THOMPSON RIVERS UNIVERSITY,
OPEN LEARNING

ANSWER KEY

PRACTICE EXAMINATION

MATH 2111 • CALCULUS III – MULTIVARIABLE CALCULUS

PART A *(80 marks total)*

1. a. $\vec{r}(t)=\langle t,\cos(at),\sin(at)\rangle$

$\vec{r}'(t)=\langle 1,-a\sin(at),a\cos(at)\rangle$

$\left|\vec{r}'(t)\right|=\sqrt{1+a^2\sin^2(at)+a^2\cos^2(at)}=\sqrt{1+a^2\left(\sin^2(at)+\cos^2(at)\right)}=\sqrt{1+a^2}$

Therefore, $\vec{T}=\dfrac{\vec{r}'(t)}{\left|\vec{r}'(t)\right|}=\dfrac{1}{\sqrt{1+a^2}}\langle 1,-a\sin(at),a\cos(at)\rangle$

$$\vec{T}'=\frac{1}{\sqrt{1+a^2}}\left\langle 0,-a^2\cos(at),-a^2\sin(at)\right\rangle$$

$$\left|\vec{T}'\right|=\frac{1}{\sqrt{1+a^2}}\sqrt{0+a^4\cos^2(at)+a^4\sin^2(at)}=\frac{1}{\sqrt{1+a^2}}\sqrt{a^4}=\frac{a^2}{\sqrt{1+a^2}}$$

Therefore,

$$\vec{N}=\frac{\vec{T}'}{\left|\vec{T}'\right|}=\frac{\sqrt{1+a^2}}{a^2}\frac{1}{\sqrt{1+a^2}}\left\langle 0,-a^2\cos(at),-a^2\sin(at)\right\rangle$$
$$=\frac{1}{a^2}\left\langle 0,-a^2\cos(at),-a^2\sin(at)\right\rangle$$
$$=\langle 0,-\cos(at),-\sin(at)\rangle$$

b. $\kappa=\dfrac{\left|\vec{T}'\right|}{\left|\vec{r}'(t)\right|}=\dfrac{\left(\dfrac{a^2}{\sqrt{1+a^2}}\right)}{\sqrt{1+a^2}}=\dfrac{a^2}{1+a^2}$

c. $L=\int_0^1\left|\vec{r}'(t)\right|dt=\int_0^1\sqrt{1+a^2}\,dt=\sqrt{1+a^2}\,\left[t\right]_0^1=\sqrt{1+a^2}$

2. a. $5x+7y=2+8z \quad \Rightarrow \quad 5x+7y-8z=2$

A normal vector for the plane is $\langle 5,7,-8\rangle$. Since the line is perpendicular to the plane, this vector is a direction vector for the line.

Parametric equations of the line are:

$$x=1+5t$$
$$y=1+7t$$
$$z=-1-8t$$

b. In vector form:

L_1: $\vec{r}_1(t) = <1,-4,1> + t<2,1,-1>$

L_2: $\vec{r}_2(t) = <-2,1,-2> + t<2,1,-1>$

A line intersecting these two lines and passing through the points $(1,-4,1)$ and $(-2,1,-2)$ has direction vector $<1-(-2),-4-1,1-(-2)> = <3,-5,3>$ and vector equation:

L_3: $\vec{r}_3(t) = <1,-4,1> + t<3,-5,3>$

The plane determined by L_1 and L_2 contains these three lines, so a normal vector for the plane is

$$\vec{n} = <2,1,-1> \times <3,-5,3> = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 2 & 1 & -1 \\ 3 & -5 & 3 \end{vmatrix} = <-2,-9,-13>$$

Using the point $(1,-4,1)$, an equation for this plane is:

$$\begin{aligned} <-2,-9,-13> \cdot (<x,y,z> - <1,-4,1>) &= 0 \\ <-2,-9,-13> \cdot <x-1,y+4,z-1> &= 0 \\ -2(x-1)+(-9)(y+4)+(-13)(z-1) &= 0 \\ -2x+2-9y-36-13z+13 &= 0 \\ -2x-9y-13z-21 &= 0 \\ 2x+9y+13z+21 &= 0 \end{aligned}$$

c. The head of the vector $\vec{u} = 2\vec{i}+3\vec{j}+\vec{k}$ is $A=(2,3,1)$; the head of the vector $\vec{v} = 4\vec{i}+\vec{j}-2\vec{k}$ is $B=(4,1,-2)$ and the head of the vector $\vec{w} = 5\vec{i}+a\vec{j}+b\vec{k}$ is $C=(5,a,b)$.

Since the heads of the three vectors must lie along a line, we must have that $\vec{AC} = k\,\vec{AB}$, for some scalar k. So,

