Undergraduate Real Analysis Facts

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Contents

1 Abstract Measure Theory

* σ-algebra $\mathcal{A}$: a collection of sets containing the whole set X and closed under taking complement or countable union & intersection

* σ-algebra $\sigma(\mathcal{C})$ generated by $\mathcal{C}$: the smallest σ-algebra containing $\mathcal{C}$

* Borel σ-algebra $\mathcal{B}(\mathbb{R})$: σ-algebra generated by $\mathcal{T} = \{\text{all open subsets of } \mathbb{R}\}$

- All open sets, closed sets, and countable sets in $\mathbb{R}$ are Borel sets

- Every open set in $\mathbb{R}$ can be expressed as a countable union of disjoint open intervals.

- $\mathcal{B}(\mathbb{R})$ can be generated by the collections above :

  i. All finite open intervals
  ii. All finite closed intervals
  iii. All finite left-closed half-open intervals
  iv. All finite right-closed half-open intervals
  v. All left-unbounded open rays
  vi. All right-unbounded open rays
  vii. All left-unbounded closed rays
  viii. All right-unbounded closed rays

√ There are many subsets of $\mathbb{R}$ which are not Borel. But there is no easy construction of non Borel sets. We can say that 'natural' sets we encounter are mostly Borel.

√ A countable intersections of open sets in $\mathbb{R}$ is called as G_δ set. A countable unions of closed sets in $\mathbb{R}$ is called as F_σ set. G_δ sets and F_σ sets are Borel sets.

* Measure μ : a set function on a σ-algebra, which is nonnegative and countably additive satisfying $\mu(\phi) = 0$

(Ex) The Lebesgues measure m on $\mathbb{R}$

  – $m(I) =$ length of I for each interval $I \subset \mathbb{R}$
  – $m(E + x) = m(E)\ \forall\ E \in \mathcal{B}(\mathbb{R}),\ x \in \mathbb{R}$ i.e. m is invariant under translation

(Ex) The counting measure

  – $(X, \mathcal{P}(X), \mu)$ where $\mu(E) = |E|\ \forall\ E \subset X$.

(Ex) Dirac measure or Point mass

  – Fix $x \in X$. $(X, \mathcal{P}(X), \delta_x)$ where $\delta_x(A) = I(x \in A)\ \forall\ A \subset X$

(Ex) Restriction of measure

  – $(X, \mathcal{A}, \mu)$: measure space. $B \in \mathcal{A}$. Then restriction $\mathcal{A}_B = \{A \cap B : A \in \mathcal{A}\}$ is a σ-algebra on B and $\mu|_{\mathcal{A}_B}$ is a measure on $(B, \mathcal{A}_B)$. $\mathcal{B}(B)$ is a restriction of Borel sigma field on a Borel set B, which is equal to a σ-field generated by open sets in subspace topology on B

- Elementary properties of a measure
  - $(X, \mathcal{A}, \mu)$: a measure space.
    i. (Monotonicity) : For any $A, B \in \mathcal{A}$, if $A \subset B$ then $\mu(A) \leq \mu(B)$
    ii. For any $A, B \in \mathcal{A}$ with $\mu(A) < \infty$, if $A \subset B$ then $\mu(B \setminus A) = \mu(B) - \mu(A)$
    iii. (Subadditivity) : For any $\{A_n\}_n \subset \mathcal{A}$, $\mu(\bigcup_n A_n) \leq \sum_n \mu(A_n)$
    iv. (Continuity from below) : For any $\{A_n\}_n \subset \mathcal{A}$, if $A_n \subset A_{n+1}\ \forall\ n$, then $\mu(\bigcup_n A_n) = \lim_n \mu(A_n)$
    v. (Continuity from above) : For any $\{A_n\}_n \subset \mathcal{A}$, if $A_n \supset A_{n+1}\ \forall\ n$ then $\mu(\bigcap_n A_n) = \lim_n \mu(A_n)$ provided $\mu(A_1) < \infty$

* Finite measure and σ-finite measure
  - If μ is a measure on $(X, \mathcal{A})$ satisfying $\mu(X) < \infty$ then μ is said to be a finite meausre. If $X = \bigcup_n X_n$ and $\mu(X_n) < \infty\ \forall n \in \mathbb{N}$ then μ is said to be σ-finite.

* Null set and complete measure
  - $(X, \mathcal{A}, \mu)$: a measure space. $E \in \mathcal{A}$. E is said to be a null set if $\mu(E) = 0$. μ is said to be complete if for any null set $E \in \mathcal{A}$, $F \subset E \Rightarrow F \in \mathcal{A}$.

- Completion of measure space
  - $(X, \mathcal{A}, \mu)$: a measure space. $\mathcal{N}$: A collection of null sets.
    Define $\overline{\mathcal{A}} = \{E \cup F : E \in \mathcal{A}, F \subset N \text{ for some } N \in \mathcal{N}\}$.
    Then (a)$\overline{\mathcal{A}}$ is a σ-algebra. (b) There is a unique measure $\overline{\mu}$ on $\overline{\mathcal{A}}$ extending μ given by $\overline{\mu}(E \cup F) = \mu(E)\ \forall\ E \in \mathcal{A}, F \subset N$ for some $N \in \mathcal{N}$.
    (c) $\overline{\mu}$ is complete measure. $(X, \mathcal{A}, \overline{\mu})$ is a completion of $(X, \mathcal{A}, \mu)$

* Lebesgue measurable sets
  - Denote $\mathcal{L}(\mathbb{R}) = \overline{\mathcal{B}(\mathbb{R})}$ for the completion $(\mathbb{R}, \overline{\mathcal{B}(\mathbb{R})}, \overline{m})$ of the measure space $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ and call it as the σ-algebra of Lebesgue measurable sets. We still denote $\overline{m}$ by m (abusing the notation) and call it the Lebesgue measure on $\mathbb{R}$. Every $E \in \mathcal{L}(\mathbb{R})$ is called as a Lebesgue measurable set.

2 Integration over a General Measure Space

2.1 Measurable Functions

* Measurable Functions
  - $(X, \mathcal{A})$: a measurable space. $f : X \to \mathbb{R} \cup \{\pm\infty\}$. f is said to be $\mathcal{A}$-measurable if $f^{-1}(B) \in \mathcal{A}\ \forall\ B \in \mathcal{B}(\mathbb{R})$

(Ex) Borel measurable functions

  i. Every continuous function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R})$ measurable. Every continuous function $f : I \to \mathbb{R}$ is $\mathcal{B}(I)$ measurable given $I \subset \mathbb{R}$ is an interval.
  ii. Every monotone function $f : \mathbb{R} \to \mathbb{R}$ is $\mathcal{B}(\mathbb{R})$ measurable. Every monotone function $f : I \to \mathbb{R}$ is $\mathcal{B}(I)$ measurable given $I \subset \mathbb{R}$ is an interval.

(Ex) Characteristic function

  - For given $A \subset X$, χ_A defined as $\chi_A(x) = I(x \in A)$ is said to be the characteristic function. χ_A is a measurable function $\Leftrightarrow$ A is a measurable set.

(Ex) Simple function

  - A real-valued function ϕ is called simple if it is measurable and has only a finite number of values. $\phi = \sum_{k=1}^{n} \alpha_k \chi_{E_k}$ for some scalaras α_k's and measurable sets E_k's.

- Any linear combination of finite number of simple functions is simple. Also, any finite product of simple functions is simple.
- Consturcting measurable functions
  - If $f, g : X \to \mathbb{R}$ are measurable then $f + c, cf, f \pm g$ and fg are measurable.
  - If $f_n : X \to \mathbb{R}, n \in \mathbb{N}$ are measurable then $\sup f_n, \inf f_n, \max\{f_1, \cdots, f_n\}, \min\{f_1, \cdots, f_n\}$, $\limsup f_n, \liminf f_n, \lim f_n$ are measurable.
- $f : I \to \mathbb{R}$ is a function with a finite number of discontinuities i.e. f is piecewise continuous. Then f is $\mathcal{B}(I)$-measurable.
- f is real-valued function. $f = f^+ - f^-$ and $f^+ f^- = 0$ and such decomposition is unique in the sense that if $f = f_1 - f_2$ for some nonnegative functions f_1, f_2 s.t. $f_1 f_2 = 0$ then $f_1 = f^+$ and $f_2 = f^-$. Also $|f| = f^+ + f^-$.
- Simple Approximation theorem
  - If $f : X \to \mathbb{R}$ is a measurable then $\exists$ a sequence $\{\phi_n\}$ of simple functions s.t.
    i. $0 \le |\phi_1| \le |\phi_2| \le \cdots \le |\phi_n| \le \cdots \le |f|$
    ii. $\phi_n \to f$ pointwisely.

