Faculty of Science

Unit 3: Partial Differentiation

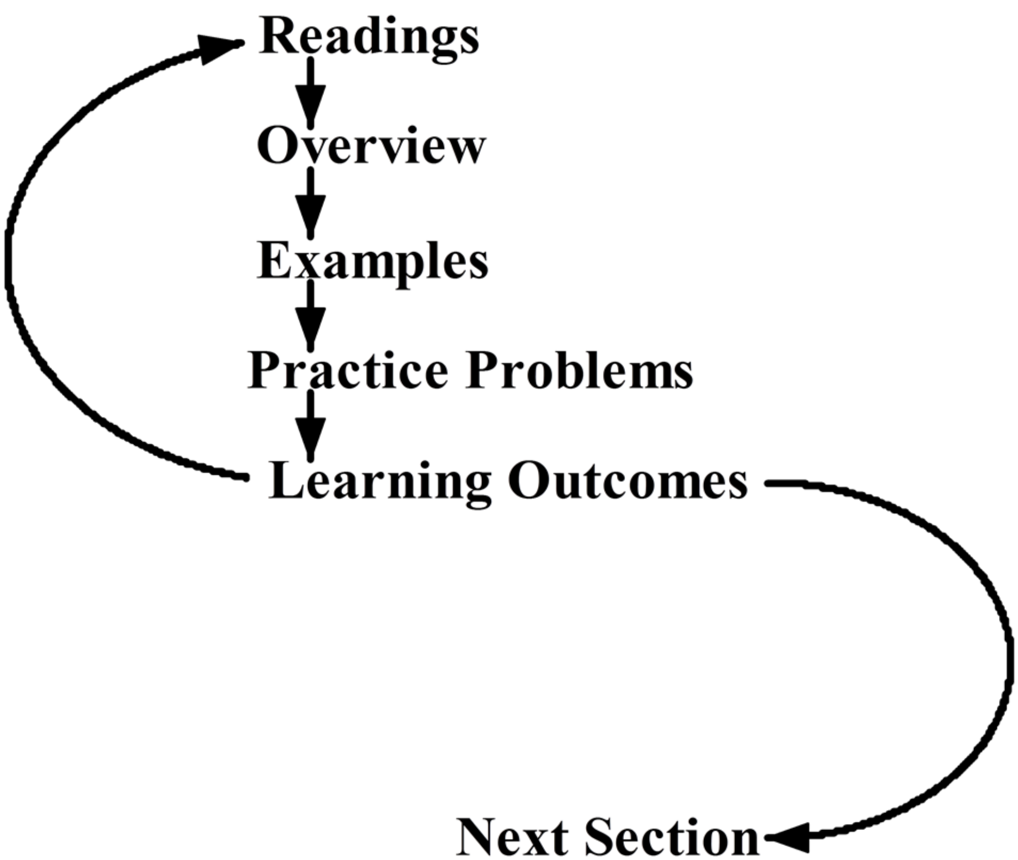
MATH 2111
Calculus III – Multivariable Calculus

Table of Contents

Instructions

The recommended procedure for working through each section of the units in this course is described in detail in your Course Guide.

This procedure is summarized below. If you are certain you have achieved the learning objectives, proceed to the next section. If you are uncertain about one or more of them, go back to the appropriate information in the section until you can complete the task listed in the objective.



Functions of Several Variables

Learning Outcomes

Upon completion of Functions of Several Variables, you should be able to:

- Evaluate a multivariable function at a given input.
- Read and interpret information displayed in a function table.
- Find and sketch the domain of a function of two or three variables.
- Find ranges of multivariable functions.
- Sketch the graph of a function of two variables.
- Determine the level curves of a function of two variables and draw a contour map.
- Find and describe the level surfaces of a function of three variables.

Readings

Read section 14.1, pages 902–911 in your textbook. Carefully study the examples worked out in the text.

Overview

In this chapter we study real-valued functions of two or more variables. Our focus will be on “functions of two variables,” but the definitions and methods we employ can naturally be extended to functions with more than two independent variables.

As an example of a function of two variables we can think of the volume formula for the volume V of a right circular cylinder with radius r and height h:

$$V(r,h)=\pi r^2 h\,,\quad r,h>0$$

As an example of a function of three variables, think of the surface area S of a closed rectangular box with length x, width y and height z:

$$S(x,y,z)=2xy+2xz+2yz$$

In each example there is a rule that assigns exactly one output, a real number, for the assigned input.

In the volume example the inputs, members of the **domain**, are ordered pairs (r,h) in the first quadrant of the rh-plane. The outputs, members of the **range**, are volume values belonging to the interval $(0,\infty)$.

In the second example the inputs are ordered triples (l, w, h) in the first octant of the lwh-coordinate system. The outputs are surface area values again belonging to the interval $(0, \infty)$.

More generally, if x and y are the independent variables (input values) and $z = f(x, y)$ denotes the dependent variable (the unique output value in the functional relationship f), then the **domain** D of the function is given by $D = \{(x, y) \mid f(x, y) \text{ is defined}\}$ and the **range** of the function is given by range $f = \{ f(x, y) \mid (x, y) \in \mathrm{D}\}$

The **graph** of $z = f(x, y)$ is then defined by $\{ (x, y, f(x, y)) \mid (x, y) \in \mathrm{D} \}$. The graph of a function of two variables is called a **surface**. It lives in 3-space.

> **Note:** The domain of a function of two variables is a subset of the xy-plane. Its graph lives in 3-space, whereas, the domain of a function of three variables is a subset of 3-space. Its graph lives in a 4-dimensional world, which we cannot visualize.

When graphing a surface by hand we use the domain, intercepts, symmetries and “level curves” to help us sketch its graph.

Level curves are curves that result from setting $f(x, y) = k, \quad k$ a constant. The intersection of the horizontal plane $z = k$ (at height k) with the surface produces a trace (curve) in that plane. If we project the level curves of our graph onto the xy-plane, we get what is called a **contour map**, a 2-dimensional picture of the shape of our surface at varying heights (see Figure 11 on page 907 of the text).

This idea can be applied to functions of three variables to get some information about the graph of the function that exists in 4 dimensions. In this case $f(x, y, z) = k, \quad k$ a constant produces a surface in 3-space, called a **level surface**.

Not all functional relationships are described using an equation. Functional information is also displayed in tabular form. Study Example 2 on page 903 of the text.

Also study Example 3 on page 903 of the text. Stewart introduces the **Cobb-Douglas production function**, an important function that arises in business and economics.

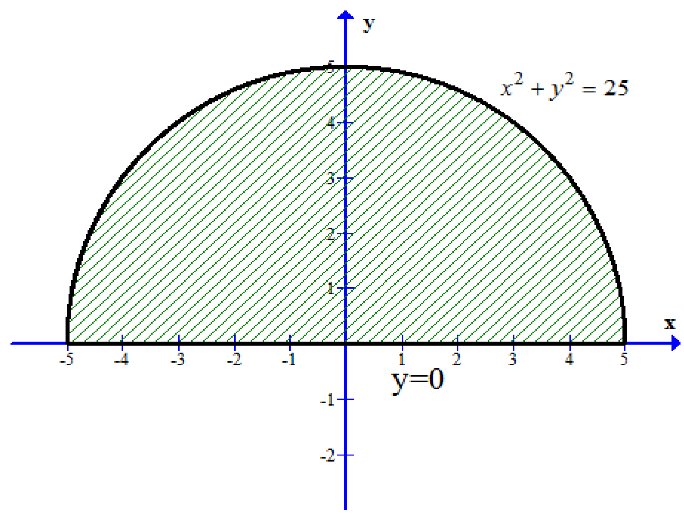
Example Exercises

1. This is problem 18 from your text on page 913.

In order that $f(x, y) = \sqrt{y} + \sqrt{25 - x^2 - y^2}$ be defined we require:

a) $y \geq 0$ and b) $25 - x^2 - y^2 \geq 0 \Rightarrow x^2 + y^2 \leq 25$

So, (x,y) must be inside or on the circle centered at the origin of radius 5 and above or on the x-axis. This region is pictured below.



Note: Algebraically we write the domain of f as

Domain $f = \{(x, y) \mid y \geq 0 \text{ and } x^2 + y^2 \leq 25\}$

Stewart, J. (2012). *Multivariable calculus* (7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.

2. Describe the level surfaces of $g(x, y, z) = x^2 - 3y^2 + 2z^2$.

Set $g(x, y, z) = k \Rightarrow x^2 - 3y^2 + 2z^2 = k$, k a constant

$k = 0$: $x^2 - 3y^2 + 2z^2 = 0$

The level surface is a double cone centered at $(0, 0, 0)$ and opening in the y-direction (axis of symmetry is the y-axis).

$k > 0$: $x^2 - 3y^2 + 2z^2 = k$

The level surface is a hyperboloid of one sheet centered at $(0,0,0)$ and opening in the y-direction (axis of symmetry is the *y*-axis).

$k<0: \quad x^2-3y^2+2z^2=k \quad \Rightarrow \quad -x^2+3y^2-2z^2=-k>0$

The level surface is a hyperboloid of two sheets centered at $(0,0,0)$ and opening in the *y*-direction (axis of symmetry is the y-axis).

Stewart, J. (2012). *Multivariable calculus* (7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.

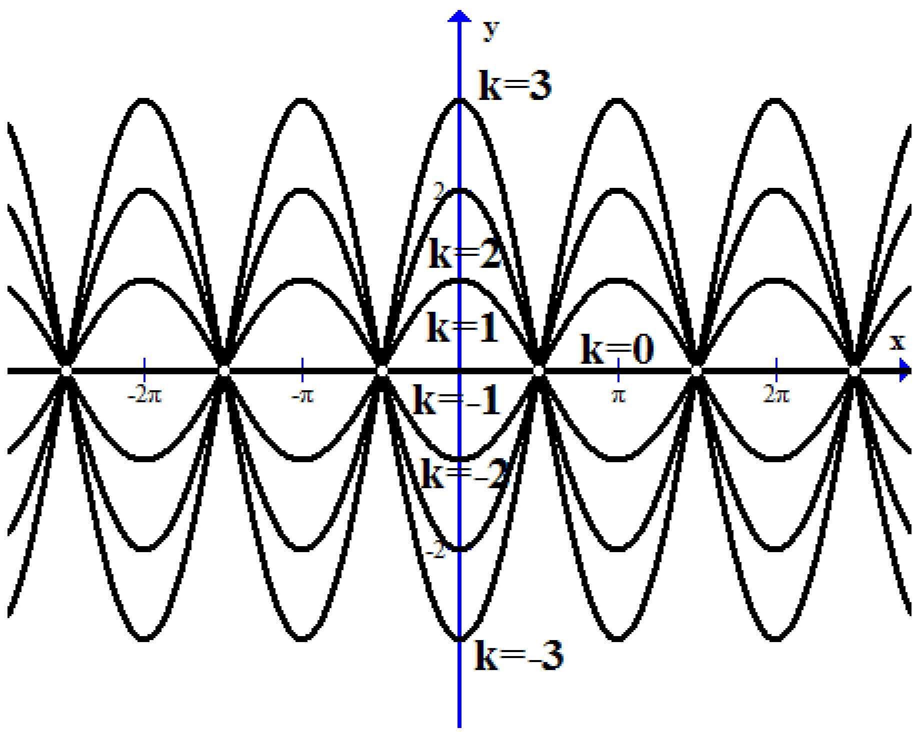
3. This is problem 48 from your text on page 914.

Draw a contour map of $f(x,y)=y\sec x$ showing several level curves.

Set

$f(x,y)=k \quad \Rightarrow \quad y\sec x=k \quad \Rightarrow \quad y=k\cos x \;; \quad x\neq\dfrac{\pi}{2}+n\pi \quad$ (n an integer)

Choose $k=-3,-2,-1,0,1,2\,3;$ the contour map is given below.



Note: Each level curved is identified by its k-value. The value $k=0$ gives the *x*-axis. When drawing a contour map be sure to label all the level curves in your graph.

Practice Exercises 14.1

From the text pages 912–916, do problems 1, 3, 7, 9, 15, 19, 25, 31, 33, 39, 47, 59, 63, and 67.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Limits and Continuity

Learning Outcomes

Upon completion of Limits and Continuity, you should be able to:

- State the informal definition of the limit of a multivariable function.
- Show that a limit does not exist by calculating limit results along different paths.
- Show that a limit does exist and compute its value by using either the Squeeze Theorem or algebraic manipulation or by converting the problem to polar coordinates.
- State what it means for a multivariable function to be continuous at a point or on a set, and be able to apply these definitions to solve problems on continuity.

Readings

Read section 14.2, pages 916–923 in your textbook. Carefully study the examples worked out in the text.

Overview

Recall that the limit of a function of one variable is a nearness concept: what happens to the values of the function as you approach a fixed number on the real line from either the left or the right?

Generalizing this concept to a function of two variables immediately raises a problem. A point in the domain of such a function is a point in the plane. There are no longer two directions of approach to this point, but infinitely many paths.

Accordingly we define $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ to mean that as (x, y) approaches (a,b) along **any path that lies in domain** f, $f(x, y)$ approaches L. This is an "intuitive definition". We omit the formal definition given in Box 1 on page 917 of the text.

Notes:

1. In order that $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exist, $f(a,b)$ need not be defined.

2. All paths into (a,b) must lie in domain f.

3. If two different paths in domain f yield two different limit results, then the limit doesn't exist (DNE).
4. Showing the limit value is the same along many different paths does not prove that the limit exists and has that value.

A technique that is sometimes useful in finding the limit of a function of two variables taken at the origin $(0,0)$ is to convert the calculation to polar coordinates. Set $x = r\cos(\theta)$ and $y = r\sin(\theta)$. The statement $(x, y) \to (0,0)$ gets replaced with $r \to 0$. Exercises 39–41 in the text on page 924 are of this type.

A useful result from differential calculus that generalizes here is the Squeeze

Theorem:

If $g(x, y) \leq f(x, y) \leq k(x, y)$ holds for all (x, y) in some open disk centered at (a,b), except possibly at (a,b), and

$$\lim_{(x,y)\to(a,b)} g(x, y) = \lim_{(x,y)\to(a,b)} k(x, y) = L \text{ , then } \lim_{(x,y)\to(a,b)} f(x, y) = L \text{ .}$$

Applying the Squeeze Theorem to the relationship $-\left|f(x, y)\right| \leq f(x, y) \leq \left|f(x, y)\right|$, we have, if $\lim_{(x,y)\to(a,b)} \left|f(x, y)\right| = 0$, then $\lim_{(x,y)\to(a,b)} f(x, y) = 0$.

The direct substitution property for limits of functions of one variable, called "continuity" is the basis for defining the concept of $z = f(x, y)$ being **continuous at** (a,b) in its domain. Specifically, f is continuous at (a,b) if the limit of f at (a,b) is the functional value at (a,b). If f is continuous at all points (a,b) in a set D, then f is **continuous on** D. Geometrically this means there are no holes, gaps or breaks in the surface over D.

As in the case of single variable functions, we can use properties of limits to show that sums, differences, products, and quotients of continuous functions are continuous on their domains. Specifically, polynomials in two variables are continuous on the entire xy-plane and rational functions, which are quotients of polynomials, are continuous at all points in their domain.

While our focus is on functions of two variables, similar definitions and results apply to functions of three or more variables.

Example Exercises

1. This is problem 14 from your text on page 923.

   Find the limit if it exists, or show that the limit doesn't exist.

$$\lim_{(x,y)\to(0,0)} \frac{x^4 - y^4}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \frac{\left(x^2 + y^2\right)\left(x^2 - y^2\right)}{x^2 + y^2} = \lim_{(x,y)\to(0,0)} \left(x^2 - y^2\right) = 0^2 - 0^2 = 0$$

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

2. This is problem 16 from your text on page 923.

$$\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2}$$

   Try different paths. Suspect the limit value is zero. To prove this we apply the Squeeze Theorem.

   Since $\sin^2 y \ge 0$ and $0 \le \frac{x^2}{x^2 + 2y^2} \le 1$, we have that

$$0 \cdot \sin^2 y \le \frac{x^2 \sin^2 y}{x^2 + 2y^2} \le 1 \cdot \sin^2 y \quad \Rightarrow \quad 0 \le \frac{x^2 \sin^2 y}{x^2 + 2y^2} \le \sin^2 y$$

   But, $\lim_{(x,y)\to(0,0)} 0 = 0$ and $\lim_{(x,y)\to(0,0)} \sin^2 y = \lim_{y\to 0} \sin^2 y = \sin^2 0 = 0$

   Therefore, by the Squeeze Theorem, $\lim_{(x,y)\to(0,0)} \frac{x^2 \sin^2 y}{x^2 + 2y^2} = 0$.

