Faculty of Science

Unit 6:
Vector Calculus

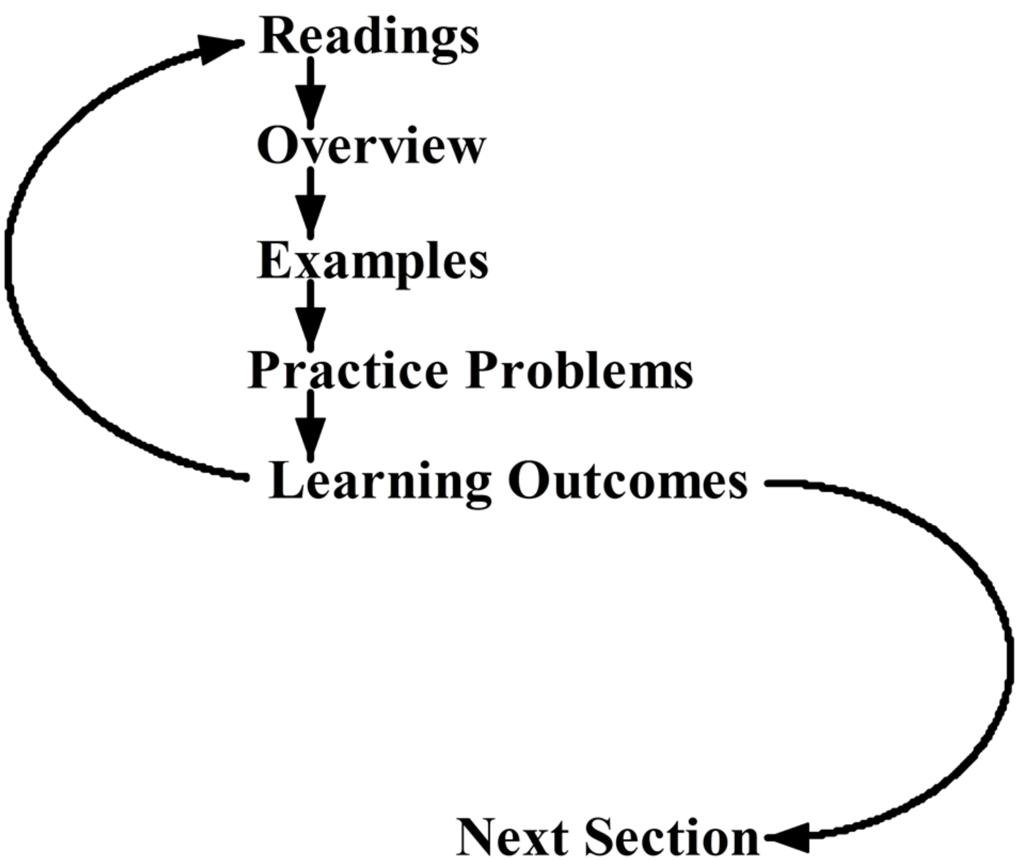
MATH 2111
Calculus III – Multivariable Calculus

Table of Contents

Instructions

The recommended procedure for working through each section of the units in this course is described in detail in your Course Guide.

This procedure is summarized below. If you are certain you have achieved the learning objectives, proceed to the next section. If you are uncertain about one or more of them, go back to the appropriate information in the section until you can complete the task listed in the objective.



Change of Variables in Multiple Integrals

Learning Outcomes

Upon completion of Change of Variables in Multiple Integrals, you should be able to:

- Find the image of a set under a given transformation.
- Find equations for a transformation that maps a given region in a 2-dimensional coordinate system to a region in another 2-dimensional coordinate system.
- Calculate the Jacobian of a given transformation $T : R^2 \to R^2$ or $T : R^3 \to R^3$.
- Calculate double and triple integrals using a change of variables.

Readings

Read section 15.10, pages 1064–1071, in your textbook. Carefully study the examples worked out in the text.

Overview

In the preceding two units, the concept of integration was extended from scalar-valued functions on R, the real line, to scalar-valued functions on R^2, 2-space, and R^3, 3-space.

The final step is to generalize the concept of integration to integrals of vector-valued functions on R^2 and R^3. Since many physical quantities are vector functions of position, this will considerably enlarge the range of applications for our analysis.

We have already seen that you should adapt the system of coordinates to your analysis of a particular problem. If you attempt to fit the problem into a system, you may experience very complicated integrals and grave difficulties with your solutions.

A change of variables that we found useful at times when working with a double integral was a conversion to polar coordinates. The equations $x = r\cos(\theta)$ and $y = r\sin(\theta)$ describe a transformation from the $r\theta$-plane to the xy-plane resulting in the change of variables formula

$$\iint_D f(x, y)\,dA = \iint_S f(r\cos(\theta), r\sin(\theta))\, r\,dr\,d\theta$$

where S is the region in the $r\theta$-plane that corresponds to the region D in the xy-plane.

More generally, a change of variables is a transformation $T : R^2 \to R^2$, from the uv-plane to the xy-plane, defined by $T(u,v) = (x, y)$, where $x = g(u,v)$ and $y = h(u,v)$ describe how x and y are related to u and v.

These relations are sometimes written as $x = x(u,v)$ and $y = y(u,v)$. In practice we require that x and y have continuous first partial derivatives with respect to u and v.

In terms of the integration problem

$$\iint_D f(x, y)\,dx\,dy$$

the situation is complicated because you have to change not only the functional relationship $z = f(x, y)$ into information expressed in terms of u and v, but also the area element $dx\,dy$.

If $x = x(u,v)$ and $y = y(u,v)$, the first step in the transformation is easy enough. The integrand becomes $f(x(u,v), y(u,v))$ a function of the new variables u and v. Equally there is no problem with the limits. However, the area element $dx\,dy$ has yet to be transformed. It becomes

$$\left|\frac{\partial(x, y)}{\partial(u,v)}\right| du\,dv$$

where the quantity inside the bars, called the Jacobian, stands for the determinant of a 2×2 matrix made up of the partial derivatives of x and y with respect to the transformed coordinates; that is,

$$\frac{\partial(x, y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$$

The bars stand for the absolute value of the determinant value.

Recall that determinants can take on negative values. Since we are replacing an area element, which is non–negative, with an area element, we need the absolute value to ensure the result is non-negative. Also, in order for this transformation to be valid, the Jacobian must be non-zero.

The full details for this transformation can be found in Box 9 on page 1068 of the text.

There is a similar change of variables formula for triple integrals. (See Boxes 12 and 13 on page 1070 of the text.)

This transformation technique can be used to find the form of the volume elements obtained in the last unit for cylindrical and spherical coordinates and, in general, for any system of coordinates that can be expressed as functions of x, y and z. The calculation of the volume element for the change to spherical coordinates is Example 4 on pages 1070–1071 of the text.

There is one more property of the Jacobian which is very useful to note:

$$\left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \frac{1}{\left|\dfrac{\partial(u,v)}{\partial(x,y)}\right|}$$

where $\dfrac{\partial(u,v)}{\partial(x,y)}$ denotes the Jacobian of the inverse mapping from the xy-plane to the uv-plane, when it exists (when the Jacobian is non-zero!).

Thus, it does not matter whether x and y are known as functions of u and v or vice versa.

The corresponding relation for three variables is:

$$\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \frac{1}{\left|\dfrac{\partial(u,v,w)}{\partial(x,y,z)}\right|}$$

Note: This relationship between the Jacobian of a mapping and its inverse is <u>not discussed in your text</u>. Because of this, problems like Example 3 on page 1069 of the text in which you are given $u = u(x, y)$ and $v = v(x, y)$, require that you solve the system of equations for $x = x(u,v)$ and $y = y(u,v)$ in order to calculate the Jacobian for the given transformation.

We will freely use this useful relationship in the examples to follow.

The question as to what is an appropriate change of variable in a given instance remains the same in multiple integration as in single integration. Practice will improve your guessing ability. In most of the examples and the practice exercises, the form of the new variables is either given or suggested by the boundaries of the regions being integrated.

Example Exercises

1. This is problem 14 on page 1071 of the text.

A region R is bounded by the hyperbolas $y=\frac{1}{x}$ and $y=\frac{4}{x}$, and the lines $y=x$ and $y=4x$ in the first quadrant. Find equations for a transformation T that maps a rectangular region S in the uv-plane onto R, where the sides of S are parallel to the u,v-axes.

Notice that $y=\frac{1}{x}$ and $y=\frac{4}{x}$ can be re-written as $xy=1$ and $xy=4$. Also, $y=x$ and $y=4x$ can be re-written $\frac{y}{x}=1$ and $\frac{y}{x}=4$. These boundaries for R suggest the transformation $u=xy$ and $v=\frac{y}{x}$, since we want the sides of S to be horizontal and vertical lines (S is a rectangle in the uv-plane).

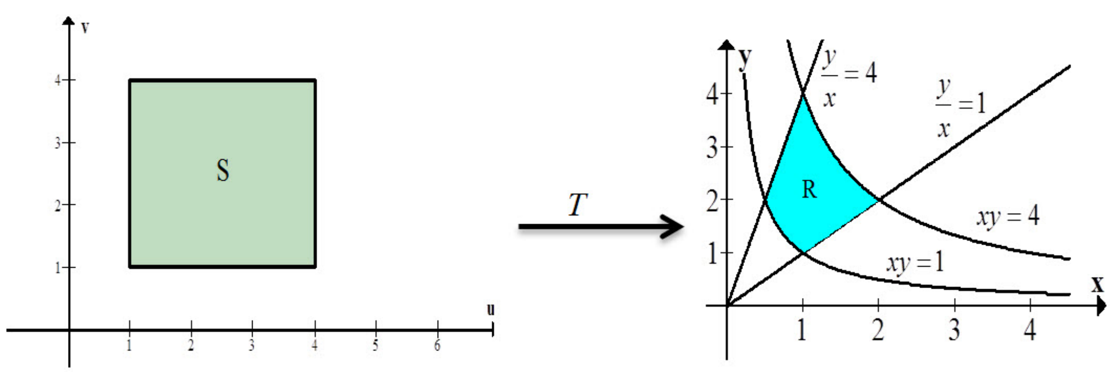
We want the transformation T from S, in the uv-plane, onto R in the xy-plane, so we need to solve for x and y in terms of u and v.

$u=xy \Rightarrow x=\frac{u}{y}$; so, $v=\frac{y}{x}=\frac{y}{(u/y)}=\frac{y^2}{u} \Rightarrow y^2=uv \Rightarrow y=\sqrt{uv}$ ($y>0$ in Quad I)

Now, $x=\frac{u}{y}=\frac{u}{\sqrt{uv}}=\frac{u}{\sqrt{uv}}\cdot\frac{\sqrt{uv}}{\sqrt{uv}}=\frac{\sqrt{uv}}{v}$

Hence, one possible transformation T is given by $x(u,v)=\frac{\sqrt{uv}}{v}$ a

$y(u,v)=\sqrt{uv}$. The picture is below.



Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

2. Evaluate $\iint_D (3x^2 + 5xy - 2y^2)\, dA$, where

$D = \{(x, y) \mid -1 \le 3x - y \le 2, -3 \le x + 2y \le -1\}$

With such an integrand and such limits, integration in Cartesian coordinates would be messy, even if possible! At least you can simplify the limits with a change of variable. Put

$3x - y = u$ and $x + 2y = v$

Then the corresponding region S in the uv-plane is:

$S = \{(u, v) \mid -1 \le u \le 2, -3 \le v \le -1\}$

You should be pleased to note that the integrand is simply $u \cdot v$, since

$3x^2 + 5xy - 2y^2 = (3x - y)(x + 2y)$

To find the transformation of $dx\,dy$ you must calculate the Jacobian

$$\frac{\partial(x, y)}{\partial(u, v)} = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix}$$

Since the transformation suggested by the problem is from the xy-plane to the uv-plane, we use

$$\frac{\partial(x, y)}{\partial(u, v)} = \frac{1}{\dfrac{\partial(u, v)}{\partial(x, y)}}$$

Now, $\dfrac{\partial u}{\partial x} = 3$, $\dfrac{\partial u}{\partial y} = -1$, $\dfrac{\partial v}{\partial x} = 1$ and $\dfrac{\partial v}{\partial y} = 2$

So, $\begin{vmatrix} \dfrac{\partial u}{\partial x} & \dfrac{\partial u}{\partial y} \\ \dfrac{\partial v}{\partial x} & \dfrac{\partial v}{\partial y} \end{vmatrix} = \begin{vmatrix} 3 & -1 \\ 1 & 2 \end{vmatrix} = (3)(2) - (1)(-1) = 7$ and $\dfrac{\partial(x, y)}{\partial(u, v)} = \dfrac{1}{\dfrac{\partial(u, v)}{\partial(x, y)}} = \dfrac{1}{7}$

Hence,

$$\iint_D (3x^2+5xy-2y^2)\,dA = \int_{-1}^{2}\int_{-3}^{-1} uv\left(\frac{1}{7}\right)du\,dv$$
$$= \frac{1}{7}\int_{-1}^{2} v\left[\frac{u^2}{2}\right]_{-3}^{-1} dv$$
$$= \frac{1}{14}\int_{-1}^{2} v\left((-1)^2-(-3)^2\right)dv$$
$$= \frac{1}{14}\int_{-1}^{2} -8v\,dv$$
$$= -\frac{8}{14}\left[\frac{v^2}{2}\right]_{-1}^{2}$$
$$= -\frac{4}{14}\left(2^2-(-1)^2\right)$$
$$= -\frac{6}{7}$$

3. Find the area enclosed by the ellipse $x^2+4xy+5y^2=4$.

We start by putting the equation in completed square form.

$$x^2+4xy+5y^2=4$$
$$\left(x^2+4xy+(2y)^2\right)+y^2=4$$
$$\left(x+2y\right)^2+y^2=4$$

Now let $x+2y=u$ and $y=v$; so, $x=u-2y=u-2v$

The ellipse $x^2+4xy+5y^2=4$ becomes $\left(x+2y\right)^2+y^2=4$, which transforms to $u^2+v^2=4$, a circle centered at (0,0) of radius 2.

Now,

$$\frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1 & -2 \\ 0 & 1 \end{vmatrix} = (1)(1)-(-2)(0)=1$$

So, $\displaystyle\iint_D 1\,dA = \iint_S 1(1)\,du\,dv = \iint_S du\,dv$

The area of the region D enclosed by the ellipse is equal to the area of the region S enclosed by the circle.

Hence, $\iint\limits_D 1 dA = \pi \cdot 2^2 = 4\pi$

Practice Exercises 15.10

From the text pages 1071–1072, do problems 3, 5, 7, 9, 11, 13, 17, 19, 21, 23, 25, and 27.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Vector Fields

Learning Outcomes

Upon completion of Vector Fields, you should be able to:

- Sketch a vector field.
- Match a vector field equation with its picture.
- Find the gradient vector field of a scalar function.
- Write down the definition of a conservative vector field.
- Solve simple applications involving vector fields.

Readings

Read section 16.1, pages 1080–1085, in your textbook. Carefully study the examples worked out in the text.

Overview

Pictorially, a "vector field" is just a map of vectors that can be useful in describing flow or movement, as in air currents, ocean currents, gravitational pull, etc.

Formally, a **vector field** on a set D in R^2 is a function F that assigns to each point (x, y) in D a 2-dimensional vector $F(x, y) =< u_1 , u_2 >$.

Notice that each of the components of the vector $F(x, y)$ can be described by scalar functions P and Q defined by

$$P(x, y) = u_1 \text{ and } Q(x, y) = u_2$$

So, $F(x, y) =< P(x, y), Q(x, y) >= P(x, y)\vec{i} + Q(x, y)\vec{j}$ or $\vec{F} = P\vec{i} + Q\vec{j}$, for short.

Correspondingly we define a **vector field** on a set D in R^3 as a function F that assigns to each point (x, y, z) in D a 3-dimensional vector $F(x, y, z)$, and we write

$$F(x, y, z) = P(x, y, z)\vec{i} + Q(x, y, z)\vec{j} + R(x, y, z)\vec{k}$$

where P, Q, R are its component functions.

Some nice examples from physics of vector fields can be found in the text on pages 1083–1084, Examples 3, 4 and 5. You should study these examples.

We have already encountered an example in our previous work of a vector field:

If $z = f(x, y)$ is a continuous function, then the gradient of f, ∇f,

defined by

$$\nabla f(x, y) = < f_x(x, y), f_y(x, y) > = f_x(x, y)\vec{i} + f_y(x, y)\vec{j}$$

describes a vector field on the domain of f, called the **gradient vector field**.

Correspondingly, the gradient of a function of three variables,

$$\nabla f(x, y, z) = < f_x(x, y, z), f_y(x, y, z), f_z(x, y, z) > = f_x(x, y, z)\vec{i} + f_y(x, y, z)\vec{j} + f_z(x, y, z)\vec{k}$$

describes a vector field on a subset of R^3.

A very important type of vector field is the "conservative vector field". A vector field F is **conservative** if it is the gradient of some scalar function; that is, there is a function f such that $F = \nabla f$, in which case the function f is called a **potential function** for F.

An important example of a vector field in physics which is conservative is the gravitational field described in Example 4 on page 1083 of the text.

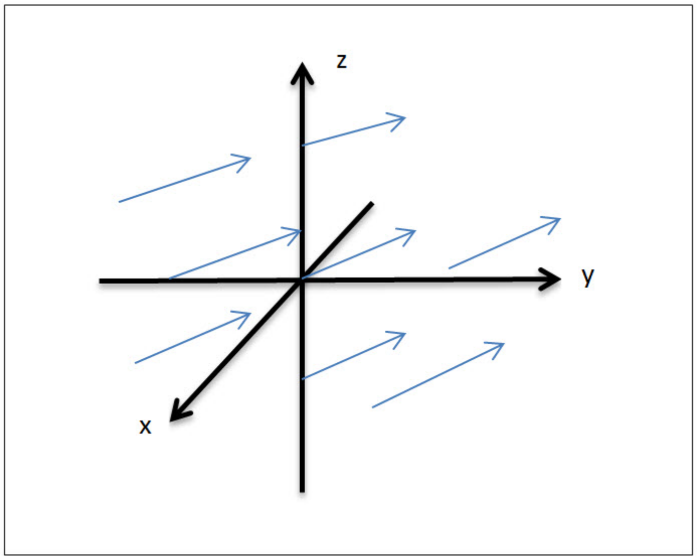
Example Exercises

1. This is problem 10 on page 1085 of the text.

   Sketch the vector field $F(x, y, z) = \vec{j} - \vec{i}$.

   The vector field is describing the vector $\vec{j} - \vec{i} = < 0,1,0 > - < 1,0,0 > = < -1,1,0 >$ constructed at any point in 3-space.

   So, all vectors in this field have length $\sqrt{2}$, are parallel to the xy-plane and point in the same direction as $< -1,1,0 >$



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2. This is problem 34 on page 1086 of the text.

   At time $t = 1$, a particle is located at position (1,3). If it moves in a velocity field $F(x, y) =< xy - 2, y^2 - 10 >$, find its approximate location at time $t = 1.05$.

   At time $t = 1$, the particle is located at position (1,3) so its velocity at that instant in time is $F(1,3) =< (1)(3) - 2, 3^2 - 10 >=< 1, -1 >$. After 0.05 units of time the particle's change in position should be approximately 0.05(1)=0.05 in the x-direction and 0.05(-1)=-0.05 in the y-direction. So, the particle should be at approximately the location (1+0.05,3-0.05)=(1.05,2.95).

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

Practice Exercises 16.1

From the text pages 1085–1086, do problems 5, 9, 11, 13, 15, 17, 23, 25, 29, and 33.

Remember: You are to attempt all of the problems carefully before checking your solutions against those given in the solutions manual.

Line Integrals

Learning Outcomes

Upon completion of Line Integrals, you should be able to:

- Evaluate line integrals in R^2 and R^3 with respect to the coordinate variables.
- Evaluate line integrals in R^2 and R^3 with respect to arc length.
- Evaluate line integrals of vector fields.
- Find the work done by a force field in moving an object along a curve.

Readings

Read section 16.2, pages 1087–1096, in your textbook. Carefully study the examples worked out in the text.

Overview

Since this section has to do with curves, you should review Unit Two where we parametrized curves, found tangents, normals and curve lengths, and so on.

In this section, we generalize the definite integral of a single variable function defined on an interval $[a,b]$ to an integral of a function of two variables defined on a "nice" curve C in the xy-plane. Such integrals are called **line integrals** and are represented by the symbolism $\int_C f(x,y)\,ds$.

Line integrals are defined using the familiar Riemann sum construction and the limiting process (See Box 2 on page 1087 of the text).

Under the assumptions that $z = f(x,y)$ is a continuous function and that the plane curve C is smooth, the line integral of f along C exists and can be evaluated in terms of a parametrization of C using the formula

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\,dt$$

Notes:

1. The value of the line integral is independent of the parametrization used for the curve, provided that the curve is traversed exactly once as t increases from a to b.

2. For the special case $f(x,y)=1$, we get:

$$\int_C ds = \int_a^b \sqrt{\left(x'(t)\right)^2 + \left(y'(t)\right)^2}\, dt = \int_a^b \left|\vec{r}\,'(t)\right| dt = L$$

where L is the length of the curve C and $\vec{r}\,'(t)$ is the derivative of the vector equation

$\vec{r}(t) = \langle x(t), y(t)\rangle$.

3. In the special case the curve C is the line segment joining $(a,0)$ to $(b,0)$ on the x-axis and we parametrize C using x as the parameter, so that $x = x$ and $y = 0$, we obtain

$$\int_C f(x,y)\, ds = \int_a^b f(x,0)\sqrt{(1)^2 + (0)^2}\, dx = \int_a^b f(x,0)\, dx$$

That is, the line integral reduces to the definite integral of a single variable function defined on $[a,b]$.

As in the case of the definite integral, we can interpret the line integral of a non-negative function f along C as area. Specifically, $\int_C f(x,y)\, ds$ represents the area of the “fence” or “curtain” whose base lies on C and whose height at the point (x,y) is $f(x,y)$. (See Figure 2 on page 1088 of the text.)

If the curve C along which we are integrating is not smooth, but can be written as a union of a finite number of smooth curves with the property that the initial point of one curve is the terminal point of the previous curve, then line integral of f along C is the sum of the line integrals along the smooth pieces. Such a curve is said to be “piece-wise smooth”.

There are two other forms of a line integral that we consider.

The line integral of f along C with respect to x:

$$\int_C f(x,y)\, dx = \int_a^b f(x(t), y(t))\, x'(t)\, dt$$

The line integral of f along C with respect to y:

$$\int_C f(x,y)\, dy = \int_a^b f(x(t), y(t))\, y'(t)\, dt$$

Notice the dx and dy symbolisms in these line integral forms that differ from the ds symbolism.

To distinguish these two line integrals from the original form, we refer to $\int_C f(x, y)\, ds$ as the **line integral with respect to arc length**.

It frequently happens that line integrals with respect to x and y occur together. When this happens we write

$$\int_C P(x, y)\, dx + \int_C Q(x, y)\, dy = \int_C P(x, y)\, dx + Q(x, y)\, dy$$

Notice that we evaluate line integrals of the form $\int_C P(x, y)\, dx + Q(x, y)\, dy$ by expressing all the information x, y, dx, dy in terms of the parameter t.

A given parametrization of a curve determines an orientation of that curve, which is the direction you move along the curve corresponding to increasing values of t. How does the orientation affect the value of a line integral?

