Faculty of Science

Unit 4: Multiple Integrals

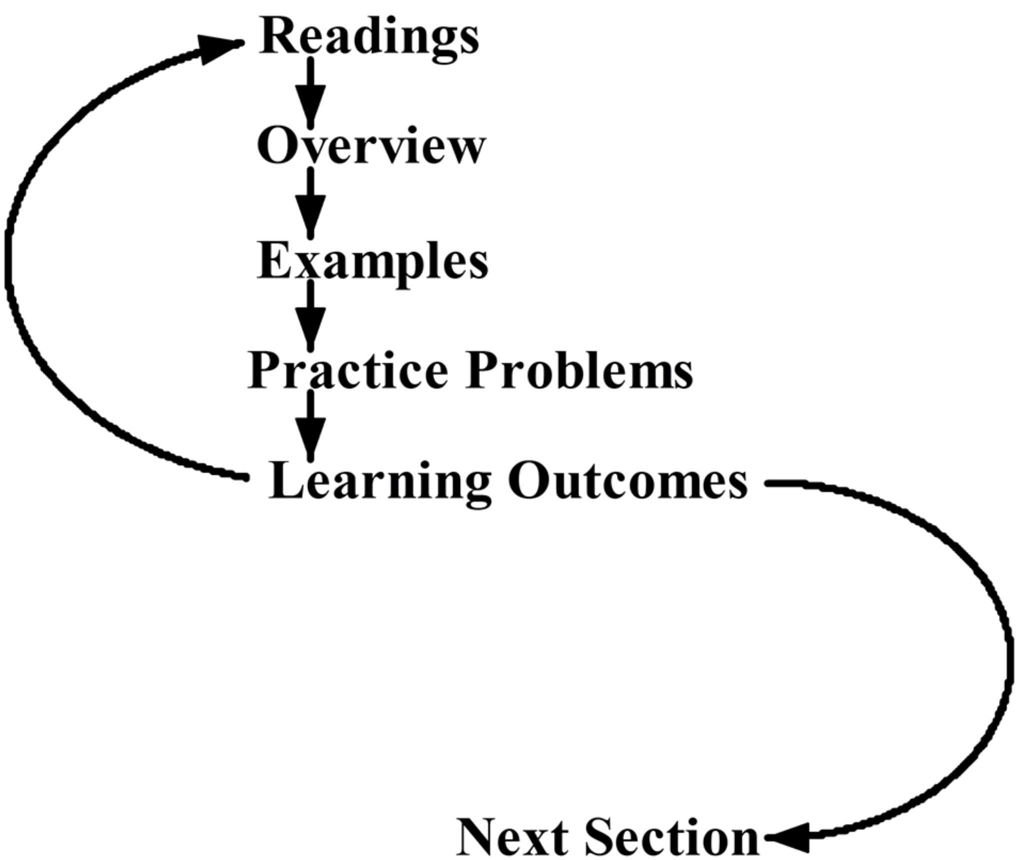
MATH 2111
Calculus III – Multivariable Calculus

Table of Contents

Instructions

The recommended procedure for working through each section of the units in this course is described in detail in your Course Guide.

This procedure is summarized below. If you are certain you have achieved the learning objectives, proceed to the next section. If you are uncertain about one or more of them, go back to the appropriate information in the section until you can complete the task listed in the objective.



Double Integrals Over Rectangles

Learning Outcomes

Upon completion of Double Integrals Over Rectangles, you should be able to:

- Define the double integral as a limit of a double Riemann sum.
- Write down a sufficient condition for a double integral to exist.
- Explain the meaning of an "integrable function".
- Evaluate a double integral using the volume interpretation.
- Estimate the value of a double integral using a double Riemann sum.
- Apply the Midpoint Rule for double integrals.
- Apply the formula for the "average value" of a function of two variables.
- Write down the three properties of a double integral listed on page 1005 of the text.

Readings

Read section 15.1, pages 981–986 in your textbook. Carefully study the examples worked out in the text.

Overview

In elementary calculus, you studied the definite integral and found that it was defined as a limit of a Riemann sum. It is now useful to review the definite integral of a function of one variable. Starting with a function $y = f(x)$ defined on a closed interval $[a, b]$, we subdivide or partition the interval into n subintervals. The partition P is the set of points $P = \{x_0, x_1, x_2, \ldots, x_n\}$ such that

$$a = x_0 < x_1 < x_2 < \ldots < x_{i-1} < x_i < \ldots < x_n = b$$

For convenience we will take the length of the subintervals to be equal and so if $\Delta x = x_i - x_{i-1}$, for any $i = 1, 2, \ldots, n$, then $\Delta x = \dfrac{b-a}{n}$.

Next we form the Riemann sum $S_P = \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$, where x_i^* are sample points chosen from each of the subintervals.

We now define the definite integral, I, in terms of the partition P by

$$I = \lim_{n\to\infty} S_P = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x$$

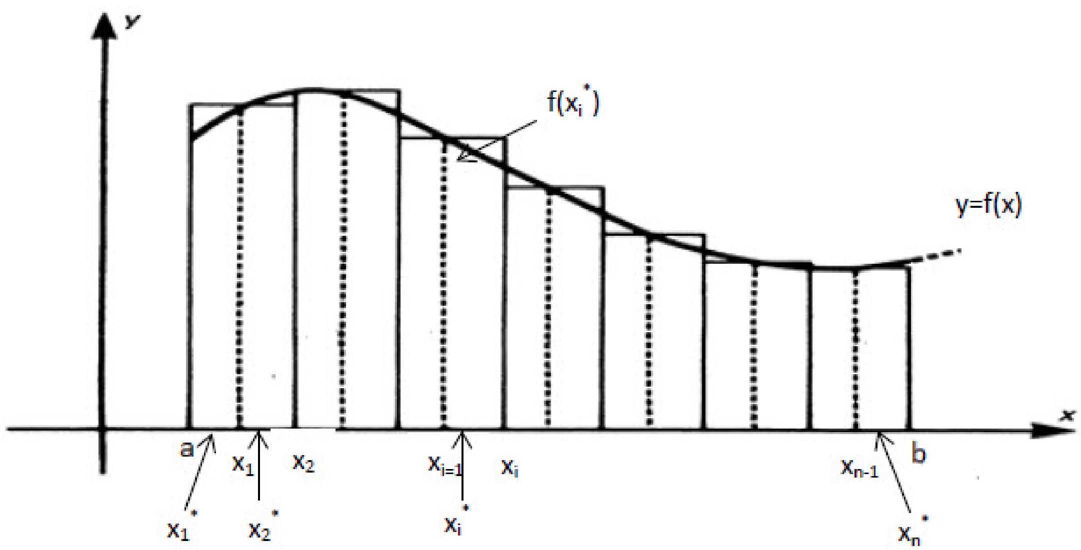
and we write

$$\int_a^b f(x)\,dx = \lim_{n\to\infty} \sum_{i=1}^{n} f\left(x_i^*\right)\Delta x \text{, if this limit exists.}$$

We call $\int_a^b f(x)\,dx$ the definite integral of f from a to b.

It can be shown that the definite integral is independent of the particular partition P and of the choice of sample points x_i^*.

As you can see from the diagram below, when the function $y = f(x)$ is positive on $[a, b]$, the definite integral, which yields the area between $y = f(x)$ and the x-axis over $[a, b]$, is approximated by the Riemann sum. This sum is geometrically a sum of areas of rectangles with bases set on the - x axis of length Δx and height $f\left(x_i^*\right)$.



If the function takes on both positive and negative values on $[a, b]$, then

$$\int_a^b f(x)\,dx = \text{Net Area} = \left(\text{Area above } x\text{-axis}\right) - \left(\text{Area below } x\text{-axis}\right)$$

We are now ready to extend the definite integral definition to a function of two variables $z = f(x, y)$ defined on a closed rectangular region $R = [a,b]\times[c,d] = \left\{(x, y) \middle| a \le x \le b, c \le y \le d\right\}$. Initially we consider $z = f(x, y)$ to be non-negative on R. (See Figure 2 on page 998 of the text.)

Start by diving R into subrectangles as follows: Partition the interval $[a, b]$ into m subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \dfrac{b-a}{m}$; similarly, partition the interval $[c, d]$ into n subintervals $[y_{j-1}, y_j]$ of equal width $\Delta y = \dfrac{d-c}{n}$. The lines parallel to the coordinate axes and through the endpoints of these subintervals form sub-rectangles with equal area $\Delta A = \Delta x \cdot \Delta y$ (see Figure 3 on page 999 of the text). Selecting an arbitrary point $\left(x_i^*, y_j^*\right)$ in each sub-rectangle $R_{i,j}$ we can approximate the volume under the surface $z = f(x, y)$ above $R_{i,j}$ by the volume of a rectangular box with base area ΔA and height $f\left(x_i^*, y_j^*\right)$. The volume of this box is $f\left(x_i^*, y_j^*\right)\Delta A = f\left(x_i^*, y_j^*\right)\Delta x \Delta y$.

Summing up the volumes of all these rectangular boxes is a double Riemann sum approximating the volume V of the solid region bounded above by the surface $z = f(x, y)$ and below by the rectangular region R in the xy-plane; that is,

$$V \approx \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_i^*, y_j^*\right)\Delta x \Delta y$$

Taking the limit as $m, n \to \infty$ gives the volume of the solid region:

$$V = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_i^*, y_j^*\right)\Delta x \Delta y$$

We now define the **double integral of** f **over** R, denoted $\iint\limits_R f(x, y)\, dA$ to be

$$\iint\limits_R f(x, y)\, dA = \lim_{m, n \to \infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_i^*, y_j^*\right)\Delta x \Delta y$$

if the limit exists. In the case the limit exists the function is said to be **integrable**. All continuous functions are examples of integrable functions.

Note: In the case the multivariable function takes on both positive and negative values on R,

$$\iint\limits_R f(x, y)\, dA = \text{Net Volume} = \left(\text{Volume above } xy\text{-plane}\right) - \left(\text{Volume below } xy\text{-plane}\right)$$

At this stage calculations of double integrals are restricted to simple non-negative functions where you can use the volume interpretation of the double integral. The extension of the Fundamental Theorem of Calculus to double integrals, called Fubini's Theorem, is discussed in the next section.

Methods used for approximating single integrals, the Midpoint Rule, the Trapezoid Rule and Simpson's Rule, extend to double integrals. Stewart discusses only the Midpoint Rule on page 1002 of the text.

The concept of the average value of a single variable function on a closed interval extends naturally to this setting. If $y = f(x)$ is a continuous function defined on the interval $[a, b]$, then

$$f_{ave} = \frac{1}{b-a}\int_a^b f(x)\,dx$$

In comparison, if $z = f(x, y)$ is a continuous function defined on the rectangular region R, we define the **average value** of f on R to be

$$f_{ave} = \frac{1}{\text{Area}(R)}\iint_R f(x,y)\,dA$$

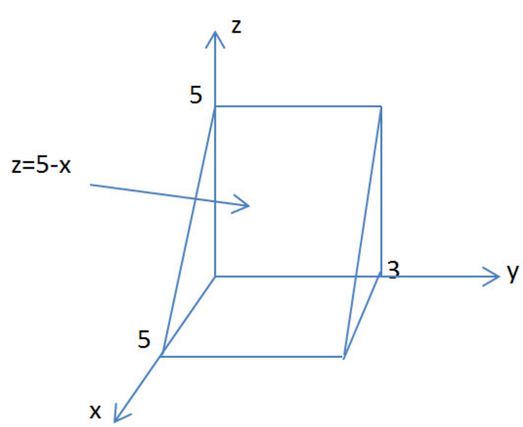
The "nice" properties of the definite integral you learned in your integral calculus course extend to double integrals. Three of these properties are given on page 1005 of the text. You need to know them.

Example Exercises

1. This is problem 12 on page 1006 of the text.

   Evaluate $\iint_R (5-x)\,dA$, where $R = [0,5]\times[0,3] = \{(x,y) \mid 0 \le x \le 5,\ 0 \le y \le 3\}$,

   using the volume interpretation of the double integral.



From the diagram we see that $z=5-x$ is a non-negative function on R. The region bounded by above by $z=5-x$ and below by R has triangular cross-sections parallel to the xz-plane.

