Probability theory II Facts

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Contents

1 Conditional Expectation

- Projection Thm for Hilbert Space
  - If E is a Hilbert space and $M \subset E$ is closed and convex, then for any $y \in E$,
    $\exists$ a unique $w \in M$ s.t. $\|y - w\| = d(y, M) := \inf\{\|y - v\| : v \in M\}$.
    Denote it as $w = proj_M y$ i.e. w is a projection of y onto M .
  - If E is a Hilbert space and $M \subset E$ is a closed vector subspace, then for any $y \in E$,
    i. $\exists$ a unique decomposition $y = w + v$ with $w = proj_M y \in M$ and $v \in M^\perp$
    ii. For $w \in M$, $\quad w = proj_M y \Leftrightarrow \langle y - w, z\rangle = 0 \quad \forall\, z \in M$

* $\mathcal{L}^2 := \{$ Random Variable $X : E(X^2) = \int X^2\, dP < \infty\}$

√ If $X \in \mathcal{L}^2$ then $E|X| < \infty$ i.e. every element of $\mathcal{L}^2$ is integrable.

  ★ Trick : $|X| \le X^2 + \frac{1}{4}$

√ $\mathcal{L}^2$ is a vector space

  ★ Trick : inequality $(aX + bY)^2 \le 2(a^2X^2 + b^2Y^2)$

- $\mathcal{L}^2$ is a Hilbert space with inner product $\langle X, Y\rangle = E(XY)$

  ★ Trick : Cauchy seq. having a subseq. converging to a point converges to the point.

- Lemma for proving $\mathcal{L}^2$ is a complete normed space.
  - If $\{X_n\} \underset{seq}{\subset} \mathcal{L}^2$ and $\|X_n - X_{n+1}\| \le 2^{-n} \quad \forall\, n \in \mathbb{N}$ then $\exists\, X \in \mathcal{L}^2$ s.t. $X_n \to X \quad a.s.$ and $\|X_n - X\| \to 0$ i.e. $X_n \to X$ in $\mathcal{L}^2$.
    ★ Lemma : If a random varaible Z satisfies $Z \ge 0$ and $E(Z) < \infty$ then $Z < \infty \quad a.s.$

* For $X \in \mathcal{L}^2$, $\quad \mathcal{L}^2(X) := \{g(X) \,|\, g : \mathbb{R} \to \mathbb{R}$ is a Borel function, $E[(g(X))^2] < \infty\}$

√ For $X \in \mathcal{L}^2$, $\quad \mathcal{L}^2(X)$ is a vector subspace of $\mathcal{L}^2$.

- For $X \in \mathcal{L}^2$, $\quad \mathcal{L}^2(X)$ is a closed vector subspace of $\mathcal{L}^2$ so that $\mathcal{L}^2(X)$ is also a Hilbert space.

* Geometric definition for conditional expectation
  - For $X, Y \in \mathcal{L}^2$, define $E[Y|X] = Proj_{\mathcal{L}^2(X)} Y$
  - $E[Y|X] = g(X)\ a.s.$ for some Borel function g
  - $\|Y - E[Y|X]\| = \min_{h(X) \in \mathcal{L}^2(X)} \|Y - h(X)\|$
    i.e. $E[(Y - E[Y|X])^2] \le E[(Y - h(X))^2] \quad \forall\, h(X) \in \mathcal{L}^2$
  - For $g(X) \in \mathcal{L}^2(X)$, $\quad g(X) = E[Y|X] \Leftrightarrow \langle Y - g(X), h(X)\rangle = 0 \quad \forall\, h(X) \in \mathcal{L}^2$
    $\Leftrightarrow E[(Y - g(X))h(X)] = 0 \quad \forall\, h(X) \in \mathcal{L}^2$

- Elementary properties of conditional expectation from geometric definition
  - If $X, Y, Z \in \mathcal{L}^2$ then the followings are true.
    i. $E[c|X] = c\ a.s. \quad \forall\, c \in \mathbb{R}$
    ii. $E[\alpha Y + \beta Z|X] = \alpha E[Y|X] + \beta E[Z|X] \quad \forall \alpha, \beta \in \mathbb{R}$
    iii. $E[Y|X] = E[Y]$ if X and Y are independent.

iv. $E[g(X)Y|X] = g(X)E[Y|X]$ if g satisfies $g(X) \in \mathcal{L}^2(X)$ and $\sup_x |g(x)| < \infty$

v. $E[E[Y|X]] = E[Y]$

√ In fact, the additional assumption about boundedness of g in (iv) is not necessary. We will see later.

- Extending the definition from $\mathcal{L}^2$ to all integrable functions

$$E[\{Y - E[Y|X]\}I(X \in A)] = 0 \quad \forall\, A \in \mathcal{B}(\mathbb{R}) \quad \because I(X \in A) \in \mathcal{L}^2(X)$$
$$\int_{(X\in A)} Y\, dP = \int_{(X\in A)} E[Y|X]\, dP \quad \forall\, A \in \mathcal{B}(\mathbb{R})$$
$$\int_B Y\, dP = \int_B E[Y|X]\, dP \quad \forall\, B \in \sigma(X)$$

  - $E[Y|X] \in \sigma(X)$ and $\int_B Y\, dP = \int_B E[Y|X]\, dP \quad \forall\, B \in \sigma(X)$. Such r.v. is unique in the sense that if any r.v. Z satisfies $Z \in \sigma(X)$ and $\int_B Y\, dP = \int_B Z\, dP \quad \forall\, B \in \sigma(X)$ then $Z = E[Y|X]$ *a.s.* provided $E|Y| < \infty$
  - From the theory on $\mathcal{L}^2$ space, we get geometric understanding about conditional expectation. But now, from the equation above, we can guess that definition for conditional expectation may be extended to all integrable random variables.

- Proof for the uniqueness mentioned above

  - $(\Omega, \mathcal{F}, P)$: a prob. space. $Y \in \mathcal{F}$ and $E|Y| < \infty$. $\mathcal{G} \subset \mathcal{F}$ is a sub σ-field. If X is a random variable satisfying (a) $X \in \mathcal{G}$ (b) $\int_A Y\, dP = \int_A X\, dP \quad \forall A \in \mathcal{G}$ then

    i. X is integrable

    ii. Such X is unique in the sense that if there is another X' then $X = X'$ *a.s.*

    ★ Trick : For any r.v. Z, $(Z > 0) = \bigcup_{\varepsilon>0}(Z \geq \varepsilon) = \bigcup_{n\in\mathbb{N}}(Z > \frac{1}{n})$

    ★ Lemma : For any $\mathcal{F}$-measurable and integrable X and Y, if $\int_A X\, dP = \int_A Y\, dP \quad \forall A \in \mathcal{F}$ then $X = Y$ *a.s.*