If $-C$ denotes the curve consisting of the same points as C but with opposite orientation, then

$$\int_{-C} f(x,y)\, dx = -\int_C f(x,y)\, dx \text{ and } \int_{-C} f(x,y)\, dy = -\int_C f(x,y)\, dy$$

But if we integrate with respect to arc length, the value of the line integral does not change

$$\int_{-C} f(x, y)\, ds = \int_C f(x, y)\, ds$$

For functions of three variables defined along a smooth space curve, line integrals with respect to x, y, z and arc length are defined in a similar manner. (See Boxes 9 and 10 on pages 1092 and 1093 of the text.)

The next important concept in this section is the line integral of a vector field. Stewart approaches this concept through the physical concept of the work done on a body as it moves along a path under the action of a force.

In Section 12.3 we saw that the work done by a constant force $\vec{F}$ in moving an object from point P to point Q in space is given by

$$W = \vec{F} \cdot \vec{D}, \text{ where } \vec{D} = \overrightarrow{PQ} \text{ is the displacement vector.}$$

Now let $\vec{F} = P\vec{i} + Q\vec{j} + R\bar{k}$ denote a continuous force field on R^3 and use the familiar Riemann construction applied to a parametrization of the curve C. At each point (x, y, z) on the space curve, the unit tangent vector $T(x, y, z)$ denotes the direction we want to move and Δs represents the distance we need to travel, so $T(x, y, z)\Delta s$ represents the displacement vector. $F(x, y, z)$ represents the force applied at point (x, y, z), so $F(x, y, z) \cdot T(x, y, z)\Delta s$ represents the work done over that small arc. Adding up the work done and taking the limit yields the result

$$W = \int_C F(x, y, z) \cdot T(x, y, z)\, ds = \int_C \vec{F} \cdot \vec{T}\, ds$$

More details on this process can be found on page 1094 of the text.

Notice that this equation tells us that the "work done is the line integral with respect to arc length of the tangential component of the force applied."

Using the relationships $\vec{T}(t) = \dfrac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|}$ and $\dfrac{ds}{dt} = |\vec{r}\,'(t)| \quad \Rightarrow \quad ds = |\vec{r}\,'(t)|\, dt$, where

$\bar{r}(t) =< x(t), y(t), z(t) >$ is a vector equation for the space curve C, we have

$W = \displaystyle\int_a^b \left[F(\vec{r}(t)) \cdot \frac{\vec{r}\,'(t)}{|\vec{r}\,'(t)|} \right] |\vec{r}\,'(t)|\, dt = \int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$ or $W = \displaystyle\int_C \vec{F} \cdot d\vec{r}$, for short.

Generalizing, if F is any continuous vector field defined on a smooth curve C defined by a vector function $\bar{r}(t) =< x(t), y(t), z(t) >$, $a \le t \le b$, then the **line integral of F along C** is:

$$\int_C \vec{F} \cdot d\vec{r} = \int_a^b F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

There is an important connection between line integrals of vector fields and line integrals of scalar fields.

If $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$, then:

$$\int_C \vec{F} \cdot d\vec{r} = \int_C P\, dx + Q\, dy + R\, dz$$

The details of this relationship can be found on page 1096 of the text.

Example Exercises

1. This is question 4 on page 1096 of the text.

Evaluate $\int_C x\sin(y)\,ds$, where C is the line segment from (0,3) to (4,6).

Recall that a vector representation of a line segment from $\vec{r}_0$ to $\vec{r}_1$ is given by: $\vec{r}(t) = (1-t)\vec{r}_0 + t\,\vec{r}_1$, $\quad 0 \le t \le 1$.

Here $\vec{r}_0 =< 0,3 >$ and $\vec{r}_1 =< 4,6 >$. So,
$\vec{r}(t) = (1-t) < 0,3 > +t < 4,6 >=< 4t,\ 3+3t >$.

Parametric equations for C are: $x = 4t$, $\quad y = 3+3t$, $0 \le t \le 1$. Now, $x'(t) = 4$ and $y'(t) = 3$

So, $\int_C xy^4\,ds = \int_0^1 4t\sin(3+3t)\sqrt{4^2+3^2}\,dt = 20\int_0^1 t\sin(3+3t)\,dt$

Using integration by parts, let $u = t$ and $dv = \sin(3+3t)dt$. Then $du = dt$ and $v = -\frac{1}{3}\cos(3+3t)$, and

$$
\begin{aligned}
\int_C xy^4\,ds &= 20\left(\left[-\frac{t}{3}\cos(3+3t)\right]_0^1 - \int_0^1 -\frac{1}{3}\cos(3+3t)\,dt\right)\\
&= -\frac{20}{3}\left(\left[t\cos(3+3t)\right]_0^1 - \int_0^1 \cos(3+3t)\,dt\right)\\
&= -\frac{20}{3}\left(\left((1)\cos(6)-0\right) + \frac{1}{3}\left[\sin(3+3t)\right]_0^1\right)\\
&= -\frac{20}{3}\left(\cos(6) + \frac{1}{3}\left(\sin(6)-\sin(3)\right)\right)\\
&= -\frac{20}{3}\cos(6) - \frac{20}{9}\left(\sin(6)-\sin(3)\right)\\
&= -\frac{20}{9}\left(3\cos(6) - \sin(6) + \sin(3)\right)
\end{aligned}
$$

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

2. Evaluate $\int_C \left(x^2 - y^2\right)dx + 2xy\,dy$, where C is the curve $y = x^2$ from (-1,1) to (2,4). .

The parabola $y = x^2$ can be parametrized by setting $x = t$ and $y = t^2$ for $-1 \le t \le 2$.

$dx = x'(t)\,dt = 1dt = dt$ and $dy = y'(t)\,dt = 2t\,dt$

So,

$$\begin{aligned}
\int_C \left(x^2 - y^2\right)dx + 2xy\,dy &= \int_{-1}^{2}\left(t^2 - \left(t^2\right)^2\right)dt + 2t\left(t^2\right)(2t\,dt) \\
&= \int_{-1}^{2}\left(t^2 - t^4\right)dt + \int_{-1}^{2} 4t^4 dt \\
&= \int_{-1}^{2}\left(t^2 + 3t^4\right)dt \\
&= \left[\frac{t^3}{3} + 3\frac{t^5}{5}\right]_{-1}^{2} \\
&= \left(\frac{2^3}{3} + \frac{3}{5}\left(2^5\right)\right) - \left(\frac{(-1)^3}{3} + \frac{3}{5}\left((-1)^5\right)\right) \\
&= \frac{8}{3} + \frac{96}{5} + \frac{1}{3} + \frac{3}{5} \\
&= \frac{114}{5}
\end{aligned}$$

3. This is question 22 on page 1097 of the text.

Evaluate the line integral $\int_C \vec{F}\cdot d\vec{r}$, where $F(x, y, z) = x\vec{i} + y\vec{j} + xy\vec{k}$ and C is described by $\vec{r}(t) = \cos(t)\vec{i} + \sin(t)\vec{j} + t\vec{k}$, $0 \le t \le \pi$.

$F(x(t), y(t), z(t)) = F(\cos(t), \sin(t), t) = <\cos(t), \sin(t), \cos(t)\sin(t)>$;

$\vec{r}\,'(t) = <-\sin(t), \cos(t), 1>$

$$\begin{aligned}
\int_C \vec{F}\cdot d\vec{r} &= \int_0^{\pi} F(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt = \int_0^{\pi} <\cos(t), \sin(t), \cos(t)\sin(t)> \cdot <-\sin(t), \cos(t), 1> dt \\
&= \int_0^{\pi} \sin(t)\cos(t)\,dt = \frac{1}{2}\sin^2(t)\Big]_0^{\pi} = \frac{1}{2}\left(\sin^2(\pi) - \sin^2(0)\right) = 0
\end{aligned}$$

Stewart, J. (2012). *Multivariable calculus* (7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.

4. This is question 42 on page 1098 of the text.

The force exerted by an electric charge at the origin on a charged particle at a point (x, y, z) with position vector $\vec{r}(t) = < x, y, z >$ is

$F(\vec{r}) = K\dfrac{\vec{r}}{|\vec{r}|^3}$, where K is a constant. Find the work done as the particle moves along the straight line from (2,0,0) to (2,1,5).

The direction vector from (2,0,0) to (2,1,5) is <0,1,5>. So, a vector equation of the line segment from (2,0,0) to (2,1,5) is

$\vec{r}(t) = < 2,0,0 > +t < 0,1,5 > = < 2, t, 5t >$, $0 \le t \le 1$. Therefore, $\vec{r}\,'(t) = < 0,1,5 >$.

So,

$$W = \int_C \vec{F} \cdot d\vec{r} = \int_0^1 F(\vec{r}(t)) \cdot \vec{r}\,'(t)\, dt$$

$$= \int_0^1 K \frac{< 2, t, 5t >}{\left(\sqrt{2^2 + t^2 + (5t)^2}\right)^3} \cdot < 0,1,5 > dt$$

$$= \int_0^1 K \frac{< 2, t, 5t >}{\left(4 + 26t^2\right)^{3/2}} \cdot < 0,1,5 > dt$$

$$= \int_0^1 \frac{K}{\left(4 + 26t^2\right)^{3/2}} \left(2(0) + t(1) + 5t(5)\right) dt$$

$$= K \int_0^1 \frac{26t}{\left(4 + 26t^2\right)^{3/2}}\, dt$$

Perform a substitution with $u = 4 + 26t^2$, $du = 52t\, dt$

$$= \frac{K}{2} \int_0^1 52t \left(4 + 26t^2\right)^{-3/2}\, dt$$

$$= \frac{K}{2}(-2)\left[\frac{1}{\sqrt{4 + 26t^2}}\right]_0^1 = -K\left(\frac{1}{\sqrt{30}} - \frac{1}{2}\right) = K\left(\frac{1}{2} - \frac{1}{\sqrt{30}}\right) = K\left(\frac{\sqrt{30} - 2}{2\sqrt{30}}\right)$$

Stewart, J. (2012). ***Multivariable calculus*** **(7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.**

Practice Exercises 16.2

From the text pages 1096–1099, do problems 3, 7, 11, 13, 17, 19, 21, 29(a), 33, 39, 43, 45, and 47.

Remember: You are to attempt all of the problems carefully before checking your solutions against those given in the solutions manual.

The Fundamental Theorem for Line Integrals

Learning Outcomes

Upon completion of The Fundamental Theorem for Line Integrals, you should be able to:

- Determine if a set of points in the plane is open, connected or simply-connected.
- Determine if a vector field is conservative.
- Find a potential function for a conservative vector field.
- Evaluate the line integral of a conservative vector field in terms of its potential function.
- Characterize, in terms of closed paths in D, when $\int_C \vec{F} \cdot d\vec{r}$ is independent of path.

Readings

Read section 16.3, pages 1099–1106, in your textbook. Carefully study the examples worked out in the text.

Overview

We have already encountered the concept of conservative vector fields. This section focuses on this concept. Tests are obtained for determining if a given vector field is conservative. If it is, a technique for finding the scalar potential function that would generate the conservative field is presented.

If the potential function is f, the vector field F is obtained by forming the gradient of f; that is,

$$F(\vec{r}(t)) = \nabla f(\vec{r}(t))$$

In Theorem 2 on page 1099 of the text, Stewart states a generalization of the Fundamental Theorem of Calculus for single variable functions. It tells us that subject to certain conditions, the line integral of such a vector field is independent of the path. Consequently, the line integral is equal to the difference in the potential function calculated at the limits. That is,

$$\int_C \vec{F} \cdot d\vec{r} = \int_C \vec{\nabla} f \cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

where C is defined by a vector function $\vec{r} = \vec{r}(t)$, $a \le t \le b$.

This result tells us that the line integral of any conservative vector field can be evaluated simply by knowing the value of the potential function at the endpoints of C.

It is apparent that this calculation is a considerable mathematical simplification. It is well worth determining if a given F is conservative and can be written as the gradient of some potential function.