Hence,

$$\iint_R (5-x)\,dA = \text{volume of region} = (\text{Area of triangle})\cdot \text{Width} = \frac{1}{2}\cdot 5\cdot 5\cdot 3 = 37.5$$

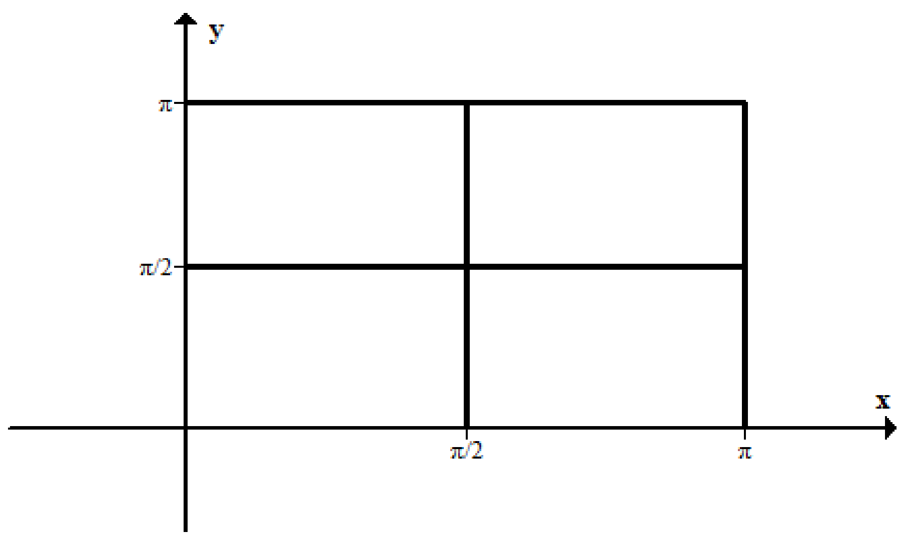
Stewart, J. (2012). Multivariable calculus (7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.

2. a) Use a Riemann sum with $m=n=2$ to estimate the value of $\iint_R \sin(x+y)\,dA$, where $R=[0,\pi]\times[0,\pi]$. Take the sample points to be the lower left corners.

   b) Use the Midpoint Rule to estimate the double integral in part (a).

Solution:

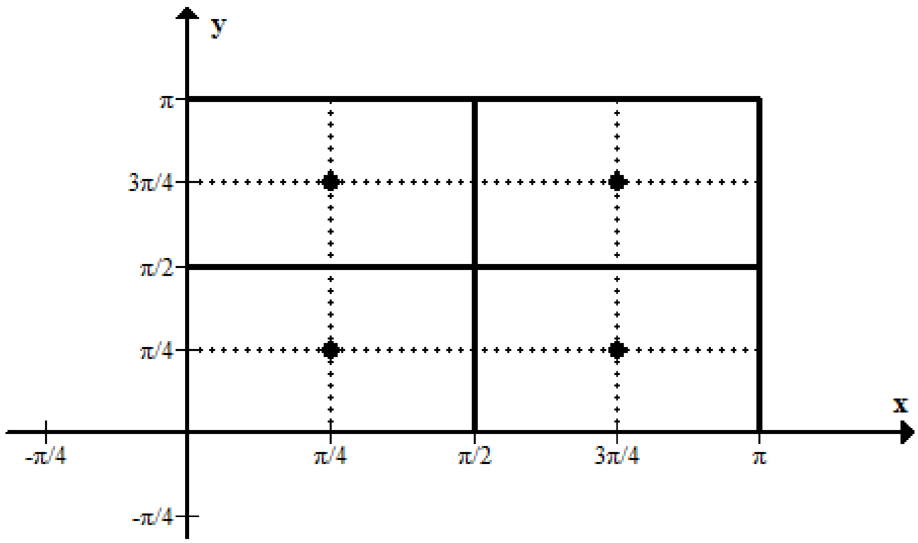
a) The sub-rectangles are shown in the figure that follows.



$$\Delta x = \frac{\pi-0}{2} = \frac{\pi}{2};\ \Delta y = \frac{\pi-0}{2} = \frac{\pi}{2};\ \text{So, } \Delta A = \Delta x\cdot\Delta y = \frac{\pi}{2}\cdot\frac{\pi}{2} = \frac{\pi^2}{4}$$

$$\iint_R \sin(x+y)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(x_i^*, y_j^*\right)\Delta A$$
$$= f(0,0)\Delta A + f(0,\pi/2)\Delta A + f(\pi/2,0)\Delta A + f(\pi/2,\pi/2)\Delta A$$
$$= [f(0,0) + f(0,\pi/2) + f(\pi/2,0) + f(\pi/2,\pi/2)]\Delta A$$
$$= [0+1+1+0]\cdot\frac{\pi^2}{4}$$
$$= \frac{\pi^2}{2} \approx 4.935$$

b) The midpoints of the sub-rectagles are pictured below.



Using the Midpoint Rule we have

$$\iint_R \sin(x+y)\,dA \approx \sum_{i=1}^{2}\sum_{j=1}^{2} f\left(\bar{x}_i, \bar{y}_j\right)\Delta A$$
$$= [f(\pi/4,\pi/4) + f(\pi/4,3\pi/4) + f(3\pi/4,\pi/4) + f(3\pi/4,3\pi/4)]\Delta A$$
$$= [1+0+0+(-1)]\cdot\frac{\pi^2}{4}$$
$$= 0$$

Practice Exercises 15.1

From the text pages 1005–1006, do problems 1, 3, 5, 7, 9, 11, 13, and 17.

Remember: You are to attempt all of the problems carefully before checking your solutions against those given in the solutions manual.

Iterated Integrals

Learning Outcomes

Upon completion of Iterated Integrals, you should be able to:

- Perform the anti-partial differentiation process.
- Evaluate iterated integrals over rectangular regions.
- Evaluate double integrals as iterated integrals.
- Sketch the solid region associated to a double integral or iterated integral expression.
- Use a double integral to calculate volumes of solid regions bounded above by a surface and below by a rectangle in the xy-plane.

Readings

Read section 15.2, pages 1006–1011 in your textbook. Carefully study the examples worked out in the text.

Overview

The first section introduced evaluation of the double integral, but only using the volume interpretation of the double integral for simple non-negative functions. This section extends the theory to more general functions, but the limits of integration are still restricted to constants, that is, we are still integrating over rectangular regions. In the next section that final restriction will be lifted.

The technique is analogous to that used in partial differentiation: One of the variables is 'frozen', that is, treated as if it were a constant, and you integrate over the other variable between its limits. That accomplished and the corresponding integral now replaced by a number, you now 'unfreeze' the remaining variable and complete the integration. Also in analogy to partial differentiation, the order of integration is immaterial for nice functions.

In multiple integration, more than ever, it is worthwhile drawing pictures of what you are doing. You might visualize the above as slicing up a loaf of bread that has been baked in a rectangular tin with dimensions $b-a$ by $d-c$. We can view the bottom of the baking tin as the reference plane; the upper crust of the loaf has a variable height z that is a function of its position (x, y) in the plane. The process of 'freezing' a variable is equivalent to slicing the loaf of bread, determining the area of the slice, multiplying that by the slice thickness to get its volume, then adding up all the volumes of the slices to get the total volume of the loaf. It doesn't matter whether we slice the loaf across its width to make traditional bread slices or across its length, the volume either way is the same.

You are now ready to learn how to evaluate double integrals over rectangles and, subsequently, over more general regions. Iterated integrals are defined as follows (see page 1007 in the text):

$$\int_a^b \int_c^d f(x,y)\,dy\,dx = \int_a^b \left[\int_c^d f(x,y)\,dy \right] dx$$

Note that in the integration $\int_c^d f(x,y)\,dy$, you freeze the variable x and integrate with respect to y only. You can think of the process as "anti-partial differentiation." After you evaluate this expression in terms of the given limits for y you obtain a definite integral of a function of x which is evaluated in the usual way.

$$\int_c^d \int_a^b f(x,y)\,dx\,dy = \int_c^d \left[\int_a^b f(x,y)\,dx \right] dy$$

Similarly in the integration $\int_a^b f(x,y)\,dx$ you freeze the variable y and integrate with respect to x only. After you evaluate this expression in terms of the given limits for x you obtain a definite integral of a function of y which is evaluated in the usual way.

Recall that if $f(x,y) \geq 0$ on R, the double integral represents volume. Now I will show you how to evaluate double integrals by using iterated integrals. First, return to the iterated integral

$$\int_a^b \int_c^d f(x,y)\,dy\,dx$$

and consider $\int_c^d f(x,y)\,dy$, where the integration is performed by keeping the variable x frozen. Recall that if $z = f(x,y)$, then $z = f(x_0, y)$ is a "slice function".

Hence, the integral

$$\int_c^d f(x,y)\,dy$$

provides the area of this slice which we represent as $A(x)$; that is,

$$A(x) = \int_c^d f(x,y)\,dy$$

Now the volume of the slice can be taken as $dV = A(x)\,dx$, so that the volume of the solid is

$$V = \int_a^b A(x)dx = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

That is,

$$\iint_R f(x,y)\,dA = \int_a^b \int_c^d f(x,y)\,dy\,dx$$

Although the above reasoning is plausible, it does not constitute a formal proof. Most textbooks at this level use plausibility arguments and so does Stewart. The results are summarized as Fubini's theorem in box 4 on page 1008 of the text.

Example Exercises

1. Evaluate $\int_2^3 \int_0^1 \left(x^2 + xy - y^2\right) dy\,dx$

$$\int_2^3 \int_0^1 \left(x^2 + xy - y^2\right) dy\,dx = \int_2^3 \left[\int_0^1 \left(x^2 + xy - y^2\right) dy \right] dx$$

Now treating x as a constant we have that

$$\int_0^1 \left(x^2 + xy - y^2\right) dy = \left[x^2 y + \frac{xy^2}{2} - \frac{y^3}{3} \right]_0^1$$

$$= x^2(1) + \frac{x(1)^2}{2} - \frac{(1)^3}{3} - \left(x^2(0) + \frac{x(0)^2}{2} - \frac{(0)^3}{3} \right)$$

$$= x^2 + \frac{x}{2} - \frac{1}{3}$$

So,

$$\begin{aligned}\int_2^3\int_0^1 \left(x^2+xy-y^2\right)dy\,dx &= \int_2^3\left(x^2+\frac{x}{2}-\frac{1}{3}\right)dx\\ &= \left[\frac{x^3}{3}+\frac{x^2}{4}-\frac{x}{3}\right]_2^3\\ &= \left[\frac{3^3}{3}+\frac{3^2}{4}-\frac{3}{3}\right]-\left[\frac{2^3}{3}+\frac{2^2}{4}-\frac{2}{3}\right]\\ &= \frac{29}{4}\end{aligned}$$

Let's return to the original question and integrate first with respect to x. We expect the value of the integral to remain unchanged. Reversing the order of integration we have

$$\begin{aligned}\int_0^1\int_2^3\left(x^2+xy-y^2\right)dx\,dy &= \int_0^1\left[\frac{x^3}{3}+\frac{x^2y}{2}-xy^2\right]_2^3 dy\\ &= \int_0^1\left[\frac{3^3}{3}+\frac{(3)^2y}{2}-(3)y^2\right]-\left[\frac{2^3}{3}+\frac{(2)^2y}{2}-(2)y^2\right]dy\\ &= \int_0^1\left(-y^2+\frac{5}{2}y+\frac{19}{3}\right)dy\\ &= \left[-\frac{y^3}{3}+\frac{5y^2}{4}+\frac{19}{3}y\right]_0^1\\ &= -\frac{1}{3}+\frac{5}{4}+\frac{19}{3}\\ &= \frac{29}{4}\end{aligned}$$

You notice that both answers are the same. This always happens if f is continuous on R.

A note on notation: since the order of multiple integration is from the inside out, the limits of integration a and b exchange places with c and d respectively, and dx and dy also exchange places.

2. Evaluate $\iint_R \frac{1}{x+y}\,dA$, where $R=[1,2]\times[0,2]$.

Since x and y appear symmetrical in the integrand, the choice of which one is to be frozen first will not affect the ease of integration. We arbitrarily freeze x. So

$$\begin{aligned}\iint_R \frac{1}{x+y}\,dA &= \int_1^2\int_0^2 \frac{1}{x+y}\,dy\,dx\\ &= \int_1^2 \left[\ln(x+y)\right]_0^2 dx\\ &= \int_1^2 \left[\ln(x+2)-\ln(x)\right]dx\end{aligned}$$

Using the integration formula $\int \ln(x)dx = x\ln(x)-x+C$ we have

$$\begin{aligned}\iint_R \frac{1}{x+y}\,dA &= \left[\big((x+2)\ln(x+2)-(x+2)\big)-\big(x\ln(x)-x\big)\right]_1^2\\ &= \left[(x+2)\ln(x+2)-x\ln(x)-2\right]_1^2\\ &= 4\ln(4)-2\ln(2)-3\ln(3)\\ &= 8\ln(2)-2\ln(2)-3\ln(3)\\ &= 6\ln(2)-3\ln(3)\end{aligned}$$

Practice Exercises 15.2

From the text pages 1011–1012, do problems 1, 3, 9, 17, 19, 23, 27, 31, 35, and 37.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Double Integrals over General Regions

Learning Outcomes

Upon completion of Double Integrals over General Regions, you should be able to:

- Determine if a region is x-simple (type II) or y-simple (type I).
- Set up and evaluate iterated integrals over simple regions.
- Sketch the solid whose volume is given by an iterated integral expression.
- Sketch the region of integration associated to an iterated integral expression.
- Use double integrals to calculate volumes of solid regions.
- Evaluate an iterated integral expression by reversing the order of integration.
- Write down and apply the properties of double integrals as listed on pages 1017 and 1018 of the text.
- Use geometry or symmetry to evaluate a double integral.