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

3. This is problem 18 from your text on page 923.

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8}$$

   Try different paths.

   Along $x = 0$:

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8} = \lim_{(0,y)\to(0,0)} \frac{0 \cdot y^4}{0^2 + y^8} = \lim_{y\to 0} \frac{0}{y^8} = \lim_{y\to 0} 0 = 0$$

Along $y = 0$:

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8} = \lim_{(x,0)\to(0,0)} \frac{x \cdot 0^4}{x^2 + 0^8} = \lim_{x\to 0} \frac{0}{x^2} = \lim_{x\to 0} 0 = 0$$

Along $y = x$:

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8} = \lim_{(x,x)\to(0,0)} \frac{x \cdot x^4}{x^2 + x^8} = \lim_{x\to 0} \frac{x^5}{x^2\left(1 + x^6\right)} = \lim_{x\to 0} \frac{x^3}{1 + x^6} = \frac{0}{1+0} = 0$$

Could the limit be zero? Notice that x^2 in the denominator can be converted to y^8 by setting $x = y^4$. The denominator now simplifies to $2y^8$. Does the numerator also simplify to a multiple of y^8? If so, the limit will not be zero.

Try $x = y^4$:

$$\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8} = \lim_{(y^4,y)\to(0,0)} \frac{y^4 \cdot y^4}{\left(y^4\right)^2 + y^8} = \lim_{y\to 0} \frac{y^8}{y^8 + y^8} = \lim_{y\to 0} \frac{y^8}{2y^8} = \lim_{y\to 0} \frac{1}{2} = \frac{1}{2}$$

We have different results along two different paths.

Therefore, $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^4}{x^2 + y^8}$ DNE (Does not exist)

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

Practice Exercises 14.2

From the text pages 923–924, do problems 7, 9, 11, 13, 17, 21, 25, 33, 35, 39, and 41.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Partial Derivatives

Learning Outcomes

Upon completion of Partial Derivatives, you should be able to:

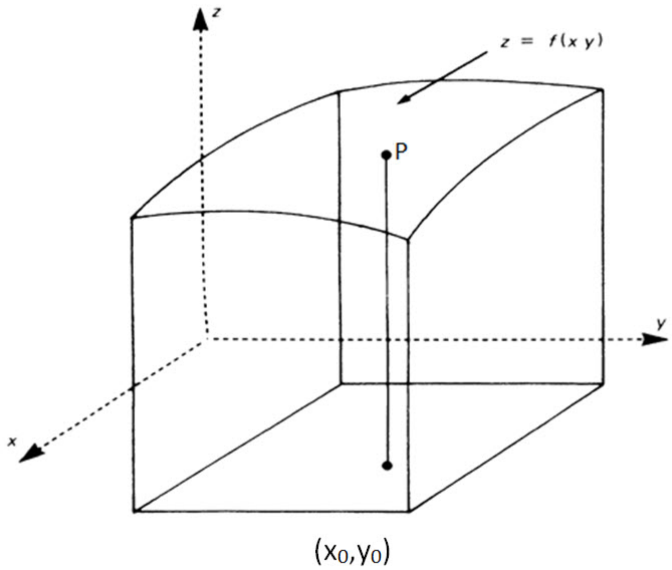
- Calculate partial derivatives using the limit definition of a partial derivative.
- Interpret a partial derivative value.
- Calculate rate of change values from a function table of two independent variables.
- Use rules of differentiation to calculate partial derivative functions.
- Recognize and use appropriate notation for partial derivatives and higher-order partial derivatives.
- Write down the statement for Clairaut's Theorem and verify that it holds for a given function.
- Use implicit differentiation to calculate partial derivatives.
- Verify that a given multivariable function is a solution of a partial differential equation.

Readings

Read section 14.3, pages 924–935 in your textbook. Carefully study the examples worked out in the text.

Overview

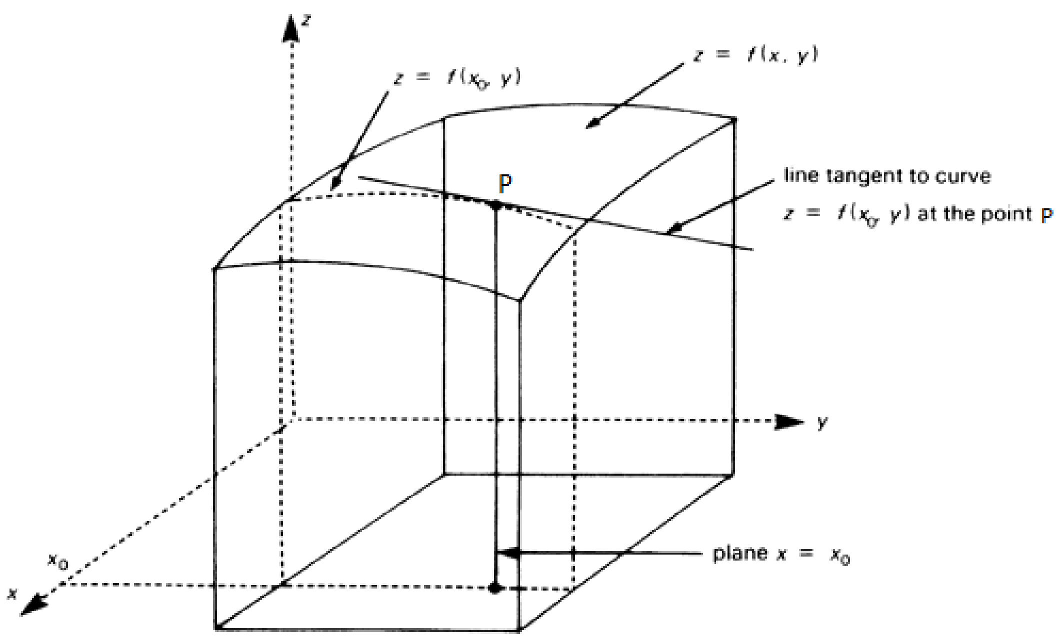
Let $z = f(x, y)$. Equations of this form, as you will recall from your work with quadric surfaces, represent surfaces in R^3.



Now consider the set of all points on the surface which have x_0 as their x-coordinate. $P = P(x_0, y_0, f(x_0, y_0))$ is one such point on the surface. This function

$$z = f(x_0, y)$$

is called a <u>slice function</u> because the points on the surface are precisely those where the vertical plane $x = x_0$ intersects the surface $z = f(x, y)$.



Since the plane $x = x_0$ intersects the surface in a plane curve, we can use the derivative to find the slope of the curve and, hence, of the tangent to the curve at P. Note that $z = f(x_0, y)$ is a function of y alone. We must indicate this by defining the slope of the tangent to this curve at the point P to be the partial derivative of f with respect to y.

Therefore we write

$$f_y(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

Similarly, if we cut the surface with the vertical plane, $y = y_0$ we obtain another slice function $z = f(x, y_0)$ and define the slope of the tangent to this curve at P to be the partial derivative of f with respect to x. We write

$$f_x(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

Letting the point (x_0, y_0) vary, we get the partial derivative functions:

$$f_y(x, y) = \lim_{h \to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

$$f_x(x, y) = \lim_{h \to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

Note:

1. Since all other independent variables are kept constant when calculating the partial derivative of the function with respect to a given variable, we can apply all the “rules of differentiation” we learned in single variable calculus!

2. Partial derivatives can be defined for functions of three or more variables. The general definition is discussed on pages 929–930 of the text.

There are a number of different notations used for partial derivatives. You need to be familiar with these notations as given on page 927 of the text.

How do we interpret partial derivatives? As in single variable calculus there are two natural interpretations.

For $f_x(x_0, y_0)$:

Geometric: slope of the line tangent to the trace curve of f in the vertical plane $y = y_0$ at $(x_0, y_0, f(x_0, y_0))$.

Rate of Change: instantaneous rate of change of $z = f(x, y)$ with respect to x at $(x_0, y_0, f(x_0, y_0))$, keeping y fixed at $y = y_0$.

We interpret $f_y(x_0, y_0)$ similarly.

As for functions of a single variable, we can iterate the partial differentiation process and produce higher order partial derivatives. Familiarize yourself with notations for second partial derivatives as given on page 930 of the text.

Pay careful attention to Clairaut's Theorem on page 931 of the text. It gives a sufficient condition for the second partials f_{xy} and f_{yx} to be equal.

Partial derivatives arise in partial differential equations. We can verify solutions to partial differential equations by computing partial derivatives. Study examples 8 and 9 on pages 932–933 of the text.

Example Exercises

1. This is problem 46 from your text on page 937.

   Use the definition of the partial derivative to find $f_x(x, y)$, given

$$f(x, y) = \frac{x}{x + y^2}$$

$$\begin{aligned}
f_x(x, y) &= \lim_{h \to 0} \frac{f(x+h, y) - f(x, y)}{h} \\
&= \lim_{h \to 0} \left[\frac{\frac{x+h}{x+h+y^2} - \frac{x}{x+y^2}}{h} \right] \\
&= \lim_{h \to 0} \left[\frac{\frac{x+h}{x+h+y^2} \cdot \frac{x+y^2}{x+y^2} - \frac{x}{x+y^2} \cdot \frac{x+h+y^2}{x+h+y^2}}{h} \right] \\
&= \lim_{h \to 0} \left[\frac{\frac{(x+h)\cdot(x+y^2) - x(x+h+y^2)}{(x+h+y^2)\cdot(x+y^2)}}{h} \right] \\
&= \lim_{h \to 0} \left[\frac{x^2 + xy^2 + xh + hy^2 - x^2 - xh - xy^2}{h(x+h+y^2)\cdot(x+y^2)} \right] \\
&= \lim_{h \to 0} \left[\frac{hy^2}{h(x+h+y^2)\cdot(x+y^2)} \right] \\
&= \lim_{h \to 0} \left[\frac{y^2}{(x+h+y^2)\cdot(x+y^2)} \right] \\
&= \lim_{h \to 0} \left[\frac{y^2}{(x+0+y^2)\cdot(x+y^2)} \right] = \frac{y^2}{(x+y^2)^2}
\end{aligned}$$

Stewart, J. (2012). *Multivariable calculus* (7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.

2. Given $f(x,y)=\sin(xy)\sqrt{x^2+y^2}$, find $f_x\left(1,\frac{\pi}{2}\right)$ and $f_y\left(1,\frac{\pi}{2}\right)$.

$$\begin{aligned}
f_x(x,y) &= \frac{\partial}{\partial x}\left(\sin(xy)\right)\cdot\sqrt{x^2+y^2}+\sin(xy)\cdot\frac{\partial}{\partial x}\left(x^2+y^2\right)^{1/2} \\
&= \cos(xy)\cdot\frac{\partial}{\partial x}\left(xy\right)\cdot\sqrt{x^2+y^2}+\sin(xy)\cdot\frac{1}{2}\left(x^2+y^2\right)^{-1/2}\cdot\frac{\partial}{\partial x}\left(x^2+y^2\right) \\
&= y\cos(xy)\sqrt{x^2+y^2}+2x\sin(xy)\cdot\frac{1}{2}\left(x^2+y^2\right)^{-1/2} \\
&= y\cos(xy)\sqrt{x^2+y^2}+\frac{x\sin(xy)}{\sqrt{x^2+y^2}}
\end{aligned}$$

So, $f_x\left(1,\frac{\pi}{2}\right)=\frac{\pi}{2}\cos\left(1\cdot\frac{\pi}{2}\right)\sqrt{1^2+\left(\frac{\pi}{2}\right)^2}+\frac{1\cdot\sin\left(1\cdot\frac{\pi}{2}\right)}{\sqrt{1^2+\left(\frac{\pi}{2}\right)^2}}=0+\frac{1}{\sqrt{1+\frac{\pi^2}{4}}}=\frac{2}{\sqrt{4+\pi^2}}$

Now, the expression for $f(x,y)$ is completely symmetrical in x and y, so we can find $f_y(x,y)$ by interchanging x and y in the expression for $f_x(x,y)$. We get

$$f_y(x,y)=x\cos(xy)\sqrt{x^2+y^2}+\frac{y\sin(xy)}{\sqrt{x^2+y^2}}$$

Hence

$$f_y\left(1,\frac{\pi}{2}\right)=1\cdot\cos\left(1\cdot\frac{\pi}{2}\right)\sqrt{1^2+\left(\frac{\pi}{2}\right)^2}+\frac{\frac{\pi}{2}\cdot\sin\left(1\cdot\frac{\pi}{2}\right)}{\sqrt{1^2+\left(\frac{\pi}{2}\right)^2}}=0+\frac{\left(\frac{\pi}{2}\right)}{\sqrt{1+\frac{\pi^2}{4}}}=\frac{\pi}{\sqrt{4+\pi^2}}$$

3. This is problem 50 from your text on page 937.

Use implicit differentiation to find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$, given $yz + x\ln y = z^2$.

$$\frac{\partial}{\partial x}\left(yz + x\ln y\right) = \frac{\partial}{\partial x}\left(z^2\right)$$

$$\frac{\partial}{\partial x}(yz) + \frac{\partial}{\partial x}(x\ln y) = 2z\frac{\partial z}{\partial x}$$

$$y\frac{\partial z}{\partial x} + \ln y = 2z\frac{\partial z}{\partial x}$$

$$2z\frac{\partial z}{\partial x} - y\frac{\partial z}{\partial x} = \ln y$$

$$\frac{\partial z}{\partial x}(2z - y) = \ln y$$

$$\frac{\partial z}{\partial x} = \frac{\ln y}{2z - y}$$

Now,

$$\frac{\partial}{\partial y}\left(yz + x\ln y\right) = \frac{\partial}{\partial y}\left(z^2\right)$$

$$\frac{\partial}{\partial y}(yz) + \frac{\partial}{\partial y}(x\ln y) = 2z\frac{\partial z}{\partial y}$$

$$1\cdot z + y\frac{\partial z}{\partial y} + x\cdot\frac{1}{y} = 2z\frac{\partial z}{\partial y}$$

$$2z\frac{\partial z}{\partial y} - y\frac{\partial z}{\partial y} = z + \frac{x}{y}$$

$$\frac{\partial z}{\partial y}(2z - y) = \frac{yz + x}{y}$$

$$\frac{\partial z}{\partial y} = \frac{yz + x}{y(2z - y)}$$

Stewart, J. (2012). *Multivariable calculus* (7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.

Practice Exercises 14.3

From the text pages 935–939, do problems 1, 3, 5, 9, 17, 21, 27, 29, 33, 41, 45, 47, 49, 55, 61, 67, 73, 75, 81, 83, and 91.

Remember: You are to attempt all of the problems carefully before checking your solutions against those given in the solutions manual.