In Theorem 3 on page 1101, Stewart states necessary and sufficient conditions for independence of path in the domain D of a continuous vector field:

$\int_C \vec{F} \cdot d\vec{r}$ is independent of the path in D if, and only if, $\int_C \vec{F} \cdot d\vec{r} = 0$ for every closed path in D.

By a <u>closed path</u> we mean one in which the terminal point and initial point coincide; that is,

$\vec{r}(b) = \vec{r}(a)$.

The next question is whether a line integral of a vector function can be independent of path <u>without</u> being the gradient of a potential function. Theorem 4 on page 1101 answers the question with a firm no, provided the region D is open and connected. The <u>only</u> functions that have line integrals independent of path are conservative vector fields.

In Theorems 5 and 6 on pages 1102 and 1103 of the text, Stewart develops a test to determine if a given vector field F is conservative.

If we start with a continuous vector field $F = P\vec{i} + Q\vec{j}$ on an open simply-connected domain D, that is, an open set D in which every simple closed curve in D encloses only points in D (there are no holes in D), and the components of F have continuous first-order partials satisfying

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

then F is conservative!

A criterion for determining whether a vector field on R^3 is conservative is not developed until later in Section 16.5.

Finally, if $\vec{F} = P\vec{i} + Q\vec{j}$ is conservative for a given D, how can you find the potential function f that would generate F through the gradient? You know that the gradient of the unknown f must equal F. Thus

$$f_x(x, y) = P(x, y)$$
$$f_y(x, y) = Q(x, y)$$

If this set of partial differential equations can be solved for $z = f(x, y)$, you have the potential. Stewart illustrates a technique in Examples 4 and 5 on pages 1103 and 1104 of the text, which sometimes works. A full-fledged method of solution must be postponed for a course on differential equations.

The final topic of this section is discussed on page 1105 of the text.

Here the basic physical "law" of conservation of energy is derived. Whenever apparent exceptions to the law have appeared, physicists have gone to great lengths to find the "missing" energy. This example considers only mechanical forces and energies. If you penetrate into physics, you will encounter many other forms. The law is so basic to theoretical physics that when energy was found missing in atomic interactions, an "invisible" entity, the neutrino, was postulated so that the law could be conserved. Neutrinos were not "observed" until about two decades later.

Example Exercises

1. This is question 8 on page 1106 of the text.

   Determine whether or not $\vec{F} = \left(2xy + y^{-2}\right)\vec{i} + \left(x^2 - 2xy^{-3}\right)\vec{j}$, $y > 0$, is a conservative vector field. If so, find a potential function for F.

   We start by observing that $D = \{(x, y) \mid y > 0\}$ is an open and simply-connected set. D is just the set of points in the first and second quadrants of the plane, including the points on the positive y-axis, but omitting the points on the x-axis. At every point in D we can construct a disk centered at that point that lies entirely inside D, which means that D is open. There are no holes in D, which means that D is simply-connected.

   Now, $P(x, y) = 2xy + y^{-2}$ and $Q(x, y) = x^2 - 2xy^{-3}$.

   So, $\dfrac{\partial P}{\partial y} = \dfrac{\partial}{\partial y}\left(2xy + y^{-2}\right) = 2x - 2y^{-3}$ and $\dfrac{\partial Q}{\partial x} = \dfrac{\partial}{\partial x}\left(x^2 - 2xy^{-3}\right) = 2x - 2y^{-3}$

   Hence, $\dfrac{\partial P}{\partial y} = \dfrac{\partial Q}{\partial x}$ and F is conservative.

To find a function f such that $\nabla f = F$, we solve: $\begin{array}{l} f_x(x,y) = 2xy + y^{-2} \\ f_y(x,y) = x^2 - 2xy^{-3} \end{array}$

for f.

$f_x(x,y) = 2xy + y^{-2} \quad \Rightarrow \quad f(x,y) = x^2 y + xy^{-2} + g(y)$,

following the method of "partial integration" outlined on page 1103–1104 of the text.

Differentiating with respect to y, we obtain $f_y(x,y) = x^2 - 2xy^{-3} + g'(y)$

But, we know $f_y(x,y) = x^2 - 2xy^{-3}$ and so $g'(y) = 0 \quad \Rightarrow \quad g(y) = k$. Hence, a potential function is $f(x,y) = x^2 y + xy^{-2} + k$, k a constant.

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2. This is question 16 on page 1107 of the text.

Given $\vec{F} = \left(y^2 z + 2xz^2\right)\vec{i} + 2xyz\,\vec{j} + \left(xy^2 + 2x^2 z\right)\vec{k}$ and C is a curve with parametrization $x = \sqrt{t}$, $y = t+1$, $z = t^2$, $0 \le t \le 1$,

a) Find a function f such that $\nabla f = F$ and b) use this potential function to evaluate $\int_C \vec{F} \cdot d\vec{r}$ along C.

(a) We need to solve: $\begin{array}{l} f_x(x,y,z) = y^2 z + 2xz^2 \\ f_y(x,y,z) = 2xyz \\ f_z(x,y,z) = xy^2 + 2x^2 z \end{array}$ for f.

Again using "partial integration," we find

$$f_x(x,y,z) = y^2 z + 2xz^2 \quad \Rightarrow \quad f(x,y,z) = xy^2 z + x^2 z^2 + g(y,z) \quad \Rightarrow \quad f_y(x,y,z) = 2xyz + g_y(y,z)$$

But, $f_y(x,y,z) = 2xyz$ and so $g_y(y,z) = 0 \quad \Rightarrow \quad g(y,z) = h(z)$. That is, it must be a function of z only.

So, $f(x,y,z) = xy^2 z + x^2 z^2 + h(z) \quad \Rightarrow \quad f_z(x,y,z) = xy^2 + 2x^2 z + h'(z)$

But, $f_z(x,y,z)=xy^2+2x^2z$ and so $h'(z)=0 \quad \Rightarrow \quad h(z)=k$.

Hence, $f(x,y,z)=xy^2z+x^2z^2+k$. Taking $k=0$, we obtain

$f(x,y,z)=xy^2z+x^2z^2$.

b) $F=\nabla f$ and so F is conservative. We only need to know the endpoints of C to evaluate the line integral.

When $t=0$, $(x,y,z)=\left(\sqrt{0},0+1,0^2\right)=(0,1,0)$; when $t=0$, $(x,y,z)=\left(\sqrt{1},1+1,1^2\right)=(1,2,1)$

Hence, $\int_C F\cdot dr = f(1,2,1)-f(0,1,0)=5-0=5$.

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3. The inverse square law of force is $F(\vec{x})=\dfrac{k}{|\vec{x}|^3}\vec{x}$, where k is a constant and $\vec{x}$ is a non-zero vector in R^3. Show that the work done in moving a particle from $\vec{x}_0$ to $\vec{x}_1$ that is very far away from the origin is approximately $\dfrac{k}{|\vec{x}_0|}$.

Let $\vec{x}=<x,y,z>$. Then

$$F(x,y,z)=\frac{k}{\left(\sqrt{x^2+y^2+z^2}\right)^3}<x.y,z>=\frac{k}{\left(x^2+y^2+z^2\right)^{3/2}}<x.y,z>$$

If $F(x,y,z)=P(x,y,z)\vec{i}+Q(x,y,z)\vec{j}+R(x,y,z)\vec{k}$ then

$$P(x,y.z)=\frac{kx}{\left(x^2+y^2+z^2\right)^{3/2}},\ Q(x,y.z)=\frac{ky}{\left(x^2+y^2+z^2\right)^{3/2}},$$

$$R(x,y.z)=\frac{kz}{\left(x^2+y^2+z^2\right)^{3/2}}$$

If F is a conservative field, we would need to solve:

$$f_x(x,y,z) = P(x,y,z) = \frac{kx}{\left(x^2+y^2+z^2\right)^{3/2}}$$

$$f_y(x,y,z) = Q(x,y,z) = \frac{ky}{\left(x^2+y^2+z^2\right)^{3/2}}$$ for a potential function f.

$$f_z(x,y,z) = R(x,y,z) = \frac{kz}{\left(x^2+y^2+z^2\right)^{3/2}}$$

Let's see if we can find such a potential function.

$$f_x(x,y,z) = kx\left(x^2+y^2+z^2\right)^{-3/2} \quad \Rightarrow \quad f(x,y,z) = \frac{k}{2}(-2)\left(x^2+y^2+z^2\right)^{-1/2} + g(y,z)$$

That is, $f(x,y,z) = -k\left(x^2+y^2+z^2\right)^{-1/2} + g(y,z)$ and so

$$f_y(x,y,z) = \frac{k}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2y) + g'(y,z) = \frac{ky}{\left(x^2+y^2+z^2\right)^{3/2}} + g'(y,z)$$

Comparing to the second equation of our system we conclude that

$g'(y,z) = 0 \quad \Rightarrow \quad g(y,z) = h(z)$. So,

$$f(x,y,z) = -k\left(x^2+y^2+z^2\right)^{-1/2} + h(z) \quad \Rightarrow \quad f_z(x,y,z) = \frac{k}{2}\left(x^2+y^2+z^2\right)^{-3/2}(2z) + h'(z)$$

$$= \frac{kz}{\left(x^2+y^2+z^2\right)^{3/2}} + h'(z)$$

Comparing to the third equation of our system we conclude that

$h'(z) = 0 \quad \Rightarrow \quad h(z) = a$, a constant. So,

$$f(x,y,z) = -k\left(x^2+y^2+z^2\right)^{-1/2} + a = -\frac{k}{|\vec{x}|} + a$$

Setting $a = 0$, $f(x,y,z) = -\frac{k}{|\vec{x}|}$ and

$$W = \int_C \vec{F}\cdot d\vec{r} = \int_{\vec{x}_0}^{\vec{x}_1} \vec{\nabla} f \cdot d\vec{r} = f(\vec{x}_1) - f(\vec{x}_0) = -\frac{k}{|\vec{x}_1|} - \left(-\frac{k}{|\vec{x}_0|}\right) = \frac{k}{|\vec{x}_0|} - \frac{k}{|\vec{x}_1|}$$

But $\vec{x}_1$ is very far from the origin and so $\frac{k}{|\vec{x}_1|}$ is a very small number.

Hence, the work done is approximately $\dfrac{k}{|\vec{x}_0|}$.

Practice Exercises 16.3

From the text pages 1106–1107, do problems 1, 5, 7, 11, 13, 15, 17, 19, 23, 25, 29, 31, 33, and 35.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Green's Theorem

Learning Outcomes

Upon completion of Green's Theorem, you should be able to:

- Use Green's Theorem to evaluate line integrals.
- Use Green's Theorem to evaluate double integrals.
- Use Green's Theorem to calculate areas of regions in the plane.

Readings

Read section 16.4, pages 1108 to 1113, in your textbook. Carefully study the examples worked out in the text.

Overview

We have seen how to evaluate vector valued functions along paths between two points. This section is concerned with the integral of such a function around a simple closed path.

Let $\vec{F} = P\vec{i} + Q\vec{j}$ be defined on a region D which is bounded by the simple closed curve C, and suppose P and Q have continuous partial derivatives on an open region containing D. We use the convention that C has a positive orientation, meaning that there is a single counter-clockwise traversal of C such that the region D is always on the left as you move along C.

Green's Theorem tells us that the line integral of such a vector field F can be calculated in terms of a double integral over D. Specifically,

$$\oint_C \vec{F} \cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Here we use the circle on the line integral sign to indicate that the line integral is calculated using the positive orientation of the closed curve C.

The symbolism

$$\int_{\partial D} P\,dx + Q\,dy$$

is also used to indicate a line integral with positively oriented boundary curve.