- Radon-Nikodym Thm

  - If μ, ν are $\sigma-$finite measures on $(\Omega, \mathcal{F})$ and $\nu \ll \mu$ ($\mu(A) = 0 \Rightarrow \nu(A) = 0 \quad \forall\, A \in \mathcal{F}$) then $\exists$ a $\mathcal{F}$-measurable nonnegative function g s.t. $\nu(A) = \int_A g\, d\mu \quad \forall\, A \in \mathcal{F}$. The function g is unique in the sense that if h is another such function then $g = h \;\; \mu - a.e.$

* Definition of conditional expectation

  - $(\Omega, \mathcal{F}_0, P)$: a prob. space. $\mathcal{F} \subset \mathcal{F}_0$: a sub σ-field.
    X is a random variable s.t. $X \geq 0$, $X \in \mathcal{F}_0$ and $E|X| < \infty$. Then $\exists$ a unique r.v. Y s.t. $Y \geq 0$, $Y \in \mathcal{F}$ and $\int_A X\, dP = \int_A Y\, dP \quad \forall A \in \mathcal{F}$. Such Y is unique in the sense that if another Y' exists then $Y = Y'$ a.s.
  - $Y = E[X|\mathcal{F}]$ is said to be conditional expectation of X given $\mathcal{F}$

    ★ Applying Radon Nikodym thm to measures $P|_{\mathcal{F}}$ and Q on $(\Omega, \mathcal{F})$ where Q is defined by $Q(A) = \int_A X\, dP \quad \forall\, A \in \mathcal{F}$. Note that $Q \ll P|_{\mathcal{F}}$ and Q is a finite measure.

  - We can extend the definition to general integrable r.v. X
    $Y = E[X|\mathcal{F}]$ is a unique random varaible s.t. $Y \in \mathcal{F}$ and $\int_A X\, dP = \int_A Y\, dP \quad \forall A \in \mathcal{F}$.
    $E[X|\mathcal{F}]$ is also integrable and the uniqueness is in the sense of *a.s.* equivalence relation.
    $Y = E[X|\mathcal{F}]$ can be derived by $Y = Y_1 - Y_2$ where $Y_1 = E[X^+|\mathcal{F}]$ and $Y_2 = E[X^-|\mathcal{F}]$

* Conditional expectation given a random variable
  - X : integrable r.v. For a random variable Y, define $E[X|Y] := E[X|\sigma(Y)]$

  √ Y need not be integrable.

  √ Since $E[X|Y] \in \sigma(Y)$, $E[X|Y] = g(Y)$ for some Borel function g. This coincides with the definition of conditional expectation in $\mathcal{L}^2$ space.

* Conditional probability
  - For $A \in \mathcal{F}_0$ and a sub σ-field $\mathcal{F} \subset \mathcal{F}_0$, define $P(A|\mathcal{F}) := E[I_A|\mathcal{F}]$
  - For $A, B \in \mathcal{F}_0$, define $P(A|B) = P(A \cap B)/P(B)$

- Elementary properties of conditional expectation
  - $(\Omega, \mathcal{F}_0, P)$: a prob. space. $\mathcal{F} \subset \mathcal{F}_0$: a sub σ-field. X, Y : integrable random variables
    1. $E[c|\mathcal{F}] = c$
    2. $E[\psi(X)|X] = \psi(X)$ given $E|\psi(X)| < \infty$
    3. If $\mathcal{F}$ is a trivial σ-field i.e. $\mathcal{F} = \{\Omega, \phi\}$ then $E[X|\mathcal{F}] = E[X]$
    4. $\Omega = \bigcup_{i=1}^{\infty} \Omega_i$ is a partition of Ω with $\Omega_i \in \mathcal{F}_0$ and $P(\Omega_i) > 0 \quad \forall\, i \in \mathbb{N}$
    $\mathcal{F} = \sigma\{\Omega_1, \Omega_2, \cdots\} = \{\bigcup_{j\in\kappa} \Omega_j : \kappa \subset \mathbb{N}\}$ ($\mathcal{F}$ is a σ-field). Then we have
    $$E[X|\mathcal{F}] = \sum_{i=1}^{\infty} a_i I_{\Omega_i} \quad with \quad a_i = \frac{E[X I_{\Omega_i}]}{P(\Omega_i)}$$
    √ For $A \in \mathcal{F}_0$, $P(A|\mathcal{F}) = P(A|\Omega_i) I_{\Omega_i}$

    ★ Lemma : If $Z \in \mathcal{F}$ for such $\mathcal{F}$, then we can write $Y = \sum_{i=1}^{\infty} c_i I_{\Omega_i}$ where $c_i \in \mathbb{R}$
    5. $E[aX + bY|\mathcal{F}] = aE[X|\mathcal{F}] + bE[Y|\mathcal{F}] \quad \forall a, b \in \mathbb{R}$
    6. $X \geq 0 \Rightarrow E[X|\mathcal{F}] \geq 0 \quad a.s.$

    ★ Lemma : If $Z > 0$ on A with $P(A) > 0$ then $\int_A Z\, dP > 0$
    7. $X \leq Y \Rightarrow E[X|\mathcal{F}] \leq E[Y|\mathcal{F}] \quad a.s.$
    8. $\big|E[X|\mathcal{F}]\big| \leq E\big[|X|\,\big|\mathcal{F}\big]$

    ☐ $|X| \leq M$ for some $M > 0 \;\Rightarrow\; \big|E[X|\mathcal{F}]\big| \leq M \;\; a.s.$
    9. $E[|X|\,|\mathcal{F}] = 0 \;\Rightarrow\; X = 0 \quad a.s.$
    10. $E[E[X|\mathcal{F}]] = E[X]$

- X, Y : integrable r.v's where $X \perp\!\!\!\perp Y$. $\psi : \mathbb{R}^2 \to \mathbb{R}$ Borel measurable s.t. $E|\psi(X,Y)| < \infty$
  Define $g : \mathbb{R} \to \mathbb{R}$ by $g(x) = E[\psi(x, Y)] \quad \forall\, x \in \mathbb{R}$. Then $E[\psi(X,Y)|X] = g(X)$

  √ $g(x) = E[\psi(x,Y)] = \int \psi(x, Y)\, dP = \int_{\mathbb{R}} \psi(x,y) dPY^{-1}(y) = \int_{\mathbb{R}} \psi_x(y)\, d\mu_Y(y) \quad \forall x \in \mathbb{R}$
  By Fubini thm in real analysis course, it is shown that g is Borel measurable & integrable.

