Probability theory II Facts

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1 Conditional Expectation

- Projection Thm for Hilbert Space
  - If E is a Hilbert space and $M \subset E$ is closed and convex, then for any $y \in E$,
    $\exists$ a unique $w \in M$ s.t. $\|y - w\| = d(y, M) := \inf\{\|y - v\| : v \in M\}$.
    Denote it as $w = proj_M y$ i.e. w is a projection of y onto M .
  - If E is a Hilbert space and $M \subset E$ is a closed vector subspace, then for any $y \in E$,
    i. $\exists$ a unique decomposition $y = w + v$ with $w = proj_M y \in M$ and $v \in M^\perp$
    ii. For $w \in M$, $\quad w = proj_M y \Leftrightarrow \langle y - w, z\rangle = 0 \quad \forall\, z \in M$

* $\mathcal{L}^2 := \{$ Random Variable $X : E(X^2) = \int X^2\, dP < \infty\}$

$\surd$ If $X \in \mathcal{L}^2$ then $E|X| < \infty$ i.e. every element of $\mathcal{L}^2$ is integrable.

★ Trick : $|X| \leq X^2 + \frac{1}{4}$

$\surd$ $\mathcal{L}^2$ is a vector space

★ Trick : inequality $(aX + bY)^2 \leq 2(a^2X^2 + b^2Y^2)$

- $\mathcal{L}^2$ is a Hilbert space with inner product $\langle X, Y\rangle = E(XY)$

★ Trick : Cauchy seq. having a subseq. converging to a point converges to the point.

- Lemma for proving $\mathcal{L}^2$ is a complete normed space.
  - If $\{X_n\} \underset{seq}{\subset} \mathcal{L}^2$ and $\|X_n - X_{n+1}\| \leq 2^{-n} \quad \forall\, n \in \mathbb{N}$ then $\exists\, X \in \mathcal{L}^2$ s.t. $X_n \to X \quad a.s.$ and $\|X_n - X\| \to 0$ i.e. $X_n \to X$ in $\mathcal{L}^2$.

    ★ Lemma : If a random varaible Z satisfies $Z \geq 0$ and $E(Z) < \infty$ then $Z < \infty \quad a.s.$

* For $X \in \mathcal{L}^2$, $\quad \mathcal{L}^2(X) := \{g(X) \,|\, g : \mathbb{R} \to \mathbb{R}$ is a Borel function, $E[(g(X))^2] < \infty\}$

$\surd$ For $X \in \mathcal{L}^2$, $\quad \mathcal{L}^2(X)$ is a vector subspace of $\mathcal{L}^2$.

- For $X \in \mathcal{L}^2$, $\quad \mathcal{L}^2(X)$ is a closed vector subspace of $\mathcal{L}^2$ so that $\mathcal{L}^2(X)$ is also a Hilbert space.

* Geometric definition for conditional expectation
  - For $X, Y \in \mathcal{L}^2$, define $E[Y|X] = Proj_{\mathcal{L}^2(X)} Y$
  - $E[Y|X] = g(X)\ a.s.$ for some Borel function g
  - $\|Y - E[Y|X]\| = \min_{h(X) \in \mathcal{L}^2(X)} \|Y - h(X)\|$
    i.e. $E[(Y - E[Y|X])^2] \leq E[(Y - h(X))^2] \quad \forall\, h(X) \in \mathcal{L}^2$
  - For $g(X) \in \mathcal{L}^2(X)$, $\quad g(X) = E[Y|X] \Leftrightarrow \langle Y - g(X), h(X)\rangle = 0 \quad \forall\, h(X) \in \mathcal{L}^2$
    $\Leftrightarrow E[(Y - g(X))h(X)] = 0 \quad \forall\, h(X) \in \mathcal{L}^2$