The theorem is an extremely important one, with many theoretical and practical applications. For example, you can transform a troublesome double integral into a single integral, or vice versa. Green's Theorem can also be used to find the area A of a region D:

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

Stewart gives a proof of Green's Theorem on pages 1109–1110 of the text for the case that D is a simple region. This theorem can be extended to the case where D is a finite union of non-overlapping simple regions with simple, positively oriented, closed boundaries.

Example Exercises

1. Evaluate $\int_C \left(x^2 - y^3\right)dx + \left(x^3 + y^2\right)dy$, where C is the circle $x^2 + y^2 = 9$ traversed in the clockwise direction.

   Let $P(x, y) = x^2 - y^3$, $Q(x, y) = x^3 + y^2$ and let D denote the region enclosed by the circle $x^2 + y^2 = 9$.

   Now, $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(x^2 - y^3\right) = -3y^2$ and $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(x^3 + y^2\right) = 3x^2$; so, P and Q have continuous partial derivatives throughout D. Hence, we can use Green's Theorem to evaluate the line integral.

   But the clockwise direction for C means that C is negatively oriented and $\int_C \left(x^2 - y^3\right)dx + \left(x^3 + y^2\right)dy = -\int_{-C} \left(x^2 - y^3\right)dx + \left(x^3 + y^2\right)dy$.

   Hence, from Green's Theorem, we have that

   $$\int_C \left(x^2 - y^3\right)dx + \left(x^3 + y^2\right)dy = -\oint_{-C} \left(x^2 - y^3\right)dx + \left(x^3 + y^2\right)dy = -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA$$

   Now,

   $$\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA = \iint_D \left(3x^2 - \left(-3y^2\right)\right)dA = 3\iint_D \left(x^2 + y^2\right)dA$$

   This double integral is best evaluated using polar coordinates.
   $D = \{(x, y) \mid x^2 + y^2 \le 9\}$ transforms to $S = \{(r, \theta) \mid 0 \le r \le 3,\ 0 \le \theta \le 2\pi\}$

So,

$$3\iint_D \left(x^2+y^2\right)dA = 3\int_0^{2\pi}\int_0^3 r^2\, r\, dr\, d\theta = 3\int_0^{2\pi}\int_0^3 r^3\, dr\, d\theta$$

$$= 3\int_0^{2\pi}\left[\frac{r^4}{4}\right]_0^3 d\theta$$

$$= \frac{3}{4}\int_0^{2\pi}\left(3^4-0\right)d\theta$$

$$= \frac{243}{4}\int_0^{2\pi} d\theta = \frac{243}{4}(2\pi-0) = \frac{243\pi}{2}$$

And hence $\int_C \left(x^2-y^3\right)dx+\left(x^3+y^2\right)dy = -\dfrac{243\pi}{2}$.

2. Use Green's Theorem to evaluate $\int_C \dfrac{x}{x^2+y^2}dx+\dfrac{y}{x^2+y^2}dy$, where $C = C_1 \cup C_2$ and where C_1 is the parabolic path from (-2,0) to (3,5) along $y = x^2-4$ and C_2 is the line segment from (3,5) to (-2,0).

Let $\vec{F} = P\vec{i}+Q\vec{j}$, where $P(x,y) = \dfrac{x}{x^2+y^2}$ and $Q(x,y) = \dfrac{y}{x^2+y^2}$. .

Now, $\dfrac{\partial P}{\partial y} = \dfrac{\partial}{\partial y}\left(\dfrac{x}{x^2+y^2}\right) = \dfrac{-2xy}{\left(x^2+y^2\right)^2}$ and $\dfrac{\partial Q}{\partial x} = \dfrac{\partial}{\partial x}\left(\dfrac{y}{x^2+y^2}\right) = \dfrac{-2xy}{\left(x^2+y^2\right)^2}$ So,

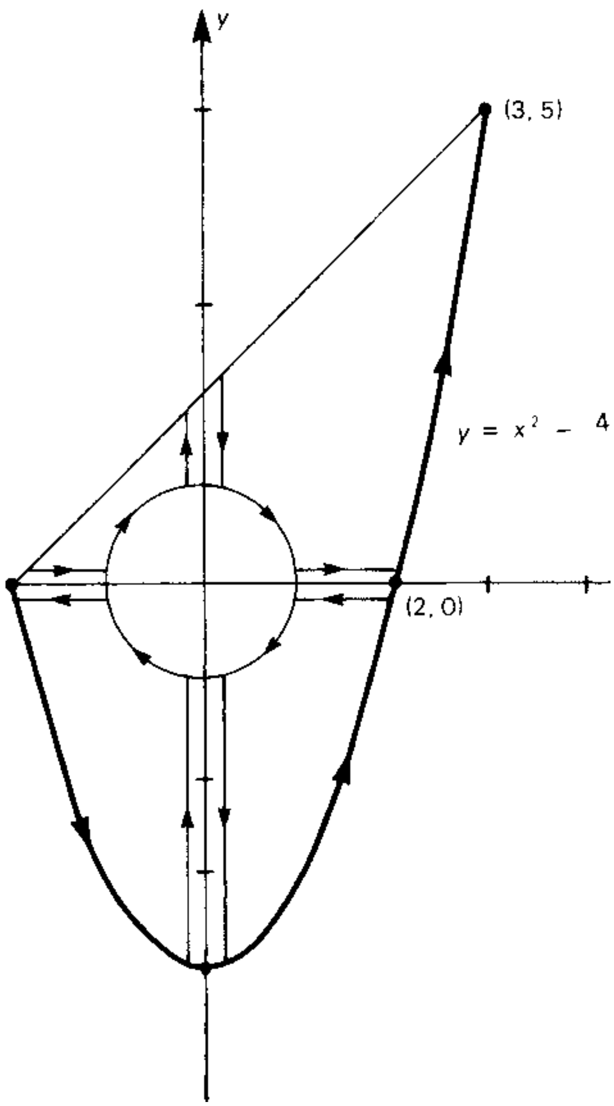
$\dfrac{\partial Q}{\partial x}-\dfrac{\partial P}{\partial y} = 0$

But, F is undefined at (0,0), so we cannot use Green's Theorem on the region enclosed by the parabola and the line segment.

To avoid this problem, let D be the region between $C = C_1 \cup C_2$ and C', the unit circle $x^2+y^2 = 1$, oriented in the clockwise direction so that the region D is on the left as we traverse the circle (as required to use Green's Theorem)

A parametrization of C' is given by: $x = \cos(t)$, $y = -\sin(t)$, $0 \le t \le 2\pi$

The region D is sketched below.



Now $C \bigcup C'$ forms the boundary of D with positive orientation. Taking D decomposed into the union of four simple sub-regions, one in each of the four quadrants of the plane, we can now apply Green's Theorem to get

$$\oint_{C \cup C'} \vec{F} \cdot d\vec{r} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA = \iint_D 0\, dA = 0$$

But,

$$\oint_{C \cup C'} \vec{F} \cdot d\vec{r} = \oint_C \vec{F} \cdot d\vec{r} + \oint_{C'} \vec{F} \cdot d\vec{r} = 0 \quad \Rightarrow \quad \oint_C \vec{F} \cdot d\vec{r} = -\oint_{C'} \vec{F} \cdot d\vec{r} = \oint_{-C'} \vec{F} \cdot d\vec{r}$$

where $-C'$ is given by: $x = \cos(t)$, $y = \sin(t)$, $0 \le t \le 2\pi$.

Now, $\vec{r}(t) = \langle \cos(t), \sin(t) \rangle \quad \Rightarrow \quad \vec{r}'(t) = \langle -\sin(t), \cos(t) \rangle$ and

$$F(x(t), y(t)) = F(\cos(t), \sin(t),) = \left\langle \frac{\cos(t)}{\cos^2(t)+\sin^2(t)}, \frac{\sin(t)}{\cos^2(t)+\sin^2(t)} \right\rangle = \langle \cos(t), \sin(t) \rangle$$

Hence,

$$\begin{aligned}
&\int_C \frac{x}{x^2+y^2}dx + \frac{y}{x^2+y^2}dy \\
&= \oint_C \vec{F} \cdot d\vec{r} \\
&= \oint_{-C'} \vec{F} \cdot d\vec{r} \\
&= \oint_{-C'} \langle \cos(t), \sin(t) \rangle \cdot \langle -\sin(t), \cos(t) \rangle \, dt \\
&= \oint_{-C'} \left(-\sin(t)\cos(t) + \sin(t)\cos(t) \right) dt \\
&= \oint_{-C'} 0 \, dt = 0
\end{aligned}$$

Note: There is nothing special about the unit circle we used in this problem. Any circle centered at the origin of sufficiently small radius so the circular region lies inside our given region could have been used to avoid the problem that our vector field was undefined at the origin.

3. The circulation of a fluid is defined as the line integral of its velocity around a closed path. Consider a fluid that is moving as in a vortex in a circular path centered on the origin of radius r. Assume the angular velocity, $\omega = \dfrac{d\theta}{dt}$, to be constant. Therefore, you can put $\theta = \omega t$. Find an expression for the circulation around the boundary of any region inside the vortex.

The speed of the fluid at position $\vec{x} =< x, y >$ is $\left| v(\vec{x}) \right| = r\dfrac{d\theta}{dt} = r\omega$.

Since the velocity is perpendicular to the radius vector drawn to the point in question, and a unit vector perpendicular to $\vec{x} =< \cos(\theta), \sin(\theta) >$ is $< -\sin(\theta), \cos(\theta) >$, we have that

$$v(\vec{x}) = r\omega < -\sin(\theta), \cos(\theta) >= \omega < -r\sin(\theta), r\cos(\theta) >= \omega < -y, x >=< -\omega y, \omega x >$$

Let $P(x, y) = -\omega y$ and $Q(x, y) = \omega x$

Then, $\dfrac{\partial P}{\partial y} = -\omega$ and $\dfrac{\partial Q}{\partial x} = \omega$.

By Green's Theorem,

$$\oint_C v(\vec{x})\cdot d\vec{x} = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA = \iint_D \left(\omega - (-\omega)\right) dA = 2\omega \iint_D dA = 2\omega A$$

where A is the area of the region enclosed by the boundary.

Practice Exercises 16.4

From the text pages 1113–1115, do problems 3, 7, 9, 13, 17, 19, 21, 25 and 29.

Remember: You are to attempt all of the problems carefully before checking your solutions against those given in the solutions manual.

Curl and Divergence

Learning Outcomes

Upon completion of Curl and Divergence, you should be able to:

- Calculate the divergence of a vector field.
- Calculate the curl of a vector field.
- Prove the identities involving the curl and divergence operators as given in exercises 23–29 on page 1122 of the text.
- Use the curl operator to determine whether a vector field is not conservative.
- State conditions on a vector field F and $curl\ F$ that guarantee that F is a conservative vector field.
- Explain what it means for a vector field to be irrotational.
- Explain what it means for a vector field to be incompressible.

Readings

Read section 16.5, pages 1115–1121, in your textbook. Carefully study the examples worked out in the text.

Overview

This section is an introduction to advanced vector analysis. One powerful tool is the vector operator. The word "operator" emphasizes its nature.

While the operator has vector properties, it only acquires numerical values after it operates on some function. (This is rather like a surgeon who specializes in the removal of tonsils. If the surgeon is all alone in the operating room, no operation occurs. The surgeon must have a patient to operate on.)

The operator in question is called del, and is defined on page 1115 of the text. It is represented by the symbol ∇, and can be expanded in terms of its components in three dimensions as

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z}$$

When del operates on a scalar function $w = f(x, y, z)$ the result is

$$\nabla f = \vec{i}\,\frac{\partial f}{\partial x} + \vec{j}\,\frac{\partial f}{\partial y} + \vec{k}\,\frac{\partial f}{\partial z} = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k}$$

When del operates on a vector function $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ the result is

$$\nabla \cdot F = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle \cdot \langle P, Q, R \rangle = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

The first operation produces a quantity you are already familiar with, the gradient of f. The second operation produces a new quantity, a scalar called the divergence of F, abbreviated $div\ F$.