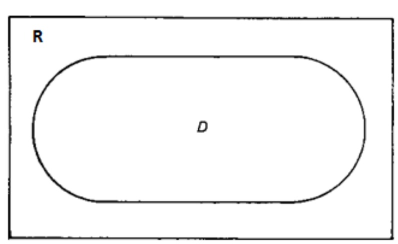
Readings

Read section 15.3, pages 1012–1019 in your textbook. Carefully study the examples worked out in the text.

Overview

In the first two sections of this unit, we defined and evaluated the double integral over rectangular regions. We now extend the definition to more general regions of the xy-plane. (See pages 1012 and 1013 of the text.)

Suppose $z = f(x, y)$ is defined on a bounded region D. By "bounded", we mean that D can be enclosed in a rectangular region R (see diagram below).



A new function F can now be defined on R as follows:

For all (x, y) in R, $F(x,y)=\begin{cases} f(x,y) & \text{if } (x,y) \text{ is in } D \\ 0 & \text{if } (x,y) \text{ is outside } D \end{cases}$

If F is integrable over R, we define the **double integral of f over D** by

$$\iint_D f(x,y)\,dA = \iint_D F(x,y)\,dA$$

Since the double integral is defined as a double Riemann sum, we can immediately see that if the sample point $\left(x_i^*, y_j^*\right)$ is in D, then we have a box with base area ΔA and height $f\left(x_i^*, y_j^*\right)$ giving volume $f\left(x_i^*, y_j^*\right)\cdot \Delta A$. However if the sample point lies outside of D in R, then $f\left(x_i^*, y_j^*\right)=0$ and the box has zero volume. Therefore, the volume of the box makes no contribution to the Riemann sum.

Knowing that a double integral exists and being able to evaluate it are two very different problems. It turns out that if the region D is y-simple (called **type I** in the text) or x-simple (called **type II** in the text), then the double integral can be evaluated, as in the case of a rectangular region, by an iterated integral expression. We use the term y-simple to describe a region whose boundaries in x are constant and whose boundaries in y are functions of x. That is, the region is bounded by two graphs of the form $y = f(x)$. The term x-simple is defined similarly. (See Figure 5 on page 1013 of the text for pictures of y-simple regions.)

Specifically, if f is continuous on a y-simple region

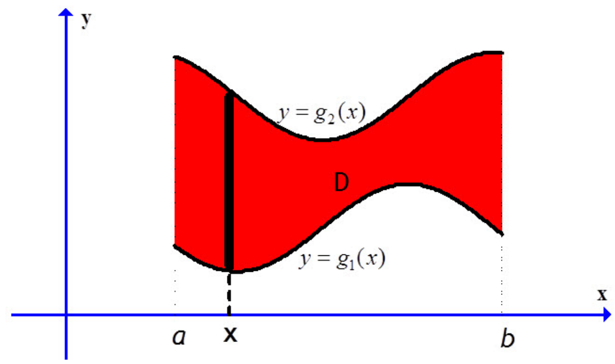
$$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\},$$

Then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy\,dx$$

Note: If the region D is y-simple, we integrate with respect to the y-variable first. If we draw a vertical strip in the region D (see diagram), then the iterated integral looks like this:

$$\int_{\substack{\text{Position of first}\\ \text{vertical strip}}}^{\substack{\text{Position of last}\\ \text{vertical strip}}} \int_{\substack{\text{bottom curve}\\ \text{where strip starts}}}^{\substack{\text{top curve}\\ \text{where strip ends}}} f(x, y)\, dy dx$$



It is important to note that while the limits of integration for the first integral may be variable, the limits for the outer integral <u>must be constant</u>.

Similarly, if f is continuous on an x-simple region

$$D = \{(x, y) \mid c \le y \le d\,,\, h_1(y) \le x \le h_2(y)\}\,,$$

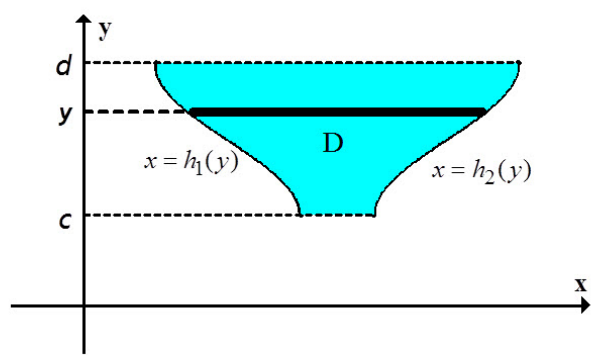
Then

$$\iint_D f(x, y)\, dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\, dx dy$$

(see Figure 7 on page 1014 of the text for pictures of x-simple regions)

Note: If the region D is x-simple, we integrate with respect to the x-variable first. If we draw a horizontal strip in the region D (see diagram), then the iterated integral looks like this:

$$\int_{\substack{\text{Position of first}\\ \text{horizontal strip}}}^{\substack{\text{Position of last}\\ \text{horizontal strip}}} \int_{\substack{\text{left curve}\\ \text{where strip starts}}}^{\substack{\text{right curve}\\ \text{where strip ends}}} f(x, y)\, dx dy$$



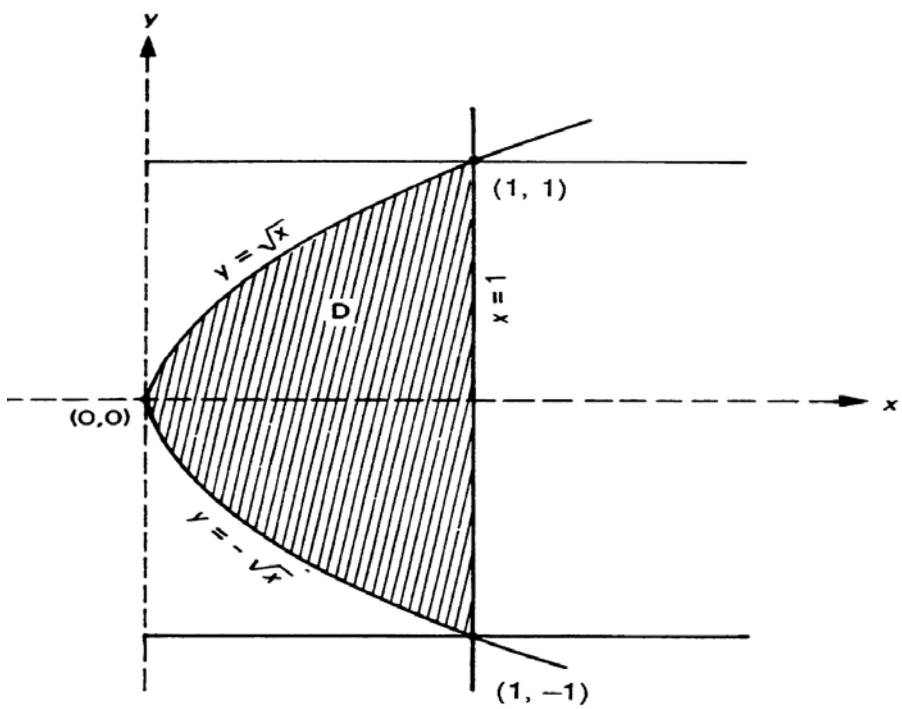
Again, it is important to note that while the limits of integration for the first integral may be variable, the limits for the outer integral <u>must be constant</u>.

A more extensive list of properties of the double integral, than previously introduced over rectangular regions, are given on pages 1017 and 1018 of the text for a general region D. Study these properties.

Example Exercises

1. Evaluate $\iint_D \left(x - y^2\right) dA$, where D is the region bounded by $x = y^2$ and $x = 1$.

   We start with a sketch of the region D.



Notice that the region D is not *x*-simple, but rather a union of two *x*-simple regions, the region above the $x-\text{axis}$ and the region below. Viewing D this way would require two integral expressions to evaluate the double integral, as described by the property of integrals in box 9 on page 1018 of the text. It is simpler to view D as *y*-simple.

Viewing D as a *y*-simple region we have

$$\iint_D \left(x - y^2\right) dA = \int_0^1 \int_{-\sqrt{x}}^{\sqrt{x}} \left(x - y^2\right) dy\, dx$$

$$= \int_0^1 \left[xy - \frac{y^3}{3} \right]_{-\sqrt{x}}^{\sqrt{x}} dx$$

$$= \int_0^1 \left[\left(x^{3/2} - \frac{x^{3/2}}{3} \right) - \left(-x^{3/2} + \frac{x^{3/2}}{3} \right) \right] dx$$

$$= \int_0^1 2\left(x^{3/2} - \frac{x^{3/2}}{3} \right) dx$$

$$= \int_0^1 \frac{4}{3} x^{3/2}\, dx$$

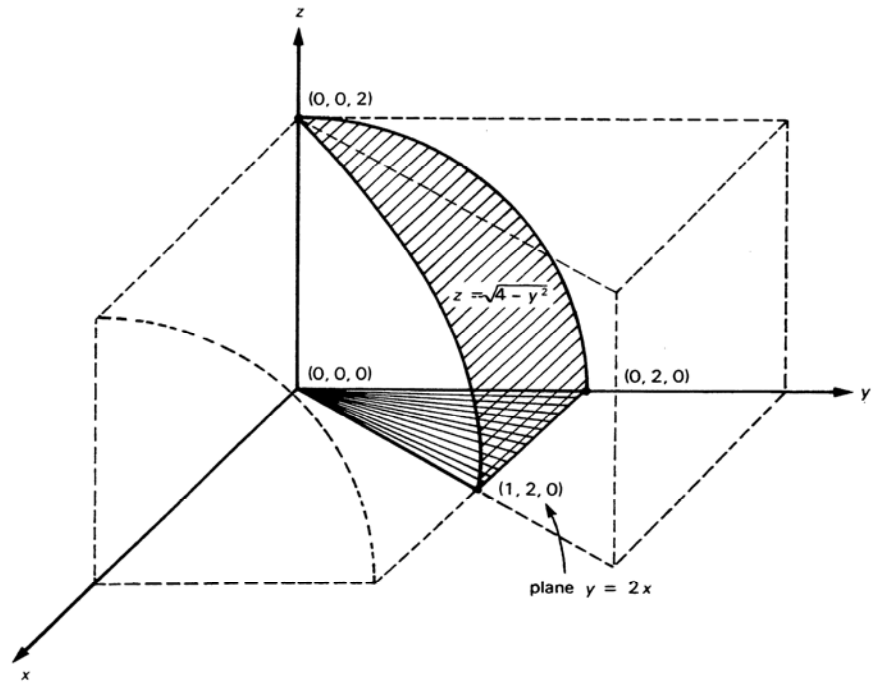
$$= \left[\frac{4}{3} \cdot \frac{2}{5} x^{5/2} \right]_0^1$$

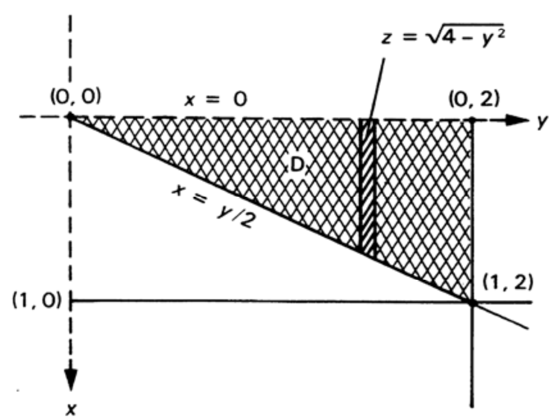
$$= \frac{8}{15}(1 - 0)$$

$$= \frac{8}{15}$$

2. Find the volume of the solid in the first octant bounded by the cylinder $y^2 + z^2 = 4$, the plane $y = 2x$ and the yz-plane

We start with a sketch of the solid and its region D of integration in the xy-plane.