- Elementary properties of conditional expectation from geometric definition
  - If $X, Y, Z \in \mathcal{L}^2$ then the followings are true.
    i. $E[c|X] = c\ a.s. \quad \forall\, c \in \mathbb{R}$
    ii. $E[\alpha Y + \beta Z|X] = \alpha E[Y|X] + \beta E[Z|X] \quad \forall \alpha, \beta \in \mathbb{R}$
    iii. $E[Y|X] = E[Y]$ if X and Y are independent.

iv. $E[g(X)Y|X] = g(X)E[Y|X]$ if g satisfies $g(X) \in \mathcal{L}^2(X)$ and $\sup_x |g(x)| < \infty$

v. $E[E[Y|X]] = E[Y]$

$\checkmark$ In fact, the additional assumption about boundedness of g in (iv) is not necessary. We will see later.

- Extending the definition from $\mathcal{L}^2$ to all integrable functions

$$E[\{Y - E[Y|X]\}I(X \in A)] = 0 \quad \forall\, A \in \mathcal{B}(\mathbb{R}) \quad \because I(X \in A) \in \mathcal{L}^2(X)$$
$$\int_{(X\in A)} Y\, dP = \int_{(X\in A)} E[Y|X]\, dP \quad \forall\, A \in \mathcal{B}(\mathbb{R})$$
$$\int_B Y\, dP = \int_B E[Y|X]\, dP \quad \forall\, B \in \sigma(X)$$

  - $E[Y|X] \in \sigma(X)$ and $\int_B Y\, dP = \int_B E[Y|X]\, dP \quad \forall\, B \in \sigma(X)$. Such r.v. is unique in the sense that if any r.v. Z satisfies $Z \in \sigma(X)$ and $\int_B Y\, dP = \int_B Z\, dP \quad \forall\, B \in \sigma(X)$ then $Z = E[Y|X]$ *a.s.* provided $E|Y| < \infty$
  - From the theory on $\mathcal{L}^2$ space, we get geometric understanding about conditional expectation. But now, from the equation above, we can guess that definition for conditional expectation may be extended to all integrable random variables.

- Proof for the uniqueness mentioned above

  - $(\Omega, \mathcal{F}, P)$: a prob. space. $Y \in \mathcal{F}$ and $E|Y| < \infty$. $\mathcal{G} \subset \mathcal{F}$ is a sub σ-field. If X is a random variable satisfying (a) $X \in \mathcal{G}$ (b) $\int_A Y\, dP = \int_A X\, dP \quad \forall A \in \mathcal{G}$ then

    i. X is integrable

    ii. Such X is unique in the sense that if there is another X' then $X = X'$ *a.s.*

    ★ Trick : For any r.v. Z, $(Z > 0) = \bigcup_{\varepsilon>0}(Z \ge \varepsilon) = \bigcup_{n\in\mathbb{N}}(Z > \frac{1}{n})$

    ★ Lemma : For any $\mathcal{F}$-measurable and integrable X and Y, if $\int_A X\, dP = \int_A Y\, dP \quad \forall A \in \mathcal{F}$ then $X = Y$ *a.s.*

- Radon-Nikodym Thm

  - If μ, ν are $\sigma-$finite measures on $(\Omega, \mathcal{F})$ and $\nu \ll \mu$ ($\mu(A) = 0 \Rightarrow \nu(A) = 0 \quad \forall\, A \in \mathcal{F}$) then $\exists$ a $\mathcal{F}$-measurable nonnegative function g s.t. $\nu(A) = \int_A g\, d\mu \quad \forall\, A \in \mathcal{F}$. The function g is unique in the sense that if h is another such function then $g = h \;\; \mu - a.e.$

* Definition of conditional expectation

  - $(\Omega, \mathcal{F}_0, P)$: a prob. space. $\mathcal{F} \subset \mathcal{F}_0$: a sub σ-field.
    X is a random variable s.t. $X \ge 0$, $X \in \mathcal{F}_0$ and $E|X| < \infty$. Then $\exists$ a unique r.v. Y s.t. $Y \ge 0$, $Y \in \mathcal{F}$ and $\int_A X\, dP = \int_A Y\, dP \quad \forall A \in \mathcal{F}$. Such Y is unique in the sense that if another Y' exists then $Y = Y'$ a.s.
  - $Y = E[X|\mathcal{F}]$ is said to be conditional expectation of X given $\mathcal{F}$

