Faculty of Science

Unit 2:
Vector Functions

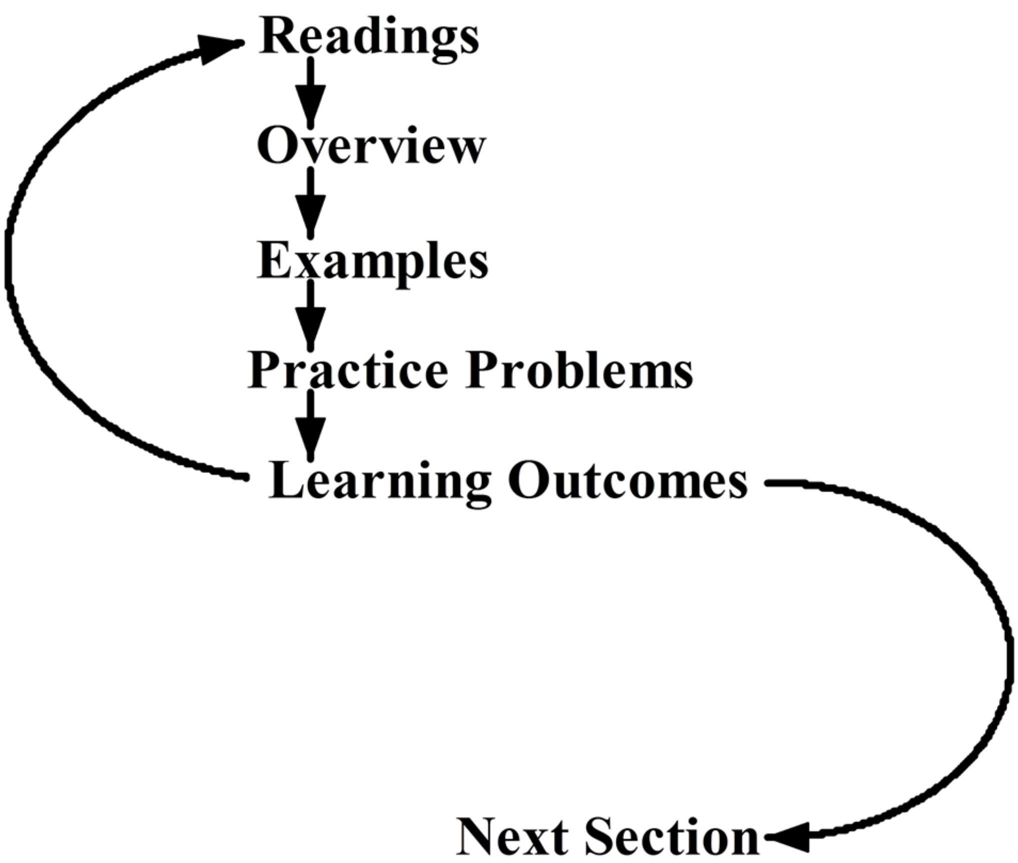
MATH 2111
Calculus III – Multivariable Calculus

Table of Contents

Instructions

The recommended procedure for working through each section of the units in this course is described in detail in your Course Guide.

This procedure is summarized below. If you are certain you have achieved the learning objectives, proceed to the next section. If you are uncertain about one or more of them, go back to the appropriate information in the section until you can complete the task listed in the objective.



Vector Functions and Space Curves

Learning Outcomes

Upon completion of Vector Functions and Space Curves, you should be able to:

- Identify the components of a vector function and determine the domains of its components.
- Calculate limits of vector functions.
- Sketch space curves given in either vector form or parametric form.
- Recognize the vector form of a line and line segment in 3-space.
- Recognize the vector or parametric form of a helix.
- Determine a vector function representing the curve of intersection of two surfaces.

Readings

Read section 13.1, pages 864–869 in your textbook. Carefully study the examples worked out in the text.

Overview

In the first unit we were concerned with vectors in 2-space and 3-space. Now we study **vector-valued functions** or **vector functions**, that is, functions defined on a set of real numbers whose values are two or three dimensional vectors. Our focus will be on vector functions with images in R^3. Symbolically we write

$$\vec{r} : I \to R^3$$

where $\vec{r}(t) = < f(t), g(t), h(t) >$ for t belonging to some set of real numbers *I*. The functions $f, g, h : I \to R$ are called the **component functions** of $\vec{r}$.

In many situations the independent variable t will represent time, but not exclusively.

In the case *I* is an interval of the real line, the collection of points (x, y, z) in 3-space defined by $x = f(t)$, $y = g(t)$, $z = h(t)$ is called a **space curve**. The component functions define **parametric equations**, and *t* is called a **parameter**.

To sketch a space curve, convert the vector function into the corresponding set of parametric equations. If the parametric equations are all linear functions, then the curve being described is a line in 3-space (see Example 3 on page 865 of the text). You should readily recognize the vector equation of a line segment joining the head of the vector $\vec{r}_0$ to the head of the vector $\vec{r}_1$:

$$\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1 \ , \quad 0 \le t \le 1$$

You should also be familiar with the "helix" described in Example 4 on page 865 of the text.

The definition in equation 1 on page 864 of the text describes the limit of a vector function in terms of the limit of its component functions, which reasonably makes the <u>continuity</u> of $\vec{r}$ dependent upon the continuity of its component functions f, g, h

$$\lim_{t\to a}\vec{r}(t) = <\lim_{t\to a} f(t), \lim_{t\to a} g(t), \lim_{t\to a} h(t)> = < f(a), g(a), h(a)> = \vec{r}(a)$$

In elementary calculus you learned some of the rules about continuity of scalar (i.e., real-valued) functions. In the most general terms, a function $y = f(t)$ is continuous at a point a if the difference $f(t) - f(a)$ can be made arbitrarily small by restricting t to a small interval around a. More precisely:

- a function is continuous at the point $t = a$ if, and only if, $f(a)$ is defined and $f(t) \to f(a)$ as $t \to a$;
- for any odd positive integer n, $f(t) = t^{1/n}$ is everywhere continuous; in the case n is even, $f(t) = t^{1/n}$ is continuous for $t \ge 0$
- any polynomial is everywhere continuous;
- any rational function $y = \dfrac{f(t)}{g(t)}$ of two polynomials is continuous, except where $g(t) = 0$;
- if $f(t) \to L$ and $g(t) \to M$ as $t \to a$, then the sum of the two functions $f(t) + g(t)$ approaches $L + M$ and the product $f(t) \cdot g(t)$ approaches $L \cdot M$ as $t \to a$;
- if $f(t) \to L$ as $t \to a$, and if $y = g(s)$ is continuous at $s = L$, then
- $g(f(t)) \to g(L)$ as $t \to a$.

Example Exercises

1. This is problem 4 from your text on page 869.

Find : $\lim\limits_{t\to 1}\left(\dfrac{t^2-t}{t-1}\vec{i}+\sqrt{t+8}\;\vec{j}+\dfrac{\sin(\pi t)}{\ln(t)}\vec{k}\right)$

$$\lim_{t\to 1}\left(\frac{t^2-t}{t-1}\vec{i}+\sqrt{t+8}\;\vec{j}+\frac{\sin(\pi t)}{\ln(t)}\vec{k}\right)$$

$$=\left(\lim_{t\to 1}\left(\frac{t^2-t}{t-1}\right)\right)\vec{i}+\left(\lim_{t\to 1}\sqrt{t+8}\right)\vec{j}+\left(\lim_{t\to 1}\frac{\sin(\pi t)}{\ln(t)}\right)\vec{k}$$

Now, $\lim\limits_{t\to 1}\left(\dfrac{t^2-t}{t-1}\right)=\lim\limits_{t\to 1}\left(\dfrac{t(t-1)}{t-1}\right)=\lim\limits_{t\to 1}t=1$;

$$\lim_{t\to 1}\sqrt{t+8}=\sqrt{1+8}=\sqrt{9}=3$$

Using L'Hopital's Rule,

$$\lim_{t\to 1}\frac{\sin(\pi t)}{\ln(t)}=\lim_{t\to 1}\left(\frac{\pi\cos(\pi t)}{1/t}\right)=\pi\lim_{t\to 1}t\cos(\pi t)=\pi(\cos(\pi))=\pi(-1)=-\pi$$

Hence,

$$\lim_{t\to 1}\left(\frac{t^2-t}{t-1}\vec{i}+\sqrt{t+8}\;\vec{j}+\frac{\sin(\pi t)}{\ln(t)}\vec{k}\right)$$

$$=\left(\lim_{t\to 1}\left(\frac{t^2-t}{t-1}\right)\right)\vec{i}+\left(\lim_{t\to 1}\sqrt{t+8}\right)\vec{j}+\left(\lim_{t\to 1}\frac{\sin(\pi t)}{\ln(t)}\right)\vec{k}$$

$$=1\vec{i}+3\vec{j}-\pi\vec{k}\quad\text{or}\quad <1,3,-\pi>$$

So as $t\to 1$, $r(t)\to <1,3,-\pi>$.