Note the orientation of the axes in the xy-plane for the region D.

Notice that the solid region is bounded above by $z = \sqrt{4 - y^2}$ and below by the shaded trianular region D. Since the function $z = \sqrt{4 - y^2}$ is non-negative over D, the volume of the solid region is represented by $\iint\limits_D z\, dA = \iint\limits_D \sqrt{4 - y^2}\, dA$.

Looking at the integrand function we see that it is easier to integrate with respect to x first, so we want to view our region D as x-simple, which it is; that is,

$$D = \{(x, y) \mid 0 \le y \le 2,\ 0 \le x \le y/2\}$$

Hence, the volume V of the solid is:

$$\begin{aligned}
V &= \iint_D \sqrt{4-y^2}\, dA = \int_0^2 \int_0^{y/2} \sqrt{4-y^2}\ dx\, dy \\
&= \int_0^2 \sqrt{4-y^2}\, [x]_0^{y/2}\, dy \\
&= \int_0^2 \sqrt{4-y^2}\,(y/2-0)\, dy \\
&= -\frac{1}{4}\int_0^2 -2y\sqrt{4-y^2}\, dy \\
&= -\frac{1}{4}\cdot\frac{2}{3}\left(4-y^2\right)^{3/2}\Big]_0^2 \\
&= -\frac{1}{6}\left(0-4^{3/2}\right) = -\frac{1}{6}(-8) = \frac{4}{3}
\end{aligned}$$

3. This is problem 50 on page 1020 of the text.

Evaluate the iterated integral expression below by reversing the order of integration.

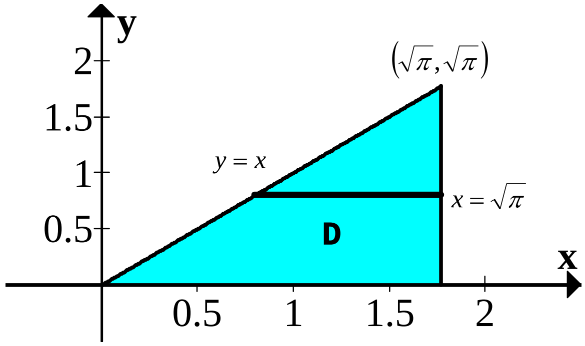
$$\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos\left(x^2\right) dx\, dy$$

Notice that we cannot find an antiderivative of the integrand function by integrating with respect to x first, so to evaluate this integral expression we must reverse the order of integration. To do this we need a picture of the region of integration D.

From the limits of integration we have that

$$D = \{(x, y) \mid 0 \le y \le \sqrt{\pi},\ y \le x \le \sqrt{\pi}\}$$

A picture of the region is drawn below:



Viewing D as y-simple we have $D = \{(x, y) \mid 0 \le x \le \sqrt{\pi}, 0 \le y \le x\}$

Hence,

$$\begin{aligned}
\int_0^{\sqrt{\pi}} \int_y^{\sqrt{\pi}} \cos\left(x^2\right) dx\, dy &= \int_0^{\sqrt{\pi}} \int_0^{x} \cos\left(x^2\right) dy\, dx \\
&= \int_0^{\sqrt{\pi}} \cos\left(x^2\right) \left[y\right]_0^x \, dx \\
&= \int_0^{\sqrt{\pi}} \cos\left(x^2\right) (x - 0)\, dx \\
&= \frac{1}{2} \int_0^{\sqrt{\pi}} 2x \cos\left(x^2\right) dx \\
&= \frac{1}{2} \left[-\sin\left(x^2\right)\right]_0^{\sqrt{\pi}} \\
&= -\frac{1}{2} \left(\sin(\pi) - \sin(0)\right) \\
&= 0
\end{aligned}$$

Stewart, J. (2012). Multivariable calculus (7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.

Practice Exercises 15.3

From the text pages 1019–1021, do problems 5, 11, 15, 19, 21, 25, 31, 35, 37, 47, 53, 55, 59. and 63.

Remember: You are to attempt all of the problems carefully **before** checking your solutions against those given in the solutions manual.

Polar Coordinates

Learning Outcomes

Upon completion of Polar Coordinates, you should be able to:

- Plot points in the polar coordinate system.
- Convert polar coordinates into rectangular coordinates and vice versa.
- Sketch polar curves.
- Recognize the polar equation and name the polar curve called "cardioid" and "n-leaved rose".
- Recognize and write the polar equation of a circle centered on the x- or y-axis.
- Write the polar equation of a line through the pole.

Readings

Read section 10.3, pages 678–683 (middle) in your textbook. Carefully study the examples worked out in the text.

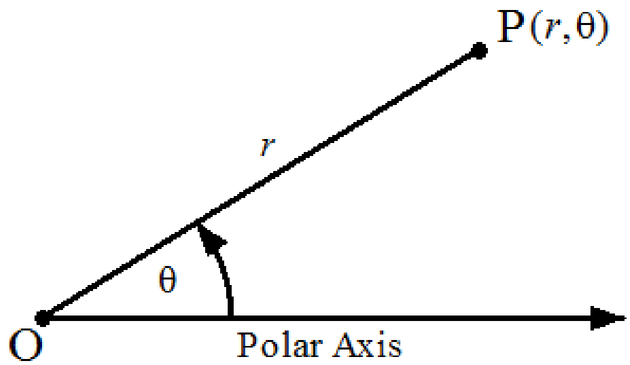
Notice that we are omitting some of the pages from this section. Our focus here is familiarity with polar coordinates, polar graphs and transformations between the rectangular (i.e., Cartesian) and the polar systems.

Overview

Before we discuss our next calculus topic, you need to review, or study, if this is your first exposure to polar coordinates, an alternate system to the Cartesian coordinate system for representing points in the plane by an ordered pair of numbers. This new system is called the **polar coordinate system**.

It is often the case that a region may be easy enough to graph, but neither x-simple nor y-simple. Then the evaluation of a double integral can become quite challenging, even though the function is integrable. The polar coordinate system may be used to simplify the description of certain regions of integration. Instead of describing a set of points in terms of the x- and y-axes, we may describe them by their distance from the origin, r, and the angle formed by the polar axis and the ray joining the point to the origin, θ. By doing so, we can describe certain boundary curves as functions not of x and y, but of r and θ. If a region is easy to describe using polar coordinates, then we may evaluate an integral associated with this region by changing the integral to polar coordinates.

In this system a point P in the plane has **polar coordinates** (r, θ), where r is the "directed distance" from the origin O of the system called the pole and θ is the "directed angle" between OP and the polar axis. (See diagram below.)



To plot the point $P(r, \theta)$ in the polar coordinate system, proceed as follows:

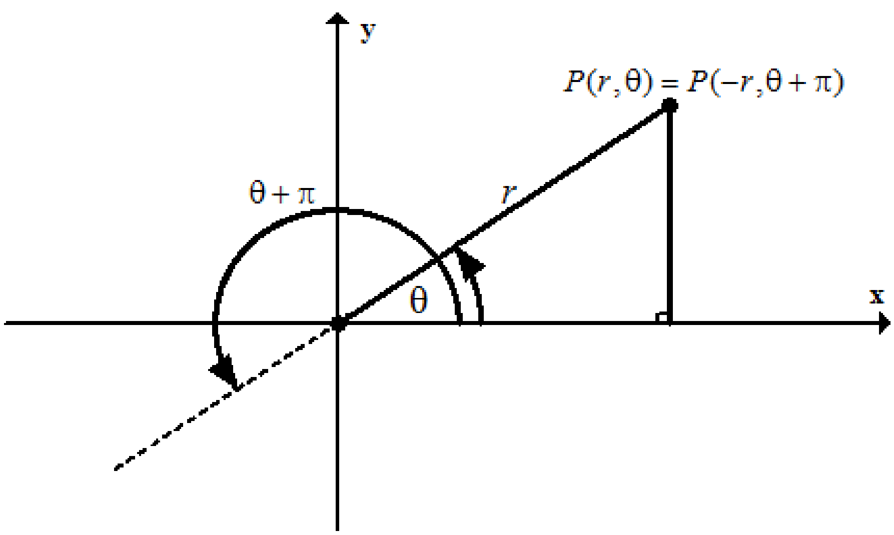
Step 1: "Stand" on the pole, i.e., the origin, and "face" in the direction of the polar axis.

Step 2: "Turn" through the angle θ starting on the polar axis and moving counter-clockwise if $\theta > 0$ and clockwise if $\theta < 0$.

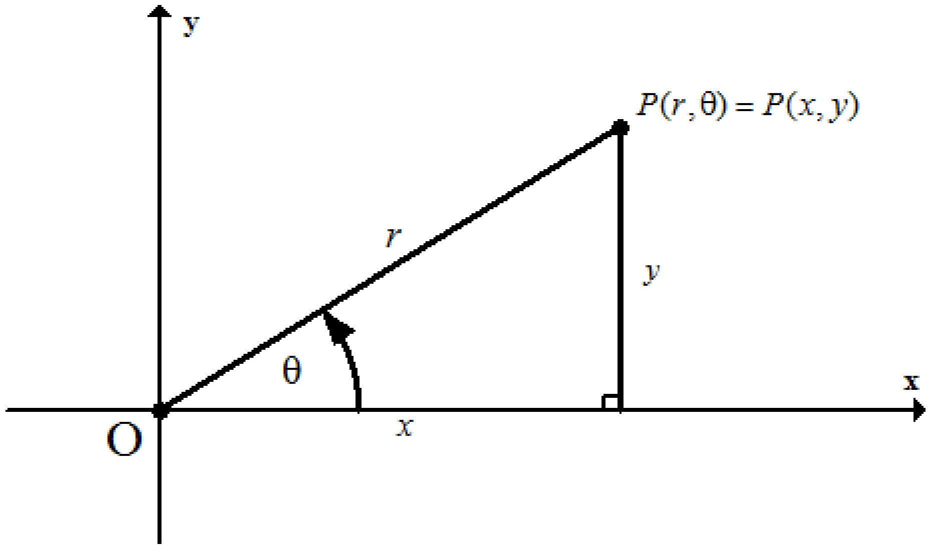
Step 3: "Move" out a distance r, forward if $r > 0$ and backward if $r < 0$

Notice that the polar coordinates of a point in the plane are not unique. If a point P has polar coordinates (r, θ), it also has polar coordinates.

$$(r, \theta + 2k\pi) \text{ and } (-r, \theta + k\pi), \text{ where } k \text{ is an integer}$$



The diagram below describes the relationship between the polar coordinate system and the familiar Cartesian coordinate system.



Using Pythagoras Theorem and basic trigonometric relationships, we have the following:

$$x = r\cos(\theta) \qquad r^2 = x^2 + y^2$$
$$\text{and}$$
$$y = r\sin(\theta) \qquad \tan(\theta) = y/x$$

So, though each (r,θ) corresponds to a unique (x, y), each (x, y) corresponds to many possible (r,θ) pairs.The **graph** of a polar equation is defined as the plot of the set of all points that have <u>at least one representation</u> (r,θ) whose coordinates satisfy the equation.

In the Cartesian coordinate system, the graphs of the basic equations $x = c$ or $y = c$, where c is a constant, describe vertical and horizontal lines; that is, lines parallel to the coordinate axes. In the polar system $\theta = c$, where c is a constant, describes a line through the pole with slope $\tan(\theta)$, whereas $r = c$, where c is a constant, describes a circle centered at the pole of radius $|r|$.

On pages 681–682 of the text Stewart does a number of examples of graphing some of the familiar polar curves. Study these examples carefully. It is important to be able to readily recognize these special curves which are described by very simple polar equations, but have more complicated equations in Cartesian coordinates. Examples 7 and 8 discuss the **cardioid** and the **n-leaved rose**.

Notice the format of the polar equations of circles with centers that lie on the x or y -axis:

$r = a\cos(\theta)$ describes a circle with centre at $\left(\frac{a}{2}, 0\right)$ and diameter a. This circle is generated for $0 \le \theta \le \pi$.

$r = a\sin(\theta)$ describes a circle with centre at $\left(0, \frac{a}{2}\right)$ and diameter a. This circle is generated for $0 \le \theta \le \pi$.

It is important to note here that these circles do not require the interval $0 \le \theta \le 2\pi$ to generate the full graph, as is the case for many polar curves that involve the sine and cosine function.