- Conditional expectation and convergence
  - $(\Omega, \mathcal{F}_0, P)$: a probability space. $\mathcal{F} \subset \mathcal{F}_0$: a sub $\sigma-$field
    1. (MCT) If $X_n \geq 0$ and $X_n \nearrow X$ $a.s.$ with $E(X) < \infty$ then $E[X_n|\mathcal{F}] \nearrow E[X|\mathcal{F}]$ $a.s.$

    ☐ If $Y_n \searrow Y$ $a.s.$ with $E|Y_1|$, $E|Y| < \infty$ then $E[Y_n|\mathcal{F}] \searrow E[Y|\mathcal{F}]$ $a.s.$
    2. (DCT) If $|X_n| \leq Y$, $E(Y) < \infty$ and $X_n \to X$ $a.s.$ then $E[X_n|\mathcal{F}] \to E[X|\mathcal{F}]$ $a.s.$

iii. (Fatou's lemma) If $X_n \geq 0$ and $X_n \to X$ *a.s.* with $E(X_n) < \infty$, $E(X) < \infty$ then $E[X|\mathcal{F}] \leq \liminf E[X_n|\mathcal{F}]$

iv. (Continuity from below) $\{B_n\} \underset{seq}{\subset} \mathcal{F}_0$ s.t. $B_n \subset B_{n+1} \quad \forall n \in \mathbb{N}$. $B := \bigcup_n B_n$

Then $P(B_n|\mathcal{F}) \nearrow P(B|\mathcal{F})$

v. (Countable additivity) If $\{C_n\} \underset{seq}{\subset} \mathcal{F}_0$ is mutually disjoint then $P(\bigcup_n C_n|\mathcal{F}) = \sum_n P(C_n|\mathcal{F})$

- Essential inequalities

  i. (Markov) $P(|X| \geq c\,|\mathcal{F}) \leq \frac{1}{c}E\big[|X|\big|\mathcal{F}\big] \quad \forall c > 0$

  ii. (Jensen) If $\phi : \mathbb{R} \to \mathbb{R}$ is convex then $\phi(E[X|\mathcal{F}]) \leq E[\phi(X)|\mathcal{F}]$ *a.s.*

  ★ Trick : For each $x \in \mathbb{R}$ and convex function $\phi : \mathbb{R} \to \mathbb{R}$, we have $\phi(x) = \sup\{ax + b : (a, b) \in S\}$ where $S = \{(a, b) \in \mathbb{R}^2 : ax + b \leq \phi(x) \; \forall x \in \mathbb{R}\}$

  iii. (Cauchy-Schwarz) For $X, Y \in \mathcal{L}^2$, we have $E^2[XY|\mathcal{F}] \leq E[X^2|\mathcal{F}]E[Y^2|\mathcal{F}]$ *a.s.*

- Smoothing property of conditional expectation

  i. If $X \in \mathcal{F}$, $E|Y| < \infty$, and $E|XY| < \infty$ then $E[XY|\mathcal{F}] = XE[Y|\mathcal{F}]$ *a.s.*

  √ $E|X| < \infty$ assumption is not required.

  ☐ If $X \in \mathcal{F}$ and $E|X| < \infty$ then $E[X|\mathcal{F}] = X$ *a.s.*

  ii. If $\mathcal{F}_1 \subset \mathcal{F}_2 \subset \mathcal{F}_0$ are sub σ-fields and $E|X| < \infty$ then

  (a) $E[E(X|\mathcal{F}_1)|\mathcal{F}_2] = E[X|\mathcal{F}_1]$

  (b) $E[E(X|\mathcal{F}_2)|\mathcal{F}_1] = E[X|\mathcal{F}_1]$

  ★ Lemma : If $\mathcal{F}_1 \subset \mathcal{F}_2$ then $Y \in \mathcal{F}_1 \Rightarrow Y \in \mathcal{F}_2$

  √ In short, "the smaller wins". In view of information, it is similar to projection onto vector subspaces $S_1 \subset S_2 \subset S$ where $Proj_{S_1}Proj_{S_2} = Proj_{S_2}Proj_{S_1} = Proj_{S_1}$

- Def. of conditional expectation by Radon-Nikodym derivative agrees with def. in $\mathcal{L}^2$ space.

  – If $E(X^2) < \infty$ then for $\mathcal{C} = \{Y : Y \in \mathcal{F},\, E(Y^2) < \infty\}$, $E[\{X - E[X|\mathcal{F}]\}^2] = \inf_{Y \in \mathcal{C}} E[\{X - Y\}^2]$ and $E[X|\mathcal{F}] = \arg\min_{Y \in \mathcal{C}} E[\{X - Y\}^2]$

  ★ Lemma : If $X \in \mathcal{L}^2$ then $E[X|\mathcal{F}] \in \mathcal{L}^2$

* Independence of a random variable and a σ-field

  – A random variable X and a σ-field $\mathcal{F}$ are said to be independent if $\sigma(X)$ and $\mathcal{F}$ are independent

- If an integrable random variable X and a σ-field $\mathcal{F}$ are independent then $E[X|\mathcal{F}] = E[X]$

☐ Two extreme cases of conditional expectations w.r.t information

  – Perfect information : If $X \in \mathcal{F}$ then $E[X|\mathcal{F}] = X$

  – No information : If $X \perp\!\!\!\perp \mathcal{F}$ then $E[X|\mathcal{F}] = E[X]$

* Conditional variance

$$Var(X|\mathcal{F}) := E[\{X - E[X|\mathcal{F}]\}^2|\mathcal{F}] = E[X^2|\mathcal{F}] - E^2[X|\mathcal{F}]$$

Conditional variance is defined for $X \in \mathcal{L}^2$

2 Martingales

* Definition needed for martingales

- Given a probability space $(\Omega, \mathcal{F}, P)$, increasing sequence of sub σ-fields $\{\mathcal{F}_n\}_{n=0}^{\infty}$ is called a filtration.
- A random sequence $\{X_n\}_{n=0}^{\infty}$ is said to be adapted to $\{\mathcal{F}_n\}$ if $X_n \in \mathcal{F}_n \quad \forall n \in \mathbb{N} \cup \{0\}$

* Definition of martingale and their cousins

- $\{X_n\}_{n=0}^{\infty}$: a random sequence. $\{\mathcal{F}_n\}_{n=0}^{\infty}$: a filtration. Assume $E|X_n| < \infty \quad \forall n \in \mathbb{N} \cup \{0\}$ and $\{X_n\}$ is adapted to $\{\mathcal{F}_n\}$. Then $\{X_n\}$ is said to be a martingale (w.r.t $\{\mathcal{F}_n\}$) if $E[X_{n+1}|\mathcal{F}_n] = X_n \quad \forall n \in \mathbb{N} \cup \{0\}$
- $\{X_n\}$ is said to be a submartingale (w.r.t $\{\mathcal{F}_n\}$) if $E[X_{n+1}|\mathcal{F}_n] \geq X_n \quad \forall n \in \mathbb{N} \cup \{0\}$
- $\{X_n\}$ is said to be a supermartingale (w.r.t $\{\mathcal{F}_n\}$) if $E[X_{n+1}|\mathcal{F}_n] \leq X_n \quad \forall n \in \mathbb{N} \cup \{0\}$

√ These are abbreviated to 'mtg', 'submtg', 'supermtg' respectively.