So, $div\ F = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$

How do we interpret $div\ F$?

If the vector function being operated on is the velocity of a fluid, the divergence of the velocity equals the net outflow of fluid per unit volume per unit of time. So, $div\ F(x, y, z)$ measures the tendency of the fluid to diverge from the point (x, y, z). If $div\ F = 0$, then F is said to be incompressible.

This result is not restricted to fluids. If the vector function being operated on is the electric field intensity, the divergence is proportional to the net flux per unit volume per unit of time. In turn, this can be shown to be proportional to the net positive charge contained per unit volume.

If the vector function has to do with neutrons, the divergence is proportional to the net production of neutrons per unit volume per unit of time.

There are two more ways ∇ may be used. First, the scalar (dot) product of the del operator with itself

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

is called the Laplace operator.

When applied to a scalar function we get

$$\nabla^2 f = \nabla \cdot \nabla f = div(\nabla f) = \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

Second, the vector (cross) product of the del operator with a vector function $\vec{F} = P\vec{i} + Q\vec{j} + R\vec{k}$ is called the <u>curl</u> of F:

$$curl\ F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z}\right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x}\right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)\vec{k}$$

Stewart shows on page 1116 of the text that if $w = f(x, y, z)$ has continuous second-order partial derivatives, then $curl(\nabla f) = \vec{0}$. Since a conservative vector field is one in which $F = \nabla f$ for some scalar function f, then it follows:

If F is a conservative vector field, then $curl\ F = \vec{0}$.

So a necessary condition for a vector field to be conservative is that its curl must be the zero vector. This gives us a way of verifying that a vector field is not conservative.

If we add to the condition $curl\ F = \vec{0}$ that F be defined on all of R^3 (or more generally defined on a simply-connected domain) with continuous partial derivatives for its components functions, then we can conclude that F is conservative.

The curl vector is associated with rotations. For example, if F represents the velocity field in fluid flow, particles near (x, y, z) in the fluid tend to rotate about the axis that points in the direction of the vector $curl\ F(x, y, z)$. The length of the curl vector is a measure of how quickly the particles move around the axis. If the velocity field is such that $curl\ F = \vec{0}$ everywhere inside a closed curve, then the circulation is zero in this region. Such a vector field is called <u>irrotational</u>.

Stewart concludes this section with a re-writing of Green's Theorem in terms of the curl and divergence operators.

Example Exercises

1. Compute the curl and divergence of $F(x,y,z) = e^{xy}\sin(z)\vec{j} + y\tan^{-1}(x/z)\vec{k}$ at $(-\pi, 1, \pi)$.

$$curl\ F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 0 & e^{xy}\sin(z) & y\tan^{-1}(x/z) \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}\left(y\tan^{-1}(x/z)\right) - \frac{\partial}{\partial z}\left(e^{xy}\sin(z)\right)\right)\vec{i} + \left(0 - \frac{\partial}{\partial x}\left(y\tan^{-1}(x/z)\right)\right)\vec{j} + \left(\frac{\partial}{\partial x}\left(e^{xy}\sin(z)\right) - 0\right)\vec{k}$$

$$= \left(\tan^{-1}(x/z) - e^{xy}\cos(z)\right)\vec{i} + \left(0 - \frac{y}{1+\left(x/z\right)^2}\cdot\frac{1}{z}\right)\vec{j} + \left(ye^{xy}\sin(z) - 0\right)\vec{k}$$

$$= \left(\tan^{-1}(x/z) - e^{xy}\cos(z)\right)\vec{i} - \frac{yz}{x^2+z^2}\vec{j} + ye^{xy}\sin(z)\vec{k}$$

At $(-\pi, 1, \pi)$,

$$curl\ F = \left(\tan^{-1}(-\pi/\pi) - e^{-\pi}\cos(\pi)\right)\vec{i} - \frac{(1)\pi}{(-\pi)^2+\pi^2}\vec{j} + (1)e^{-\pi}\sin(\pi)\vec{k}$$

$$= \left(\tan^{-1}(-1) - e^{-\pi}(-1)\right)\vec{i} - \frac{\pi}{2\pi^2}\vec{j} + 0\vec{k}$$

$$= \left(-\frac{\pi}{4} + \frac{1}{e^{\pi}}\right)\vec{i} - \frac{1}{2\pi}\vec{j}$$

$$div\ F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z} = \frac{\partial}{\partial x}(0) + \frac{\partial}{\partial y}\left(e^{xy}\sin(z)\right) + \frac{\partial}{\partial z}\left(y\tan^{-1}(x/z)\right)$$

$$= 0 + xe^{xy}\sin(z) + y\frac{1}{1+(x/z)^2}\left(-xz^{-2}\right)$$

$$= xe^{xy}\sin(z) - \frac{xy}{x^2+z^2}$$

At $(-\pi, 1, \pi)$,

$$div\ F = (-\pi)e^{-\pi}\sin(\pi) - \frac{-\pi}{(-\pi)^2+\pi^2} = 0 + \frac{\pi}{2\pi^2} = \frac{1}{2\pi}$$

2. This is problem 28 on page 1122 of the text.

Prove that $div\left(\nabla f \times \nabla g\right) = 0$.

$$\nabla f \times \nabla g = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} & \frac{\partial g}{\partial z} \end{vmatrix} = \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right)\vec{i} - \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\right)\vec{j} + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)\vec{k}$$

$$= \left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right)\vec{i} + \left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial z}\right)\vec{j} + \left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right)\vec{k}$$

So,

$$\begin{aligned}
div\left(\nabla f \times \nabla g\right) &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial x}\frac{\partial g}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) \\
&= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial z}\right) - \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial y}\right) + \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial z}\frac{\partial g}{\partial x}\right) - \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial z}\right) + \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial x}\frac{\partial g}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial f}{\partial y}\frac{\partial g}{\partial x}\right) \\
&= \frac{\partial^2 f}{\partial x\partial y}\frac{\partial g}{\partial z} + \frac{\partial f}{\partial y}\frac{\partial^2 g}{\partial x\partial z} - \frac{\partial^2 f}{\partial x\partial z}\frac{\partial g}{\partial y} - \frac{\partial f}{\partial z}\frac{\partial^2 g}{\partial x\partial y} + \frac{\partial^2 f}{\partial y\partial z}\frac{\partial g}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial^2 g}{\partial y\partial x} - \frac{\partial^2 f}{\partial y\partial x}\frac{\partial g}{\partial z} - \frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial y\partial z} \\
&\quad + \frac{\partial^2 f}{\partial z\partial x}\frac{\partial g}{\partial y} + \frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial z\partial y} - \frac{\partial^2 f}{\partial z\partial y}\frac{\partial g}{\partial x} - \frac{\partial f}{\partial y}\frac{\partial^2 g}{\partial z\partial x} \\
&= \left(\frac{\partial^2 f}{\partial x\partial y}\frac{\partial g}{\partial z} - \frac{\partial^2 f}{\partial y\partial x}\frac{\partial g}{\partial z}\right) + \left(\frac{\partial f}{\partial y}\frac{\partial^2 g}{\partial x\partial z} - \frac{\partial f}{\partial y}\frac{\partial^2 g}{\partial z\partial x}\right) + \left(\frac{\partial^2 f}{\partial y\partial z}\frac{\partial g}{\partial x} - \frac{\partial^2 f}{\partial z\partial y}\frac{\partial g}{\partial x}\right) + \left(\frac{\partial f}{\partial z}\frac{\partial^2 g}{\partial y\partial x} - \frac{\partial f}{\partial z}\frac{\partial^2 g}{\partial x\partial y}\right) \\
&\quad + \left(\frac{\partial^2 f}{\partial z\partial x}\frac{\partial g}{\partial y} - \frac{\partial^2 f}{\partial x\partial z}\frac{\partial g}{\partial y}\right) + \left(\frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial z\partial y} - \frac{\partial f}{\partial x}\frac{\partial^2 g}{\partial y\partial z}\right) \\
&= \left(\frac{\partial^2 f}{\partial x\partial y} - \frac{\partial^2 f}{\partial y\partial x}\right)\frac{\partial g}{\partial z} + \left(\frac{\partial^2 g}{\partial x\partial z} - \frac{\partial^2 g}{\partial z\partial x}\right)\frac{\partial f}{\partial y} + \left(\frac{\partial^2 f}{\partial y\partial z} - \frac{\partial^2 f}{\partial z\partial y}\right)\frac{\partial g}{\partial x} + \left(\frac{\partial^2 g}{\partial y\partial x} - \frac{\partial^2 g}{\partial x\partial y}\right)\frac{\partial f}{\partial z} \\
&\quad + \left(\frac{\partial^2 f}{\partial z\partial x} - \frac{\partial^2 f}{\partial x\partial z}\right)\frac{\partial g}{\partial y} + \left(\frac{\partial^2 g}{\partial z\partial y} - \frac{\partial^2 g}{\partial y\partial z}\right)\frac{\partial f}{\partial x} \\
&= 0 + 0 + 0 + 0 + 0 + 0 = 0 \quad \left(\text{using Clairaut's Theorem}\right)
\end{aligned}$$

Stewart, J. (2012). *Multivariable calculus* (7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.

3. This is problem 16 on page 1121 of the text.

Determine whether or not the vector field $F(x,y,z)=\vec{i}+\sin(z)\vec{j}+y\cos(z)\vec{k}$ is a conservative vector field. If so, find a potential function for F.

$$curl\ F=\nabla\times F=\begin{vmatrix}\vec{i} & \vec{j} & \vec{k}\\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z}\\ 1 & \sin(z) & y\cos(z)\end{vmatrix}$$
$$=\left(\frac{\partial}{\partial y}\big(y\cos(z)\big)-\frac{\partial}{\partial z}\big(\sin(z)\big)\right)\vec{i}+\left(\frac{\partial}{\partial z}(1)-\frac{\partial}{\partial x}\big(y\cos(z)\big)\right)\vec{j}+\left(\frac{\partial}{\partial x}\big(\sin(z)\big)-\frac{\partial}{\partial y}(1)\right)\vec{k}$$
$$=\big(\cos(z)-\cos(z)\big)\vec{i}+(0-0)\vec{j}+(0-0)\vec{k}$$
$$=\vec{0}$$

F is defined on R^3 and its component functions have continuous partial derivatives of all orders. Hence, F is conservative and so there exists a scalar function f such that $F=\nabla f$. To find f we must solve:

$$\begin{aligned}f_x(x,y,z)&=1\\ f_y(x,y,z)&=\sin(z) \quad \text{for } f.\\ f_z(x,y,z)&=y\cos(z)\end{aligned}$$

$f_x(x,y,z)=1 \quad\Rightarrow\quad f(x,y,z)=x+g(y,z) \quad\Rightarrow\quad f_y(x,y,z)=g_y(y,z)$

But, $f_y(x,y,z)=\sin(z)$ and so $g_y(y,z)=\sin(z) \quad\Rightarrow\quad g(y,z)=y\sin(z)+h(z)$

So, $f(x,y,z)=x+y\sin(z)+h(z) \quad\Rightarrow\quad f_z(x,y,z)=y\cos(z)+h'(z)$

But, $f_z(x,y,z)=y\cos(z)$ and so $h'(z)=0 \quad\Rightarrow\quad h(z)=k$, k a constant.

Hence, $f(x,y,z)=x+y\sin(z)+k$.

Stewart, J. (2012). *Multivariable calculus* (7th ed.). Belmont, CA: Brooks/Cole Cengage Learning.

Practice Exercises 16.5

From the text pages 1121–1123, do problems 5, 7, 13, 17, 19, 21, 23, 25, 27, 29, 31, and 37.

Remember: You are to attempt all of the problems carefully <u>before</u> checking your solutions against those given in the solutions manual.