Example Exercises

1. a) Convert $(r.\theta) = \left(2, \frac{5\pi}{6}\right)$ from polar to rectangular coordinates. Give an exact answer.

$$x = r\cos(\theta) = 2\cos\left(\frac{5\pi}{6}\right) = 2\left(-\frac{\sqrt{3}}{2}\right) = -\sqrt{3}$$

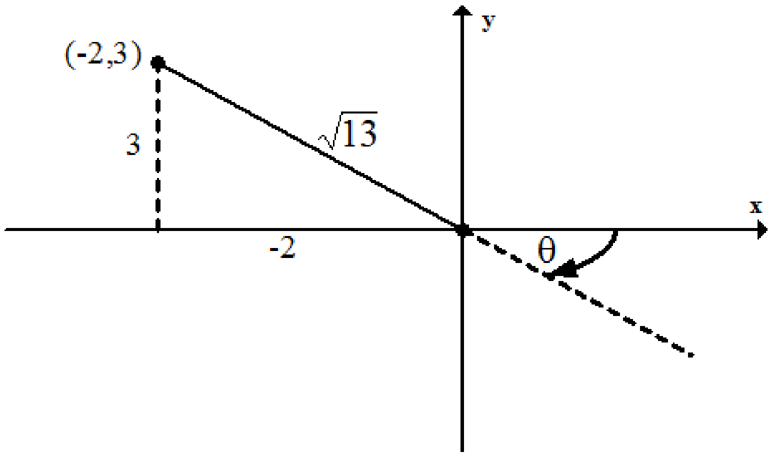
$$y = r\sin(\theta) = 2\sin\left(\frac{5\pi}{6}\right) = 2\left(\frac{1}{2}\right) = 1$$

So, $(x, y) = (-\sqrt{3}, 1)$

b) Convert $(x, y) = (-2, 3)$ from rectangular to polar coordinates. Give an exact answer.

$$r^2 = x^2 + y^2 = (-2)^2 + 3^2 = 13 \quad \Rightarrow \quad r = \pm\sqrt{13}$$

$$\tan(\theta) = \frac{3}{-2} = -\frac{3}{2} \quad \Rightarrow \quad \theta = \tan^{-1}\left(-\frac{3}{2}\right) \approx -0.98 \text{ rads}$$



From the picture above we see that one exact set of coordinates is $(r,\theta)=\left(-\sqrt{13},\ \tan^{-1}(-3/2)\right)$.

Another set would be $(r,\theta)=\left(\sqrt{13},\ \tan^{-1}(-3/2)+\pi\right)$.

2. This is problem 20 on page 687 of the text.

Identify the polar curve $r=\tan(\theta)\sec(\theta)$ by finding a Cartesian equation for the curve.

$$r=\tan(\theta)\sec(\theta)$$

$$r=\tan(\theta)\cdot\frac{1}{\cos(\theta)}$$

$$r\cos(\theta)=\tan(\theta)$$

Now, $r\cos(\theta)=x$ and $\tan(\theta)=y/x$

So, $r\cos(\theta)=\tan(\theta) \quad\Rightarrow\quad x=\dfrac{y}{x} \quad\Rightarrow\quad y=x^2$

The curve is a parabola with vertex at the origin and opening in the direction of the positive y-axis.

Stewart, J. (2012). Multivariable calculus (7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.

Practice Exercises 10.3

From the text pages 686–687, do problems 1, 3, 5, 9, 11, 17, 19, 23, 25, 27, 31, 33, 37, 41, and 45.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Areas and Lengths in Polar Coordinates

Learning Outcomes

Upon completion of Areas and Lengths in Polar Coordinates, you should be able to:

- Determine the area enclosed by a polar curve.
- Find the area that lies inside one region of a polar curve and outside the other curve, or inside both curves.
- Find the length of a polar curve.
- Find the simultaneous intersection points of two polar curves.

Readings

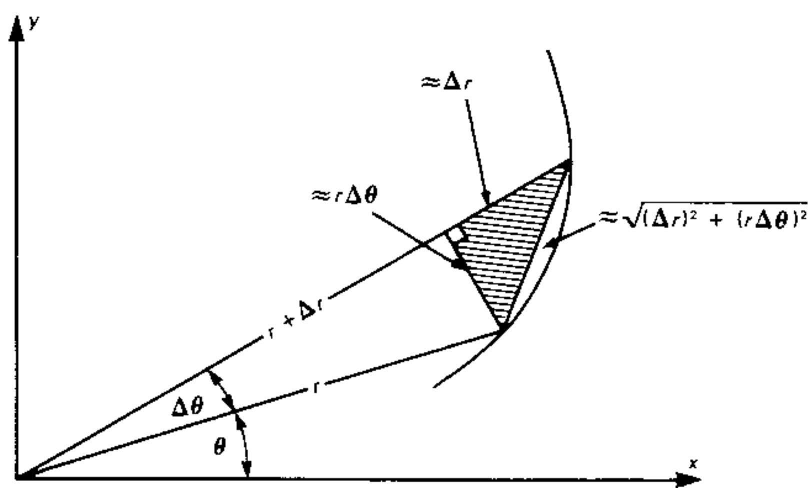
Read section 10.4, pages 689–692 in your textbook. Carefully study the examples worked out in the text.

Overview

In this course what we are particularly interested in is the use of polar coordinates in measuring arc lengths, areas of surfaces, and volumes. In our study of calculus arc length will be important in the definition of a “line integral”. We have already seen the importance of calculating areas of plane regions in determining the average value of a function of two variables. Calculations of volumes of solid regions in 3-space can be done through the use of a “triple integral” and is required in the calculation of the average value of a function of three variables.

Stewart derives formulae for the area of a polar region in the plane bounded by $r = f(\theta),\ a \le \theta \le b$, and the length of the curve $r = f(\theta),\ a \le \theta \le b$.

Below is an elementary derivation of these formulae.



In the drawing, we are approximating a small portion of the region bounded by the curve $r = f(\theta)$ over the interval $[\theta_1, \theta_2]$ by a sector of a circle. The sector of the arc is enormously exaggerated so that the details are more easily seen. In your imagination, you should extrapolate to the situation where the angle of the sector, $\Delta\theta$, is so small that the radius vector, r, and the radius vector, $r + \Delta r$ are practically parallel to each other. In this limit, the triangle shown cross-hatched is right-angled, and the hypotenuse is equal in length to the arc. Thus,

$$ds = \lim_{\Delta\theta \to 0} \sqrt{(\Delta r)^2 + (r\Delta\theta)^2} = \sqrt{(dr)^2 + r^2 (d\theta)^2}$$

The length of an arc from θ_1 to θ_2 becomes

$$L = \int_{s_1}^{s_2} ds = \int_{\theta_1}^{\theta_2} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} \, d\theta$$

Integrating over the entire length of the curve $r = f(\theta),\ a \le \theta \le b$, we havethe arclength formula:

$$L = \int_a^b \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2} \, d\theta$$

The area of the large triangle shown in the drawing can be calculated in the usual way as the product of 1/2 the base, $r\Delta\theta$, times the altitude, r:

$$\Delta A = \frac{1}{2} r^2 \Delta\theta$$

Taking the limit in a similar way, you can obtain the formula for the area of a sector cut off by a section of a curve from θ_1 to θ_2:

$$A = \int_{\theta_1}^{\theta_2} \frac{1}{2} r^2 \, d\theta$$

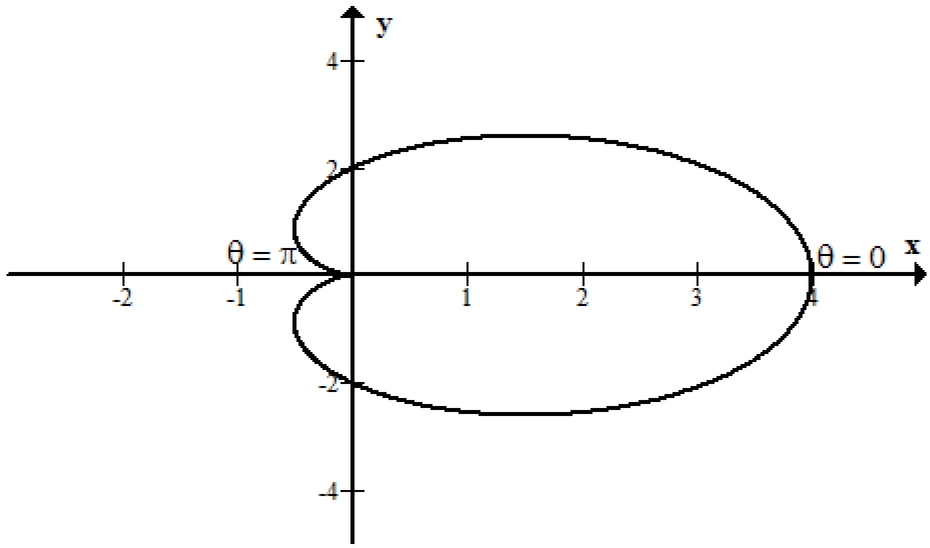
Integrating over the entire region bounded by $r = f(\theta),\ a \le \theta \le b$, we have the area formula:

$$A = \int_a^b \frac{1}{2} r^2 \, d\theta = \int_a^b \frac{1}{2} \left(f(\theta)\right)^2 \, d\theta$$

Example Exercises

1. This is problem 48 on page 693 of the text.

Find the <u>exact</u> length of the polar curve $r = 2(1+\cos(\theta))$. This is the cardioid centered on the x-axis.



Using the symmetry in the polar curve, we know its length L is determined by:

$$\begin{aligned}
L &= 2\int_0^{\pi} \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\ d\theta \\
&= 2\int_0^{\pi} \sqrt{\left(-2\sin(\theta)\right)^2 + \left(2(1+\cos(\theta))\right)^2}\ d\theta \\
&= 2\int_0^{\pi} \sqrt{4\sin^2(\theta) + 4(1+2\cos(\theta)+\cos^2(\theta))}\ d\theta \\
&= 2\int_0^{\pi} \sqrt{4\sin^2(\theta) + 4\cos^2(\theta) + 8\cos(\theta) + 4}\ d\theta \\
&= 2\int_0^{\pi} \sqrt{4\left(\sin^2(\theta)+\cos^2(\theta)\right) + 8\cos(\theta) + 4}\ d\theta \\
&= 2\int_0^{\pi} \sqrt{8+8\cos(\theta))}\ d\theta
\end{aligned}$$

This integral is now solved using the following algebraic trick:

$$\begin{aligned}
\sqrt{8+8\cos(\theta)} &= \sqrt{8+8\cos(\theta)} \cdot \frac{\sqrt{8-8\cos(\theta)}}{\sqrt{8-8\cos(\theta)}} = \frac{\sqrt{64-64\cos^2(\theta)}}{\sqrt{8(1-\cos(\theta))}} \\
&= \frac{\sqrt{64\left(1-\cos^2(\theta)\right)}}{\sqrt{8}\sqrt{(1-\cos(\theta))}} = \frac{8\sqrt{\sin^2(\theta)}}{\sqrt{8}\sqrt{(1-\cos(\theta))}} = \frac{\sqrt{8}\sin(\theta)}{\sqrt{1-\cos(\theta)}}
\end{aligned}$$

Notice that $\sqrt{\sin^2(\theta)} = |\sin(\theta)| = \sin(\theta), \quad \text{since } 0 \le \theta \le \pi$.

Now using the substitution $u = 1-\cos(\theta), \quad du = \sin(\theta)d\theta$, with $u = 0$ when $\theta = 0$ and $u = 2$ when $\theta = \pi$ we have:

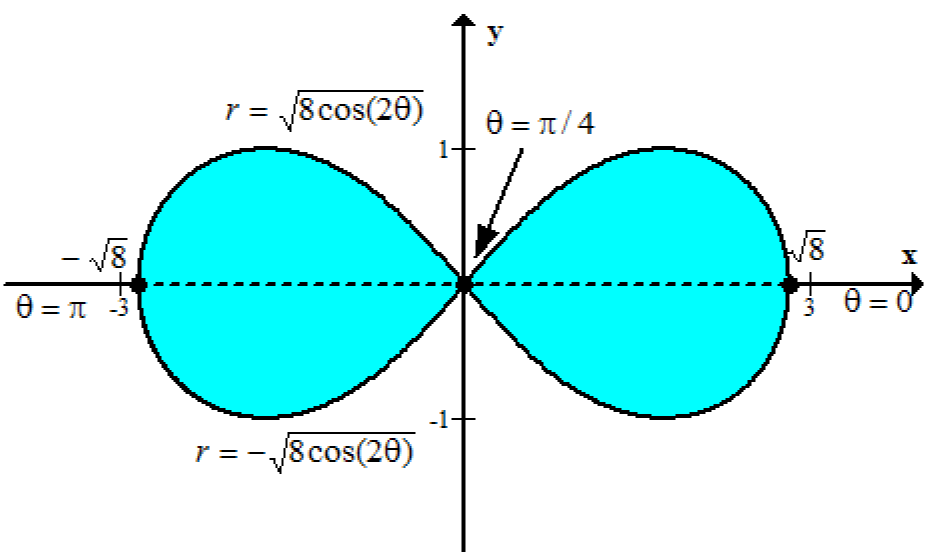
$$\begin{aligned}
L &= 2\int_0^{\pi} \sqrt{8+8\cos(\theta))}\ d\theta \\
&= 2\int_0^{\pi} \frac{\sqrt{8}\sin(\theta)}{\sqrt{1-\cos(\theta)}}\ d\theta \\
&= 2\sqrt{8}\int_0^{2} \frac{1}{\sqrt{u}}\ du \\
&= 2\sqrt{8}\int_0^{2} u^{-1/2}\ du \\
&= 2\sqrt{8}\left[2u^{1/2}\right]_0^2 \\
&= 4\sqrt{8}\left(\sqrt{2}-0\right) \\
&= 4\sqrt{16} \\
&= 16
\end{aligned}$$

Stewart, J. (2012). Multivariable calculus (7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.

