Faculty of Science

Unit 1: Vectors and the Geometry of Space

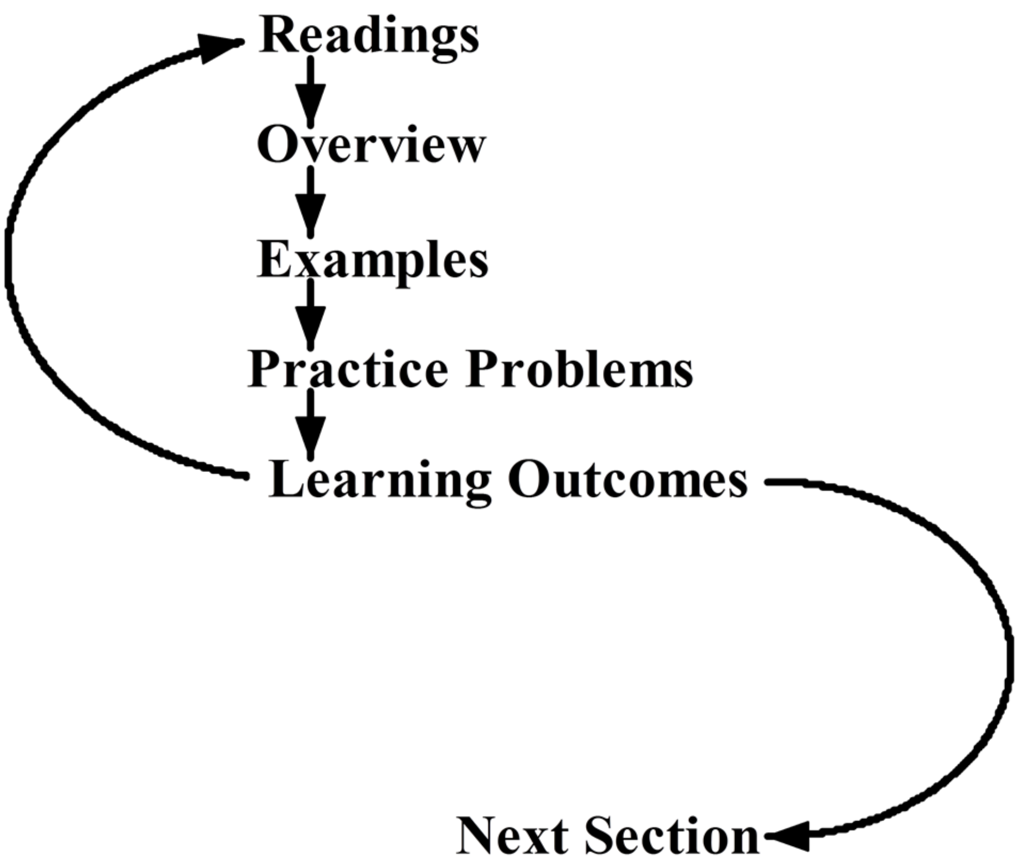
MATH 2111
Calculus III – Multivariable Calculus

Table of Contents

Instructions

The recommended procedure for working through each section of the units in this course is described in detail in your Course Guide.

This procedure is summarized below. If you are certain you have achieved the learning objectives, proceed to the next section. If you are uncertain about one or more of them, go back to the appropriate information in the section until you can complete the task listed in the objective.



Three-Dimensional Coordinate Systems

Learning Outcomes

Upon completion of Three-Dimensional Coordinate Systems, you should be able to:

- Plot points in a 3-dimensional coordinate system.
- Sketch and describe in words surfaces given by simple equations.
- Distinguish between a curve and a surface.
- Determine basic regions of 3-space determined by a system of inequalities.
- Calculate distances in 3-space.
- Recognize an equation of a sphere, be able to put it in standard form and identify the centre and radius of the sphere.
- It is essential that you complete the readings before continuing with overview and examples that follow. As you work through the section you should refer back to the readings if you require clarification of specific points.

Readings

Read section 12. 1, pages 810 to 814 in your textbook. Carefully study the examples worked out in the text.

Overview

Recall that the *xy*-plane or R^2 is described by two copies of the real line that intersect at right angles. The point of intersection is called the origin and is assigned the coordinates (0,0). Points in the plane are then identified by an ordered pair of real numbers (x, y), where x is the directed distance from the origin along the horizontal line called the *x*-axis, and y is the directed distance from the origin along the vertical line called the *y*-axis.

To identify the position of points in 3-space, we add a third axis. Now we have three mutually perpendicular copies of the real line, called the coordinate *x*, *y*, and *z*-axes, intersecting at the origin O with coordinates (0,0,0). Points are now identified and located through an ordered triple (x, y, z) The system so described is called a "rectangular coordinate system" and is denoted by R^3.

Pictures of points plotted in R^3 are found on page 811 of the text. Study these pictures. You need to be comfortable with plotting points in 3-space.

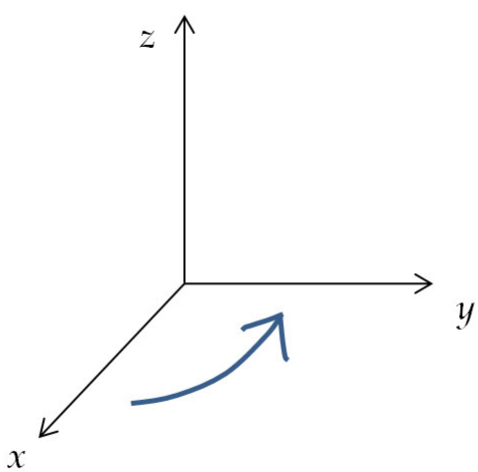
When drawing three mutually perpendicular axes there are two distinct systems you can construct: a right-hand system and a left-hand system. As the name suggests a right-hand system is determined by the use of your right hand and it is called the right-hand rule.

<u>Right-hand Rule</u>: If you curl the fingers of your right hand around the z-axis with your fingers pointing in the direction from the positive x-axis to the positive y-axis, then your thumb points in the direction of the positive z-axis. (See text page 810 for a picture.)

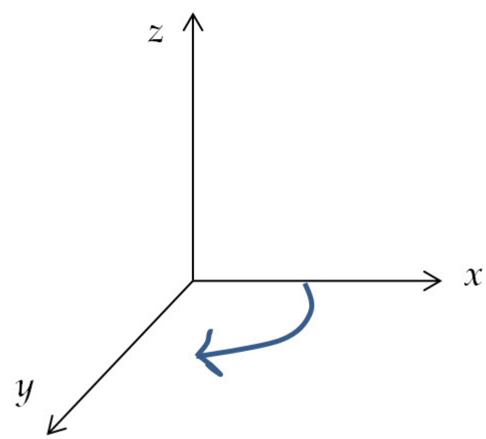
Correspondingly, a left-hand system is determined using your left hand.

Below are orientations that give right-hand and left-hand systems.

RIGHT-HAND SYSTEM:



LEFT-HAND SYSTEM:

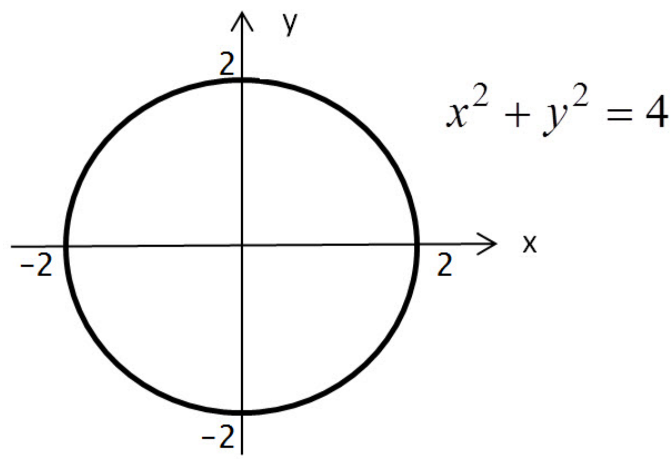


Note: All graphs must be drawn using a right-hand system. This is a convention we will use throughout this course.

Just as the x-axis and y-axis in the plane divide the region into 4 sub-regions called quadrants, the intersection of any pair of coordinate axes in 3-space form a **coordinate plane**. These three coordinate planes, the xy-plane, the xz-plane and the yz-plane, divide 3-space into 8 sub-regions called **octants**. The **first octant** refers to the region where $x > 0$, $y > 0$ and $z > 0$.

In the xy-plane the graph of an equation is called a **curve**. In three dimensions the graph is called a **surface**.

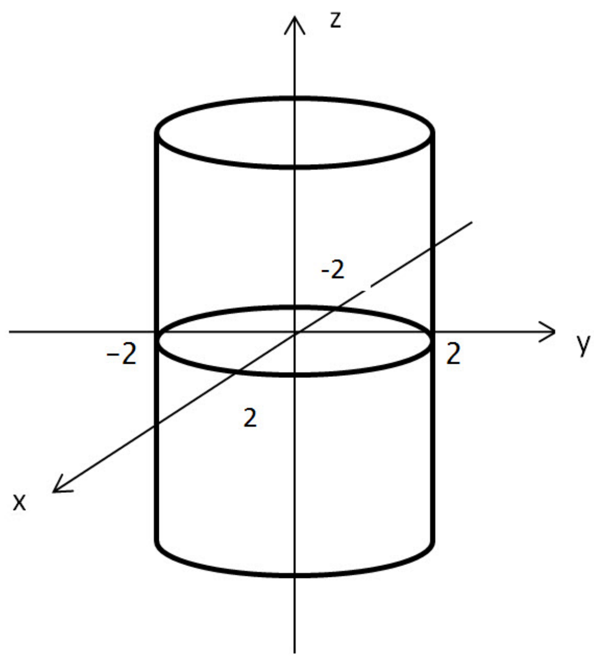
What does the graph of the equation $x^2 + y^2 = 4$ look like? In 2-space this is the familiar circle centered at the origin of radius 2.



But if we view this equation in 3-space, this equation is really

$x^2 + y^2 + 0z^2 = 4$ or $x^2 + y^2 = 4$

and the graph is a surface, the familiar circular right cylinder.



Important: A single equation in *x*, *y*, *z* (whether one or two variables are missing from the equation) always represents a surface in 3-space, never a curve or line.

The distance formula in the plane naturally generalizes to 3-space. If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two points in 3-space the distance d between them is calculated by

$$d = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Just as the distance formula in the plane is used to derive the equation of a circle centered at a point and of a fixed radius, so too the distance formula in 3-space is used to derive the equation of a sphere centered at a point and of a fixed radius, the generalization to 3-space of the circle.

The equation form $(x-h)^2+(y-k)^2+(z-l)^2=r^2$ describing a sphere with centre (h,k,l) and radius r $\left(\text{not } r^2 \,!\right)$ is called the **standard form**. If we expand and collect like terms we get the following **general form** of the equation of a sphere

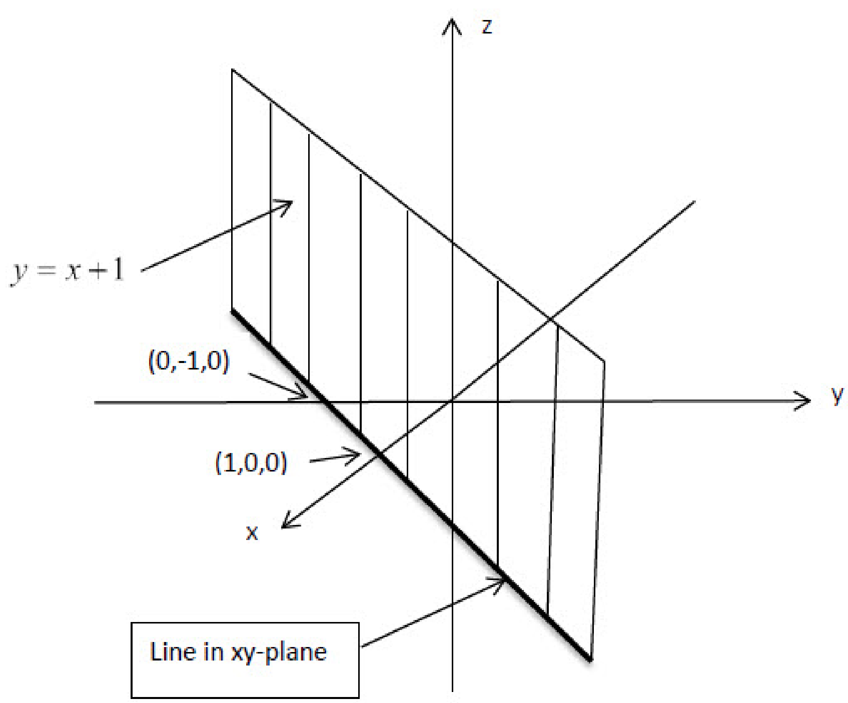
$$x^2+y^2+z^2+ax+by+cz+d=0$$

Notice that the coefficients of the squared terms are equal and have the value 1. If the equation is written with equal coefficients, not 1, then simply divide both sides of the equation by that non-zero coefficient to create coefficient 1. This general equation form can be converted into the more convenient standard form by completing the square in the variables *x*, *y*, and *z*.

Example Exercises

1. What does the equation $y=x+1$ represent in 3-space? Draw its graph.

   In three dimensions the equation $y=x+1$ is really $y=x+0z+1$. Its graph is a vertical plane intersecting the *xy*-plane in the line $y=x+1$ (see graph below).



2. This is problem 18 from your text on page 814.

Show that the equation $3x^2+3y^2+3z^2=10+6y+12z$ represents a sphere and find its centre and radius.