    ★ Applying Radon Nikodym thm to measures $P|_{\mathcal{F}}$ and Q on $(\Omega, \mathcal{F})$ where Q is defined by $Q(A) = \int_A X\, dP \quad \forall\, A \in \mathcal{F}$. Note that $Q \ll P|_{\mathcal{F}}$ and Q is a finite measure.
  - We can extend the definition to general integrable r.v. X
    $Y = E[X|\mathcal{F}]$ is a unique random varaible s.t. $Y \in \mathcal{F}$ and $\int_A X\, dP = \int_A Y\, dP \quad \forall A \in \mathcal{F}$. $E[X|\mathcal{F}]$ is also integrable and the uniqueness is in the sense of *a.s.* equivalence relation. $Y = E[X|\mathcal{F}]$ can be derived by $Y = Y_1 - Y_2$ where $Y_1 = E[X^+|\mathcal{F}]$ and $Y_2 = E[X^-|\mathcal{F}]$

* Conditional expectation given a random variable

  - X : integrable r.v. For a random variable Y, define $E[X|Y] := E[X|\sigma(Y)]$

  $\surd$ Y need not be integrable.

  $\surd$ Since $E[X|Y] \in \sigma(Y)$, $E[X|Y] = g(Y)$ for some Borel function g. This coincides with the definition of conditional expectation in $\mathcal{L}^2$ space.

* Conditional probability

  - For $A \in \mathcal{F}_0$ and a sub σ-field $\mathcal{F} \subset \mathcal{F}_0$, define $P(A|\mathcal{F}) := E[I_A|\mathcal{F}]$
  - For $A, B \in \mathcal{F}_0$, define $P(A|B) = P(A \cap B)/P(B)$

- Elementary properties of conditional expectation

  - $(\Omega, \mathcal{F}_0, P)$: a prob. space. $\mathcal{F} \subset \mathcal{F}_0$: a sub σ-field. X, Y : integrable random variables

    i. $E[c|\mathcal{F}] = c$

    ii. $E[\psi(X)|X] = \psi(X)$ given $E|\psi(X)| < \infty$

    iii. If $\mathcal{F}$ is a trivial σ-field i.e. $\mathcal{F} = \{\Omega, \phi\}$ then $E[X|\mathcal{F}] = E[X]$

    iv. $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$ is a partition of Ω with $\Omega_i \in \mathcal{F}_0$ and $P(\Omega_i) > 0 \quad \forall\, i \in \mathbb{N}$
    $\mathcal{F} = \sigma\{\Omega_1, \Omega_2, \cdots\} = \{\bigcup_{j \in \kappa} \Omega_j : \kappa \subset \mathbb{N}\}$ ($\mathcal{F}$ is a σ-field). Then we have

    $$E[X|\mathcal{F}] = \sum_{i=1}^{\infty} a_i I_{\Omega_i} \quad with \quad a_i = \frac{E[XI_{\Omega_i}]}{P(\Omega_i)}$$

    $\surd$ For $A \in \mathcal{F}_0$, $P(A|\mathcal{F}) = P(A|\Omega_i) I_{\Omega_i}$