$<3, a-3, b-1> = k<2,-2,-3> = <2k, -2k, -3k>$

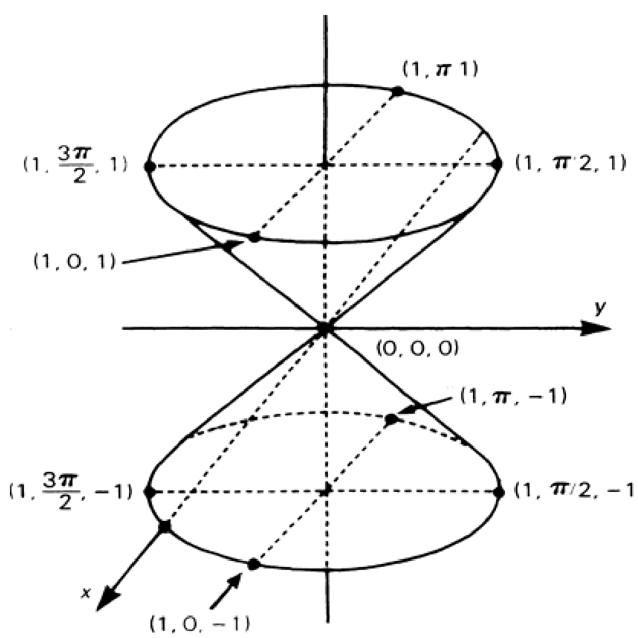
$\therefore\quad 2k = 3,\quad -2k = a-3,\quad -3k = b-1$

$2k = 3 \quad\Rightarrow\quad k = \dfrac{3}{2}$

$\therefore\quad a-3 = -2\left(\dfrac{3}{2}\right) = -3 \quad\Rightarrow\quad a = 0 \quad$ and $\quad b-1 = -3\left(\dfrac{3}{2}\right) = -\dfrac{9}{2} \quad\Rightarrow\quad b = 1-\dfrac{9}{2} = -\dfrac{7}{2}$

3. $z^2 = x^2 + y^2$

Trace curves:
Set: $z = k$: $x^2 + y^2 = k^2$
For all $k \neq 0$, the traces are circles centered at $(0,0,k)$ of radius k.
Next set: $x = k$: $z^2 = k^2 + y^2 \quad\Rightarrow\quad z^2 - y^2 = k^2$
When: $k > 0$: Traces are hyperbolas centered at $(k,0,0)$ opening in the $z-$direction.
$k = 0$: Traces are two lines through $(0,0,0)$
$k < 0$: Traces are hyperbolas centered at $(k,0,0)$ opening in the $y-$direction.
$y = k$: Similar to $x = k$ traces
The surface is the double cone. Its graph is given below.



b. Set $w = e^x \sin(y) - z = 0$

$\nabla w = \left\langle e^x \sin(y), e^x \cos(y), -1 \right\rangle$ is a normal vector to the surface $z = e^x \sin(y)$ at an arbitrary point.

So, a normal vector at the point $(\ln(3), \frac{\pi}{2}, 3)$ is:

$$\nabla w\big|_{(\ln(3),\frac{\pi}{2},3)} = \left\langle e^{\ln(3)} \sin(\frac{\pi}{2}), e^{\ln(3)} \cos(\frac{\pi}{2}), -1 \right\rangle = \langle 3(1), 0, -1 \rangle = \langle 3, 0, -1 \rangle$$

An equation for the tangent plane at $(\ln(3), \frac{\pi}{2}, 3)$ is:

$$< 3,0,-1 > \cdot \left(< x, y, z > - < \ln(3), \frac{\pi}{2}, 3 > \right) = 0$$

$$< 3,0,-1 > \cdot < x - \ln(3), y - \frac{\pi}{2}, z - 3 >= 0$$

$$3(x - \ln(3)) + (0)(y - \frac{\pi}{2}) + (-1)(z - 3) = 0$$

$$3x - 3\ln(3) - z + 3 = 0$$

$$3x - z = 3\ln(3) - 3$$

4. a. $f(x, y) = x^2 + xy + y^2$

$\nabla f = \langle 2x + y, x + 2y \rangle$; $\nabla f(1,1) = \langle 2(1) + 1, 1 + 2(1) \rangle = \langle 3,3 \rangle$;

$|\nabla f(1,1)| = |\langle 3,3 \rangle| = \sqrt{3^2 + 3^2} = \sqrt{18}$ or $3\sqrt{2}$

Hence, the direction of most rapid increase of $z = f(x, y)$ is in the direction of the gradient vector $\langle 3,3 \rangle$. The rate of change in this direction is $|\nabla f(1,1)| = 3\sqrt{2}$.

b. $w = g(x,y)$; $\nabla g(1,2) = \left\langle g_x(1,2), g_y(1,2) \right\rangle = \langle a,b \rangle$

A vector in the direction towards $(2,2)$ from $(1,2)$ is
$\vec{u} = <2-1, 2-2> = <1,0>$; $|\vec{u}| = 1$, so $\vec{u}$ is a unit vector.

A vector in the direction towards $(1,1)$ from $(1,2)$ is
$\vec{v} = <1-1, 1-2> = <0,-1>$; $|\vec{v}| = 1$, so $\vec{v}$ is also a unit vector.

$D_{\vec{u}} g(1,2) = \nabla g(1,2) \cdot \vec{u} = \langle a,b \rangle \cdot \langle 1,0 \rangle = a$; but we are given that $D_{\vec{u}} g(1,2) = 2$.
Hence, $a = 2$.

Similarly, $D_{\vec{v}} g(1,2) = \nabla g(1,2) \cdot \vec{v} = \langle a,b \rangle \cdot \langle 0,-1 \rangle = -b$ and
$D_{\vec{v}} g(1,2) = -2 \quad \Rightarrow \quad -b = -2 \quad \Rightarrow \quad b = 2$

Hence, $\nabla g(1,2) = \langle 2,2 \rangle$.