    Additionally, if f is nonnegative measurable then above (i) is replaced by
    $0 \le \phi_1 \le \phi_2 \le \cdots \le \phi_n \le \cdots \le f$

2.2 Integration on Nonnegative Functions

* Integration of nonnegative simple functions
  - For a nonenegative simple function $\phi = \sum_{k=1}^{n} \alpha_k \chi_{E_k}$, we define the interal by
$$\int_X \phi \, d\mu = \sum_{k=1}^{n} \alpha_k \mu(E_k)$$
    . For meaurable set $E \in \mathcal{A}$, we define $\int_E \phi \, d\mu = \int_X \phi \chi_E \, d\mu$

  √ The definition above is well defined i.e. if a nonnegative simple function ϕ is written as $\phi = \sum_{k=1}^{n} \alpha_k \chi_{E_k}$ and $\phi = \sum_{j=1}^{m} \beta_j \chi_{F_j}$ then integral is same for both expressions.
  - Integral might have the value of ∞.

- Elementary properties of integral of nonnegative simple functions.
  - Let φ and ψ be nonnegative simple functions
    i. If $\alpha, \beta \geq 0$ then $\int (\alpha\varphi + \beta\psi) \, d\mu = \alpha \int \varphi \, d\mu + \beta \int \psi \, d\mu$
    ii. If $\varphi \leq \psi$ then $\int \varphi \, d\mu \leq \int \psi \, d\mu$
    iii. If $\nu : \mathcal{A} \to [0, \infty]$ is defined by $E \mapsto \int_E \varphi \, d\mu$ then ν is a measure.

* Integration of nonnegative measurable functions
  - For a nonnegative measurable function $f : X \to [0, \infty]$, we define the integral by
$$\int_X f \, d\mu = \sup\{\int \phi \, d\mu : 0 \leq \phi \leq f, simple\}$$
    . For meaurable set $E \in \mathcal{A}$, we define $\int_E f \, d\mu = \int f \chi_E \, d\mu$
  - We denote the space of nonnegative measurable functions by $\mathcal{L}^+$

- Elementary properties of integral of nonnegative measurable functions.
  - Let $f, g \in \mathcal{L}^+$
    i. If $\alpha, \beta \geq 0$ then $\int (\alpha f + \beta g) \, d\mu = \alpha \int f \, d\mu + \beta \int g \, d\mu$
    ii. If $f \leq g$ then $\int f \, d\mu \leq \int g \, d\mu$
    iii. If $\nu : \mathcal{A} \to [0, \infty]$ is defined by $E \mapsto \int_E f \, d\mu$ then ν is a measure.

- Monotone convergence theorem (MCT)
  - If $\{f_n\}$ is a sequence in $\mathcal{L}^+$ with $f_n \leq f_{n+1}$ $\forall n$ then we have
$$\lim_n \int f_n \, d\mu = \int \lim_n f_n \, d\mu$$

☐ If $\{f_n\}$ is a sequence in $\mathcal{L}^+$ then we have
$$\int \sum_{n=1}^{\infty} f_n \, d\mu = \sum_{n=1}^{\infty} \int f_n \, d\mu$$

- If $f \in \mathcal{L}^+$ and E is a measure zero set then $\int_E f \, d\mu = 0$.

* Almost everywhere
  - A statement $P(x)$ depending on $x \in X$ is said to hold almost everywhere if the set $\{x \in X : P(x) \text{ does not hold }\}$ is a subset of a measure zero set.

□ Suppose $\{f_n\}$ is a sequence in $\mathcal{L}^+$ and $f \in \mathcal{L}^+$. If $f_n \nearrow f$ *a.e.* then $\int f\, d\mu = \lim_n \int f_n\, d\mu$

- If $f \in \mathcal{L}^+$ then $\int f\, d\mu = 0 \Leftrightarrow f = 0$ *a.e.*
- Fatou's lemma
  - If $\{f_n\}$ is a sequence in $\mathcal{L}^+$ then $\int \liminf f_n\, d\mu \leq \liminf \int f_n\, d\mu$
  - If $\{f_n\}$ is a sequence in $\mathcal{L}^+$ and $f_n \to f$ *a.e.* then $\int f\, d\mu = \liminf \int f_n\, d\mu$

2.3 Integration for the General Case

* Integrability
  - For $f \in \mathcal{L}^+$, f is said to be integrable if $\int f\, d\mu < \infty$
  - For a measurable function f, f is said to be integrable if both f^+ and f^- are integrable, or equivalently, $|f|$ is integrable.
  - We denote the space of integrable functions by $\mathcal{L}^1$

- If $f \in \mathcal{L}^+$ is integrable then $f < \infty$ *a.e.* and $\{f > 0\}$ is a σ-finite set.
- If $f \in \mathcal{L}^+$ is integrable then $\forall\, \epsilon > 0$, $\exists\, E$ measurable set s.t. $\mu(E) < \infty$ and $\int f\, d\mu - \varepsilon < \int_E f\, d\mu$
- Elementary properties of integral of integrable functions
  - Let $f, g \in \mathcal{L}^1$
    1. For $\alpha, \beta \in \mathbb{R}$, $\alpha f + \beta g \in \mathcal{L}^1$ and $\int (\alpha f + \beta g)\, d\mu = \alpha \int f\, d\mu + \beta \int g\, d\mu$
    2. If $f = f_1 - f_2$ with $f_1, f_2 \in \mathcal{L}^+ \cap \mathcal{L}^1$ then $\int f\, d\mu = \int f_1\, d\mu - \int f_2\, d\mu$
    3. If $f \leq g$ then $\int f\, d\mu \leq \int g\, d\mu$
    4. $|\int f\, d\mu| \leq \int |f|\, d\mu$
- Given $f \in \mathcal{L}^1$, a map defined by $E \mapsto \int_E f\, d\mu$ has a countable additivity.
- If $f \in \mathcal{L}^1$ and E is a measure zero set then $\int_E f\, d\mu = 0$
- If $f \in \mathcal{L}^1$ then $f = 0$ *a.e* $\Rightarrow \int f\, d\mu = 0$
- If $f, g \in \mathcal{L}^+$ or $f, g \in \mathcal{L}^1$ then $f \leq g$ *a.e.* $\Rightarrow \int f\, d\mu \leq \int g\, d\mu$ and $f = g$ *a.e.* $\Rightarrow \int f\, d\mu = \int g\, d\mu$

(Note) Summary for facts about $\mathcal{L}^+$ and $\mathcal{L}^1$

Statement about functions	Functions in $\mathcal{L}^+$	Functions in $\mathcal{L}^1$
Closed under linear combi.	Yes (with positive scalars)	Yes
$f \leq g$ *a.e.* $\Rightarrow \int f\, d\mu \leq \int g\, d\mu$	Yes	Yes
$f = g$ *a.e.* $\Rightarrow \int f\, d\mu = \int g\, d\mu$	Yes	Yes
$E \mapsto \int_E f\, d\mu$ is a measure	Yes	No (But countably additive)
$\mu(E) = 0 \Rightarrow \int_E f\, d\mu = 0$	Yes	Yes
$f = 0$ *a.e.* $\Rightarrow \int f\, d\mu = 0$	Yes	Yes
$\int f\, d\mu = 0 \Rightarrow f = 0$ *a.e.*	Yes	No

- Dominated convergence theorem (DCT)
  - $\{f_n\}$: sequence in $\mathcal{L}^1$. let f be a measurable function. If $f_n \to f$ *a.e.* and $|f_n| \le g$ *a.e.* $\forall\, n \in \mathbb{N}$ for some $g \in \mathcal{L}^1$, then f is integrable and $\int f\, d\mu = \lim_n \int f_n\, d\mu$

- Generalized DCT : $\{f_n\}, \{g_n\}$: sequences in $\mathcal{L}^1$. let f, g be integrable functions. If $f_n \to f$ *a.e.* , $g_n \to g$ *a.e.* , $\int g_n\, d\mu \to \int g\, d\mu$ and $|f_n| \le g_n\ \forall\, n \in \mathbb{N}$ then $\int f\, d\mu = \lim_n \int f_n\, d\mu$

(Note) Summary for classical results about interchanging limit and integral
(All Functions are at least measurable on the following statements)