    ★ Lemma : If $Z \in \mathcal{F}$ for such $\mathcal{F}$, then we can write $Y = \sum_{i=1}^{\infty} c_i I_{\Omega_i}$ where $c_i \in \mathbb{R}$

    v. $E[aX + bY|\mathcal{F}] = aE[X|\mathcal{F}] + bE[Y|\mathcal{F}] \quad \forall a, b \in \mathbb{R}$

    vi. $X \geq 0 \Rightarrow E[X|\mathcal{F}] \geq 0 \quad a.s.$

    ★ Lemma : If $Z > 0$ on A with $P(A) > 0$ then $\int_A Z\, dP > 0$

    vii. $X \leq Y \Rightarrow E[X|\mathcal{F}] \leq E[Y|\mathcal{F}] \quad a.s.$

    viii. $\big|E[X|\mathcal{F}]\big| \leq E\big[|X|\,\big|\mathcal{F}\big]$

    ix. $E[|X|\,|\mathcal{F}] = 0 \;\Rightarrow\; X = 0 \quad a.s.$

    x. $E[E[X|\mathcal{F}]] = E[X]$

- X, Y : integrable r.v's where $X \perp\!\!\!\perp Y$. $\psi : \mathbb{R}^2 \to \mathbb{R}$ Borel measurable s.t. $E|\psi(X, Y)| < \infty$
  Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = E[\psi(x, Y)] \quad \forall\, x \in \mathbb{R}$. Then $E[\psi(X, Y)|X] = g(X)$

  $\surd$ $g(x) = E[\psi(x, Y)] = \int \psi(x, Y)\, dP = \int_{\mathbb{R}} \psi(x, y) dPY^{-1}(y) = \int_{\mathbb{R}} \psi_x(y)\, d\mu_Y(y) \quad \forall x \in \mathbb{R}$
  By Fubini thm in real analysis course, it is shown that g is Borel measurable & integrable.

- Conditional expectation and convergence

  - $(\Omega, \mathcal{F}_0, P)$: a probability space. $\mathcal{F} \subset \mathcal{F}_0$: a sub $\sigma-$field

    i. (MCT) If $X_n \geq 0$ and $X_n \nearrow X$ $a.s.$ with $E(X) < \infty$ then $E[X_n|\mathcal{F}] \nearrow E[X|\mathcal{F}]$ $a.s.$

    ☐ If $Y_n \searrow Y$ $a.s.$ with $E|Y_1|$, $E|Y| < \infty$ then $E[Y_n|\mathcal{F}] \searrow E[Y|\mathcal{F}]$ $a.s.$

    ii. (DCT) If $|X_n| \leq Y$, $E(Y) < \infty$ and $X_n \to X$ $a.s.$ then $E[X_n|\mathcal{F}] \to E[X|\mathcal{F}]$ $a.s.$

    iii. (Continuity from below) $\{B_n\} \underset{seq}{\subset} \mathcal{F}_0$ s.t. $B_n \subset B_{n+1} \quad \forall\, n \in \mathbb{N}$. $B := \bigcup_n B_n$
    Then $P(B_n|\mathcal{F}) \nearrow P(B|\mathcal{F})$

iv. (Countable additivity) If $\{C_n\} \underset{seq}{\subset} \mathcal{F}_0$ is mutually disjoint then $P(\bigcup_n C_n|\mathcal{F}) = \sum_n P(C_n|\mathcal{F})$

- Essential inequalities

  i. (Markov) $P(|X| \geq c\,|\mathcal{F}) \leq \frac{1}{c}E\big[|X|\big|\mathcal{F}\big] \quad \forall\, c > 0$

  ii. (Jensen) If $\phi : \mathbb{R} \to \mathbb{R}$ is convex then $\phi(E[X|\mathcal{F}]) \leq E[\phi(X)|\mathcal{F}]$ $a.s.$

  ★ Trick : For each $x \in \mathbb{R}$ and convex function $\phi : \mathbb{R} \to \mathbb{R}$, we have $\phi(x) = \sup\{ax + b : (a, b) \in S\}$ where $S = \{(a, b) \in \mathbb{R}^2 : ax + b \leq \phi(x)\ \forall\, x \in \mathbb{R}\}$

  iii. (Cauchy-Schwarz) For $X, Y \in \mathcal{L}^2$, we have $E^2[XY|\mathcal{F}] \leq E[X^2|\mathcal{F}]E[Y^2|\mathcal{F}]$ $a.s.$

- Smoothing property of conditional expectation

  i. If $X \in \mathcal{F}$, $E|Y| < \infty$, and $E|XY| < \infty$ then $E[XY|\mathcal{F}] = XE[Y|\mathcal{F}]$ $a.s.$

  √ $E|X| < \infty$ assumption is not required.