A vector in the direction towards $(4,6)$ from $(1,2)$ is $\langle 4-1, 6-2 \rangle = \langle 3,4 \rangle$
A unit vector in this direction is $\vec{w} = \dfrac{1}{\left|\langle 3,4 \rangle\right|} \langle 3,4 \rangle = \dfrac{1}{\sqrt{3^2+4^2}} \langle 3,4 \rangle = \dfrac{1}{5} \langle 3,4 \rangle$
or $\left\langle \dfrac{3}{5}, \dfrac{4}{5} \right\rangle$

Therefore, $D_{\vec{w}} g(1,2) = \nabla g(1,2) \cdot \vec{w} = \langle 2,2 \rangle \cdot \dfrac{1}{5} \langle 3,4 \rangle = \dfrac{1}{5}(6+8) = \dfrac{14}{5}$

5. a. $f(x,y)=2x^3-24xy+16y^3$

Solve: $f_x(x,y)=6x^2-24y=0$ ....(1)
$f_y(x,y)=-24x+48y^2=0$ ....(2)

(1): Solve for y to get $y=\dfrac{6x^2}{24}=\dfrac{1}{4}x^2$.

(2): Substituting for y we have,

$$-24x+48\left(\frac{1}{4}x^2\right)^2=0$$
$$-24x+\frac{48}{16}x^4=0$$
$$-24x+3x^4=0$$
$$3x\left(-8+x^3\right)=0$$

$3x=0\Rightarrow \quad x=0$ or $-8+x^3=0 \quad\Rightarrow\quad x^3=8 \quad\Rightarrow\quad x=2$

Now, when $x=0$, $y=\frac{1}{4}(0)^2=0$; when $x=2$, $y=\frac{1}{4}(2)^2=1$.

There are two critical points: $(0,0)$, $(2,1)$

$f_{xx}(x,y)=\dfrac{\partial}{\partial x}\left(6x^2-24y\right)=12x$; $f_{yy}(x,y)=\dfrac{\partial}{\partial y}\left(-24x+48y^2\right)=96y$;
$f_{xy}(x,y)=\dfrac{\partial}{\partial y}\left(6x^2-24y\right)=-24$

$$D(x,y)=f_{xx}(x,y)\cdot f_{yy}(x,y)-\left[f_{xy}(x,y)\right]^2=12x(96y)-(-24)^2=1152xy-576$$

At $(0,0)$: $D(0,0)=1152(0)(0)-576=-576<0$; so, $(0,0)$ yields a saddle point.

At $(2,1)$: $D(2,1)=1152(2)(1)-576=1728>0$ and $f_{xx}(2,1)=12(2)=24>0$; so, $(2,1)$ yields a relative minimum.

b. Optimize: $f(x,y) = z = xy$
Subject to: $x^2 + y^2 = 1$

Let $g(x,y) = x^2 + y^2$. Then $f_x(x,y) = y$; $f_y(x,y) = x$;
$g_x(x,y) = 2x$; $g_y(x,y) = 2y$

Solve the Lagrange system:

$$\begin{aligned} f_x(x,y) &= \lambda \cdot g_x(x,y) & & y = \lambda(2x) \text{ ....(1)} \\ f_y(x,y) &= \lambda \cdot g_y(x,y) & \Rightarrow \quad & x = \lambda(2y) \text{ ....(2)} \\ g(x,y) &= 1 & & x^2 + y^2 = 1 \text{ ....(3)} \end{aligned}$$

(1): Multiply through by y: $y^2 = 2\lambda xy$
(2): Multiply through by x: $x^2 = 2\lambda xy$
Equating we have, $x^2 = y^2$

(3): $x^2 + y^2 = 1 \quad \Rightarrow \quad x^2 + x^2 = 1 \quad \Rightarrow \quad x^2 = \frac{1}{2} \quad \Rightarrow \quad x = \pm\frac{1}{\sqrt{2}}$

Now, when $x = \frac{1}{\sqrt{2}}$, $y^2 = \frac{1}{2} \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{2}}$; when $x = -\frac{1}{\sqrt{2}}$,
$y^2 = \frac{1}{2} \quad \Rightarrow \quad y = \pm\frac{1}{\sqrt{2}}$

(x,y)	$f(x,y)=xy$
$\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$\frac{1}{2}$
$\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$-\frac{1}{2}$
$\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$	$-\frac{1}{2}$
$\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$	$\frac{1}{2}$

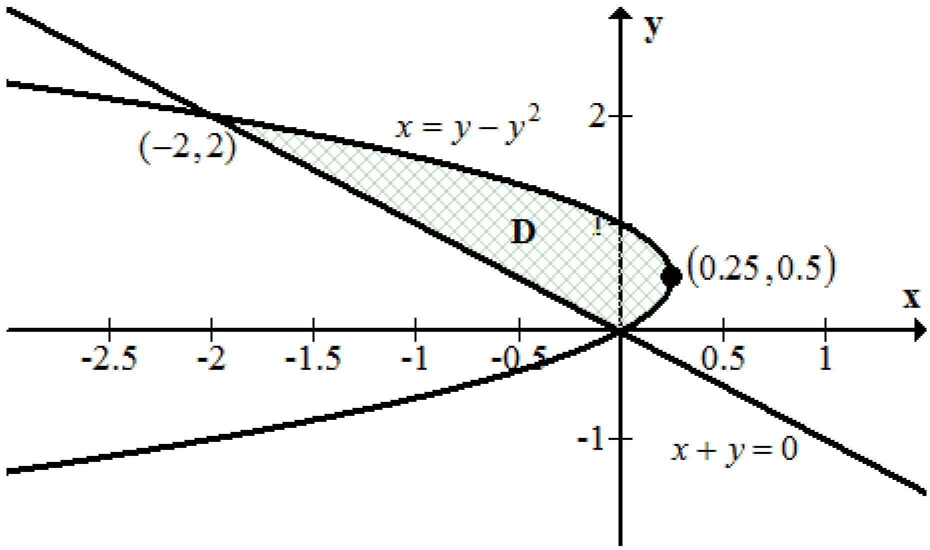
<u>Conclusion</u>:

The constrained maximum is $\frac{1}{2}$. It occurs at $\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$.