Collecting all terms on the left side of the equation and dividing through by 3 to make the coefficients of the squared terms 1, we have

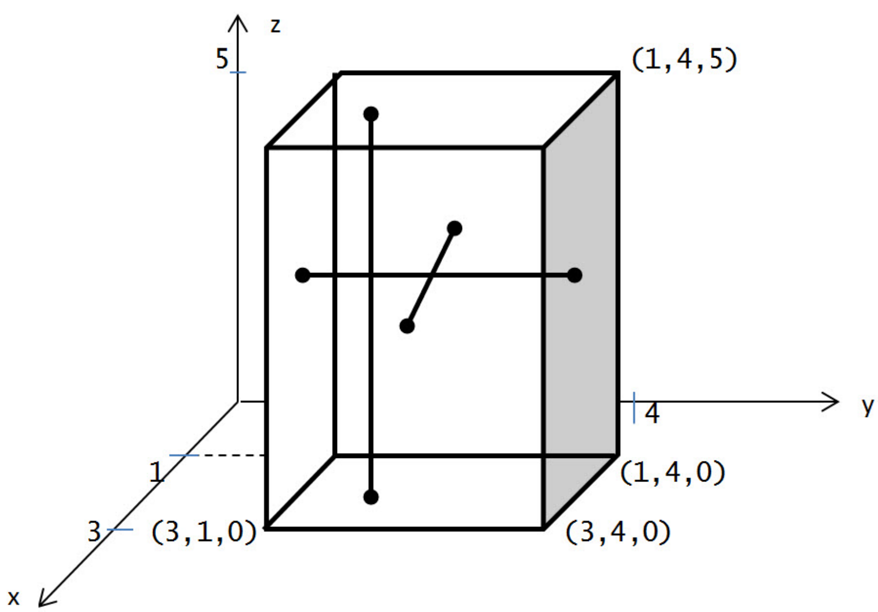
$$x^2+y^2+z^2-2y-4z+10=0$$
$$x^2+\left(y^2-2y\right)+\left(z^2-4z\right)+10=0$$
$$x^2+\left(y^2-2y+1\right)+\left(z^2-4z+4\right)+10=0+1+4$$
$$(x-0)^2+(y-1)^2+(z-2)^2=5=\left(\sqrt{5}\right)^2$$

This is the standard form of an equation for the sphere centered at (0,1,2) with radius $\sqrt{5}$.

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

3. Write a system of inequalities that describes the solid rectangular box in the first octant with vertices (3,4,0), (1,4,5), (1,4,0) and (3,1,0).

Start by drawing a picture.



In the x-direction we move from 1 to 3, so $1 \le x \le 3$.

In the y-direction we move from 1 to 4, so $1 \le y \le 4$.

In the z-direction we move from 0 to 5, so $0 \le z \le 5$.

So any point inside or on the face of the box has coordinates (x, y, z) satisfying these constraints. The system of inequalities is:

$1 \le x \le 3$

$1 \le y \le 4$

$0 \le z \le 5$

Practice Exercises 12.1

From the text pages 814–815, do problems 3, 7, 9, 13, 17, 19, 21, 25, 33, 35, 37, 39, 41, and 43.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Vectors

Learning Outcomes

Upon completion of Vectors, you should be able to:

- Add, subtract and perform the scalar multiplication operation on vectors, both algebraically and geometrically.
- Write down the eight properties of vectors listed on page 819 of the text.
- Determine the components and length of a vector.
- Solve application type problems using vectors, like Example 7 on page 821 of the text and Exercises 30–52 on pages 823 and 824.

Readings

Read section 12. 2, pages 815–822 in your textbook. Carefully study the examples worked out in the text.

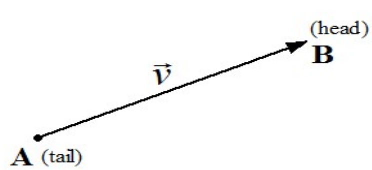
Overview

What is a vector?

A **vector** is a quantity that has both magnitude (length) and direction, an arrow, if you wish. The length of the arrow is the magnitude of the vector and its direction is indicated by the arrow.

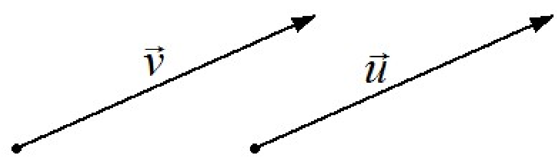
In many textbooks, vectors are indicated by bold face type. Since it is difficult to handwrite in bold face, we will always write a vector by putting an arrow above the symbol to distinguish it from a scalar or real number. So, $\vec{v}$ denotes a vector, whereas k denotes a scalar.

The initial point of a vector $\vec{v}$ is called its **tail** and the terminal point is called the **tip** or **head** of the vector (see diagram below).

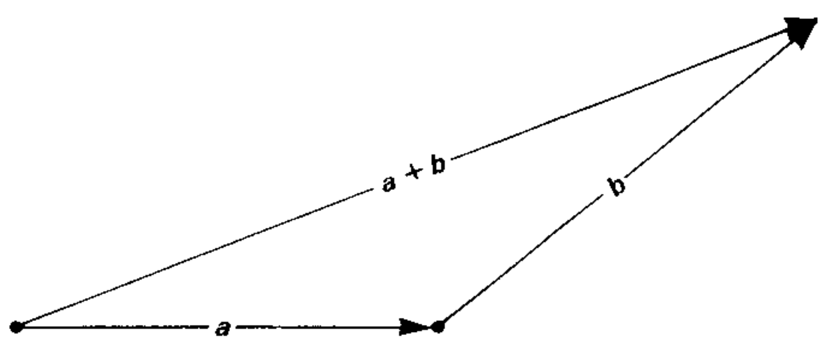


Notice that the tail and head of $\vec{v}$ are identified in the diagram above. In this case we can write $\vec{v} = \overrightarrow{AB}$.

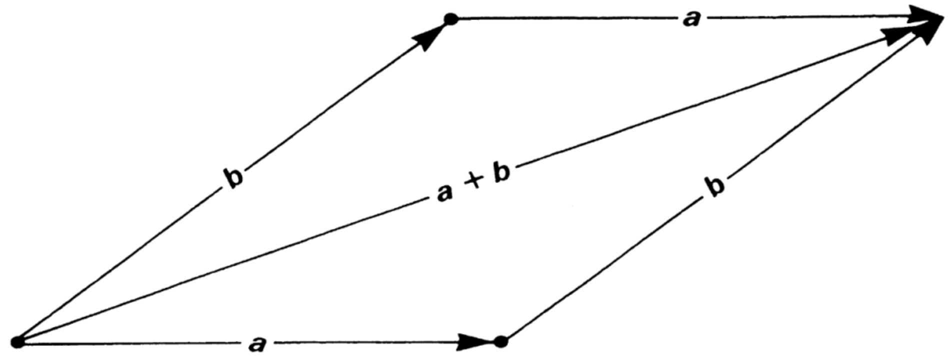
The beauty of vectors is that all arrows that have the same length and point the same way represent the same vector. They are **equivalent**. In the diagram below the vectors $\vec{u}$ and $\vec{v}$ are equivalent and we write $\vec{u} = \vec{v}$.



We can add vectors geometrically by putting them tail to head. This could be represented graphically as shown below, where ***a*** and ***b*** are the two vectors, and ***a* + *b*** is the sum.

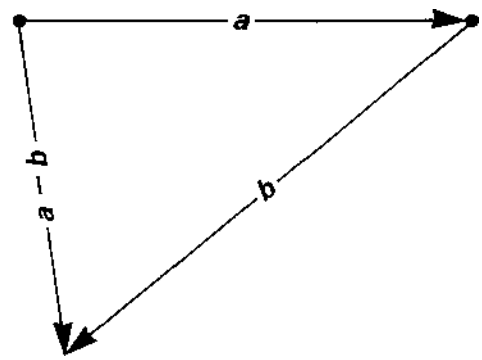


There is another way of adding vectors geometrically. It is called the Parallelogram Law. Below is an illustration using the same two vectors ***a*** and ***b***.



If the **negative of a vector** is defined as simply that vector equal in magnitude, but opposite in direction, the rule for subtraction follows quite straightforwardly.

You write ***a* – *b* = *a* +(-*b*)** which is represented in the illustration below.



The operation of multiplying a vector by a scalar is found on page 817 of the text. Taking the negative of a vector is the same as multiplying the vector by the scalar -1.

Vectors can also be treated algebraically. In 2-space, any vector $\vec{v}$ that starts at the origin (0,0) and ends at the point (x_0, y_0) has coordinates written $\vec{v} = < x_0, y_0 >$. Similarly, in 3-space, any vector $\vec{u}$ that starts at the origin (0,0,0) and ends at the point (x_0, y_0, z_0) has coordinates written $\vec{u} = < x_0, y_0, z_0 >$. The coordinates of a vector are called its **components**.

If $P(x_1, y_1, z_1)$ and $Q(x_2, y_2, z_2)$ are two given points in 3-space, the components of the vector that run from P to Q are given by $PQ = < x_2 - x_1, y_2 - y_1, z_2 - z_1 >$.

Vertical bars placed around a symbol for a vector mean that you are dealing with its magnitude, that is, its length.

If $\vec{v} =< x, y >$, $|\vec{v}| = \sqrt{x^2 + y^2}$; if $\vec{u} =< x, y, z >$, $|\vec{u}| = \sqrt{x^2 + y^2 + z^2}$

The symbol $\|\vec{v}\|$ is also used to represent the length of the vector $\vec{v}$.

Vectors can be added algebraically, by adding their components, subtracted, by subtracting their components and multiplied by a scalar by multiplying each component by that scalar.

See page 819 in the text for these definitions and the corresponding properties of the operations.

The symbolism shown on page 820 of the text for representing a vector in 3-space is very commonly used:

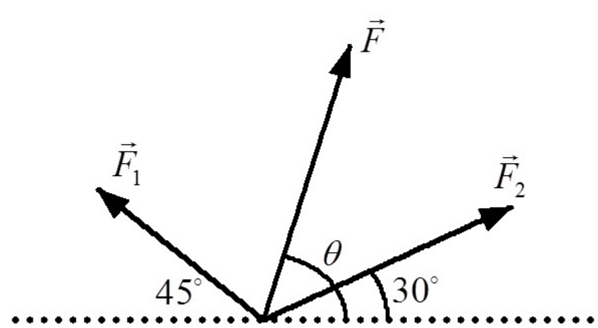
$\vec{v} =< x_1, x_2, x_3 >= x_1\vec{i} + x_2\vec{j} + x_3\vec{k}$

where $\vec{i} =< 1,0,0 >$, $\vec{j} =< 0,1,0 >$ and $\vec{k} =< 0,0,1 >$ are unit vectors along the x, y, and z-axes, respectively. The corresponding representation for vectors in the plane in terms of the unit vectors is $\vec{i} =< 1,0 >$ and $\vec{j} =< 0,1 >$.

Example Exercises

Your general practice should be to take a second look at examples worked out in the text and the one below. This time cover up the solutions to the worked examples and try to work them out for yourself.

Two forces $\vec{F}_1$ and $\vec{F}_2$ with magnitudes 10 pounds and 12 pounds act on an object at a point P as shown in the figure. Find the resultant force $\vec{F}$ acting at P as well as its magnitude and direction. (Indicate the direction by finding the angle θ in the figure.)



We are given that $|\vec{F}_1| = 10\,\text{lb}$ and $|\vec{F}_2| = 12\,\text{lb}$.

Using trigonometry,

$$F_1 = -\left|\vec{F}_1\right|\cos(45^\circ)\vec{i} + \left|\vec{F}_1\right|\sin(45^\circ)\vec{j} = -10\left(\frac{\sqrt{2}}{2}\right)\vec{i} + 10\left(\frac{\sqrt{2}}{2}\right)\vec{j}$$

$$= -5\sqrt{2}\,\vec{i} + 5\sqrt{2}\,\vec{j}$$

Similarly,

$$F_2 = -\left|\vec{F}_2\right|\cos(30^\circ)\vec{i} + \left|\vec{F}_1\right|\sin(30^\circ)\vec{j} = 12\left(\frac{\sqrt{3}}{2}\right)\vec{i} + 12\left(\frac{1}{2}\right)\vec{j}$$

$$= 6\sqrt{3}\,\vec{i} + 6\,\vec{j}$$

Since $\vec{F}$ is the resultant force,

$$\vec{F} = F_1 + F_2 = \left(-5\sqrt{2}\,\vec{i} + 5\sqrt{2}\,\vec{j}\right) + \left(6\sqrt{3}\,\vec{i} + 6\,\vec{j}\right)$$

$$= \left(6\sqrt{3} - 5\sqrt{2}\right)\vec{i} + \left(6 + 5\sqrt{2}\right)\vec{j}$$

Now,

$$\left|\vec{F}\right| = \sqrt{\left(6\sqrt{3} - 5\sqrt{2}\right)^2 + \left(6 + 5\sqrt{2}\right)^2} \approx 13.5 \text{ lb}$$

$$\tan(\theta) = \frac{6 + 5\sqrt{2}}{6\sqrt{3} - 5\sqrt{2}} \Rightarrow \theta = \tan^{-1}\left(\frac{6 + 5\sqrt{2}}{6\sqrt{3} - 5\sqrt{2}}\right) \approx 76^\circ$$

Practice Exercises 12.2

From the text pages 822–824, do problems 5, 9, 13, 21, 25, 27, 33, 37, and 45.