- MCT
  i. $\{f_n\}$ nonnegative and increasing. $\Rightarrow \lim_n \int f_n\, d\mu = \int \lim_n f_n\, d\mu$
  ii. $\{f_n\}$ nonnegative and $f_n \nearrow f$ a.e. $\Rightarrow \lim_n \int f_n\, d\mu = \int f\, d\mu$
- Fatou's lemma
  i. $\{f_n\}$ nonnegative $\Rightarrow \int \liminf f_n\, d\mu \le \liminf \int f_n\, d\mu$
  ii. $\{f_n\}$ nonnegative and $f_n \to f$ *a.e.* $\Rightarrow \int f\, d\mu \le \liminf \int f_n\, d\mu$
- DCT
  i. $\{f_n\}$ integrable, $f_n \to f$ *a.e.* and $|f_n| \le g$ *a.e.* $\forall\, n$ for some integrable g
  $\Rightarrow f$ is integrable and $\int f\, d\mu = \lim_n \int f_n\, d\mu$
- Additional Results
  i. $\{f_n\}$ integrable, $f_n \ge 0$ *a.e.* $\forall\, n$. and $f_n \nearrow f$ a.e. $\Rightarrow \lim_n \int f_n\, d\mu = \int f\, d\mu$
  ii. $\{f_n\}$ increasing. $f_n \nearrow f$ a.e. and $f_n \ge g\ \forall\, n$ for some integrable g
  $\Rightarrow \lim_n \int f_n\, d\mu = \int f\, d\mu$
  iii. $\{f_n\}$ integrable and $f_n \ge 0$ *a.e.* $\forall\, n$. $\Rightarrow \int \liminf f_n\, d\mu \le \liminf \int f_n\, d\mu$
  iv. $\{f_n\}$ integrable, $f_n \ge 0$ *a.e.* $\forall\, n$. and $f_n \to f$ *a.e.* $\Rightarrow \int f\, d\mu \le \liminf \int f_n\, d\mu$
  v. $\{f_n\}$ integrable, $f_n \to f$ *a.e.* for some integrable f, and $\exists\ \{g_n\}$ integrable s.t. $|f_n| \le g_n\ \forall\, n$ where $g_n \to g$ *a.e.* & $\int g_n\, d\mu \to \int g\, d\mu$ for some integrable g
  $\Rightarrow \int f\, d\mu = \lim_n \int f_n\, d\mu$

- Approximation in $\mathcal{L}^1$
  - $f \in \mathcal{L}^1$. $\{\phi_n\}$ is the sequence of simple functions obtained by the Simple approximation theorem. ($\phi_n \to f$ and $|\phi_n| \le |f|$). Then we get
  $\int f\, d\mu = \lim_n \int \phi_n\, d\mu$ and $\lim_n \int |f - \phi_n|\, d\mu = 0$

- $\{f_n\}$: seq. in $\mathcal{L}^1$. and $f \in \mathcal{L}^1$ s.t. $f_n \to f$ *a.e.*
  Then $\int |f_n - f|\, d\mu \to 0 \Leftrightarrow \int |f_n|\, d\mu \to \int |f|\, d\mu$

2.4 Concrete Examples

- The case of counting measures on $\mathbb{N}$
  - Consider measure space $(\mathbb{N}, \mathcal{P}(\mathbb{N})), \mu$ where μ is the counting measure. For any measurable function $f : \mathbb{N} \to \mathbb{R}$, f can be regarded as a sequence $a_n = f(n)$
    i. If $f \geq 0$ then $\int f \, d\mu = \sum_n a_n$
    ii. If $f \in \mathcal{L}^1$ then $\int f \, d\mu = \sum_n a_n$ and the series on RHS converges absolutely.
- The case of Dirac measures
  - Consider measure space $(X, \mathcal{P}(X), \delta_x)$ for a fixed $x \in X$.
    For any $f : X \to \mathbb{R}$, we have $\int f \, d\delta_x = f(x)$
- The case of the Lebesgue measure on $\mathbb{R}$
  - √ Every Borel measurable function is Lebesgue measurable
  - Every Riemann integrable function $f : [a, b] \to \mathbb{R}$ is Lebesgue intebrable and the result of integrals for both integrations are the same. i.e. $\int_{[a,b]} f \, dm = \int_a^b f(x) \, dx$
  - (Characterization of Riemann integrability) For a bounded function $f : [a, b] \to \mathbb{R}$, f is Riemann integrable $\Leftrightarrow$ f is continuous $m-a.e.$ where m is the Lebesgue measure
  - (Generalize to improper integral) If $f : [0, \infty) \to \mathbb{R}$ satisfies that f is Riemann integrable on $[0, b]$ $\forall$ $b > 0$ and $lim_{b\to\infty} \int_0^b f(x)dx$ exists, i.e. the improper integral $\int_0^\infty f(x)dx$ exists, then f is Lebesgue measurable and $\int_{[0,\infty)} f \, dm = \int_0^\infty f(x)dx$ provided f is nonnegative or Lebesgue integrable.
  - (Interchanging partial differentiation and integration) Consider a bivariate function $f : [0, \infty) \times [a, b] \to \mathbb{R}$ s.t. $f(\cdot, y) : [0, \infty) \to \mathbb{R}$ is integrable for each $y \in [a, b]$. Define $F(y) = \int_0^\infty f(x, y)dx$
    i. Suppose $\exists$ $g \in \mathcal{L}^1$ s.t. $|f(x, y)| \leq g(x)$ $\forall$ x, y.
    Then $lim_{y\to y_0} f(x, y) = f(x, y_0)$ $\forall$ $x \Rightarrow lim_{y\to y_0} F(y) = F(y_0)$.
    In particular, $f(x, \cdot)$ is continuous $\forall$ $x \Rightarrow F$ is continuous.
    ii. Suppose $\frac{\partial f}{\partial y}$ exists on (a, b) and $\exists$ $g \in \mathcal{L}^1$ s.t. $|\frac{\partial f}{\partial y}(x, y)| \leq g(x)$ $\forall$ x, y
    Then F is diff.able on (a, b) and $F'(y) = \frac{\partial}{\partial y} \int_0^\infty f(x, y)dx = \int_0^\infty \frac{\partial}{\partial y} f(x, y)dx$
- The case of measure coming from function called density
  - $(X, \mathcal{A}, \mu)$: measure space. ν is the measure given by $\nu(E) = \int_E f \, d\mu$ $\forall$ $E \in \mathcal{A}$ for some $f \in \mathcal{L}^+$ which is called as the density.
    For any $g \in \mathcal{L}^+$ or $g \in \mathcal{L}^1(\nu)$, we have $\int g \, d\nu = \int fg \, d\mu$
- A remark on the completeness of measure
  - Suppose$(X, \overline{\mathcal{A}}, \overline{\mu})$ is the completion of $(X, \mathcal{A}, \mu)$. If $f : X \to \mathbb{R}$ is $\overline{\mathcal{A}}$-measurable then $\exists$ $g : X \to \mathbb{R}$ s.t. $g = f$ $\overline{\mu} - a.e.$ and g is $\mathcal{A}$-measurable.

3 Construction of Measures

* Algebra $\mathcal{A}$: a collection of sets containing the whole set X and closed under taking complement or finite union & intersection

* Premeasure μ : a set function on an an algebra, which is nonnegative and countably additive $\mu(\phi) = 0$.

√ Algebra is not closed under taking countable union, so countable additivity of premeasure μ is represented as $\mu(\bigcup_n A_n) = \sum_n \mu(A_n)$ for a disjoint $\{A_n\} \underset{seq}{\subset} \mathcal{A}$ s.t. $\bigcup_n A_n \in \mathcal{A}$.

√ A premeasure μ is called σ-finite if $X = \bigcup_n X_n$ with $\{X_n\} \subset \mathcal{A}$ and $\mu(X_n) < \infty \ \forall \ n \in \mathbb{N}$

* Outer measure μ^*
  - For a premeasure μ on an algebra $\mathcal{A} \subset \mathcal{P}(X)$, the outer measure $\mu^* : \mathcal{P}(X) \to [0, \infty]$ is defined by $\mu^*(E) = \inf\{\sum_n \mu(A_n) : A_n \in \mathcal{A}, \ \bigcup_n A_n \text{ covers } E\}$

• Elementary properties of outer measure

  i. $\mu^*(\phi) = 0$
  ii. (Monotonicity) For any $A, B \subset X$, if $A \subset B$ then $\mu^*(A) \leq \mu^*(B)$
  iii. (Countable Subadditivity) For any $\{A_n\} \subset \mathcal{P}(X)$, $\mu^*(\bigcup_n A_n) \leq \sum_n \mu^*(A_n)$

* Caratheodory condition
  - μ^* : the outer measure associated to a premeasure μ on an algebra $\mathcal{A} \subset \mathcal{P}(X)$. $E \subset X$ is said to be μ^*-measurable if E satisfies the 'Catheodory condition' below : $\mu^*(A) = \mu^*(A \cap E) + \mu^*(A \cap E^C)$ for any $A \subset X$
  - The collection of all μ^*-measurable sets is denoted by $\mathcal{M}^*$