The constrained minimum is $-\frac{1}{2}$. It occurs at $\left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}\right)$ and $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

6. a. $x = y - y^2 \Rightarrow x = -\left(y^2 - y\right) = -\left(y^2 - y + \frac{1}{4}\right) + \frac{1}{4} = -\left(y - \frac{1}{2}\right)^2 + \frac{1}{4}$

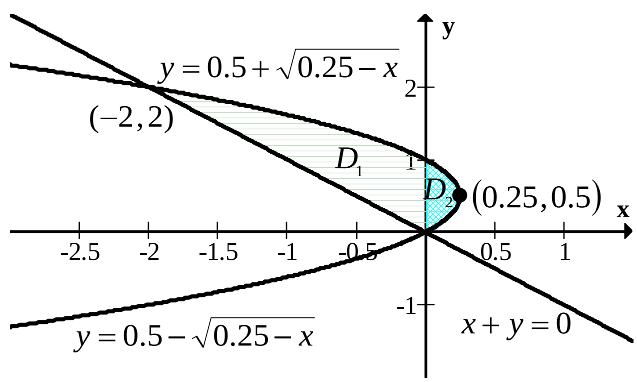
This vertex of the parabola is at $(0.25, 0.5)$. The shaded region is graphed below.



i) Since the shaded region is $x-$simple,

$$A = \iint_D dA = \int_0^2 \int_{-y}^{y-y^2} 1\, dx\, dy$$

ii) The region is not $y-$simple, but can be written as the union of two $y-$simple regions D_1 and D_2 as diagrammed.



$$A = \iint_{D_1} dA + \iint_{D_2} dA = \int_{-2}^{0} \int_{-x}^{\frac{1}{2}+\sqrt{\frac{1}{4}-x}} dy\, dx + \int_0^{0.25} \int_{\frac{1}{2}-\sqrt{\frac{1}{4}-x}}^{\frac{1}{2}+\sqrt{\frac{1}{4}-x}} dy\, dx$$

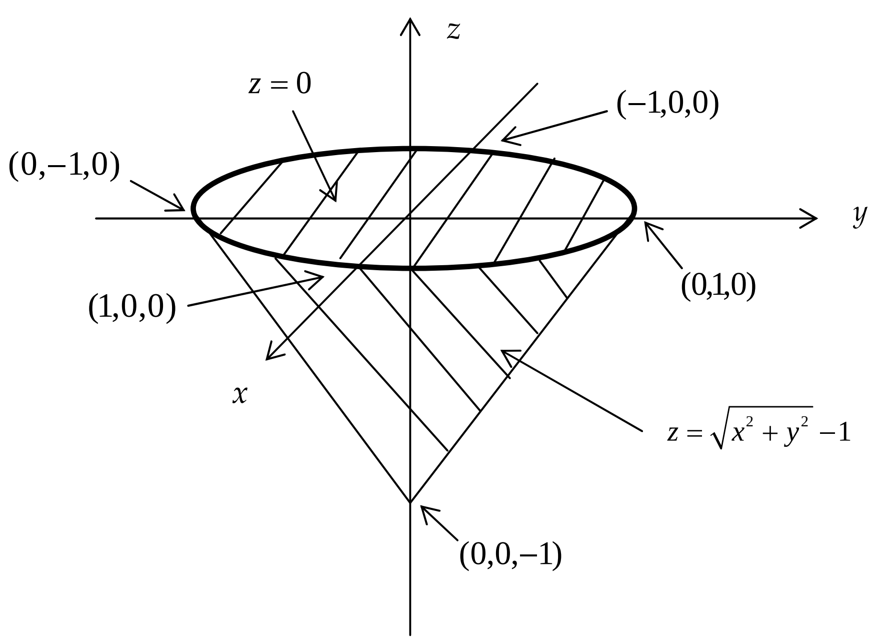
b. $\int_{-2}^{2} \int_0^{\sqrt{4-x^2}} 2y^2\left(x^2 + y^2\right)^2 dy\, dx$

The region of integration D is the semi-circle centered at the origin of radius 2 and lying in the first and second quadrants. So, in polar coordinates $D = \{(r,\theta) | 0 \le r \le 2,\, 0 \ge \theta \le \pi\}$.

