Faculty of Science

Unit 5: Applications––Triple Integration

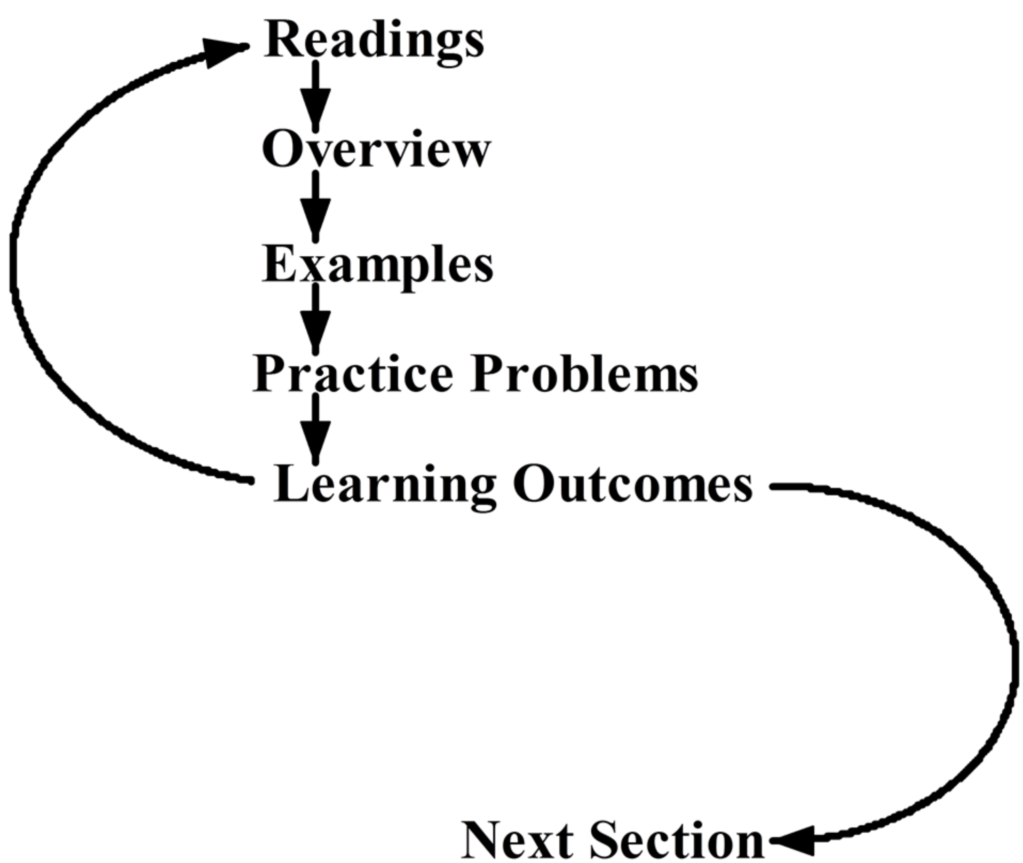
MATH 2111
Calculus III – Multivariable Calculus

Table of Contents

Instructions

The recommended procedure for working through each section of the units in this course is described in detail in your Course Guide.

This procedure is summarized below. If you are certain you have achieved the learning objectives, proceed to the next section. If you are uncertain about one or more of them, go back to the appropriate information in the section until you can complete the task listed in the objective.



Applications of Double Integrals

Learning Outcomes

Upon completion of Applications of Double Integrals, you should be able to:

- Identify what the double integral of a mass density or charge density function yields for a plane lamina.
- Calculate the mass, the centre of mass, the moment about the x-axis and the moment about the y-axis for a given lamina with a given density function.
- Determine density functions described by proportionality relationships, as in question 11 on page 1036 of the text.
- Calculate the moment of inertia about the x-axis, the moment of inertia about the y-axis and the moment of inertia about the origin for a given lamina with a given density function.
- Calculate the radius of gyration of a lamina about an axis.

Readings

Read section 15.5, pages 1027 –1032 (middle), in your textbook. Carefully study the examples worked out in the text. Notice that we are omitting the topics "Probability" and "Expected Values" from this section.

Overview

In this section we explore how the double integral can be used to calculate some fundamental quantities from physics, such as mass, electric charge, centre of mass and moment of inertia.

We start by considering a thin plate or **lamina** with variable density that occupies a region D of the xy-plane. Assuming $\rho(x, y)$ defines a continuous function on D that gives the density (in units of mass per unit area) at a point (x, y) in D, Stewart shows (on page 1028 of the text) that the total mass m of the lamina is given by the double integral

$$m = \iint_D \rho(x, y)\, dA$$

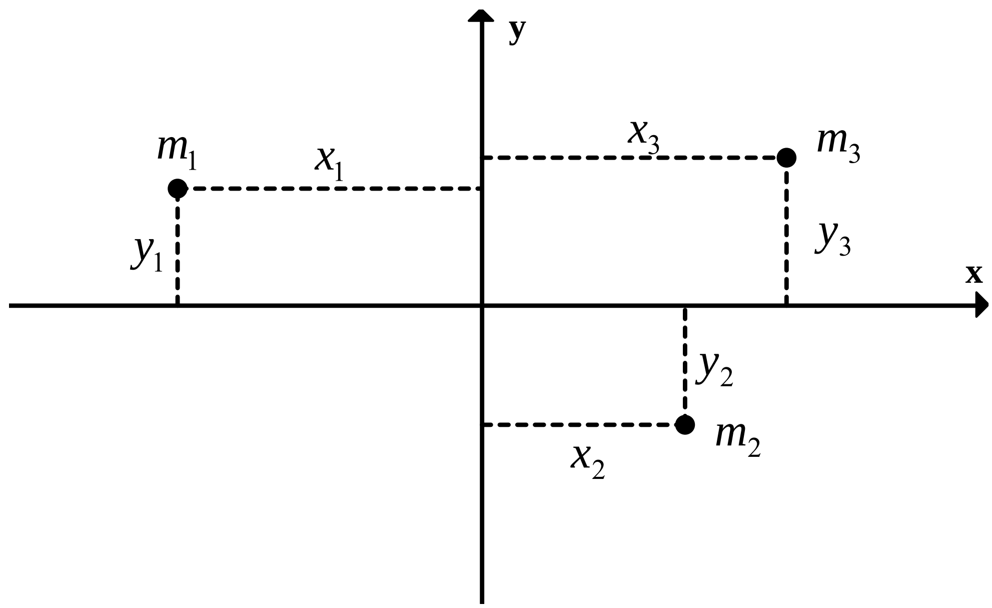
Notice the units of $\rho(x, y)\, dA$: $\dfrac{\text{unit of mass}}{\text{unit of area}} \cdot \left(\text{unit of area}\right) = \text{unit of mass}$

Other density functions are of interest to physicists. For example, if an electric charge is distributed over the lamina D according to the charge density function $\sigma(x, y)$ (in units of charge per unit area), then the total charge Q on D is

$$Q = \iint_D \sigma(x, y)\, dA$$

Recall that the moment of a particle about an axis is defined as the product of its mass and its "directed distance" from the axis. Its units are a unit of mass times a unit of distance.

For example, consider a system of three particles with masses m_1, m_2, m_3 at locations (x_1, y_1), (x_2, y_2), (x_3, y_3), respectively (see diagram below).



The moment of the system about the y-axis is calculated as:

$$M_y = m_1 x_1 + m_2 x_2 + m_3 x_3$$

while the moment of the system about the x-axis is calculated as:

$$M_x = m_1 y_1 + m_2 y_2 + m_3 y_3$$

If we want the moment of an entire lamina D about the x-axis or y-axis we need the double integral.

About the x-axis: $$M_x = \iint_D y\,\rho(x, y)\,dA$$

About the y-axis: $$M_y = \iint_D x\,\rho(x, y)\,dA$$

Again, notice the units of $y\,\rho(x, y)\,dA$ and $x\,\rho(x, y)\,dA$:

$$(\text{unit of distance})\cdot\frac{\text{unit of mass}}{\text{unit of area}}\cdot(\text{unit of area}) = (\text{unit of mass})\cdot(\text{unit of distance})$$

The point on the lamina where the lamina behaves as if all of its mass is concentrated there and so balances horizontally when supported at this place is called the **centre of mass** or the **centre of gravity**. In the case the density function is constant, this point is also called the **centroid**. See page 1029 in the text for the formula for calculating these coordinates.

On pages 1030 and 1031 in the text Stewart discusses the concept of **moment of inertia** (or **second moment**) and the corresponding integration formulas for the moments of inertia about the x-axis, the y-axis and the origin (see Boxes 6, 7, and 8). More generally, the moment of inertia of the lamina about any axis is given by:

$$I_a = \iint_D d_a^2(x, y)\,\rho(x, y)\,dA$$

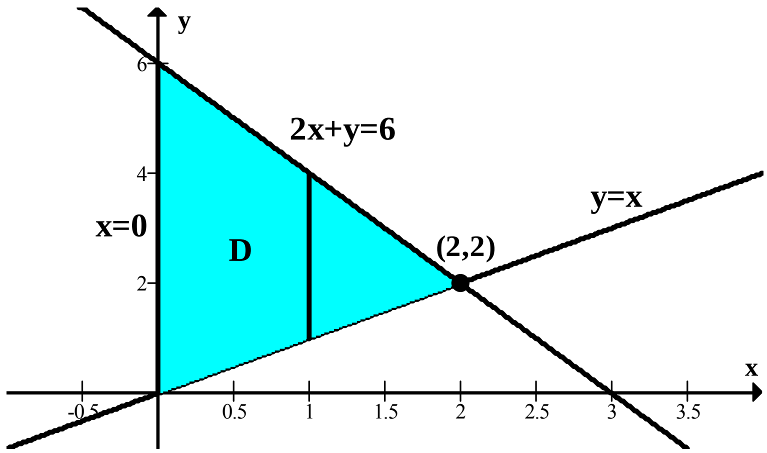
where $d_a(x, y) =$ the distance from (x, y) to the axis.

An examination of the expression $d_a^2(x, y)\,\rho(x, y)\,dA$ shows that it is calculating the quantity $\text{mass}\cdot(\text{distance})^2$.

Example Exercises

1. Find the mass and centre of mass of the lamina that occupies the triangular region enclosed by the lines $x = 0$, $y = x$, $2x + y = 6$ and has the density function $\rho(x, y) = x^2$.

   We start with a picture of the region.



Notice the region D is y-simple. All vertical strips start on the line $y = x$ and end on the line $2x + y = 6$. The mass of the region is:

$$
\begin{aligned}
m = \iint_D \rho(x,y)\,dA &= \int_0^2 \int_x^{6-2x} x^2\,dy\,dx \\
&= \int_0^2 x^2 \left[y\right]_x^{6-2x} dx \\
&= \int_0^2 x^2 \left(6-2x-x\right) dx \\
&= \int_0^2 \left(6x^2 - 3x^3\right) dx \\
&= \left[2x^3 - \frac{3}{4}x^4\right]_0^2 = (16-12) - 0 = 4
\end{aligned}
$$

The moments about the $x-$ and y-axes are:

$$
\begin{aligned}
M_y = \iint_D x\,\rho(x,y)\,dA &= \int_0^2 \int_x^{6-2x} x \cdot x^2\,dy\,dx \\
&= \int_0^2 \int_x^{6-2x} x^3\,dy\,dx \\
&= \int_0^2 x^3 \left[y\right]_x^{6-2x} dx \\
&= \int_0^2 x^3 \left(6-3x\right) dx \\
&= \int_0^2 \left(6x^3 - 3x^4\right) dx \\
&= \left[\frac{6}{4}x^4 - \frac{3}{5}x^5\right]_0^2 = \left(24 - \frac{96}{5}\right) - 0 = \frac{24}{5}
\end{aligned}
$$

$$M_x = \iint_D y\,\rho(x,y)\,dA = \int_0^2 \int_x^{6-2x} y \cdot x^2\,dy\,dx$$
$$= \int_0^2 x^2 \left[\frac{y^2}{2}\right]_x^{6-2x} dx$$
$$= \int_0^2 \frac{x^2}{2}\left[(6-2x)^2 - x^2\right]dx$$
$$= \frac{1}{2}\int_0^2 x^2\left(36 - 24x + 3x^2\right)dx$$
$$= \frac{1}{2}\int_0^2 \left(36x^2 - 24x^3 + 3x^4\right)dx$$
$$= \frac{1}{2}\left[12x^3 - 6x^4 + \frac{3}{5}x^5\right]_0^2 = \frac{1}{2}\left[\left(96 - 96 + \frac{96}{5}\right) - 0\right] = \frac{48}{5}$$

Calculating the centre of mass we have:

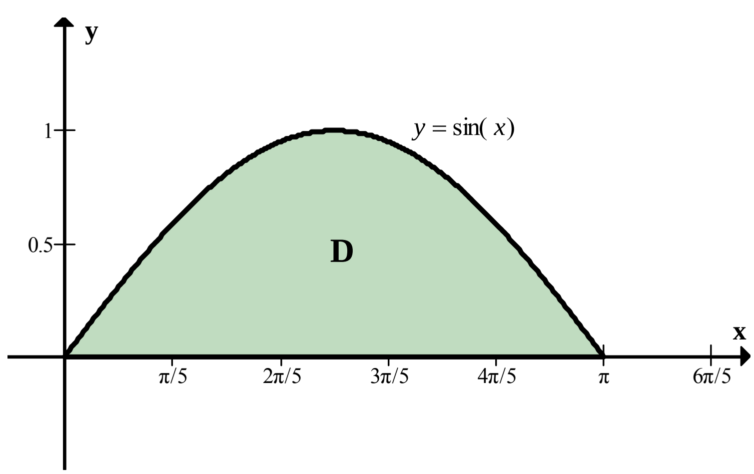
$$\overline{x} = \frac{M_y}{m} = \frac{(24/5)}{4} = \frac{6}{5} \text{ and } \overline{y} = \frac{M_x}{m} = \frac{(48/5)}{4} = \frac{12}{5}$$

The centre of mass is $\left(\frac{6}{5}, \frac{12}{5}\right)$.