Tangent Planes and Linear Approximations

Learning Outcomes

Upon completion of Tangent Planes and Linear Approximations, you should be able to:

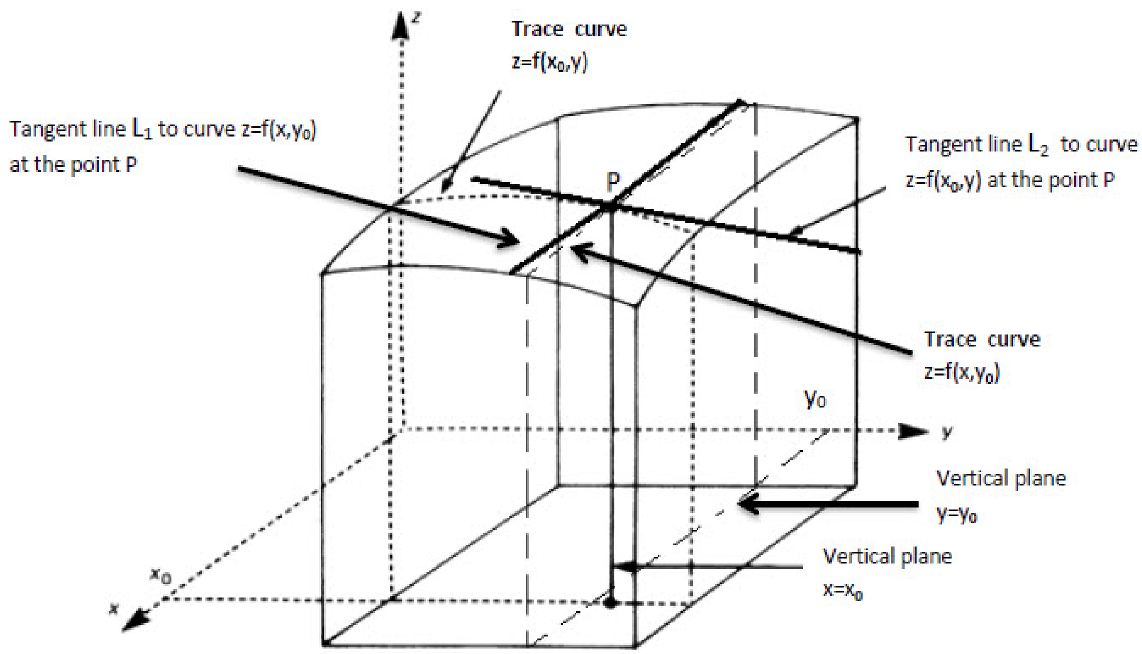
- Determine an equation for the tangent plane to the graph of a function at a given point.
- Find the linearization of a multivariable function and use it to approximate a function value.
- Calculate the total differential of a multivariable function and use it to approximate a change in functional values.
- Determine if a functional is differentiable at a given point using the theorem in box 8 on page 942 of the text.
- Solve application problems using differentials.

Readings

Read section 14.4, pages 939–946 in your textbook. Carefully study the examples worked out in the text.

Overview

If you zoom in toward a point P on the graph of a "nice function" of two variables, the surface starts to look like a plane. The plane that is tangent to the surface at P is called the **tangent plane**, which is a linear function of two variables. The approximation of the function $z = f(x, y)$ by its tangent plane in a neighbourhood of P is called the **linearization of** f **at** P**.**



Notice that the two tangent lines L1 and L2 to the graph of $z = f(x, y)$ above at P determine a unique plane. This is the tangent plane. If L1 has direction vector $\vec{v}_1$ and L2 has direction vector $\vec{v}_2$, then

$$\vec{v}_1 = \left\langle 0, 1, f_y(x_0, y_0) \right\rangle \text{ and } \vec{v}_2 = \left\langle 1, 0, f_x(x_0, y_0) \right\rangle$$

A normal vector to the tangent plane is given by

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 1 & f_y(x_0, y_0) \\ 1 & 0 & f_x(x_0, y_0) \end{vmatrix} = \left\langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \right\rangle$$

Using the point $P = P(x_0, y_0, z_0)$ which lives on both tangent lines, we have an equation for the tangent plane as follows:

$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + (-1)(z - z_0) = 0$$

Rearranging we obtain,

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

Note: Here $z_0 = f(x_0, y_0)$.

When approximating the function $z = f(x, y)$ by its tangent plane we let $z = L(x, y)$ and write the equation for the tangent plane in the form:

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

For (x, y) near (x_0, y_0), $f(x, y) \approx L(x, y)$.

The extension of the concept of the "differentiability" of single variable functions to functions of two variables is described in box 7 on page 942 of the text. This is not a workable definition. Fortunately, there is a sufficient condition for the differentiability of $z = f(x, y)$ that is often easy to check. It is given as a theorem in Box 8 on page 942 of the text. You need to know this result.

As in the case of a single variable function, if $z = f(x, y)$ is differentiable at (x_0, y_0), then f is continuous at (x_0, y_0).

Recall that for $y = f(x)$ the differential of the independent variable, dx, is taken to be any real number. The differential of the dependent variable is then defined by

$$dy = f'(x)dx$$

For a function with two independent variables x and y, dx and dy can be assigned any real values. Then we define the **total differential** of the dependent variable z by

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy$$

Figure 7 on page 944 of the text gives a graphical illustration of the relationship between the change in the functional value, $\Delta f = \Delta z$, and the total differential, $df = dz$, in a neighbourhood of the point (x_0, y_0) where the tangent plane is constructed. Carefully study the elements of this diagram.

From this diagram we see that if $\Delta z = f(x, y) - f(x_0, y_0)$, then

$$\Delta z \approx dz = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$

for (x, y) near (x_0, y_0).

Alternatively, we can write

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$

But $dx = x - x_0$ and $dy = y - y_0$.

So, $f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)\left(x - x_0\right) + f_y(x_0, y_0)\left(y - y_0\right) = L(x, y)$

This is just the linearization of f at (x_0, y_0) discussed earlier.

The concepts discussed in this section naturally generalize to functions of more than two variables.

Example Exercises

1. This is problem 14 from your text on page 946.

Given $f(x, y) = \sqrt{x + e^{4y}}$, explain why f is differentiable at $(3, 0)$ and find the linearization of f at that point.

We start by calculating the partial derivatives of f.

$$f_x(x, y) = \frac{1}{2}\left(x + e^{4y}\right)^{-1/2} \cdot (1 + 0) = \frac{1}{2\sqrt{x + e^{4y}}}$$

$$f_y(x, y) = \frac{1}{2}\left(x + e^{4y}\right)^{-1/2} \cdot (0 + 4e^{4y}) = \frac{2e^{4y}}{\sqrt{x + e^{4y}}}$$

Using the theorem in Box 8 on page 942 of the text, we only have to find a neighbourhood of $(3, 0)$ where f_x and f_y are defined and show f_x and f_y are continuous at $(3, 0)$.

Observe that $x + e^{4y} > 0$ for $x \geq 0$ and so the square root function is defined and is continuous for points (x, y) in the first and fourth quadrants. Take an open disk centered at $(3, 0)$ of radius, say, r=1.Then f_x and f_y are continuous functions on this open disk, as quotients of continuous functions, and so by the theorem f is differentiable at $(3, 0)$.

Now,

$$\begin{aligned} L(x, y) &= f(3,0) + f_x(3,0)(x - 3) + f_y(3,0)(y - 0) \\ &= \sqrt{3 + e^0} + \frac{1}{2\sqrt{3 + e^0}}(x - 3) + \frac{2e^0}{\sqrt{3 + e^0}}(y - 0) \\ &= \sqrt{4} + \frac{1}{2\sqrt{4}}(x - 3) + \frac{2}{\sqrt{4}}y \\ &= 2 + \frac{1}{4}(x - 3) + y \\ &= \frac{1}{4}x + y + \frac{5}{4} \end{aligned}$$

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

2. Use a linear approximation to find an approximate value for $\sqrt{12.1^2+4.97^2}$. Express the answer rounded to 2 decimal places.

Let $f(x,y)=\sqrt{x^2+y^2}$

$$f(x,y)\approx f(x_0,y_0)+f_x(x_0,y_0)\,dx+f_y(x_0,y_0)\,dy$$

Now, $f_x(x,y)=\dfrac{x}{\sqrt{x^2+y^2}}$ and $f_y(x,y)=\dfrac{y}{\sqrt{x^2+y^2}}$

Take $(x_0,y_0)=(12,5)$. Then $(dx,dy)=(0.1,-0.03)$

So, $f(12.1,4.97)\approx f(12,5)+f_x(12,5)\,dx+f_y(12,5)\,dy$

$$\sqrt{12.1^2+4.97^2}\approx\sqrt{12^2+5^2}+\frac{12}{\sqrt{12^2+5^2}}(0.1)+\frac{5}{\sqrt{12^2+5^2}}(-0.03)$$

$$\sqrt{12.1^2+4.97^2}\approx 13+\frac{12}{13}(0.1)+\frac{5}{13}(-0.03)$$

$$\sqrt{12.1^2+4.97^2}\approx 13.08$$

Checking with a calculator we find this approximation is good to two decimal places: $\sqrt{1.21^2+4.97^2}\approx 13.0809$

3. This is problem 34 from your text on page 947.

Use differentials to estimate to 2 decimal places the amount of metal in a closed cylindrical can with outside dimensions 10 cm high and 4 cm in diameter, if the metal in the top and bottom is 0.1 cm thick and the metal in the sides is 0.05 cm thick.

Let V denote the volume of a can with radius r and height h.

Then $V(r,h)=\pi r^2 h$ and $\Delta V\approx dV=V_r(r,h)\,dr+V_h(r,h)\,dh$ is an estimate of the amount of metal in the can.

Now, $(r,h)=(2-0.05,10-2(0.1))=(1.95,9.8)$, with $dr=0.05$ and $dh=0.2$,and $V_r(r,h)=2\pi rh$, $V_h(r,h)=\pi r^2$. So,

$$\begin{aligned}\Delta V\approx dV&=V_r(1.95,9.8)\,dr+V_h(1.95,9.8)\,dh\\&=2\pi(1.95)(9.8)(0.05)+\pi(1.95)^2(0.2)\\&\approx 8.39\ \text{cm}^3\end{aligned}$$

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

Practice Exercises 14.4

From the text pages 946–948, do problems 5, 11, 15, 19, 21, 23, 25, 31, 33, 39, and 41.

Remember: You are to attempt all of the problems carefully before checking your solutions against those given in the solutions manual.

The Chain Rule

Learning Outcomes

Upon completion of the Chain Rule, you should be able to:

- Use the substitution process or the Chain Rule to find single variable derivatives or multivariable partial derivatives.
- Use tree diagrams to write out the Chain Rule for a given case.
- Use the implicit function theorem to find derivatives or partial derivatives.
- Use the Chain Rule to solve applied problems.

Readings

Read section 14.5, pages 948–954 in your textbook. Carefully study the examples worked out in the text.

Overview

In elementary calculus, you mastered the chain rule in learning how to differentiate a function of a function, for example

$$y = \sin\left(x^2 + 4\right) \text{ or } y = \ln(\sin(x))$$

We will now develop the equivalent rule for functions of several variables. We will first develop the chain rule for a function f of two variables and then extend the result to other functions.

Let's start by looking at an example.

Let $z = f(x, y) = x^2 + y^2 + e^{xy}$, where $x = \cos(t)$ and $y = \sin(t)$

Now $z = f(x, y)$ is really a function of a single variable t and we can write

$$z = F(t) = \cos^2(t) + \sin^2(t) + e^{\sin(t)\cdot\cos(t)} = 1 + e^{\sin(t)\cdot\cos(t)}$$

Differentiating with respect to t, we have

$$\frac{dz}{dt} = F'(t) = e^{\sin(t)\cdot\cos(t)}\left(\cos^2(t) - \sin^2(t)\right)$$

If we were now asked to evaluate at $t = \dfrac{\pi}{2}$, we have

$$\left.\frac{dz}{dt}\right|_{t=\pi/2} = e^{\sin(\pi/2)\cdot\cos(\pi/2)}\left(\cos^2(\pi/2) - \sin^2(\pi/2)\right) = e^0(0-1) = -1$$

The above process required that we replace x and y in the expression for $f(x, y)$ by the equivalent expressions in t. Is there a way you can get the desired result without performing the substitution?

The answer is yes. The result is called the **Chain Rule-Case 1**, as seen in Box 2 on page 948 of the text.

It states: If $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t, then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}$$

If we apply it to the previous example where $z = f(x, y) = x^2 + y^2 + e^{xy}$ and $x = \cos(t)$, $y = \sin(t)$ we get

$$\begin{aligned}\frac{dz}{dt} &= \frac{\partial z}{\partial x}\cdot\frac{dx}{dt} + \frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\\ &= \left(2x + ye^{xy}\right)\cdot(-\sin(t)) + \left(2y + xe^{xy}\right)\cdot\cos(t)\\ &= \left(2\cos(t) + \sin(t)e^{\cos(t)\sin(t)}\right)\cdot(-\sin(t)) + \left(2\sin(t) + \cos(t)e^{\cos(t)\sin(t)}\right)\cdot\cos(t)\\ &= -2\cos(t)\sin(t) - \sin^2(t)e^{\cos(t)\sin(t)} + 2\sin(t)\cos(t) + \cos^2(t)e^{\cos(t)\sin(t)}\\ &= e^{\sin(t)\cos(t)}\left(\cos^2(t) - \sin^2(t)\right)\end{aligned}$$

This is the same result.

Notice that $t=\dfrac{\pi}{2}$ gives the point $(x,y)=(\cos\left(\dfrac{\pi}{2}\right),\sin\left(\dfrac{\pi}{2}\right)=(0,1)$. If we evaluate the derivative right after we apply the chain rule we get

$$
\begin{aligned}
\left.\frac{dz}{dt}\right|_{t=\frac{\pi}{2}} &= \left.\frac{\partial z}{\partial x}\cdot\frac{dx}{dt}+\frac{\partial z}{\partial y}\cdot\frac{dy}{dt}\right|_{t=\frac{\pi}{2}} \\
&= \left.\left(2x+ye^{xy}\right)\cdot(-\sin(t))+\left(2y+xe^{xy}\right)\cdot\cos(t)\right|_{t=\pi/2} \\
&= \left(2(0)+1\cdot e^{0}\right)\cdot(-\sin\left(\frac{\pi}{2}\right))+\left(2(1)+0e^{0}\right)\cdot\cos\left(\frac{\pi}{2}\right) \\
&= 1(-1)+0 \\
&= -1
\end{aligned}
$$

Case 2 of the Chain Rule, as found in Box 3 on page 950 of the text, involves the situation where the intermediate functions are themselves functions of two variables; that is, $x=g(s,t)$ and $y=h(s,t)$. In this case, the function $z=f(x,y)$ is really a function of s and t.

The rule now becomes

$$\frac{\partial z}{\partial s}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial s}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial s} \quad \text{and} \quad \frac{\partial z}{\partial t}=\frac{\partial z}{\partial x}\cdot\frac{\partial x}{\partial t}+\frac{\partial z}{\partial y}\cdot\frac{\partial y}{\partial t}$$

When differentiating a multivariable function with intermediate variables that are themselves functions of independent variables, you always have the option of either doing the substitution process first and then differentiating as usual or using the chain rule. Tree diagrams are a useful way of remembering the statement of the Chain Rule. See Figures 2, 3, 4 and 5 in the text, pages 950–952.

The general version of the Chain Rule is described in Box 4 on page 951 of the text.

The Chain Rule can be used to derive a formula for the derivative of a single variable function defined implicitly by $F(x,y)=0$ using partial derivatives of the function $z=F(x,y)$. This is described in Box 6 on page 953 of the text. The result is called the "Implicit Function Theorem." It can be extended to calculating partial derivatives of multivariable functions. A version for a function of two variables is given in Box 7 on page 954 of the text.

Example Exercises

1. If $w=f(x,y,z)=x^3+y^3-3xyz$, where $x=t^2$, $y=2t$, $z=e^t$, find $\left.\dfrac{dw}{dt}\right|_{t=1}$

Using the Chain Rule,

$$\frac{dw}{dt} = \frac{\partial w}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial w}{\partial y} \cdot \frac{dy}{dt} + \frac{\partial w}{\partial z} \cdot \frac{dz}{dt}$$
$$= \left(3x^2 - 3yz\right) \cdot 2t + \left(3y^2 - 3xz\right) \cdot 2 + (-3xy) \cdot e^t$$
$$= 2t\left(3x^2 - 3yz\right) + 2\left(3y^2 - 3xz\right) - 3xye^t$$

When $t = 1$, $(x, y, z) = \left(1^2, 2 \cdot 1, e^1\right) = (1, 2, e)$. So,

$$\left.\frac{dw}{dt}\right|_{t=1} = 2(1)\left(3(1)^2 - 3(2)(e)\right) + 2\left(3(2)^2 - 3(1)(e)\right) - 3(1)(2)(e)$$
$$= 6 - 12e + 24 - 6e - 6e = 30 - 24e$$

2. If $z = f(x, y) = x^2 + 3y^2$, where $x = s^2 + t^2$, $y = 4st$, find $\frac{\partial z}{\partial s}$ and $\frac{\partial z}{\partial t}$ as functions of t.