- Examples of martingales

i. $\{\xi_n\}_n$ *i.i.d* with $E(\xi_1) = 0$. $X_0 = 0$. $X_n = \xi_1 + \cdots + \xi_n$ and $\mathcal{F}_0 = \{\phi, \Omega\}$. $\mathcal{F}_n = \sigma(\xi_1, \cdots, \xi_n)$. Then $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$

★ Trick : $E[Z]$ is finite $\Leftrightarrow$ Z is integrable. ($\because$ the definition of expectation)

ii. Adding assumption $Var(\xi_1) = \sigma^2 < \infty$ to i. above.
Then $\{X_n - n\sigma^2\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$

iii. $\{\varepsilon_n\}_n$ *i.i.d* $\sim (0, 1)$. $X_0 = 0$. $X_{n+1} = X_n + h(X_n)\varepsilon_{n+1}$ with $h : \mathbb{R} \to \mathbb{R}$ Borel function s.t. $E|h(X_n)| < \infty \quad \forall n \in \mathbb{N} \cup \{0\}$ and $\mathcal{F}_0 = \{\phi, \Omega\}$. $\mathcal{F}_n = \sigma(\varepsilon_1, \cdots, \varepsilon_n)$
Then $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$

iv. $\{\varepsilon_n\}_n$ *i.i.d* $\sim (0, 1)$. $Y_0 = 0$. $Y_{n+1} = \phi(Y_n)\varepsilon_{n+1}$ with $\phi(y) = w + \alpha y^2$ $(w > 0,\ 0 \leq \alpha < 1)$ and $E[\phi(Y_n)] < \infty \quad \forall n \in \mathbb{N}$. and $\mathcal{F}_0 = \{\phi, \Omega\}$. $\mathcal{F}_n = \sigma(\varepsilon_1, \cdots, \varepsilon_n)$.
Let $X_0 = 0$. $X_n = Y_1 + \cdots Y_n$. Then $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$

√ Such $\{Y_n\}$ is called as ARCH (autoregressive conditional heteroskedasticity) process

- Elementary facts about Martingales

  - Every martingale is a submartingale and a supermartingale
  - If $\{X_n\}$ is a submartingale then $\{-X_n\}$ is a supermartingale

    √ We develop theory about martingales often assuming submartingale since every martingale is submartingale and every supermartingale is negative version of submartingale

  - If $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$ then $E[X_n|\mathcal{F}_m] = X_m$ whenever $n \geq m$
  - If $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$ then $E[X_n|\mathcal{F}_m] \geq X_m$ whenever $n \geq m$
  - If $\{X_n\}$ is a supermartingale w.r.t $\{\mathcal{F}_n\}$ then $E[X_n|\mathcal{F}_m] \leq X_m$ whenever $n \geq m$
  - If $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$ then $\{E[X_n]\}$ is constant.
  - If $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$ then $\{E[X_n]\}$ is increasing.
  - If $\{X_n\}$ is a supermartingale w.r.t $\{\mathcal{F}_n\}$ then $\{E[X_n]\}$ is decreasing.

- Convex transformation of martingale
  - If $\{X_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a convex function s.t. $E|\phi(X_n)| < \infty \quad \forall n \in \mathbb{N}$ then $\{\phi(X_n)\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$
  - If $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a convex and increasing function s.t. $E|\phi(X_n)| < \infty \quad \forall n \in \mathbb{N}$ then $\{\phi(X_n)\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$
  - If $\{X_n\}$ is a supermartingale w.r.t $\{\mathcal{F}_n\}$ and $\phi : \mathbb{R} \to \mathbb{R}$ is a concave and increasing function s.t. $E|\phi(X_n)| < \infty \quad \forall n \in \mathbb{N}$ then $\{\phi(X_n)\}$ is a supermartingale w.r.t $\{\mathcal{F}_n\}$

  (Ex) If $\{X_n\}$ is a martingale and $E[|X_n|^p] < \infty$ for some $p \geq 1$, then $\{|X_n|^p\}$ is a submartingale

  (Ex) If $\{X_n\}$ is a submartingale then for any $a \in \mathbb{R}$, $\{(X_n - a)^+\}$ is a submartingale

  (Ex) If $\{X_n\}$ is a supermartingale then for any $a \in \mathbb{R}$, $\{X_n \wedge a\}$ is a supermartingale

  (Ex) If $\{X_n\}$ is a submartingale then $\{X_n^+\}$ is a submartingale and $\{X_n^-\}$ is a supermartingale

* Predicatable sequence and a process using it
  - For a filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$, a random sequence $\{H_n\}_{n=1}^{\infty}$ is said to be a predicatable sequence (w.r.t $\{\mathcal{F}_n\}$) if $H_n \in \mathcal{F}_{n-1} \quad \forall n \in \mathbb{N}$
    - √ A letter H stands for a 'height'
  - Suppose $\{X_n\}$ is adapted to $\{\mathcal{F}_n\}$. For a predicatable sequence $\{H_n\}$ (w.r.t $\{\mathcal{F}_n\}$), we define a process $\{(H \cdot X)_n\}$ by

$$(H \cdot X)_n = \sum_{m=1}^{n} H_m(X_m - X_{m-1})$$

    - √ Note that $\{(H \cdot X)_n\}$ is adapted to $\{\mathcal{F}_n\}$
    - √ The definition above can be extended from $\{(H \cdot X)_n\}_{n \in \mathbb{N}}$ to $\{(H \cdot X)_n\}_{n \in \mathbb{N} \cup \{0\}}$ with additionally defining $(H \cdot X)_0 = 0$. Obviously $(H \cdot X)_0 \in \mathcal{F}_0$. For the following theorems using this process, we can regard it as $\{(H \cdot X)_n\}_{n \in \mathbb{N} \cup \{0\}}$

- Elementary facts about martingale transform with predicatable sequence
  - Let $\{X_n\}_{n=0}^{\infty}$ and $\{H_n\}_{n=1}^{\infty}$ be a random sequence and $\{H_n\}$ is a predicatable sequence w.r.t. a filtration $\{\mathcal{F}_n\}_{n=0}^{\infty}$. Assume $E|X_n H_n| < \infty$, $E|X_{n-1} H_n| < \infty \quad \forall n \in \mathbb{N}$
    1. If $\{X_n\}$ is a martingale (w.r.t $\{\mathcal{F}_n\}$) then $\{(H \cdot X)_n\}$ is also a martingale
    2. If $\{X_n\}$ is a submartingale (w.r.t $\{\mathcal{F}_n\}$) and $H_n \geq 0$ then $\{(H \cdot X)_n\}$ is also a submartingale
    3. If $\{X_n\}$ is a supermartingale (w.r.t $\{\mathcal{F}_n\}$) and $H_n \geq 0$ then $\{(H \cdot X)_n\}$ is also a supermartingale
  - √ The condition "$E|X_n H_n| < \infty$, $E|X_{n-1} H_n| < \infty \quad \forall n \in \mathbb{N}$" can be replaced with "For each $n \in \mathbb{N}$, H_n is bounded".