• Caratheodory Extension Theorem
  - μ^* : the outer measure associated to a premeasure μ on an algebra $\mathcal{A} \subset \mathcal{P}(X)$. The followings are true. (a) $\mathcal{M}^*$ is a σ-algebra. (b) $\mu^*|_{\mathcal{M}^*}$ is a measure. (c) $\mu^*|_{\mathcal{A}} = \mu$ i.e. μ^* is indeed an extension of μ (d) $\mathcal{A} \subset \mathcal{M}^*$
  - In particular, if we denote $\mathcal{M} = \sigma(\mathcal{A})$, then $\mathcal{M} \subset \mathcal{M}^*$ by the result of (d). Define a measure $\tilde{\mu}$ on $\mathcal{M}$ by $\tilde{\mu} = \mu^*|_{\mathcal{M}}$. If μ is a σ-finite premeasure then $\tilde{\mu}$ is the unique extension of μ which is a measure on $\mathcal{M}$

3.1 The Lebesgue Measure on $\mathbb{R}$

- Building an appropriate algebra to construct Lebesgue measure on $\mathbb{R}$
  - $\mathcal{I} = \{(a,b] : -\infty < a < b < \infty\}$, $\mathcal{J} = \{(-\infty, b] : b \in \mathbb{R}\}$, $\mathcal{K} = \{(a,\infty) : a \in \mathbb{R}\}$
    Every element of $\mathcal{I} \cup \mathcal{J} \cup \mathcal{K}$ is said to be h-interval. ('h' stands for 'half-open')
    $\mathcal{A}$ is defined as the collection of finite unions of disjoint h-intervals.
  - $\mathcal{A}$ above is an algebra.
  - $\mathcal{A}$ generates a Borel σ-algebra i.e. $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$
- Constructing a premeasure which extends a length function.
  - Define $\mu : \mathcal{A} \to [0,\infty]$ by $\mu(\bigcup_{k=1}^{n} I_k) = \sum_{k=1}^{n} length(I_k)$ where I_k's are disjoint h-intervals. Note that length of every ray is ∞.
  - μ is well-defined premeasure on $\mathcal{A}$
- Lebesgue measure on $\mathbb{R}$
  - The Lebesgue measure m on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ is the unique extension of μ on $\mathcal{B}(\mathbb{R})$, which is guaranteed by the Caratheodory Extension Thm.
- Elementary properties of Lebesgue measure
  - $m(I) = length(I)$ for any interval $I \subset \mathbb{R}$ (For any ray, length is measured as ∞)
  - $m(E + x) = m(E)$ $\forall\, E \in \mathcal{B}(\mathbb{R})$ and $\forall\, x \in \mathbb{R}$ i.e. m is translation invariant.

3.2 The σ-algebra of Lebesgue Measurable sets $\mathcal{L}(\mathbb{R})$ and $\mathcal{M}^*$

- Assume same condition with the Caratheodory Extension Thm ;
  μ^* : the outer measure associated to a premeasure μ on an algebra $\mathcal{A} \subset \mathcal{P}(X)$.
  Then $(X, \mathcal{M}^*, \mu^*|_{\mathcal{M}^*})$ is a complete measure space.
- Assume that we're in the situation of constructing Lebesgue measure with premeasure μ.
  Take $\varepsilon > 0$. For any $E \in \mathcal{M}^*$, $\exists\, F \underset{closed}{\subset} \mathbb{R}, \mathcal{U} \underset{open}{\subset} \mathbb{R}$ s.t. $F \subset E \subset \mathcal{U}$ and $\mu^*(\mathcal{U} \setminus F) < \varepsilon$.
  Moreover, $\exists$ a F_σ set F and a G_δ set G s.t. $F \subset E \subset G$ and $\mu^*(G \setminus F) = 0$.
- The σ-algebra $\mathcal{M}^*$ appearing in the construction of Lebesgue measure is the same as the σ-algebra of Lebesgue measurable sets $\mathcal{L}(\mathbb{R})$ defined by a completion of $\mathcal{B}(\mathbb{R})$. The outer measure μ^* restricted to $\mathcal{M}^*$ in the construction of Lebesgue measure is indeed the extended Lebesgue measure $\overline{m}$ on $\mathcal{L}(\mathbb{R})$.

☐ Regularity of Lebesgue measure m

  - Let $\varepsilon > 0$. For $E \in \mathcal{B}(\mathbb{R})$, $\exists\, F \underset{closed}{\subset} \mathbb{R}, \mathcal{U} \underset{open}{\subset} \mathbb{R}$ s.t. $F \subset E \subset \mathcal{U}$ and $m(\mathcal{U} \setminus F) < \varepsilon$.
    Moreover, $\exists\, F, G \in \mathcal{B}(\mathbb{R})$ s.t. $F \subset E \subset G$ and $m(G \setminus F) = 0$.

3.3 Probability Borel Measure and Distribution Function

- Construction of Probability Borel Measure μ_F from a distribution function F
  - Let $F : \mathbb{R} \to \mathbb{R}$ be a monotone increasing right-continuous function with $F(-\infty) = \lim_{x\to-\infty} F(x) = 0$ and $F(\infty) = \lim_{xto\infty} F(x) = 1$. Also consider algebra $\mathcal{A}$ used in the construction of Lebesgue measure.
  - Define $\mu : \mathcal{A} \to [0, 1]$ by $\mu(\phi) = 0$ and $\mu(\bigcup_{k=1}^{n}(a_k, b_k]) = \sum_{k=1}^{n} F(b_k) - F(a_k)$ Here, if $b_k = \infty$ then regard $(a_k, b_k]$ as (a_k, ∞).
  - μ above is a premeasure on $\mathcal{A}$. Since μ is a finite premeasure and $\sigma(\mathcal{A}) = \mathcal{B}(\mathbb{R})$, thanks to the Caratheodory Extension Thm, there is a unique extension of μ which is a measure on $\mathcal{B}(\mathbb{R})$.
  - We denote such measure as μ_F, which is a probability Borel measure. $\mu_F(a, b] = F(b) - F(a) \;\; \forall \; -\infty < a < b < \infty$ and $\mu_F(-\infty, x] = F(x) \;\; \forall\, x \in \mathbb{R}$.
  - Using same logic, we can construct a unique Borel Measure from a distribution-like function F s.t. $F(\infty) = \lim_{xto\infty} F(x) \leq 1$.

4 Product Measures and The Fubini-Tonelli Theorem

4.1 Construction of Product Measure

* Product σ-algebra
  - $(X, \mathcal{A}), (Y, \mathcal{B})$: measurable spaces. The product σ-algebra $\mathcal{A} \otimes \mathcal{B}$ is defined by the σ-algebra on $X \times Y$ generated by $\mathcal{A} \times \mathcal{B} = \{A \times B : A \in \mathcal{A},\; B \in \mathcal{B}\}$
    i.e. $\mathcal{A} \otimes \mathcal{B} = \sigma(\mathcal{A} \times \mathcal{B})$
  - Every set of the form $A \times B : A \in \mathcal{A},\; B \in \mathcal{B}$ is called as (measurable) rectangles.

- If X and Y are topological spaces satisfying second countability axiom i.e. each X and Y has a countable basis, then $\mathcal{B}(X) \times \mathcal{B}(Y) = \mathcal{B}(X \times Y)$ (Note that for a topological space X, $\mathcal{B}(X)$ is defined as a σ-algebra generated by a collection of all open sets in X)

☐ $\mathcal{B}(\mathbb{R}) \otimes \mathcal{B}(\mathbb{R}) = \mathcal{B}(\mathbb{R}^2)$ and $\mathcal{P}(\mathbb{N}) \otimes \mathcal{P}(\mathbb{N}) = \mathcal{P}(\mathbb{N}^2)$

- Product measure
  - $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$: measure spaces. A product measure $\mu \times \nu$ is a measure on $\mathcal{A} \otimes \mathcal{B}$ satisfying $\mu \times \nu(A \times B) = \mu(A)\nu(B) \; \forall\; A \in \mathcal{A}, B \in \mathcal{B}$ (existence of such measure is guaranteed by the Caratheodory Extension Thm.)
  - In addition, if μ and ν are both σ-finite, then $\mu \times \nu$ is uniquely determined.