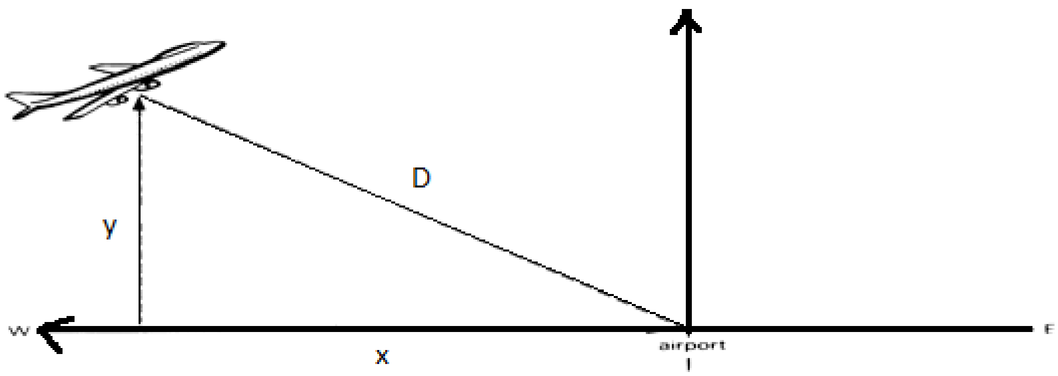
Using the Chain Rule,

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} = 2x \cdot 2s + 6y \cdot 4t = 4xs + 24yt$$
$$= 4(s^2 + t^2)s + 24(4st)t = 4s^3 + 4st^2 + 96st^2 = 4s^3 + 100st^2$$
$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t} = 2x \cdot 2t + 6y \cdot 4s = 4(s^2 + t^2)t + 24(4st)s$$
$$= 4s^2t + 4t^3 + 96s^2t = 4t^3 + 100s^2t$$

3. A plane is flying due east with a speed of 400 miles per hour and is climbing at the rate of 528 feet per minute. It is being tracked by an airport tower. Find how fast the plane is approaching the tower when it is 3 miles above the ground over a point exactly 4 miles due west of the tower.

You should first solve this problem by direct substitution, then compare the work involved with that below using the chain rule.

In solving a real problem, it is essential to use a consistent set of units, and directions. In the diagram below, East is arbitrarily chosen to be the negative x-direction so the speed $\frac{dx}{dt} = -400$ mph indicates the plane is flying towards the airport.



We are told the plane is climbing at the rate of 528 feet per minute, so $\frac{dy}{dt} = \frac{528 \text{ ft/min}}{5280 \text{ ft/mi}} \cdot 60 \text{ min/hr} = 6 \text{ mph}$

We want $\frac{dD}{dt}$ when $x = 4$ miles and $y = 3$ miles.

Now $D(x, y) = \sqrt{x^2 + y^2}$ and using the Chain Rule we have

$$\frac{dD}{dt} = \frac{\partial D}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial D}{\partial y} \cdot \frac{dy}{dt} = \frac{x}{\sqrt{x^2 + y^2}} \cdot (-400) + \frac{y}{\sqrt{x^2 + y^2}} \cdot (6)$$

At $(x, y) = (4,3)$, we have

$$\frac{dD}{dt} = \frac{4}{\sqrt{4^2 + 3^2}} \cdot (-400) + \frac{3}{\sqrt{4^2 + 3^2}} \cdot (6) \approx -316 \text{ mph}$$

Therefore the plane is approaching from the west (since $\frac{dD}{dt}$ is negative) at about 316 mph.

Practice Exercises 14.5

From the text pages 954–956, do problems 3, 5, 13, 15, 17, 25, 29, 33, 35, 39, 45, 47, 49, and 53.

Remember: You are to attempt all of the problems carefully before checking your solutions against those given in the solutions manual.

Directional Derivatives and the Gradient Vector

Learning Outcomes

Upon completion of Directional Derivatives and the Gradient Vector, you should be able to:

- Find the directional derivative of a function at a given point and in a specified direction.
- Calculate the gradient of a given function at a given point and explain its significance.
- Determine the maximum rate of increase or decrease for a given function at a point and the direction in which it occurs.
- Use the gradient to determine equations for the tangent plane and normal line to a level surface at a given point.

Readings

Read section 14.6, pages 957–966 in your textbook. Carefully study the examples worked out in the text.

Overview

Imagine a number of people standing on the side of a mountain. Clearly, the steepness will vary from one position to another. Now imagine yourself standing in some chosen spot. You are going to start climbing; the direction you choose will determine whether you go up or down, and how steep your path will be.

Now apply this example to a surface $z = f(x, y)$. The rate of change will depend upon the position of the fixed point $P(x_0, y_0, f(x_0, y_0))$ and the direction of travel. This direction of travel can be expressed by specifying a vector $\vec{u} = < a, b >$ in the xy-plane. (See Figure 3 on page 957 of the text.) Since we want the rate of change to depend only on the direction $\vec{u}$ and not on its magnitude, we must always use a unit vector in the given direction we want to go.

We define the **directional derivative of** f at (x_0, y_0) in the direction of a unit vector $\vec{u} = < a, b >$ to be

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

There are several standard notations for the directional derivative. You may also encounter $\frac{\partial f}{\partial \vec{u}}(\vec{x}_0)$.

Notice that if we take the direction $\vec{i} = <1,0>$ along the positive x-axis we get the familiar definition of the partial derivative $f_x(x_0, y_0)$.

Correspondingly, if we take the direction $\vec{j} = <0,1>$ along the positive y-axis we get the familiar definition of the partial derivative $f_y(x_0, y_0)$.

Obviously the limit calculation for the directional derivative is in general a tough and tedious calculation. Fortunately there is a nice formula we can use for "nice" functions.

The result is: If f is a differentiable function, then f has a directional directive in every direction and

$$D_{\vec{u}} f(x_0, y_0) = a\ f_x(x, y) + b\ f_y(x, y)$$

Where $\vec{u} = <a, b>$ is a unit vector in the desired direction.

Notice that that this formula can be expressed in terms of a dot product of vectors.

Let $\nabla f(x,y) = \left\langle f_x(x,y),\ f_y(x,y) \right\rangle = f_x(x,y)\vec{i} + f_y(x,y)\vec{j}$.

Then $D_{\vec{u}} f(x_0, y_0) = \left\langle f_x(x,y),\ f_y(x,y) \right\rangle \cdot <a,b> = \nabla f(x,y) \cdot \vec{u}$

The vector $\nabla f(x, y)$ is called the **gradient vector** of f .

Recall from earlier work that the cosine of the angle between two vectors is

$$\cos(\theta) = \frac{\vec{x} \cdot \vec{y}}{|\vec{x}|\,|\vec{y}|}$$

From this formula we see that

$$\vec{x} \cdot \vec{y} = |\vec{x}|\,|\vec{y}| \cos(\theta)$$

Since,

$$-1 \le \cos(\theta) \le 1$$

We have that

$$-\left|\nabla f(x_0, y_0)\right| \cdot |\vec{u}| \le \nabla f(x_0, y_0) \cdot \vec{u} \le \left|\nabla f(x_0, y_0)\right| \cdot |\vec{u}|$$

But, $\vec{u}$ is a unit vector and so

$$|\vec{u}| = 1.$$

Hence,

$$-\left|\nabla f(x_0, y_0)\right| \le \nabla f(x_0, y_0) \cdot \vec{u} \le \left|\nabla f(x_0, y_0)\right|$$

That is,

$$-\left|\nabla f(x_0, y_0)\right| \le D_{\vec{u}}\, f(x_0, y_0) \le \left|\nabla f(x_0, y_0)\right|$$

Observe that the maximum value of the directional derivative is equal to the magnitude of the gradient at (x_0, y_0), $\left|\nabla f(x_0, y_0)\right|$, and that this occurs when $\cos(\theta) = 1$, i.e., when $\theta = 0$ and both $\vec{u}$ and $\nabla f(x_0, y_0)$ are in the same direction. The minimum value is - $\left|\nabla f(x_0, y_0)\right|$ and this occurs when $\vec{u}$ is in the opposite direction to $\nabla f(x_0, y_0)$. The directional derivative is zero when $\vec{u}$ is at right angles to the gradient of the function.

The directional derivative, of course, gives the rate of increase or decrease of the function. The maximum rate of increase of a differentiable function at a point (x_0, y_0) is in the direction of $\nabla f(x_0, y_0)$; the maximum rate of decrease is in the direction -$\nabla f(x_0, y_0)$. A function f is instantaneously constant in the direction at right angles to the gradient.

Applications of the above will come out in the examples and problems, but you might consider the following:

Picture an area of the countryside described by the function $z = f(x, y)$ where z is the altitude above the xy-plane. Beginning at any point $P(x_0, y_0, z_0)$ and proceeding in the instantaneous direction $\vec{u}$ everywhere perpendicular to the gradient of f, a path can be plotted along which z is a constant. This might be called a level curve. On a geodetic map this is projected on to the xy-plane and is called a <u>contour curve</u>. If a series of these curves spaced at regular intervals are plotted, they provide a good representation of the three-dimensional surface. Where the curves are close together, the gradient, i.e., the slope, is steep; where they are far apart, the terrain is flatter. The gradient vector at any point is perpendicular to the adjacent contour line, and its magnitude is inversely proportional to the spacing.

While our focus is on functions of two variables, similar definitions of the directional derivative and gradient vector apply to functions of three or more variables.

In the case of a level surface defined by $F(x, y, z) = k$, the gradient vector $\nabla f(x_0, y_0, z_0) = \left\langle f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0)\right\rangle$ is perpendicular to this level surface at the point $P(x_0, y_0, z_0)$ and so is a normal vector to the tangent plane of the surface at that point. It is also a direction vector for the normal line to the surface at P. We can use the gradient vector in this situation to help us determine equations for the tangent plane and normal line at a point on the given surface. Study Example 8 on page 965 of the text.

Example Exercises

1. Find the directional derivative of $f(x, y) = x^2 + 2xy + y^3$ at the point $(1, 3)$ and in the direction (a) <2,-1> and (b) towards the point $(-1, 6)$.

   a) We need a unit vector for our direction.

   $$\vec{u} = \frac{1}{|<2,-1>|} \cdot <2,-1> = \frac{1}{\sqrt{5}} <2,-1>$$

   Now, $f_x(x, y) = 2x + 2y$ and $f_y(x, y) = 2x + 3y^2$

   So,

   $$\nabla f(1,3) = \left\langle f_x(1,3), f_y(1,3)\right\rangle = \left\langle 2(1) + 2(3), 2(1) + 3(3)^2\right\rangle = <8, 29>$$

   Hence,

   $$D_{\vec{u}} f(1,3) = \nabla f(1,3) \cdot \frac{1}{\sqrt{5}} <2,-1>$$

   $$= \frac{1}{\sqrt{5}} <8,29> \cdot <2,-1> = \frac{1}{\sqrt{5}} \cdot (16-29) = -\frac{13}{\sqrt{5}}$$

   b) The vector from (1,3) to (-1,6) is <-1-1,6-3>=<-2,3>

   Again, we need a unit vector.

   $$\vec{u} = \frac{1}{|<-2,3>|} \cdot <-2,3> = \frac{1}{\sqrt{13}} <-2,3>$$

Now,

$$D_{\vec{u}} f(1,3) = \nabla f(1,3) \cdot \frac{1}{\sqrt{13}} < -2, 3 >$$

$$= \frac{1}{\sqrt{13}} < 8,29 > \cdot < -2, 3 > = \frac{1}{\sqrt{13}} \cdot (-16 + 87) = \frac{71}{\sqrt{13}}$$

2. For the function $f(x, y) = x^2 + 2xy + y^3$, find the direction and rate of maximum increase and decrease at the point $(1, 3)$.

Using the calculation from question #1, $\nabla f(1,3) = < 8, 29 >$.

The rate of maximum increase is

$|\nabla f(1,3)| = |< 8, 29 >| = \sqrt{8^2 + 29^2} = \sqrt{905}$.

The direction of maximum increase is $\nabla f(1,3) = < 8, 29 >$.

The rate of maximum decrease is

$$-|\nabla f(1,3)| = -|< 8, 29 >| = -\sqrt{8^2 + 29^2} = -\sqrt{905}.$$

The direction of maximum decrease is $-\nabla f(1,3) = - < 8, 29 >$.

3. Find equations of the tangent plane and the normal line to $x - z = 4\arctan(yz)$ at $(1 + \pi, 1, 1)$.

Let $F(x, y, z) = x - z - 4\arctan(yz)$. Then $x - z = 4\arctan(yz)$ is the level surface of $F(x, y, z) = 0$ and $\nabla F(1 + \pi, 1, 1)$ is a normal vector to the tangent plane at $(1 + \pi, 1, 1)$.

Now, $F_x(x, y, z) = 1$, $F_y(x, y, z) = -\dfrac{4z}{1 + y^2 z^2}$ and

$F_z(x, y, z) = -1 - \dfrac{4y}{1 + y^2 z^2}$

So,

$$\nabla F(1+\pi,1,1) = \left\langle F_x(1+\pi,1,1),\, F_y(1+\pi,1,1),\, F_z(1+\pi,1,1) \right\rangle$$

$$= \left\langle 1, -\frac{4(1)}{1+(1)^2(1)^2}, -1-\frac{4(1)}{1+(1)^2(1)^2} \right\rangle = \langle 1,-2,-3 \rangle$$

An equation for the tangent plane at $(1+\pi,1,1)$ is:

$$1(x-(1+\pi)) + (-2)(y-1) + (-3)(z-1) = 0$$

or $x - 2y - 3z = \pi - 4$

$\nabla F(1+\pi,1,1) = < 1,-2,-3 >$ is a direction vector for the normal line through $(1+\pi,1,1)$.

So, parametric equations for this line are:

$$x = 1+\pi+t$$
$$y = 1-2t$$
$$z = 1-3t$$

Practice Exercises 14.6

From the text pages 967–969, do problems 1, 5, 9, 11, 17, 19, 23, 29, 33, 35, 41, 45, 55, and 61.

Remember: You are to attempt all of the problems carefully before checking your solutions against those given in the solutions manual.

Maximum and Minimum Values

Learning Outcomes

Upon completion of Maximum and Minimum Values, you should be able to:

- Find and classify all the critical points of a function of two variables using the Second Derivative Test.
- Use a contour map to determine the location of critical points of a function of two variables and classify these points.
- Write down what it means for a set to be closed and bounded.
- Write down the statement of the Extreme Value Theorem for a function of two variables.
- Find the absolute maximum and minimum of a function of two variables on a closed and bounded set.
- Solve optimization problems by finding and classifying critical points.

Readings

Read section 14.7, pages 970–977 in your textbook. Carefully study the examples worked out in the text.

Overview

In elementary calculus, one of the first applications of differentiation was to locate relative maxima and minima of a function of one variable. Many of the ideas you used earlier have their analogue in functions of two variables.

Read the definitions on page 970 of your text for relative maximum and minimum, and for absolute maximum and minimum. Note that Stewart uses the words <u>local maximum</u> and <u>local minimum</u> for relative extreme values.

It should be intuitively clear to you that if a function of two variables (which is <u>differentiable</u> on an open disk containing (x_0, y_0)) has a local extremum at (x_0, y_0), then it must have a horizontal tangent plane at that point.

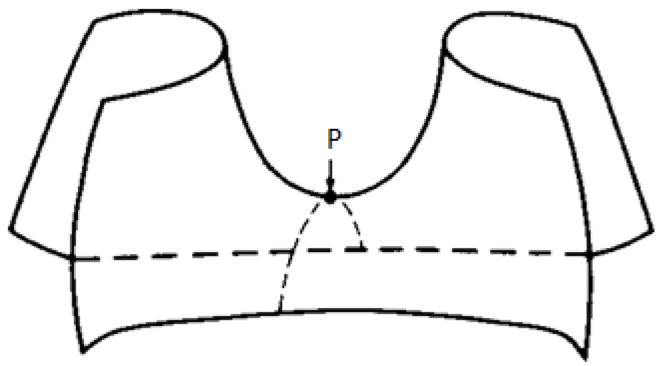
Since $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$ represent the slopes of the tangent lines to the "slice" curves, it must be true that these slopes are zero; that is,

$$f_x(x_0, y_0) = 0 \text{ and } f_y(x_0, y_0) = 0$$

We can summarize this by writing

$\nabla f(x_0, y_0) = <0,0>$, if f has an extremum at (x_0, y_0)

You should note that, as for functions of one variable, if f is differentiable and if $\nabla f(x_0, y_0) = <0,0>$, then (x_0, y_0) is a critical point. Further tests are needed to determine whether (x_0, y_0) yields a local maximum, (that is, $f(x_0, y_0)$ is a local maximum value) or yields a local minimum, or neither. The sketch below illustrates a critical point that yields neither a local maximum nor a local minimum. It is called a saddle point. You should recognize the surface as the hyperbolic paraboloid.