Unit 6: Summary and Self-Test

You have now worked through Unit 6 in MATH 2111. It is time to take stock of what you have learned, review all of the material, and bring your shorthand notes up to date. A summary of the material covered so far is provided in the following pages. This summary should be modified, added to, and fleshed out to form a solid body of knowledge.

When you have completed your review, you should test your comprehension of the material with a closed book self-administered examination. Put all your notes aside, find a quiet place where you will not be disturbed, and take the examination provided at the end of this unit. You will find some questions straightforward and easy, but others will test your ingenuity.

You will find the solutions to the Unit 6 exam questions, and the point value for each question in the Answer Key provided at the end of this unit. Become your own examiner. If you have done well, according to your personal standards, it is time to prepare for the final exam. If not, then more review and practice is obviously called for.

Summary

Change of Variables

Transformation of coordinates in R^2:

$T : S \to R$, defined by $T(u,v) = (x,y)$, where $x = x(u,v)$, $\quad y = y(u,v)$

$$\text{Jacobian} = \left|\frac{\partial(x,y)}{\partial(u,v)}\right| = \begin{vmatrix} \dfrac{\partial x}{\partial u} & \dfrac{\partial x}{\partial v} \\ \dfrac{\partial y}{\partial u} & \dfrac{\partial y}{\partial v} \end{vmatrix} = \frac{1}{\dfrac{\partial(u,v)}{\partial(x,y)}}$$

$$\iint\limits_{T(S)} f(x,y)\,dx\,dy = \iint\limits_{S} f(x(u,v), y(u,v)) \left|\frac{\partial(x,y)}{\partial(u,v)}\right| du\,dv$$

Transformation of coordinates in R^3:

$T: S \to R$, defined by $T(u,v,w) = (x,y,z)$, where
$x = x(u,v,w)$, $y = y(u,v,w)$, $z = z(u,v,w)$

$$\text{Jacobian} = \left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} & \frac{\partial x}{\partial w} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} & \frac{\partial y}{\partial w} \\ \frac{\partial z}{\partial u} & \frac{\partial z}{\partial v} & \frac{\partial z}{\partial w} \end{vmatrix} = \frac{1}{\frac{\partial(u,v,w)}{\partial(x,y,z)}}$$

$$\iiint_{T(S)} f(x,y,z)\,dx\,dy\,dz = \iiint_{S} f(x(u,v,w), y(u,v,w), z(u,v,w))\left|\frac{\partial(x,y,z)}{\partial(u,v,w)}\right| du\,dv\,dw$$

Line Integrals

If $z = f(x,y)$ is a continuous function and the plane curve C is smooth, the line integral with respect to arc length exists and can be evaluated in terms of a parametrization of C using the formula

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t))\sqrt{(x'(t))^2 + (y'(t))^2}\,dt$$

Notes:

1. The value of the line integral is independent of the parametrization used for the curve, provided that the curve is traversed exactly once as t increases from a to b.

2. For the special case $f(x,y) = 1$, we get:

$$\int_C ds = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2}\,dt = \int_a^b |\vec{r}\,'(t)|\,dt = L$$

where L is the length of the curve C.

3. In the special case the curve C is the line segment joining $(a,0)$ to $(0,b)$ on the x-axis and we parametrize C using x as the parameter, so that $x = x$ and $y = 0$, we obtain

$$\int_C f(x,y)\,ds = \int_a^b f(x,0)\sqrt{(1)^2+(0)^2}\,dx = \int_a^b f(x,0)\,dx$$

That is, the line integral reduces to the definite integral of a single variable function defined on $[a,b]$.

The line integral of f along C with respect to x:

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t))\,x'(t)\,dt$$

The **line integral of f along C with respect to y:**

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t))\,y'(t)\,dt$$

$$\int_C P(x,y)\,dx + \int_C Q(x,y)\,dy = \int_C P(x,y)\,dx + Q(x,y)\,dy$$

A vector representation of the line segment starting at the point which is the head of the vector $\vec{r}_0$ and ending at the point which is the head of the vector $\vec{r}_1$ is $\vec{r}(t) = (1-t)\vec{r}_0 + t\vec{r}_1$.

If $-C$ denotes the curve consisting of the same points as C but with opposite orientation, then

$$\int_{-C} f(x,y)\,dx = -\int_C f(x,y)\,dx \quad \text{and} \quad \int_{-C} f(x,y)\,dy = -\int_C f(x,y)\,dy$$

But if we integrate with respect to arc length, the value of the line integral does not change:

$$\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds$$

Let $\vec{F} = P\vec{i} + Q\vec{j} + R\bar{k}$ denote a continuous force field on R^3. Let C denote a smooth curve defined by a vector function $\bar{r}(t) = < x(t), y(t), z(t) >$, $a \le t \le b$.

$$\int_C \vec{F}\cdot d\vec{r} = \int_a^b F(\vec{r}(t))\cdot \vec{r}\,'(t)\,dt = \int_C P\,dx + Q\,dy + R\,dz$$

Conservative Vector Fields

Fundamental Theorem: If F is a conservative vector field, then

$$\int_C \vec{F}\cdot d\vec{r} = \int_C \vec{\nabla} f\cdot d\vec{r} = f(\vec{r}(b)) - f(\vec{r}(a))$$

where f is a potential function of F; that is, $\nabla f = F$.

That is, a line integral of a conservative vector field is independent of the path joining the two endpoints.

If $F = P\vec{i} + Q\vec{j}$ is a conservative vector field, where P and Q have continuous first partial derivatives, then

$$\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$$

Conversely, if $F = P\vec{i} + Q\vec{j}$ is a continuous vector field on an open simply-connected domain D and $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, then F is conservative.

Green's Theorem

Let $\vec{F} = P\vec{i} + Q\vec{j}$ be defined on a region D that is bounded by the simple closed curve C, and suppose P and Q have continuous partial derivatives on an open region containing D. We use the convention that C has a <u>positive orientation</u>, meaning that there is a single counter-clockwise traversal of C such that the region D is always on the left as you move along C. Then

$$\oint_C \vec{F}\cdot d\vec{r} = \oint_C P\,dx + Q\,dy = \iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right) dA$$

Green's Theorem gives us a formula for the area of a plane region D bounded by a simple closed curve C.

$$A = \oint_C x\,dy = -\oint_C y\,dx = \frac{1}{2}\oint_C x\,dy - y\,dx$$

Curl and Divergence

Vector differential operators:

$$\nabla = \left\langle \frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right\rangle = \vec{i}\,\frac{\partial}{\partial x} + \vec{j}\,\frac{\partial}{\partial y} + \vec{k}\,\frac{\partial}{\partial z} \qquad \text{(del operator)}$$

$$\nabla \cdot \nabla = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \qquad \text{(Laplace operator)}$$

$$curl\ F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ P & Q & R \end{vmatrix} = \left(\frac{\partial R}{\partial y} - \frac{\partial Q}{\partial z} \right)\vec{i} + \left(\frac{\partial P}{\partial z} - \frac{\partial R}{\partial x} \right)\vec{j} + \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right)\vec{k}$$

$curl(\nabla f) = \vec{0}$

If F is conservative, $curl\ F = \vec{0}$.

Conversely, if $F = P\vec{i} + Q\vec{j}$ is a vector field defined on all of R^3 (or more generally defined on a simply-connected domain) with continuous partial derivatives for its components functions and $curl\ F = \vec{0}$, then F is conservative.

$$div\ F = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

Self-Test *(23 marks)*

Treat this as a real test. Do not refer to any course materials. The time for this test is 1.5 hours. Use the answer key provided to mark your test. The point value for each question is posted in the left margin.

[3] 1. Evaluate the double integral $\iint_D xy\,dA$ by making an appropriate change of variables, where

$D = \{(x, y) \mid -1 \le x - y \le 1,\ 0 \le x + y \le 2\}$.

2. Evaluate the line integrals:

[3] a) $\int_C xy\,ds$, where $C:\quad x = t^2,\ y = 2t,\quad 0 \le t \le 1$.

[4] b) $\int_C \vec{F} \cdot d\vec{r}$, where $F(x,y) = \left\langle x+2y, x^2-y^2 \right\rangle$ and C is the triangular path from $(0,0)$ to $(1,0)$ to $(1,1)$ to $(0,0)$.

3. Determine whether or not F is a conservative vector field on the given domain

D. If it is, find a potential function for F.

[2] a) $F(x,y) = \left\langle 3xy^2 + 2y, 3x^2 y + x^2 \right\rangle$, $D = \{(x,y) \mid x^2 + y^2 \leq 5\}$

[2] b) $F(x,y) = e^x \sin(y)\vec{i} + e^x \cos(y)\vec{j}$, $D = R^2$

[2] 4. Evaluate $\int_C y^2\,dx + 2xy\,dy$, where C is the path from $(0,0)$ to $(2,4)$ along the parabola $y = x^2$.

[3] 5. Use Green's Theorem to evaluate

$\int_C \left(e^x + y - 2x^2\right)dx + \left(7x - \sin(y)\right)dy$, where C is the triangular path from $(0,0)$ to $(0,1)$ to $(1,0)$ to $(0,0)$.

[2] 6. Show that the vector field $F(x,y,z) = \left\langle x, xz+3, yz+x \right\rangle$ is not conservative.

[2] 7. Find the divergence of $F(x,y,z) = xy^2 z\,\vec{i} + y^2 \sin(x)\,\vec{j} + ze^y\,\vec{k}$ at $(\pi,1,3)$.

Answer Key

1. $\iint_D xy\,dA$, $D = \{(x, y) \mid -1 \le x - y \le 1, 0 \le x + y \le 2\}$

Let $u = x - y$ and $v = x + y$. Then $u + v = 2x \Rightarrow x = \frac{1}{2}(u+v)$ and

$u - v = -2y \Rightarrow y = -\frac{1}{2}(u-v)$. So, $xy = \frac{1}{2}(u+v)\left(-\frac{1}{2}(u-v)\right) = -\frac{1}{4}(u^2 - v^2)$

Now, $S = \{(u, v) \mid -1 \le u \le 1, 0 \le v \le 2\}$.

So,

$$\begin{aligned}
\iint_D xy\,dA &= \int_{-1}^{1}\int_{0}^{2} -\frac{1}{4}(u^2 - v^2)\,dv\,du \\
&= -\frac{1}{4}\int_{-1}^{1}\left[u^2 v - \frac{v^3}{3}\right]_0^2 du \\
&= -\frac{1}{4}\int_{-1}^{1}\left(2u^2 - \frac{2^3}{3} - 0\right) du \\
&= -\frac{1}{4}\int_{-1}^{1}\left(2u^2 - \frac{8}{3}\right) du \\
&= -\frac{1}{4}\left[\frac{2u^3}{3} - \frac{8u}{3}\right]_{-1}^{1} \\
&= -\frac{1}{4}\left(\frac{2}{3} - \frac{8}{3} - \left(-\frac{2}{3} + \frac{8}{3}\right)\right) \\
&= 1
\end{aligned}$$

2. a) $\int_C xy\,ds$, where $C: \quad x = t^2, \; y = 2t, \quad 0 \le t \le 1$.

$x'(t) = 2t$; $y'(t) = 2$

So,

$$\begin{aligned}
\int_C xy\,ds &= \int_0^1 x(t)y(t)\sqrt{(x'(t))^2 + (y'(t))^2}\,dt \\
&= \int_0^1 t^2(2t)\sqrt{(2t)^2 + (2)^2}\,dt = 2\int_0^1 t^3\sqrt{4(t^2+1)}\,dt = 4\int_0^1 t^3\sqrt{t^2+1}\,dt
\end{aligned}$$

Use the substitution: $u = t^2 + 1 \Rightarrow t^2 = u - 1$, $du = 2t\,dt$

When $t = 0$, $u = 0^2 + 1 = 1$; when $t = 1$, $u = 1^2 + 1 = 2$.