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

2. This is problem 40 from your text on page 871.

Find a vector function that represents the curve of intersection of $x^2 + y^2 = 4$ and $z = xy$.

Remember the equation $x^2 + y^2 = 4$ represents a right circular cylinder in 3-space. Since any point (x,y) that satisfies the equation $x^2 + y^2 = 4$ will give a point on the surface $z = xy$, the projection into the *xy*-plane of the curve of intersection will be the circle $x^2 + y^2 = 4$, centered at the origin (0,0) of radius 2.

In parametric form we can describe this circle by

$$x = 2\cos(t),\; y = 2\sin(t),\; 0 \le t \le 2\pi$$

But the curve also lies on the surface $z = xy$, so we have

$$z = xy = (2\cos(t))(2\sin(t)) = 4\cos(t)\sin(t)$$

Hence, parametric equations for this space curve are:

$$x = 2\cos(t),\; y = 2\sin(t),\; z = 4\cos(t)\sin(t);\; 0 \le t \le 2\pi$$

and the vector function defining this curve of intersection is:

$$\vec{r}(t) = \langle 2\cos(t), 2\sin(t), 4\cos(t)\sin(t) \rangle,\; 0 \le t \le 2\pi$$

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

Practice Exercises 13.1

From the text pages 869–871, do problems 1, 5, 11, 13, 19, 21, 23, 27, 39, 41, and 47.

Remember: You are to attempt all of the problems carefully before checking your solutions against those given in the solutions manual.

Derivatives and Integrals of Vector Functions

Learning Outcomes

Upon completion of Derivatives and Integrals of Vector Functions, you should be able to:

- Differentiate vector functions.
- Calculate and sketch the tangent vector and unit tangent vector to a curve at a given point.
- Find parametric equations or a vector equation for a tangent line to a curve at a given point.
- State and apply the 6 rules of differentiation as described in box 3 on page 874 of the text.
- Calculate integrals of vector functions.

Readings

Read section 13.2, pages 871–875 in your textbook. Carefully study the examples worked out in the text.

Overview

The concept of the **derivative** $\vec{r}'$ of a vector function $\vec{r}$ is defined in the same manner as for real valued functions. (See equation 1 on page 871.)

Stewart then proceeds to prove that the **differentiability** of $\vec{r}$ boils down to the differentiability of its component functions; specifically, if $\vec{r}(t) = < f(t), g(t), h(t) >$, where $f, g, h : R \to R$ are differentiable functions, then $\vec{r}'(t) = < f'(t), g'(t), h'(t) >$ (see equation 2 on page 872).

Geometrically, if the vector $\vec{r}'(t)$ exists and $\vec{r}'(t) \neq \vec{0}$, then $\vec{r}'(t)$ is always tangent to the space curve defined by $\vec{r}$ at the point P on the curve where it is drawn. For this reason it is called a **tangent vector**, and is parallel to the tangent line to the curve at P. If we require a unit vector, then the **unit tangent** $\vec{T}(t)$ is calculated as

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

The six rules for differentiation of vector functions given in Box 3 on page 874 should become permanent parts of your memory bank, and should be immediately at hand for the solution of any problem.

The chain rule (6) is a straightforward extension from elementary calculus: If $\vec{u} = \vec{r}(s)$ is a differentiable vector function of the real variable s and $s = f(t)$ is a differentiable real-valued function, then $\vec{u} = \vec{r}(f(t))$ is a differentiable vector function of t and

$$\frac{d\vec{u}}{dt} = \frac{d}{dt}\left(\vec{r}(f(t))\right) = \vec{r}\,'(f(t)) \cdot f'(t)$$

or in Leibnitz notation,

$$\frac{d\vec{u}}{dt} = \frac{d\vec{u}}{ds} \cdot \frac{ds}{dt}$$

On the other side of the coin, the integration process for vector functions can be defined in terms of the integration of the real-valued components of the vector function.

If $\vec{r}(t) = < f(t), g(t), h(t) >$, where $f, g, h : R \to R$ are integrable, real-valued functions, then

$$\begin{aligned}\int_a^b \vec{r}(t)\,dt &= \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle \\ &= \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}\end{aligned}$$

Example Exercises

1. You can use the differentiation laws to derive the vector equation of a line.

   A line can be defined as a vector whose direction $\vec{v}$ is constant, but whose length is variable. This means that the derivative with respect to the parameter t is a constant. Letting $\vec{x}(t)$ represent the vector to an arbitrary point on the line we have,

   $$\frac{d\vec{x}}{dt} = \vec{v} \quad \Rightarrow \quad \frac{d}{dt}\left(\vec{x} - t\vec{v}\right) = \vec{0} \quad \Rightarrow \vec{x} - t\vec{v} = \vec{x}_0 \quad \text{(a constant)}$$

   yielding the vector equation of the line

   $$\vec{x} = \vec{x}_0 + t\vec{v}$$

2. To take the opposite approach, find an equation for the surface in R^3 described by the condition that a vector to a variable point on the surface has a constant length but a variable direction.

This is expressed vectorially as

$$\frac{d}{dt}|\vec{x}| = 0$$

Writing in component form,

$$\frac{d}{dt}\left(\sqrt{x^2+y^2+z^2}\right) = 0 \quad \Rightarrow \quad \sqrt{x^2+y^2+z^2} = r \quad \text{(a constant)}$$

or $x^2+y^2+z^2=r^2$, the familiar equation of a sphere centered at the origin with radius r.

3. Sketch the plane curve with the vector equation $\vec{r}(t) = (1+2\cos t)\vec{i} + (2+\sin t)\vec{j}$. Find $\vec{r}'(t)$. Sketch the position vector $\vec{r}(t)$ and the vector $\vec{r}'(t)$ for $t=\frac{\pi}{3}$.