2. Find the exact area of the region enclosed by the "lemniscate" $r^2 = 8\cos(2\theta)$

It is helpful in area problems to have a sketch of the region.

We are plotting two polar curves here, $r = \sqrt{8\cos(2\theta)}$ and $r = -\sqrt{8\cos(2\theta)}$



Important: Since $\cos(2\theta) < 0 \Rightarrow \pi/2 < 2\theta < 3\pi/2 \Rightarrow \pi/4 < \theta < 3\pi/4$, both square root expressions are undefined for these values of θ

Using the symmetry in the polar curve and the fact that $r = 0$ at $\theta = \pi/4$, we determine the area of the shaded region by:

$$\begin{aligned}
A &= 4\int_0^{\pi/4} \frac{1}{2}\cdot 8\cos(2\theta)\,d\theta \\
&= 16\int_0^{\pi/4} \cos(2\theta)\,d\theta \\
&= 16\left[\frac{1}{2}\sin(2\theta)\right]_0^{\pi/4} \\
&= 8(\sin(\pi/2) - \sin(0)) \\
&= 8(1-0) \\
&= 8
\end{aligned}$$

Practice Exercises 10.4

From the text pages 692–694, do problems 3, 7, 11, 17, 21, 27, 31, 41, 45 and 47.

Remember: You are to attempt all of the problems carefully before checking your solutions against those given in the solutions manual.

Double Integrals in Polar Coordinates

Learning Outcomes

Upon completion of Double Integrals in Polar Coordinates, you should be able to:

- Decide whether a given double integral should be evaluated in rectangular coordinates or polar coordinates.
- Transform a double integral in rectangular coordinates to an equivalent integral expression in polar coordinates and calculate its value.
- Set up and calculate double integrals in polar coordinates representing the area of a given polar region in the plane or the volume of a solid region in 3-space.

Readings

Read section 15.4, pages 1021–1025 in your textbook. Carefully study the examples worked out in the text.

Overview

Back to calculus!

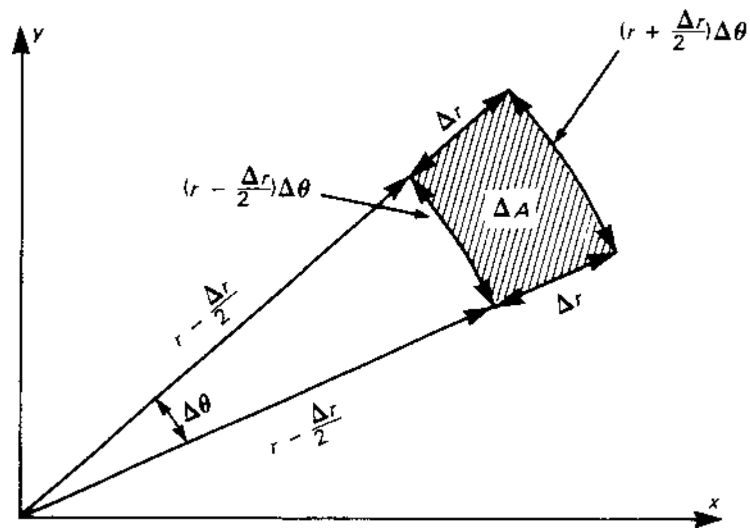
You might infer from your readings that polar coordinates are mainly used to simplify difficult integrals in the Cartesian form. While this is an important use, you must learn to work with polar coordinates with the same facility you have with the Cartesian form. There are many different systems of coordinates. The system that is most useful for a particular problem depends on the problem's physical parameters. If the boundaries are most easily expressed in Cartesian form, those coordinates will be most apt. If the boundaries are better expressed in polar form, the work will be simplified if done in polar coordinates. Later on you will be working with cylindrical and spherical systems.

You might be interested in learning that a skewed system of coordinates makes calculations in special relativity much easier. Now back to polar coordinates.

Suppose we want to evaluate the double integral $\iint\limits_D f(x, y)\,dA$ over some r-simple polar region $D = \{(r, \theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$.

Note: The concept of r-simple is similar to the concept of x-simple discussed in section 15.3.

Subdivide this region into polar rectangles. A picture of a typical polar rectangle is given below.



As in the previous section, the drawing exaggerates the separation, $\Delta\theta$, of the two radius vectors so that the details are more apparent. The polar rectangle is shown cross-hatched, and its area is equal to the difference in area between the larger circular sector with radius $\left(r - \frac{\Delta r}{2}\right) + \Delta r = r + \frac{\Delta r}{2}$ and the smaller sector with radius $r - \frac{\Delta r}{2}$. Using the standard formula for the area of such sectors, each is equal to 1/2 the product of its base and the radius, we have

$$\Delta A = \frac{1}{2}\left(r + \frac{\Delta r}{2}\right)^2 \Delta\theta - \frac{1}{2}\left(r - \frac{\Delta r}{2}\right)^2 \Delta\theta$$

This equation reduces to

$$\Delta A = r\Delta r\,\Delta\theta$$

So, the volume element in rectangular coordinates, $f(x, y)\Delta A$, becomes in polar coordinates

$$f(r\cos(\theta), r\sin(\theta))r\Delta r\Delta\theta = g(r,\theta)\Delta r\Delta\theta$$

where $g(r,\theta) = rf(r\cos(\theta), r\sin(\theta))$.

Forming the familiar double Riemann sum for this function $z = g(r, \theta)$ and taking the limit as the number of polar rectangles becomes infinite, we obtain

$$\begin{aligned}\iint_D g(r,\theta)\,dr\,d\theta &= \iint_D f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta \\ &= \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta\end{aligned}$$

It can be shown that for continuous functions f, the double integral defined in terms of ordinary rectangles yields the same answer as defined using polar rectangles, so we have the general result given in box 3 on page 1024 of the text:

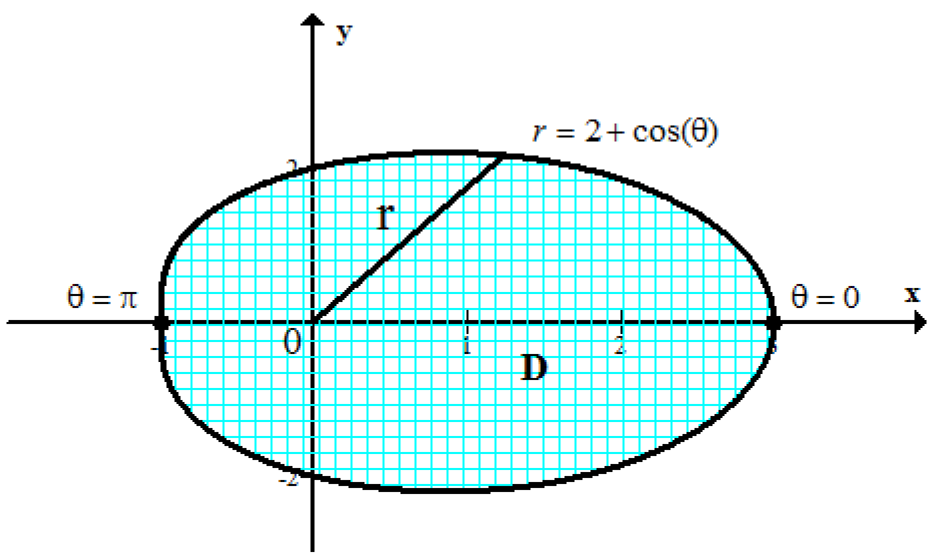
$$\iint_D f(x,y)\,dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))\,r\,dr\,d\theta$$

Important: When transforming a double integral in rectangular coordinates to polar coordinates, always remember to replace dA by $r\,dr\,d\theta$. Heed the reminder on page 1023 of the text to include the additional factor r.

Example Exercises

1. Find the exact area of the region inside the "limaçon" $r = 2 + \cos(\theta)$.

We start with a picture of the region D.



Notice that D is r-simple and can be described b

$$D = \{(r,\theta) \mid 0 \le \theta \le 2\pi,\ \ 0 \le r \le 2 + \cos(\theta)\}$$

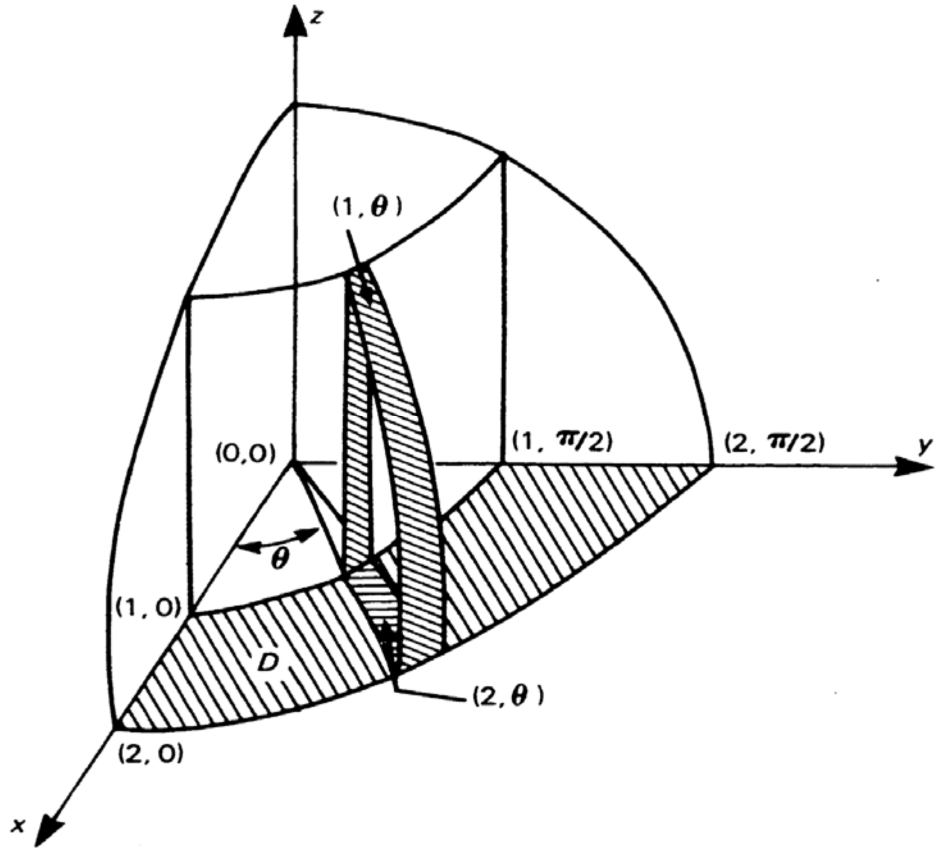
Since $\cos(\theta) = \cos(-\theta)$,the curve is symmetric about the x-axis and we need only evaluate the area A from $\theta = 0$ to $\theta = \pi$ and multiply by 2.