2. This is problem 24 on page 1037 of the text.

A lamina with a constant density $\rho(x, y) = \rho$ occupies the region under the curve $y = \sin(x)$ from $x = 0$ to $x = \pi$. Find the moments of inertia I_x and I_y and the radii of gyration $\overline{\overline{x}}$ and $\overline{\overline{y}}$.

We start with a picture of the region.



The mass of the lamina is:

$$\begin{aligned} m &= \iint_D \rho(x,y)\,dA = \int_0^{\pi}\int_0^{\sin(x)} \rho\,dy\,dx \\ &= \rho\int_0^{\pi}\left[y\right]_0^{\sin(x)}dx \\ &= \rho\int_0^{\pi}\sin(x)\,dx \\ &= \rho\left[-\cos(x)\right]_0^{\pi} \\ &= -\rho(\cos(\pi)-\cos(0)) \\ &= -\rho(-1-1) \\ &= 2\rho \end{aligned}$$

The moments of inertia about the $x-$ and y-axes are:

$$\begin{aligned} I_y &= \iint_D \rho x^2\,dA = \rho\int_0^{\pi}\int_0^{\sin(x)} x^2\,dy\,dx \\ &= \rho\int_0^{\pi} x^2\left[y\right]_0^{\sin(x)}dx \\ &= \rho\int_0^{\pi} x^2\sin(x)\,dx \end{aligned}$$

Now using integration by parts twice, we obtain

$$\begin{aligned} \int x^2\sin(x)\,dx &= -x^2\cos(x) - \int -\cos(x)\cdot 2x\,dx \\ &= -x^2\cos(x) + 2\int x\cos(x)\,dx \\ &= -x^2\cos(x) + 2\left[x\sin(x) - \int \sin(x)\,dx\right] \\ &= -x^2\cos(x) + 2x\sin(x) + 2\cos(x) + C \end{aligned}$$

$$\boxed{\begin{aligned} u &= x^2, & dv &= \sin(x)\,dx \\ du &= 2x\,dx, & v &= -\cos(x) \end{aligned}}$$

$$\boxed{\begin{aligned} u &= x, & dv &= \cos(x)\,dx \\ du &= dx, & v &= \sin(x) \end{aligned}}$$

Hence,

$$\begin{aligned} I_y &= \rho\int_0^{\pi} x^2\sin(x)\,dx \\ &= \rho\left[-x^2\cos(x) + 2x\sin(x) + 2\cos(x)\right]_0^{\pi} \\ &= \rho\left[\left(-\pi^2\cos(\pi) + 2\pi\sin(\pi) + 2\cos(\pi)\right) - \left(0+0+2\cos(0)\right)\right] \\ &= \rho\left(-\pi^2(-1) + 0 + 2(-1) - 2\right) \\ &= \rho\left(\pi^2 - 4\right) \end{aligned}$$

$$I_x = \iint_D \rho y^2 \, dA = \rho \int_0^{\pi} \int_0^{\sin(x)} y^2 \, dy \, dx$$

$$= \rho \int_0^{\pi} \left[\frac{y^3}{3} \right]_0^{\sin(x)} dx$$

$$= \frac{\rho}{3} \int_0^{\pi} \sin^3(x) \, dx$$

$$= \frac{\rho}{3} \int_0^{\pi} \sin^2(x) \cdot \sin(x) \, dx$$

$$= \frac{\rho}{3} \int_0^{\pi} \left(1 - \cos^2(x)\right) \sin(x) \, dx$$

$$= \frac{\rho}{3} \int_0^{\pi} \left(\sin(x) + \left(-\sin(x)\right) \cos^2(x) \right) dx$$

$$= \frac{\rho}{3} \left[-\cos(x) + \frac{\cos^3(x)}{3} \right]_0^{\pi}$$

$$= \frac{\rho}{3} \left[\left(-\cos(\pi) + \frac{\cos^3(\pi)}{3} \right) - \left(-\cos(0) + \frac{\cos^3(0)}{3} \right) \right] = \frac{\rho}{3} \left(1 - \frac{1}{3} + 1 - \frac{1}{3} \right) = \frac{4}{9} \rho$$

Now, the radii of gyration are determined as follows:

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}} = \sqrt{\frac{4\rho/9}{2\rho}} = \sqrt{\frac{2}{9}} = \frac{\sqrt{2}}{3} \text{ and } \bar{\bar{x}} = \sqrt{\frac{I_y}{m}} = \sqrt{\frac{\rho\left(\pi^2 - 4\right)}{2\rho}} = \sqrt{\frac{\pi^2 - 4}{2}}$$

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

Practice Exercises 15.5

From the text pages 1036–1037, do problems 1, 5, 9, 11, 15, 17, 19, and 23.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Triple Integrals

Learning Outcomes

Upon completion of Triple Integrals, you should be able to:

- Set up and evaluate triple integrals as iterated integral expressions.
- Use triple integrals to calculate volumes, mass, moments about a coordinate plane, moments of inertia about a coordinate axis and the average value of a function of three variables defined on a solid region of 3-space. (See problems 53, 54 on page 1051 of the text.)
- Sketch the solid region whose volume is given by an iterated integral expression and write down the equivalent iterated integrals for the five other orders of integration.

Readings

Read section 15.7, pages 1041–1048, in your textbook. Carefully study the examples worked out in the text.

Overview

How do we extend the integration process to a function of 3 independent variables? We mimic what we did for a function of two variables!

Let $w = f(x, y, z)$ be a continuous function defined on a rectangular box B in 3-space, where $B = \{(x, y, z) \mid a \le x \le b,\ c \le y \le d,\ e \le z \le f\}$. Using the familiar Riemann sum construction we:

1. Partition B into rectangular sub-boxes of equal volume $\Delta V = \Delta x \cdot \Delta y \cdot \Delta z$ using planes parallel to the coordinate planes. (See Figure 1 on page 1041 of the text.)
2. Choose an arbitrary point $\left(x_i^*, y_j^*, z_k^*\right)$ in each sub-box.
3. Form the triple Riemann sum

$$\sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} f\left(x_i^*, y_j^*, z_k^*\right) \Delta V$$

where $m, n,\ p$ are the number of subintervals of $[a, b], [c, d]$ and $[e, f]$, respectively.

4. Take the limit as the number of subintervals becomes infinite.

We define the triple integral of f over B to be the value of this limit, if it exists; that is,

$$\iiint_B f(x,y,z)\,dV = \lim_{m,n,p\to\infty} \sum_{i=1}^{m}\sum_{j=1}^{n}\sum_{k=1}^{p} f\left(x_i^*, y_j^*, z_k^*\right)\Delta V$$

As for double integrals, triple integrals exist for continuous functions.

We extend the definition of the triple integral of $w = f(x,y,z)$ to a bounded region E in 3-space, that is, a region which can be enclosed in a rectangular box B, by defining a new function F on B as follows:

For all (x,y,z) in B, $F(x,y,z) = \begin{cases} f(x,y,z) & \text{if } (x,y,z) \text{ is in } E \\ 0 & \text{if } (x,y,z) \text{ is outside } E \end{cases}$

If F is integrable over B, we define the **triple integral of f over E** by

$$\iiint_E f(x,y,z)\,dV = \iiint_B F(x,y,z)\,dV$$

Fortunately, Fubini's Theorem extends to higher dimensions and triple integrals can be evaluated as iterated integral expressions over rectangular boxes or over more general "simple" regions.

<u>Over rectangular boxes</u>: $B = \{(x,y,z) \mid a \le x \le b,\ c \le y \le d,\ e \le z \le f\}$

$$\iiint_B f(x,y,z)\,dV = \int_a^b \int_c^d \int_e^f f(x,y,z)\,dz\,dy\,dx$$

Notice there are $3 \cdot 2 \cdot 1 = 6$ possible orders of integration here, all of which will give the same value for the triple integral.

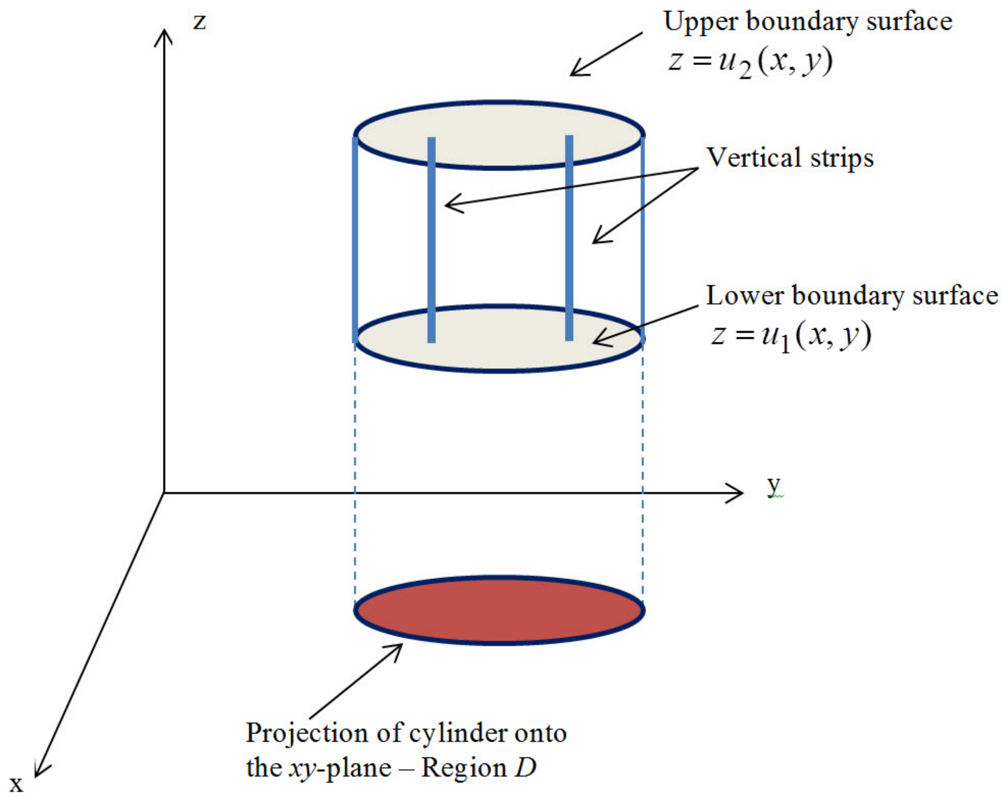
Over "simple" regions:

Stewart calls these regions of "Type I", "Type II" or "Type III". I prefer to call them "$z-\text{simple}$", $x-\text{simple}$ or $y-\text{simple}$, respectively, because they indicate the direction for the first integration process.

Let's look at the description of a $z-\text{simple}$, or Type I, region. The definitions for $x-\text{simple}$ and $y-\text{simple}$ are analogous.

Basically, a $z-\text{simple}$ region is one in which every strip in the solid region that is parallel to the *z*-axis starts on the same lower boundary surface and ends on the same upper boundary surface. (See Figure 2 on page 1042 of the text.)

For example, consider the cylindrical region in 3-space below. This is a $z-\text{simple}$ region. Any strip inside this region which is parallel to the *z*-axis (there are two such strips shown) always starts on the lower boundary surface $z=u_1(x,y)$, which is a closed disk here, and ends on the upper boundary surface $z=u_2(x,y)$, which is also a closed disk.



Notice that the projection of this solid region onto the xy-plane gives a closed disk D.

Formally, we could describe this solid region E by

$$E=\{(x,y,z)\,|\,(x,y)\in D,\, u_1(x,y)\le z\le u_2(x,y)\}$$

Compare with Equation 5 on page 1042 of the text. With this description of E we define

$$\iiint_E f(x,y,z)\,dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz \right] dA$$

The first integration is a partial integration with respect to z, holding x and y constant. The result is a function of x and y that can now be integrated over the region D in the xy-plane as a double integral.

Viewing the projection D of the solid into the xy-plane as either $x-\text{simple}$ or $y-\text{simple}$ allows us to express the triple integral as an iterated integral expression.