- Examples of product measure

  i. The Lebesgue measure on $\mathbb{R}^2$

  - The product measure of Lebesgue measure $m^2 = m \times m$ is an extension of area function on $\mathcal{B}(\mathbb{R}^2)$ in the sense that for any real interval I and J, we get $m^2(I \times J) = m(I)m(J) = length(I) \times length(J) = area(I \times J)$

  ii. The Counting measure on $\mathbb{N}^2$

  - The product measure $\mu^2 = \mu \times \mu$ where μ is a counting measure on $\mathcal{P}(\mathbb{N})$ is also a counting measure on $\mathbb{N}^2$

  iii. The product of probability Borel measures

  - If each μ and ν is a probability Borel measures on $(\mathbb{R}, \mathcal{B}(\mathbb{R}))$ then there is a uniquely determined probability Borel measure $\mu \times \nu$ on $(\mathbb{R}^2, \mathcal{B}(\mathbb{R}^2))$
  - If X and Y are independent random variables with distribution $X \sim \mu$ and $Y \sim \nu$ respectively, i.e. $P(X \in B) = \mu(B)$ and $P(Y \in B) = \nu(B)\ \ \forall\, B \in \mathcal{B}(\mathbb{R})$ then a random vector (X, Y) has a distribution $(X, Y) \sim \mu \times \nu$ so that $P((X, Y) \in A \times B) = P(X \in A)P(Y \in B)\ \ \forall\, A, B \in \mathcal{B}(\mathbb{R})$

4.2 Fubini-Tonelli Theorem

* The concept of sections

  i. X, Y : sets. For any $E \subset X \times Y$ and $x \in X, y \in Y$, x-section and y-section of E are defined by $E_x = \{y \in Y : (x, y) \in E\}$, $E^y = \{x \in X : (x, y) \in E\}$

  ii. $f : X \times Y \to \mathbb{R}$. x-section and y-section of f are defined by $f_x : Y \to \mathbb{R}$, $f^y : X \to \mathbb{R}$ and $f_x(y) = f(x, y)\ \forall\, y \in Y$, $f^y(x) = f(x, y)\ \forall\, x \in X$

  √ $(\chi_E)_x = \chi_{E_x}$ (as a map from Y to $\mathbb{R}$) / $(\chi_E)^y = \chi_{E^y}$ (as a map from X to $\mathbb{R}$)

- Measurability of sections

  - $(X, \mathcal{A}), (Y, \mathcal{B})$: measurable spaces

    i. If $E \in \mathcal{A} \otimes \mathcal{B}$ then $E_x \in \mathcal{B}$ and $E^y \in \mathcal{A}\ \ \forall\, x \in X, y \in Y$

    ii. If $f : X \times Y \to \mathbb{R}$ is $\mathcal{A} \otimes \mathcal{B}$-measurable then f_x is $\mathcal{B}$-measurable and f^y is $\mathcal{A}$-measurable $\ \forall\, x \in X, y \in Y$

* Monotone Class

  - X : a set. $\mathcal{A} \subset \mathcal{P}(X)$ is said to be a monotone class if $E_n \in \mathcal{A}, E_n \subset E_{n+1}\ \forall\, n \Rightarrow \bigcup_n E_n \in \mathcal{A}$ / $E_n \in \mathcal{A}, E_n \supset E_{n+1}\ \forall\, n \Rightarrow \bigcap_n E_n \in \mathcal{A}$

  √ Every σ-algebra is a monotone class. An intersection of monotone classes on the same set is a monotone class.

- The monotone class lemma

  - If $\mathcal{A} \subset \mathcal{P}(X)$ is an algebra then $\mathcal{C}(\mathcal{A}) = \sigma(\mathcal{A})$ where $\mathcal{C}(\mathcal{A})$ denotes the smallest monotone class containing $\mathcal{A}$.

- $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$: σ-finite measure spaces. For $E \in \mathcal{A} \otimes \mathcal{B}$, the followings are satisfied.

  i. $x \mapsto \nu(E_x)$ is a $\mathcal{A}$-measurable function / $y \mapsto \mu(E^y)$ is a $\mathcal{B}$-measurable function

  ii. $\mu \times \nu(E) = \int \nu(E_x)\, d\mu(x) = \int \mu(E^y)\, d\nu(y)$

- Fubini-Tonelli Theorem

  – $(X, \mathcal{A}, \mu), (Y, \mathcal{B}, \nu)$: σ-finite measure spaces. $f : X \times Y \to \mathbb{R}$

    i. (Tonelli)
    If f is nonnegative $A \otimes B$-measurable function, then $g(x) = \int f_x\, d\nu$ is nonnegative $\mathcal{A}$-measurable, $h(y) = \int f^y\, d\mu$ is nonnegative $\mathcal{B}$-measurable, and the equation $(*)$ holds true.

    ii. (Fubini)
    If f is integrable function then f_x is integrable for almost all $x \in X$, f^y is integrable for almost all $y \in Y$, and $g(x) = \int f_x\, d\nu$ & $h(y) = \int f^y\, d\mu$ are integrable. Moreover the equation $(*)$ holds true.

    $$(*) \quad \int_{X \times Y} f\, d(\mu \times \nu) = \int_X \int_Y f(x,y)\, d\nu(y) d\mu(x) = \int_Y \int_X f(x,y)\, d\mu(x) d\nu(y)$$

    √ The meaning of statement of Fubini Thm is :
    If $f \in \mathcal{L}^1(\mu \times \nu)$ i.e. $\int |f|\, d(\mu \times \nu) < \infty$ then $f_x \in \mathcal{L}^1(\nu)$ i.e. $\int |f_x|\, d\nu < \infty$ for $\mu - a.e.\ x$ and $f^y \in \mathcal{L}^1(\mu)$ i.e. $\int |f^y| d\mu < \infty$ for $\nu - a.e.\ y$

☐ Fubini-Tonelli Thm for Probability Theory

  – X, Y : independent random variables with distribution $X \sim \mu$ and $Y \sim \nu$
  If a Borel measurable function $f : \mathbb{R}^2 \to \mathbb{R}$ satisfies $f \geq 0$ or $E|f(X,Y)| < \infty$ then

  $$E[f(X,Y)] = \int_Y \int_X f(x,y)\, d\mu(x) d\nu(y) = \int_X \int_Y f(x,y)\, d\nu(y) d\mu(x)$$

- Useful result about Lebesgue integral

  – If Borel measurable $f : \mathbb{R} \to \mathbb{R}$ is nonnegative or Lebesgue integrable function, then $\int f(x+\alpha)\, dm(x) = \int f(x)\, dm(x)$ and $\int f(\alpha x)\, dm(x) = \alpha^{-1} \int f(x)\, dm(x) \ \ \forall\, \alpha \neq 0$

- If $T : \mathbb{R}^2 \to \mathbb{R}^2$ is an invertible linear map and $f : \mathbb{R}^2 \to \mathbb{R}$ is Borel measurable function, then we have $\int f\, dm^2 = |det(T)| \int f \circ T\, dm^2$

- Properties of the Lebesgue measure m^2 on $\mathbb{R}^2$

  – Rotation invariance ; $m^2(R(E)) = m^2(E) \ \ \forall\, E \in \mathcal{B}(\mathbb{R}^2)$ where R is a rotation map

  – Translation invaraince ; $m^2(E + x) = m^2(E) \ \ \forall\, E \in \mathcal{B}(\mathbb{R}^2), x \in \mathbb{R}^2$

5 The Spaces L^1 and L^2

* Banach Space
  - A complete normed vector space is said to be a Banach space
* L^1-norm
  - $(X, \mathcal{A}, \mu)$: a measure space. $\|\cdot\|_1$ on $\mathcal{L}^1 = \mathcal{L}^1(X, \mathcal{A}, \mu)$ is defined as $\|f\|_1 = \int |f| \, d\mu$
  - In order to satisfy the defining properties of norm, introduce equivalence relation on $\mathcal{L}^1$ given as $f \sim g \Leftrightarrow f = g$ *a.e.*
* The space L^1
  - $(X, \mathcal{A}, \mu)$: a measure space. Define $L^1 = L^1(X, \mathcal{A}, \mu)$ by $L^1 = \mathcal{L}^1/\sim$ i.e. L^1 is the space of equivalence classes in $\mathcal{L}^1$ w.r.t. $\sim$ above.
  - √ L^1 is a normed space with $\|\cdot\|_1$. L^1 identifies $f, g \in \mathcal{L}^1$ whenever $f = g$ *a.e.*
* Riesz-Fisher Theorem
  - L^1 is a Banach space.
* The effect of completion of measure spaces for L^1
  - $(X, \overline{A}, \overline{\mu})$ is the completion of $(X, \mathcal{A}, \mu)$. Then we have $L^1(X, \overline{\mu}) = L^1(X, \mu)$ as Banach spaces. i.e. there is a norm preserving linear bijection between two spaces.
  - √ We cannot distinguish $L^1(\mathbb{R}, \mathcal{L}(\mathbb{R}), m)$ from $L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$, which is why we do not need to consider the σ-algebra of Lebesgue measurable sets in reality.
* Hilbert space
  - An inner product space being complete as a normed space is called as a Hilbert space.
* The space L^2
  - $(X, \mathcal{A}, \mu)$: a measure space. Define $\mathcal{L}^2 = \big\{ f : X \to \mathbb{R} \cup \{\pm\infty\} \mid \int |f|^2 \, d\mu < \infty \big\}$
  - √ $\mathcal{L}^2$ is a vector space.
  - L^2 is defined as $L^2 = \mathcal{L}^2/\sim$ where $\sim$ is the equivalence relation of being equal almost everywhere.
  - Inner product on L^2 and the induced L^2-norm is given as $\langle f, g \rangle = \int fg \, d\mu$ and $\|f\|_2 = \big(\int |f|^2 \, d\mu \big)^{1/2}$
* Approximation in L^1 and L^2
  1. For any $f \in L^1$ and $\varepsilon > 0$, $\exists$ a simple function $\phi \in L^1$ with $\|f - \phi\|_1 < \varepsilon$
  2. For any $f \in L^2$ and $\varepsilon > 0$, $\exists$ a simple function $\phi \in L^2$ with $\|f - \phi\|_2 < \varepsilon$