Now a test must be developed (the equivalent of the second derivative test in elementary calculus) which will help you to determine the nature of the critical point at (x_0, y_0). This test is described in Box 3 on page 971 of the text. It is aptly called the Second Derivative Test.

Note:

1. The discriminant

$$D(x_0, y_0) = f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - \left[f_{xy}(x_0, y_0)\right]^2$$

is easily remembered with the aid of determinant notation:

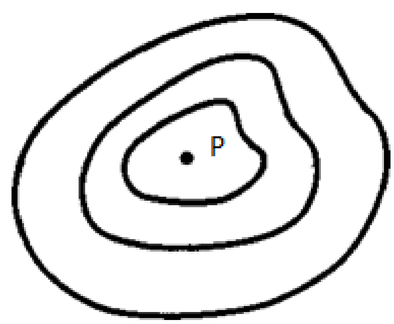
$$D = \begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix} = f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - \left[f_{xy}(x_0, y_0)\right]^2$$

2. If $D = 0$ the test is inconclusive. Further analysis is needed. Try checking the graph or a contour map. Maybe a geometrical or algebraic argument is applicable.

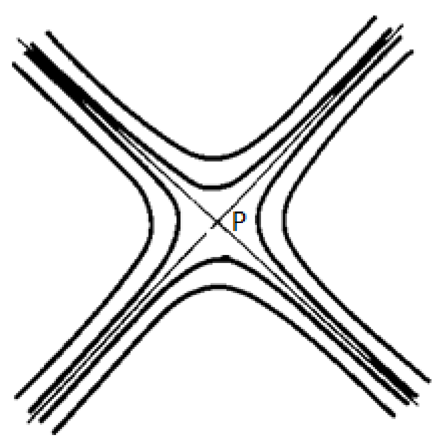
3. If $f_{xx}(x_0, y_0)$ and $f_{yy}(x_0, y_0)$ have opposite signs, then $D < 0$ and there is a saddle point at (x_0, y_0).

4. In the case a local extreme value exists, the critical point (x_0, y_0) is not the local extreme value. The critical point <u>yields</u> or <u>gives rise to</u> the local extreme value. The local extreme value is the functional value $f(x_0, y_0)$.

5. If f has a relative maximum or a relative minimum at the point P, then the level curves of f will look like the contour curves on a map that show a peak or a low lake.



6. If f has a saddle point at P, then the level curves will look like this:



What can we say about absolute extreme values?

Recall the Extreme Value Theorem for a single variable function:

"If f is continuous on [a,b], then f has an absolute maximum and absolute minimum value on that interval."

This result generalizes for functions of two variables and beyond.

For functions of two variables the closed interval gets replaced by the concept of a closed and bounded set which is discussed on page 975 of the text.

The Extreme Value Theorem for a function of two variables now states:

"If f is continuous on a closed and bounded set in the xy-plane, then f has an absolute maximum and absolute minimum value on that set."

The procedure for finding these extreme values is an extension of the method used for single variable functions. It goes as follows:

To find the absolute max and min of a continuous function on the closed and bounded set D:

- Find the values of f at the critical points of D.
- Find the extreme values of f on the boundary of D.

The largest number from the lists in steps 1 and 2 is the max; the smallest of these values is the min.

Example Exercises

1. Find and classify all critical points of the function

$$f(x, y) = x^3 + 2y^3 - 3x^2 + 3y^2 + 4$$

Solve: $\nabla f(x, y) = < 0,0 >$

Here, $f_x(x, y) = 3x^2 - 6x$ and $f_y(x, y) = 6y^2 + 6y$

The system of equations we are solving is:

$3x^2 - 6x = 0$ ………(1)

$6y^2 + 6y = 0$ ………(2)

(1): $3x^2 - 6x = 0 \quad \Rightarrow \quad 3x(x-2) = 0 \quad \Rightarrow \quad x = 0$ or $x = 2$

(2): $6y^2 + 6y = 0 \quad \Rightarrow \quad 6y(y+1) = 0 \quad \Rightarrow \quad y = 0$ or $y = -1$

The critical points are: (0,0), (2,0), (0,-1), (2,-1)

Now we classify the critical points.

$f_{xx}(x, y) = 6x - 6$, $f_{yy}(x, y) = 12y + 6$, $f_{xy}(x, y) = 0$

So,

$$D = \begin{vmatrix} 6x-6 & 0 \\ 0 & 12y+6 \end{vmatrix} = (6x-6)(12y+6)$$

For convenience the results are tabulated.

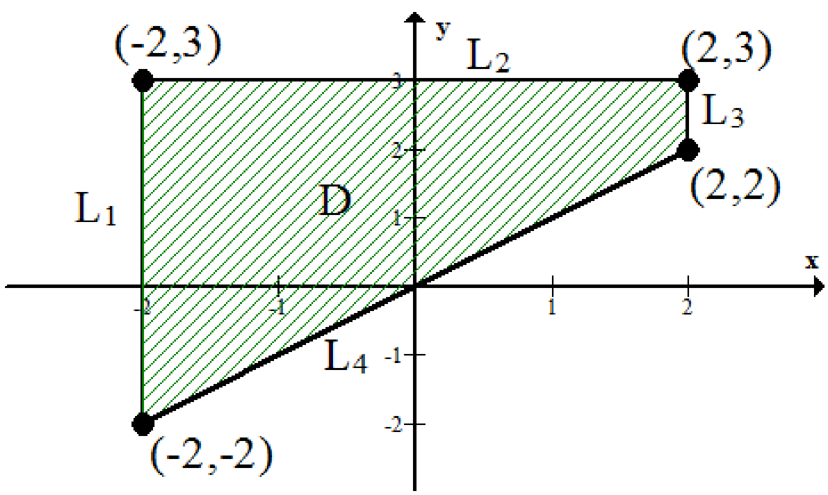
Point	$D(x,y)$	$f_{xx}(x,y)$	**Conclusion**
(0,0)	–36<0		Saddle point at (0,0)
(2,0)	36>0	6>0	Local minimum at (2,0) • Local min = $f(2,0)=0$
(0,-1)	36>0	–6<0	Local maximum at (0,-1) • Local max = $f(0,-1)=5$
(2,-1)	–36<0		Saddle point at (2,-1)

2. This is problem 36 from your text on page 978.

Find the maximum and minimum values of $f(x,y) = x^3 - 3x - y^3 + 12y$ on the set D which is the quadrilateral with vertices (-2,3), (2,3), (2,2) and (-2,-2).

A picture of the region D is given below.

Notice that D is a closed and bounded set, so by the Extreme Value Theorem f has an absolute maximum and minimum on this set.



We start by looking for critical points of f on the interior of D.

$f_x(x,y) = 3x^2 - 3$ and $f_y(x,y) = -3y^2 + 12$

Solve: $\nabla f(x, y) = <0,0>$

This gives the following system of equations:

$3x^2 - 3 = 0$ ....(1)

$-3y^2 + 12 = 0$ ....(2)

(1): $3x^2 - 3 = 0 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1$

(2): $-3y^2 + 12 = 0 \quad \Rightarrow \quad y^2 = 4 \quad \Rightarrow \quad y = \pm 2$

There are four critical points (-1,-2), (-1,2), (1,-2) and (1,2), but only (-1,2) and (1,2) are in D.

We now look for critical points on the boundary of D.

Along L_1: $x = -2$ and $g(y) = f(-2, y) = -2 - y^3 + 12y, \quad -2 \le y \le 3$
$g'(y) = -3y^2 + 12 = 0 \quad \Rightarrow \quad y^2 = 4 \quad \Rightarrow y = \pm 2$

The candidates for extrema of g are the critical numbers of g and the endpoints of the interval, so the candidates for extrema of f are

(-2,-2), (-2,2) and (-2,3).

Along L_2: $y = 3$ and $h(x) = f(x,3) = x^3 - 3x + 9, \quad -2 \le x \le 2$
$h'(x) = 3x^2 - 3 = 0 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow x = \pm 1$

So the candidates for extrema of f are (-2,3), (-1,3), (1,3) and (2,3).

Along L_3: $x = 2$ and $k(y) = f(2, y) = 2 - y^3 + 12y, \quad 2 \le y \le 3$

$k(y)$ is similar to $g(y)$ along L_1.

The candidates for extrema of f are (2,2) and (2,3).

Along L_4: $y = x$ and $m(x) = f(x, x) = 9x, \quad -2 \le x \le 2$

So the candidates for extrema of f are (-2,-2) and (2,2).

Putting all the candidates together into a function table, we have

(x,y)	***f(x,y)***
(-1,2)	18
(1,2)	14
(-2,-2)	-18
(-2,2)	14
(-2,3)	7
(-1,3)	11
(1.3)	7
(2,3)	11
(2,2)	18

Conclusion:

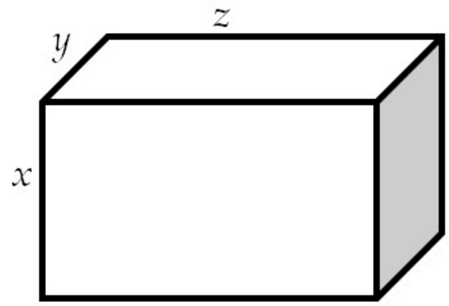
1. f has a maximum value of 18. It occurs at the points (-1,2) and (2,2).

2. f has a minimum value of -18. It occurs at the point (-2,-2).

Important Note: We were able to determine the candidates on the boundary of D by incorporating the boundary equation of the curve into our multivariable function, thereby reducing it to a function of one variable. You cannot always expect to be able to do this. We will discuss in the next section a method which will help us to deal more effectively with the boundary of a closed set. This is called the "Method of Lagrange Multipliers."

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

3. A manufacturer wants to make boxes without lids from two kinds of cardboard. The cardboard for the bottom of the box is three times as thick and as expensive as the cardboard for the sides. Find the dimensions of the most economical box of 4 cubic feet capacity that can be made.



Let C denote the cost of the box. Arbitrarily assigning a cost of \$1 per square foot for the cardboard in the sides of the box, the cost of the cardboard for the base is \$3 per square foot. The cost function is then:

$$C(x, y, z) = 3xy + 2xz + 2yz$$

There is a constraint on the volume of this box of 4 ft^3.

So, Volume = $4\text{ ft}^3 \Rightarrow xyz = 4$. Solving for z and incorporating this condition into our cost function, we have

$$z = \frac{4}{xy}$$

and so

$$C(x, y) = 3xy + 2x \cdot \frac{4}{xy} + 2y \cdot \frac{4}{xy} = 3xy + \frac{8}{y} + \frac{8}{x}$$

The problem is now to minimize $C(x, y) = 3xy + \frac{8}{y} + \frac{8}{x}, \quad x, y > 0$.

$C_x(x, y) = 3y - \frac{8}{x^2}; \; C_y(x, y) = 3x - \frac{8}{y^2}$

Solve: $\nabla C(x, y) = < 0,0 >$

This gives the following system of equations:

$$3y - \frac{8}{x^2} = 0 \ldots\ldots..(1)$$

$$3x - \frac{8}{y^2} = 0 \ldots\ldots...(2)$$

(1): $3y = \frac{8}{x^2} \quad \Rightarrow \quad 3yx^2 = 8 \quad \Rightarrow y = \frac{8}{3x^2}$

(2): $3x = \frac{8}{\left(\frac{8}{3x^2}\right)^2} \quad \Rightarrow \quad 3x = \frac{8}{\left(\frac{64}{9x^4}\right)} \quad \Rightarrow 3x = \frac{9x^4}{8} \quad \Rightarrow \quad 24x - 9x^4 = 0$

$$24x - 9x^4 = 0 \quad \Rightarrow \quad 3x\left(8 - 3x^3\right) = 0 \quad \Rightarrow \quad x = 0 \text{ or } x = \sqrt[3]{\frac{8}{3}} = \frac{2}{\sqrt[3]{3}}$$

We discard $x = 0$. It is an extraneous root, i.e., it does not satisfy equation 1. It is also not in the domain of our cost function.

$$x = \frac{2}{\sqrt[3]{3}} \quad \Rightarrow \quad y = \frac{8}{3\left(\frac{2}{\sqrt[3]{3}}\right)^2} = \frac{8}{\left(\frac{12}{\sqrt[3]{9}}\right)} = \frac{2\sqrt[3]{9}}{3} = \frac{2(3)^{2/3}}{3} = \frac{2}{\sqrt[3]{3}}$$

There is only one critical point. It must give rise to a minimum value for cost.

When $x = \frac{2}{\sqrt[3]{3}}$ and $y = \frac{2}{\sqrt[3]{3}}$, $z = \frac{4}{xy} = \frac{4}{\left(\frac{2}{\sqrt[3]{3}}\right) \cdot \left(\frac{2}{\sqrt[3]{3}}\right)} = \sqrt[3]{9}$

Conclusion:

The most economical box has dimensions: $\frac{2}{\sqrt[3]{3}}$ ft by $\frac{2}{\sqrt[3]{3}}$ ft by $\sqrt[3]{9}$ ft. In practical terms, this is approximately 1.39 ft by 1.39 ft by 2.08 ft.

Practice Exercises 14.7

From the text pages 977–979, do problems 3, 7, 13, 17, 29, 31, 35, 39, 41, 45, 47, 49, and 51.

Remember: You are to attempt all of the problems carefully before checking your solutions against those given in the solutions manual.

Lagrange Multipliers

Learning Outcomes

Upon completion of Lagrange Multipliers, you should be able to:

- Estimate constrained optimal values from a contour map.
- Properly set up a constrained optimization problem, including the Lagrange system of equations to be solved.
- Find maximum and/or minimum values of functions of two or three variables, subject to a given constraint.

Readings

Read section 14.8, pages 981–986 in your textbook. Carefully study the examples worked out in the text.

Overview

In the previous section you learned how to find the extreme and relative extreme values of a function of two variables. The method can be extended to more than two, but it requires matrix methods beyond the scope of this course.

However, there are situations in which a function of more than two variables can be reduced to a function of two, at least implicitly. This will be the case, for example, where you wish to find the extreme values of $w = F(x, y, z)$ where a condition, that is, a constraint $G(x, y, z) = 0$ is imposed upon the degrees of freedom of the three variables. Generally speaking, the extreme values of the constrained problem will not be the same as the unconstrained problem; that is, optimizing $w = F(x, y, z)$ considered alone.

This condition of constraint implies that there are effectively only two **independent** variables. You are free to choose which they will be. Following the previous pattern of choosing z as the variable implicitly dependent upon x and y, we set $z = f(x, y)$.
If you are actually able to solve $G(x, y, z) = 0$ for z as a function of x and y, you could then substitute it into $w = F(x, y, z)$, which would then become $w = F(x, y, f(x, y))$. Since you are now dealing with a problem in two variables, this can be handled by the methods of the previous section.

This was the case of the cost problem involving a cardboard box solved in the previous section.

In some cases this may not be practicable, in many cases not even possible. Fortunately we can have recourse to another method, called the Method of Lagrange Multipliers.

This method is based upon the observation by the mathematician Joseph-Louis Lagrange that if you look at the level curves of a function of two variables, $z = f(x, y)$, and the constraint equation in two variables, $g(x, y) = 0$, you may notice that at the points where the constrained extremum exists, the constraint curve is tangent to a level curve of the function; that is, the constraint curve and the level curve of the function share a common tangent line at their point of intersection. (See Figure 1 on page 981 of the text.)

Extending to the constrained optimization problem in three independent variables, the observation is that where the constrained extremum exists, the constraint surface, $G(x, y, z) = 0$, is tangent to a level surface of the function $w = F(x, y, z)$, meaning they share a common tangent plane at their intersection point.

For the problem in three independent variables, this observation translates to solving the system of equations given by the statements:

- $\nabla F(x,y,z) = \lambda \cdot G(x,y,z)$
- and $G(x,y,z) = 0$

Here λ is a scalar called the "Lagrange multiplier".