We start by converting the parametric form into rectangular form. From the vector equation we have

$$x = 1+2\cos t \quad \Rightarrow \quad \frac{x-1}{2} = \cos t$$

$$y = 2+\sin t \quad \Rightarrow \quad y-2 = \sin t$$

Squaring both sides of the equations and adding, we have

$$\left(\frac{x-1}{2}\right)^2 + \left(y-2\right)^2 = \sin^2 t + \cos^2 t = 1$$

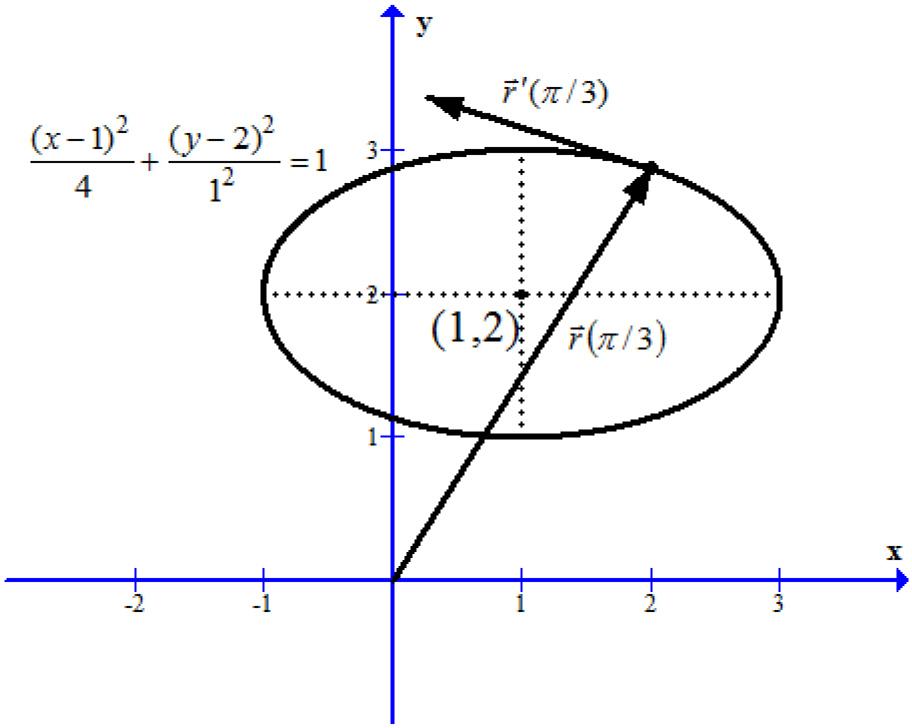
$$\frac{(x-1)^2}{4} + \frac{(y-2)^2}{1^2} = 1$$

The curve is an ellipse centered at (1,2) with a horizontal major axis.

$$\vec{r}'(t) = \frac{d}{dt}(1+2\cos t)\vec{i} + \frac{d}{dt}(2+\sin t)\vec{j} = \left(-2\sin t\right)\vec{i} + \left(\cos t\right)\vec{j}$$

$$\begin{aligned}\vec{r}\left(\frac{\pi}{3}\right) &= \left(1+2\cos\left(\frac{\pi}{3}\right)\right)\vec{i} + \left(2+\sin\left(\frac{\pi}{3}\right)\right)\vec{j} \\ &= (1+2\cdot(1/2))\vec{i} + (2+\sqrt{3}/2)\vec{j} \\ &= 2\vec{i} + (2+\sqrt{3}/2)\vec{j}\end{aligned}$$

$$\vec{r}'\left(\frac{\pi}{3}\right) = \left(-2\sin\left(\frac{\pi}{3}\right)\right)\vec{i} + \left(\cos\left(\frac{\pi}{3}\right)\right)\vec{j} = -\sqrt{3}\,\vec{i} + \frac{1}{2}\vec{j}$$



4. Find parametric equations for the tangent line to the curve with vector equation $\vec{x}(t) =< \ln(3t), \sqrt{12t}, 9t^2 >$ at the point (0,2,1).

The parameter value that gives the point (0,2,1) comes from solving one of

$\ln(3t) = 0$ or $\sqrt{12t} = 2$ or $9t^2 = 1$

We get $t = \frac{1}{3}$.

Now, $\vec{x}'(t) = \left\langle \frac{d}{dt}(\ln(3t)), \frac{d}{dt}\left(\sqrt{12t}\right), \frac{d}{dt}\left(9t^2\right) \right\rangle = \left\langle \frac{3}{3t}, \frac{12}{2\sqrt{12t}}, 18t \right\rangle = \left\langle \frac{1}{t}, \frac{6}{\sqrt{12t}}, 18t \right\rangle$

$$\vec{x}'\left(\frac{1}{3}\right) = \left\langle \frac{1}{(1/3)}, \frac{6}{\sqrt{12\cdot(1/3)}}, 18\cdot\frac{1}{3} \right\rangle = \langle 3,3,6 \rangle$$

So, the vector equation of the tangent line at (0,2,1) in the direction <3,3,6> is

$$\vec{x} = \vec{x}_0 = t\vec{v} =< 0,2,1 > +t < 3,3,6 >$$

Parametric equations are now given by $x = 0 + 3t = 3t$, $y = 2 + 3t$, $z = 1 + 6t$

Practice Exercises 13.2

From the text pages 876–877, do problems 1, 5, 7, 15, 19, 25, 27, 33, 39, 49, and 53.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Arc Length and Curvature

Learning Outcomes

Upon completion of Arc Length and Curvature, you should be able to:

- Find the arc length of a space curve.
- Calculate the arc length function and re-parametrize the curve in terms of its arc length.
- Calculate the curvature of a curve.
- Find the unit normal and binormal vectors to a curve at a given point.
- Find equations for the normal plane and osculating plane at a point on a curve.
- Find and graph the osculating circle of a curve at a point on the curve.

Readings

Read section 13.3, pages 877–883 in your textbook. Carefully study the examples worked out in the text.

Overview

Recall the problem introduced in integral calculus of determining the length of a curve in the plane. In this problem the graph of an arc is approximated by a sequence of chords. Then, as the length of the longest of these chords approaches zero, the Riemann sum/definite integral is defined as the arc length.

From the Pythagorean theorem, the length of one of these chords can be written as

$$\Delta s = \sqrt{(\Delta x)^2 + (\Delta y)^2}$$

If the coordinates x and y for a point on the set defining the arc are each functions of the parameter t, then as $\Delta t \to 0$, $\Delta x \to 0$ and $\Delta y \to 0$, we have the familiar relationship

$$\frac{ds}{dt} = \lim_{\Delta t \to 0} \frac{\Delta s}{\Delta t} = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2} \quad \ldots\ldots\ldots\ldots\ldots(1)$$

Using the language of vectors,

$$\frac{d\vec{r}}{dt} = \left\langle \frac{dx}{dt}, \frac{dy}{dt} \right\rangle$$

and so the equation (1) above becomes

$$\frac{ds}{dt} = \left| \frac{d\vec{r}}{dt} \right| = |\vec{r}\,'(t)|$$

Put in words, the magnitude of the tangent to the path equals the rate of change of the path arc length with respect to the parameter t.

To find the total path length over some interval, we integrate to get

$$L = \int_a^b ds = \int_a^b |\, \vec{r}\,'(t) \,|\, dt$$

(See equation 3 on page 878 of the text.)

This arc length formula is valid for a space curve in R^3. This arc length formula gives the same answer independent of the parametrization used to define the curve.

The <u>arc length function</u> is defined by replacing the upper limit in the arc length formula by t:

$$s(t) = \int_a^t |\, \vec{r}\,'(u) \,|\, du = \int_a^t \sqrt{\left(\frac{dx}{du}\right)^2 + \left(\frac{dy}{du}\right)^2 + \left(\frac{dz}{du}\right)^2}\; du$$

where the dummy variable u is used to avoid confusion between the differential and the limit.

Differentiating the above we get, by the Fundamental Theorem of Calculus,

$$\frac{ds}{dt} = |\vec{r}\,'(t)|$$

which states the obvious: the speed is equal to the time rate of change of arc length.

If the parameter t is equal to the arc length s, then

$$\frac{ds}{dt} = \frac{d}{ds}(s) = 1 = |\vec{r}\,'(t)|$$

Thus, the speed is 1, and the tangent vector is always a unit vector when the parameter is equal to the arc length.

Conversely, it is obvious that if the tangent vector is always a unit vector, then the arc length equals the parameter t. If this is the case, the curve is said to be parametrized by arc length.

It is sometimes useful to parametrize a curve with respect to its arc length. If a curve $\vec{r} = \vec{r}(t)$ is given and $s = s(t)$, the arc length function is calculated, it may be possible to solve the arc length function for t in terms of s: $t = t(s)$. The curve can then be re-parametrized in terms of s by substituting for t in the vector function to get $\vec{r} = \vec{r}(t(s))$.