If $D = \{(x,y) \mid a \le x \le b,\, g_1(x) \le y \le g_2(x)\}$, then we have

$$\iiint_E f(x,y,z)\,dV = \int_a^b \int_{g_1(x)}^{g_2(x)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dy\,dx$$

If $D = \{(x,y) \mid c \le y \le d,\, h_1(y) \le x \le h_2(y)\}$, then we have

$$\iiint_E f(x,y,z)\,dV = \int_c^d \int_{h_1(y)}^{h_2(y)} \int_{u_1(x,y)}^{u_2(x,y)} f(x,y,z)\,dz\,dx\,dy$$

Interpretations of the triple integral:

1. If $f(x,y,z)=1$, then $\iiint_E 1\,dV = \iiint_E dV$ equals the volume of the solid region E.
2. If $w = f(x,y,z)$ is a **mass density function** in units of mass per unit volume, then $\iiint_E f(x,y,z)\,dV$ represents the total mass of E.
3. If $w = f(x,y,z)$ is a **charge density function** in units of charge per unit volume, then $\iiint_E f(x,y,z)\,dV$ equals the total charge on E.

   The concept of "moment about a coordinate axis" in the xy-plane extends to the concept of "moment about a coordinate plane" in 3-space.

The moment about an axis was defined as the product of mass with the directed distance to the axis. In 3-space, we define the concept of **moment about a coordinate plane** to be the product of mass with the directed distance to that plane.

If E is a solid object with mass density function $\rho(x, y, z)$, in units of mass per unit volume, then the directed distance to the yz-plane is represented by x, and so the moment of E about the yz-plane is given by the triple integral

$$M_{yz} = \iiint_E x\,\rho(x, y, z)\,dV$$

Notice how the symbol and integral expression for moment about the y-axis in the plane

$$M_y = \iint_D x\,\rho(x, y)\,dA$$

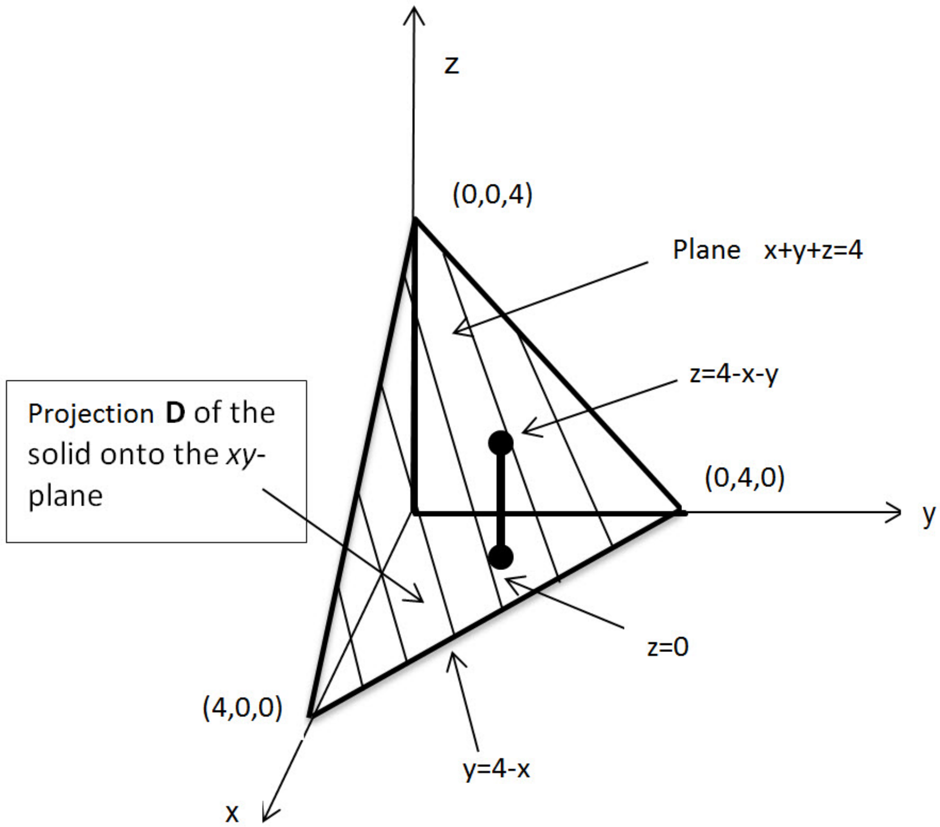
has changed for moment about the yz-plane.

Similar integral expressions exist for the moments M_{xy} and M_{xz} about the xy and xz-planes, respectively. See page 1047 in the text for these formulas, as well as integral expressions for calculating the coordinates of the **centre of mass** and **moments of inertia** about the coordinate axes.

Example Exercises

1. Evaluate $\iiint_E e^{x+y+z}\,dV$, where E is the solid region bounded in the first octant by $x + y + z = 4$ and the coordinate planes.

   It is important to start with a picture of the solid region E, the region of integration.



In this example we can view E as $x-\text{simple}$, $y-\text{simple}$ or $z-\text{simple}$. If we view E as $z-\text{simple}$, then every strip parallel to the z-axis in this region starts on the xy-plane($z=0$) and ends on the plane $x+y+z=4$ or $z=4-x-y$. The projection of the solid onto the xy-plane yields the triangular region D identified in the diagram.

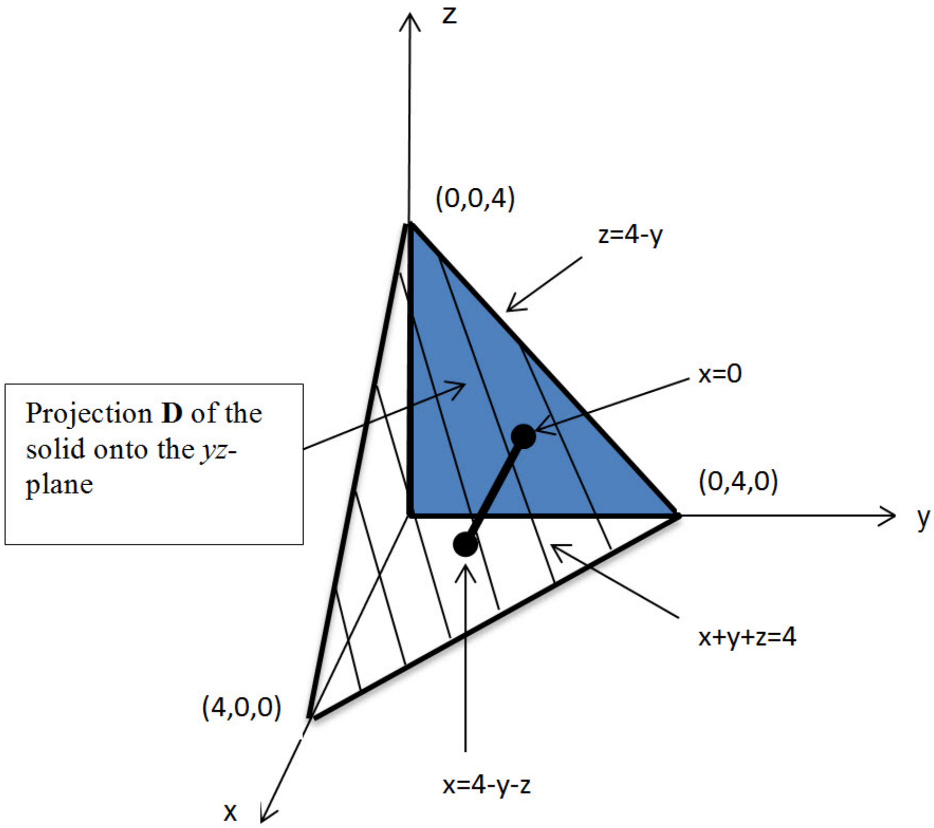
Viewing D as a $y-\text{simple}$ region in the xy-plane, we have

$$D=\{(x,y)\,|\,0\le x\le 4\,,0\le y\le 4-x\}$$

Hence, $E=\{(x,y,z)\,|\,0\le x\le 4\,,0\le y\le 4-x,0\le z\le 4-x-y\}$ and so

$$
\begin{aligned}
\iiint_E e^{x+y+z}\,dV &= \int_0^4\int_0^{4-x}\int_0^{4-x-y} e^{x+y+z}\,dz\,dy\,dx \\
&= \int_0^4\int_0^{4-x} \left[e^{x+y+z}\right]_{z=0}^{z=4-x-y} dy\,dx \\
&= \int_0^4\int_0^{4-x} \left[e^{x+y+(4-x-y)} - e^{x+y+0}\right] dy\,dx \\
&= \int_0^4\int_0^{4-x} \left(e^4 - e^{x+y}\right) dy\,dx \\
&= \int_0^4 \left[e^4 y - e^{x+y}\right]_{y=0}^{y=4-x} dx \\
&= \int_0^4 \left[\left(e^4(4-x) - e^{x+(4-x)}\right) - \left(0 - e^{x+0}\right)\right] dx \\
&= \int_0^4 \left(e^4(4-x) - e^4 + e^x\right) dx \\
&= \int_0^4 \left[3e^4 - e^4 x + e^x\right] dx \\
&= \left[3e^4 x - e^4\frac{x^2}{2} + e^x\right]_0^4 \\
&= \left(3e^4(4) - e^4\frac{4^2}{2} + e^4\right) - \left(0 - 0 + e^0\right) \\
&= 12e^4 - 8e^4 + e^4 - 1 \\
&= 5e^4 - 1
\end{aligned}
$$

As a check on this calculation, let's redo the calculation viewing E as an $x-\text{simple}$ region.



Again we start with a picture of the solid E and its projection D onto the yz-plane. Notice that every strip parallel to the x-axis starts on the yz-plane ($x=0$) and ends on the plane $x+y+z=4$ or $x=4-y-z$. The projection of the solid onto the yz-plane yields the triangular region D identified in the diagram.

Viewing D as a $z-\text{simple}$ region in the yz-plane, we have

$$D=\{(y,z)\,|\,0\le y\le 4\,,\,0\le z\le 4-y\}$$

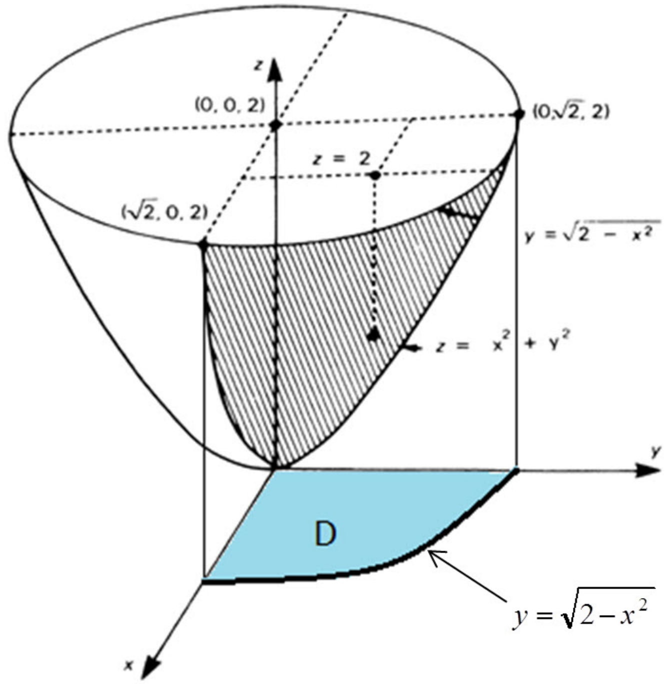
Hence, $E = \{(x, y, z) \mid 0 \le y \le 4, 0 \le z \le 4 - y, 0 \le x \le 4 - y - z\}$ and so

$$\begin{aligned}
\iiint_E e^{x+y+z}\,dV &= \int_0^4 \int_0^{4-y} \int_0^{4-y-z} e^{x+y+z}\,dx\,dz\,dy \\
&= \int_0^4 \int_0^{4-y} \left[e^{x+y+z}\right]_{x=0}^{x=4-y-z} dz\,dy \\
&= \int_0^4 \int_0^{4-y} \left[e^{(4-y-z)+y+z} - e^{0+y+z}\right] dz\,dy \\
&= \int_0^4 \int_0^{4-y} \left(e^4 - e^{y+z}\right) dz\,dy \\
&= \int_0^4 \left[e^4 z - e^{y+z}\right]_{z=0}^{z=4-y} dy \\
&= \int_0^4 \left[\left(e^4(4-y) - e^{y+(4-y)}\right) - \left(0 - e^{y+0}\right)\right] dy \\
&= \int_0^4 \left(e^4(4-y) - e^4 + e^y\right) dy \\
&= \int_0^4 \left[3e^4 - e^4 y + e^y\right] dy \\
&= \left[3e^4 y - e^4 \frac{y^2}{2} + e^y\right]_0^4 \\
&= \left(3e^4(4) - e^4 \frac{4^2}{2} + e^4\right) - \left(0 - 0 + e^0\right) \\
&= 12e^4 - 8e^4 + e^4 - 1 \\
&= 5e^4 - 1
\end{aligned}$$

Same answer as before!

2. Find the volume of the solid bounded by the plane $z = 2$ and the surface $z = x^2 + y^2$.

   We start with a picture of the solid region below. Note that because of symmetry we can focus on the portion E of the depicted region that lives in the first octant, calculate its volume, and then multiply the answer by 4.



The volume $V(E)$ of this region can be represented by the triple integral

$$V(E) = \iiint_E 1 \, dV$$

Now the region E is $x-\text{simple}$, $y-\text{simple}$ and $z-\text{simple}$, so we have choices to make. Viewing E as $z-\text{simple}$, we notice that every strip parallel to the *z*-axis in this region starts on the surface $z = x^2 + y^2$ and ends on the plane $z = 2$. The projection of E onto the xy-plane yields the region D which is a quarter-disk of radius $\sqrt{2}$.