5.1 Concrete Cases

- The case of $([0,1], \mathcal{B}([0,1]), m)$, which is a probability space.
  - $L^2([0,1], m) \subset L^1([0,1], m)$ with $\|f\|_1 \le \|f\|_2 \ \forall f \in L^2([0,1], m)$
- The case of $(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ where μ is a counting measure.
  - Denote $L^p(\mathbb{N}, \mu)$ by ℓ^p and call it the little L^p space or the sequential L^p space.
  - $\ell^1 \subset \ell^2$ with $\|\{a_n\}\|_2 \le \|\{a_n\}\|_1 \ \forall \{a_n\} \in \ell^1$
- The case of $(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$
  - There is no inclusion between the spaces $L^1(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$ and $L^2(\mathbb{R}, \mathcal{B}(\mathbb{R}), m)$

6 Basic Fourier Analysis

6.1 Integration of complex-valued functions

- Componentwise measurability implies measurability of vector-valued function
  - $(X, \mathcal{A})$: a measurable space. $f : X \to \mathbb{R}^2$. Componenetwise representation of f is (f_1, f_2). Then f_1, f_2 are $\mathcal{A}$-measurable $\Leftrightarrow f$ is measurable.
    i.e. $f_1^{-1}(B_1),\ f_2^{-1}(B_2) \in \mathcal{A} \ \forall B_1, B_2 \in \mathcal{B}(\mathbb{R}) \Leftrightarrow f^{-1}(B) \in \mathcal{A} \ \forall B \in \mathcal{B}(\mathbb{R}^2)$

* Measurability of complex-valued function
  - $(X, \mathcal{A}, \mu)$: a measure space. $f : X \to \mathbb{C}$
    i. f is said to be $\mathcal{A}$-measurable if $Re(f)$ and $Im(f)$ are $\mathcal{A}$-measurable.
    ii. f is said to be integrable if $Re(f)$ and $Im(f)$ are integrable.
    In this case we define integral of f by $\int f\, d\mu := \int Re(f)\, d\mu + i \int Im(f)\, d\mu$
    $\surd$ If f is a measurable complex-valued function,
    then f is integrable $\Leftrightarrow |f|$ is integrable.

- Elementary properties of integral of complex-valued functions
  - Let $f, g : X \to \mathbb{C}$ be integrable functions
    i. For $\alpha, \beta \in \mathbb{C}$, we have $\int \alpha f + \beta g\, d\mu = \alpha \int f\, d\mu + \beta \int g\, d\mu$
    ii. $|\int f\, d\mu| \le \int |f|\, d\mu$

* L^1 and L^2 spaces of complex-valued functions.
  - $(X, \mathcal{A}, \mu)$: a measure space. we define $\mathcal{L}^1_{\mathbb{C}}$ and $\mathcal{L}^2_{\mathbb{C}}$ spaces as below

$$\mathcal{L}^1_{\mathbb{C}}(X, \mathcal{A}, \mu) = \{f : X \to \mathbb{C} \mid \int |f|\, d\mu < \infty\}, \quad \mathcal{L}^2_{\mathbb{C}}(X, \mathcal{A}, \mu) = \{f : X \to \mathbb{C} \mid \int |f|^2\, d\mu < \infty\}$$

  - We also define $L^1_{\mathbb{C}}$ and $L^2_{\mathbb{C}}$ by $L^1_{\mathbb{C}} := \mathcal{L}^1_{\mathbb{C}}/\sim$ and $L^2_{\mathbb{C}} := \mathcal{L}^2_{\mathbb{C}}/\sim$

- L^1 and L^2 spaces are complex normed spaces with the norms

$$\|f\|_1 = \int |f| \, d\mu, \quad \|f\|_2 = \Big(\int |f|^2 \, d\mu \Big)^2$$

- Note that L^2 is also a complex inner product space with inner product

$$\langle f, g \rangle = \int f\overline{g} \, d\mu$$

- Orthonormal family in L^2

  -
$$\frac{1}{2\pi} \int_{-\pi}^{\pi} e^{inx} e^{-imx} \, dx = \begin{cases} 1 & (n = m) \\ 0 & (n \neq m) \end{cases}$$

  - $\{e^{inx} : n \in \mathbb{Z}\}$ is an orthonormal family in $L^2([-\pi, \pi], \frac{1}{2\pi} m)$ where $\frac{1}{2\pi} m$ is the normalized Lebesgue measure on $[-\pi, \pi]$

6.2 Fourier series of periodic funcitons

* Fourier coefficient and Fourier series

- Let $f : [-\pi, \pi] \to \mathbb{C}$ be an integrable funtion w.r.t. the Lebesgue measure.
  i. For each $n \in \mathbb{Z}$, the n-th Fourier coefficient of f is defined by

$$\hat{f}(n) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) e^{-inx} \, dx = \langle f, e_n \rangle$$

  ii. Fourier series of f is defined as the following formal series

$$\sum_{n=-\infty}^{\infty} \hat{f}(n) e^{inx}$$

  iii. Partial sums of the Fourier series of f is denoted by $S_N(f)$, which is given as

$$S_N(f)(x) = \sum_{n=-N}^{N} \hat{f}(n) e^{inx}$$

√ Given the integrability of f, $\hat{f}(n)$ is well-defined for every $n \in \mathbb{Z}$

√ Fourier series is called 'formal' since we don't know its convergence at this moment.

* Trigonometric series and polynomial

- Trigonometric series is the series of the following form ; $\sum_{n=-\infty}^{\infty} c_n e^{inx}$
- Trigonometric polynomial is a special case of trigonometric series with $c_n = 0$ for $|n| > N$, i.e. $\sum_{n=-N}^{N} c_n e^{inx}$
- If $c_N \neq 0$ or $c_{-N} \neq 0$ then N is called the degree of the trigonometric polynomial

√ The main problem of this section is "In what sense $S_N(f)$ converges to f as $N \to \infty$"

6.3 Convolutions and good kernels

* Convolution

- For 2π-periodic square-integrable functions f and g on $\mathbb{R}$, the convolution $f * g$ on $[-\pi, \pi]$ is defined as

$$f * g(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(y) g(x-y)\, dy$$

√ By Cauchy-Schwarz inequality, convolution is well-defined.

√ Loosely speaking, convolution is a kind of weighted average.

- Properties of convolutions
  - For 2π-periodic $f, g, h \in L^2$ and $\alpha, \beta \in \mathbb{C}$
    i. $f * g = g * f$
    ii. $(\alpha f + \beta g) * h = \alpha f * h + \beta g * h$
    iii. $\widehat{f * g} = \hat{f} \cdot \hat{g}$ i.e. $\widehat{f * g}(n) = \hat{f}(n)\hat{g}(n) \quad \forall\, n \in \mathbb{Z}$

* Dirichlet kernel

- $D_N(x) = \sum_{n=-N}^{N} e^{inx}$ is called as the N-th Dirichlet kernel.
- Simplified form is $D_N(x) = \frac{sin((N+1/2)x)}{sin(x/2)}$

- For 2π-periodic $f \in L^2$, we have $S_N(f) = f * D_N$

* A family of good kernels

- A family of good kernels is a family $\{K_n\}_{n\in\mathbb{N}}$ of functions on $[-\pi, \pi]$ satisfying
  i. (Normalized) $\frac{1}{2\pi} \int_{-\pi}^{\pi} K_n(x)\, dx = 1 \quad \forall\, n \in \mathbb{N}$
  ii. (Boundedness) $\{\|K_n\|_1\}_{n\in\mathbb{N}}$ is a bounded sequence of positive numbers.
  iii. (Concentration at zero) $\forall\, \delta > 0$, we have $\int_{\delta \le |x| \le \pi} |K_n(x)|\, dx \to 0$ as $n \to \infty$

- Good kernels and convergence of convolution
  - If $\{K_n\}_{n\in\mathbb{N}}$ is a family of good kernels and f is continuous function on $[-\pi, \pi]$ then $f * K_n$ converges to f uniformly on $[-\pi, \pi]$

- Dirichlet kernels $\{D_n\}_{n\in\mathbb{N}}$ is not a family of good kernels. Hence it is difficult for us to hope that the partial sums $S_N(f)$ converges to f uniformly.