  □ If $X \in \mathcal{F}$ and $E|X| < \infty$ then $E[X|\mathcal{F}] = X$ $a.s.$

  ii. If $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_0$ are sub σ-fields and $E|X| < \infty$ then

  (a) $E[E(X|\mathcal{F}_1)|\mathcal{F}_2] = E[X|\mathcal{F}_1]$

  (b) $E[E(X|\mathcal{F}_2)|\mathcal{F}_1] = E[X|\mathcal{F}_1]$

  ★ Lemma : If $\mathcal{F}_1 \subset \mathcal{F}_2$ then $Y \in \mathcal{F}_1 \Rightarrow Y \in \mathcal{F}_2$

  √ In short, "the smaller wins". In view of information, it is similar to projection onto vector subspaces $S_1 \subset S_2 \subset S$ where $Proj_{S_1}Proj_{S_2} = Proj_{S_2}Proj_{S_1} = Proj_{S_1}$

- Def. of conditional expectation by Radon-Nikodym derivative agrees with def. in $\mathcal{L}^2$ space.

  – If $E(X^2) < \infty$ then for $\mathcal{C} = \{Y : Y \in \mathcal{F},\ E(Y^2) < \infty\}$, $E[\{X - E[X|\mathcal{F}]\}^2] = \inf_{Y\in\mathcal{C}} E[\{X - Y\}^2]$ and $E[X|\mathcal{F}] = \arg\min_{Y\in\mathcal{C}} E[\{X - Y\}^2]$

  ★ Lemma : If $X \in \mathcal{L}^2$ then $E[X|\mathcal{F}] \in \mathcal{L}^2$

* Independence of a random variable and a σ-field

  – A random variable X and a σ-field $\mathcal{F}$ are said to be independent if $\sigma(X)$ and $\mathcal{F}$ are independent

- If an integrable random variable X and a σ-field $\mathcal{F}$ are independent then $E[X|\mathcal{F}] = E[X]$

□ Two extreme cases of conditional expectations w.r.t information

  – Perfect information : If $X \in \mathcal{F}$ then $E[X|\mathcal{F}] = X$

  – No information : If $X \perp\!\!\!\perp \mathcal{F}$ then $E[X|\mathcal{F}] = E[X]$

* Conditional variance

$$Var(X|\mathcal{F}) := E[\{X - E[X|\mathcal{F}]\}^2|\mathcal{F}] = E[X^2|\mathcal{F}] - E^2[X|\mathcal{F}]$$

Conditional variance is defined for $X \in \mathcal{L}^2$

2 Martingales

* Definition needed for martingales

- Given a probability space $(\Omega, \mathcal{F}, P)$, increasing sequence of sub σ-fields $\{\mathcal{F}_n\}_{n=0}^{\infty}$ is called a filtration.
- A random sequence $\{X_n\}_{n=0}^{\infty}$ is said to be adapted to $\{\mathcal{F}_n\}$ if $X_n \in \mathcal{F}_n \quad \forall n \in \mathbb{N} \cup \{0\}$

* Definition of martingale and their cousins

- $\{X_n\}_{n=0}^{\infty}$: a random sequence. $\{\mathcal{F}_n\}_{n=0}^{\infty}$: a filtration. Assume $E|X_n| < \infty \quad \forall n \in \mathbb{N} \cup \{0\}$ and $\{X_n\}$ is adapted to $\{\mathcal{F}_n\}$. Then $\{X_n\}$ is said to be a martingale (w.r.t $\{\mathcal{F}_n\}$) if $E[X_{n+1}|\mathcal{F}_n] = X_n \quad \forall n \in \mathbb{N} \cup \{0\}$
- $\{X_n\}$ is said to be a submartingale (w.r.t $\{\mathcal{F}_n\}$) if $E[X_{n+1}|\mathcal{F}_n] \geq X_n \quad \forall n \in \mathbb{N} \cup \{0\}$
- $\{X_n\}$ is said to be a supermartingale (w.r.t $\{\mathcal{F}_n\}$) if $E[X_{n+1}|\mathcal{F}_n] \leq X_n \quad \forall n \in \mathbb{N} \cup \{0\}$