Viewing D as a $y-\text{simple}$ region in the xy-plane, we have

$$D = \{(x, y) \,|\, 0 \le x \le \sqrt{2}, 0 \le y \le \sqrt{2 - x^2}\,\}$$

Hence, $E = \{(x, y, z) \mid 0 \le x \le \sqrt{2}, 0 \le y \le \sqrt{2 - x^2}, x^2 + y^2 \le z \le 2\}$ and so

$$\begin{aligned}
\iiint_E 1\,dV &= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \int_{x^2+y^2}^{2} dz\,dy\,dx \\
&= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} [z]_{x^2+y^2}^{2}\,dy\,dx \\
&= \int_0^{\sqrt{2}} \int_0^{\sqrt{2-x^2}} \left(2 - x^2 - y^2\right) dy\,dx \\
&= \int_0^{\sqrt{2}} \left[\left(2 - x^2\right) y - \frac{y^3}{3}\right]_0^{\sqrt{2-x^2}} dx \\
&= \int_0^{\sqrt{2}} \left[\left(2 - x^2\right)\sqrt{2 - x^2} - \frac{\left(\sqrt{2 - x^2}\right)^3}{3}\right] dx \\
&= \int_0^{\sqrt{2}} \left[\left(2 - x^2\right)^{3/2} - \frac{1}{3}\left(2 - x^2\right)^{3/2}\right] dx \\
&= \int_0^{\sqrt{2}} \left(2 - x^2\right)^{3/2} \left(1 - \frac{1}{3}\right) dx \\
&= \frac{2}{3} \int_0^{\sqrt{2}} \left(2 - x^2\right)^{3/2} dx
\end{aligned}$$

This is not a standard integral but can be solved using the trig substitution $x = \sqrt{2}\sin(\theta)$, $dx = \sqrt{2}\cos(\theta)\,d\theta$.

When $x = 0$, $\theta = \sin^{-1}(0) = 0$; when $x = \sqrt{2}$, $\theta = \sin^{-1}(1) = \pi/2$.

$$
\begin{aligned}
V(E) &= \frac{2}{3}\int_0^{\sqrt{2}} \left(2 - x^2\right)^{3/2} dx = \frac{2}{3}\int_0^{\pi/2} 2^{3/2}\left(1 - \sin^2(\theta)\right)^{3/2} \sqrt{2}\cos(\theta)\,d\theta \\
&= \frac{8}{3}\int_0^{\pi/2} \left(\cos^2(\theta)\right)^{3/2} \cos(\theta)\,d\theta \\
&= \frac{8}{3}\int_0^{\pi/2} \cos^4(\theta)\,d\theta \\
&= \frac{8}{3}\int_0^{\pi/2} \left(\cos^2(\theta)\right)^2 d\theta \\
&= \frac{8}{3}\int_0^{\pi/2} \left(\frac{1+\cos(2\theta)}{2}\right)^2 d\theta \\
&= \frac{2}{3}\int_0^{\pi/2} \left(1 + 2\cos(2\theta) + \cos^2(2\theta)\right) d\theta \\
&= \frac{2}{3}\int_0^{\pi/2} \left(1 + 2\cos(2\theta) + \frac{1+\cos(4\theta)}{2}\right) d\theta \\
&= \frac{2}{3}\left[\theta + 2\cdot\frac{1}{2}\sin(2\theta) + \frac{1}{2}\theta + \frac{1}{2}\cdot\frac{1}{4}\sin(4\theta)\right]_0^{\pi/2} \\
&= \frac{2}{3}\left[\frac{3}{2}\theta + \sin(2\theta) + \frac{1}{8}\sin(4\theta)\right]_0^{\pi/2} \\
&= \frac{2}{3}\left[\left(\frac{3}{2}\cdot\frac{\pi}{2} + \sin(\pi) + \frac{1}{8}\sin(2\pi)\right) - 0\right] \\
&= \frac{2}{3}\left(\frac{3\pi}{4} + 0 + 0\right) \\
&= \frac{\pi}{2}
\end{aligned}
$$

Now use the trig identity:

$$\cos^2(\theta) = \frac{1+\cos(2\theta)}{2}$$

The volume of the full solid is therefore $4 \cdot \frac{\pi}{2} = 2\pi$.

Notice that a triple integral was not necessary to determine the volume of this solid region. We could have used a double integral:

$$\text{Volume} = 4 \cdot \iint_D \left(2 - x^2 - y^2\right) dA$$

The double integral $\iint_D \left(2 - x^2 - y^2\right) dA$ represents the difference of two volumes. The first volume is that of the region bounded above by the plane

$z = 2$ and below by the xy-plane; the second is that of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the xy-plane. The difference therefore describes the volume of the region between the two surfaces.

3. This is problem 28 on page 1049 in your text.

Sketch the solid whose volume is given by the iterated integral

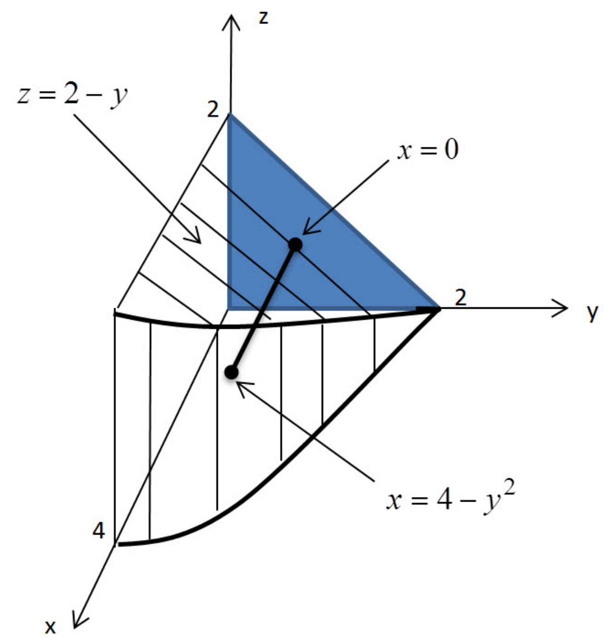
$$\int_0^2 \int_0^{2-y} \int_0^{4-y^2} dx\, dz\, dy$$

From the order of integration and the limits of integration we have the following description of the solid as an $x-\text{simple}$ region:

$$E = \{(x, y, z) \mid 0 \le y \le 2,\, 0 \le z \le 2 - y,\, 0 \le x \le 4 - y^2\}$$

Since $x \ge 0,\ y \ge 0$ and $z \ge 0$, the solid lives in the first octant. The surface $x = 4 - y^2$ is a parabolic cylinder intersecting the x-axis at $x = 4$ and the positive y-axis at $y = 2$. The surface $z = 2 - y$ is a plane intersecting the

z-axis at $z = 2$ and the positive y-axis at $y = 2$. A picture of the solid is drawn below.



Stewart, J. (2012). *Multivariable calculus* (7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.

Practice Exercises 15.7

From the text pages 1049–1051, do problems 5, 11, 13, 21, 23, 25, 27, 31, 33, 35, 37, 41, 45, 47, and 53.

Remember: You are to attempt all of the problems carefully before checking your solutions against those given in the solutions manual.

Triple Integrals in Cylindrical Coordinates

Learning Outcomes

Upon completion of Triple Integrals in Cylindrical Coordinates, you should be able to:

- Plot points in cylindrical coordinates and find their corresponding rectangular coordinates.
- Transform equations in rectangular form to cylindrical form and conversely.
- Sketch the solid region whose volume is given by an iterated integral expression in cylindrical coordinates.
- Set up and evaluate triple integrals using cylindrical coordinates.
- Use triple integrals in cylindrical coordinates to calculate volumes, mass, moments about a coordinate plane, centre of mass and moments of inertia about a coordinate axis.
- Transform a given triple iterated integral expression in rectangular coordinates to cylindrical coordinates and evaluate the integral expression.

Readings

Read section 15.8, pages 1051–1054, in your textbook. Carefully study the examples worked out in the text.

Overview

In the **cylindrical coordinate system**, a point P in 3-space is identified by the ordered triple (r, θ, z), where r and θ are the polar coordinates of the projection of P onto the xy-plane and z is the usual directed distance of P from the xy-plane (see Figure 2 on page 1052 of the text). The conversion from cylindrical to rectangular coordinates is unique. It is accomplished using the relationships:

$$x = r\cos(\theta),\ y = r\sin(\theta) \text{ and } z = z$$

To go the other way, the results are not unique. There are infinitely many cylindrical coordinates (r, θ, z) associated to a given set of rectangular coordinates (x, y, z) using the relationships:

$$r^2 = x^2 + y^2,\ \tan(\theta) = \frac{y}{x} \text{ and } z = z$$

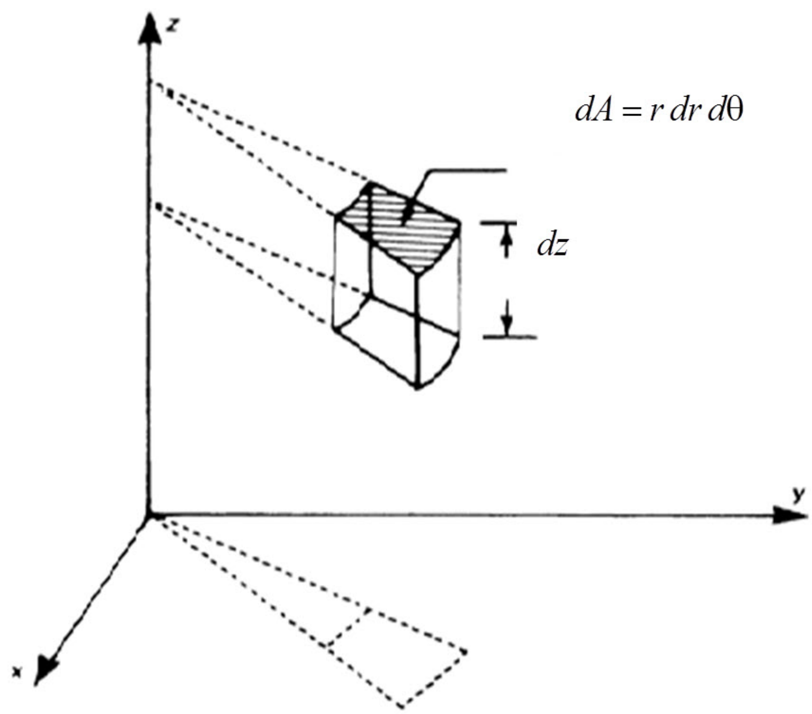
Note: Cylindrical coordinates are just polar coordinates with a third coordinate perpendicular to the polar plane. In the case the solid region has boundary surfaces described by $x = g(y,z)$, then that third coordinate would be x. So, the cylindrical coordinates are (r,θ,x) and the polar plane is the $yz-$plane. Similarly, if the solid region has boundary surfaces described by $y = g(x,z)$, then that third coordinate would be y. So, the cylindrical coordinates are (r,θ,y) and the polar plane is the $xz-$plane.

In Section 15.4 we considered double integrals in polar coordinates. We saw that the area element associated with the double integral is

$$dA = r\,dr\,d\theta$$

In cylindrical coordinates the volume element is obtained by multiplying this area element by the height dz.

We get: $dV = r\,dr\,d\theta\,dz$ (see picture below)



The decision to use cylindrical rather than Cartesian coordinates is automatic when the boundaries of the region in question have cylindrical symmetry. In some cases, the transformation from Cartesian to cylindrical will produce a more easily managed integral even without the presence of such symmetry. This technique should always be considered when terms like $x^2 + y^2$ appear. In other words, if an integral expressed in Cartesian coordinates appears tricky, it's often a good strategy to transform it to cylindrical coordinates and see if the integral becomes more easily managed. As before, you can simply choose the order of integration that is easiest to use.

Example Exercises

1. To illustrate the power of cylindrical coordinates we will redo Example 2 in the previous section—it required three pages of work and some nasty integration.

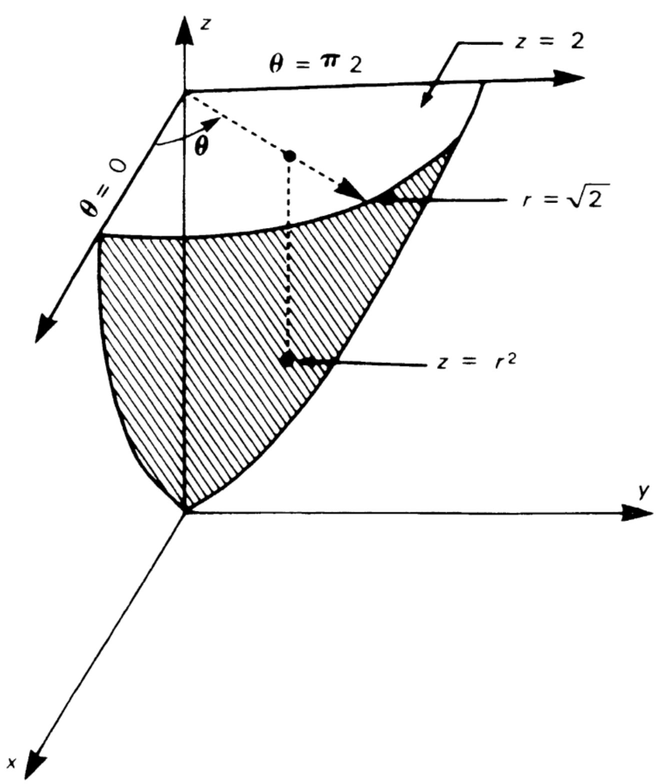
   Find the volume of the solid bounded by the plane $z = 2$ and the surface $z = x^2 + y^2$.