* Cesaro Summable

- A sequence of complex numbers $\{c_n\}_n$ is said to be Cesaro summable to $c \in \mathbb{C}$ if the arithmetic mean of their partial sums converges to c
- For the case of Fourier series of f, "the Fourier series of f is Cesaro summalbe to f" means that $\sigma_N(f) = \frac{S_0(f)+S_1(f)+\cdots+S_{N-1}(f)}{N}$ converges to f as $N \to \infty$

* Fejer kernel

- $F_N = \frac{D_0+D+1+\cdots+D_{N-1}}{N}$ is called as the N-th Fejer kernel.
- Simplified form is $F_N(x) = \frac{\sin^2(Nx/2)}{N\sin^2(x/2)}$

- Fejer kernels $\{F_n\}_{n\in\mathbb{N}}$ is a fmily of good kernels.

- If f is a continuous function on $[-\pi,\pi]$ with $f(-\pi)=f(\pi)$ then $f*F_n$ converges to f uniformly on $[-\pi,\pi]$ i.e. the Fourier series of f is uniformly Cesaro summable to f.

□ If f is a continuous function on $[-\pi,\pi]$ with $f(-\pi)=f(\pi)$ then f can be uniformly approximated by trigonometric polynomials. i.e. $\forall\ \varepsilon>0$, there is a trigonometric polynomial $p=f*F_N$ with large enough N s.t. $|f(x)-p(x)|<\varepsilon \quad \forall\ -\pi\le x\le\pi$

6.4 Convergence of Fourier series in L^2 space and Plancherel Thm

- Here we consider $L^2([-\pi,\pi])$ having L^2-norm defined as $\|f\|_2^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx$. Note that for $f\in L^2([-\pi,\pi])$, we can always assume $f(-\pi)=f(\pi)$ since we identify functions which coincide almost everywhere.

- Fourier partial sum as the Best approximtaion
  - If $f\in L^2([-\pi,\pi])$ then we have $\langle f-S_N(f),e_n\rangle=0\ \ \forall\ |n|\le N$ and

$$\|f-S_N(f)\|_2\le\left\|f-\sum_{|n|\le N}c_ne_n\right\|_2\qquad\forall\ c_n\in\mathbb{C}$$

  √ It tells us that Fourier partial sum is the best approximation for a function in $L^2([-\pi,\pi])$ space among all trigonometric polynomials with same order in the sense of L^2-distance.

- For any $f\in L^2([-\pi,\pi])$ and $\varepsilon>0$, $\exists$ a 2π-periodic continuous function g on $[-\pi,\pi]$ s.t. $\|f-g\|_2<\varepsilon$

- Convergence of Fourier series in L^2
  - For any $f\in L^2([-\pi,\pi])$ we have $\|f-S_N(f)\|_2\to 0$ as $N\to\infty$

- Parseval's identity
  - For any $f\in L^2([-\pi,\pi])$ we have the identity below

$$\|f\|_2^2=\frac{1}{2\pi}\int_{-\pi}^{\pi}|f(x)|^2\,dx=\sum_{n\in\mathbb{Z}}|\hat{f}(n)|^2$$

(Ex) Using Parseval's identity with $f(x)=x$ on $[-\pi,\pi]$, we can show $\sum_{n=1}^{\infty}\frac{1}{n^2}=\frac{\pi^2}{6}$

□ Riemann-Lebesgue Lemma (Easy version)
  - For any $f\in L^2([-\pi,\pi])$ we have $\hat{f}(n)\to 0$ as $|n|\to\infty$

- Plancherel Theorem
  - $\Phi: L^2([-\pi,\pi])\to\ell^2(\mathbb{Z})$ given by $f\mapsto\{\hat{f}(n)\}_{n\in\mathbb{Z}}$ is a linear isometric bijection

7 The space L^p

* L^p space for $p \geq 1$
  - $\mathcal{L}^p := \{f : X \to \mathbb{R} \cup \{\pm\infty\} \mid \int |f|^p \, d\mu < \infty\}$

  √ $\mathcal{L}^p$ is a vector space since $|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p)$ ($\because$ convexity of $x \mapsto |x|^p$ given $p \geq 1$) and $|\alpha f|^p = |\alpha|^p |f|^p \ \ \forall \, \alpha \in \mathbb{R}$
  - $L^p := \mathcal{L}^p / \sim$. $\quad$ L^p-norm is $\|f\|_p = \left(\int |f|^p \, d\mu\right)^{\frac{1}{p}}$

  √ To show L^p-norm is indeed a norm, we need to show the triangle inequality.

* Conjugate exponents
  - For $p, q \geq 1$, p and q are called as conjugate exponents if $\frac{1}{p} + \frac{1}{q} = 1$

- Young's inequality
  - $a, b > 0$ and $1 < p, q < \infty$. If p and q are conjugate exponents then $ab \leq \frac{a^p}{p} + \frac{b^q}{q}$
  - Equality holds iff $a^p = b^q$

- Hölder's inequality
  - If $1 < p, q < \infty$ are conjugate exponents and $f \in L^p$, $g \in L^q$ then $fg \in L^1$ and

$$\|fg\|_1 = \int |fg| \, d\mu \leq \|f\|_p \|g\|_q$$

  - Equality holds iff $\alpha |f|^p = \beta |g|^q$ *a.e.* for some $\alpha, \beta \in \mathbb{R}$ s.t. $(\alpha, \beta) \neq (0, 0)$
    If $\|f\|_p > 0$ and $\|g\|_q > 0$ then $\alpha = \|g\|_q^q$ and $\beta = \|f\|_p^p$

- Minkowski's inequality
  - For $1 < p < \infty$, if $f, g \in L^p$ then $\|f + g\|_p \leq \|f\|_p + \|g\|_p$

  √ L^p-norm is indeed a norm for $p \geq 1$

* L^∞ space
  - For a measurable function $f : X \to \mathbb{R} \cup \{\pm\infty\}$, the essential supremum norm is defined as

$$\|f\|_\infty := \inf\{M > 0 : \mu(\{|f| > M\}) = 0 \quad i.e. \quad |f| \leq M \ \ \mu - a.e.\}$$

  - $\mathcal{L}^\infty := \{f : X \to \mathbb{R} \cup \{\pm\infty\} \mid \|f\|_\infty < \infty\}$ where we say such f is essentially bounded. $\quad$ $\mathcal{L}^\infty$ is a vector space. $\quad$ $L^\infty := \mathcal{L}^\infty / \sim$

- Both Hölder's and Minkowski's inequality can be extended to the case of $p = \infty$
  - (Hölder) If $f \in L^\infty$, $g \in L^1$ then $\|fg\|_1 \leq \|f\|_\infty \|g\|_1$
  - (Minkowski) If $f, g \in L^\infty$ then $\|f + g\|_\infty \leq \|f\|_\infty + \|g\|_\infty$

√ L^∞-norm (essential supremum norm) is indeed a norm on L^∞

- $\|f\|_\infty = \inf\{\sup_{x \in X} |g(x)| : g \in \mathcal{L}^\infty, \ f = g \ \ a.e.\}$

- Riesz-Fisher Theorem
  - For $1 \le p \le \infty$, L^p is a Banach space.
- Approximation in L^p
  - $1 \le p < \infty$. For any $f \in L^p$ and $\varepsilon > 0$, $\exists$ a simple function $\phi \in L^p$ with $\|f - \phi\|_p < \varepsilon$

√ Why L^p spaces ?

- Among all L^p spaces, $p = 1, 2, \infty$ are the special ones.
- L^2 is a Hilbert space, which is easier to anlyze. L^1 is a natural one consisting of all integrable functions.
- One might ask whether we really need to consider L^p spaces for $p \ge 1$ other than $p = 1, 2, \infty$. The answer is Yes.
  i. In Fourier anlysis, there are many operators continuous on L^p for $1 < p < \infty$ but not on L^1 or L^∞, such as the Hilbert transform.
  ii. In probability theory, the most important distribution is the gaussian distribution which belongs to L^p for $1 \le p < \infty$. The second most importnat one could be the p-stable distribution for $0 < p < 2$ which belongs to L^q for $q < p$ but not to L^p. This yields a heavy tailed process.