$\surd$ These are abbreviated to 'mtg', 'submtg', 'supermtg' respectively.

- Examples of martingales

  i. $\{\xi_n\}_n$ $i.i.d$ with $E(\xi_1) = 0$. $X_0 = 0$. $X_n = \xi_1 + \cdots + \xi_n$ and $\mathcal{F}_0 = \{\phi, \Omega\}$.
  $\mathcal{F}_n = \sigma(\xi_1, \cdots, \xi_n)$. Then $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$

  ★ Trick : $E[Z]$ is finite $\Leftrightarrow$ Z is integrable. ($\because$ the definition of expectation)

  ii. Adding assumption $Var(\xi_1) = \sigma^2 < \infty$ to i. above.
  Then $\{X_n - n\sigma^2\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$

  iii. $\{\varepsilon_n\}_n$ $i.i.d$ $\sim (0, 1)$. $X_0 = 0$. $X_{n+1} = X_n + h(X_n)\varepsilon_{n+1}$ with $h : \mathbb{R} \to \mathbb{R}$ Borel function s.t. $E|h(X_n)| < \infty \quad \forall n \in \mathbb{N} \cup \{0\}$ and $\mathcal{F}_0 = \{\phi, \Omega\}$. $\mathcal{F}_n = \sigma(\varepsilon_1, \cdots, \varepsilon_n)$
  Then $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$

  iv. $\{\varepsilon_n\}_n$ $i.i.d$ $\sim (0, 1)$. $Y_0 = 0$. $Y_{n+1} = \phi(Y_n)\varepsilon_{n+1}$ with $\phi(y) = w + \alpha y^2$ ($w > 0$, $0 \leq \alpha < 1$) and $E[\phi(Y_n)] < \infty \quad \forall n \in \mathbb{N}$. and $\mathcal{F}_0 = \{\phi, \Omega\}$. $\mathcal{F}_n = \sigma(\varepsilon_1, \cdots, \varepsilon_n)$.
  Let $X_0 = 0$. $X_n = Y_1 + \cdots Y_n$. Then $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$

  $\surd$ Such $\{Y_n\}$ is called as ARCH (autoregressive conditional heteroskedasticity) process

- Elementary facts about Martingales

  - Every martingale is a submartingale and a supermartingale
  - If $\{X_n\}$ is a submartingale then $\{-X_n\}$ is a supermartingale

    $\surd$ We develop theory about martingales often assuming submartingale since every martingale is submartingale and every supermartingale is negative version of submartingale

  - If $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$ then $E[X_n|\mathcal{F}_m] = X_m$ whenever $n \geq m$
  - If $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$ then $E[X_n|\mathcal{F}_m] \geq X_m$ whenever $n \geq m$
  - If $\{X_n\}$ is a supermartingale w.r.t $\{\mathcal{F}_n\}$ then $E[X_n|\mathcal{F}_m] \leq X_m$ whenever $n \geq m$
  - If $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$ then $\{E[X_n]\}$ is constant.
  - If $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$ then $\{E[X_n]\}$ is increasing.
  - If $\{X_n\}$ is a supermartingale w.r.t $\{\mathcal{F}_n\}$ then $\{E[X_n]\}$ is decreasing.