Another important concept discussed in this section is "curvature". Curvature of a curve at a given point is a measure of how quickly the curve changes direction.

Recall that the unit tangent vector is given by

$$\vec{T}(t) = \frac{\vec{r}'(t)}{|\vec{r}'(t)|}$$

and it indicates the direction of the curve.

We define **curvature** to be the magnitude of the rate of change of the unit tangent vector with respect to arc length. Arc length is used here so that the curvature will be independent of the parametrization used for the curve.

So, the **curvature** κ of a curve is defined by

$$\kappa = \left| \frac{d\overline{T}}{ds} \right|$$

where $\vec{T}$ is the unit tangent vector.

It turns out that curvature is easier to compute in terms of the parameter t instead of the arc length s.

So, $\kappa = \left| \frac{d\overline{T}}{ds} \right| = \left| \frac{d\overline{T}}{dt} \cdot \frac{ds}{dt} \right| = \left| \frac{d\overline{T} / dt}{ds / dt} \right| = \left| \frac{\vec{T}'(t)}{\vec{r}'(t)} \right|$

(See equation 9 on page 880 in the text.)

Study Example 3, page 880 of the text. It demonstrates that the curvature of a circle is simply the reciprocal of the radius-of curvature.

Equation 10 on page 880 of the text gives another formula for curvature that is often easier to apply.

At any point on a curve you can construct a vector that is normal or perpendicular to the curve meaning that it is perpendicular to the unit tangent vector $\vec{T}(t)$ at that point. There are many such vectors but we single out a special one, called the **unit normal**, defined by

$$\vec{N}(t) = \frac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|}$$

The vector $\vec{B} = \vec{T}(t) \times \vec{N}(t)$ constructed from the unit tangent and normal vectors is also a unit vector, called the **binormal vector**. By the cross product construction it is a vector that is perpendicular to both $\vec{T}(t)$ and $\vec{N}(t)$.

The **normal plane** determined by the unit normal and binormal vectors and the **osculating plane** determined by the unit tangent and normal vectors are discussed at the top of page 883 in the text. Read these two paragraphs and pay attention to the description of the **osculating circle** that lies in the osculating plane.

Example Exercises

1. Parametrize the curve $\vec{x}(t) =< 2t, t, 1-t >, \quad 0 \le t \le 1$, by arc length.

Differentiating with respect to *t*,

$$\vec{x}'(t) =< 2, 1, -1 >$$

So, $\left|\frac{d\vec{x}}{dt}\right| = \sqrt{4+1+1} = \sqrt{6}$

Integrating to get the arc length as a function of t, using the dummy variable u to avoid confusion with the limit

$$s = \int_0^t \left|\frac{d\vec{x}}{du}\right| du = \int_0^t \sqrt{6}\ du = \sqrt{6}\,t$$

Thus, to parametrize in terms of arc length, we put $t = \frac{s}{\sqrt{6}}$ into the vector function to get the re-parametrization in terms of s

$$\vec{x}(s) = \left\langle \frac{2s}{\sqrt{6}}, \frac{s}{\sqrt{6}}, 1 - \frac{s}{\sqrt{6}} \right\rangle$$

2. Find the curvature and the unit tangent and normal vectors to the curve $\vec{x}(t) = <3\sin t, 3\cos t, 4t>$ at $t = \frac{\pi}{2}$.

$$\vec{x}'(t) = <3\cos t, -3\sin t, 4>$$

$$|\vec{x}'(t)| = \sqrt{9\cos^2 t + 9\sin^2 t + 16} = \sqrt{9\left(\cos^2 t + \sin^2 t\right) + 16} = \sqrt{9 \cdot 1 + 16} = 5$$

$$\vec{T}(t) = \frac{\vec{x}'(t)}{|\vec{x}'(t)|} = \frac{<3\cos t, -3\sin t, 4>}{5} = \left\langle \frac{3}{5}\cos t, -\frac{3}{5}\sin t, \frac{4}{5} \right\rangle$$

At $t = \frac{\pi}{2}$ the unit tangent vector is

$$\vec{T}\left(\frac{\pi}{2}\right) = \left\langle \frac{3}{5}\cos\left(\frac{\pi}{2}\right), -\frac{3}{5}\sin\left(\frac{\pi}{2}\right), \frac{4}{5} \right\rangle = \left\langle \frac{3}{5} \cdot 0, -\frac{3}{5} \cdot 1, \frac{4}{5} \right\rangle = \left\langle 0, -\frac{3}{5}, \frac{4}{5} \right\rangle$$

$$\vec{N}(t) = \frac{\vec{T}'(t)}{|\vec{T}'(t)|} = \frac{\left\langle -\frac{3}{5}\sin t, -\frac{3}{5}\cos t, 0 \right\rangle}{\sqrt{\frac{9}{25}\sin^2 t + \frac{9}{25}\cos^2 t + 0}} = \frac{\left\langle -\frac{3}{5}\sin t, -\frac{3}{5}\cos t, 0 \right\rangle}{\left(\frac{3}{5}\right)} = \left\langle -\sin t, -\cos t, 0 \right\rangle$$

At $t = \frac{\pi}{2}$ the unit normal vector is

$$\vec{N}\left(\frac{\pi}{2}\right) = \left\langle -\sin\left(\frac{\pi}{2}\right), -\cos\left(\frac{\pi}{2}\right), 0 \right\rangle = \left\langle -1, 0, 0 \right\rangle$$

Now, $\kappa = \left| \frac{\vec{T}'(t)}{\vec{x}'(t)} \right| = \frac{(3/5)}{5} = \frac{3}{25}$ (a constant for all t)

At $t = \frac{\pi}{2}$ the curvature is $\kappa = \frac{3}{25}$.

Practice Exercises 13.3

From the text pages 884–886, do problems 3, 11, 13, 19, 23, 29, 33, 39, 47, 49, 53, and 57.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Motion in Space: Velocity and Acceleration

Learning Outcomes

Upon completion of Motion in Space: Velocity and Acceleration, you should be able to:

- Calculate the velocity, speed and acceleration of an object with a given position vector.
- Write down Newton's Second Law of Motion and apply it to solve motion problems.
- Solve projectile problems.
- Calculate the tangential and normal components of acceleration and be able to write the acceleration vector in terms of these components.
- Use the integration process to recover velocity when acceleration is known and recover position when velocity is known.

Readings

Read section 13.4, pages 886–891 in your textbook. Omit Kepler's Laws of Planetary Motion. Carefully study the examples worked out in the text.

Overview

If $\vec{r} = \vec{r}(t)$ describes the **position** of some object at time t, then $\vec{v}(t) = \vec{r}'(t)$ describes the velocity of the object at time t and is called the **velocity vector**. The velocity vector is a tangent vector to the curve described by the vector function $\vec{r} = \vec{r}(t)$ and points in the direction of the tangent line. The **speed** of the object is magnitude of the velocity vector, $\left|\vec{v}(t)\right|$, and describes the rate of change of distance with respect to time.

The rate of change of velocity with respect to time is called **acceleration**. It is calculated as the first derivative of velocity or the second derivative of position; that is,

$$\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$$

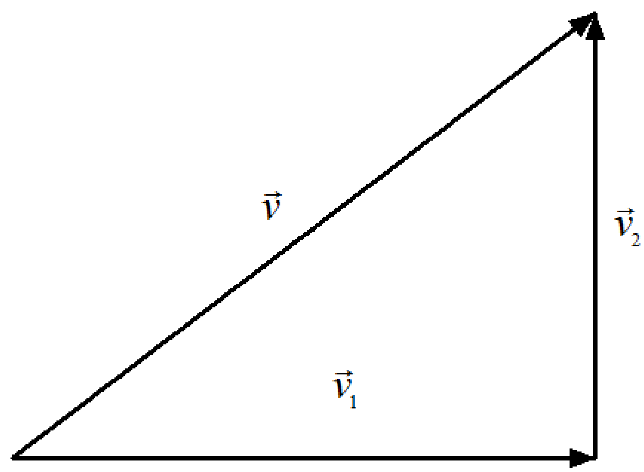
Because of the derivative relationships, we can use the vector integration process to retrieve velocity when acceleration is known or retrieve position when velocity is known.