Remember: You are to attempt all of the problems carefully before checking your solutions against those given in the solutions manual.

The Dot Product

Learning Outcomes

Upon completion of The Dot Product, you should be able to:

- Define the dot (scalar) product directly in terms of the vectors.
- Define the dot product in terms of the components of the two vectors.
- Calculate the angle between two vectors.
- State the 5 rules as given in box 2 on page 825 of the text and be able to use them to solve problems.
- Calculate the vector projection and scalar projection of one vector on to another.
- Use the dot product to determine when two vectors are orthogonal or parallel.
- Solve application type problems using the dot product.

Readings

Read section 12. 3, pages 824–829 in your textbook. Carefully study the examples worked out in the text.

Overview

The dot product is also known as the scalar product or the inner product.

There is a real advantage in using the words <u>scalar product</u> (common among physicists); it serves as a reminder that the result of this form of multiplication is a scalar quantity. In the sequel you will be encountering another form multiplication where the resulting product is a vector. Each form of multiplication has its application, as you will see.

The concept of dot product is introduced in box 1 on page 824 of the text, where it is expressed in terms of the <u>components</u> of the two vectors being multiplied.

If $\vec{u} = <u_1, u_2, u_3>$ and $\vec{v} = <v_1, v_2, v_3>$, then $\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + u_3 v_3$

Then this definition is used to derive the results shown in box 2 on page 825. This order of development is advantageous because the written out component expression is one that will frequently be used in the final numerical solution of many problems.

However, one shouldn't lose sight of the forest (the vector) in being overwhelmed by the "trees" (the components). Particularly in theoretical work, where general formulae are being deduced, it is vital to conceptualize and manipulate the vectors and their combinations directly.

Thus an alternative approach would be to define the dot product in the first instance directly in terms of the vectors being multiplied as done in box 3 on page 825.

If θ is the angle between two vectors $\vec{u}$ and $\vec{v}$ $(0 \le \theta \le \pi)$, then

$$\vec{u} \cdot \vec{v} = |u||\vec{v}|\cos(\theta)$$

That is, the scalar product of two vectors is equal to the ordinary product of their two magnitudes, times the cosine of the angle between them.

Solving this equation for $\cos(\theta)$ gives a way of calculating the angle between two vectors if you know the components of the vectors.

Pay close attention to the result presented in box 7 on page 827. This characterizes when two vectors are orthogonal or perpendicular and it has many applications.

On page 828 the author describes a very important construction that of projecting one vector onto another. This is called vector projection. Do not confuse the concepts of vector projection and scalar projection. When you project the vector $\vec{u}$ onto $\vec{v}$, the projection vector, denoted $proj_{\vec{v}}\vec{u}$, is parallel to $\vec{v}$. If you construct a unit vector in the direction of $\vec{v}$, the scalar projection of $\vec{u}$ onto $\vec{v}$ is precisely that scalar which multiplies the unit vector to give $proj_{\vec{v}}\vec{u}$.

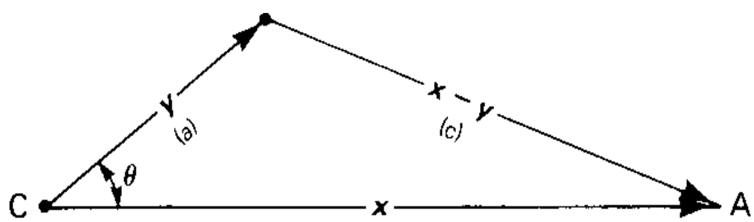
Example Exercises

1. a) The dot product can be used to derive the law of cosines, familiar to you from trigonometry. Using the distributive laws for vectors you can write

$$(\vec{x}-\vec{y})\cdot(\vec{x}-\vec{y}) = \vec{x}\cdot\vec{x} - \vec{y}\cdot\vec{x} - \vec{x}\cdot\vec{y} + \vec{y}\cdot\vec{y}$$

One of the rules you have learned about the dot product is that multiplication is commutative. Therefore, $\vec{x}\cdot\vec{y} = \vec{y}\cdot\vec{x}$, and the above becomes

$$|\vec{x}-\vec{y}|^2 = |\vec{x}|^2 + |\vec{y}|^2 - 2|\vec{x}||\vec{y}|\cos(\theta)$$



b) In trigonometry, the usual labelling of the sides of a triangle would put a for $\vec{y}$, b for $\vec{x}$, c for $\vec{x}-\vec{y}$, and $\angle C$ for θ, and you have the familiar expression

$$c^2 = a^2 + b^2 - 2ab\cos(\theta)$$

2. This is problem 50 from your text on page 831.

A tow truck drags a stalled car along a road. The chain makes an angle of 30° with the road and the tension in the chain is 1500 N. How much work is done by the truck in pulling the car 1 km?

The dot product is used to calculate the work done by a force acting upon a body during a displacement.

A few definitions are in order. Most of these terms are used in the text and it is essential to be able to distinguish between pairs that are often used interchangeably in popular language. The others are included for completeness.

<u>Displacement</u> is the vector whose magnitude is equal to the distance a body moves in the direction moved. Note that <u>distance</u> is a scalar.

<u>Velocity</u> is the vector whose magnitude is equal to the speed, and whose direction is the direction of motion. Note that speed is a <u>scalar.</u>

In the internationally agreed upon system of units, <u>force,</u> a vector, is measured in <u>newtons.</u> For example, a mass of m kilograms hanging at the end of a rope exerts a force on the rope approximately equal to 9.8 times m newtons.

The <u>work done</u> (W) when a body undergoes a displacement $\vec{S}$, when acted upon by a force $\vec{F}$, is equal to the scalar product of the two:

$$W = \vec{F}\cdot\vec{S} = \left|\vec{F}\right|\left|\vec{S}\right|\cos(\theta)$$

In the International System of Units (S. I. U.), if the force is measured in <u>newtons</u> and the displacement in <u>metres,</u> the work done is in <u>joules</u> and the work done per second, the <u>power,</u> is in <u>watts.</u> Note that if the force acts upon the body at right angles, the cosine is zero, and no work is done. Such forces are called <u>forces</u> of <u>constraint.</u>

Note that work is a scalar.

In the present problem you are given that the force has magnitude 1500 N, and the body undergoes a displacement of 1 km = 1000 m. The work done is then

$$\begin{aligned} W &= \left|\vec{F}\right|\left|\vec{S}\right|\cos(\theta) = (1500N)(1000m)\cos(30^\circ) \\ &= (1500000)\left(\frac{\sqrt{3}}{2}\right)Nm \approx 1299038\ J \end{aligned}$$

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

3. This is problem 64 from your text on page 831.

Show that if $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ are orthogonal, then $\vec{u}$ and $\vec{v}$ must have the same length.

Two vectors are orthogonal precisely when their dot product is zero.

So we must have that

$$\begin{aligned} \left(\vec{u}+\vec{v}\right)\cdot\left(\vec{u}-\vec{v}\right) &= 0 \\ \vec{u}\cdot\vec{u}+\vec{v}\cdot\vec{u}-\vec{u}\cdot\vec{v}-\vec{v}\cdot\vec{v} &= 0 \\ \left|\vec{u}\right|^2+\vec{u}\cdot\vec{v}-\vec{u}\cdot\vec{v}-\left|\vec{v}\right|^2 &= 0 \\ \left|\vec{u}\right|^2-\left|\vec{v}\right|^2 &= 0 \\ \left|\vec{u}\right|^2 &= \left|\vec{v}\right|^2 \\ \left|\vec{u}\right| &= \left|\vec{v}\right|, \text{ since } \left|\vec{v}\right| \geq 0 \end{aligned}$$

Hence, $\vec{u}$ and $\vec{v}$ have the same length.

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

Practice Exercises 12.3

From the text pages 830–831, do problems 1, 7, 9, 19, 21, 27, 31, 39, 43, 49, 55, 61, and 63.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

The Cross Product

Learning Outcomes

Upon completion of The Cross Product, you should be able to:

- Calculate the cross product vector using both component form and vector form, and be able to draw this vector according to the right-hand rule.
- Write down the properties of the cross product operation described in boxes 10 and 11 on pages 835 and 826 of the text and be able to use these properties to solve problems.
- Calculate the area of a parallelogram using the magnitude of a cross product vector.
- Calculate triple scalar products and be able to interpret the magnitude of the value.
- Determine the torque in an application.

Readings

Read section 12. 4, pages 832–838 in your textbook. Carefully study the examples worked out in the text.

Overview

As you fix the concept of cross product in your memory, be certain to attach the equally common (to a physicist) label, vector product. In the previous section, the terms "scalar product" and "dot product" were used interchangeably. Using the term "vector product" emphasizes the functional nature of the form of vector multiplication presented here.

Stewart develops the cross product in terms of its components in box 4 on page 832 of the text. Then he derives the vector form in the theorem found in box 9 on page 834 of the text. While I won't belabor the point, you should note that you can use this theorem as the fundamental definition. Then the statement in box 4 would become the derived theorem, with the vectors expanded in their components with respect to some now assumed system of coordinates. In any event it is essential that you be equally fluent in working with the vector product from either viewpoint.

The vector product can be treated as a vector equal in magnitude to the product of the magnitudes of the two vectors being multiplied, times the sine of the angle between them:

$$|\vec{u}\times\vec{v}| = |\vec{u}||\vec{v}|\sin(\theta)$$

and forms **a right-handed system** with the vectors in the order $\vec{u}$, $\vec{v}$, and $\vec{u}\times\vec{v}$; in particular, $\vec{u}\times\vec{v}$ is orthogonal to $\vec{u}$ and $\vec{v}$.

Because the component form of the cross product vector is cumbersome to remember, we make use of **determinant notation** to easily calculate the coordinates. Stewart introduces the concepts of a **determinant of order 2** and a **determinant of order 3** on pages 832 and 833 of the text.

Formally, if $\vec{u} =< u_1, u_2, u_3 >$ and $\vec{v} =< v_1, v_2, v_3 >$

$$\begin{aligned}\vec{u}\times\vec{v} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}\vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}\vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}\vec{k} \\ &= (u_2v_3 - u_3v_2)\vec{i} - (u_1v_3 - u_3v_1)\vec{j} + (u_1v_2 - u_2v_1)\vec{k} \\ &=< u_2v_3 - u_3v_2,\ u_3v_1 - u_1v_3, u_1v_2 - u_2v_1 >\end{aligned}$$

Determinant notation is often abused by students. Omitting the row of vectors $\vec{i}, \vec{j}, \vec{k}$ is not acceptable, as is using square bracket notation when vertical line notation $|\ \ |$ is required. Be careful!

The cross product operation can be used to characterize parallel vectors.

$\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u}\times\vec{v} = \vec{0}$

Notice the arrow above the number 0. This is the zero vector, not the scalar zero, since $\vec{u}\times\vec{v}$ is a vector.

The magnitude of the cross product vector is related to area:

$|\vec{u}\times\vec{v}|$ is equal to the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$

In box 11 on page 836 of the text is a list of properties of vector products. Notice that some of the usual laws of algebra do not hold here. For example, when dealing with the multiplication operation on the set of real numbers, we know that $a\cdot b = b\cdot a$. But with the cross product operation on vectors, this property does not hold. In fact, we have $\vec{a}\times\vec{b} = -\vec{b}\times\vec{a}$.

A combination of the cross product operation with the dot product has a special name. It is called the **scalar triple product**. It turns out that this special product can be calculated using a determinant of order 3 and it has a geometric significance.

If $\vec{u} = < u_1, u_2, u_3 >$, $\vec{v} = < v_1, v_2, v_3 >$ and $\vec{w} = < w_1, w_2, w_3 >$, then

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

and the magnitude of this vector gives the volume of the parallelepiped (a box whose faces are parallelograms) determined by $\vec{u}$, $\vec{v}$ and $\vec{w}$.

Notice that if the three vectors lie in the same plane, then the scalar triple product is zero and there is no parallelepiped.

Example Exercises

1. A constant force $\vec{F} = < 3, -1, 2 >$ acts on a particle located at (2,1,1). Find the resulting torque at (-1,1,0).

   The concept of **torque** is discussed in the text on page 837. You should read this first.

   Start by calculating the vector for the lever arm. Since the force acts at R(2,1,1) and the torque acts at P(-1,1,0), the position vector $\vec{r} = PR = < 2-(-1), 1-1, 1-0 > = < 3,0,1 >$

   By the definition of <u>torque</u> (also called the <u>moment of force</u>), we obtain

$$\begin{aligned} \tau &= \vec{r} \times \vec{F} = < 3,0,1 > \times < 3,-1,2 > \\ &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 3 & 0 & 1 \\ 3 & -1 & 2 \end{vmatrix} = \vec{i} - 3\vec{j} - 3\vec{k} \end{aligned}$$

   The torque at (-1,1,0) is <1,-3,-3> with magnitude

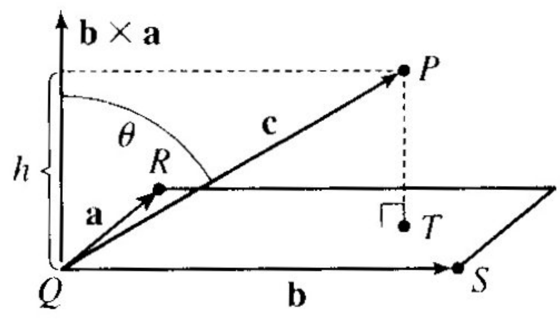
$$|\tau| = \sqrt{1^2 + (-3)^2 + (-3)^2} = \sqrt{19} \approx 4.4 \; Nm$$

2. This is problem 46 from your text on page 839.

a) Let P be a point not on the plane that passes through the points Q, R, S. Show that the distance d from P to the plane is

$$d = \frac{\left|\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right|}{\left|\vec{a}\times\vec{b}\right|}$$, where $\vec{a} = \overrightarrow{QR}$, $\vec{b} = \overrightarrow{QS}$ and $\vec{c} = \overrightarrow{QP}$.