Now, $y = r\sin(\theta)$ and $x^2 + y^2 = r^2$. Transforming the iterated integral into polar coordinates, we have

$$\begin{aligned}
\int_{-2}^{2}\int_{0}^{\sqrt{4-x^2}} 2y^2\left(x^2+y^2\right)^2 dy\,dx &= \int_0^{\pi}\int_0^2 2\left(r\sin(\theta)\right)^2\left(r^2\right)^2 r\,dr\,d\theta \\
&= 2\int_0^{\pi}\int_0^2 r^7\sin^2(\theta)\,dr\,d\theta \\
&= 2\int_0^{\pi}\sin^2(\theta)\left[\frac{r^8}{8}\right]_0^2 d\theta \\
&= \frac{1}{4}\int_0^{\pi}\sin^2(\theta)\left(2^8-0\right)d\theta \\
&= 64\int_0^{\pi}\sin^2(\theta)\,d\theta \\
&= 64\int_0^{\pi}\frac{1-\cos(2\theta)}{2}\,d\theta \\
&= 32\int_0^{\pi}\left(1-\cos(2\theta)\right)d\theta \\
&= 32\left[\theta-\frac{1}{2}\sin(2\theta)\right]_0^{\pi} \\
&= 32\left(\pi-0-0\right) = 32\pi
\end{aligned}$$

7. a.



Rectangular coordinates: $V = 4\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{x^2+y^2}-1} 1\,dz\,dy\,dx$ (using symmetry)

Spherical coordinates: $z = \rho\cos(\varphi)$; $\sqrt{x^2+y^2} - 1 = r - 1 = \rho\sin(\varphi) - 1$

$$z = \sqrt{x^2+y^2} - 1 \quad \Rightarrow \quad \rho\cos(\varphi) = \rho\sin(\varphi) - 1 \quad \Rightarrow \quad \rho\left(\sin(\varphi) - \cos(\varphi)\right) = 1 \quad \Rightarrow \quad \rho = \frac{1}{\sin(\varphi) - \cos(\varphi)}$$

Therefore, $0 \le \rho \le \dfrac{1}{\sin(\varphi) - \cos(\varphi)}$.

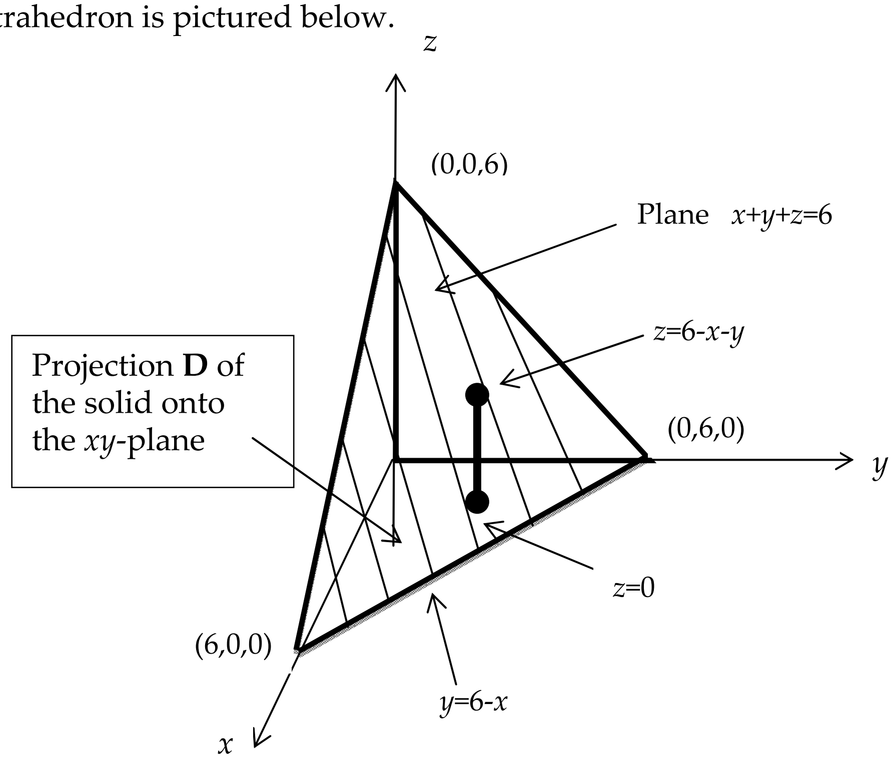
Since the region lies below the $xy-$plane, $\dfrac{\pi}{2} \le \varphi \le \pi$.

Transforming into spherical coordinates and replacing dV by $dV = \rho^2 \sin(\varphi)\,d\rho\,d\varphi\,d\theta$, we have $V = 4\int_0^{\pi/2} \int_{\pi/2}^{\pi} \int_0^{\frac{1}{\sin(\varphi)-\cos(\varphi)}} \rho^2 \sin(\varphi)\,d\rho\,d\varphi\,d\theta$ (using symmetry)

b. $x+y+z=6,\ x\geq 0, y\geq 0, z\geq 0$

Since the distance to the $xz-$plane is y in the first octant, $\rho(x,y,z)=k\,y$, for some constant k.