* Stopping time
  - A (extended) random variable N taking values of $\mathbb{N} \cup \{0, \infty\}$ is said to be a stopping time (w.r.t a filtration $\{\mathcal{F}_n\}$) if an event $(N = n) \in \mathcal{F}_n \quad \forall n \in \mathbb{N}$

$$(N \leq n) = \bigcup_{j=0}^{n} (N = j) \in \mathcal{F}_n \qquad (N > n) = (N \leq n)^C \in \mathcal{F}_n$$

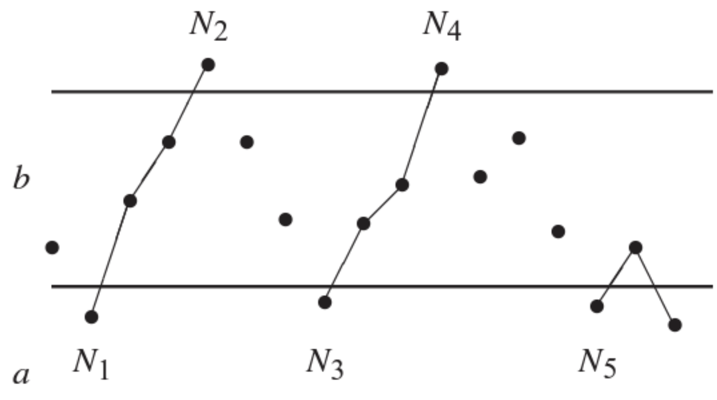
$$(N < n) = \bigcup_{j=0}^{n-1} (N = j) \in \mathcal{F}_{n-1} \qquad (N \geq n) = (N < n)^C \in \mathcal{F}_{n-1}$$

- $(N \geq n)$ is a $\mathcal{F}_{n-1}$-measurable event. $I(N \geq n)$ is $\mathcal{F}_{n-1}$-measurable random variable. Hence, $\{I(N \geq n)\}_n$ is a predictable sequence given N is a stopping time.

- Martingale stopped by stopping time
  - Let $\{X_n\}$ be a random sequence adapted to $\{\mathcal{F}_n\}$. Let N be a stopping time w.r.t $\{\mathcal{F}_n\}$ and put $H_n = I(N \geq n) \quad \forall n \in \mathbb{N}$. Then $(H \cdot X)_n = X_{N \wedge n} - X_0$.
  - The process $\{X_{N \wedge n}\}_n$ is said to be a martingale stopped by stopping time N , provided $\{X_n\}$ is a martingale.
    - ★ If $\{X_n\}$ and $\{Y_n\}$ are martingales (w.r.t. $\{\mathcal{F}_n\}$) then $\{X_n + Y_n\}$ is also a martingale. The same holds for submartingales and supermartingales too.
  - If $\{X_n\}$ is a martingale and N is a stopping time then $\{X_{N \wedge n}\}$ is martingale.
  - If $\{X_n\}$ is a submartingale and N is a stopping time then $\{X_{N \wedge n}\}$ is submartingale.
  - If $\{X_n\}$ is a supermartingale and N is a stopping time then $\{X_{N \wedge n}\}$ is supermartingale.

- Stopping time and Upcrossing
  - Suppose $\{X_n\}_{n=0}^{\infty}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$. Let $a < b$. Define N_j's as below :

$$
\begin{array}{cc}
N_1 = \inf\{m \geq 0 \,:\, X_m \leq a\} & N_2 = \inf\{m > N_1 \,:\, X_m \geq b\} \\
N_3 = \inf\{m > N_2 \,:\, X_m \leq a\} & N_4 = \inf\{m > N_3 \,:\, X_m \geq b\} \\
\vdots & \vdots \\
N_{2k-1} = \inf\{m > N_{2k-2} \,:\, X_m \leq a\} & N_{2k} = \inf\{m > N_{2k-1} \,:\, X_m \geq b\} \\
\vdots & \vdots
\end{array}
$$



  - Every N_j for $j \in \mathbb{N}$ is stopping time w.r.t $\{\mathcal{F}_n\}$. $N_1 < N_2 < N_3 \cdots$ provided all N_j's are finite. (It is possible that $N_j = \infty$ provided it has a form of $\inf(\phi)$)
  - 'Upcrossing' is a case where the submartingale $\{X_n\}$ crosses from below a to above b.
  - $\mathcal{U}_n := \sup\{k \,:\, N_{2k} \leq n\}$ is the number of upcommings completed by time n

- Upcrossing inequality
  - Suppose $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$. If stopping time N_j and the number of upcrossings $\mathcal{U}_n$ are defined as above then

$$(b-a)E[\mathcal{U}_n] \leq E[(X_n - a)^+] - E[(X_0 - a)^+]$$

- Submartingale convergence theorem
  - If $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\}$ with $\sup_n E(X_n^+) < \infty$ then $X_n \to X$ $a.s.$ for some integrable random variable X
    - ★ Trick : If $X_n \to X$ $a.s.$ then $X_n^+ \to X^+$ $a.s.$ and $X_n^- \to X^-$ $a.s.$
    - ★ Lemma : If the number of upcrossings of $[a,b]$ by submartingale $\{X_n\}$ is finite for any $a, b \in \mathbb{Q}$, then $\lim_n X_n$ exists. i.e. X_n converges to some r.v. almost surely.
  - □ If $\{X_n\}$ is a nonnegative supermartingale w.r.t $\{\mathcal{F}_n\}$ then $X_n \to X$ $a.s.$ for some integrable random variable X s.t. $E(X) \le E(X_0)$
- Example of martingale which converges almost surely but not in L^1
  - $\{\xi_n\}_n$ $i.i.d$ with $P(\xi_1 = 1) = P(\xi_1 = -1) = 1/2$. Let $S_0 = 1$, $S_n = S_{n-1} + \xi_n$ and $\mathcal{F}_0 = \{\phi, \Omega\}$, $\mathcal{F}_n = \sigma(\xi_1, \cdots, \xi_n)$. Then $\{S_n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$
    Let $N = \inf\{n \in \mathbb{N} : S_n = 0\}$. Then N is a stopping time.
    $X_n := S_{N \wedge n}$ so that $X_n = S_n$ if $n < N$ and $X_n = 0$ if $n \ge N$. $\{X_n\}$ is a nonnegative integer valued martingale w.r.t $\{\mathcal{F}_n\}$. $X_n \to 0$ $a.s.$ but $X_n \nrightarrow 0$ in $\mathcal{L}^1$.
- If $\{X_n\}_{n \in \mathbb{N} \cup \{0\}}$ is a negative submartingale w.r.t $\{\mathcal{F}_n\}_{n \in \mathbb{N} \cup \{0\}}$ then so is $\{X_n\}_{n \in \mathbb{N} \cup \{0, \infty\}}$ w.r.t $\{\mathcal{F}_n\}_{n \in \mathbb{N} \cup \{0, \infty\}}$ where $X_\infty = \lim_n X_n$ and $\mathcal{F}_\infty = \sigma\big(\bigcup_{n=0}^\infty \mathcal{F}_n\big)$
  - If $\{X_n\}_{n \in \mathbb{N}}$ is a martingale w.r.t $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ and $X_n \to X_\infty$ $a.s.$ then $X_\infty \in \mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F}_n)$
- Doob's decomposition
  - Any submartingale $\{X_n\}$ can be written as $X_n = M_n + A_n$ where $\{M_n\}$ is a martingale and $\{A_n\}$ is a predictable increasing sequence with $A_0 = 0$. Also, this expression is unique in the sense that if $X_n = M_n' + A_n'$ is another expression then $M_n = M_n'$ and $A_n = A_n'$ $a.s.$
  - √ The exact form of M_n, A_n for given X_n is $A_n = A_{n-1} + E[X_n|\mathcal{F}_{n-1}] - X_{n-1}$ $\forall n \in \mathbb{N}$ and $M_n = X_n - A_n$ $\forall n \in \mathbb{N} \cup \{0\}$ (Since $A_0 = 0$, $A_n = \sum_{k=1}^n (E[X_k|\mathcal{F}_{k-1}] - X_{k-1})$)
- Martingales with bounded increments either converge or oscillate between ∞ and $-\infty$
  - Let $\{X_n\}$ be a martingale with $|X_n - X_{n-1}| \le M < \infty$ $\forall n \in \mathbb{N}$ for some $M > 0$.
    Define disjoint subsets C, $D \subset \Omega$ by