Formally,

$$\vec{v}(t) = v(t_0) + \int_{t_0}^{t} \vec{a}(u)\,du \text{ and } \vec{r}(t) = \overline{r}(t_0) + \int_{t_0}^{t} \vec{v}(u)\,du$$

Read carefully through Examples 4, 5 and 6 on pages 888 and 889 of the text. These problems are based upon Newton's Second Law of Motion.

Using the geometric definition for adding vectors, any vector can be broken down as the vector sum of two components, one in the direction of the horizontal and the other in the direction of the vertical.



Adding the arrows we get $\vec{v} = \vec{v}_1 + \vec{v}_2$.

This is particularly useful in motion problems when working with the acceleration vector. Here the magnitude of the components are referred to as the **tangential and normal components of acceleration**.

Stewart develops a representation for the tangential and normal components of the acceleration vector in terms of the unit tangent and normal vectors. (See equation 7 on page 890 of the text.)

This representation is not always easy to use. It is desirable to have an expression for each component in terms of the vector function $\vec{r} = \vec{r}(t)$ and its derivatives. In equations 9 and 10 on page 891, Stewart gives such a representation for the tangential and normal components of acceleration in terms of $\vec{r}(t)$ and its first two derivatives.

Example Exercises

1. This is problem 22 from your text on page 894.

   Show that if a particle moves with constant speed, then the velocity and acceleration vectors are orthogonal.

   Since $\text{speed} = \left|\vec{v}(t)\right| = c$ (a constant) and $\left|\vec{v}(t)\right|^2 = \vec{v}(t)\cdot\vec{v}(t)$, we have that

   $\vec{v}(t)\cdot\vec{v}(t) = c^2$

   Differentiating both sides of the equation with respect to *t* and using Rule 4 in equation 3 on page 874 of the text, we have

$$\begin{aligned}
&\frac{d}{dt}\left[\vec{v}(t)\cdot\vec{v}(t)\right] = \frac{d}{dt}\left(c^2\right) = 0\\
&\vec{v}'(t)\cdot\vec{v}(t) + \vec{v}(t)\cdot\vec{v}'(t) = 0\\
&2\,\vec{v}'(t)\cdot\vec{v}(t) = 0\\
&\vec{v}'(t)\cdot\vec{v}(t) = 0\\
&\vec{a}(t)\cdot\vec{v}(t) = 0
\end{aligned}$$

   Since the dot product of the velocity and acceleration vectors is zero, the vectors are orthogonal.

   Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

2. This is problem 40 from your text on page 895.

   Find the normal and tangential components of the acceleration vector for a particle moving with position vector $\vec{r}(t) = t\,\vec{i} + t^2\,\vec{j} + 3t\,\vec{k}$.

   We will use the formulas in equations 9 and 10 on page 891 of the text.

$$\vec{r}\,'(t) = \vec{i} + 2t\vec{j} + 3\vec{k}$$

$$\left|\vec{r}\,'(t)\right| = \sqrt{1^2 + (2t)^2 + 3^2} = \sqrt{4t^2 + 10}$$

$$\vec{r}\,''(t) = 0\vec{i} + 2\vec{j} + 0\vec{k} = 2\vec{j}$$

$$\vec{r}\,'(t)\times\vec{r}\,''(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k}\\ 1 & 2t & 3\\ 0 & 2 & 0\end{vmatrix} = -6\vec{i} - 0\vec{j} + 2\vec{k}$$

$$\left|\vec{r}\,'(t)\times\vec{r}\,''(t)\right| = \sqrt{(-6)^2 + 2^2} = \sqrt{40} = 2\sqrt{10}$$

Now, $a_T = \dfrac{\vec{r}'(t)\cdot\vec{r}''(t)}{|\vec{r}'(t)|} = \dfrac{<1,2t,3>\cdot<0,2,0>}{\sqrt{4t^2+10}} = \dfrac{4t}{\sqrt{4t^2+10}}$

and $a_N = \dfrac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|} = \dfrac{|<-6,0,2>|}{\sqrt{4t^2+10}} = \dfrac{2\sqrt{10}}{\sqrt{4t^2+10}}$

Note: the acceleration vector would be written in terms of its components as

$$\vec{a}(t) = a_T\,\vec{T} + a_N\,\vec{N} = \left(\frac{4t}{\sqrt{4t^2+10}}\right)\vec{T} + \left(\frac{2\sqrt{10}}{\sqrt{4t^2+10}}\right)\vec{N}$$

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

Practice Exercises 13.4

From the text pages 894–895, do problems 5, 11, 15, 19, 23, 27, 31, 39, and 45.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Unit 2: Summary and Self-Test

You have now worked through Unit 2 in MATH 2111. It is time to take stock of what you have learned, review all of the material, and bring your shorthand notes up to date. A summary of the material covered so far is provided in the following pages. This summary should be modified, added to, and fleshed out to form a solid body of knowledge.

When you have completed your review, you should test your comprehension of the material with a closed book self-administered examination. Put all your notes aside, find a quiet place where you will not be disturbed, and take the examination provided at the end of this unit. You will find some questions straightforward and easy, but others will test your ingenuity.

You will find the solutions to the Unit 2 exam questions, and the point value for each question in the Answer Key provided at the end of this unit. Become your own examiner. If you have done well, according to your personal standards, go on to Unit 3. If not, then more review and practice is obviously called for.

Summary

Vector Functions and Space Curves

$\vec{r}: I \to R^3$, defined by $\vec{r}(t) =< f(t), g(t), h(t) >$, for t belonging to some set of real numbers I , is a vector function with component functions $f, g, h: I \to R$

In the case I is an interval of the real line, the collection of points (x, y, z) in 3-space defined by $x = f(t)$, $y = g(t)$, $z = h(t)$ is called a space curve. The component functions define parametric equations, and t is called a parameter.

If $\vec{r}(t) =< f(t), g(t), h(t) >$, then $\lim_{t \to a} \vec{r}(t) =< \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) >$, provided the limit of the component functions exist.

A vector function $\vec{r}(t) =< f(t), g(t), h(t) >$ is continuous at a if

$\lim_{t \to a} \vec{r}(t) =< \lim_{t \to a} f(t), \lim_{t \to a} g(t), \lim_{t \to a} h(t) >=< f(a), g(a), h(a) >= \vec{r}(a)$

Derivatives and Integrals of Vector Functions

If $\vec{r}(t) =< f(t), g(t), h(t) >$, where $f, g, h: R \to R$ are differentiable functions, then $\vec{r}'(t) =< f'(t), g'(t), h'(t) >$.

$\vec{r}'(t)$ is always tangent to the space curve defined by $\vec{r}$ at the point P on the curve where it is drawn.

A vector equation for the tangent line to $\vec{r} = \vec{r}(t)$ at $P = \vec{r}(t_0)$ is:

$\vec{x}(t) = \vec{r}(t_0) + t\,\vec{r}\,'(t_0)$

$\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ is the unit tangent vector.