- Convex transformation of martingale

- If $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function s.t. $E|\phi(X_n)| < \infty \quad \forall n \in \mathbb{N}$ then $\{\phi(X_n)\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$
- If $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a convex and increasing function s.t. $E|\phi(X_n)| < \infty \quad \forall n \in \mathbb{N}$ then $\{\phi(X_n)\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$
- If $\{X_n\}$ is a supermartingale w.r.t $\{\mathcal{F}_n\}$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a concave and increasing function s.t. $E|\phi(X_n)| < \infty \quad \forall n \in \mathbb{N}$ then $\{\phi(X_n)\}$ is a supermartingale w.r.t $\{\mathcal{F}_n\}$

(Ex) If $\{X_n\}$ is a martingale and $E[|X_n|^p] < \infty$ for some $p \geq 1$, then $\{|X_n|^p\}$ is a submartingale

(Ex) If $\{X_n\}$ is a submartingale then for any $a \in \mathbb{R}$, $\{(X_n - a)^+\}$ is a submartingale

(Ex) If $\{X_n\}$ is a supermartingale then for any $a \in \mathbb{R}$, $\{X_n \wedge a\}$ is a supermartingale

* Predicatable sequence and a process using it

- For a filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$, a random sequence $\{H_n\}_{n=1}^{\infty}$ is said to be a predicatable sequence (w.r.t $\{\mathcal{F}_n\}$) if $H_n \in \mathcal{F}_{n-1} \quad \forall n \in \mathbb{N}$

  √ A letter H stands for a 'height'
- Suppose $\{X_n\}$ is adapted to $\{\mathcal{F}_n\}$. For a predicatable sequence $\{H_n\}$ (w.r.t $\{\mathcal{F}_n\}$), we define a process $\{(H \cdot X)_n\}$ by

$$(H \cdot X)_n = \sum_{m=1}^{n} H_m (X_m - X_{m-1})$$

  √ Note that $\{(H \cdot X)_n\}$ is adapted to $\{\mathcal{F}_n\}$

- Elementary facts about martingale transform
  - Let $\{X_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=1}^{\infty}$ be a random sequence and $\{H_n\}$ is a predicatable sequence w.r.t. a filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$. Assume $E|X_n H_n| < \infty$, $E|X_{n-1} H_n| < \infty \quad \forall n \in \mathbb{N}$
    i. If $\{X_n\}$ is a martingale (w.r.t $\{\mathcal{F}_n\}$) then $\{(H \cdot X)_n\}$ is also a martingale
    ii. If $\{X_n\}$ is a submartingale (w.r.t $\{\mathcal{F}_n\}$) and $H_n \geq 0$ then $\{(H \cdot X)_n\}$ is also a submartingale
    iii. If $\{X_n\}$ is a supermartingale (w.r.t $\{\mathcal{F}_n\}$) and $H_n \geq 0$ then $\{(H \cdot X)_n\}$ is also a supermartingale

  √ The condition "$E|X_n H_n| < \infty$, $E|X_{n-1} H_n| < \infty \quad \forall n \in \mathbb{N}$" can be replaced with "For each $n \in \mathbb{N}$, H_n is bounded".

* Stopping time

- A (extended) random variable N taking values of $\mathbb{N} \cup \{0, \infty\}$ is said to be a stopping time (w.r.t a filtration $\{\mathcal{F}_n\}$) if an event $(N = n) \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$

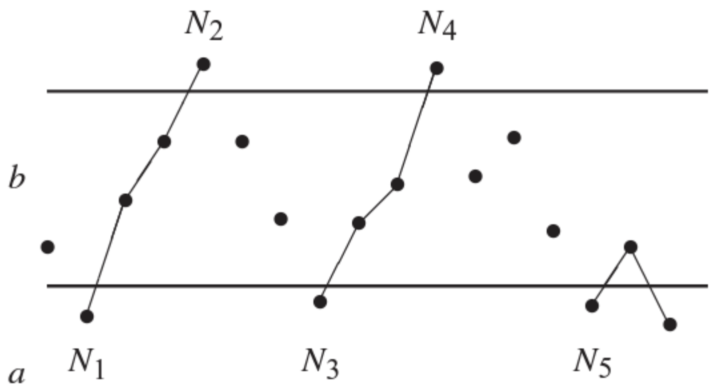
$$(N \leq n) = \bigcup_{j=0}^{n} (N = j) \in \mathcal{F}_n \qquad (N > n) = (N \leq n)^C \in \mathcal{F}_n$$

$$(N < n) = \bigcup_{j=0}^{n-1} (N = j) \in \mathcal{F}_{n-1} \qquad (N \geq n) = (N < n)^C \in \mathcal{F}_{n-1}$$