    $$C = (\ \lim_n X_n \text{ exists and } -\infty < \lim_n X_n < \infty\ )$$
    $$D = (\ \limsup X_n = \infty \text{ and } \liminf X_n = -\infty\ )$$

    Then $P(C \cup D) = 1$
    - ★ Define "$X_n \to X$ $a.s.$ on B" for measurable set B as $P((X_n \to X) \cap B) = P(B)$
    - ★ Trick : $X_n \to X$ $a.s.$ on B $\Rightarrow$ $X_n \to X$ $a.s.$ on A whenever $A \subset B$
- Conditional Borel-Cantelli second lemma
  - Let $\{\mathcal{F}_n\}_{n \in N \cup \{0\}}$ be a filtration with $\mathcal{F}_0 = \{\phi, \Omega\}$. If $A_n \in \mathcal{F}_n$ $\forall n \in \mathbb{N}$ then

    $$(A_n\ i.o.) = \Big(\sum_{n=1}^\infty P(A_n|\mathcal{F}_{n-1}) = \infty\Big)\ a.s.$$

    - ★ Define "$A = B$ $a.s.$" for measurable sets A and B by $P(A \Delta B) = 0$ where $A \Delta B$ denotes the symmetric difference of two sets.

★ $\sum_{k=1}^{n} I_{A_k}$ is a submartingale whose martingale compoenent of Doob's decomposition is

$$\sum_{k=1}^{n} I_{A_k} - \sum_{k=1}^{n}\Big(E\Big[\sum_{j=1}^{k} I_{A_j}|\mathcal{F}_{k-1}\Big] - \sum_{j=1}^{k-1} I_{A_j}\Big) = \sum_{k=1}^{n} I_{A_k} - \sum_{k=1}^{n} P(A_k|\mathcal{F}_{k-1})$$

and this is the martingale we exploit in the proof of conditional B-C 2nd lemma

★ Trick : $(A_n\ i.o.) = \big(\sum_{n=1}^{\infty} I_{A_n} = \infty\big)$

√ Given $\{A_n\}$ is independent, by setting $\mathcal{F}_n = \sigma(A_1, \cdots, A_n)$, conditional Borel-Cantelli second lemma implies original Borel-Cantelli second lemma which is given by

$$\sum_{n} P(A_n) = \infty \ \Rightarrow\ P(A_n\ i.o.) = 1$$

* Branching process (Galton-Watson process)

  – Let $\{\xi_i^n\}_{i\in\mathbb{N}, n\in\mathbb{N}}$ be i.i.d nonnegative integer-valued random variables. Define a Galton-Watson process $\{Z_n\}_{n\in\mathbb{N}\cup\{0\}}$ as below :

$$Z_0 = 1$$
$$Z_{n+1} = \begin{cases} \xi_1^{n+1} + \cdots + \xi_{Z_n}^{n+1} = \sum_{j=1}^{Z_n} \xi_j^{n+1} & \text{if} \quad Z_n > 0 \\ 0 & \text{if} \quad Z_n = 0 \end{cases}$$

  √ The idea behind the definitions is that Z_n is the population in the n-th generation and each member of the n-th generation gives birth independently to an identically distributed number of offspring.

  – $P(\xi_1^1 = k) \quad \forall\, k \in \mathbb{N} \cup \{0\}$ is called the offspring distribution. $\mu = E(\xi_1^1)$ is the expected number of offspring per individual.

- Properties of the branching process

  – Let $\mathcal{F}_n = \sigma(\{\xi_i^m : i \in \mathbb{N},\, 1 \le m \le n\}) \quad \forall\, n \in \mathbb{N}$, $\mathcal{F}_0 = \{\phi, \Omega\}$. If $\mu = E(\xi_1^1) \in (0, \infty)$ then $\{Z_n/\mu^n\}$ is a martingale w.r.t $\{\mathcal{F}_n\}$ and $E(Z_n) = \mu^n \quad \forall\, n \in \mathbb{N}$

  – If $\mu = E(\xi_1^1) \in (0, 1)$ then $Z_n = 0$ for large enough n's $a.s.$ i.e. the species goes extinct.

- Inequality for bounded stopping time

  – If $\{X_n\}$ is a submartingale and N is a stopping time with $P(N \le K) = 1$ for some $K \in \mathbb{N}$ then

$$E(X_0) \le E(X_N) \le E(X_K)$$

  √ Since $\{X_n\}$ is a submartingale, $E(X_0) \le E(X_j) \le E(X_K)$ whenever $0 \le j \le K$. This thm tells us that similar inequality still holds true when the index is random.