The system of equations can be written

$$
\begin{aligned}
&F_x(x, y, z) = \lambda \cdot G_x(x, y, z) \\
&F_y(x, y, z) = \lambda \cdot G_y(x, y, z) \\
&F_z(x, y, z) = \lambda \cdot G_z(x, y, z) \\
&G(x, y, z) = 0
\end{aligned}
$$

This is a system of 4 equations in 4 unknowns x, y, z and λ. We are normally interested only in the candidate point (x, y, z) and so it is not necessary to solve for λ.

While Stewart does discuss the constrained optimization problem with two constraints on pages 985–986, we will focus only on one constraint.

Important Notes:

1. When solving a constrained optimization problem using this method you must start by setting up the problem. Identify the function to be optimized (maximized, minimized or both) and the constraint equation. A good way to start is to write

   Optimize: $F(x, y, z) = ...$

   Subject to: $G(x, y, z) = ...$

   Then you must write down the entire system of equations you are going to solve <u>before</u> you start the process of solving the system!

   Finish up the problem with a concluding statement. The format of your solution is all important here.

2. The Method of Lagrange Multipliers is based upon two very important assumptions. First, we assume that an optimal value(s) exists for our function, and second, that $\nabla G \neq \vec{0}$ along the constraint curve.
   Once you solve the Lagrange system of equations, all you have are candidate points that <u>could</u> give rise to an optimal value for your function. Once you know that your function does indeed have a constrained maximum and/or minimum, then the largest functional value coming from the list of candidate points is the constrained maximum and/or the smallest functional value is the constrained minimum.

Example Exercises

1. Find the maximum of the function $f(x, y) = x + 2y$ subject to the constraint $x^2 + y^2 = 5$.

   We start by setting up the problem.

   Maximize: $f(x, y) = x + 2y$

   Subject to: $g(x, y) = x^2 + y^2 = 5$

   Now, $\nabla f(x, y) =< 1, 2 >$ and $\nabla g(x, y) =< 2x, 2y >$

   So taking $\nabla f(x, y) = \lambda \cdot \nabla g(x, y)$

   We obtain

   $$< 1,2 >= \lambda \cdot < 2x, 2y >=< 2\lambda x, 2\lambda y >$$

The Lagrange system of equations is

$$2\lambda x = 1$$
$$2\lambda y = 2$$
$$x^2 + y^2 = 5$$

Since we know from our constraint that $x \neq 0$ and $y \neq 0$, we have $\lambda = \frac{1}{2x}$ and $\lambda = \frac{2}{2y} = \frac{1}{y}$

Equating we get $\frac{1}{2x} = \frac{1}{y} \quad \Rightarrow \quad y = 2x$

Substituting this expression into the constraint equation, we get

$$x^2 + y^2 = 5 \;\Rightarrow\; x^2 + (2x)^2 = 5 \;\Rightarrow\; 5x^2 = 5 \;\Rightarrow\; x^2 = 1 \;\Rightarrow\; x = \pm 1$$

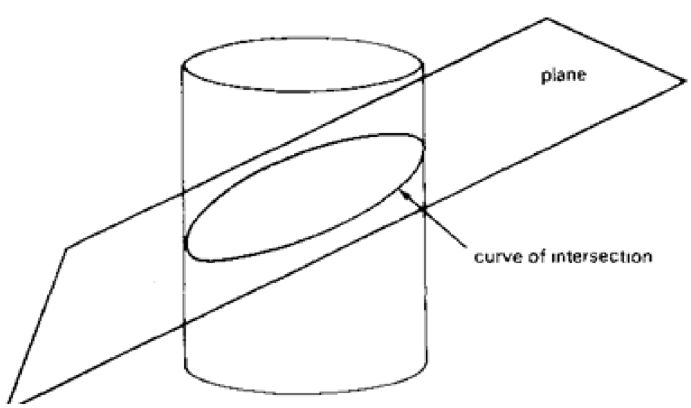
When $x = 1, \; y = 2 \cdot 1 = 2$; so, (1,2)

When $x = -1, \; y = 2 \cdot (-1) = -2$; so, (-1,-2)

Candidate points: (1,2), (-1.-2)

Since $f(1,2) = 1 + 2(2) = 5$ and $f(-1,-2) = -1 + 2(-2) = -5$, the maximum value of f subject to the given constraint is 5, assuming the maximum exists.

A geometrical look at this problem is helpful. $z = f(x, y) = x + 2y$ is a plane passing through the origin. The constraint $x^2 + y^2 = 5$ is a right circular cylinder. The maximum value of $f(x, y) = x + 2y$ exists and occurs at the highest point on the curve of intersection of the cylinder and the plane (see picture below).



2. Find the point on the curve $5x^2 + 6xy + 5y^2 = 4$ that is closest to the origin. This does not sound like a Lagrange multiplier problem, but it is. We can reword it as follows:

Minimize $d = \sqrt{(x-0)^2 + (y-0)^2} = \sqrt{x^2 + y^2}$

Subject to $5x^2 + 6xy + 5y^2 = 4$

The problem can be simplified slightly by noting that it is sufficient to minimize d^2, since $d > 0$. Therefore, you can take

$f(x, y) = d^2 = x^2 + y^2$ and the problem becomes

Minimize $f(x, y) = x^2 + y^2$

Subject to $g(x, y) = 5x^2 + 6xy + 5y^2 = 4$

Now, $\nabla f(x, y) = < 2x, 2y >$ and $\nabla g(x, y) = < 10x + 6y, 6x + 10y >$

So taking $\nabla f(x, y) = \lambda \cdot \nabla g(x, y)$

We obtain $< 2x, 2y > = \lambda \cdot < 10x + 6y, 6x + 10y > = < \lambda(10x + 6y), \lambda(6x + 10y) >$

The Lagrange system of equations is:

$$2x = \lambda(10x + 6y)$$
$$2y = \lambda(6x + 10y)$$
$$5x^2 + 6xy + 5y^2 = 4$$

Solving the first two equations for λ we have:

$$\lambda = \frac{2x}{10x + 6y} = \frac{x}{5x + 3y}, \quad 5x + 3y \neq 0 \text{ and}$$

$$\lambda = \frac{2y}{6x + 10y} = \frac{y}{3x + 5y}, \quad 3x + 5y \neq 0$$

What happens if $5x + 3y = 0$ or $3x + 5y = 0$? We will come back to this at the end of the problem.

Equating, we have

$$\frac{x}{5x + 3y} = \frac{y}{3x + 5y} \quad \Rightarrow \quad 3x^2 + 5xy = 5xy + 3y^2 \quad \Rightarrow \quad x^2 = y^2 \quad \Rightarrow \quad x = \pm y$$

Replacing y by x in the constraint equation, we have

$$5x^2 + 6x \cdot x + 5x^2 = 4 \quad \Rightarrow \quad 16x^2 = 4 \quad \Rightarrow \quad x^2 = \frac{1}{4} \quad \Rightarrow \quad x = \pm\frac{1}{2}$$

Now since $y = x$ we have the points $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{2}\right)$

Using $y = -x$ in the constraint equation we have

$$5x^2 + 6x \cdot (-x) + 5x^2 = 4 \quad \Rightarrow \quad 4x^2 = 4 \quad \Rightarrow \quad x^2 = 1 \quad \Rightarrow \quad x = \pm 1$$

Now since $y = -x$ we have the points $(1, -1)$ and $(-1, 1)$.

Candidate points: $\left(\frac{1}{2}, \frac{1}{2}\right)$, $\left(-\frac{1}{2}, -\frac{1}{2}\right)$, $(1, -1)$ and $(-1, 1)$

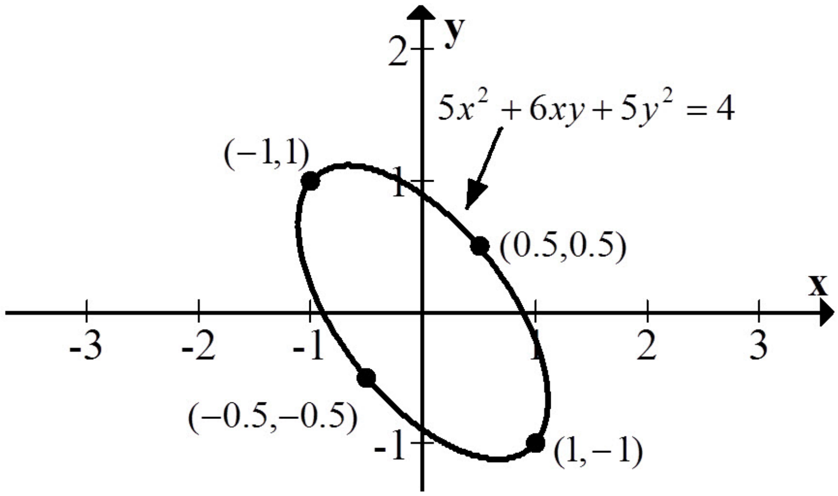
Putting all the candidates together into a table, we have

(x,y)	*f(x,y)*
$\left(\frac{1}{2}, \frac{1}{2}\right)$	$\frac{1}{2}$
$\left(-\frac{1}{2}, -\frac{1}{2}\right)$	$\frac{1}{2}$
$(1, -1)$	2
$(-1, 1)$	2

Conclusion:

Obviously there should be a solution to this problem. From the table we see that the minimum value for the distance squared is

$\frac{1}{2}$. It occurs at two points, $\left(\frac{1}{2}, \frac{1}{2}\right)$ and $\left(-\frac{1}{2}, -\frac{1}{2}\right)$. The minimum distance is $\sqrt{\frac{1}{2}} = \frac{1}{\sqrt{2}}$ or $\frac{\sqrt{2}}{2}$ .The graph of the constraint curve is given below. Notice the symmetry of this graph about the lines $y = \pm x$. Since the set of points defining this curve is a closed set and our distance function is continuous on this curve, the Extreme Value Theorem guarantees that there is be a maximum and minimum value for the distance function. We can see from the graph that the maximum distance from the origin occurs at the points $(1, -1)$ and $(-1, 1)$.



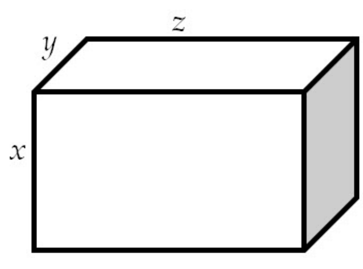
Note:

In solving for λ in the first two equations of the Lagrange system, we assumed that $3x+5y\neq 0$ and $5x+3y\neq 0$.If $5x+3y=0$, then $x=0$ since $2x=\lambda(10x+6y) \quad\Rightarrow\quad x=\lambda(5x+3y)$. It now follows that $3\lambda y=0 \quad\Rightarrow\quad \lambda=0$ or $y=0$. Since $(0,0)$ does not satisfy the constraint equation $5x^2+6xy+5y^2=4$ and $\lambda=0$ or $y=0$ both yield the point $(0,0)$, we were correct to assume that $5x+3y\neq 0$. Similarly, we were correct to assume that $3x+5y\neq 0$. You should check the details.

3. A manufacturer wants to make boxes without lids from two kinds of cardboard. The cardboard for the bottom of the box is three times as thick and as expensive as the cardboard for the sides. Find the dimensions of the most economical box of 4 cubic feet capacity that can be made.

This problem was solved in the previous section by incorporating the constraint on volume into the cost function thereby reducing it to an unconstrained optimization problem in two independent variables.

We will now solve this problem as a constrained optimization problem.



Let C denote the cost of the box. Arbitrarily assigning a cost of $1 per square foot for the cardboard in the sides of the box, the cost of the cardboard for the base is $3 per square foot. The cost function is then:

$$C(x, y, z) = 3xy + 2xz + 2yz$$

Let V denote the volume of the box.

The problem is

Minimize: $C(x, y, z) = 3xy + 2xz + 2yz$, $x, y, z > 0$

Subject to: $V(x, y, z) = xyz = 4$

Now, $\nabla C(x, y, z) =< 3y + 2z, 3x + 2z, 2x + 2y >$ and $\nabla V(x, y, z) =< yz, xz, xy >$

So taking $\nabla C(x, y) = \lambda \cdot \nabla V(x, y)$

We obtain $< 3y + 2z, 3x + 2z, 2x + 2y >= \lambda \cdot < yz, xz, xy >=< \lambda yz, \lambda xz, \lambda xy >$

The Lagrange system of equations is:

$$\begin{aligned} 3y + 2z &= \lambda yz \\ 3x + 2z &= \lambda xz \\ 2x + 2y &= \lambda xy \\ xyz &= 4 \end{aligned}$$

Using the symmetry in the first 3 equations we obtain

$$\begin{aligned} x(3y + 2z) &= \lambda xyz \\ y(3x + 2z) &= \lambda xyz \\ z(2x + 2y) &= \lambda xyz \end{aligned}$$

Equating equations 1 & 2:

$$\begin{aligned} x(3y + 2z) &= y(3x + 2z) \\ 3xy + 2xz &= 3xy + 2yz \\ xz &= yz \\ z(x - y) &= 0 \\ \Rightarrow \quad z &= 0 \text{ or } x = y \end{aligned}$$

Since $z > 0$, we conclude $x = y$.

Equating equations 2 and 3:

$$\begin{aligned} y(3x+2z) &= z(2x+2y) \\ 3xy+2yz &= 2xz+2yz \\ 3xy &= 2xz \\ x(3y-2z) &= 0 \end{aligned}$$

$$\Rightarrow \quad x=0 \text{ or } y=\frac{2}{3}z$$

Again, since $x>0$, we conclude $y=\frac{2}{3}z$.

Now substituting into the constraint equation we get

$$\begin{aligned} &xyz = 4 \\ &\frac{2z}{3}\cdot\frac{2z}{3}\cdot z = 4 \\ &4z^3 = 36 \\ &z^3 = 9 \\ &z = \sqrt[3]{9} \end{aligned}$$

Hence $x=y=\frac{2}{3}\cdot\sqrt[3]{9}=\frac{2\cdot 3^{\frac{2}{3}}}{3}=\frac{2}{\sqrt[3]{3}}$

There is only one candidate point. Since the problem must have a solution, the minimum cost must lie at $\left(\frac{2}{\sqrt[3]{3}},\frac{2}{\sqrt[3]{3}},\sqrt[3]{9}\right)$.

Conclusion:

The most economical box has dimensions: $\frac{2}{\sqrt[3]{3}}$ ft by $\frac{2}{\sqrt[3]{3}}$ ft by $\sqrt[3]{9}$ ft.

Practice Exercises 14.8

From the text pages 987–988, do problems 1, 3, 5, 9, 11, 21, 27, 29, 37, and 39.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Unit 3: Summary and Self-Test

You have now worked through Unit 3 in MATH 2111. It is time to take stock of what you have learned, review all of the material, and bring your shorthand notes up to date. A summary of the material covered so far is provided in the following pages. This summary should be modified, added to, and fleshed out to form a solid body of knowledge.

When you have completed your review, you should test your comprehension of the material with a closed book self-administered examination. Put all your notes aside, find a quiet place where you will not be disturbed, and take the examination provided at the end of this unit. You will find some questions straightforward and easy, but others will test your ingenuity.
You will find the solutions to the Unit 3 exam questions, and the point value for each question in the Answer Key provided at the end of this unit. Become your own examiner. If you have done well, according to your personal standards, go on to Unit 4. If not, then more review and practice is obviously called for.

Summary

Multivariable Functions

The **domain** D of the function $z = f(x, y)$ of two independent variables is given by $D = \{(x, y) \mid f(x, y) \text{ is defined}\}$. It is a subset of the xy-plane. The **range** of the function $z = f(x, y)$ is given by $\text{range } f = \{ f(x, y) \mid (x, y) \in D\}$. It is a subset of the real line $\Re$.

The **graph** of $z = f(x, y)$ is then defined by $\{ (x, y, f(x, y)) \mid (x, y) \in \mathrm{D} \}$. The graph of a function of two variables is called a **surface**. It lives in 3-space.

Level curves are curves that result from setting $f(x, y) = k, \quad k$ a constant. The intersection of the horizontal plane $z = k$ (at height k) with the surface produces a trace (curve) in that plane. If we project the level curves of our graph onto the $xy-$plane, we get what is called a **contour map**, a 2-dimensional picture of the shape of our surface at varying heights.

For a function of three independent variables its domain is a subset of 3-space. Its range is a subset of the real line R. The surface in 3-space produced by setting $f(x, y, z) = k, \quad k$ a constant, is called a **level surface**.