   As before we focus on the portion E of the depicted region that lives in the first octant, calculate its volume, and then multiply the answer by 4.

   First, transform the surfaces from Cartesian to cylindrical coordinates:

   $$z = x^2 + y^2 = r^2 \text{ and } z = 2 \text{ remains as is.}$$

The picture looks as follows:



The quarter-disk of radius $\sqrt{2}$, which is the projection of the solid region in the first octant on to the xy-plane, can be described in polar coordinates by

$$D = \left\{(r,\theta) \mid 0 \le r \le \sqrt{2}, 0 \le \theta \le \pi/2\right\}$$

So, in cylindrical coordinates the solid region E can be described by

$$E = \{(r,\theta,z) \mid 0 \le r \le \sqrt{2}, 0 \le \theta \le \pi/2, \ r^2 \le z \le 2\}$$

As before, begin by integrating with respect to z. It ranges from the point where it touches the surface $z = r^2$ up to the upper plane $z = 2$. The integral becomes

$$\begin{aligned} V(E) &= \iiint_E 1\, dV \\ &= \int_0^{\pi/2}\int_0^{\sqrt{2}}\int_{r^2}^{2} 1 \ r\, dz\, dr\, d\theta \\ &= \int_0^{\pi/2}\int_0^{\sqrt{2}} r\left[z\right]_{r^2}^{2} dr\, d\theta \\ &= \int_0^{\pi/2}\int_0^{\sqrt{2}} r\left(2-r^2\right) dr\, d\theta \\ &= \int_0^{\pi/2}\int_0^{\sqrt{2}} \left(2r-r^3\right) dr\, d\theta \\ &= \int_0^{\pi/2}\left[r^2-\frac{r^4}{4}\right]_0^{\sqrt{2}} d\theta \\ &= \int_0^{\pi/2}\left(\left(\sqrt{2}\right)^2-\frac{\left(\sqrt{2}\right)^4}{4}-0\right) d\theta = \int_0^{\pi/2} 1\, d\theta = \frac{\pi}{2} \end{aligned}$$

The volume of the full solid is therefore $4 \cdot \dfrac{\pi}{2} = 2\pi$.

You'll agree that this is by far the easier method!

2. Evaluate: $\displaystyle\iiint_E yz\, dV$, where E is the region bounded by the planes $y = 0$, $z = 0$ and $z = y$, and the surface $x^2 + y^2 = 1$.

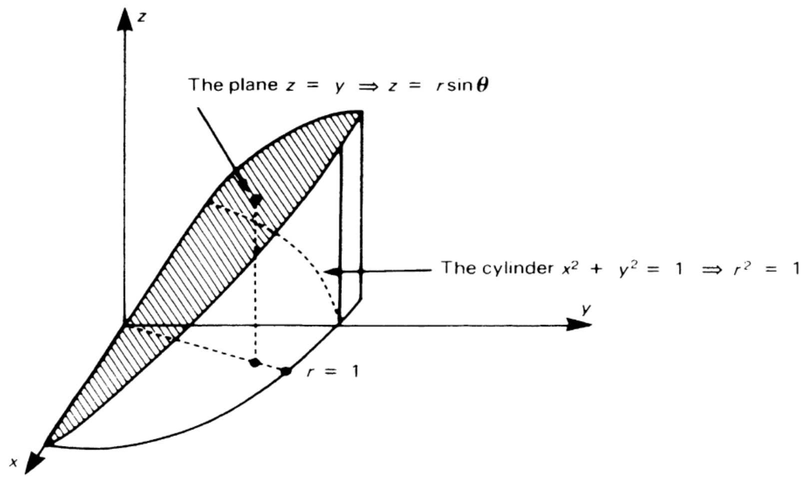
The presence of the cylinder $x^2 + y^2 = 1$ strongly suggests that cylindrical coordinates would be the best approach here.

Transforming to cylindrical coordinates we have:

$$z = y = r\sin(\theta) \text{ and } x^2 + y^2 = 1 \quad \Rightarrow \quad r^2 = 1 \quad \Rightarrow \quad r = 1$$

Note: In polar coordinates $r = 1$ and $r = -1$ both generate the unit circle.

We start with a picture of the solid region.



Notice that strips in the region E that are parallel to the z-axis always start on the surface $z = 0$, i.e., the xy-plane, and end on the plane $z = r\sin(\theta)$.

The semi-circle of radius 1 in the top half of the xy-plane, which is the projection of the solid region onto the xy-plane, can be described in polar coordinates by

$$D = \left\{ (r, \theta) \,|\, 0 \le r \le 1,\, 0 \le \theta \le \pi \right\}$$

So, in cylindrical coordinates the solid region E can be described by

$$E = \{(r, \theta, z) \,|\, 0 \le r \le 1,\, 0 \le \theta \le \pi,\; 0 \le z \le r\sin(\theta)\}$$

The integral becomes

$$\begin{aligned}
&\iiint\limits_E yz\,dV \\
&= \int_0^{\pi}\int_0^1\int_0^{r\sin(\theta)} r\sin(\theta)\,z\,r\,dz\,dr\,d\theta \\
&= \int_0^{\pi}\int_0^1\int_0^{r\sin(\theta)} r^2\sin(\theta)\;z\,dz\,dr\,d\theta \\
&= \int_0^{\pi}\int_0^1 r^2\sin(\theta)\left[\frac{z^2}{2}\right]_0^{r\sin(\theta)} dr\,d\theta \\
&= \int_0^{\pi}\int_0^1 r^2\sin(\theta)\left(\frac{r^2\sin^2(\theta)}{2}-0\right)dr\,d\theta \\
&= \frac{1}{2}\int_0^{\pi}\int_0^1 r^4\sin^3(\theta)\,dr\,d\theta \\
&= \frac{1}{2}\int_0^{\pi} \sin^3(\theta)\left[\frac{r^5}{5}\right]_0^1 d\theta \\
&= \frac{1}{2}\int_0^{\pi} \sin^3(\theta)\left(\frac{1}{5}\right)d\theta \\
&= \frac{1}{10}\int_0^{\pi} \sin^2(\theta)\sin(\theta)\;d\theta \\
&= \frac{1}{10}\int_0^{\pi} \left(1-\cos^2(\theta)\right)\sin(\theta)\;d\theta
\end{aligned}$$

Using the substitution $u=\cos(\theta)$, $du=-\sin(\theta)\,d\theta$, $\theta=0 \Rightarrow u=1$ and $\theta=\pi \Rightarrow u=-1$

We have:

$$
\begin{aligned}
&\frac{1}{10}\int_0^{\pi} \left(1-\cos^2(\theta)\right)\sin(\theta)\ d\theta \\
&= \frac{1}{10}\int_1^{-1} \left(1-u^2\right)(-du) \\
&= -\frac{1}{10}\int_1^{-1} \left(1-u^2\right)du \\
&= \frac{1}{10}\int_{-1}^{1} \left(1-u^2\right)du \\
&= \frac{1}{10}\left[u-\frac{u^3}{3}\right]_{-1}^{1} \\
&= \frac{1}{10}\left(1-\frac{1}{3}-\left(-1-\frac{-1}{3}\right)\right) \\
&= \frac{1}{10}\left(\frac{2}{3}+\frac{2}{3}\right) \\
&= \frac{4}{30}=\frac{2}{15}
\end{aligned}
$$

As an exercise, you might set this problem up as a Cartesian integral and try to solve it.

Practice Exercises 15.8

From the text pages 1055–1056, do problems 1, 3, 5, 7, 9, 13, 15, 19, 21, 25, 27, and 29.

Remember: You are to attempt all of the problems carefully before checking your solutions against those given in the solutions manual.

Triple Integrals in Spherical Coordinates

Learning Outcomes

Upon completion of Triple Integrals in Spherical Coordinates, you should be able to:

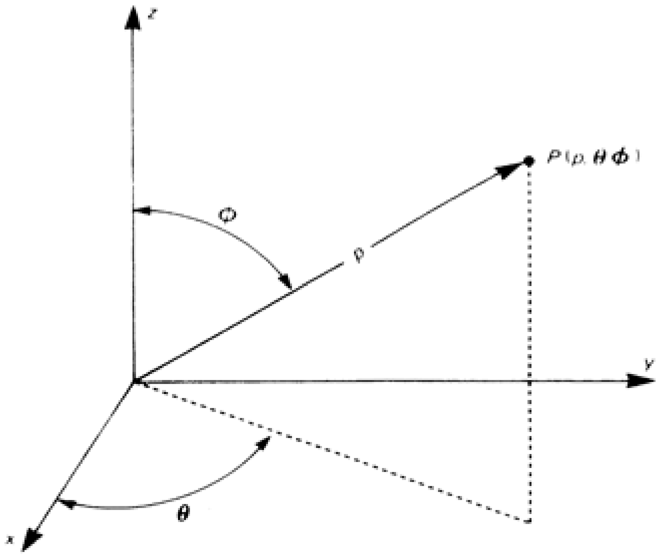
- Plot points in spherical coordinates and find their corresponding rectangular coordinates.
- Transform equations in rectangular form to spherical form and conversely.
- Set up and evaluate triple integrals using spherical coordinates.
- Sketch the solid region whose volume is given by an iterated integral expression in spherical coordinates.
- Use triple integrals in spherical coordinates to calculate volumes, mass, moments about a coordinate plane, centre of mass, and moments of inertia about a coordinate axis.
- Transform a given triple iterated integral expression in rectangular coordinates to spherical coordinates and evaluate the integral expression.

Readings

Read section 15.9, pages 1057–1061, in your textbook. Carefully study the examples worked out in the text.

Overview

In the spherical coordinate system, the three-dimensional space is sliced into a grid by concentric spheres centered at the origin (see pictures below). Planes pass through the z-axis, and the apex of cones is at the origin with axes along the z-axis. A point in 3-space P is described by the radius ρ of the sphere, the angle θ the vertical plane makes with the xz-plane (i.e., the familiar polar angle), and the angle ϕ the cone makes with the positive z-axis.

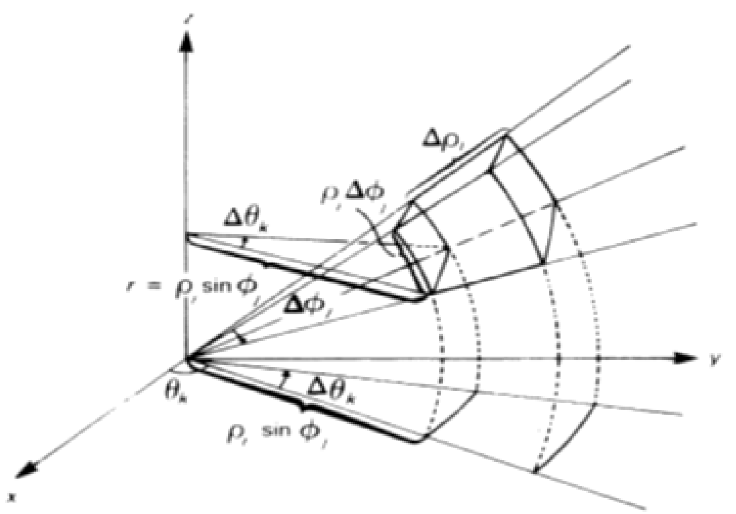


We write the spherical coordinates of P as the ordered triple (ρ, θ, ϕ). The Greek letter ρ is pronounced "rho" and the letter ϕ is pronounced "phi".

Note: In spherical coordinates $\rho \geq 0$ and $0 \leq \phi \leq \pi$.

Refer to page 1057 of the text for the relationships between rectangular and spherical coordinates.

The volume element here is difficult to illustrate.



As usual, the "spherical box" is enormously exaggerated in size. It is constructed using two cones, two vertical planes and two spheres. You should conceptually shrink it to infinitesimal dimensions, so that the sides become perpendicular. Then you can calculate its volume by forming the product of the sides. The length of one side is simply $\Delta\rho$, the difference in the radii of the inner and outer spheres. Since the angle between the two cones is $\Delta\phi$, and the radius is ρ, using the formula for arc length, the length of the second side is $\rho\Delta\phi$. Again using the formula for arc length, the length of the third side in the drawing is $\rho\sin(\phi)\Delta\theta$, the product of the radius $\rho\sin(\phi)$ with the angle $\Delta\theta$ between the two vertical planes.

Thus, the element of volume is

$$dV = d\rho\left(\rho\, d\phi\right)\left(\rho\sin(\phi)\, d\theta\right) = \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

Using the relationships

$$x = \rho\sin(\phi)\cos(\theta),\ y = \rho\sin(\phi)\sin(\theta),\ z = \rho\cos(\phi)$$

and remembering to replace dV by $\rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$

we have the triple integral in spherical coordinates given by

$$\iiint_E f(x, y, z)\, dV$$

$$= \iiint_E f(\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))\, \rho^2 \sin(\phi)\, d\rho\, d\phi\, d\theta$$

where the limits, in general, may be functions of the coordinates.

