} < \infty)$

★ Trick : If N is a stopping time, then $X_{N \wedge n}$ is a square integrable martingale and $A_{N \wedge n}$ is the increasing process associated with $X_{N \wedge n}$

ii. If $f : \mathbb{R} \rightarrow \mathbb{R}$ be an increasing function with $f(t) \geq 1 \quad \forall t \in \mathbb{R}$ and $\int_0^{\infty} 1/f^2(t) dt < \infty$ then $X_n/f(A_n) \rightarrow 0$ *a.s.* on $(A_{\infty} = \infty)$

★ Trick : For such function f , we have $f(t) \nearrow \infty$ as $t \rightarrow \infty$

★ Lemma : [Kronecker's lemma] If $0 < a_n \nearrow \infty$ and $\sum_{n=1}^{\infty} \frac{1}{a_n} x_n$ converges then $\frac{1}{a_n} \sum_{j=1}^n x_j \rightarrow 0$

• Conditional Borel-Cantelli second lemma (2nd version)

- Suppose $B_n \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$ and $\mathcal{F}_0 = \{\phi, \Omega\}$. Let $p_n = P(B_n | \mathcal{F}_{n-1}) \quad \forall n \in \mathbb{N}$. Then,

$$\frac{\sum_{m=1}^n I_{B_m}}{\sum_{m=1}^n p_m} \rightarrow 1 \quad \text{a.s.} \quad \text{on} \quad \left(\sum_n p_n = \infty \right)$$

✓ This extends the original Borel-Cantelli second lemma.