- Doob's inequality

  – Let $\{X_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a submartingale. Take $n \in \mathbb{N}$ and define $\overline{X}_n = \max_{0\le m\le n} X_m$. Let $\lambda > 0$ and define an event $A = (\overline{X}_n \ge \lambda)$. Then the inequality below holds true.

$$\lambda P(A) \le E[X_n I_A] \le E[X_n^+ I_A] \le E[X_n^+]$$

  □ Let $\{X_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a supermartingale. Take $n \in \mathbb{N}$ and define $\overline{X}_n = \max_{0\le m\le n} X_m$. Let $\lambda > 0$ and define an event $A = (\overline{X}_n \ge \lambda)$. Then the inequality below holds true.

$$\lambda P(A) \le E[X_0] - E[X_n I_{A^C}] \le E[X_0] + E[X_n^-]$$

✓ Note that $P(A)$ involves $\max_{0\le m\le n}$ term while $E[X_n^+]$ or $E[X_n^-]$ only depends on n

- Doob's L^p maximal inequality

  - If $\{X_n\}_{n\in\mathbb{N}\cup\{0\}}$ is a nonnegative submartingale, then for $1<p<\infty$ and $\overline{X}_n = \max_{0\le m\le n} X_m$, the inequality belows holds true.

$$E\big(\overline{X}_n^p\big) \le \Big(\frac{p}{p-1}\Big)^p E[X_n^p]$$

    ★ Lemma : If $X\ge 0$ then $E(X)=\int_0^\infty P(X>t)\,dt$

- L^p convergence thm

  - If $\{X_n\}$ is a martingale with $\sup_n E|X_n|^p<\infty$ for some $p>1$ then $X_n\to X$ $a.s.$ and $X_n\to X$ in L^p for some integrable r.v. X

  ✓ For a martingale convergence thm, the condition was $\sup_n E(X_n^+)<\infty$

    ★ Trick : $a,b\in\mathbb{R}$ and $p\ge 1 \Rightarrow |a+b|^p \le 2^p(|a|^p+|b|^p)$

* σ-field generated by a stopping time

  - Let τ be a stopping time w.r.t. a filtration $\{\mathcal{F}_n\}$. Then we define $\mathcal{F}_\tau$ as the following :

$$\mathcal{F}_\tau = \{A\in\mathcal{F} : A\cap(\tau=n)\in\mathcal{F}_n \quad \forall n\in\mathbb{N}\}$$

    ✓ Note that $\mathcal{F}_\tau$ is indeed a σ-field.
    ✓ τ is $\mathcal{F}_\tau$-measurable
    ✓ If $\{X_n\}$ is adapted to $\{\mathcal{F}_n\}$ then X_τ is $\mathcal{F}_\tau$-measurable

- Bounded optional stopping thm

  - Let $\{X_n\}$ be a submartingale . Let σ and τ be two bounded stopping times s.t. $\sigma\le\tau\le B$ a.s. for some $B\in\mathbb{N}$. Then $E[X_\tau|\mathcal{F}_\sigma]\ge X_\sigma$ $a.s.$

  ✓ $X_\tau=\sum_{n=0}^B X_n I(\tau=n)$ is well-defined and integrable.

  ✓ By defining property of submartingale, $E[X_m|\mathcal{F}_n]\ge X_n \quad \forall\, m\ge n$. The thm tells us that this property is preserved even when indices are stopping times if they are bounded.

    ★ Trick : For a random variable X and a σ-field $\mathcal{F}$,
      i. $(X\le a)\in\mathcal{F} \quad \forall\, a\in\mathbb{R} \Rightarrow (X\in A)\in\mathcal{F} \quad \forall\, A\in\mathcal{B}(\mathbb{R})$
      ii. For $S\in\mathcal{F}$, $(X\le a)\cap S\in\mathcal{F} \quad \forall\, a\in\mathbb{R} \Rightarrow (X\in A)\cap S\in\mathcal{F} \quad \forall\, A\in\mathcal{B}(\mathbb{R})$
    ★ Lemma : For any $\mathcal{F}$-measurable and integrable X and Y ,
      i. If $\int_A X\,dP=\int_A Y\,dP \quad \forall A\in\mathcal{F}$ then $X=Y$ $a.s.$
      ii. If $\int_A X\,dP\le\int_A Y\,dP \quad \forall A\in\mathcal{F}$ then $X\le Y$ $a.s.$
    ★ Lemma : $\{X_n\}$ is a submartingale w.r.t $\{\mathcal{F}_n\} \Rightarrow \int_A X_n\,dP \le \int_A X_{n+1}\,dP \quad \forall\, A\in\mathcal{F}_n$

* Uniform integrability

  - A collection of r.v.'s $\{X_t : t\in T\}$ is said to be uniformly integrable if

$$\lim_{a\to\infty}\sup_{t\in T}\int_{|X_t|\ge a}|X_t|\,dP = \lim_{a\to\infty}\sup_{t\in T} E|X_t|I(|X_t|\ge a)=0$$

- ✓ Denote it as $\{X_t\}_{t\in T}$ $u.i.$
- ✓ A uniformly integrable family is well-controlled in the sense that if $\{X_t\}_{t\in T}$ $u.i.$ then $\exists\, M > 0$ s.t. $\sup_{t\in T} E|X_t| \le M + 1 < \infty$
- ✓ If $\{X_t\}_{t\in T}$ is uniformly integrable then each X_t is integrable .

- If $\{X_t\}_{t\in T}$ is dominated by a nonnegative integrable r.v. X i.e. $|X_t| \le X$ $a.s.$ $\forall\, t \in T$ then $\{X_t\}_{t\in T}$ is uniformly integrable.

  ★ Lemma: If X is integrable then $\int_{|X|\ge a} |X|\, dP = E|X|I(|X| \ge a) \to 0$ as $a \to \infty$

- Equivalent condition for uniform integrability
  - $\{X_t\}_{t\in T}$ is uniformly integrable iff both of two conditions below are satified.
    i. $\sup_t E|X_t| < \infty$
    ii. $\forall\, \varepsilon > 0$, $\exists\, \delta > 0$ s.t. $\sup_t \int_A |X_t|\, dP < \varepsilon$ whenever $A \in \mathcal{F}$ and $P(A) < \delta$

- Elementary properties of uniform integrable family
  - If $\{X_n\}_{n\in\mathbb{N}}$ and $\{Y_n\}_{n\in\mathbb{N}}$ are both uniformly integrable then $\{X_n + Y_n\}_{n\in\mathbb{N}}$ $u.i.$
  - If $|X_n| \le |Y_n|$ $\forall\, n \in \mathbb{N}$ and $\{Y_n\}_{n\in\mathbb{N}}$ is uniformly integrable then $\{X_n\}_{n\in\mathbb{N}}$ $u.i.$

- Vitali's lemma
  - For $p \ge 1$, if $\{X_n\} \underset{seq}{\subset} L^p$ and $X_n \xrightarrow{P} X$ then the followings are equivalent.
    i. $\{{X_n}^p\}_{n\in\mathbb{N}}$ is uniformly integrable.
    ii. $X \in L^p$ and $X_n \to X$ in L^p
    iii. $E|X_n|^p \to E|X|^p < \infty$
    ★ Lemma : For a r.v. Z, continuity set $\{z \in \mathbb{R} : P(Z = z) = 0\}$ is dense in $\mathbb{R}$