Both volumes defined by spherical and conical boundaries, and planes through the z-axis, are particularly amenable to resolution by triple integrals expressed in spherical coordinates. As you have already seen, problems in Cartesian coordinates, containing expressions like $x^2 + y^2 + z^2$ may be more easily managed when converted to spherical coordinates. Of course the order of integration is immaterial and should be chosen on the basis of mathematical simplicity.

Example Exercises

1. Find spherical coordinates for the point with rectangular coordinates (2,-1,3).

$$\rho = \sqrt{x^2 + y^2 + z^2} = \sqrt{2^2 + (-1)^2 + 3^2} = \sqrt{14}$$

$$z = \rho\cos(\phi) \quad \Rightarrow \quad 3 = \sqrt{14}\cos(\phi) \quad \Rightarrow \quad \cos(\phi) = \frac{3}{\sqrt{14}} \quad \Rightarrow \quad \phi = \cos^{-1}\left(\frac{3}{\sqrt{14}}\right)$$

$$\tan(\theta) = \frac{y}{x} = -\frac{1}{2} \quad \Rightarrow \quad \theta = \tan^{-1}\left(-\frac{1}{2}\right)$$

So, $(\rho,\theta,\varphi) = \left(\sqrt{14}, \tan^{-1}\left(-\frac{1}{2}\right), \cos^{-1}\left(\frac{3}{\sqrt{14}}\right)\right)$

2. Obtain a spherical coordinate equation for the surface

$$2x^2 + 2y^2 + z^2 - 6z = 0$$

$$\begin{aligned}
&2x^2 + 2y^2 + z^2 - 6z = 0 \\
&2\left(x^2 + y^2\right) + z^2 - 6z = 0 \\
&2r^2 + z^2 - 6z = 0 \\
&2\left(\rho\sin(\varphi)\right)^2 + \left(\rho\cos(\varphi)\right)^2 - 6\rho\cos(\varphi) = 0 \\
&2\rho^2\sin^2(\varphi) + \rho^2\cos^2(\varphi) - 6\rho\cos(\varphi) = 0 \\
&\rho\left(2\rho\sin^2(\varphi) + \rho\cos^2(\varphi) - 6\cos(\varphi)\right) = 0 \\
&\rho = 0 \ \text{ or } \ 2\rho\sin^2(\varphi) + \rho\cos^2(\varphi) - 6\cos(\varphi) = 0
\end{aligned}$$

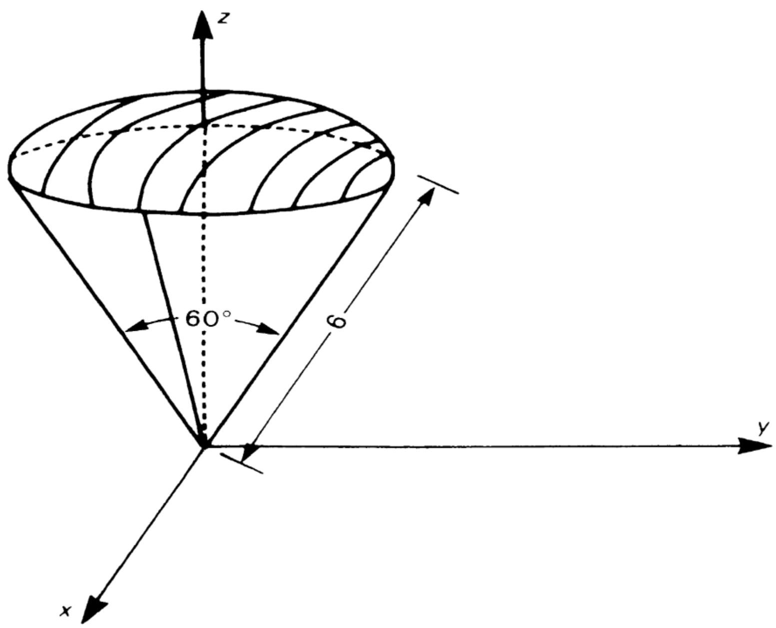
$\rho = 0$ describes the origin (0,0,0), which is also on the surface
$2\rho\sin^2(\phi) + \rho\cos^2(\phi) - 6\cos(\phi) = 0$.

Solving for ρ in this equation and simplifying, we have

$$\begin{aligned}
2\rho\sin^2(\varphi) + \rho\cos^2(\varphi) - 6\cos(\varphi) &= 0 \\
\rho\left(2\sin^2(\varphi) + \cos^2(\varphi)\right) &= 6\cos(\varphi) \\
\rho\left(\sin^2(\varphi) + \left(\sin^2(\varphi) + \cos^2(\varphi)\right)\right) &= 6\cos(\varphi) \\
\rho\left(\sin^2(\varphi) + 1\right) &= 6\cos(\varphi) \\
\rho &= \frac{6\cos(\varphi)}{\sin^2(\varphi) + 1}
\end{aligned}$$

This is a spherical coordinate equation for the surface.

3. Find the volume of a "recession" ice cream cone cut from a sphere of radius 6 with an apex angle of 60°.



This problem is ideally suited for spherical coordinates. If E denotes this solid region, then in spherical coordinates

$$E = \{(\rho, \theta, \phi) \mid 0 \le \rho \le 6,\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/6\}$$

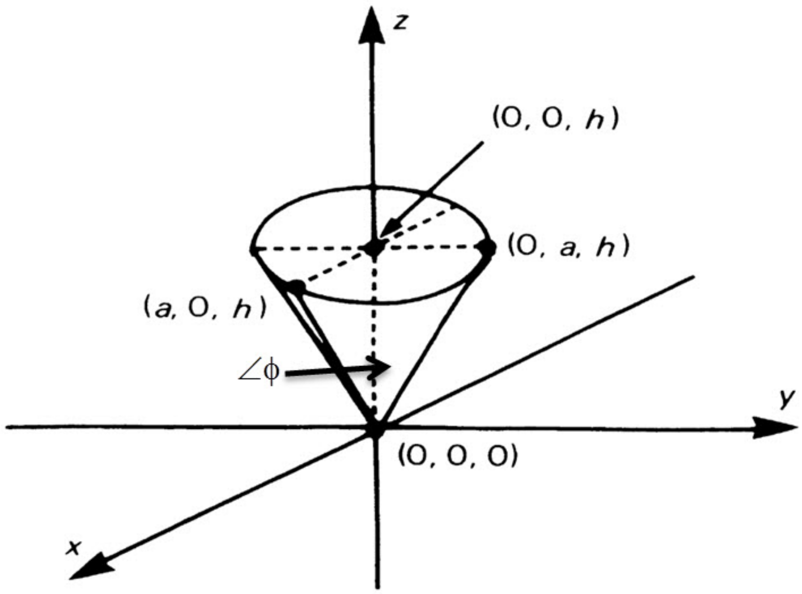
The volume integral becomes

$$\begin{aligned}
V(E) &= \iiint_E 1\, dV \\
&= \int_0^{2\pi}\int_0^{\pi/6}\int_0^{6} \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/6} \sin(\varphi)\left[\frac{\rho^3}{3}\right]_0^6 d\varphi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\pi/6} \sin(\varphi)\left[\frac{6^3}{3}-0\right] d\varphi\, d\theta \\
&= 72\int_0^{2\pi}\int_0^{\pi/6} \sin(\varphi)\, d\varphi\, d\theta \\
&= 72\int_0^{2\pi} \left[-\cos(\varphi)\right]_0^{\pi/6} d\theta \\
&= -72\int_0^{2\pi}\left(\cos\left(\frac{\pi}{6}\right)-\cos(0)\right) d\theta \\
&= -72\int_0^{2\pi}\left(\frac{\sqrt{3}}{2}-1\right) d\theta \\
&= -72\left(\frac{\sqrt{3}}{2}-1\right)\int_0^{2\pi} d\theta \\
&= -72\left(\frac{\sqrt{3}}{2}-1\right)(2\pi-0) \\
&= 72\pi\left(2-\sqrt{3}\right)
\end{aligned}$$

Set up the triple integral for this problem in rectangular coordinates and consider the complexity of the mathematics required in the solution!

4. Use spherical coordinates to find a formula for the volume enclosed by a right circular cone of base radius a and height h.

We start with a picture.



Notice that the largest value for the angle ϕ in this problem is calculated from basic trigonometry: $\tan(\phi) = \dfrac{a}{h} \quad \Rightarrow \quad \phi = \tan^{-1}\left(\dfrac{a}{h}\right)$

So, $0 \le \phi \le \tan^{-1}\left(\dfrac{a}{h}\right)$

Also, any point in the region E enclosed by the cone lies on some radial line that starts at the origin and ends on the plane $z = h$. In spherical coordinates this plane is described by

$$\rho\cos(\phi) = h \quad \Rightarrow \rho = \frac{h}{\cos(\phi)} = h\sec(\phi)\,; \quad \text{So, } 0 \le \rho \le h\sec(\phi)$$

Hence, $E = \left\{(\rho,\theta,\varphi) \mid 0 \le \rho \le h\sec(\varphi),\, 0 \le \theta \le 2\pi,\, 0 \le \varphi \le \tan^{-1}\left(\dfrac{a}{h}\right)\right\}$

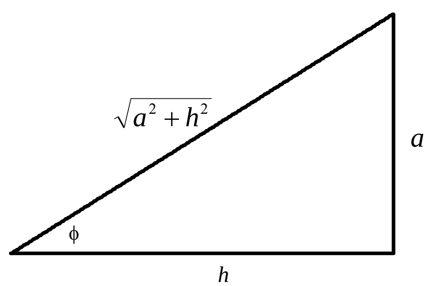
The volume integral becomes

$$\begin{aligned}
V(E) &= \iiint_E 1\, dV \\
&= \int_0^{2\pi}\int_0^{\tan^{-1}\left(\frac{a}{h}\right)}\int_0^{h\sec(\varphi)} \rho^2 \sin(\varphi)\, d\rho\, d\varphi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\tan^{-1}\left(\frac{a}{h}\right)} \sin(\varphi)\left[\frac{\rho^3}{3}\right]_0^{h\sec(\varphi)} d\varphi\, d\theta \\
&= \int_0^{2\pi}\int_0^{\tan^{-1}\left(\frac{a}{h}\right)} \sin(\varphi)\left[\frac{h^3\sec^3(\varphi)}{3} - 0\right] d\varphi\, d\theta \\
&= \frac{h^3}{3}\int_0^{2\pi}\int_0^{\tan^{-1}\left(\frac{a}{h}\right)} \frac{\sin(\varphi)}{\cos^3(\varphi)}\, d\varphi\, d\theta
\end{aligned}$$

We now use the substitution $u = \cos(\phi)$, $du = -\sin(\phi)\, d\phi$;

when $\phi = 0$, $u = \cos(0) = 1$;

when $\phi = \tan^{-1}\left(\frac{a}{h}\right)$, $u = \cos(\phi) = \frac{h}{\sqrt{a^2 + h^2}}$



So, continuing with our calculation

$$
\begin{aligned}
&\frac{h^3}{3}\int_0^{2\pi}\int_0^{\tan^{-1}\left(\frac{a}{h}\right)} \frac{\sin(\varphi)}{\cos^3(\varphi)}\,d\varphi\,d\theta \\
&= \frac{h^3}{3}\int_0^{2\pi}\int_1^{\frac{h}{\sqrt{a^2+h^2}}} u^{-3}\,(-du)\,d\theta \\
&= -\frac{h^3}{3}\int_0^{2\pi}\int_1^{\frac{h}{\sqrt{a^2+h^2}}} u^{-3}\,du\,d\theta \\
&= -\frac{h^3}{3}\int_0^{2\pi}\left[\frac{u^{-2}}{-2}\right]_1^{\frac{h}{\sqrt{a^2+h^2}}} d\theta \\
&= \frac{h^3}{6}\int_0^{2\pi}\left[\frac{1}{u^2}\right]_1^{\frac{h}{\sqrt{a^2+h^2}}} d\theta \\
&= \frac{h^3}{6}\int_0^{2\pi}\left(\frac{a^2+h^2}{h^2}-1\right)d\theta \\
&= \frac{h^3}{6}\left(\frac{a^2}{h^2}+1-1\right)\int_0^{2\pi} d\theta \\
&= \frac{a^2h}{6}(2\pi-0) \\
&= \frac{1}{3}\pi a^2 h
\end{aligned}
$$

Hence, $V(E)=\frac{1}{3}\pi a^2 h$.

Practice Exercises 15.9

From the text pages 1061–1063, do problems 1, 3, 5, 7, 9, 13, 15, 17, 19, 23, 25, 27, 29, 33, 35, 39, and 41.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Unit 5: Summary and Self-Test

You have now worked through Unit 5 in MATH 2111. It is time to take stock of what you have learned, review all of the material, and bring your shorthand notes up to date. A summary of the material covered so far is provided in the following pages. This summary should be modified, added to, and fleshed out to form a solid body of knowledge.

When you have completed your review, you should test your comprehension of the material with a closed book self-administered examination. Put all your notes aside, find a quiet place where you will not be disturbed, and take the examination provided at the end of this unit. You will find some questions straightforward and easy, but others will test your ingenuity.