- $(N \geq n)$ is a $\mathcal{F}_{n-1}$-measurable event. $I(N \geq n)$ is $\mathcal{F}_{n-1}$-measurable random variable. Hence, $\{I(N \geq n)\}_n$ is a predictable sequence given N is a stopping time.

- Martingale stopped by stopping time
  - Let $\{X_n\}$ be a random sequence adapted to $\{\mathcal{F}_n\}$. Let N be a stopping time w.r.t $\{\mathcal{F}_n\}$ and put $H_n = I(N \geq n) \quad \forall n \in \mathbb{N}$. Then $(H \cdot X)_n = X_{N \wedge n} - X_0$.
  - The process $\{X_{N\wedge n}\}_n$ is said to be a martingale stopped by stopping time N , provided $\{X_n\}$ is a martingale.
    - ★ If $\{X_n\}$ and $\{Y_n\}$ are martingales (w.r.t. $\{\mathcal{F}_n\}$) then $\{X_n + Y_n\}$ is also a martingale. The same holds for submartingales and supermartingales too.
  - If $\{X_n\}$ is a martingale and N is a stopping time then $\{X_{N\wedge n}\}$ is martingale.
  - If $\{X_n\}$ is a submartingale and N is a stopping time then $\{X_{N\wedge n}\}$ is submartingale.
  - If $\{X_n\}$ is a supermartingale and N is a stopping time then $\{X_{N\wedge n}\}$ is supermartingale.
- Stopping time and Upcrossing
  - Suppose $\{X_n\}_{n=0}^{\infty}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$. Let $a < b$. Define N_j's as below :

$$
\begin{array}{ll}
N_1 = \inf\{m \geq 0 \,:\, X_m \leq a\} & N_2 = \inf\{m > N_1 \,:\, X_m \geq b\} \\
N_3 = \inf\{m > N_2 \,:\, X_m \leq a\} & N_4 = \inf\{m > N_3 \,:\, X_m \geq b\} \\
\vdots & \vdots \\
N_{2k-1} = \inf\{m > N_{2k-2} \,:\, X_m \leq a\} & N_{2k} = \inf\{m > N_{2k-1} \,:\, X_m \geq b\} \\
\vdots & \vdots
\end{array}
$$



  - Every N_j for $j \in \mathbb{N}$ is stopping time w.r.t $\{\mathcal{F}_n\}$. $N_1 < N_2 < N_3 \cdots$ provided all N_j's are finite. (It is possible that $N_j = \infty$ provided it has a form of $\inf(\phi)$)
  - 'Upcrossing' is a case where the submartingale $\{X_n\}$ crosses from below a to above b.
  - $\mathcal{U}_n := \sup\{k \,:\, N_{2k} \leq n\}$ is the number of upcommings completed by time n
- Upcrossing inequality
  - Suppose $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$. If stopping time N_j and the number of upcrossings $\mathcal{U}_n$ are defined as above then

$$(b-a)E[\mathcal{U}_n] \leq E[(X_n - a)^+] - E[(X_0 - a)^+]$$

- Submartingale convergence theorem
  - If $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$ with $\sup_n E(X_n^+) < \infty$ then $X_n \to X$ $a.s.$ for some integrable random varialbe X
    - ★ If $X_n \to X$ $a.s.$ then $X_n^+ \to X^+$ $a.s.$ and $X_n^- \to X^-$ $a.s.$