Let us start with a picture.



Notice that the direction of the cross product vector is determined by the right-hand rule.

The distance from the point P to the plane is $d = \left|\overrightarrow{TP}\right|$. From the diagram we see that $d = h$, the length of the vector projection of $\vec{c}$ onto $\vec{b}\times\vec{a}$.

So, $d = \left|\left(\dfrac{\vec{c}\cdot\left(\vec{b}\times\vec{a}\right)}{\left(\vec{b}\times\vec{a}\right)\cdot\left(\vec{b}\times\vec{a}\right)}\right)\left(\vec{b}\times\vec{a}\right)\right| = \dfrac{\left|\vec{c}\cdot\left(\vec{b}\times\vec{a}\right)\right|}{\left|\vec{b}\times\vec{a}\right|^2}\left|\vec{b}\times\vec{a}\right| = \dfrac{\left|\vec{c}\cdot\left(\vec{b}\times\vec{a}\right)\right|}{\left|\vec{b}\times\vec{a}\right|}$

Pay careful attention to the symbolism here. The expression $\left|\vec{b}\times\vec{a}\right|$ means length of vector, whereas the expression $\left|\vec{c}\cdot\left(\vec{b}\times\vec{a}\right)\right|$ means the absolute value of the real number given by the dot product operation.

Now, using properties of the dot and cross product operations we have $\vec{c}\cdot\left(\vec{b}\times\vec{a}\right) = \left(\vec{c}\times\vec{b}\right)\cdot\vec{a} = \vec{a}\cdot\left(\vec{c}\times\vec{b}\right) = -\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)$ and $\vec{b}\times\vec{a} = -\vec{a}\times\vec{b}$

But, $\left|\vec{c}\cdot\left(\vec{b}\times\vec{a}\right)\right| = \left|-\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right| = \left|\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right|$ and $\left|\vec{b}\times\vec{a}\right| = \left|-\vec{a}\times\vec{b}\right| = \left|\vec{a}\times\vec{b}\right|$

So, $d = \dfrac{\left|\vec{c}\cdot\left(\vec{b}\times\vec{a}\right)\right|}{\left|\vec{b}\times\vec{a}\right|} = \dfrac{\left|\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right|}{\left|\vec{a}\times\vec{b}\right|}$

b) Use the distance formula to find the distance from the point P(2,1,4) to the plane through the points Q(1,0,0), R(0,2,0) and S(0,0,3).

$\vec{a} = \overrightarrow{QR} = <-1,2,0>$, $\vec{b} = \overrightarrow{QS} = <-1,0,3>$ and $\vec{c} = \overrightarrow{QP} = <1,1,4>$

We want to calculate $d = \dfrac{\left|\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right|}{\left|\vec{a}\times\vec{b}\right|}$. We need to find $\vec{b}\times\vec{c}$ and $\vec{a}\times\vec{b}$.

$$\vec{b}\times\vec{c} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 0 & 3 \\ 1 & 1 & 4 \end{vmatrix} = -3\vec{i} + 7\vec{j} - \vec{k} = <-3,7,-1>$$

$$\vec{a}\times\vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 2 & 0 \\ -1 & 0 & 3 \end{vmatrix} = 6\vec{i} + 3\vec{j} + 2\vec{k} = <6,3,2>$$

So, $d = \dfrac{\left|\vec{a}\cdot\left(\vec{b}\times\vec{c}\right)\right|}{\left|\vec{a}\times\vec{b}\right|} = \dfrac{\left|<-1,2,0>\cdot<-3,7,-1>\right|}{\left|<6,3,2>\right|} = \dfrac{|17|}{\sqrt{6^2+3^2+2^2}} = \dfrac{17}{7}$

Stewart, J. (2012). *Multivariable calculus* (7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.

3. This is problem 52 from your text on page 840.

Prove that: $\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right)=\begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c}\\ a\cdot d & b\cdot d\end{vmatrix}$

Let $\vec{v}=\vec{c}\times\vec{d}$. Then

$$\left(\vec{a}\times\vec{b}\right)\cdot\left(\vec{c}\times\vec{d}\right)$$
$$=\left(\vec{a}\times\vec{b}\right)\cdot\vec{v}$$
$$=\vec{a}\cdot\left(\vec{b}\times\vec{v}\right)$$
$$=\vec{a}\cdot\left(\vec{b}\times\left(\vec{c}\times\vec{d}\right)\right)$$
$$=\vec{a}\cdot\left(\left(\vec{b}\cdot\vec{d}\right)\vec{c}-\left(\vec{b}\cdot\vec{c}\right)\vec{d}\right)$$
$$=\left(\vec{b}\cdot\vec{d}\right)\cdot\left(\vec{a}\cdot\vec{c}\right)-\left(\vec{b}\cdot\vec{c}\right)\cdot\left(\vec{a}\cdot\vec{d}\right)$$
$$=\begin{vmatrix}\vec{a}\cdot\vec{c} & \vec{b}\cdot\vec{c}\\ \vec{a}\cdot\vec{d} & \vec{b}\cdot\vec{d}\end{vmatrix}$$

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

Practice Exercises 12.4

From the text pages 838–840, do problems 5, 11, 13, 19, 23, 31, 35, 39, 43, 45, 47, 49, and 53.

Remember: You are to attempt all of the problems carefully before checking your solutions against those given in the solutions manual.

Equations of Lines and Planes

Learning Outcomes

Upon completion of Equations of Lines and Planes, you should be able to:

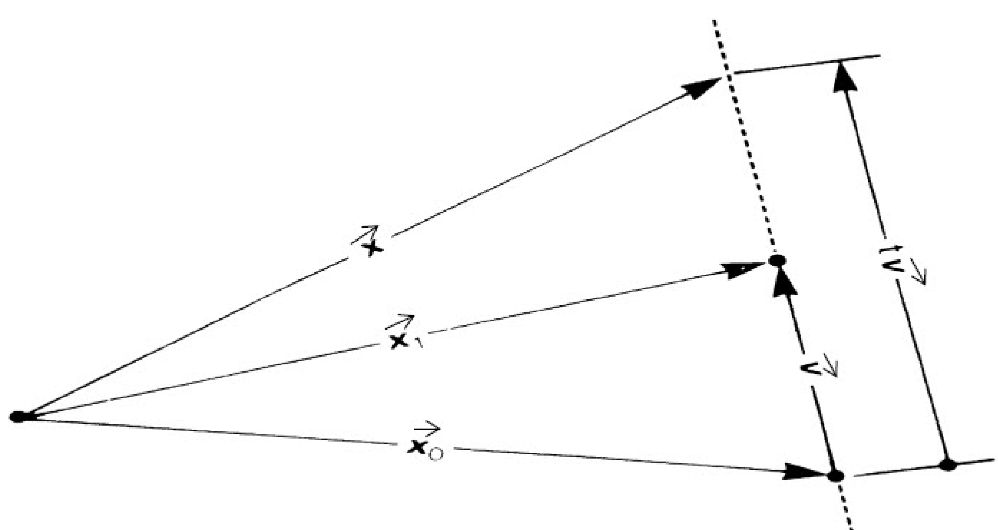
- Determine the vector equation of a line and of a line segment.
- Determine parametric equations for a line.
- Determine symmetric equations for a line.
- Write the equation of a plane in the normal vector form.
- Write the equation of a plane in scalar form.
- Examine the equations of two planes and determine whether they are parallel, perpendicular, or intersecting at an angle.
- Find the angle between two planes.
- Find the line of intersection of two non-parallel planes.
- Find the perpendicular distance from a point to a plane.
- Write the equation of a plane containing three non-collinear points.

Readings

Read section 12. 5, pages 840–847 in your textbook. Carefully study the examples worked out in the text.

Overview

Stewart begins by deriving the vector equation of a straight line. You can do this directly.



$\vec{x}_0$ is a fixed vector whose head determines a point on the line. $\vec{v}$ is a free vector translated so that its tail is at the head of $\vec{x}_0$ and its direction determines the direction of the line. $\vec{x}$ is a vector whose head lies on the line. Any point on the line through the head of $\vec{x}_0$ in the direction of $\vec{v}$ is determined by adding to $\vec{x}_0$ a vector $t\vec{v}$, where t is some scalar parameter. Using the rules for vector addition, you can write

$$\vec{x} = \vec{x}_0 + t\vec{v}$$

the equation Stewart obtains in box 1 on page 841 of the text.

If we write $\vec{x}_0 = < x_0, y_0, z_0 >$, $\vec{x} = < x, y, z >$ and $\vec{v} = < a, b, c >$ then the vector equation becomes $< x, y, z > = < x_0, y_0, z_0 > + t < a, b, c > = < x_0 + ta, y_0 + tb, z_0 + tc >$

Equating components we obtain a set of equations, called **parametric equations**, of the line through $P(x_0, y_0, z_0)$ with direction vector $\vec{v} = < a, b, c >$:

$$x = x_0 + ta \quad y = y_0 + tb \quad z = z_0 + tc$$

Here $t \in R$, that is, t is any real number, to generate all the points on the line.

Solving these equations for t and equating, we obtain **symmetric equations** of a line with the parameter eliminated:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

Note: It is possible that one of the coordinates of the direction vector $\vec{v} = < a, b, c >$ could be zero. For example, if $c = 0$,then $z = z_0$ and we would eliminate t from the other two equations to obtain the symmetric equations

$$\frac{x - x_0}{a} = \frac{y - y_0}{b}; \ z = z_0$$

If you look at the diagram that was drawn above to determine the vector equation of a line, you will see that the direction vector $\vec{v}$ can be expressed as $\vec{v} = \vec{x}_0 - \vec{x}_1$, where the heads of the vectors $\vec{x}_0$ and $\vec{x}_1$ lie on the line. The vector equation now becomes

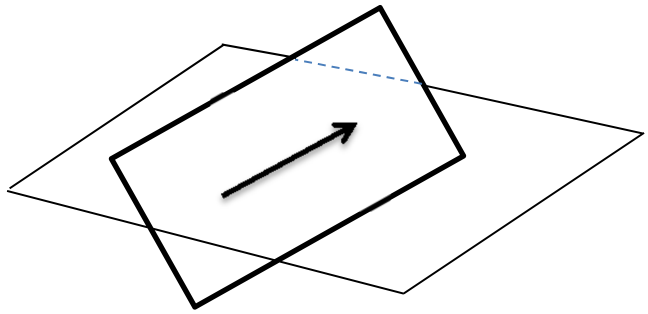
$$\vec{x} = \vec{x}_0 + t\vec{v} = \vec{x}_0 + t\left(\vec{x}_0 - \vec{x}_1\right) = (1+t)\vec{x}_0 - t\vec{x}_1$$

If we restrict t to $0 \le t \le 1$, we obtain the line segment joining the heads of the two vectors that lie on the line. Notice that if we set $s = -t$ in the equation above we obtain the version given in box 4 of the text on page 843:

$$\vec{x} = (1+t)\vec{x}_0 - t\vec{x}_1 = (1-s)\vec{x}_0 + s\vec{x}_1, \ 0 \le s \le 1$$

Either form can be used to determine the line segment joining any two points.

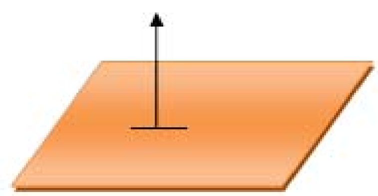
While the minimum information required to describe a line is a point on the line and a vector parallel, this is not enough information to determine a plane. Below are two different planes that intersect in a line. They share common points and vectors that are parallel to these planes.



The orientation of a plane in 3-space is not determined by a vector parallel to the plane, but rather by a vector perpendicular to it, called a **normal vector**.

A plane is uniquely determined if we know

- a point on the plan $P(x_0, y_0, z_0)$ and
- a normal vector $\vec{n} =< a,b,c >$ to the plane



On page 844 Stewart gives a number of ways of describing an equation of a plane. In scalar form we have

$$a(x-x_0)+b(y-y_0)+c(z-z_0)=0$$

This is called the **scalar equation** of a plane through $P(x_0, y_0, z_0)$ with normal vector $\vec{n} =< a,b,c >$. If we remove the brackets in this equation and combine the constant terms together as a single constant d we obtain the **linear equation** of the plane

$$ax+by+cz+d=0$$

The problem of determining the distance from a point to a plane is discussed on pages 846 and 847 of the text. Pay attention to the distance formula described in box 9 on page 847.

Example Exercises

1. During your studies of plane analytic geometry, you have often been asked to find the equation of a line perpendicular to a given line and passing through a given point. What happens when you try to solve the same problem in 3-space? Do you obtain a unique line?

   Restricting our attention to 3-space, attempt to find the equation of a line perpendicular to a given line.

   We will begin with the general theoretical problem, and then restrict to a particular given line. Let the given line have the direction of the vector $\vec{v}$. We must find the equation of a line perpendicular to $\vec{v}$ that passes through the head of a fixed vector $\vec{x}_0$.

   Let $\vec{x}$ be the variable vector with its head on the required line. Then $\vec{x}-\vec{x}_0$ is perpendicular to $\vec{v}$ and we can write

   $$\left(\vec{x}-\vec{x}_0\right)\cdot\vec{v}=0$$

   as the equation of the line.