So, setting up the problem in rectangular coordinates and transforming we have

$$\begin{aligned}
A &= \iint_D dA = \iint_D r\,dr\,d\theta = 2\int_0^{\pi}\int_0^{2+\cos(\theta)} r\,dr\,d\theta \\
&= 2\int_0^{\pi} \left[\frac{r^2}{2}\right]_0^{2+\cos(\theta)} d\theta \\
&= \int_0^{\pi} \left[r^2\right]_0^{2+\cos(\theta)} d\theta \\
&= \int_0^{\pi} \left[\left(2+\cos(\theta)\right)^2 - 0\right] d\theta \\
&= \int_0^{\pi} \left(4+4\cos(\theta)+\cos^2(\theta)\right) d\theta \\
&= \int_0^{\pi} 4\,d\theta + \int_0^{\pi} 4\cos(\theta)\,d\theta + \int_0^{\pi} \frac{1+\cos(2\theta)}{2}\,d\theta \\
&= 4\left[\theta\right]_0^{\pi} + 4\left[\sin(\theta)\right]_0^{\pi} + \frac{1}{2}\left[\theta + \frac{1}{2}\sin(2\theta)\right]_0^{\pi} \\
&= 4(\pi-0) + 4(\sin(\pi)-\sin(0)) + \frac{1}{2}\left(\pi + \frac{1}{2}\sin(2\pi) - 0\right) \\
&= 4\pi + \frac{\pi}{2} \\
&= \frac{9\pi}{2}
\end{aligned}$$

2. A cylindrical drill of radius 1 drills through a sphere of radius 2, passing straight through the centre of the sphere. Find the volume of the resulting solid.

We start with a picture of the solid. Because of symmetry our picture will focus on the first octant and we will calculate the volume of the solid region there. The total volume V will then be calculated by multiplying this result by eight.



Notice that the region D in the first quadrant of the xy-plane is the annular region between two circles of radii 1 and 2. This region is nicely described in polar coordinates by

$$D = \{(r, \theta) \,|\, 0 \le \theta \le \pi/2,\ 1 \le r \le 2\}$$

The height of the wedge in the picture can be found from the equation of the sphere centered at (0,0,0) with radius 2:

$$\begin{aligned} x^2 + y^2 + z^2 &= 4 \\ z^2 &= 4 - \left(x^2 + y^2\right) \\ z &= \sqrt{4 - \left(x^2 + y^2\right)}, \quad z > 0 \end{aligned}$$

Transforming to polar coordinates, we have:

$$z = \sqrt{4 - \left(x^2 + y^2\right)} = \sqrt{4 - r^2}$$

So,

$$
\begin{aligned}
V &= 8\iint_D z\,dA = 8\iint_D \sqrt{4-r^2}\, r\,dr\,d\theta = 8\int_0^{\pi/2}\int_1^2 \sqrt{4-r^2}\, r\,dr\,d\theta \\
&= 8\int_0^{\pi/2} -\frac{1}{2}\int_1^2 (-2r)\left(4-r^2\right)^{1/2} dr\,d\theta \\
&= -4\int_0^{\pi/2} \frac{2}{3}\left[\left(4-r^2\right)^{3/2}\right]_1^2 d\theta \\
&= -\frac{8}{3}\int_0^{\pi/2} \left(0-3^{3/2}\right) d\theta \\
&= 8\left(\sqrt{3}\right)\int_0^{\pi/2} d\theta \\
&= 8\sqrt{3}\left(\frac{\pi}{2}-0\right) \\
&= 4\pi\sqrt{3}
\end{aligned}
$$

Practice Exercises 15.4

From the text pages 1026–1027, do problems 1, 3, 5, 9, 11, 13, 15, 21, 25, 31, 35, and 39.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Unit 4: Summary and Self-Test

You have now worked through Unit 4 in MATH 2111. It is time to take stock of what you have learned, review all of the material, and bring your shorthand notes up to date. A summary of the material covered so far is provided in the following pages. This summary should be modified, added to, and fleshed out to form a solid body of knowledge.

When you have completed your review, you should test your comprehension of the material with a closed book self-administered examination. Put all your notes aside, find a quiet place where you will not be disturbed, and take the examination provided at the end of this unit. You will find some questions straightforward and easy, but others will test your ingenuity.

You will find the solutions to the Unit 4 exam questions, and the point value for each question in the **Answer Key** provided at the end of this unit. Become your own examiner. If you have done well, according to your personal standards, go on to Unit 5. If not, then more review and practice is obviously called for.

Summary

Double Integrals

If R denotes a rectangular region in the plane, we define the double integral of f over R by

$$\iint_R f(x,y)\,dA = \lim_{m,n\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} f\left(x_i^*, y_j^*\right)\Delta x \Delta y$$

if this limit exists. This limit exists if f is continuous.

If f is continuous on a y-simple region

$D = \{(x,y) \mid a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$, then

$$\iint_D f(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)\,dy dx = \int_a^b I(x,y)\Big]_{y=g_1(x)}^{y=g_2(x)} dx = \int_a^b \left(I(x,g_2(x)) - I(x,g_1(x))\right) dx,$$

where $I(x,y)$ is a partial anti-derivative with respect to y of $f(x,y)$.

If f is continuous on an x-simple region

$D = \{(x, y) \mid c \le y \le d\,,\, h_1(y) \le x \le h_2(y)\}$, then

$$\iint_D f(x, y)\,dA = \int_c^d \int_{h_1(y)}^{h_2(y)} f(x, y)\,dx dy = \int_c^d I(x, y)\Big]_{x=h_1(y)}^{x=h_2(y)}\,dy = \int_a^b \big(I(h_2(y), y) - I(h_1(y), y)\big)\,dy$$

where $I(x, y)$ is a partial anti-derivative with respect to x of $f(x, y)$.

If D is both x-simple and y-simple, as in a rectangular region, either order of integration can be used.

Interpretations:

1. If $z = f(x, y) \ge 0$ on D, then $\iint_D f(x, y)\,dA$ is the volume of the solid region bounded above by $z = f(x, y)$ and below by the region D in the xy-plane.

2. $\iint_D dA = \iint_D 1\,dA =$ area of the region D.

Properties:

1. $\iint_D \big(f(x, y) + g(x, y)\big)\,dA = \iint_D f(x, y)\,dA + \iint_D g(x, y)\,dA$

2. $\iint_D k\,f(x, y)\,dA = k\iint_D f(x, y)\,dA$, k a constant

3. $\iint_D f(x, y)\,dA = \iint_{D_1} f(x, y)\,dA + \iint_{D_2} f(x, y)\,dA$, where $D = D_1 \cup D_2$ and D_1, D_2 don't overlap, except possibly on their boundaries.

Polar Coordinates:

Polar to rectangular coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$

Rectangular to polar coordinates: $r^2 = x^2 + y^2$, $\tan(\theta) = \dfrac{y}{x}$

Arc length in polar coordinates:

Given a polar curve $r = f(\theta)$, $a \le \theta \le b$, the length L of the curve is

$$L = \int_a^b \sqrt{\left(\frac{dr}{d\theta}\right)^2 + r^2}\;d\theta$$

Area in polar coordinates:

The area of the region bounded by $r = f(\theta),\ a \le \theta \le b$ is

$$A = \int_a^b \frac{1}{2} r^2 \, d\theta = \int_a^b \frac{1}{2}\left(f(\theta)\right)^2 \, d\theta$$

If f is continuous on an $r-$simple polar region of the form

$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$, then

$$\iint_D f(x,y)\, dA = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} f(r\cos(\theta), r\sin(\theta))\ r\, dr\, d\theta$$

Important: When transforming a double integral in rectangular coordinates to polar coordinates, always remember to replace dA by $r\, dr\, d\theta$.

Self-Test *(25 marks)*

Treat this as a real test. Do not refer to any course materials. The time for this test is 1.5 hours. Use the answer key provided to mark your test. The point value for each question is posted in the left margin.

[2] 1. Use the Midpoint Rule with $m = n = 2$ to estimate the volume of the solid that lies below the surface $z = x^2 + 2y + 3$ and above the rectangle $R = [1,3] \times [0,2]$

[2] 2. Use a <u>geometric interpretation</u> of the double integral to evaluate exactly $\iint_R x\, dA$, where $R = [0,2] \times [0,2]$. Do not evaluate using iterated integrals.

3. Evaluate the following. Give exact answers.

[3] a) $\int_0^1 \int_0^\pi y\cos(xy)\ dy\, dx$

[3] b) $\iint_D xy^2\ dA$, where D is the region enclosed by $x = 0$ and $x = \sqrt{1-y^2}$

[4] c) $\displaystyle\iint_D \frac{y^2}{x^2+y^2}\,dA$, where $D = \{(x, y) \mid x^2 + y^2 \le 2\}$

[2] 4. Carefully sketch the region of integration for

$\displaystyle\int_1^2 \int_0^{\ln(y)} f(x, y)\, dx\, dy$ and change the order of integration.

[3] 5. Set up, but do not evaluate, an iterated integral expression for the area of the region enclosed by $x = y^2$ and $y = x - 2$ using the order of integration $dy\,dx$.

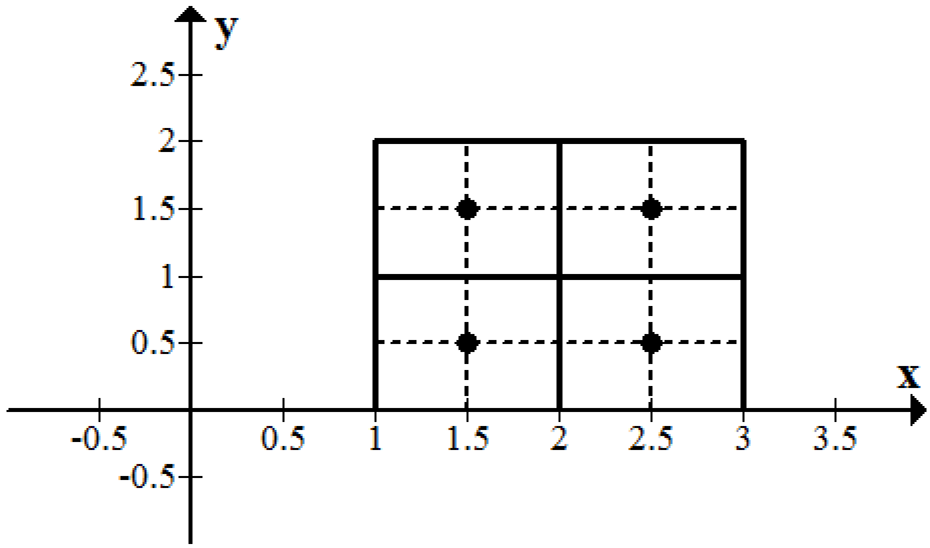
[3] 6. Use an integration formula to find the total length of the curve $r = 2a\cos(\theta)$ for $a > 0$.

[3] 7. Use polar coordinates to find the volume of the region inside the sphere $x^2 + y^2 + z^2 = 16$ and outside the cylinder $x^2 + y^2 = 4$.

Answer Key

1. Since $f(x, y) = x^2 + 2y + 3$ is non-negative on R, the volume of this region is represented by $\iint_R \left(x^2 + 2y + 3\right) dA$.

The midpoints of the sub-rectangles are pictured below.

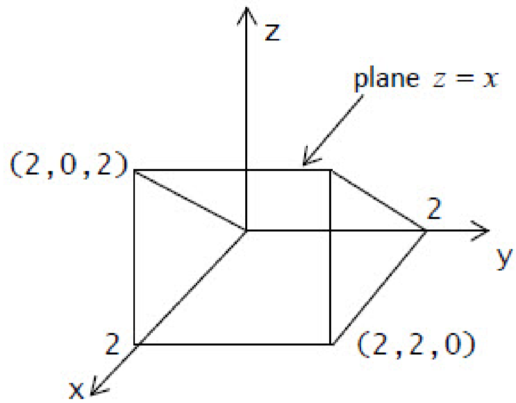


Using the Midpoint Rule we have

$$\begin{aligned}\iint_R \left(x^2 + 2y + 3\right) dA &\approx \sum_{i=1}^{2} \sum_{j=1}^{2} f\left(\bar{x}_i, \bar{y}_j\right) \Delta A \\ &= [f(1.5, 0.5) + f(2.5, 0.5) + f(1.5, 1.5) + f(2.5, 1.5)]\Delta A \\ &= [6.25 + 10.25 + 8.25 + 12.25] \cdot 1 \\ &= 37\end{aligned}$$

Therefore the volume is approximately 37.

2. The solid region bounded by $z = x$ and $R = [0,2]\times[0,2]$ is pictured below.



Because this is a cylindrical region, the volume of this solid can be readily obtained as the area of the triangular cross-section times the width.

But, because the integrand function $z = x$ is non-negative on R, we can interpret $\iint_R x\,dA$ as the volume of this region.