You will find the solutions to the Unit 5 exam questions, and the point value for each question in the Answer Key provided at the end of this unit. Become your own examiner. If you have done well, according to your personal standards, go on to Unit 6, the final one! If not, then more review and practice is obviously called for.

Summary

Applications of Double Integrals

The mass m of a plane lamina is obtained by putting $f(x, y)$ equal to the surface density $\rho(x, y)$ in the double integral

$$m = \iint_D \rho(x, y)\, dA$$

The moment of the lamina about the $y-$axis is

$$M_y = \iint_D x\rho(x, y)\, dA$$

The moment of the lamina about the $x-$axis is

$$M_x = \iint_D y\rho(x, y)\, dA$$

More generally, the moment of inertia of the lamina about any axis is given by:

$$I_a = \iint_D d_a^2(x, y)\,\rho(x, y)\,dA$$

where $d_a(x, y) =$ the distance from (x, y) to the axis.

The centre of mass of the lamina is $(\bar{x}, \bar{y})$, where

$$\bar{x} = \frac{M_y}{m} = \frac{\iint_D x\,\rho(x, y)\,dA}{\iint_D \rho(x, y)\,dA}, \quad \bar{y} = \frac{M_x}{m} = \frac{\iint_D y\,\rho(x, y)\,dA}{\iint_D \rho(x, y)\,dA}$$

The moment of inertia of the lamina about the $y-$axis is

$$I_y = \iint_D x^2\,\rho(x, y)\,dA$$

The moment of inertia of the lamina about the $x-$axis is

$$I_x = \iint_D y^2\,\rho(x, y)\,dA$$

The moment of inertia of the lamina about the origin is

$$I_0 = \iint_D \left(x^2 + y^2\right)\rho(x, y)\,dA$$

The radius of gyration with respect to the $x-$axis is

$$\bar{\bar{y}} = \sqrt{\frac{I_x}{m}}$$

The radius of gyration with respect to the $y-$axis is

$$\bar{\bar{x}} = \sqrt{\frac{I_y}{m}}$$

Triple Integrals

Let $w = f(x, y, z)$ be a continuous function defined on a rectangular box in 3-space, $B = \{(x, y, z) \mid a \le x \le b,\ c \le y \le d,\ e \le z \le f\}$. We define the **triple integral of f over** B to be the value of the limit

$$\iiint_B f(x, y, z)\, dV = \lim_{m,n,p\to\infty} \sum_{i=1}^{m} \sum_{j=1}^{n} \sum_{k=1}^{p} f\left(x_i^*, y_j^*, z_k^*\right)\Delta V$$

if it exists. This limit exists if f is continuous.

If f is continuous on a z-simple region
$E = \{(x, y, z) \mid (x, y) \in D, u_1(x, y) \le z \le u_2(x, y)\}$, where D is the vertical projection of E onto the $xy-$plane, then

$$\iiint_E f(x, y, z)\, dV = \iint_D \left[\int_{u_1(x,y)}^{u_2(x,y)} f(x, y, z)\, dz\right] dA$$

Similar definitions are given for triple integrals over x-simple and y-simple regions.

Interpretations of the triple integral:

1. If $f(x, y, z) = 1$, then $\iiint_E 1\, dV = \iiint_E dV$ = the volume of the solid region E.

2. If $w = f(x, y, z)$ is a **mass density function** in units of mass per unit volume, then $\iiint_E f(x, y, z)\, dV$ = the total mass of E.

3. If $w = f(x, y, z)$ is a **charge density function** in units of charge per unit volume, then $\iiint_E f(x, y, z)\, dV$ = the total charge on E.

   The mass m of a solid object occupying the region E in 3-space is obtained by putting $f(x, y, z)$ equal to the volume density $\rho(x, y, z)$ in the triple integral

$$m = \iiint_E \rho(x, y, z)\, dV$$

The moment of the solid about the three coordinate planes is

$$M_{yz} = \iiint_E x\,\rho(x,y,z)\,dV\,,\ M_{xz} = \iiint_E y\,\rho(x,y,z)\,dV\,,$$
$$M_{xy} = \iiint_E z\,\rho(x,y,z)\,dV$$

The centre of mass is located at the point $(\bar{x}, \bar{y}, \bar{z})$, where

$$\bar{x} = \frac{M_{yz}}{m}\,,\quad \bar{y} = \frac{M_{xz}}{m}\,,\quad \bar{z} = \frac{M_{xy}}{m}$$

The moments of inertia about the three coordinate axes are:

$$I_x = \iiint_E \left(y^2 + z^2\right)\rho(x,y,z)\,dV\,,\ I_y = \iiint_E \left(x^2 + z^2\right)\rho(x,y,z)\,dV\,,$$

$$I_z = \iiint_E \left(x^2 + y^2\right)\rho(x,y,z)\,dV$$

Cylindrical Coordinates

Cylindrical to rectangular coordinates: $x = r\cos(\theta)$, $y = r\sin(\theta)$, $z = z$

Rectangular to cylindrical coordinates: $r^2 = x^2 + y^2$, $\tan(\theta) = \dfrac{y}{x}$, $z = z$

Note: Cylindrical coordinates are just polar coordinates with a third z-coordinate. In the case the solid region has boundary surfaces described by $x = g(y,z)$, then that third coordinate could be x. So, the cylindrical coordinates are (r, θ, x) and the polar plane is the $yz-$plane. Similarly, if the solid region has boundary surfaces described by $y = g(x,z)$, then that third coordinate could be y. So, the cylindrical coordinates are (r, θ, y) and the polar plane is the $xz-$plane.

If f is continuous on a z-simple region of the form
$E = \{(x,y,z) \mid (x,y) \in D, u_1(x,y) \le z \le u_2(x,y)\}$,

where D is given in polar coordinates by
$D = \{(r,\theta) \mid \alpha \le \theta \le \beta,\ h_1(\theta) \le r \le h_2(\theta)\}$,

then $\displaystyle\iiint_E f(x,y,z)\,dV = \int_\alpha^\beta \int_{h_1(\theta)}^{h_2(\theta)} \int_{u_1(r\cos(\theta), r\sin(\theta))}^{u_2(r\cos(\theta), r\sin(\theta))} f(r\cos(\theta), r\sin(\theta), z)\; r\,dz\,dr\,d\theta$

IMPORTANT: When transforming a triple integral in rectangular coordinates to cylindrical coordinates, always remember to replace dV by $r\,dz\,dr\,d\theta$.

Spherical Coordinates

Spherical to rectangular coordinates:

$$x = r\cos(\theta) = \rho\sin(\phi)\cos(\theta),\quad y = r\sin(\theta) = \rho\sin(\phi)\sin(\theta),\quad z = \rho\cos(\phi)$$

Rectangular to spherical coordinates:

$$\rho = \sqrt{x^2 + y^2 + z^2},\ \tan(\theta) = \frac{y}{x},\ \phi = \cos^{-1}\left(\frac{z}{\rho}\right),\ 0 \le \phi \le \pi$$

If f is continuous on a solid region E which transforms to the region S in spherical coordinates, then

$$\iiint_E f(x,y,z)\,dV = \iiint_S f(\rho\sin(\varphi)\cos(\theta), \rho\sin(\varphi)\sin(\theta), \rho\cos(\varphi))\,\rho^2\sin(\varphi)\,d\rho\,d\varphi\,d\theta$$

Important: When transforming a triple integral in rectangular coordinates to spherical coordinates, always remember to replace dV by $\rho^2\sin(\phi)\,d\rho\,d\phi\,d\theta$.

Self-Test *(26 marks)*

Treat this as a real test. Do not refer to any course materials. The time for this test is 1.5 hours. Use the answer key provided to mark your test. The point value for each question is posted in the left margin.

[5] 1. Find the exact centre of mass of the plane lamina bounded by $y = 1 - x^2$ and the $x-$axis, assuming a mass density of $\rho(x,y) = y$.

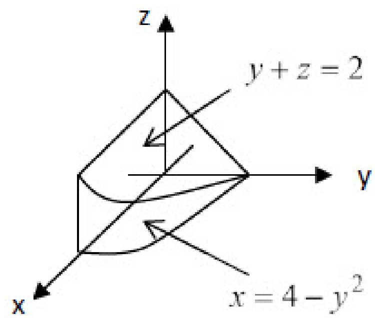
2. Evaluate the following. Give exact answers.

[3] a) $\int_0^1\int_1^2\int_0^{\pi} 4xe^y\sin(z)\,dz\,dx\,dy$

[3] b) $\iiint_E xyz\,dV$, where E is the solid tetrahedron with vertices (0,0,0), (1,0,0), (1,1,0) and (0,0,1)

[4] 3. Consider the region in the first octant bounded by the coordinate planes, the plane $y + z = 2$ and the cylinder $x = 4 - y^2$ (see diagram). Set up, but do not evaluate, triple integrals for the volume of the solid region using the order

i) $dz\,dy\,dx$ ii) $dy\,dx\,dz$



[2] 4. Set up only a triple integral expression in cylindrical coordinates for

$$\int_{-3}^{3}\int_{0}^{\sqrt{9-x^2}}\int_{0}^{9-x^2-y^2} \sqrt{x^2+y^2}\,dz\,dy\,dx$$

.

5. Each equation below represents a surface in 3-space. Convert each equation into rectangular coordinates. Your answer must be in simplified form. Name the surface.

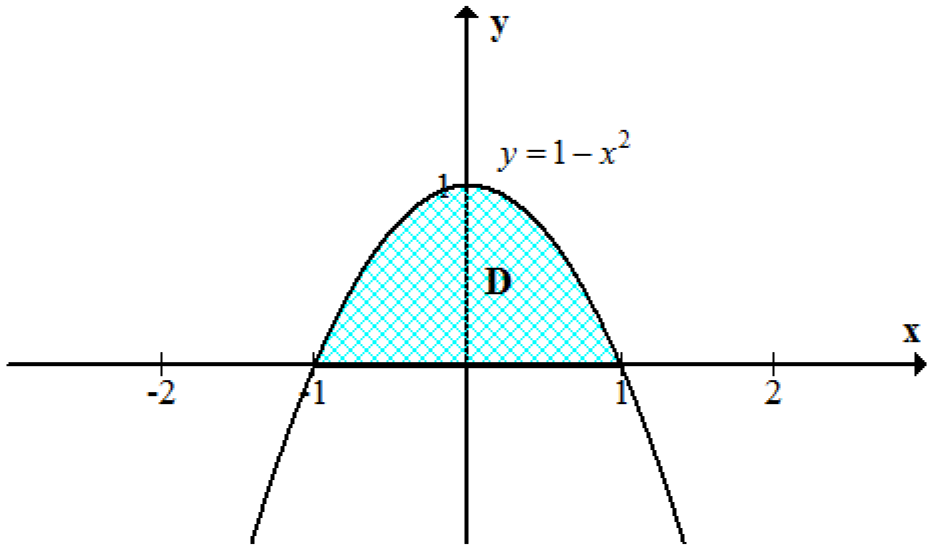
[2] a) $\sin(\theta)=2\cos(\theta)$

[2] b) $\rho=2\csc(\varphi)$

[5] 6. Use spherical coordinates to find the volume of the solid that is enclosed by the cone $z=\sqrt{x^2+y^2}$ and the sphere $x^2+y^2+z^2=z$.

Answer Key

1. The region is pictured below.



Because the region is symmetric about the y – axis and the density function $\rho(x, y) = y$ has the even property, that is, $\rho(-x, y) = y = \rho(x, y)$, the first coordinate of the centre of mass $\bar{x} = 0$.

Notice that the region is y – simple so

$$\text{mass} = \iint_D \rho(x,y)\,dA = \int_{-1}^{1}\int_{0}^{1-x^2} y\,dy\,dx = \int_{-1}^{1} \frac{y^2}{2}\Bigg]_0^{1-x^2} dx = \frac{1}{2}\int_{-1}^{1} \left(\left(1-x^2\right)^2 - 0\right)dx$$

$$= \frac{1}{2}\int_{-1}^{1} \left(1 - 2x^2 + x^4\right)dx = \frac{1}{2}\left(\left[x - \frac{2x^3}{3} + \frac{x^5}{5}\right]_{-1}^{1}\right) = \frac{1}{2}\left(1 - \frac{2}{3} + \frac{1}{5} - \left(-1 + \frac{2}{3} - \frac{1}{5}\right)\right)$$

$$= \frac{1}{2}\left(\frac{16}{15}\right) = \frac{8}{15}$$

$$M_x = \iint_D y\,\rho(x,y)\,dA = \int_{-1}^{1}\int_{0}^{1-x^2} y^2\,dy\,dx = \int_{-1}^{1} \frac{y^3}{3}\Bigg]_0^{1-x^2} dx = \frac{1}{3}\int_{-1}^{1} \left(\left(1-x^2\right)^3 - 0\right)dx$$

$$= \frac{1}{3}\int_{-1}^{1} \left(1 - 3x^2 + 3x^4 - x^6\right)dx = \frac{1}{3}\left(\left[x - x^3 + \frac{3x^5}{5} - \frac{x^7}{7}\right]_{-1}^{1}\right) = \frac{1}{3}\left(1 - 1 + \frac{3}{5} - \frac{1}{7} - \left(-1 + 1 - \frac{3}{5} + \frac{1}{7}\right)\right)$$

$$= \frac{1}{3}\left(\frac{32}{35}\right) = \frac{32}{105}$$

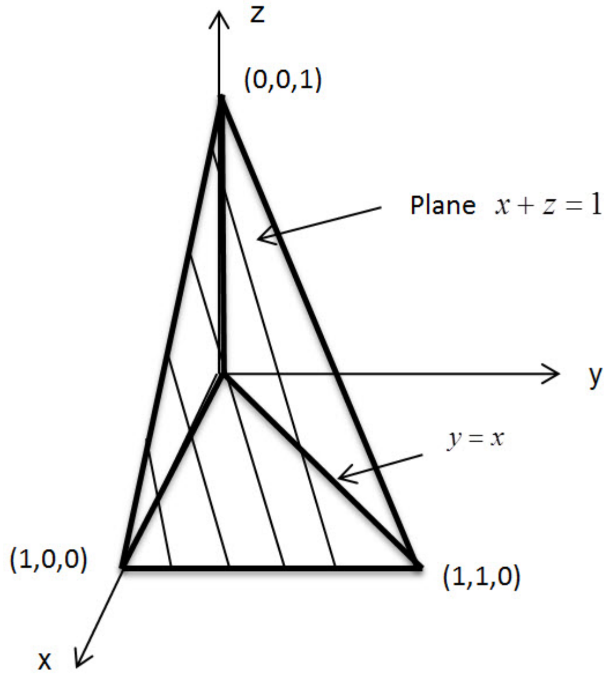
Hence, $\bar{y} = \dfrac{M_x}{\text{mass}} = \left(\dfrac{32/105}{8/15}\right) = \dfrac{4}{7}$.