Limits

Intuitive Definition of Limit:

$\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$ means that as (x,y) approaches (a,b) along <u>any path that lies in domain</u> f, $f(x,y)$ approaches L.

NOTE: Omit formal definition of limit

1. In order that $\lim\limits_{(x,y)\to(a,b)} f(x,y)$ exist, $f(a,b)$ need not be defined.
2. All paths into (a,b) must lie in domain f.
3. If two different paths in domain f yield two different limit results, then the limit doesn't exist (DNE).
4. Showing the limit value is the same along many different paths does not prove that the limit exists and has that value.

<u>Squeeze Theorem</u>: If $g(x,y) \le f(x,y) \le k(x,y)$ holds for all (x,y) in some open disk centered at (a,b), except possibly at (a,b), and $\lim\limits_{(x,y)\to(a,b)} g(x,y) = \lim\limits_{(x,y)\to(a,b)} k(x,y) = L$, then $\lim\limits_{(x,y)\to(a,b)} f(x,y) = L$.

Continuity

$z = f(x,y)$ is **continuous** at (a,b) if $\lim\limits_{(x,y)\to(a,b)} f(x,y) = f(a,b)$.

If f is continuous at all points (a,b) in a set D, then f is **continuous on** D. Geometrically this means there are no holes, gaps or breaks in the surface over D.

Sums, differences, products and quotients of continuous functions are continuous on their domains.

Polynomials in two variables are continuous on the entire *xy*-plane.

Rational functions, i.e., quotients of polynomials, are continuous at all points in their domain.

Partial Derivatives

Partial derivative at a point:

$$f_x(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) - f(x_0, y_0)}{h}$$

$$f_y(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0, y_0 + h) - f(x_0, y_0)}{h}$$

Partial derivative functions:

$$f_x(x, y) = \lim_{h\to 0} \frac{f(x + h, y) - f(x, y)}{h}$$

$$f_y(x, y) = \lim_{h\to 0} \frac{f(x, y + h) - f(x, y)}{h}$$

Interpretations:

Geometric: $f_x(x_0, y_0)$ is the slope of the line tangent to the trace curve of f in the vertical plane $y = y_0$ at $(x_0, y_0, f(x_0, y_0))$.

Rate of Change: $f_x(x_0, y_0)$ is the instantaneous rate of change of $z = f(x, y)$ with respect to x at $(x_0, y_0, f(x_0, y_0))$, keeping y fixed at $y = y_0$.

Similar interpretations for $f_y(x_0, y_0)$

Notations: If $z = f(x, y)$, we write

$$f_x(x, y) = \frac{\partial f}{\partial x}(x, y) = \frac{\partial z}{\partial x} = D_x f(x, y)$$

$$f_y(x, y) = \frac{\partial f}{\partial y}(x, y) = \frac{\partial z}{\partial y} = D_y f(x, y)$$

Partial derivatives can be defined for functions of more than two variables.

In general, if $u = f(x_1, x_2, ..., x_n)$, its partial derivative with respect to x_i is

$$\frac{\partial u}{\partial x_i} = \lim_{h\to 0} \frac{f(x_1, x_2, ..., x_{i-1}, x_i + h, x_{i+1} .., x_n) - f(x_1, x_2, ..., x_i, .., x_n)}{h}$$

To calculate the partial derivative $\frac{\partial u}{\partial x_i}$, treat x_i as the variable and differentiate with respect to that variable, holding all other symbols constant.

Notations for second partial derivative functions of $z = f(x, y)$:

$$f_{xx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial x^2} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial x^2}$$

$$f_{yy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial y^2}$$

$$f_{xy} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right) = \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y}\left(\frac{\partial z}{\partial x}\right) = \frac{\partial^2 z}{\partial y \partial x}$$

$$f_{yx} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x}\left(\frac{\partial z}{\partial y}\right) = \frac{\partial^2 z}{\partial x \partial y}$$

Clairaut's Theorem: Let $z = f(x, y)$ be defined on a disk D that contains the point (a, b). If the second partial functions f_{xy} and f_{yx} are both continuous on D, then

$$f_{xy}(a,b) = f_{yx}(a,b)$$

Tangent Planes

If $z = f(x, y)$ has continuous partial derivatives, an equation for the tangent plane to the graph of f at the point $P(x_0, y_0, z_0)$ is

$$z - z_0 = f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

or
$$f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0) + (-1)(z - z_0) = 0$$

$$\vec{n} = \vec{v}_1 \times \vec{v}_2 = \begin{vmatrix} \vec{i} & \vec{j} & \overline{k} \\ 0 & 1 & f_y(x_0, y_0) \\ 1 & 0 & f_x(x_0, y_0) \end{vmatrix} = \left\langle f_x(x_0, y_0), f_y(x_0, y_0), -1 \right\rangle$$

is a normal vector to the tangent plane.

When approximating the function $z = f(x, y)$ by its tangent plane we let $z = L(x, y)$ and write the equation for the tangent plane in the form

$$L(x, y) = f(x_0, y_0) + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

This is called the **linearization of** f **at** (a, b).

For (x, y) near (x_0, y_0), $f(x, y) \approx L(x, y)$.

This is called the **tangent plane approximation of** f **at** (a, b).

Differentiability

$z = f(x, y)$ is **differentiable** at (a, b) if $\Delta z = f(x, y) - f(a, b)$ can be expressed in the form

$$\Delta z = f_x(a, b)\Delta x + f_y(a, b)\Delta y + \varepsilon_1 \Delta x + \varepsilon_2 \Delta y$$

Where $\varepsilon_1 \to 0$ and $\varepsilon_2 \to 0$ as $(\Delta x, \Delta y) \to (0,0)$.

This definition is extremely hard to check. Use the following sufficient condition for differentiability:

If the partial derivative functions f_x and f_y exist near (a, b) and are continuous at (a, b), then f is differentiable at (a, b).

As in the case of a single variable function, if $z = f(x, y)$ is differentiable at (a, b), then f is continuous at (a, b).

Differentials

For a function with two independent variables x and y, dx and dy can be assigned any real values. Then we define the **total differential** of the dependent variable z by

$$dz = f_x(x, y)\,dx + f_y(x, y)\,dy$$

For (x, y) near (x_0, y_0),

$$\Delta z \approx dz = f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$

Alternatively, we can write

$$f(x, y) \approx f(x_0, y_0) + f_x(x_0, y_0)\,dx + f_y(x_0, y_0)\,dy$$

Chain Rule

Chain Rule – Case 1:

If $z = f(x, y)$ is a differentiable function of x and y, where $x = g(t)$ and $y = h(t)$ are both differentiable functions of t, then z is a differentiable function of t and

$$\frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

Chain Rule - Case 2:

If $z = f(x, y)$ is a differentiable function of x and y, where $x = g(s, t)$ and $y = h(s, t)$ are both differentiable functions of s and t, then z is a differentiable function of s and t and

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial s} \text{ and } \frac{\partial z}{\partial t} = \frac{\partial z}{\partial x} \cdot \frac{\partial x}{\partial t} + \frac{\partial z}{\partial y} \cdot \frac{\partial y}{\partial t}$$

This rule can be generalized to functions of many variables.

Implicit Differentiation

Partial derivatives of single variable functions defined implicitly by $F(x, y) = 0$ can be calculated using the Implicit Function Theorem:

If $z = F(x, y)$ is differentiable and $y = f(x)$ is defined implicitly by $F(x, y) = 0$, then

$$\frac{dy}{dx} = -\frac{F_x(x, y)}{F_y(x, y)}$$

This theorem can be extended to calculate partial derivatives of multivariable functions.

If $w = F(x, y, z)$ is differentiable and $z = f(x, y)$ is defined implicitly by $F(x, y, z) = 0$, then

$$\frac{\partial z}{\partial x} = -\frac{F_x(x, y, z)}{F_z(x, y, z)} \text{ and } \frac{\partial z}{\partial y} = -\frac{F_y(x, y, z)}{F_z(x, y, z)}$$

Directional Derivatives and the Gradient Vector

The directional derivative of f at (x_0, y_0) in the direction of a unit vector $\vec{u} = \langle a, b \rangle$ is

$$D_{\vec{u}} f(x_0, y_0) = \lim_{h \to 0} \frac{f(x_0 + ha, y_0 + hb) - f(x_0, y_0)}{h}$$

if this limit exists.

The directional derivative of f at (x_0, y_0) in the direction of $\vec{i} = \langle 1, 0 \rangle$ is $f_x(x_0, y_0)$.

The directional derivative of f at (x_0, y_0) in the direction of $\vec{j} = \langle 0, 1 \rangle$ is $f_y(x_0, y_0)$.

Computational Result:

If f is a differentiable function, then f has a directional directive in every direction and

$$D_{\vec{u}} f(x_0, y_0) = a \cdot f_x(x, y) + b \cdot f_y(x, y)$$

Where $\vec{u} = < a, b >$ is a unit vector in the desired direction.

Given $z = f(x, y)$:

1. The **gradient vector** of f can be expressed as
   $\nabla f(x, y) = \left\langle f_x(x, y),\ f_y(x, y) \right\rangle$ (a vector).

$D_{\vec{u}} f(x_0, y_0) = \left\langle f_x(x, y),\ f_y(x, y) \right\rangle \cdot < a, b > = \nabla f(x, y) \cdot \vec{u}$ (a scalar).

If $\nabla f(x, y) = \vec{0}$, then $D_{\vec{u}} f(x_0, y_0) = 0$, for all unit direction vectors $\vec{u}$.

If $\nabla f(x, y) \neq \vec{0}$, then

i) The <u>STEEPEST ASCENT</u> or the <u>GREATEST RATE of CHANGE</u> at (x_0, y_0) is in the direction of $\nabla f(x_0, y_0)$.

   The <u>RATE of STEEPEST ASCENT</u> or the <u>MAXIMUM RATE of CHANGE</u> at (x_0, y_0) is $\left| \nabla f(x_0, y_0) \right|$.

ii) The <u>STEEPEST DESCENT</u> or the <u>MINIMUM RATE of CHANGE</u> at (x_0, y_0) is in the direction $-\nabla f(x_0, y_0)$.

   The <u>RATE of STEEPEST DESCENT</u> is $\left| \nabla f(x_0, y_0) \right|$ or the <u>MINIMUM RATE of CHANGE</u> at (x_0, y_0) is $-\left| \nabla f(x_0, y_0) \right|$.

   (Notice the wording here. The word “descent” incorporates the negative derivative value.)

(iii) $\nabla f(x_0, y_0)$ is perpendicular to the level curve of f through (x_0, y_0).

   In the case of a function of three variables, $w = f(x, y, z)$,

   the gradient vector

$$\nabla f(x_0, y_0, z_0) = \left\langle f_x(x_0, y_0, z_0), f_y(x_0, y_0, z_0), f_z(x_0, y_0, z_0) \right\rangle$$

is perpendicular to the level surface $f(x, y, z) = k$ at the point $P(x_0, y_0, z_0)$ and so it is a normal vector to the tangent plane of the surface at that point. It is also a direction vector for the normal line to the surface at P. We can use the gradient vector in this situation to help us determine equations for the tangent plane and normal line at a point on the given surface.

Extreme Values

The local extreme values of $z = f(x, y)$ are found by solving $\nabla f(x_0, y_0) = < 0,0 >$. The solutions of this 2x2 system are the critical points of the function. We test the critical points using the Second Derivative Test:

Let $D = D(x_0, y_0) = f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2$.

If $D > 0$ and $f_{xx}(x_0, y_0) > 0$, then $f(x_0, y_0)$ is a local minimum.

If $D > 0$ and $f_{xx}(x_0, y_0) < 0$, then $f(x_0, y_0)$ is a local maximum.

If $D < 0$, then there is a saddle point at (x_0, y_0).

If $D = 0$, the test is inconclusive.

Notes:

1. The discriminant D is easily remembered with the aid of determinant notation:

$$D = \begin{vmatrix} f_{xx}(x_0, y_0) & f_{xy}(x_0, y_0) \\ f_{xy}(x_0, y_0) & f_{yy}(x_0, y_0) \end{vmatrix} = f_{xx}(x_0, y_0) \cdot f_{yy}(x_0, y_0) - [f_{xy}(x_0, y_0)]^2$$

2. If $f_{xx}(x_0, y_0)$ and $f_{yy}(x_0, y_0)$ have opposite signs, then $D < 0$ and there is a saddle point at (x_0, y_0).

3. In the case a local extreme value exists, the critical point (x_0, y_0) is not the local extreme value. The critical point yields or gives rise to the local extreme value. The local extreme value is the functional value, $f(x_0, y_0)$.

4. The Second Derivative Test is about local extrema, not global extrema.

The guaranteed existence of absolute extreme values is given by the **Extreme Value Theorem**: If f is continuous on a closed and bounded set in the xy – plane, then f has an absolute maximum and absolute minimum value on that set.

By a **closed set** in R^2 we mean a set that contains all its boundary points. A **boundary point** of a set D is ia point (x_0, y_0) with the property that every disk with centre (x_0, y_0) contains points in D and points not in D.

A **bounded set** in R^2 is one that is contained inside some disk.

To find the absolute max and min of a continuous function on the closed and bounded set D:

1. Find the values of f at the critical points of D.
2. Find the extreme values of f on the boundary of D.
3. The largest number from the lists in steps 1 and 2 is the max, and the smallest of these values is the min.

Lagrange Multipliers

If $w = F(x, y, z)$ is subjected to the constraint $G(x, y, z) = 0$, then its extreme values are found by solving the system

$$\nabla F(x, y, z) = \lambda \cdot G(x, y, z)$$

$$G(x, y, z) = 0$$

Here λ is a scalar called the "Lagrange multiplier".

The system of equations can be written

$$\begin{aligned} F_x(x,y,z) &= \lambda \cdot G_x(x,y,z) \\ F_y(x,y,z) &= \lambda \cdot G_y(x,y,z) \\ F_z(x,y,z) &= \lambda \cdot G_z(x,y,z) \\ G(x,y,z) &= 0 \end{aligned}$$

This is a system of 4 equations in 4 unknowns, x, y, z and λ. We are normally interested only in the candidate point (x, y, z) and so it is not necessary to solve for λ.

Important Note: When solving a constrained optimization problem using this method you must start by setting up the problem. Define any variables you introduce into the problem. Identify the function to be optimized (maximized or minimized or both) and the constraint equation. A good way to start is to write

Optimize: $F(x, y, z) =$ ....

Subject to: $G(x, y, z) =$ ....

Then you must write down the entire system of equations you are going to solve **before** you start the process of solving the system!

Self-Test *(40 marks)*

Treat this as a real test. Do not refer to any course materials. The time for this test is 2.0 hours. Use the answer key provided to mark your test. The point value for each question is posted in the left margin.

1. Given $f(x, y) = x^2 e^{xy} + y\sin(x)$, find

[2] a) $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$ and evaluate each at the point (0,2).

[2] b) $\dfrac{\partial^2 f}{\partial y^2}$ and $\dfrac{\partial^2 f}{\partial x\,\partial y}$

[4] 2. Find an equation of the tangent plane to the surface

$$z^2 = x^2 + 2y^2 + 1 \text{ at the point (1,1,2).}$$

[3] 3. Find and sketch the domain of $g(x, y) = \arcsin\left(x^2 + y^2 - 2\right)$.

4. Evaluate the following limits:

[2] a) $\displaystyle\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2 + y^2}$

[2] b) $\displaystyle\lim_{(x,y,z)\to(0,0,0)} \frac{yz}{x^2 + 4y^2 + 9z^2}$

[4] 5. Given $f(x, y, z) = x^2 + xy + xz + xyz$ find the directional derivative at the point (1,-1, 2) in the direction $\vec{v} = \langle 1, 0, 2\rangle$. In what direction is the rate of increase the greatest and what is this greatest value?

[3] 6. Given $w = \ln\left(x^2 + y^2\right)$, where $x = s\cos(t)$ and $y = st$, use the Chain Rule to find $\dfrac{\partial w}{\partial s}$ at $(s,t) = (2,0)$.

[2] 7. Use the Implicit Function Theorem to find $\frac{dy}{dx}$, given

$$x^3 y + xy^2 = 2 - y^3.$$

[8] 8. Find and classify all critical points of the function

$$f(x, y) = x^3 + 2y^3 - 12x + 4y^2 + 2$$

[8] 9. Find all points on the ellipse

$$x^2 + \frac{(y-1)^2}{4} = 1$$

which are closest to the origin. Which point is farthest from the origin?