Hence, $\iint_R x\,dA = \frac{1}{2}(2)(2)\cdot 2 = 4$

3. a) $\int_0^1 \int_0^\pi y\cos(xy)\ dy\,dx$

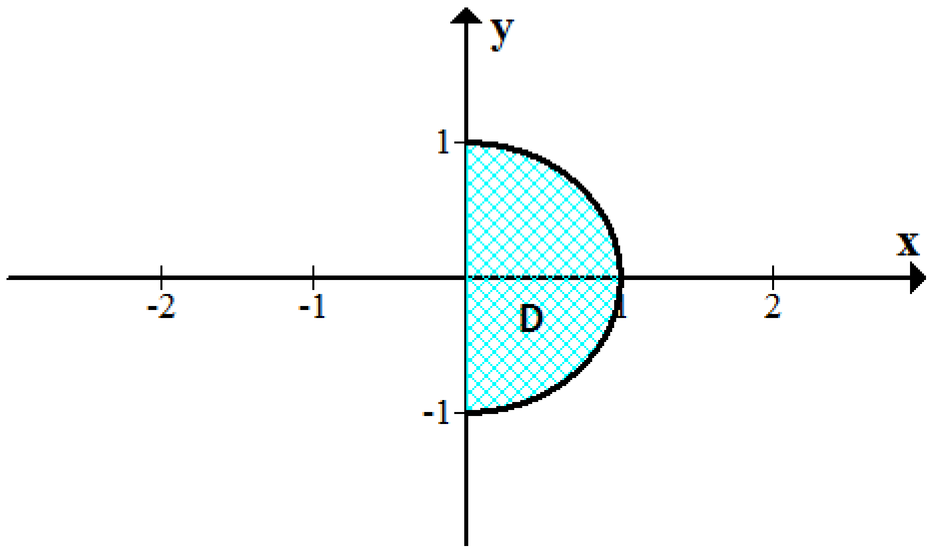
Notice that to integrate with respect to y first would require the technique of integration by parts. This complicates the calculation. Integrating with respect to x first is more desirable.

Changing the order of integration we have

$$\int_0^1 \int_0^\pi y\cos(xy)\ dy\,dx = \int_0^\pi \int_0^1 y\cos(xy)\ dx\,dy = \int_0^\pi \sin(xy)\Big]_0^1 dy = \int_0^\pi \sin(y)\,dy = -\cos(y)\Big]_0^\pi$$
$$= -\big(\cos(\pi) - \cos(0)\big) = -\big(-1-1\big) = 2$$

b) $\iint_D xy^2\,dA$, D is the region enclosed by $x = 0$ and $x = \sqrt{1-y^2}$.

The region is pictured below.



This region is $x-$simple. As an $x-$simple region, $D=\{(x,y)\,|\,0\le x\le\sqrt{1-y^2}\,,-1\le y\le 1\}$.

Hence,

$$\iint_D xy^2\,dA=\int_{-1}^{1}\int_0^{\sqrt{1-y^2}} xy^2\;dx\,dy=\int_{-1}^{1}y^2\left[\frac{x^2}{2}\right]_0^{\sqrt{1-y^2}}dy=\int_{-1}^{1}\frac{y^2}{2}\left(\left(\sqrt{1-y^2}\right)^2-0\right)dy$$

$$=\frac{1}{2}\int_{-1}^{1}y^2\left(1-y^2\right)dy=\frac{1}{2}\int_{-1}^{1}\left(y^2-y^4\right)dy=\frac{1}{2}\cdot\left[\frac{y^3}{3}-\frac{y^5}{5}\right]_{-1}^{1}=\frac{1}{2}\left(\frac{1}{3}-\frac{1}{5}-\left(-\frac{1}{3}+\frac{1}{5}\right)\right)=\frac{2}{15}$$

c) $\displaystyle\iint_D\frac{y^2}{x^2+y^2}dA$, where $D=\{(x,y)\,|\,x^2+y^2\le 2\}$

The presence of the term x^2+y^2 in the integrand and the fact that D is a nice polar region (a circle centered at the origin of radius $\sqrt{2}$) strongly suggest we use polar coordinates to evaluate this double integral.

$y=r\sin(\theta)$, $x^2+y^2=r^2$; $D=\{(r,\theta)\,|\,0\le r\le\sqrt{2}\,,\quad 0\le\theta\le 2\pi\}$

Transforming to polar coordinates and replacing dA by $r\,dr\,d\theta$, we have

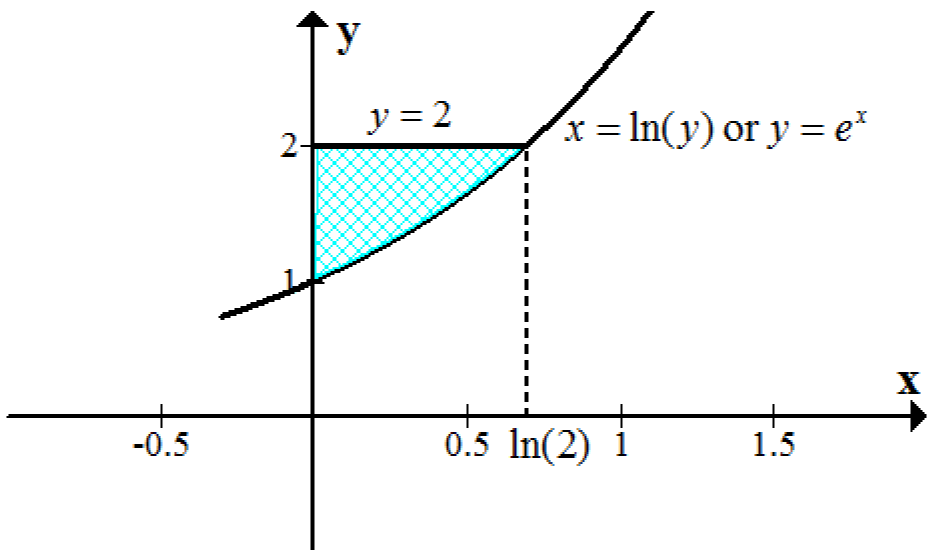
$$\iint_D \frac{y^2}{x^2+y^2}\,dA = \int_0^{2\pi}\int_0^{\sqrt{2}} \frac{(r\sin(\theta))^2}{r^2}\,r\,dr\,d\theta = \int_0^{2\pi}\int_0^{\sqrt{2}} r\sin^2(\theta)\,dr\,d\theta = \int_0^{2\pi}\sin^2(\theta)\left[\frac{r^2}{2}\right]_0^{\sqrt{2}} d\theta$$

$$= \frac{1}{2}\int_0^{2\pi}\sin^2(\theta)\left(\left(\sqrt{2}\right)^2-0\right)_0^{\sqrt{2}} d\theta = \int_0^{2\pi}\sin^2(\theta)\,d\theta = \int_0^{2\pi}\frac{1-\cos(2\theta)}{2}\,d\theta = \left[\frac{\theta}{2}-\frac{1}{4}\sin(2\theta)\right]_0^{2\pi}$$

$$= \pi - 0 - 0 = \pi$$

4. $\int_1^2\int_0^{\ln(y)} f(x,y)\,dx\,dy$; $D = \{(x,y)\,|\,0 \le x \le \ln(y),\, 1 \le y \le 2\}$

The region D is sketched below.



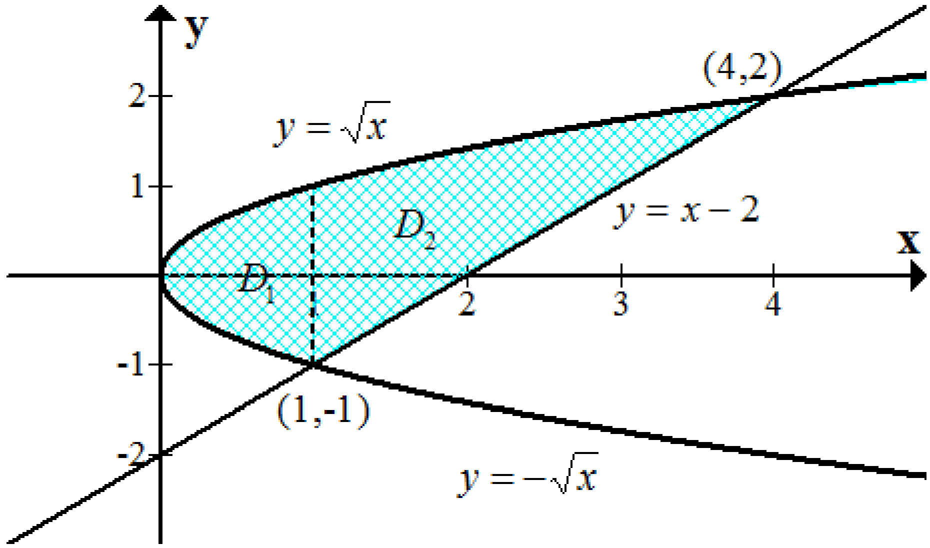
Viewing the shaded region of integration as $y-$simple, we have that

$$D = \{(x,y)\,|\,e^x \le y \le 2,\, 0 \le x \le \ln(2)\}.$$

Hence, $\int_1^2\int_0^{\ln(y)} f(x,y)\,dx\,dy = \int_0^{\ln(2)}\int_{e^x}^2 f(x,y)\,dy\,dx$.

5. The area A of the enclosed region D is represented by $\iint_D dA$. Since the order of integration is $dy\,dx$ we need to view D as the union of two regions, $D = D_1 \cup D_2$, where $D_1 = \{(x,y)\,|\,-\sqrt{x} \le y \le \sqrt{x},\, 0 \le x \le 1\}$ and $D_2 = \{(x,y)\,|\,x-2 \le y \le \sqrt{x},\, 1 \le x \le 4\}$

The region D is sketched below.



Hence, $A=\iint\limits_{D_1} dA+\iint\limits_{D_2} dA=\int_0^1\int_{-\sqrt{x}}^{\sqrt{x}} dy\,dx+\int_1^4\int_{x-2}^{\sqrt{x}} dy\,dx$

6. $r=2a\cos(\theta)$, $a>0$

This polar curve is a circle centered on the $x-$axis at $(a,0)$ of radius a. Its graph is generated for $0\le\theta\le\pi$.

Now, $\dfrac{dr}{d\theta}=-2a\sin(\theta)$. So,

$$\begin{aligned}
L &= \int_0^{\pi}\sqrt{r^2+\left(\frac{dr}{d\theta}\right)^2}\,d\theta=\int_0^{\pi}\sqrt{\left(2a\cos(\theta)\right)^2+\left(-2a\sin(\theta)\right)^2}\,d\theta\\
&= \int_0^{\pi}\sqrt{4a^2\cos^2(\theta)+4a^2\sin^2(\theta)}\,d\theta\\
&= \int_0^{\pi}\sqrt{4a^2\left(\cos^2(\theta)+\sin^2(\theta)\right)}\,d\theta\\
&= \int_0^{\pi}\sqrt{4a^2}\,d\theta\\
&= \int_0^{\pi}2a\,d\theta \quad (\text{since } a>0)\\
&= 2a\left(\pi-0\right)\\
&= 2\pi a
\end{aligned}$$

7. The cylinder $x^2 + y^2 = 4$ intersects the sphere $x^2 + y^2 + z^2 = 16$ when

$$x^2 + y^2 + z^2 = 16 \quad \Rightarrow \quad 4 + z^2 = 16 \quad \Rightarrow z^2 = 12 \quad \Rightarrow \quad z = \pm\sqrt{12}$$

At $z = \pm\sqrt{12}$, the curve of intersection is $x^2 + y^2 = 4$. Since the region is inside the sphere, but outside the cylinder, it lives over the annular region between the circle centered at the origin of radius 2 and the circle centered at the origin of radius 4 in the $xy-$plane. That is, the region lies between the lower hemisphere $z = -\sqrt{16 - x^2 - y^2}$ and the top hemisphere $z = \sqrt{16 - x^2 - y^2}$ over this annular region.

Using symmetry the volume is

$$V = 2\iint_D \sqrt{16 - x^2 - y^2}\, dA$$

Converting to polar coordinates, we have

$$\sqrt{16 - x^2 - y^2} = \sqrt{16 - r^2}\ ;\ D = \{(r,\theta) \mid 2 \le r \le 4, \quad 0 \le \theta \le 2\pi\}$$

So,

$$\begin{aligned}
V &= 2\iint_D \sqrt{16 - x^2 - y^2}\, dA = 2\int_0^{2\pi}\int_2^4 \sqrt{16 - r^2}\ r\,dr\,d\theta \\
&= -\int_0^{2\pi}\int_2^4 -2r\sqrt{16 - r^2}\ dr\,d\theta \\
&= -\int_0^{2\pi} \left[\frac{2}{3}\left(16 - r^2\right)^{3/2}\right]_2^4 d\theta \\
&= -\frac{2}{3}\int_0^{2\pi} \left(0 - 12^{3/2}\right) d\theta \\
&= \frac{2\left(24\sqrt{3}\right)}{3}(2\pi - 0) \\
&= 32\pi\sqrt{3}
\end{aligned}$$