The centre of mass is $\left(0, \dfrac{4}{7}\right)$.

2. a)

$$\begin{aligned}
&\int_0^1\int_1^2\int_0^\pi 4xe^y \sin(z)\,dz\,dx\,dy \\
&= \int_0^1\int_1^2 4xe^y\left[-\cos(z)\right]_0^\pi dx\,dy \\
&= \int_0^1\int_1^2 -4xe^y\left(\cos(\pi)-\cos(0)\right)dx\,dy \\
&= \int_0^1\int_1^2 -4xe^y\left(-1-1\right)dx\,dy \\
&= 8\int_0^1\int_1^2 xe^y\,dx\,dy \\
&= 8\int_0^1 e^y\left[\frac{x^2}{2}\right]_1^2 dy \\
&= 4\int_0^1 e^y\left(2^2-1\right)dy \\
&= 12\left[e^y\right]_0^1 = 12\left(e^1-e^0\right) = 12(e-1)
\end{aligned}$$

b) The solid region E is pictured below. It is $z-$simple and the projection of E onto the $xy-$plane is $D=\{(x,y)\,|\,0\le y\le x,\,0\le x\le 1\}$.



So, $E=\{(x,y,z)\,|\,0\le z\le 1-x,0\le y\le x,\,0\le x\le 1\}$

Hence,

$$\iiint_E xyz\,dV=\int_0^1\int_0^x\int_0^{1-x} xyz\,dz\,dy\,dx\;=\int_0^1\int_0^x xy\left[\frac{z^2}{2}\right]_0^{1-x}dy\,dx=\frac{1}{2}\int_0^1\int_0^x xy\left((1-x)^2-0\right)dy\,dx$$

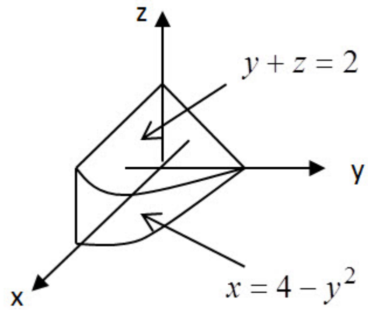
$$=\frac{1}{2}\int_0^1\int_0^x x\left(1-2x+x^2\right)y\,dy\,dx=\frac{1}{2}\int_0^1 x\left(1-2x+x^2\right)\left[\frac{y^2}{2}\right]_0^x dx=\frac{1}{4}\int_0^1 x\left(1-2x+x^2\right)\left(x^2-0\right)dx$$

$$=\frac{1}{4}\int_0^1\left(x^3-2x^4+x^5\right)dx=\frac{1}{4}\left[\frac{x^4}{4}-\frac{2x^5}{5}+\frac{x^6}{6}\right]_0^1=\frac{1}{4}\left(\frac{1}{4}-\frac{2}{5}+\frac{1}{6}-0\right)=\frac{1}{240}$$

3. i) order $dz\,dy\,dx$

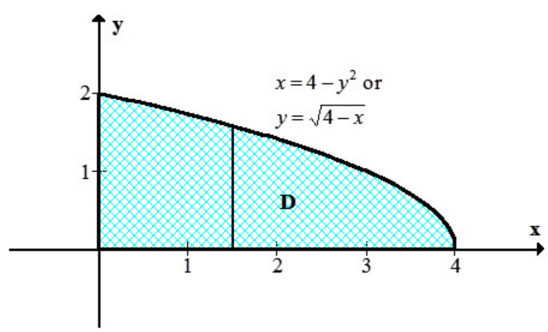
The projection of the solid E onto the $xy-$plane is a parabolic region D which must be viewed as a $y-$simple region according to the order $dy\,dx$. So, $D=\{(x,y)\,|\,0\le y\le\sqrt{4-x}\,,0\le x\le 4\}$

and $E=\{(x,y,z)\,|\,0\le z\le 2-y\,,0\le y\le\sqrt{4-x}\,,0\le x\le 4\}$



Hence, the volume of the solid is

$$V=\int_0^4\int_0^{\sqrt{4-x}}\int_0^{2-y}\,dz\,dy\,dx$$



ii) order $dy\,dx\,dz$

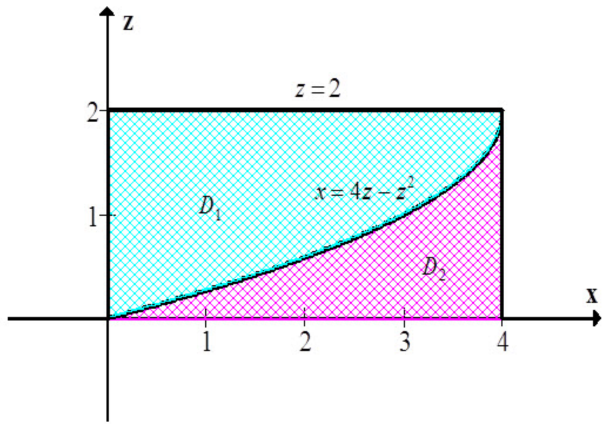
In this case the projection of the solid E onto the $xz-$plane is a region D which must be viewed as the union of two $x-$simple regions D_1 and D_2 according to the order $dx\,dz$.

The intersection of the plane $y+z=2$

with the cylinder $x=4-y^2$ is

$x=4-(2-z)^2=4-\left(4-4z+z^2\right)=4z-z^2$.

The region $D = D_1 \cup D_2$ is pictured below.



Now, $D_1 = \{(x, y) \,|\, 0 \le x \le 4z - z^2, 0 \le z \le 2\}$ and
$D_2 = \{(x, y) \,|\, 4z - z^2 \le x \le 4, 0 \le z \le 2\}$.

So,

$$E = E_1 \cup E_2$$
$$= \{(x, y, z) \,|\, 0 \le y \le 2 - z,\ 0 \le x \le 4z - z^2, 0 \le z \le 2\} \cup \{(x, y, z) \,|\, 0 \le y \le \sqrt{4 - x},\ 4z - z^2 \le x \le 4, 0 \le z \le 2\}$$

Hence, the volume of the solid is

$$V = \int_0^2 \int_0^{4z-z^2} \int_0^{2-z} dy\,dx\,dz + \int_0^2 \int_{4z-z^2}^{4} \int_0^{\sqrt{4-x}} dy\,dx\,dz$$

4. $\displaystyle\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2+y^2}\, dz\,dy\,dx$

The projection of the solid region onto the $xy-$plane is the circular half-disk D centered at the origin of radius 3 that lies above the $x-$axis. In polar coordinates $D = \{(r, \theta) \,|\, 0 \le r \le 3, \quad 0 \le \theta \le \pi\}$.

Now, in polar form $\sqrt{x^2 + y^2} = r$ and $z = 9 - x^2 - y^2 = 9 - r^2$. Replacing $dz\,dy\,dx$ by $r\,dz\,dr\,d\theta$, we have that

$$\int_{-3}^{3} \int_0^{\sqrt{9-x^2}} \int_0^{9-x^2-y^2} \sqrt{x^2+y^2}\, dz\,dy\,dx = \int_0^{\pi} \int_0^{3} \int_0^{9-r^2} r\, r\,dz\,dr\,d\theta = \int_0^{\pi} \int_0^{3} \int_0^{9-r^2} r^2 dz\,dr\,d\theta$$

5. a) $\sin(\theta) = 2\cos(\theta)$

Multiplying both sides of the equation by r gives: $r\sin(\theta) = 2r\cos(\theta)$

But, $r\sin(\theta) = y$ and $r\cos(\theta) = x$.

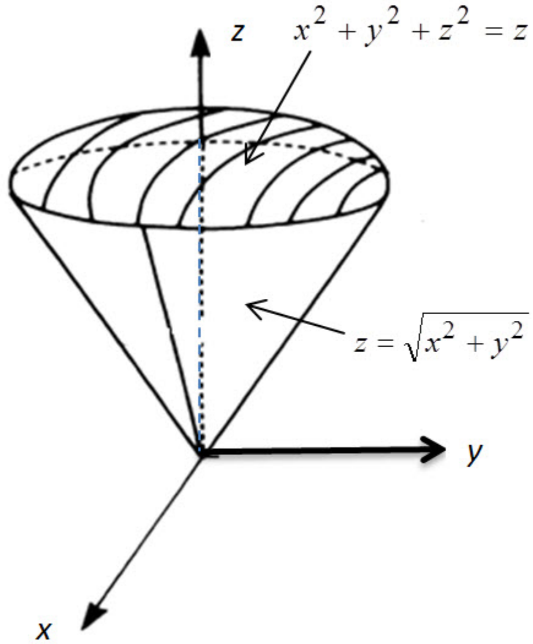
So, the equation becomes $y = 2x$.

This is a vertical plane intersecting the xy – plane in the line $y = 2x$.

b) $\rho = 2\csc(\varphi) = \dfrac{2}{\sin(\varphi)} \Rightarrow \rho\sin(\varphi) = 2 \Rightarrow r = 2$

But, $r = 2$ describes the circle $x^2 + y^2 = 4$, which as a surface in 3-space is the right circular cylinder.

6. The solid E is pictured below.



In spherical coordinates:

$$x^2 + y^2 + z^2 = z \quad \Rightarrow \quad \rho^2 = \rho\cos(\phi)$$
$$\Rightarrow \quad \rho = \cos(\phi)$$

$$z = \sqrt{x^2 + y^2} \quad \Rightarrow \quad \rho\cos(\phi) = r = \rho\sin(\phi)$$
$$\Rightarrow \quad \cos(\phi) = \sin(\phi)$$
$$\Rightarrow \quad \tan(\phi) = 1$$
$$\Rightarrow \quad \phi = \tan^{-1}(1) = \frac{\pi}{4}$$

So, $E = \{(\rho, \theta, \phi) \,|\, 0 \le p \le \cos(\phi),\ 0 \le \theta \le 2\pi,\ 0 \le \phi \le \pi/4\}$. Replacing dV with $\rho^2\sin(\phi)\,d\rho\,d\theta\,d\phi$, we have that the volume of E is

$$V = \iiint_E dV = \int_0^{\pi/4}\int_0^{2\pi}\int_0^{\cos(\varphi)} \rho^2\sin(\varphi)\,d\rho\,d\theta\,d\varphi$$
$$= \int_0^{\pi/4}\int_0^{2\pi}\sin(\varphi)\left[\frac{\rho^3}{3}\right]_0^{\cos(\varphi)} d\theta\,d\varphi = \frac{1}{3}\int_0^{\pi/4}\int_0^{2\pi}\sin(\varphi)\left(\cos^3(\varphi) - 0\right)d\theta\,d\varphi$$
$$= \frac{1}{3}\int_0^{\pi/4}\int_0^{2\pi}\cos^3(\varphi)\sin(\varphi)\,d\theta\,d\varphi = \frac{1}{3}\int_0^{\pi/4}\cos^3(\varphi)\sin(\varphi)\left[\theta\right]_0^{2\pi} d\varphi$$
$$= \frac{1}{3}\int_0^{\pi/4}\cos^3(\varphi)\sin(\varphi)\left(2\pi - 0\right)d\varphi = \frac{2\pi}{3}\int_0^{\pi/4}\cos^3(\varphi)\sin(\varphi)\,d\varphi$$
$$= -\frac{2\pi}{3}\left[\frac{\cos^4(\varphi)}{4}\right]_0^{\pi/4} = -\frac{\pi}{6}\left(\cos^4(\pi/4) - \cos^4(0)\right) = -\frac{\pi}{6}\left(\left(\frac{1}{\sqrt{2}}\right)^4 - 1\right) = \frac{\pi}{8}$$

Use the substitution: $u = \cos(\phi)$,
$du = -\sin(\phi)\,d\phi$