Suppose $\vec{u}$ and $\vec{v}$ are differentiable vector functions, c is a scalar, and f is a real-valued function. Then the following properties hold:

$$\frac{d}{dt}\left(\vec{u}(t) + \vec{v}(t)\right) = \vec{u}\,'(t) + \vec{v}\,'(t)$$

$$\frac{d}{dt}\left(c\vec{u}(t)\right) = c\,\vec{u}\,'(t)$$

$$\frac{d}{dt}\left(f(t)\vec{u}(t)\right) = f\,'(t)\vec{u}(t) + f(t)\vec{u}\,'(t)$$

$$\frac{d}{dt}\left(\vec{u}(t)\cdot\vec{v}(t)\right) = \vec{u}\,'(t)\cdot\vec{v}(t) + \vec{u}(t)\cdot\vec{v}\,'(t)$$

$$\frac{d}{dt}\left(\vec{u}(t)\times\vec{v}(t)\right) = \vec{u}\,'(t)\times\vec{v}(t) + \vec{u}(t)\times\vec{v}\,'(t)$$

$$\frac{d}{dt}\left(\vec{u}(f(t))\right) = \vec{u}\,'(f(t))\,f\,'(t)$$

If $\vec{r}$ is a continuous vector function,

$$\int_a^b \vec{r}(t)\,dt = \left\langle \int_a^b f(t)\,dt, \int_a^b g(t)\,dt, \int_a^b h(t)\,dt \right\rangle = \left(\int_a^b f(t)\,dt\right)\vec{i} + \left(\int_a^b g(t)\,dt\right)\vec{j} + \left(\int_a^b h(t)\,dt\right)\vec{k}$$

Arc Length

If the curve $\vec{r}(t) =< f(t), g(t), h(t) >$ is traversed exactly once as t increases from a to b, then the length of the curve is

$$L = \int_a^b \sqrt{\left(f\,'(t)\right)^2 + \left(g'(t)\right)^2 + \left(h'(t)\right)^2}\;dt = \int_a^b \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2 + \left(\frac{dz}{dt}\right)^2}\;dt$$

or more compactly,

$$L = \int_a^b ds = \int_a^b |\vec{r}\,'(t)|\,dt$$

The arc length function is

$$s(t)=\int_a^t \left|\vec{r}'(u)\right| du=\int_a^t \sqrt{\left(\frac{dx}{du}\right)^2+\left(\frac{dy}{du}\right)^2+\left(\frac{dz}{du}\right)^2}\, du$$

The curve can be parametrized by arc length, s, by replacing the parameter t in $\vec{r}(t)$ by s through solving the arc length function above for t as a function of s.

Curvature

The curvature of a curve is given by $\kappa=\left|\frac{d\vec{T}}{ds}\right|$.

Curvature is easier to compute in terms of the parameter t instead of the arc length s.

$$\kappa=\frac{\left|\vec{T}'(t)\right|}{\left|\vec{r}'(t)\right|}$$

Another convenient formula is: $\kappa(t)=\dfrac{\left|\vec{r}'(t)\times\vec{r}''(t)\right|}{\left|\vec{r}'(t)\right|^3}$

In the case of a plane curve with equation $y=f(x)$, choose x as the parameter and the curvature formula reduces to

$$\kappa(t)=\frac{\left|f''(x)\right|}{\left(1+\left(f'(x)\right)^2\right)^{3/2}}$$

The unit normal vector to the curve is: $\vec{N}(t)=\dfrac{\vec{T}'(t)}{\left|\vec{T}'(t)\right|}$.

The vector $\vec{B}=\vec{T}(t)\times\vec{N}(t)$ constructed from the unit tangent and normal vectors is also a unit vector, called the binormal vector. By the cross product construction it is a vector that is perpendicular to both $\vec{T}(t)$ and $\vec{N}(t)$.

Velocity and Acceleration

For an object moving on a curve $\vec{r}=\vec{r}(t)$, where t represents time, the velocity of the object at time t is

$$\vec{v}(t)=\vec{r}'(t)$$

and its speed is

$$\left|\vec{v}(t)\right|=\left|\vec{r}'(t)\right|$$

The acceleration of the object is $\vec{a}(t) = \vec{v}'(t) = \vec{r}''(t)$.

In terms of its normal component a_N and tangential component a_T, acceleration can be written:

$$\vec{a} = a_T T + a_N N = v'T + \kappa v^2 N$$

where $v = |\vec{v}|$ is the speed of the object.

In terms of $\vec{r}$, $\vec{r}'$ and $\vec{r}''$,

$$a_T = \frac{\vec{r}'(t)\cdot\vec{r}''(t)}{|\vec{r}'(t)|} \quad \text{and} \quad a_N = \frac{|\vec{r}'(t)\times\vec{r}''(t)|}{|\vec{r}'(t)|}$$

Self-Test (31 marks)

Treat this as a real test. Do not refer to any course materials. The time for this test is 1.5 hours. Use the answer key provided to mark your test. The point value for each question is posted in the left margin.

1. Given $\vec{r}(t) = \left\langle \sec(t), \ln(1+t), e^{2t} \right\rangle$ find

[2] a) $\vec{r}'(t)$ at $t = 0$.

[2] b) $\vec{T}(t)$, the unit tangent vector, at $t = 0$.

[2] c) An equation in vector form of the tangent line to the curve at $t = 0$.

[2] 2. Evaluate $\int_0^{\pi} \vec{r}(t)\,dt$, where $\vec{r}(t) = \left\langle 1, \sin(2t), e^{-t} \right\rangle$.

[2] 3. Show that the curve $\vec{r}_1(t) = \left\langle \sec(t), \ln(1+t), e^{2t} \right\rangle$ and the curve $\vec{r}_2(t) = \left\langle t+\cos(t), t^2+2t, e^{-t} \right\rangle$ intersect orthogonally at $t = 0$.

[2] 4. Find the length of the curve $\vec{r}(t) = \left\langle \frac{2t^3}{3}, \sqrt{3}t^2, 3t \right\rangle$ from the origin to the point $(18, 9\sqrt{3}, 9)$.

[4] 5. On the same axes sketch the two surfaces $x^2 + y^2 + z^2 = 8$, $z \geq 0$, and $z = \sqrt{x^2 + y^2}$.

Show the volume they contain and determine a vector function that represents their curve of intersection.

6. Given the space curve $\vec{r}(t) = \langle \cos(2t), \sin(2t), 4t \rangle$, $0 \le t \le \pi$

[4] a) Find the curvature, $\kappa(t)$, the unit tangent, $\vec{T}(t)$ and the unit the unit normal, $\vec{N}(t)$.

[2] b) Find the binormal vector $\vec{B}(t)$ at the point $(-1/2, \sqrt{3}/2, 4\pi/3)$

[2] 7. Given $\vec{r}(t) = \vec{u}(t) \times v(t)$ where $\vec{u}(3) = \langle 2, -2, 1 \rangle$, $\vec{u}'(3) = \langle 1, -3, 0 \rangle$ and $\vec{v}(t) = \left\langle t^2 - t, 2t^3, 2t - 3 \right\rangle$, find $\vec{r}'(3)$.

[3] 8. Find the curvature of $y = 2x^3 + 3x - 5$ at (0,-5).

[4] 9. Find the velocity, acceleration and speed of the particle with the position function $\vec{r}(t) = \langle 4\sin(t), 3\cos(t) \rangle$.

Sketch the path of the particle and draw the velocity and acceleration vectors at the point corresponding to $t = 3\pi/4$.

Answer Key

1. a) $\vec{r}(t) = \left\langle \sec(t), \ln(1+t), e^{2t} \right\rangle$

$$\vec{r}'(t) = \left\langle \sec(t)\tan(t), \frac{1}{1+t}, 2e^{2t} \right\rangle$$

$$\vec{r}'(0) = \left\langle \sec(0)\tan(0), \frac{1}{1+0}, 2e^{2(0)} \right\rangle = \langle 0, 1, 2 \rangle$$

b) $\vec{T}(0) = \dfrac{\vec{r}'(0)}{|\vec{r}'(0)|} = \dfrac{\langle 0,1,2\rangle}{\sqrt{0+1+4}} = \dfrac{1}{\sqrt{5}}\langle 0,1,2\rangle = \left\langle 0, \dfrac{1}{\sqrt{5}}, \dfrac{2}{\sqrt{5}} \right\rangle$

c) The tangent line is given by:

$\vec{x}(t) = \vec{r}(t_0) + t\,\vec{r}'(t_0) = \vec{r}(0) + t\,\vec{r}'(0)$

Now, $\vec{r}(0) = \left\langle \sec(0), \ln(1+0), e^{2(0)} \right\rangle = \langle 1,0,1 \rangle$ and $\vec{r}'(0) = \langle 0,1,2 \rangle$.