The solid tetrahedron is pictured below.



mass $=\iiint\limits_E \rho(x,y,z)\,dV$, where $E=\{(x,y,z)\,|\,0\leq z\leq 6-x-y,\ 0\leq y\leq 6-x,\ 0\leq x\leq 6\}$

Hence, mass $=\int_0^6\int_0^{6-x}\int_0^{6-x-y} k\,y\,dz\,dy\,dx = k\int_0^6\int_0^{6-x} y\big[z\big]_0^{6-x-y}dy\,dx = k\int_0^6\int_0^{6-x} y\left(6-x-y\right)dy\,dx$

$$=k\int_0^6\int_0^{6-x}\left(y\left(6-x\right)-y^2\right)dy\,dx = k\int_0^6\left[\frac{y^2}{2}\left(6-x\right)-\frac{y^3}{3}\right]_0^{6-x}dx$$

$$=k\int_0^6\left(\frac{\left(6-x\right)^3}{2}-\frac{\left(\left(6-x\right)\right)^3}{3}-0\right)dx = k\int_0^6\frac{\left(6-x\right)^3}{6}dx = k\int_0^6\left(-\frac{x^3}{6}+3x^2-18x+36\right)dx$$

$$=k\left[-\frac{x^4}{24}+x^3-9x^2+36x\right]_0^6 = k\left(-\frac{6^4}{24}+6^3-9\left(6^2\right)+36(6)-0\right)=54k$$

8. a. $\int_C (3x-2y+z)\,ds$; C is the line segment from $(0,4,-4)$ to $(3,1,2)$.

We start by parametrizing the line segment C.

$\vec{r}(t)=(1-t)\langle 0,4,-4\rangle + t\langle 3,1,2\rangle = \langle 3t, 4-3t, 6t-4\rangle,\ 0\le t\le 1$

So, $x(t)=3t$, $y(t)=4-3t$, $z(t)=6t-4$ and $x'(t)=3$, $y'(t)=-3$, $z'(t)=6$.

Now,

$$\int_C f(x,y,z)\,ds = \int_a^b f(x(t),y(t),z(t))\sqrt{\left(x'(t)\right)^2+\left(y'(t)\right)^2+\left(z'(t)\right)^2}\,dt$$

$$=\int_0^1 \left(3(3t)-2(4-3t)+(6t-4)\right)\sqrt{3^2+(-3)^2+6^2}\,dt$$

$$=\sqrt{54}\int_0^1 (21t-12)\,dt$$

$$=3\sqrt{6}\left[21\frac{t^2}{2}-12t\right]_0^1$$

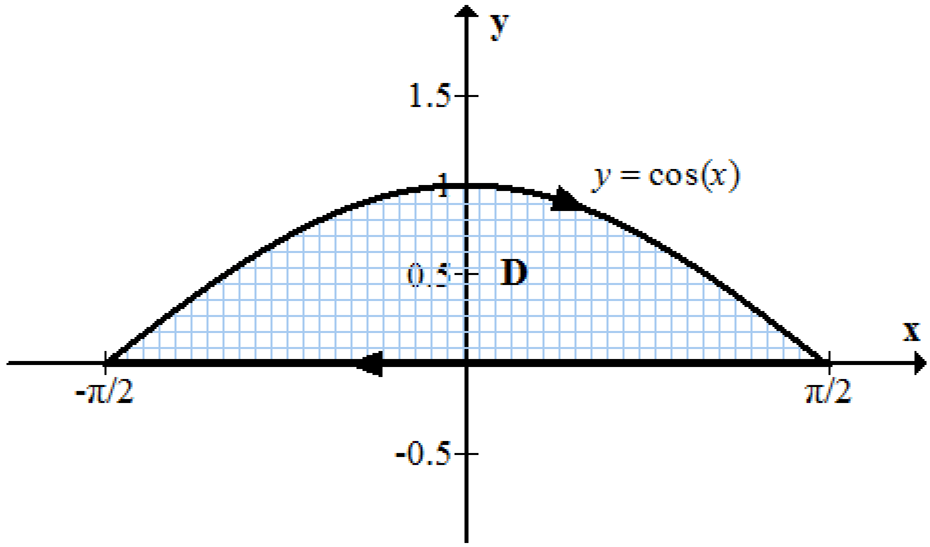
$$=3\sqrt{6}\left(\frac{21}{2}-12-0\right)$$

$$=-\frac{9\sqrt{6}}{2}$$

b. $F(x,y)=\langle e^{-x}+y^2, e^{-y}+x^2\rangle$ and C is the closed curve consisting of the curve $y=\cos(x)$ from $(-\frac{\pi}{2},0)$ to $(\frac{\pi}{2},0)$ and the line segment from $(\frac{\pi}{2},0)$ to $(-\frac{\pi}{2},0)$.

Let $P(x,y)=e^{-x}+y^2$ and $Q(x,y)=e^{-y}+x^2$. Then $\frac{\partial P}{\partial y}=2y$ and $\frac{\partial Q}{\partial x}=2x$

Let D be the region bounded by C.



From the diagram we see that C is negatively oriented. Both P and Q have continuous partial derivatives on R^2. Hence, by Green's Theorem,

$$\int_C \vec{F}\cdot d\vec{r} = -\int_{-C}\vec{F}\cdot d\vec{r} = -\int_{-C}\left(e^{-x}+y^2\right)dx+\left(e^{-y}+x^2\right)dy = -\iint_D\left(\frac{\partial Q}{\partial x}-\frac{\partial P}{\partial y}\right)dA$$

$$= -\int_{-\pi/2}^{\pi/2}\int_0^{\cos(x)}\left(2x-2y\right)dy\,dx$$

$$= -\int_{-\pi/2}^{\pi/2}\left[2xy-y^2\right]_0^{\cos(x)}dx$$

$$= -\int_{-\pi/2}^{\pi/2}\left(2x\cos(x)-\cos^2(x)\right)dx$$

Using integration by parts, let $u = 2x$ and $dv = \cos(x)\,dx$. Then $du = 2dx$ and $v = \sin(x)$.