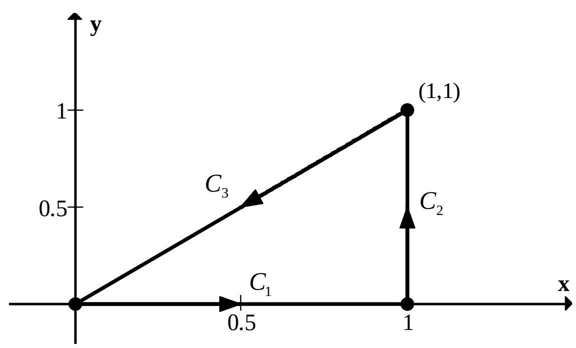
Hence,

$$\begin{aligned}
\int_C xy\,ds &= 4\int_0^1 t^3\sqrt{t^2+1}\,dt \\
&= 2\int_0^1 t^2\sqrt{t^2+1}\;2t\,dt \\
&= 2\int_0^1 (u-1)\sqrt{u}\,du \\
&= 2\int_0^1 (u^{3/2}-u^{1/2})\,du \\
&= 2\left[\frac{2}{5}u^{5/2}-\frac{2}{3}u^{3/2}\right]_1^2 \\
&= 2\left(\frac{2}{5}(2)^{5/2}-\frac{2}{3}(2)^{3/2}-\left(\frac{2}{5}-\frac{2}{3}\right)\right) \\
&= 2\left(\frac{8\sqrt{2}}{5}-\frac{4\sqrt{2}}{3}+\frac{4}{15}\right) \\
&= 2\left(\frac{4\sqrt{2}}{15}+\frac{4}{15}\right)=\frac{8}{15}\left(\sqrt{2}+1\right)
\end{aligned}$$

b) $\displaystyle\int_C \vec{F}\cdot d\vec{r}$, where $F(x,y)=\left\langle x+2y, x^2-y^2\right\rangle$ and C is the triangular path from $(0,0)$ to $(1,0)$ to $(1,1)$ to $(0,0)$.

A picture of the path C is shown to the right. It is composed of three smooth paths

C_1, C_2, C_3.



C_1: $\vec{r}(t)=(1-t)<0,0>+t<1,0>=<t,0>$, $0\le t\le 1$

C_2: $\vec{r}(t)=(1-t)<1,0>+t<1,1>=<1,t>$, $0\le t\le 1$

C_3: $\vec{r}(t)=(1-t)<1,1>+t<0,0>=<1-t,1-t>$, $0\le t\le 1$

$$\int_C \vec{F}\cdot d\vec{r}=\int_a^b F(\vec{r}(t))\cdot\vec{r}\,'(t)\,dt$$

Setting up a table of information:

Path	$\vec{r}(t)$	$\vec{r}\,'(t)$	$F(\vec{r}(t))$	$F(\vec{r}(t))\cdot\vec{r}\,'(t)$
C_1	$<t,0>$	$<1,0>$	$<t,t^2>$	t
C_2	$<1,t>$	$<0,1>$	$<1+2t,1-t^2>$	$1-t^2$
C_3	$<1-t,1-t>$	$<-1,-1>$	$<3-3t,0>$	$3t-3$

Hence,

$$\begin{aligned}\int_C \vec{F}\cdot d\vec{r} &= \int_{C_1}\vec{F}\cdot d\vec{r}+\int_{C_2}\vec{F}\cdot d\vec{r}+\int_{C_3}\vec{F}\cdot d\vec{r}\\ &= \int_0^1 t\,dt+\int_0^1\left(1-t^2\right)dt+\int_0^1(3t-3)\,dt\\ &= \int_0^1\left(-2+4t-t^2\right)dt\\ &= -2t\Big]_0^1+2t^2\Big]_0^1-\frac{t^3}{3}\Bigg]_0^1\\ &= -2(1-0)+2(1-0)-\frac{1}{3}(1-0)=-\frac{1}{3}\end{aligned}$$

3. a) $F(x,y)=\left\langle 3xy^2+2y, 3x^2y+x^2\right\rangle$, $D=\{(x,y)\mid x^2+y^2\le 5\}$

Let $P(x,y)=3xy^2+2y$ and $Q(x,y)=3x^2y+x^2$

Since the domain of the vector field is simply connected, F is conservative if and only if $\dfrac{\partial P}{\partial y}=\dfrac{\partial Q}{\partial x}$.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(3xy^2 + 2y\right) = 6xy + 2\,;\ \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(3x^2 y + x^2\right) = 6xy + 2x$$

Since $\frac{\partial P}{\partial y} \neq \frac{\partial Q}{\partial x}$, F is not conservative.

b) $F(x, y) = e^x \sin(y)\vec{i} + e^x \cos(y)\vec{j}$, $D = R^2$

Let $P(x, y) = e^x \sin(y)$ and $Q(x, y) = e^x \cos(y)$

Since the domain of the vector field is simply connected, F is conservative if and only if $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.

$$\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(e^x \sin(y)\right) = e^x \cos(y)\,;\ \frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(e^x \cos(y)\right) = e^x \cos(y)$$

Since $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$, F is conservative.

4. $\int_C y^2\,dx + 2xy\,dy$, where C is the path from $(0,0)$ to $(2,4)$ along the parabola $y = x^2$.

We parametrize the curve C as follows:

Let $x = t$. Then $y = t^2$.

So, $C: \vec{r}(t) = <t, t^2>$, $0 \le t \le 2$

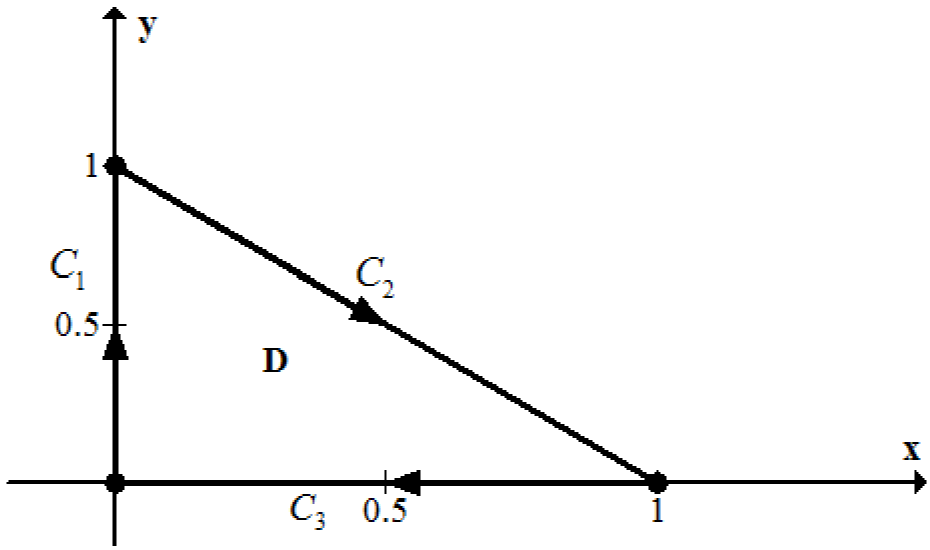
Now, $x = t \quad \Rightarrow \quad dx = dt$, $y = t^2 \quad \Rightarrow \quad dy = 2t\,dt$

Hence,

$$\begin{aligned}
\int_C y^2\,dx + 2xy\,dy &= \int_0^2 \left(t^2\right)^2 dt + \int_0^2 2t\left(t^2\right)2t\,dt \\
&= \int_0^2 t^4 dt + \int_0^2 4t^4\,dt \\
&= \int_0^2 5t^4\,dt \\
&= 5\left[\frac{t^5}{5}\right]_0^2 \\
&= 2^5 - 0 = 32
\end{aligned}$$

5. $\int_C \left(e^x + y - 2x^2\right)dx + \left(7x - \sin(y)\right)dy$, where C is the triangular path from $(0,0)$ to $(0,1)$ to $(1,0)$ to $(0,0)$.

A picture of the path C is shown below. It is composed of three smooth paths C_1, C_2, C_3.



Notice this is a negatively oriented path $C = C_1 \cup C_2 \cup C_3$. The path $-C$ has the positive orientation required for Green's Theorem.

Let $P(x,y) = e^x + y - 2x^2$ and $Q(x,y) = 7x + \sin(y)$

Then $\frac{\partial P}{\partial y} = \frac{\partial}{\partial y}\left(e^x + y - 2x^2\right) = 1$ and $\frac{\partial Q}{\partial x} = \frac{\partial}{\partial x}\left(7x + \sin(y)\right) = 7$

Since P and Q have continuous partial derivatives on R^2, by Green's Theorem,

$$\begin{aligned}
\int_C \left(e^x + y - 2x^2\right)dx + \left(7x - \sin(y)\right)dy &= -\int_{-C} \left(e^x + y - 2x^2\right)dx + \left(7x - \sin(y)\right)dy \\
&= -\iint_D \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)dA \\
&= -\iint_D (7-1)\,dA \\
&= -6\iint_D dA = -6(\text{Area of D})
\end{aligned}$$

Since the region D is triangular, its area is $A = \frac{1}{2}(1)(1) = \frac{1}{2}$.

Hence,

$$\int_C \left(e^x + y - 2x^2\right)dx + \left(7x - \sin(y)\right)dy = -6(0.5) = -3$$

6. $F(x,y,z) = \langle x, xz+3, yz+x\rangle$ has continuous second partial derivatives since its component functions are polynomial. Recall that, if F is conservative, then $curl\ F = \vec{0}$.

But,

$$curl\ F = \nabla \times F = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ \dfrac{\partial}{\partial x} & \dfrac{\partial}{\partial y} & \dfrac{\partial}{\partial z} \\ x & xz+3 & yz+x \end{vmatrix}$$

$$= \left(\frac{\partial}{\partial y}(yz+x) - \frac{\partial}{\partial z}(xz+3)\right)\vec{i} - \left(\frac{\partial}{\partial x}(yz+x) - \frac{\partial}{\partial z}(x)\right)\vec{j} + \left(\frac{\partial}{\partial x}(xz+3) - \frac{\partial}{\partial y}(x)\right)\vec{k}$$

$$= (z-x)\vec{i} - (1-0)\,\vec{j} + (z-0)\vec{k}$$

$$= \langle z-x, -1, z\rangle \neq \vec{0}$$

Therefore, F is not conservative.

7. $F(x,y,z) = xy^2z\,\vec{i} + y^2\sin(x)\,\vec{j} + ze^y\,\vec{k}$

Let $P(x,y,z) = xy^2z$, $Q(x,y,z) = y^2\sin(x)$ and $R(x,y,z) = ze^y$

$$div\ F = \nabla \cdot F = \frac{\partial P}{\partial x} + \frac{\partial Q}{\partial y} + \frac{\partial R}{\partial z}$$

$$= y^2z + 2y\sin(x) + e^y$$

At $(\pi, 1, 3)$,

$$div\ F = 1^2(3) + 2(1)\sin(\pi) + e^1 = 3 + 0 + e = 3 + e$$

Epilogue

Within every epilogue lies a prologue. You have reached the point where you have mathematical tools for many physical applications… and you have reached the end of MATH 2111. It would be interesting to go on to the remaining sections of Chapter 16 in Stewart's textbook. There you would learn how to integrate scalar functions over surfaces

$$\iint_S f(x,y,z)\,dS = \iint_D f(\vec{r}(u,v))\left|\vec{r}_u \times \vec{r}_v\right| dA$$

and vector functions over surfaces

$$\iint_S \vec{F}\cdot\vec{n}\,dS = \iint_D \vec{F}\cdot\left(\vec{r}_u \times \vec{r}_v\right) dA$$

This would permit you to calculate the flow of water out of a surface, the flux of electric lines of force out of a surface, etc.

You would learn about Stokes's Theorem. You would also learn about Gauss's Theorem, which permits you to transform a volume integral into a surface integral. But we have to call a halt somewhere. In your special discipline you may encounter many applications of the theory, and you will have Stewart to refer to if you need something not covered in this course.

Good hunting!