- If $\{X_n\}_{n\in\mathbb{N}}$ is uniformly integrable and $X_n \xrightarrow{D} X$ then $E|X_n| \to E|X|$ and $E(X_n) \to E(X)$

  ★ Lemma : If $Y_n \to Y$ in L^1 then $E|Y_n| \to E|Y|$ and $E(Y_n) \to E(Y)$

* Regular martingale and closable martingale
  - Let $\{X_n\}_{n\in\mathbb{N}\cup\{0\}}$ be a martingale.
    i. $\{X_n\}$ is said to be regular if $\exists\, X \in L^1$ s.t. $X_n = E[X|\mathcal{F}_n]$ $a.s.$ $\forall\, n \in \mathbb{N}$
    ii. $\{X_n\}$ is said to be closable if $\exists\, X_\infty \in L^1$ s.t. $X_n \to X_\infty$ $a.s.$, $X_\infty \in \mathcal{F}_\infty$ where $\mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F}_n)$ and $E[X_\infty|\mathcal{F}_n] = X_n$ $\forall\, n \in \mathbb{N}$ so that $\{X_n\}_{n\in\mathbb{N}\cup\{0,\infty\}}$ is a martingale w.r.t $\{\mathcal{F}_n\}_{n\in\mathbb{N}\cup\{0,\infty\}}$
    ✓ Every closable martingale is regular.

- For a martingale $\{X_n\}_{n\in\mathbb{N}}$, the followings are equivalent.
  i. $\{X_n\}$ is regular.
  ii. $\{X_n\}$ is uniformly integrable and converges $a.s.$
  iii. $\{X_n\}$ converges in L^1
  iv. $\{X_n\}$ is closable.

□ For a martingale $\{X_n\}_{n\in\mathbb{N}}$ w.r.t $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$

- If $X_n \to X$ in L^1 then $X_n \to X$ $a.s.$ and $X_n = E[X|\mathcal{F}_n]$ $\forall n \in \mathbb{N}$
- If $\{X_n\}$ is uniformly integrable then $X_n \to X$ $a.s.$ for some integrable r.v. X and $X_n = E[X|\mathcal{F}_n]$ $\forall n \in \mathbb{N}$
- If $X_n = E[X|\mathcal{F}_n]$ for some integrable r.v. X then $\{X_n\}$ is uniformly integrable and $\exists$ integrable r.v. $X_\infty \in \mathcal{F}_\infty$ s.t. $E[X_\infty|\mathcal{F}_n] = X_n$ $\forall n \in \mathbb{N}$ and $X_n \to X_\infty$ $a.s.$ and in L^1.

- Levy's thm
  - If $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ is a filtration and $\mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F}_n)$ then for an integrable r.v. X, $E[X|\mathcal{F}_n] \to E[X|\mathcal{F}_\infty]$ $a.s.$ and in L^1.

- Conditional DCT (generalized version)
  - Let $\{\mathcal{F}_n\}_{n\in\mathbb{N}}$ be a filtration and $\mathcal{F}_\infty = \sigma(\bigcup_n \mathcal{F}_n)$. If $X_n \to X$ $a.s.$ and $|X_n| \le Z$ for some integrable r.v. Z , then $E[X_n|\mathcal{F}_n] \to E[X|\mathcal{F}_\infty]$ $a.s.$

* Potential
  - A nonnegative supermartingale $\{X_n\}$ is said to be potential if $E(X_n) \to 0$

  √ If $\{X_n\}$ is potential then $\{X_n\}$ is uniformly integrable and $X_n \to 0$ $a.s.$

- Riesz decomposition
  - Let $\{X_n\}$ be a uniformly integrable nonnegative supermartingale. Then we can express X_n as $X_n = M_n + V_n$ where $\{M_n\}$ is uniformly integrable martingale and $\{V_n\}$ is potential. Furthermore, such decomposition is unique.

- If $\{X_n\}$ is uniformly integrable submartingale, then for any stopping time N, stopped process $\{X_{N\wedge n}\}$ is also uniformly integrable submartingale.

  ★ Lemma : If $X_n \to X$ $a.s.$ then $X_n^+ \to X^+$ $a.s.$ and $X_n^- \to X^-$ $a.s.$

- Inequality for unbounded stopping time
  - If $\{X_n\}$ is uniformly integrable submartingale then for any stopping time N, we have $E(X_0) \le E(X_N) \le E(X_\infty)$ where $X_n \to X_\infty$ $a.s.$

- Optional stopping thm
  - If $L \le M$ are stopping times and $\{X_n\}$ is uniformly integrable submartingale then $E[X_L] \le E[X_M]$ and $X_L \le E[X_M|\mathcal{F}_L]$ $a.s.$

- Suppose $\{X_n\}$ is a submartingale and $E\big[|X_{n+1} - X_n|\big|\mathcal{F}_n\big] \le B$ $a.s.$ $\forall n \in \mathbb{N}$. If N is a stopping time with $E(N) < \infty$ then $\{X_{N\wedge n}\}$ is unifomly integrable and $E(X_0) \le E(X_N)$

  √ Note that $E(N) < \infty$ condition implies that N is almost surely finite.

  ★ Lemma : $E|X| < \infty \Leftrightarrow \sum_n P(|X| \ge n) < \infty$

- If $\{X_n\}$ is a nonnegative supermartingale and N is a stopping time then $E(X_0) \ge E(X_N)$

- Comment for X_N with stopping time N and (sub)martingale $\{X_n\}$
  - $X_N = \sum_{n=0}^{\infty} X_n I(N = n)$

- Note that N can take value of $N = \infty$. Thus, for X_N to make sense, N should be almost surely bounded or X_∞ is well-defined.
- If X_∞ is well-defined such that $X_n \to X_\infty$ $a.s.$ then $X_{N\wedge n} \to X_N$ $a.s.$
- How can we figure out integrability of X_N ?
  i. If N is bounded $a.s.$
     - $N \le K$ $a.s.$ for some $K \in \mathbb{N}$. Hence $E|X_N| \le \sum_{n=0}^{K} E|X_n| < \infty$
  ii. If $\{X_n\}$ is uniformly integrable submartingale
     - $X_n \to X_\infty$ $a.s. \Rightarrow X_{N\wedge n} \to X_N$ $a.s.$ Since $\{X_{N\wedge n}\}$ is also uniformly integrable submartingale, by Vitali lemma, $X_N \in L^1$ i.e. X_N is integrable.
  iii. If $\{X_n\}$ is nonengative supermartingale
     - $X_n \to X_\infty$ $a.s. \Rightarrow X_{N\wedge n} \to X_N$ $a.s.$ By inequality for bounded stopping time, $E[X_{N\wedge n}] \le E[X_0]$ and using Fatou's lemma, we have $0 \le E[X_N] \le E[X_0] < \infty$