Hence,

$$\int 2x\cos(x)\,dx = 2x\sin(x) - \int \sin(x)\,dx = x\sin(x) - (-\cos(x)) + C = 2x\sin(x) + \cos(x) + C$$

Therefore,

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= -\int_{-\pi/2}^{\pi/2} \left(2x\cos(x) - \cos^2(x)\right)dx \\
&= -\int_{-\pi/2}^{\pi/2} \left(2x\cos(x) - \frac{1+\cos(2x)}{2}\right)dx \\
&= -\left[2x\sin(x) + \cos(x) - \frac{1}{2}x - \frac{1}{4}\sin(2x)\right]_{-\pi/2}^{\pi/2} \\
&= -\left[2\left(\frac{\pi}{2}\right)\sin\left(\frac{\pi}{2}\right) + \cos\left(\frac{\pi}{2}\right) - \frac{\pi}{4} - \frac{1}{4}\sin(\pi) - \left(2\left(-\frac{\pi}{2}\right)\sin\left(-\frac{\pi}{2}\right) + \cos\left(-\frac{\pi}{2}\right) + \frac{\pi}{4} - \frac{1}{4}\sin(-\pi)\right)\right] \\
&= -\left(\pi + 0 - \frac{\pi}{4} - 0 - \pi - 0 - \frac{\pi}{4} - 0\right) \\
&= \frac{\pi}{2}
\end{aligned}$$

PART B *(20 marks total)*

9. a. $f(x,y)=g(r)$, where $r=\sqrt{x^2+y^2}$

$$\frac{\partial f}{\partial x}=g'(r)\cdot\frac{\partial r}{\partial x}=g'(r)\cdot\frac{2x}{2\sqrt{x^2+y^2}}=g'(r)\cdot\frac{x}{r}$$

$$\frac{\partial^2 f}{\partial x^2}=\frac{\partial}{\partial x}\left(\frac{g'(r)}{r}\cdot x\right)=\frac{\partial}{\partial x}\left(\frac{g'(r)}{r}\right)\cdot x+\frac{g'(r)}{r}\frac{\partial}{\partial x}(x)$$

$$=\frac{\frac{\partial}{\partial x}\left(g'(r)\right)\cdot r-g'(r)\frac{\partial}{\partial x}(r)}{r^2}\cdot x+\frac{g'(r)}{r}(1)$$

$$=\left(\frac{g''(r)\left(\frac{x}{r}\right)\cdot r-g'(r)\left(\frac{x}{r}\right)}{r^2}\right)x+\frac{g'(r)}{r}$$

$$=\left(\frac{g''(r)\cdot r-g'(r)}{r^3}\right)x^2+\frac{g'(r)}{r}$$

Similarly,

$$\frac{\partial f}{\partial y}=g'(r)\cdot\frac{\partial r}{\partial y}=g'(r)\cdot\frac{y}{r}\ \text{ and }\ \frac{\partial^2 f}{\partial y^2}=\left(\frac{g''(r)\cdot r-g'(r)}{r^3}\right)y^2+\frac{g'(r)}{r}$$

Now,

$$\frac{\partial^2 f}{\partial x^2}+\frac{\partial^2 f}{\partial y^2}=\left(\frac{g''(r)\cdot r-g'(r)}{r^3}\right)x^2+\frac{g'(r)}{r}+\left(\frac{g''(r)\cdot r-g'(r)}{r^3}\right)y^2+\frac{g'(r)}{r}$$

$$=\left(\frac{g''(r)\cdot r-g'(r)}{r^3}\right)\left(x^2+y^2\right)+\frac{2g'(r)}{r}=\left(\frac{g''(r)\cdot r-g'(r)}{r^3}\right)r^2+\frac{2g'(r)}{r}$$

$$=\frac{g''(r)\cdot r-g'(r)}{r}+\frac{2g'(r)}{r}=g''(r)-\frac{g'(r)}{r}+\frac{2g'(r)}{r}$$

$$=\frac{d^2g}{dr^2}+\frac{1}{r}\frac{dg}{dr}$$

b. $z^2 - \cos(x^2 z) = 2xy^2 + 3y$

Let $F(x,y,z) = z^2 - \cos(x^2 z) - 2xy^2 - 3y = 0$.

Then $F_x(x,y,z) = -(-\sin(x^2 z)(2xz)) - 2y^2 = 2xz\sin(x^2 z) - 2y^2$ and $F_z(x,y,z) = 2z - (-\sin(x^2 z)(x^2)) = 2z + x^2\sin(x^2 z)$.

By the Implicit Function Theorem,

$$\frac{\partial z}{\partial x} = -\frac{F_x(x,y,z)}{F_z(x,y,z)} = -\frac{2xz\sin(x^2 z) - 2y^2}{2z + x^2\sin(x^2 z)} = \frac{2y^2 - 2xz\sin(x^2 z)}{2z + x^2\sin(x^2 z)}$$

10. $F(x,y,z) = \left\langle ye^x, 2yz + e^x, y^2 \right\rangle$

F has continuous partial derivatives on R^3, so F will be conservative if and only if $curl\ F = \vec{0}$.

$$curl\ F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ ye^x & 2yz + e^x & y^2 \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left(y^2\right) - \frac{\partial}{\partial z}\left(2yz + e^x\right)\right)\vec{i} - \left(\frac{\partial}{\partial x}\left(y^2\right) - \frac{\partial}{\partial z}\left(2yz + e^x\right)\right)\vec{j} + \left(\frac{\partial}{\partial x}\left(2yz + e^x\right) - \frac{\partial}{\partial y}\left(ye^x\right)\right)\vec{k}$$

$$= \left(2y - 2y\right)\vec{i} - \left(0 - 0\right)\vec{j} + \left(e^x - e^x\right)\vec{k} = \vec{0}$$

Therefore, F is conservative and so $F = \nabla f$, for some potential function $w = f(x,y,z)$.