   Now let the particular point vector $\vec{x}_0$ be $<1,0,4>$ and the given line have the direction $\vec{v}=<2,-1,4>$ Substituting in the equation above, we obtain

   $$\begin{aligned}\left(<x,y,z>-<1,0,4>\right)\cdot<2,-1,4>&=0\\ <x-1,y-0,z-4>\cdot<2,-1,4>&=0\\ 2(x-1)+(-1)(y)+4(z-4)&=0\\ 2x-y+4z-18&=0\end{aligned}$$

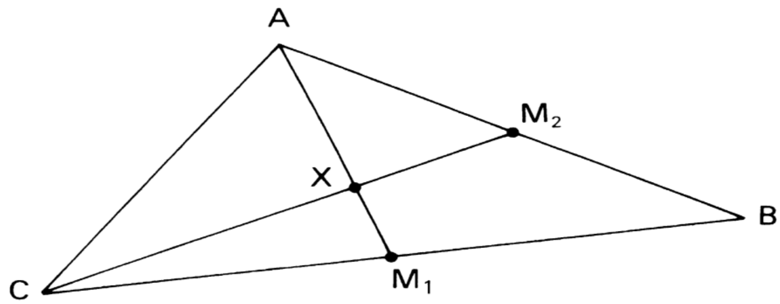
   You have your answer, but there is a problem. This scalar equation is a single one, unlike the pair of equations you obtained for the symmetric form (see box 3 on page 842 of the text). Is this a new form of the equation of a line, or has something gone wrong in our theory? I leave it to you to ponder this question.

   Hint: Do you obtain a unique line?

2. An interesting application of the equation

   $$\vec{x}=\left(1+t\right)\vec{x}_0-t\vec{x}_1=\left(1-s\right)\vec{x}_0+s\vec{x}_1\ ,\ 0\le s\le 1$$

   is to prove that the medians of any triangle are concurrent, and to find the location of that point.



AM_1 is the median drawn to the side *BC*, and CM_2 is the median drawn to the side *AB*. Since *x* lies on AM_1

$X = tM_1 + (1-t)A$ for some $0 \le t \le 1$

and, since M_1 is the midpoint of the line BC,

$$M_1 = \frac{1}{2}(B + C)$$

Therefore,

$$X = (1-t)A + \frac{1}{2}tB + \frac{1}{2}tC$$

Similarly,

$$X = sM_2 + (1-s)C \text{ for some } 0 \le s \le 1$$

and, substituting for $M_2 = \frac{1}{2}(A + B)$

$$X = (1-s)C + \frac{1}{2}sA + \frac{1}{2}sB$$

Since these two values for *X* must be the same for all values of *A*, *B*, and *C*, we can equate the coefficients to find

$$s = t = \frac{2}{3}$$

Substituting in either of the above equations for X we find

$$X = \frac{A + B + C}{3}$$

Since this is symmetric in *A*, *B*, and *C*, *X* lies on all of the medians and is located 2/3 of the way from each of the vertices towards the middle point of the opposite side.

3. Find the line of intersection of the two planes

$x+2y-z=0;\ 3x-y+2z=0$

We have seen that the coefficients of the coordinate variables give the components of the normal vector to a plane. Thus, the normals to the two planes are given by $\vec{n}_1 = <1,2,-1>$ and $\vec{n}_2 = <3,-1,2>$.

Since the line of intersection is in both planes, it must be perpendicular to both of the normals. Therefore, its vector direction will be given by the vector product of $\vec{n}_1 = <1,2,-1>$ and $\vec{n}_2 = <3,-1,2>$

$$\vec{n}_1 \times \vec{n}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & -1 \\ 3 & -1 & 2 \end{vmatrix} = 3\vec{i} - 5\vec{j} - 7\vec{k} = <3,-5,-7>$$

If you examine the equations of the planes, you will note that they are satisfied jointly by (0,0,0). Therefore, this point must lie on the line of intersection. From the above you can see that the direction of the line of intersection is $<3,-5,-7>$. Therefore, the line of intersection in vector form is

$$\vec{x} = <0,0,0> + t<3,-5,-7>$$

or in parametric form

$$x = 3t\,,\quad y = -5t\,,\quad z = -7t$$

4. If $\vec{x} = \vec{x}_0 + s\vec{u}$ and $\vec{x} = \vec{x}_1 + t\vec{v}$ are two skew lines (non-intersecting, non-parallel lines) in 3-space, show that

a) $\vec{N} = \dfrac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|}$ is a unit vector perpendicular to both lines

b) The distance between the lines is $\left|\vec{N} \cdot (\vec{x}_0 - \vec{x}_1)\right|$

We are now getting into three-dimensional problems that are almost impossible to show intelligibly on two-dimensional paper, and very difficult to visualize. Now is as good a time as any to begin to train your mind to work in three dimensions. You might attempt an actual construction of the situation in real three dimensions. One of the lines could be ruled on your desk. The other line, skewed, is some centimetres above the desk. If you are not an octopus, you might call in an extra pair of hands. Try to carry out the construction step by step.

Since the lines are nonintersecting and nonparallel, the sine of the angle between them is not equal to zero. You can write

$$\vec{u} \times \vec{v} \neq 0$$

where $\vec{u}$ and $\vec{v}$ are vectors specifying the directions of the two lines. By the definition of cross product, $\vec{u} \times \vec{v}$ is a vector perpendicular to both $\vec{u}$ and $\vec{v}$.

Dividing by the magnitude of $\vec{u} \times \vec{v}$, we obtain

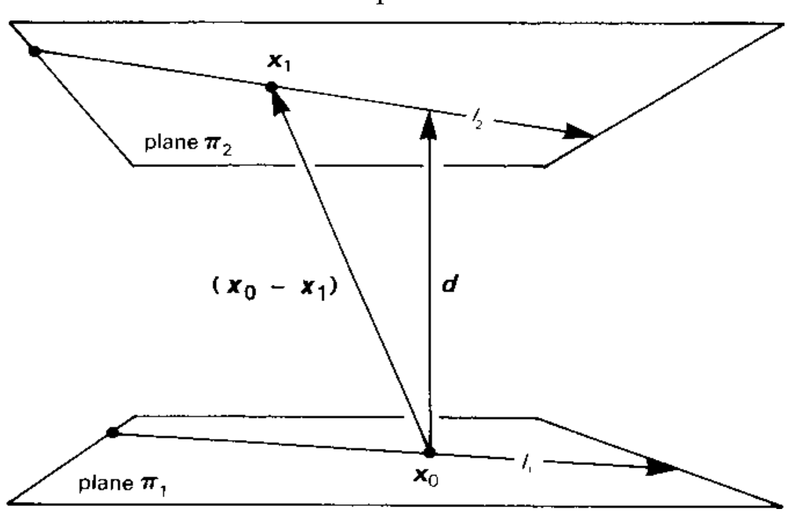
$$\vec{N} = \frac{\vec{u} \times \vec{v}}{|\vec{u} \times \vec{v}|}$$

which is a unit vector in the direction of the common perpendicular to the two lines. The problem, then, is to find the length of this perpendicular.

A student with a particularly well-developed right brain hemisphere can probably jump straight to the answer, but the rest of us need more help. I am going to go through a rather involved construction which should lead to the solution.

Where the common perpendicular between the two lines, representing the distance between them, meets line one, l_1, you should construct a plane, π_1, perpendicular to it and, naturally, containing line one. (You can do this in your imagination, or with one of those extra pairs of hands.) Do the same with line two, l_2, and plane two, π_2. The two resulting planes are parallel to each other and will contain, respectively, the head of the vector, $\vec{x}_0$, on line one, and the head of the vector $\vec{x}_1$, on line two. The perpendicular distance between them is what you are trying to find.

At the head of the vector $\vec{x}_0$, erect a vector $\vec{d}$, in the direction $\vec{n}$, with length equal to the distance between the planes.



We now have a right-angle triangle consisting of $\vec{x}_0 - \vec{x}_1$, and the second skew line. (It might help to think of π_1 as the floor of a room, and π_2 as the ceiling.)

From elementary trigonometry, you know that the length of $\vec{d}$ will be equal to the length of $\vec{x}_0 - \vec{x}_1$, multiplied by the cosine of the angle between them. With $\vec{N}$ a unit vector in the direction of $\vec{n}$, this is nothing other than the dot product

$$\left|\vec{d}\right| = \left|\vec{N} \cdot (\vec{x}_0 - \vec{x}_1)\right|$$

Practice Exercises 12.5

From the text pages 848–850, do problems 1, 5, 7, 11, 17, 19, 29, 31, 37, 43, 47, 51, 59, 61, 65, 69, 71, and 77.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Cylinders and Quadric Surfaces

Learning Outcomes

Upon completion of Cylinders and Quadric Surfaces, you should be able to:

- Identify and sketch the graph of a cylinder.
- Classify the quadric surfaces by name and by their equation.
- Graph the quadric surfaces and be able to give a detailed description of all horizontal and vertical traces.

Readings

Read section 12. 6, pages 851–856 in your textbook. Carefully study the examples worked out in the text.

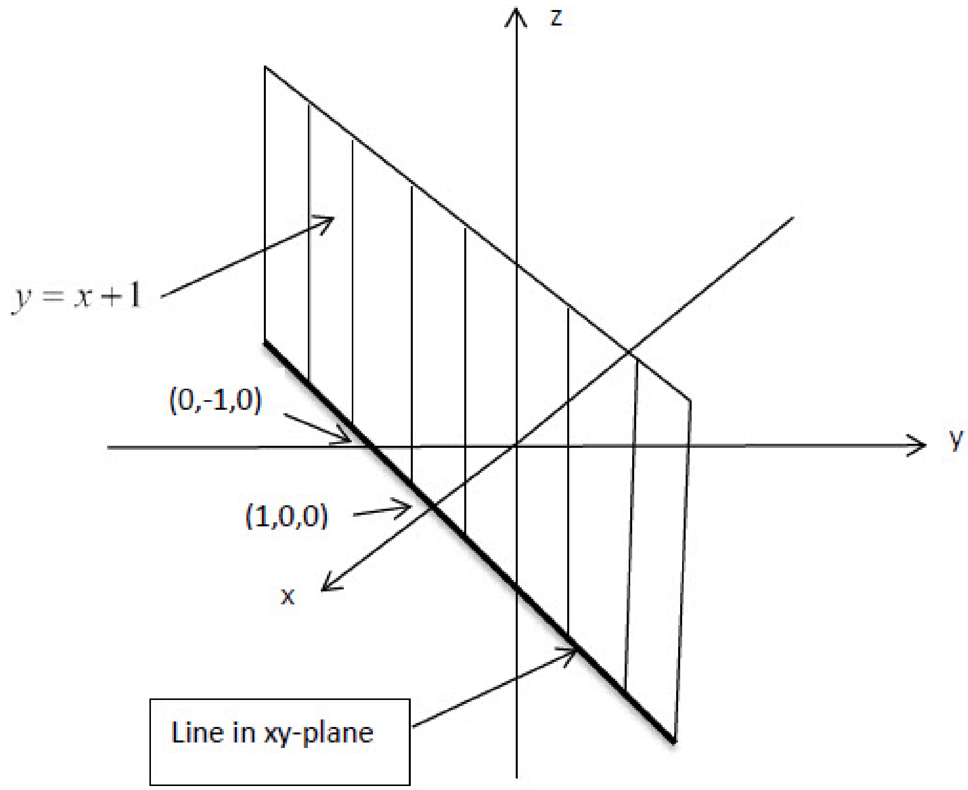
Overview

This section introduces two classes of surfaces, one called **cylinders**, and the other, **quadric surfaces**, located on the axes so that their equations are in their simplest forms.

A **cylinder** is generated when a line, called the "generator" line, traverses a plane curve always moving parallel to some fixed line. When the generator line is parallel to the *z*-axis and traverses a circle in the xy-plane, the resulting surface is the familiar right circle cylinder.

NOTE: If the generator line moves parallel to one of the coordinate axes, the equation of the resulting surface will not contain that coordinate variable.

Recall that the equation $y = x + 1$ in 3-space represents a vertical plane that intersects the xy-plane in the line $y = x + 1$ (see the diagram that follows).



A line parallel to the $y = x + 1$-axis travelling along the line $y = x + 1$ in the plane generates this plane. Notice that the *z*-variable is absent from the equation of the plane.

The class of surfaces, called the quadric surfaces, is the result of generalizing a second degree equation in two variables to three variables. By the process of rotation and translation the general equation

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + gx + Hy + Iz + J = 0$$

can be transformed to the following two standard forms

$$Ax^2 + By^2 + Cz^2 + J = 0 \qquad \text{or} \qquad Ax^2 + By^2 + Iz = 0$$

You need to know the name of each quadric surface, its equation form and its graph. You also need to know how to determine the “horizontal and vertical traces” for each of these surfaces and be able to give a detailed description of each trace. In general, a **trace** is a curve of intersection of one surface with another surface (usually a plane). When the trace is in a plane parallel to one of the coordinate planes, it is referred to as a “horizontal” or “vertical” trace.

In order that you will be able to effectively describe the trace curves for the quadratic surfaces, we need to do a quick review of the “conic sections”.

All of the conics can be represented by the general quadratic equation

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0 \quad \text{............} (1)$$

where the coefficients of the variables have certain defined characteristics. We will consider the different conics one by one, so located with respect to the coordinate axes that their equations are in the simplest forms.