So, $\vec{x}(t) = \vec{r}(0) + t\,\vec{r}'(0) = \langle 1,0,1 \rangle + t\langle 0,1,2 \rangle$

2.

$$\begin{aligned}
\int_0^{\pi} \vec{r}(t)\,dt &= \int_0^{\pi} \left\langle 1, \sin(2t), e^{-t} \right\rangle dt \\
&= \left\langle \int_0^{\pi} 1\,dt, \int_0^{\pi} \sin(2t)\,dt, \int_0^{\pi} e^{-t}\,dt \right\rangle \\
&= \left\langle t\Big]_0^{\pi}, -\frac{1}{2}\cos(2t)\Big]_0^{\pi}, -e^{-t}\Big]_0^{\pi} \right\rangle \\
&= \left\langle \pi - 0, -\frac{1}{2}\left(\cos(2\pi) - \cos(0)\right), -\left(e^{-\pi} - e^{0}\right) \right\rangle \\
&= \left\langle \pi, 0, \frac{e^{\pi}-1}{e^{\pi}} \right\rangle
\end{aligned}$$

3. From question 1, $\vec{r}_1(0) = \langle 1,0,1 \rangle$.

Now, $\vec{r}_2(0) = \left\langle 0+\cos(0), 0^2+2(0), e^{-0} \right\rangle = \langle 1,0,1 \rangle$. So, the two curves intersect at the point (1,0,1).

$\vec{r}_2'(t) = \left\langle 1-\sin(t), 2t+2, -e^{-t} \right\rangle$; $\vec{r}_2'(0) = \left\langle 1-\sin(0), 2(0)+2, -e^{-0} \right\rangle = \langle 1,2,-1 \rangle$

Now, $\vec{r}_1'(0)\cdot\vec{r}_2'(0) = \langle 0,1,2 \rangle \cdot \langle 1,2,-1 \rangle = 0+2-2 = 0$

Hence, the curves intersect orthogonally at $t=0$.

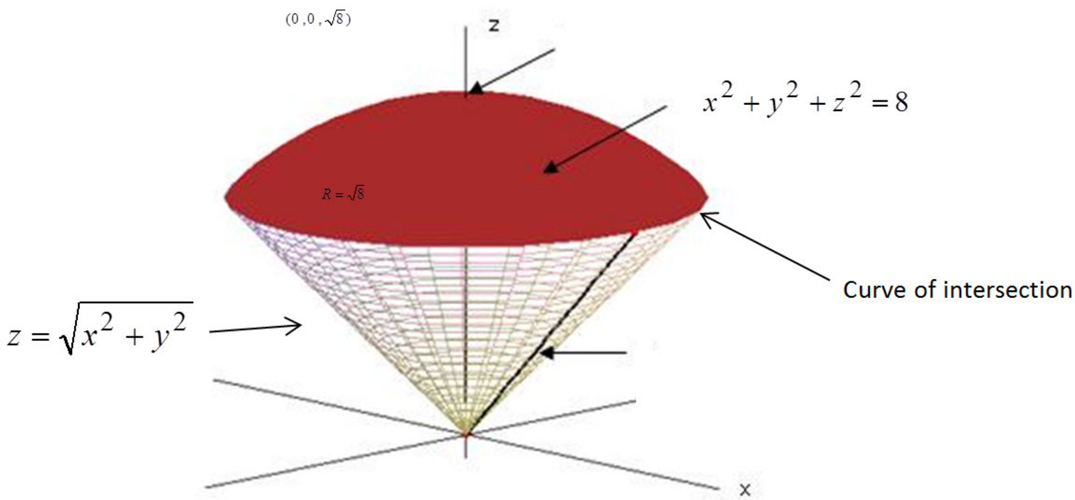
4. $\vec{r}(t) = \left\langle \frac{2t^3}{3}, \sqrt{3}t^2, 3t \right\rangle \Rightarrow \vec{r}'(t) = \left\langle 2t^2, 2\sqrt{3}t, 3 \right\rangle$; The curve runs from $(0,0,0)$, corresponding to $t=0$, to $(18, 9\sqrt{3}, 9)$, corresponding to $t=3$.

The length of the curve is:

$$\begin{aligned} L &= \int_0^3 \left| \vec{r}'(t) \right| dt = \int_0^3 \left| \left\langle 2t^2, 2\sqrt{3}t, 3 \right\rangle \right| dt \\ &= \int_0^3 \sqrt{4t^4 + 12t^2 + 9}\, dt \\ &= \int_0^3 \sqrt{\left(2t^2+3\right)^2}\, dt \\ &= \int_0^3 \left(2t^2+3\right) dt \\ &= \frac{2t^3}{3} + 3t \Big]_0^3 \\ &= \left(\frac{2(3)^3}{3} + 3(3) - 0 \right) = 18 + 9 = 27 \end{aligned}$$

5. The surface $x^2+y^2+z^2=8$, $z \ge 0$, is the top half of the sphere centered at the origin of radius $\sqrt{8}$. The surface $z=\sqrt{x^2+y^2}$ is a cone with vertex at $(0,0,0)$ and opening in the positive $z-$direction.

The picture of the region bounded by these two surfaces is given below.



Intersecting the two surfaces, we have

$$x^2+y^2+z^2=8 \text{ and } z^2=x^2+y^2$$
$$\Rightarrow x^2+y^2+\left(x^2+y^2\right)=8$$
$$\Rightarrow 2\left(x^2+y^2\right)=8$$
$$\Rightarrow x^2+y^2=4$$

Now, $z^2=x^2+y^2=4 \quad \Rightarrow \quad z=2 \ (z>0)$. The curve of intersection is a circle centered at (0,0,2) with radius 2, lying in the plane $z=2$.

The circle can be represented parametrically by

$x=2\cos(t)$, $y=2\sin(t)$, $0\le t\le 2\pi$

So, a vector equation for this space curve is:

$$\vec{r}(t)=<2\cos(t),2\sin(t),2>$$

6. a) $\vec{r}(t)=\langle\cos(2t),\sin(2t),4t\rangle,\ 0\le t\le\pi$

$$\vec{r}\,'(t)=\langle -2\sin(2t),2\cos(2t),4\rangle$$

Hence,

$$\vec{T}(t)=\frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}=\frac{\langle -2\sin(2t),2\cos(2t),4\rangle}{|\langle -2\sin(2t),2\cos(2t),4\rangle|}=\frac{\langle -2\sin(2t),2\cos(2t),4\rangle}{\sqrt{4\sin^2(2t)+4\cos^2(2t)+16}}$$

$$=\frac{\langle -2\sin(2t),2\cos(2t),4\rangle}{\sqrt{4\left(\sin^2(2t)+\cos^2(2t)\right)+16}}=\frac{\langle -2\sin(2t),2\cos(2t),4\rangle}{\sqrt{20}}=\left\langle -\frac{\sin(2t)}{\sqrt{5}},\frac{\cos(2t)}{\sqrt{5}},\frac{2}{\sqrt{5}}\right\rangle$$