Since $\frac{\partial f}{\partial x} = ye^x$, $f(x,y,z) = \int ye^x\,dx = ye^x + g(y,z)$.

Now, since $\frac{\partial f}{\partial y} = 2yz + e^x$, $\frac{\partial}{\partial y}\left(ye^x + g(y,z)\right) = 2yz + e^x$

But, $\frac{\partial}{\partial y}\left(ye^x + g(y,z)\right) = e^x + g_y(y,z)$. So, $e^x + g_y(y,z) = 2yz + e^x \quad \Rightarrow \quad g_y(y,z) = 2yz$

Hence, $g(y,z)=\int 2yz\,dy = y^2z+h(z)$.

This gives $f(x,y,z)=ye^x+g(y,z)=ye^x+y^2z+h(z)$

Since $\dfrac{\partial f}{\partial z}=y^2$ and $\dfrac{\partial f}{\partial z}=\dfrac{\partial}{\partial z}\left(ye^x+y^2z+h(z)\right)=y^2+h'(z)$, we conclude that $h'(z)=0 \quad \Rightarrow \quad h(z)=C$, a constant.

Therefore, $f(x,y,z)=ye^x+y^2z+C$.

Setting $C=0$ gives a particular potential function of F, namely, $f(x,y,z)=ye^x+y^2z$.

11. $E=\{(r,\theta,z)\,|\,0\le z\le 3+r,\ 0\le r\le 1+\sin(\theta),\ 0\le\theta\le 2\pi\}$

So,

$$\begin{aligned}
V=\iiint_E dV &= \int_0^{2\pi}\int_0^{1+\sin(\theta)}\int_0^{3+r} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{1+\sin(\theta)} r\big[z\big]_0^{3+r}\,dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{1+\sin(\theta)} r\left(3+r-0\right)dr\,d\theta \\
&= \int_0^{2\pi}\int_0^{1+\sin(\theta)} \left(3r+r^2\right)dr\,d\theta \\
&= \int_0^{2\pi}\left[\frac{3r^2}{2}+\frac{r^3}{3}\right]_0^{1+\sin(\theta)} dr\,d\theta \\
&= \int_0^{2\pi}\left(\frac{3\left(1+\sin(\theta)\right)^2}{2}+\frac{\left(1+\sin(\theta)\right)^3}{3}\right)d\theta \\
&= \int_0^{2\pi}\left(\frac{3}{2}\left(1+2\sin(\theta)+\sin^2(\theta)\right)+\frac{1}{3}\left(1+3\sin(\theta)+3\sin^2(\theta)+\sin^3(\theta)\right)\right)d\theta \\
&= \int_0^{2\pi}\left(\frac{11}{6}+4\sin(\theta)+\frac{5}{2}\sin^2(\theta)+\frac{1}{3}\sin^3(\theta)\right)d\theta
\end{aligned}$$

Now write $\sin^2(\theta)=\frac{1}{2}-\frac{1}{2}\cos(2\theta)$ and $\sin^3(\theta)=\sin(\theta)\cdot\sin^2(\theta)=\sin(\theta)\left(1-\cos^2(\theta)\right)$

$$V=\int_0^{2\pi}\left(\frac{11}{6}+4\sin(\theta)+\frac{5}{2}\left(\frac{1}{2}-\frac{1}{2}\cos(2\theta)\right)+\frac{1}{3}\sin(\theta)\left(1-\cos^2(\theta)\right)\right)d\theta$$

$$=\int_0^{2\pi}\left(\frac{11}{6}+4\sin(\theta)+\frac{5}{4}-\frac{5}{4}\cos(2\theta)\right)d\theta+\frac{1}{3}\int_0^{2\pi}\sin(\theta)\left(1-\cos^2(\theta)\right)d\theta$$

$$=\int_0^{2\pi}\left(\frac{37}{12}+4\sin(\theta)-\frac{5}{4}\cos(2\theta)\right)d\theta+\frac{1}{3}\int_0^{2\pi}\sin(\theta)\left(1-\cos^2(\theta)\right)d\theta$$

Use the substitution: $u=\cos(\theta)$, $du=-\sin(\theta)\,d\theta$

$$=\left[\frac{37}{12}\theta-4\cos(\theta)-\frac{5}{8}\sin(2\theta)\right]_0^{2\pi}-\frac{1}{3}\left[\cos(\theta)-\frac{\cos^3(\theta)}{3}\right]_0^{2\pi}$$

$$=\left(\frac{37}{12}(2\pi)-4\cos(2\pi)-\frac{5}{8}\sin(4\pi)-(0-4(1)-0)\right)-\frac{1}{3}\left(\cos(2\pi)-\frac{\cos^3(2\pi)}{3}-\left(1-\frac{1}{3}\right)\right)$$

$$=\frac{74\pi}{12}-4+4-\frac{1}{3}\left(1-\frac{1}{3}-\left(1-\frac{1}{3}\right)\right)=\frac{37\pi}{6}$$