Classically, it can be shown that the conics can be defined in terms of a fixed point called the <u>focus,</u> and a fixed straight line called the <u>directrix</u>. A conic is the path traced by a point which moves so that

$$\frac{\text{distance from a fixed point (focus)}}{\text{distance from a fixed line}} = \text{constant} \quad \text{(eccentricity)}$$

The magnitude of the eccentricity e determines the shape of the conic.

When $e = 0$, the conic is a circle

When $e < 1$, the conic is an ellipse

When $e = 1$, the conic is a parabola

When $e > 1$, the conic is an hyperbola.

The first conic to consider is the circle. Referring to the above general equation of a conic, the condition that it represents a circle is

$$A = C \text{ and } B = 0$$

If, in addition, it is centered at the origin then $D = E = 0$.

Next is the ellipse. (This category also includes circles, since a circle may be considered to be a degenerate ellipse.) Once more, referring to the general equation of a conic, the condition that it represents an ellipse is

$$A \text{ and } C \text{ have the same sign}$$

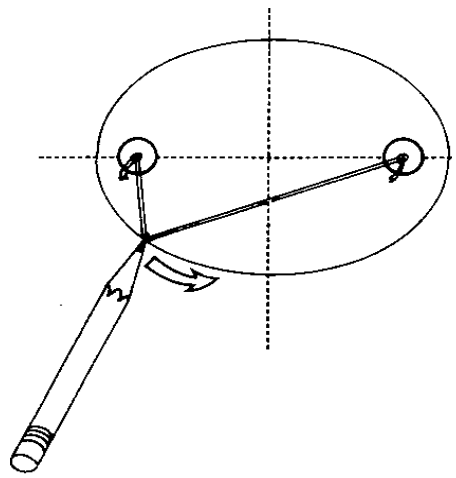
In addition, if the graph is centered at the origin and its major and minor axes lie along the two coordinate axes, you have the additional condition that

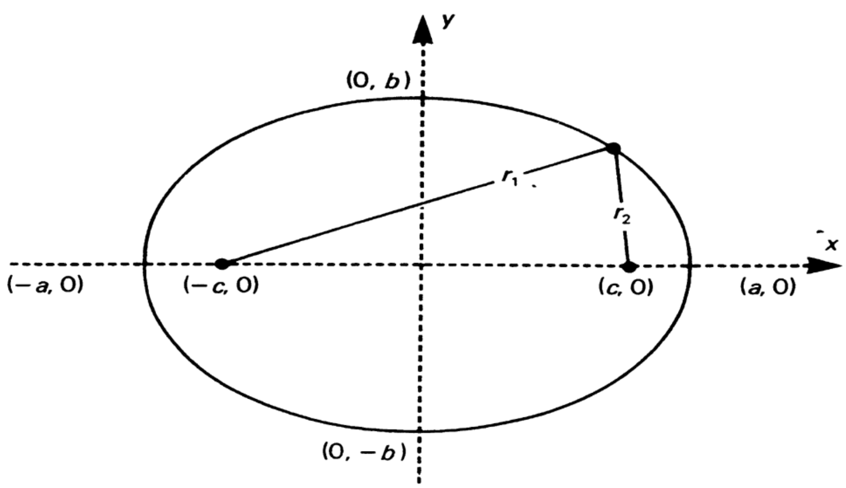
$$B = D = E = 0$$

The ellipse is centered at the origin, with its major axis 2a units long, both lying along the x axis, and its two foci are located at

(-c,0) and (c,0), respectively.

The ellipse can be drawn by taking a piece of string $2a$ units long with a small loop at each end. Thumb tacks are placed through the two loops and inserted at the two foci. A pencil point is pressed against the string and, with the string kept taut, moved along the string to trace out the ellipse.





The equation of an ellipse in standard form centered at the point (h,k) is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

At the origin this becomes $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

a is the semi-major axis, b the semi-minor axis, and the points $(-a,0)$ and $(a,0)$ are the vertices.

The case $a=b$ gives the circle.

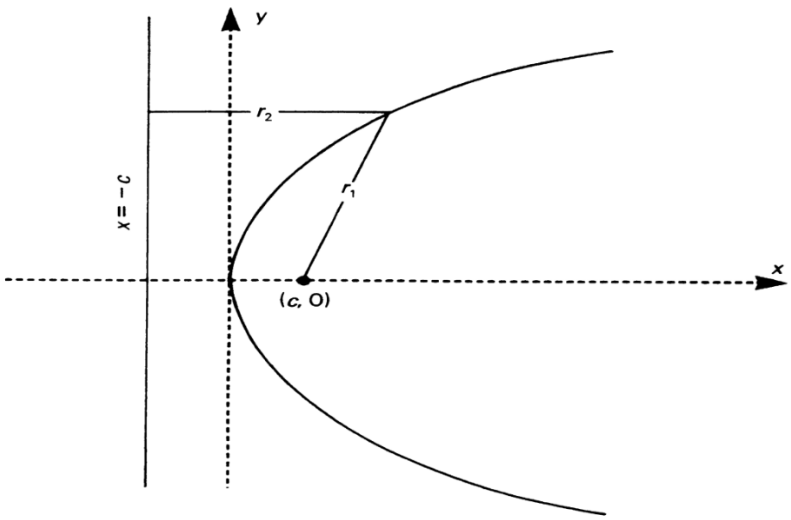
Next is the parabola. The general quadratic equation (1) will represent a parabola in standard form if

$B=0$, and either $A=0$ or $C=0$

The equation of the parabola in its standard form with vertex at (h,k) is:

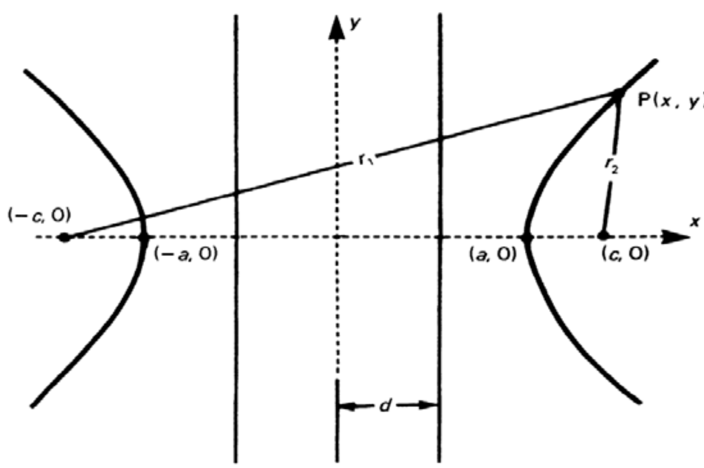
$(y-k)^2=4c(x-h)$ or $(x-h)^2=4c(y-k)$

At the origin this becomes $y^2=4cx$ or $x^2=4cy$



The last of the conics is the hyperbola. The general quadratic equation (1) will represent a hyperbola if A and C have opposite signs. If, in addition, it is centered at the origin and oriented so that its axis of symmetry is along one of the coordinate axes,

then $B = D = E = 0$.



The equation of the hyperbola in its standard form with vertex at (h,k) is:

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \quad \text{or} \quad \frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$

At the origin this becomes $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ or $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$

The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ has x-intercepts $(\pm a,0)$, opens in the x- direction and has asymptotes $y=\pm\frac{b}{a}x$.

The hyperbola $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ has y-intercepts $(0,\pm b)$, opens in the y- direction and has asymptotes $y=\pm\frac{b}{a}x$.

Example Exercises

1. Identify and sketch the graph of each equation. Give the coordinates of all intercepts. Identify the equation of and give a detailed description of the horizontal and vertical traces for the surface.

   a) $z=x^2+y^2$

   Notice the two squared variable terms with the same coefficients and the one linear term. This identifies the surface as a circular paraboloid. (See Table 1 on page 854 of the text.)

   The only intercept is the origin (0,0,0). It is the vertex of the paraboloid.

   Traces:
   Set: $z=k$: $x^2+y^2=k$

   When:

   $k>0$: Traces are circles centered at $(0,0,k)$ with radius $\sqrt{k}$.

   $k=0$: Trace is the point (0,0,0)

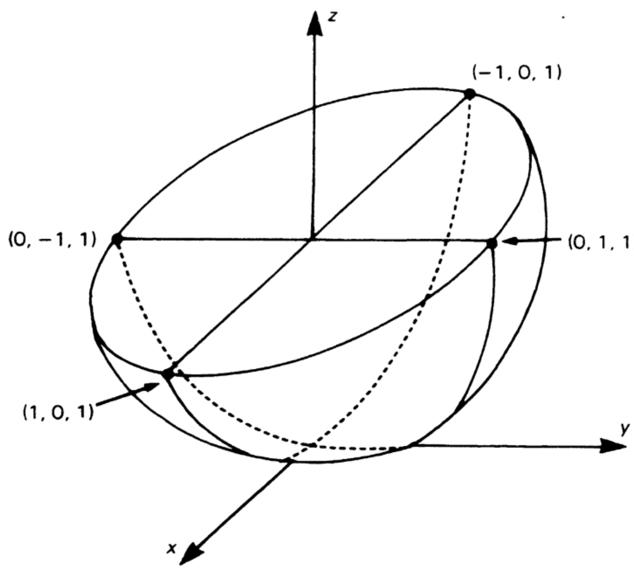
   $k<0$: No traces

   Next set: $x=k$: $z=k^2+y^2=(y-0)^2+k^2$

For all k, the traces are parabolas with vertex at $(k,0,k^2)$ that open in the positive z-direction.

$y=k$: Similar to $x=k$ traces

The graph:



b) $x^2+y^2-2z^2=4$

Re-writing we have $\frac{x^2}{4}+\frac{y^2}{4}-\frac{z^2}{2}=1$

Notice the three squared variable terms, the constant 1 and the one negative sign. This identifies the surface as a hyperboloid of one sheet. (See Table 1 on page 854 of the text.)

Setting $y=z=0$ in the equation above and solving gives the x-intercepts $(\pm 2,0,0)$. Similarly, Setting $x=z=0$ in the equation and solving gives the y-intercepts $(0,\pm 2,0)$. Setting $x=z=0$ in the equation gives no solution, so there are no z-intercepts.

Traces:

Set: $z=k$: $x^2+y^2-2k^2=4 \;\Rightarrow\; x^2+y^2=2k^2+4>0$

For all k, the traces are circles centered at $(0,0,k)$ with radius $\sqrt{2k^2+4}$

Next set: $x=k$: $k^2+y^2-2z^2=4 \;\Rightarrow\; y^2-2z^2=4-k^2$

Now, $4-k^2>0 \Rightarrow k^2<4 \Rightarrow -2<k<2$

When:

$-2<k<2$: Traces are hyperbolas centered at $(k,0,0)$ opening in the y-direction.

$k=\pm 2$: $y^2-2z^2=0 \Rightarrow y^2=2z^2 \Rightarrow y=\pm\sqrt{2}z$

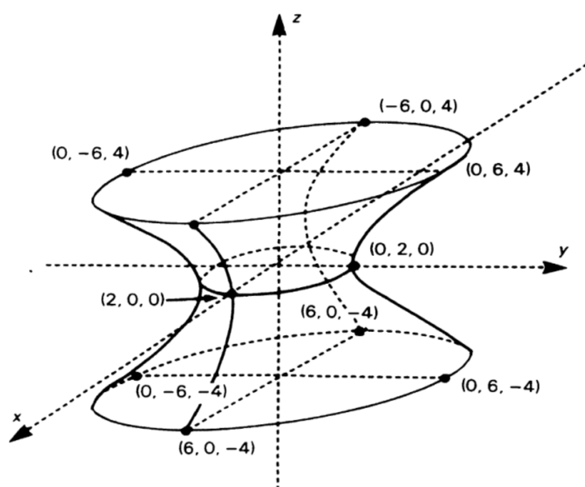
Traces are the lines $y=\pm\sqrt{2}z$ through $(k,0,0)$

$k>2$ or $k<-2$: $y^2-2z^2=4-k^2<0 \Rightarrow 2z^2-y^2=k^2-4>0$

Traces are hyperbolas centered at $(k,0,0)$ opening in the z-direction.

$y=k$: Similar to $x=k$ traces.

The graph:



2. This is problem 50 from your text on page 858.

Show that the curve of intersection of the surfaces

$x^2+2y^2-z^2+3x=1$ and $2x^2+4y^2-2z^2-5y=0$

lies in a plane.

If we double the first equation and subtract the second we obtain

$$\begin{array}{rl} & 2x^2+4y^2-2z^2+6x=2 \\ - & 2x^2+4y^2-2z^2-5y=0 \\ \hline & 6x+5y=2 \end{array}$$

> Any point on the curve of intersection of these two surfaces must have the property that its first two coordinates satisfy the equation $6x+5y=2$. But this is an equation of a vertical plane, so the curve of intersection must lie in this plane.

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

Practice Exercises 12.6

From the text pages 856–858, do problems 1, 5, 9, 13, 19, 21, 23, 27, 33, 35, 41, 43, 45

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Unit 1: Summary and Self-Test

You have now worked through Unit 1 in MATH 2111. It is time to take stock of what you have learned, review all of the material, and bring your shorthand notes up to date. A summary of the material covered so far is provided in the following pages. This summary should be modified, added to, and fleshed out to form a solid body of knowledge.