Now, $\vec{T}'(t)=\left\langle -\frac{2\cos(2t)}{\sqrt{5}},-\frac{2\sin(2t)}{\sqrt{5}},0\right\rangle$. So,

$$\vec{N}(t)=\frac{\vec{T}'(t)}{|\vec{T}'(t)|}=\frac{\left(1/\sqrt{5}\right)\langle -2\cos(2t),-2\sin(2t),0\rangle}{\left(1/\sqrt{5}\right)\sqrt{4\cos^2(2t)+4\sin^2(2t)}}=\frac{\langle -2\cos(2t),-2\sin(2t),0\rangle}{\sqrt{4\left(\cos^2(2t)+\sin^2(2t)\right)}}$$

$$=\frac{\langle -2\cos(2t),-2\sin(2t),0\rangle}{2}=\langle -\cos(2t),-\sin(2t),0\rangle$$

$$\kappa=\frac{|\vec{T}'(t)|}{|\vec{r}\,'(t)|}=\frac{\left|\left(1/\sqrt{5}\right)\langle -2\cos(2t),-2\sin(2t),0\rangle\right|}{|\langle -2\sin(2t),2\cos(2t),4\rangle|}=\frac{\left(1/\sqrt{5}\right)\sqrt{4\cos^2(2t)+4\sin^2(2t)}}{\sqrt{4\sin^2(2t)+4\cos^2(2t)+16}}=\frac{2/\sqrt{5}}{2\sqrt{5}}=\frac{2}{10}=\frac{1}{5}$$

b)

$$B(t) = T(t) \times N(t) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\frac{\sin(2t)}{\sqrt{5}} & \frac{\cos(2t)}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ -\cos(2t) & -\sin(2t) & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{5}} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -\sin(2t) & \cos(2t) & 2 \\ -\cos(2t) & -\sin(2t) & 0 \end{vmatrix}$$

$$= \frac{1}{\sqrt{5}} \left(2\sin(2t)\vec{i} - 2\cos(2t)\vec{j} + \left(\sin^2(2t) + \cos^2(2t) \right) \vec{k} \right)$$

$$= \left\langle \frac{2\sin(2t)}{\sqrt{5}}, -\frac{2\cos(2t)}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

Now $\langle \cos(2t), \sin(2t), 4t \rangle = (-1/2, \sqrt{3}/2, 4\pi/3)$ yields

$\cos(2t) = -\frac{1}{2}$ and $\sin(2t) = \frac{\sqrt{3}}{2}$ for $0 \le t \le \pi$; $4t = \frac{4\pi}{3}$

Solving either equation we get $t = \frac{\pi}{3}$. So,

$$B(\pi/3) = \left\langle \frac{2\sin(2\pi/3)}{\sqrt{5}}, -\frac{2\cos(2\pi/3)}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle = \left\langle \frac{2\left(\sqrt{3}/2\right)}{\sqrt{5}}, -\frac{2(1/2)}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

$$= \left\langle \frac{\sqrt{3}}{\sqrt{5}}, -\frac{1}{\sqrt{5}}, \frac{1}{\sqrt{5}} \right\rangle$$

7. $\vec{r}(t) = \vec{u}(t) \times v(t)$, where $\vec{u}(3) = \langle 2, -2, 1 \rangle$, $\vec{u}'(3) = \langle 1, -3, 0 \rangle$ and

$$\vec{v}(t) = \left\langle t^2 - t, 2t^3, 2t - 3 \right\rangle.$$

$$\vec{r}'(t) = \frac{d}{dt}\left(\vec{u}(t) \times v(t) \right) = \vec{u}'(t) \times \vec{v}(t) + \vec{u}(t) \times \vec{v}'(t);$$

So, $\vec{r}\,'(3) = \vec{u}\,'(3) \times \vec{v}(3) + \vec{u}(3) \times \vec{v}\,'(3)$

Now, $\vec{v}'(t) = \langle 2t - 1, 6t^2, 2 \rangle$, $\vec{v}'(3) = \langle 2(3) - 1, 6(3)^2, 2 \rangle = \langle 5, 54, 2 \rangle$,

$\vec{v}(3) = \langle 3^2 - 3, 2(3)^3, 2(3) - 3 \rangle = \langle 6, 54, 3 \rangle$

Hence,

$$\begin{aligned}
\vec{r}\,'(3) &= \vec{u}\,'(3)\times\vec{v}(3)+\vec{u}(3)\times\vec{v}\,'(3)\\
&= \langle 1,-3,0\rangle\times\langle 6,54,3\rangle+\langle 2,-2,1\rangle\times\langle 5,54,2\rangle\\
&= \begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 1 & -3 & 0\\ 6 & 54 & 3\end{vmatrix}+\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 2 & -2 & 1\\ 5 & 54 & 2\end{vmatrix}\\
&= <-9,-3,72>+<-58,1,118>\\
&= <-67,-2,190>
\end{aligned}$$

8. Parametrize the function $y=2x^3+3x-5$ by using x as the parameter.

Then $\vec{r}(x)=\langle x,2x^3+3x-5\rangle$. Now $y'(x)=6x^2+3$ and $y''(x)=12x$.

So,

$$\kappa(x)=\frac{|y''(x)|}{\left(1+(y'(x))^2\right)^{3/2}}=\frac{|12x|}{\left(1+\left(6x^2+3\right)^2\right)^{3/2}}=\frac{12|x|}{\left(36x^4+36x^2+10\right)^{3/2}}$$

At the point $(0,-5)$, $\kappa(0)=\dfrac{12|0|}{\left(36(0)^4+36(0)^2+10\right)^{3/2}}=0$.

9. $\vec{r}(t)=\langle 4\sin(t),3\cos(t)\rangle$

The velocity of the particle is $\vec{v}(t)=\vec{r}\,'(t)=\langle 4\cos(t),-3\sin(t)\rangle$.

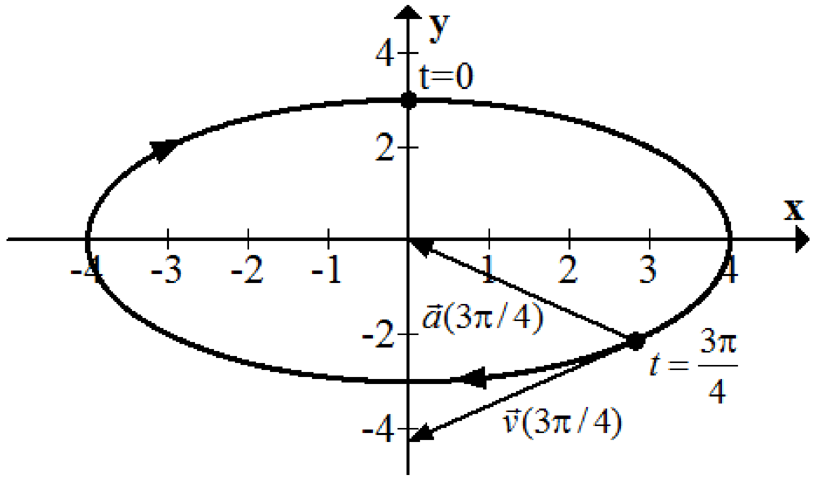
The speed of the particle is $|\vec{r}\,'(t)|=|\langle 4\cos(t),-3\sin(t)\rangle|=\sqrt{16\cos^2(t)+9\sin^2(t)}$

The acceleration of the particle is $\vec{a}(t)=\vec{r}\,''(t)=\langle -4\sin(t),-3\cos(t)\rangle$.

At $t=\dfrac{3\pi}{4}$, $\vec{v}(3\pi/4)=\langle 4\cos(3\pi/4),-3\sin(3\pi/4)\rangle=\left\langle -\dfrac{4}{\sqrt{2}},-\dfrac{3}{\sqrt{2}}\right\rangle$

and $\vec{a}(3\pi/4)=\langle -4\sin(3\pi/4),-3\cos(3\pi/4)\rangle=\left\langle -\dfrac{4}{\sqrt{2}},\dfrac{3}{\sqrt{2}}\right\rangle$

A sketch of the curve and of the velocity and acceleration vectors at $t=\dfrac{3\pi}{4}$ is given below.


y
x
4
2
-2
-4
t=0
-4
-3
-2
-1
1
2
3
4
$\vec{a}(3\pi/4)$
$t = \frac{3\pi}{4}$
$\vec{v}(3\pi/4)$