Answer Key

1. a) $f(x, y) = x^2 e^{xy} + y\sin(x)$

$$f_x(x, y) = 2xe^{xy} + x^2 e^{xy} y + y\cos(x) = 2xe^{xy} + x^2 ye^{xy} + y\cos(x)$$

$$f_x(0,2) = 2(0)e^0 + (0)^2 (2)e^0 + 2\cos(0) = 2(1) = 2$$

$$f_y(x, y) = x^2 e^{xy} x + \sin(x) = x^3 e^{xy} + \sin(x)$$

$$f_y(0,2) = (0)^3 e^0 + \sin(0) = 0$$

b) $$\frac{\partial^2 f}{\partial y^2} = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial y}\left(x^3 e^{xy} + \sin(x)\right) = x^3 e^{xy} x + 0 = x^4 e^{xy}$$

$$\frac{\partial^2 f}{\partial x\,\partial y} = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right) = \frac{\partial}{\partial x}\left(x^3 e^{xy} + \sin(x)\right) = 3x^2 e^{xy} + x^3 e^{xy} y + \cos(x) = e^{xy}\left(3x^2 + x^3 y\right) + \cos(x)$$

2. Let $f(x, y, z) = x^2 + 2y^2 - z^2 + 1$. Then $\nabla f(x, y, z) = < 2x, 4y, -2z >$ and

$\nabla f(1,1,2) = < 2(1), 4(1), -2(2) > = < 2, 4, -4 > = 2 < 1, 2, -2 >$.

A normal vector for the tangent plane at (1,1,2) is $\vec{n} = < 1, 2, -2 >$.

An equation for the tangent plane at (1,1,2) is:

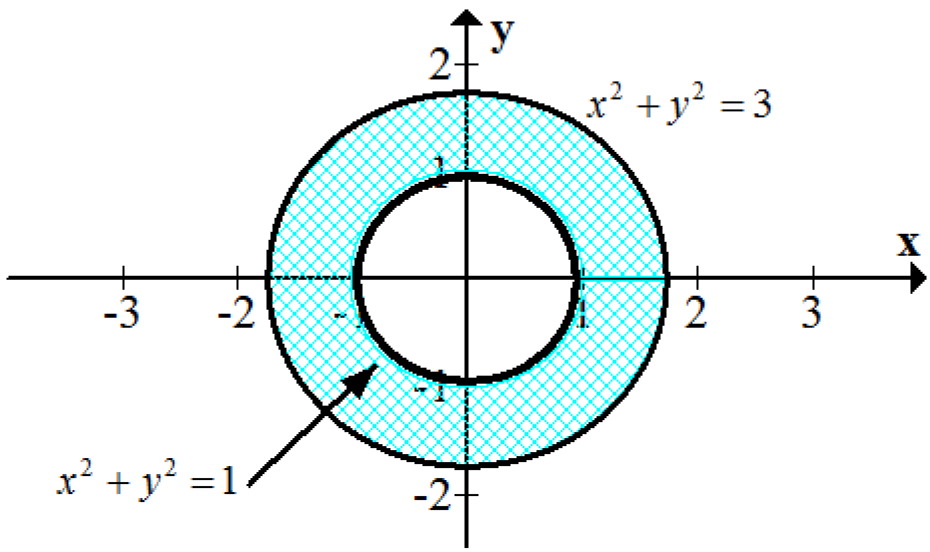
$$\vec{n} \cdot \left(\vec{x} - \vec{x}_0\right) = 0$$
$$< 1, 2, -2 > \cdot < x-1, y-1, z-2 > = 0$$
$$x - 1 + 2y - 2 - 2z + 4 = 0$$
$$x + 2y - 2z = -1$$

3. $g(x, y) = \arcsin\left(x^2 + y^2 - 2\right)$

In order that $g(x, y)$ be defined we require that

$$-1 \le x^2 + y^2 - 2 \le 1 \quad \Rightarrow \quad 1 \le x^2 + y^2 \le 3$$

So domain $g = \{(x, y) \mid 1 \le x^2 + y^2 \le 3\}$. Geometrically this is the annular region between the circle centred at the origin of radius 1 and the circle centred at the origin of radius $\sqrt{3}$.



4. a) $$\lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2}$$

Try different paths. Suspect the limit value is zero. To prove this we use the Squeeze Theorem.

$$0 \le \left|\frac{xy^2}{x^2+y^2}\right| \le |x|\frac{y^2}{x^2+y^2} \le |x|\cdot 1 \le |x|$$

Therefore,

$$\lim_{(x,y)\to(0,0)} 0 \le \lim_{(x,y)\to(0,0)} \left|\frac{xy^2}{x^2+y^2}\right| \le \lim_{(x,y)\to(0,0)} |x|$$

$$0 \le \lim_{(x,y)\to(0,0)} \left|\frac{xy^2}{x^2+y^2}\right| \le \lim_{x\to 0} |x|$$

$$0 \le \lim_{(x,y)\to(0,0)} \left|\frac{xy^2}{x^2+y^2}\right| \le 0$$

By the Squeeze Theorem,

$$\lim_{(x,y)\to(0,0)} \left|\frac{xy^2}{x^2+y^2}\right| = 0 \quad \Rightarrow \quad \lim_{(x,y)\to(0,0)} \frac{xy^2}{x^2+y^2} = 0$$

b) $$\lim_{(x,y,z)\to(0,0,0)} \frac{yz}{x^2+4y^2+9z^2}$$

Try different paths.

Along $y = z = 0$ (x – axis):

$$\lim_{(x,y,z)\to(0,0,0)} \frac{yz}{x^2 + 4y^2 + 9z^2} = \lim_{x\to 0} \frac{0}{x^2 + 0 + 0} = \lim_{x\to 0} 0 = 0$$

Along $x = y = z$

$$\lim_{(x,y,z)\to(0,0,0)} \frac{yz}{x^2 + 4y^2 + 9z^2} = \lim_{(x,x,x)\to(0,0,0)} \frac{xx}{x^2 + 4x^2 + 9x^2} = \lim_{x\to 0} \frac{x^2}{14x^2} = \lim_{x\to 0} \frac{1}{14} = \frac{1}{14}$$

We have different results along two different paths. Therefore,

$$\lim_{(x,y,z)\to(0,0,0)} \frac{yz}{x^2 + 4y^2 + 9z^2} \text{ does not exist.}$$

5. $f(x, y, z) = x^2 + xy + xz + xyz$

A unit vector in the direction $\vec{v} = \langle 1, 0, 2 \rangle$ is

$$\vec{u} = \frac{1}{|\vec{v}|}\vec{v} = \frac{1}{\sqrt{1+4}}\langle 1, 0, 2 \rangle = \frac{1}{\sqrt{5}}\langle 1, 0, 2 \rangle$$

$\nabla f(x, y, z) = < 2x + y + z + yz, x + xz, x + xy >$

$$\nabla f(1, -1, 2) = < 2(1) + (-1) + 2 + (-1)(2), 1 + +(1)(2), 1 + (1)(-1) > = < 1, 3, 0 >$$

So, $D_{\vec{u}} f(1, -1, 2) = \nabla f(1, -1, 2) \cdot \vec{u} = < 1, 3, 0 > \cdot \frac{1}{\sqrt{5}}\langle 1, 0, 2 \rangle = \frac{1}{\sqrt{5}}(1 + 0 + 0) = \frac{1}{\sqrt{5}}$

The maximum rate of increase is in the direction of the gradient vector $\nabla f(1, -1, 2) = < 1, 3, 0 >$ and its greatest value is
$|\nabla f(1, -1, 2)| = |< 1, 3, 0 >| = \sqrt{1 + 9 + 0} = \sqrt{10}$.

6. $w = \ln(x^2 + y^2)$, where $x = s\cos(t)$ and $y = st$

By the Chain Rule,

$$\frac{\partial w}{\partial s} = \frac{\partial w}{\partial x} \cdot \frac{\partial x}{\partial s} + \frac{\partial w}{\partial y} \cdot \frac{\partial y}{\partial s} = \frac{2x}{x^2 + y^2} \cdot \cos(t) + \frac{2y}{x^2 + y^2} \cdot t = \frac{2x\cos(t)}{x^2 + y^2} + \frac{2yt}{x^2 + y^2}$$

When $(s, t) = (2, 0)$, $x = 2\cos(0) = 2$ and $y = 2 \cdot 0 = 0$

Therefore,

$$\left.\frac{\partial w}{\partial s}\right|_{(s,t)=(2,0)} = \frac{2(2)\cos(0)}{2^2 + 0^2} + \frac{2(0)(0)}{2^2 + 0^2} = \frac{4}{4} + 0 = 1$$

7. Let $f(x, y) = x^3 y + xy^2 + y^3 - 2 = 0$

$\frac{\partial f}{\partial x} = 3x^2 y + y^2$; $\frac{\partial f}{\partial y} = x^3 + 2xy + 3y^2$

By the Implicit Function Theorem,

$$\frac{dy}{dx} = -\frac{\left(\frac{\partial f}{\partial x}\right)}{\left(\frac{\partial f}{\partial y}\right)} = -\frac{3x^2 y + y^2}{x^3 + 2xy + 3y^2} \text{ or } \frac{-3x^2 y - y^2}{x^3 + 2xy + 3y^2}$$

8. $f(x, y) = x^3 + 2y^3 - 12x + 4y^2 + 2$

$f_x(x, y) = 3x^2 - 12$; $f_y(x, y) = 6y^2 + 8y$

Solve: $\begin{array}{l} f_x(x, y) = 0 \\ f_y(x, y) = 0 \end{array} \Rightarrow \begin{array}{l} 3x^2 - 12 = 0 \text{..............(1)} \\ 6y^2 + 8y = 0 \text{.............(2)} \end{array}$

(1): $3x^2 - 12 = 0 \quad \Rightarrow \quad x^2 = 4 \quad \Rightarrow \quad x = \pm 2$

(2): $6y^2 + 8y = 0 \quad \Rightarrow \quad 2y(3y + 4) = 0 \quad \Rightarrow \quad y = 0$ or $3y + 4 = 0$

$$3y + 4 = 0 \quad \Rightarrow \quad y = -\frac{4}{3}$$

The critical points are: $(2,0)$, $(-2,0)$, $\left(2, -\frac{4}{3}\right)$, $\left(-2, -\frac{4}{3}\right)$

Now, $f_{xx}(x, y) = \frac{\partial}{\partial x}\left(3x^2 - 12\right) = 6x$; $f_{yy}(x, y) = \frac{\partial}{\partial y}\left(6y^2 + 8y\right) = 12y + 8$

$f_{xy}(x, y) = \frac{\partial}{\partial y}\left(3x^2 - 12\right) = 0$

So,

$$\begin{aligned} D(x, y) &= f_{xx}(x, y) \cdot f_{yy}(x, y) - \left[f_{xy}(x, y) \right]^2 \\ &= 6x \cdot (12y + 8) - 0^2 \\ &= 72xy + 48x \end{aligned}$$

At $(2,0)$, $D(2,0)=72(2)(0)+48(2)=96>0$ and $f_{xx}(2,0)=6(2)=12>0$. Hence, $(2,0)$ yields a local minimum.

At $(-2,0)$, $D(-2,0)=72(-2)(0)+48(-2)=-96<0$; hence, $(-2,0)$ yields a saddle point.

At $\left(2,-\frac{4}{3}\right)$, $D\left(2,-\frac{4}{3}\right)=72(2)\left(-\frac{4}{3}\right)+48(2)=-96<0$; hence, $\left(2,-\frac{4}{3}\right)$ yields a saddle point.

At $\left(-2,-\frac{4}{3}\right)$, $D\left(-2,-\frac{4}{3}\right)=72(-2)\left(-\frac{4}{3}\right)+48(-2)=96>0$ and $f_{xx}\left(-2,-\frac{4}{3}\right)=6(-2)=-12<0$; hence, $\left(-2,-\frac{4}{3}\right)$ yields a local maximum.

9. The problem is to find the extreme values for the distance $d=\sqrt{x^2+y^2}$, or equivalently the distance squared, $s=d^2=x^2+y^2$,

from the origin (0,0) to a point on the ellipse $x^2+\frac{(y-1)^2}{4}=1$.

Let $g(x,y)=x^2+\frac{(y-1)^2}{4}$. The problem is now:

$$\text{Optimize: } s(x,y)=x^2+y^2$$

$$\text{Subject to: } g(x,y)=x^2+\frac{(y-1)^2}{4}=1$$

Now, $s_x(x,y)=2x$; $s_y(x,y)=2y$; $g_x(x,y)=2x$; $g_y(x,y)=\frac{1}{2}(y-1)$

We solve the Lagrange system:

$$\begin{array}{lcl} s_x(x,y)=\lambda\cdot g_x(x,y) & & 2x=\lambda(2x)\ \ldots\ldots\ldots\ldots\ldots(1) \\ s_y(x,y)=\lambda\cdot g_y(x,y) & \Rightarrow & 2y=\lambda\frac{1}{2}(y-1)\ \ldots\ldots\ldots(2) \\ g(x,y)=1 & & x^2+\frac{(y-1)^2}{4}=1\ \ldots\ldots(3) \end{array}$$

(1): $2x=\lambda(2x) \quad\Rightarrow\quad 2x-2\lambda x=0 \quad\Rightarrow\quad 2x(1-\lambda)=0 \quad\Rightarrow\quad x=0$ or $\lambda=1$

(2): $2y = \lambda \frac{1}{2}(y-1) \quad \Rightarrow \quad 4y = \lambda(y-1) \quad \Rightarrow \quad \lambda = \frac{4y}{y-1} \quad , \quad y \neq 1$

<u>Case 1</u>: $x = 0$

(3): $0^2 + \frac{(y-1)^2}{4} = 1 \quad \Rightarrow \quad (y-1)^2 = 4 \quad \Rightarrow \quad y-1 = \pm 2 \quad \Rightarrow \quad y = -1 \text{ or } 3$

So we have the points (0,-1) and (0,3)

<u>Case 2</u>: $\lambda = 1$

$$\lambda = \frac{4y}{y-1} \quad \Rightarrow \quad 1 = \frac{4y}{y-1} \quad \Rightarrow \quad 4y = y-1 \quad \Rightarrow \quad y = -\frac{1}{3}$$

$$(3): x^2 + \frac{\left(-\frac{1}{3}-1\right)^2}{4} = 1 \quad \Rightarrow \quad x^2 + \frac{\left(\frac{16}{9}\right)}{4} = 1 \quad \Rightarrow \quad x^2 + \frac{4}{9} = 1$$

$$\Rightarrow \quad x^2 = \frac{5}{9} \quad \Rightarrow \quad x = \pm\frac{\sqrt{5}}{3}$$

So we have the points $\left(\frac{\sqrt{5}}{3}, -\frac{1}{3}\right)$, $\left(-\frac{\sqrt{5}}{3}, -\frac{1}{3}\right)$

The candidate points are: (0,-1), (0,3), $\left(\frac{\sqrt{5}}{3}, -\frac{1}{3}\right)$, $\left(-\frac{\sqrt{5}}{3}, -\frac{1}{3}\right)$

At (0,-1), $s(0,-1) = 0^2 + (-1)^2 = 1$

At (0,3), $s(0,3) = 0^2 + (3)^2 = 9$

At $\left(\frac{\sqrt{5}}{3}, -\frac{1}{3}\right)$, $s\left(\frac{\sqrt{5}}{3}, -\frac{1}{3}\right) = \left(\frac{\sqrt{5}}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 = \frac{5}{9} + \frac{1}{9} = \frac{2}{3}$

At $\left(-\frac{\sqrt{5}}{3}, -\frac{1}{3}\right)$, $s\left(-\frac{\sqrt{5}}{3}, -\frac{1}{3}\right) = \left(-\frac{\sqrt{5}}{3}\right)^2 + \left(-\frac{1}{3}\right)^2 = \frac{5}{9} + \frac{1}{9} = \frac{2}{3}$

Conclusion: The closest points to the origin are

$\left(\frac{\sqrt{5}}{3}, -\frac{1}{3}\right)$ and $\left(-\frac{\sqrt{5}}{3}, -\frac{1}{3}\right)$ and the farthest point is (0,3).