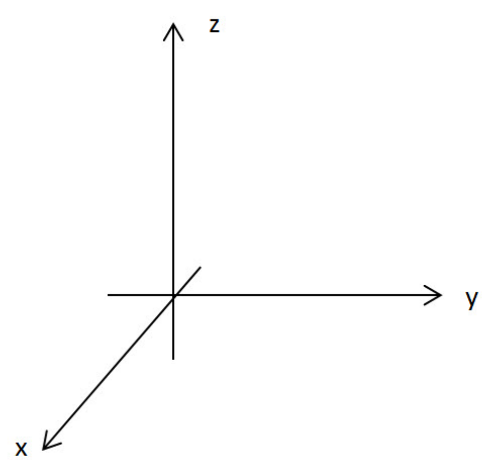
When you have completed your review, you should test your comprehension of the material with a closed book self-administered examination. Put all your notes aside, find a quiet place where you will not be disturbed, and take the examination provided at the end of this unit. You will find some questions straightforward and easy, but others will test your ingenuity.

You will find the solutions to the Unit 1 exam questions, and the point value for each question in the Answer Key provided at the end of this unit. Become your own examiner. If you have done well, according to your personal standards, go on to Unit 2. If not, then more review and practice is obviously called for.

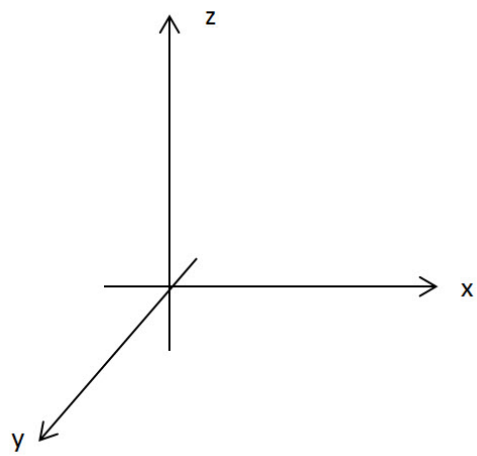
Summary

Three-Dimensional Coordinate Systems

Right-hand system:



Left-hand system:



Convention: Always graph with a right-hand system.

Important Note: When graphing an equation in one, two or three variables, it is important to note the context in which you are graphing. Does the equation represent a graph in R^2 or R^3? For example, the equation $x^2 + y^2 = 1$ represents a circle in R^2, but a right circular cylinder in R^3.

Distance Formula in 3-Space:

The distance $|P_1P_2|$ between the points $P_1(x_1, y_1, z_1)$ and $P_2(x_2, y_2, z_2)$ is

$$|P_1P_2| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}$$

Equation of a Sphere:

An equation of a sphere in standard form with centre (x_0, y_0, z_0) and radius r is

$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2$$

If the centre is at the origin, the equation becomes $x^2 + y^2 + z^2 = r^2$

The general form is $x^2 + y^2 + z^2 + ax + by + cz + d = 0$.

Vector Arithmetic: Addition, Subtraction and Scalar Multiplication

$\vec{u} + \vec{v} = \vec{v} + \vec{u}$

$(\vec{u} + \vec{v}) + \vec{w} = \vec{u} + (\vec{v} + \vec{w})$

$\vec{u} + \vec{0} = \vec{u}$

$\vec{u} + (-\vec{u}) = \vec{0}$

$c(\vec{u}+\vec{v}) = c\vec{u}+c\vec{v}$

$(c+d)\vec{u} = c\vec{u}+d\vec{u}$

$(cd)\vec{u} = c(d\vec{u})$

$1\cdot\vec{u} = \vec{u}$

If $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$,

$\vec{u}+\vec{v} = \langle u_1+v_1, u_2+v_2, u_3+v_3\rangle$

$c\vec{u} = c\langle u_1, u_2, u_3\rangle = \langle cu_1, cu_2, cu_3\rangle$

$\vec{u}-\vec{v} = \vec{u}+(-\vec{v}) = \langle u_1-v_1, u_2-v_2, u_3-v_3\rangle$

$|\vec{u}| = \sqrt{u_1^2+u_2^2+u_3^2}$

In 3-space, $\vec{u} = \langle u_1, u_2, u_3\rangle = u_1\vec{i}+u_2\vec{j}+u_3\vec{k}$

Vector Arithmetic: The Dot/Scalar Product

$\vec{u}\cdot\vec{v} = |\vec{u}||\vec{v}|\cos(\theta)$

If $\vec{u} = \langle u_1, u_2, u_3\rangle$ and $\vec{v} = \langle v_1, v_2, v_3\rangle$, $\vec{u}\cdot\vec{v} = u_1v_1+u_2v_2+u_3v_3$

$\vec{u}\cdot\vec{v} = \vec{v}\cdot\vec{u}$

$\vec{u}\cdot(\vec{v}+\vec{w}) = \vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{w}$

$(c\vec{u})\cdot\vec{v} = c(\vec{u}\cdot\vec{v}) = \vec{u}\cdot(c\vec{v})$

If $\vec{u} = \langle u_1, u_2, u_3\rangle$, $\vec{u}\cdot\vec{u} = |\vec{u}|^2 = u_1^2+u_2^2+u_3^2$

$\dfrac{\vec{u}}{|\vec{u}|}$ is a unit vector in the direction of $\vec{u}$.

$-\dfrac{\vec{u}}{|\vec{u}|}$ is a unit vector in the opposite direction of $\vec{u}$.

$\cos(\theta) = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}$

Two vectors $\vec{u}$ and $\vec{v}$ are <u>orthogonal</u> if and only if $\vec{u}\cdot\vec{v} = 0$

Vector projection of $\vec{u}$ onto $\vec{v}$: $proj_{\vec{v}}\vec{u} = \dfrac{\vec{u}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = \dfrac{\vec{u}\cdot\vec{v}}{|\vec{v}|^2}\vec{v}$

Vector Arithmetic: The Cross/Vector Product

If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$,

$$\vec{u} \times \vec{v} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} \vec{i} - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} \vec{j} + \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \vec{k}$$

$$= (u_2 v_3 - u_3 v_2)\vec{i} - (u_1 v_3 - u_3 v_1)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k}$$

$$= < u_2 v_3 - u_3 v_2,\ u_3 v_1 - u_1 v_3, u_1 v_2 - u_2 v_1 >$$

The vector $\vec{u} \times \vec{v}$ is orthogonal to both $\vec{u}$ and $\vec{v}$.

Two non-zero vectors $\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$.

$|\vec{u} \times \vec{v}| = |\vec{u}||\vec{v}|\sin(\theta)$

$\vec{u} \times \vec{v} = \left(|\vec{u}||\vec{v}|\sin(\theta)\right)\vec{w}$, where $\vec{w}$ is a unit vector perpendicular to $\vec{u}$ and $\vec{v}$ and forms a right-handed system in the order $\vec{u}, \vec{v}$ and $\vec{w}$.

$\vec{u} \times \vec{v} = -\vec{v} \times \vec{u}$

$(c\vec{u}) \times \vec{v} = c(\vec{u} \times \vec{v}) = \vec{u} \times (c\vec{v})$

$\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$

$(\vec{v} + \vec{w}) \times \vec{u} = \vec{v} \times \vec{u} + \vec{w} \times \vec{u}$

$\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$

$\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

The length of the cross product vector $\vec{u} \times \vec{v}$ is equal to the area of the parallelogram determined by $\vec{u}$ and $\vec{v}$.

The triple scalar product is $\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$

The volume of the parallelepiped determined by the vectors $\vec{u}, \vec{v}$ and $\vec{w}$ is the magnitude of their triple scalar product: $V = |\vec{u} \cdot (\vec{v} \times \vec{w})|$

Lines

If we write $\vec{x}_0 = <x_0, y_0, z_0>$, $\vec{x} = <x, y, z>$ and $\vec{v} = <a, b, c>$ then the vector equation of a line in the direction $\vec{v}$ through the point $P(x_0, y_0, z_0)$ defined by the vector $\vec{x}_0$ is

$$<x, y, z> = <x_0, y_0, z_0> + t<a, b, c> = <x_0 + ta, y_0 + tb, z_0 + tc>$$

The vector equation of a line through the points $P(x_0, y_0, z_0)$ and $Q(x_1, y_1, z_1)$ defined by the vectors $\vec{x}_0 = <x_0, y_0, z_0>$ and $\vec{x}_1 = <x_1, y_1, z_1>$ is

$$<x, y, z> = <x_0, y_0, z_0> + t<x_1 - x_0, y_1 - y_0, z_1 - z_0>$$

The parametric equations of a line are:

$$x = x_0 + ta; \quad y = y_0 + tb; \quad z = z_0 + tc$$

The symmetric equations of a line are:

$$\frac{x - x_0}{a} = \frac{y - y_0}{b} = \frac{z - z_0}{c}$$

The line segment PQ that extends from the point $P(x_0, y_0, z_0)$, defined by the vector $\vec{x}_0$, to the point $Q(x_1, y_1, z_1)$, defined by the vector $\vec{x}_1$, is given by the vector equation

$$\vec{x} = (1 + t)\vec{x}_0 - t\vec{x}_1 \text{, where } 0 \le t \le 1$$

Planes

The vector equation of a plane with normal $\vec{n}$ through the point $P(x_0, y_0, z_0)$ defined by the vector $\vec{x}_0$ is

$$\vec{n} \cdot (\vec{x} - \vec{x}_0) = 0$$

The scalar equation of a plane through $P(x_0, y_0, z_0)$ with normal vector $\vec{n} = <a, b, c>$ is

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0 \text{ or } ax + by + cz + d = 0$$

The perpendicular distance d from a point $P(x_1, y_1, z_1)$ to a plane with normal vector $\vec{n} = <a, b, c>$ is

$$d = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2 + b^2 + c^2}}$$

Cylinders

A cylinder is a surface that consists of lines (called rulings) that are parallel to a given line and pass through a given plane curve.

If the generator line moves parallel to one of the coordinate axes, the equation of the resulting surface will not contain that coordinate variable.

So, if one of the variables x, y or z is missing from the equation of the surface then that surface is a cylinder.

The Conics

The equation of an ellipse in standard form centered at the point (h,k) is

$$\frac{(x-h)^2}{a^2}+\frac{(y-k)^2}{b^2}=1$$

At the origin this becomes $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

a is the semi-major axis, b the semi-minor axis, and the points $(-a,0)$ and $(a,0)$ are the vertices. The case $a=b$ gives the circle.

The equation of the parabola in its standard form with vertex at (h,k) is
$(y-k)^2=4c(x-h)$ or $(x-h)^2=4c(y-k)$

At the origin this becomes $y^2=4cx$ or $x^2=4cy$.

The equation of the hyperbola in its standard form with vertex at (h,k) is

$$\frac{(x-h)^2}{a^2}-\frac{(y-k)^2}{b^2}=1 \text{ or } \frac{(y-k)^2}{b^2}-\frac{(x-h)^2}{a^2}=1$$

At the origin this becomes $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ or $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$.

The hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ has $x-$intercepts $(\pm a,0)$, opens in the $x-$direction and has asymptotes $y=\pm\frac{b}{a}x$.

The hyperbola $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ has $y-$intercepts $(0,\pm b)$, opens in the $y-$direction and has asymptotes $y=\pm\frac{b}{a}x$.

The Quadric Surfaces

A quadric surface is the graph of a second degree equation in three variables x, y and z, generalizing the conics from the xy-plane.

The most general equation is

$$Ax^2 + By^2 + Cz^2 + Dxy + Eyz + Fxz + Gx + Hy + Iz + J = 0$$

The standard forms are:

ELLIPSOID $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ ($a=b=c$ gives the sphere)

CONE $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=0$

(The negative term determines the direction in which the cone opens.)

HYPERBOLOID OF ONE SHEET $\frac{x^2}{a^2}+\frac{y^2}{b^2}-\frac{z^2}{c^2}=1$

(One negative term translates to one sheet. The negative term determines the direction in which the hyperboloid opens.)

HYPERBOLOID OF TWO SHEETS $-\frac{x^2}{a^2}-\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$

(Two negative terms translate to two sheets. The positive term determines the direction in which the hyperboloid opens.)

ELLIPTIC PARABOLOID $z=\frac{x^2}{a^2}+\frac{y^2}{b^2}$

(One non-squared term translates to a paraboloid. Two squared terms of the same sign determines to the elliptic paraboloid. The linear term determines the direction in which the paraboloid opens)

HYPERBOLIC PARABOLOID $z=\frac{x^2}{a^2}-\frac{y^2}{b^2}$

(One non-squared term translates to a paraboloid. Two squared terms of the opposite sign determines to the hyperbolic paraboloid. The linear term determines the top of the "saddle". The positive squared term determines the direction in which the "rider moves". The negative squared term determines the direction in which the "rider's legs dangle".

The quadric surfaces are built from the basic conics. Their nature can be ascertained by considering their projections on, or parallel to, each of the coordinate planes; that is, by determining their horizontal and vertical traces.

Table 1 in Section 12. 6 of your textbook contains a side-by-side summary of quadric surfaces and their graphs. Study it carefully so that you can easily determine quadric surface types from their equations.

Self-Test *(35 marks)*

Treat this as a real test. Do not refer to any course materials. The time for this test is 1. 5 hours. Use the answer key provided to mark your test. The point value for each question is posted in the left margin.

1. Let $A=(4,2,1)$ and $B=(-1,1,3)$ be two points.

[2] a) Find $\vec{AB}$ and calculate its length.

[1] b) Find a unit vector in the opposite direction to $\vec{AB}$.

[2] c) Given $C=(3,1,z)$, find z so that $\vec{AB}$ is perpendicular to $\vec{BC}$.

[2] d) Find parametric equations for the line through A and B.

[5] 2. Determine whether or not the line $\frac{x-2}{3}=\frac{y+1}{-1}=\frac{z-1}{2}$ intersects the plane $2x+3y-z=6$ and, if so, find the coordinates of the point of intersection.