8 Signed Measures and The Radon-Nikodym Theorem

* Signed measure

- $(X, \mathcal{A})$: a measurable space. A set function $\nu : \mathcal{A} \to \mathbb{R} \cup \{\pm\infty\}$ is said to be a signed measure on $(X, \mathcal{A})$ if
  i. $\nu(\phi) = 0$
  ii. ν assumes at most one of the values $\pm\infty$.
  iii. $\nu(\bigcup_n E_n) = \sum_n \nu(E_n)$ for any disjoint $\{E_n\} \underset{seq}{\subset} \mathcal{A}$, where the RHS sum is absolutely convergent if LHS is finite.
  √ For the third condition, note that when the indices of $\{E_n\}_n$ change, LHS does not change. To prevent the change of RHS, the additional condition is added.

√ Every measure is a signed measure. For emphasizing the difference, we sometimes call a measure as a positive measure.

- μ_1, μ_2 : positive measures on $(X, \mathcal{A})$. If at least one of them is finite measure, then $\nu = \mu_1 - \mu_2$ is a signed measure.

* Extended μ-integrable

- $f : X \to \mathbb{R} \cup \{\pm\infty\}$ is said to be extended μ-integrable if either one of $\int f^+ \, d\mu$ or $\int f^- \, d\mu$ is finite.

- $(X, \mathcal{A}, \mu)$: a measure space. If $f \in L^1(\mu)$ or f is extended μ-integrable, then $\nu : \mathcal{A} \to \mathbb{R} \cup \{\pm\infty\}$ defined by $\nu(E) := \int_E f \, d\mu = \int_E f^+ \, d\mu - \int_E f^- \, d\mu$ is a signed measure. (Signed measure can be understood as a generalization of function in this sense)

- A signed measure satiesfies continuity from above and from below.

* Positive set, Negative set and Null set
  - ν : a signed measure on $(X, \mathcal{A})$. Let $E \in \mathcal{A}$
    i. E is a positive set (w.r.t. ν) if $\nu(F) \geq 0 \ \forall F \subset E, F \in \mathcal{A}$. Denote $E \geq_\nu 0$
    ii. E is a negative set (w.r.t. ν) if $\nu(F) \leq 0 \ \forall F \subset E, F \in \mathcal{A}$ Denote $E \leq_\nu 0$
    iii. E is a null set (w.r.t. ν) if $\nu(F) = 0 \ \forall F \subset E, F \in \mathcal{A}$ Denote $E =_\nu 0$

- Elementary properties of positive sets
  i. If $E \geq_\nu 0$ then for any $F \subset E, F \in \mathcal{A}$, we have $F \geq_\nu 0$
  ii. If $E_n \geq_\nu 0 \ \forall n \in \mathbb{N}$ then $\bigcup_n E_n \geq_\nu 0$

- Hahn decomposition thm
  - If ν is a signed measure on $(X, \mathcal{A})$, then $\exists \ P \geq_\nu 0$, $N \leq_\nu 0$ s.t. $X = P \cup N$ is a partition. The choice of (P, N) is unique up to null sets.
  - √ Uniqueness of Hahn decomposition upto null sets means that if (P, N) and (P', N') are two Hahn decompositions of $(X, \mathcal{A}, \nu)$ then $P \cap N' =_\nu 0$ and $P' \cap N =_\nu 0$

* Mutually singular signed measure
  - Two signed measures μ and ν are said to be mutually signular if $\exists$ a partition $X = E \cup F$ with $E =_\mu 0$ and $F =_\nu 0$. Denote it as $\mu \perp \nu$

- Jordan decomposition thm
  - If ν : a signed measure then, there are unique positive measures ν^+ and ν^- s.t. $\nu = \nu^+ - \nu^-$ and $\nu^+ \perp \nu^-$
  - Given a Hahn decomposition $X = P \cup N$ for ν,
    ν^+ and ν^- are given as $\nu^+(E) = \nu(E \cap P)$ and $\nu^-(E) = \nu(E \cap N)$
  - √ Jordan decomposition is very similar to a unique decomposition of a measurable function $f = f^+ - f^-$ where f^+, f^- are both nonnegative and have disjoint supports.

- Hahn decomposition and Jordan decomposition for ν defined by $\nu(E) = \int_E f \, d\mu$
  - For a measure space $(X, \mathcal{A}, \mu)$ and (extended) μ-integrable f, we have a signed measure ν defined by $\nu(E) = \int_E f \, d\mu$
  - Hahn decomposition for ν is $X = P \cup N$ with $P = \{f \geq 0\}$ and $N = \{f < 0\}$
    (Other possible choice is $P = \{f > 0\}$ and $N = \{f \leq 0\}$)
  - Jordan decomposition for ν is $\nu = \nu^+ - \nu^-$ with
    $\nu^+(E) = \int_E f^+ \, d\mu$ and $\nu^-(E) = \int_E f^- \, d\mu$

- If μ is a positive measure and λ_1, λ_2 are signed measures on $(X, \mathcal{A})$ with $\lambda_1 \perp \mu$, $\lambda_2 \perp \mu$, then $(\lambda_1 + \lambda_2) \perp \mu$

* Total variation of signed measure & Finiteness of signed measure
  - For a signed measure ν, the total variation of ν is defined by the positive measure $|\nu| = \nu^+ + \nu^-$. (This is similar to $|f| = f^+ + f^-$)

- A signed measure ν is said to be finite (or σ-finite) if $|\nu|$ is finite (or σ-finite)

- If ν is a signed measure and μ is a positive measure then

  i. $E =_\nu 0 \Leftrightarrow |\nu|(E) = 0 \quad \forall\, E \in \mathcal{A}$

  ii. $\nu \perp \mu \Leftrightarrow \nu^+ \perp \mu,\, \nu^- \perp \mu \Leftrightarrow |\nu| \perp \mu$

* Absolutely continuity of a signed measure w.r.t. a positive measure.

  - ν is a signed measure and μ is a positive measure on $(X, \mathcal{A})$. We say ν is absolutely continuous with respect to μ if $\mu(E) = 0 \Rightarrow \nu(E) = 0 \ \ \forall\, E \in \mathcal{A}$. Denote it as $\nu \ll \mu$

- If $(X, \mathcal{A}, \mu)$ is a measure space and f is extended μ-integrable function, then the signed measure ν defined by $\nu(E) = \int_E f\, d\mu$ is absolutely continuous w.r.t. μ

- Lebesgue decomposition

  - If μ is a σ-finite positive measure on $(X, \mathcal{A})$, then a σ-finite signed measure ν on $(X, \mathcal{A})$ is uniquely decomposed as $\nu = \nu_1 + \nu_2$ with $\nu_1 \ll \mu$ and $\nu_2 \perp \mu$ where ν_1, ν_2 are signed measures.
  - Moreover, $\exists$ a μ-null set $B \in \mathcal{A}$ s.t. $\nu_1(E) = \nu(E \setminus B),\ \nu_2(E) = \nu(E \cap B) \ \ \forall\, E \in \mathcal{A}$

  √ This decomposition tells us that absolute continuity is, in a sense, "opposite" to mutual singularity.

- Radon-Nikodym thm

  - If μ is a σ-finite positive measure and ν is a σ-finite signed measure on $(X, \mathcal{A})$ s.t. $\nu \ll \mu$, then $\exists$ a unique extended μ-integrable function $g : X \to \mathbb{R}$ satisfying $\nu(E) = \int_E g\, d\mu \ \ \forall\, E \in \mathcal{A}$
  - The function g is called as the Radon-Nikodym derivative of ν w.r.t. μ. Denote as

$$d\nu = g\, d\mu, \quad g = \frac{d\nu}{d\mu}$$

- Conditional expectation

  - $(X, \mathcal{A}, \mu)$: a finite measure space. $\mathcal{B}$: a sub σ-algebra of $\mathcal{A}$. Let $\nu = \mu|_{\mathcal{B}}$.
    Then for any $f \in L^1(X, \mathcal{A}, \mu)$, $\exists\, g \in L^1(X, \mathcal{B}, \nu)$ s.t. $\int_E f\, d\mu = \int_E g\, d\nu \quad \forall\, E \in \mathcal{B}$

  √ For $\mathcal{B} \subset \mathcal{A}$, '$f$ is $\mathcal{A}$-measurable' $\not\Rightarrow$ 'f is $\mathcal{B}$-measurable'. Thus we should find
    $\mathcal{B}$- measurable g satisfying $\int_E f\, d\mu = \int_E g\, d\nu \quad \forall\, E \in \mathcal{B}$

  - Such g is unique in the sense that if $\exists$ another such function g', then $g = g'\ \nu - a.e.$
  - In probability theory, g is called as the conditional expectation of f.
    Denote it as $g = E\,[\, f \,|\, \mathcal{B}\,]$