[2] 3. a) Find an equation of the plane through (1,-1,2) parallel to the plane $x-y-z=6$.

[2] b) Find an equation of the line of intersection of the planes $x-y-z=6$ an $2x+y+z=3$. Describe geometrically the position of this line of intersection.

[2] c) Find the perpendicular distance from the point $(2,1,1)$ to the plane $x+y+z=0$.

[2] d) Find the equation of the plane through the three points $A=(1,1,-1)$, $B=(4,-1,2)$ and $C=(2,-1,1)$.

[5] 4. Use vectors to show that the diagonals of a rhombus (a figure with four equal sides) intersect at right angles.

5. Draw a neat sketch of the surfaces in R^3 whose equations are given below. Identify on your graph any intercepts or special points. Name each surface.

[2] a) $x^2 - y^2 + z^2 = 2$

[2] b) $x^2 + y^2 + z = 1$

[2] c) $x^2 = 6y$

[4] 6. Draw the level curves of the surface $z = x^2 - 4y$ for $z = 0, 1, 2, 3$. Identify the level curves by name and by equation.

Answer Key

1. a) $\vec{AB} = < -1-4, 1-2, 3-1 > = < -5, -1, 2 >$

$\left|\vec{AB}\right| = \left|< -5, -1, 2 >\right| = \sqrt{25+1+4} = \sqrt{30}$

b) $\vec{u} = \dfrac{\vec{BA}}{\left|\vec{BA}\right|} = -\dfrac{1}{\sqrt{30}} < -5, -1, 2 > = \left\langle \dfrac{5}{\sqrt{30}}, \dfrac{1}{\sqrt{30}}, -\dfrac{2}{\sqrt{30}} \right\rangle$

c) $\vec{BC} = < 3+1, 1-1, z-3 > = < 4, 0, z-3 >$

$\vec{AB}$ is perpendicular to $\vec{BC}$ if, and only if, $\vec{AB} \cdot \vec{BC} = 0$

$\vec{AB} \cdot \vec{BC} = < -5, -1, 2 > \cdot < 4, 0, z-3 > = -20 + 0 + 2z - 6 = 2z - 26$

$\vec{AB} \cdot \vec{BC} = 0 \quad \Rightarrow \quad 2z - 26 = 0 \quad \Rightarrow \quad z = 13$

d) $\vec{r}(t) = \vec{OA} + t \cdot \vec{AB} = < 4, 2, 1 > + t < -5, -1, 2 >$

So,

$x = 4 - 5t$
$y = 2 - t$
$z = 1 + 2t$

2. The line and the plane will intersect if the line is not parallel to the plane, that is, if the line is not perpendicular to the normal to the plane.

Normal for plane: $\vec{n} = < 2, 3, -1 >$

Direction for line: $\vec{v} = < 3, -1, 2 >$

$\vec{n} \cdot \vec{v} = < 2, 3, -1 > \cdot < 3, -1, 2 > = 6 - 3 - 2 = 1 \neq 0$

Therefore the line and plane intersect.

To find the point of intersection, substitute the general point on the line, $x = 2 + 3t$, $y = -1 - t$, $z = 1 + 2t$ in the equation of the plane:

$$\begin{aligned} 2(2+3t) + 3(-1-t) - (1+2t) &= 6 \\ 4 + 6t - 3 - 3t - 1 - 2t &= 6 \\ t &= 6 \end{aligned}$$

Substituting into the parametric equations above, we find the point of intersection is (20,-7,13).

3. a) Two planes are parallel if, and only if, their normal vectors are parallel.

Therefore the plane we seek has normal $\vec{n} =< 1,-1,-1 >$ and passes through (1,1,2).

$$\vec{n} \cdot \left(\vec{x} - \vec{x}_0\right) = 0$$
$$< 1,-1,-1 > \cdot < x-1, y+1, z-2 >= 0$$
$$x-1-y-1-z+2=0$$
$$x-y-z=0$$

b)
$$x-y-z=6 \ldots\ldots\ldots\ldots\ldots (1)$$
$$2x+y+z=3 \ldots\ldots\ldots\ldots (2)$$

Adding (1) and (2)

$3x = 9 \quad \Rightarrow \quad x = 3$

Then $y+z=-3$. Setting $z=t$, we obtain $y=-3-t$.

Therefore, a vector equation of the line of intersection is:

$$\vec{r}(t) =< 3,-3-t, t >=< 3,-3,0 > +t < 0,-1,1 >$$

In parametric form we have:

$$x = 3$$
$$y = -3-t$$
$$z = t$$

The line lies in the plane $x = 3$, which is parallel to the yz-plane (See Example 1 on page 811 of the text).

c) A normal vector for the plane is $< a,b,c >=< 1,1,1 >$.

The given point is $(x_1, y_1, z_1) = (2,1,1)$. Using the formula

$$D = \frac{|ax_1 + by_1 + cz_1 + d|}{\sqrt{a^2+b^2+c^2}}$$
$$= \frac{|1(2)+1(1)+1(1)+0|}{\sqrt{1^2+1^2+1^2}}$$
$$= \frac{|4|}{\sqrt{3}} = \frac{4}{\sqrt{3}} \text{ or } \frac{4\sqrt{3}}{3}$$

d) $\overrightarrow{AB} =< 3,-2,3 >$, $\overrightarrow{AC} =< 1-2,2 >$

So,

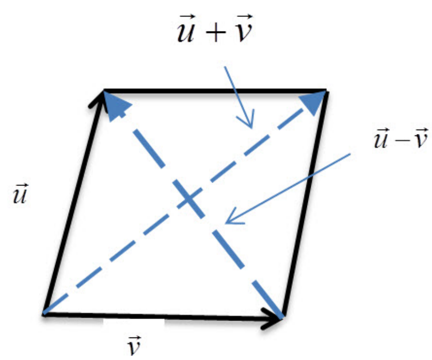
$$\vec{AB}\times\vec{AC}=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ 3 & -2 & 3\\ 1 & -2 & 2\end{vmatrix}=<-4+6,3-6,-6+2>=<2,-3,-4>$$

A normal for the plane is $\vec{n}=<2,-3,-4>$; a point on the plane is A, so $x_0=\vec{OA}=<1,1,-1>$.

An equation for the plane containing the points A, B, C is:

$$\begin{aligned}\vec{n}\cdot\left(\vec{x}-\vec{x}_0\right)&=0\\ <2,-3,-4>\cdot<x-1,y-1,z+1>&=0\\ 2x-2-3y+3-4z-4&=0\\ 2x-3y-4z&=3\end{aligned}$$

4. Let the vectors representing the two sides be $\vec{u}$ and $\vec{v}$. Then the diagonals will be $\vec{u}+\vec{v}$ and $\vec{u}-\vec{v}$ (See diagram).



Now,

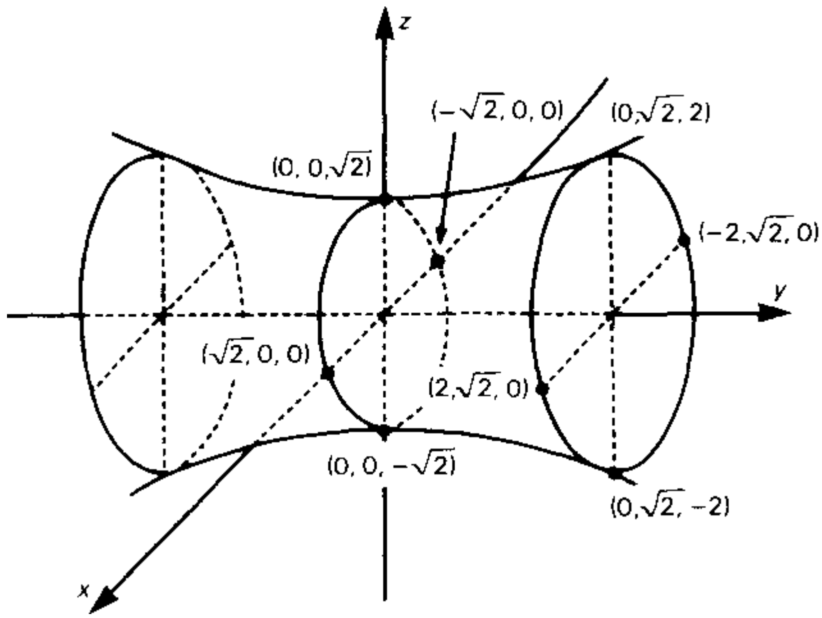
$$\begin{aligned}&\left(\vec{u}+\vec{v}\right)\cdot\left(\vec{u}-\vec{v}\right)\\ &=\vec{u}\cdot\vec{u}-\vec{u}\cdot\vec{v}+\vec{v}\cdot\vec{u}-\vec{v}\cdot\vec{v}\\ &=\vec{u}\cdot\vec{u}-\vec{u}\cdot\vec{v}+\vec{u}\cdot\vec{v}-\vec{v}\cdot\vec{v}\\ &=\vec{u}\cdot\vec{u}-\vec{v}\cdot\vec{v}\\ &=\left|\vec{u}\right|^2-\left|\vec{v}\right|^2\end{aligned}$$

But, $\left|\vec{u}\right|=\left|\vec{v}\right|$ since all sides of the rhombus are equal.

Hence, $\left(\vec{u}+\vec{v}\right)\cdot\left(\vec{u}-\vec{v}\right)=\left|\vec{u}\right|^2-\left|\vec{v}\right|^2=\left|\vec{u}\right|^2-\left|\vec{u}\right|^2=0$ and the diagonals of the rhombus intersect at right angles.

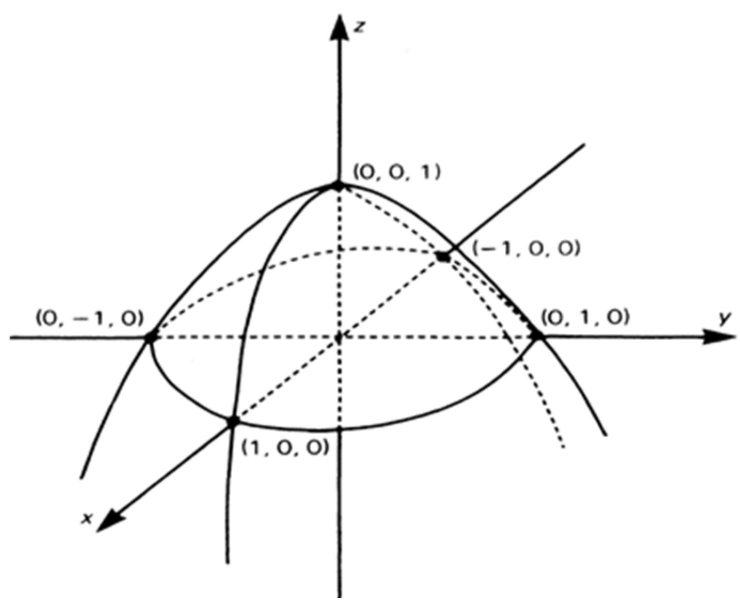
5. a) The equation $x^2 - y^2 + z^2 = 2$ in standard form becomes

$$\frac{x^2}{\left(\sqrt{2}\right)^2} - \frac{y^2}{\left(\sqrt{2}\right)^2} + \frac{z^2}{\left(\sqrt{2}\right)^2} = 1.$$ This is a hyperboloid of one sheet, opening in the y-direction.

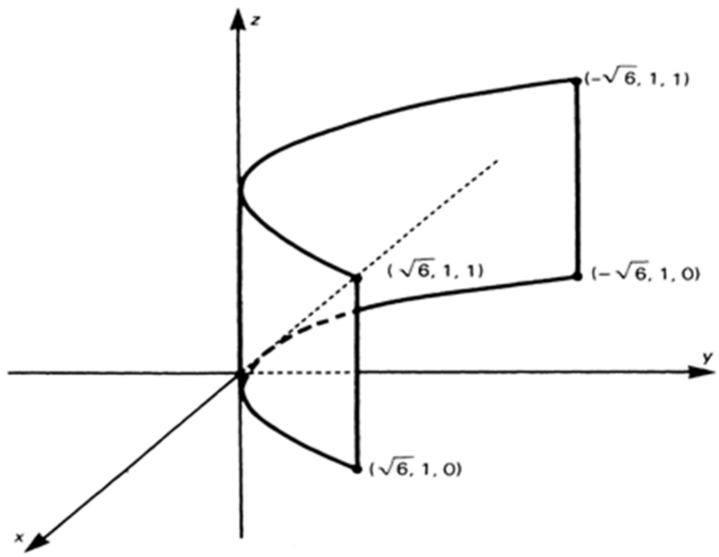


b) $x^2 + y^2 + z = 1 \quad \Rightarrow \quad z = 1 - \left(x^2 + y^2\right)$

This is the circular paraboloid with vertex at (0,0,1)and opening in the negative z-direction.



c) $x^2 = 6y$. This is a parabolic cylinder with generator line parallel to the z-axis.



6. $z = k \quad \Rightarrow \quad x^2 - 4y = k \quad \Rightarrow \quad y = \frac{x^2}{4} - \frac{k}{4},\ k = 0,1,2,3$

The level curves are parabolas opening in the positive y-direction with vertex at $\left(0, -\frac{k}{4}\right)$.

$k = 0: \quad y = \frac{x^2}{4}$ $k = 1: \quad y = \frac{x^2}{4} - \frac{1}{4}$ $k = 2: \quad y = \frac{x^2}{4} - \frac{1}{2}$ $k = 1: \quad y = \frac{x^2}{4} - \frac{3}{4}$