Advanced Mathematical Logic - Exercises

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1 Exercise 18

Show that if t is a ground term, then there is a $k \in \mathbb{N}$ such that $\mathbf{Q} \vdash \ t = k$.

Solution. The theory $\mathbf{Q}$ and the here used language $L = (\mathbb{N}, \{0\backslash 0, s\backslash 1, +\backslash 2, \cdot\backslash 2, = \backslash 2, < \backslash 2\})$ are defined in section 7 "Formal arithmetic". We prove the proposition via structural induction. To avoid confusion, we'll denote $\mathbf{Q}$'s language-level equality as $=$ and our syntactic equality as $\equiv$. Note that, in addition to $\mathbf{Q}$'s axioms, we also need the equality axioms Refl, Symm, Trans and Ext. Ext is the axiom schema of extensionality and allows us to replace a subterm x_i of $f(x_1, \ldots, x_n)$ with a subterm y_i if $x_i = y_i$ (for all $f \in L$ and all $1 \le i \le n$).

$$\begin{aligned}
&\text{Refl} \equiv [\forall x]\, x = x, \\
&\text{Symm} \equiv [\forall x, y]\, x = y \Rightarrow y = x, \\
&\text{Trans} \equiv [\forall x, y, z]\, x = y \wedge y = z \Rightarrow x = z, \\
&\text{Ext}_{f,i} \equiv [\forall x_1, \ldots, x_n, y_i]\, x_i = y_i \Rightarrow f(x_1, \ldots, x_i, \ldots, x_n) = f(x_1, \ldots, y_i, \ldots, x_n).
\end{aligned}$$

Base case. $t \equiv s(\cdots s(0) \cdots) \equiv s^n(0)$. This follows from Refl: $s^n(0) = s^n(0)$.

Step case "$+_1$". $t \equiv t' + 0$. IH: $t' = s^n(0)$.
Per (3), $t' + 0 = t'$.
Per Trans, $(t' + 0 = t') \wedge (t' = s^n(0)) \Rightarrow (t' + 0 = s^n(0))$.
Therefore, $t' + 0 = s^n(0)$.

Step case "$+_2$". $t \equiv t' + s(r)$. IH: $t' = s^n(0)$ and $r = s^m(0)$ and $s^n(0) + s^m(0) = s^{n+m}(0)$.
Per (4), $t' + s(r) = s(t' + r)$.
Now we apply Symm two times, followed by $\text{Ext}_{+,1}$ and $\text{Ext}_{+,2}$, instantiating x_i,y_i with the parts of the IH:

$$\begin{array}{lcl}
t' = s^n(0) & \Rightarrow & s^n(0) = t' \\
r = s^m(0) & \Rightarrow & s^m(0) = r \\
s^n(0) = t' & \Rightarrow & s^n(0) + s^m(0) = t' + s^m(0) \\
s^m(0) = r & \Rightarrow & s^n(0) + s^m(0) = t' + r
\end{array}$$

We now know that $s^n(0) + s^m(0) = t' + r$. Applying Symm, we get $t' + r = s^n(0) + s^m(0)$. We again apply $\text{Ext}_s, 1$ to the term $s(t' + r)$:

$$t' + r = s^n(0) + s^m(0) \quad \Rightarrow \quad s(t' + r) = s(s^n(0) + s^m(0))$$

We apply $\text{Ext}_s, 1$ again to this, using the third part of the IH:

$$s^n(0) + s^m(0) = s^{n+m}(0) \quad \Rightarrow \quad s(s^n(0) + s^m(0)) = s(s^{n+m}(0))$$

Through repeated application of Trans, we get

$$t \equiv t' + s(r) = s(t' + r) = s(s^n(0) + s^m(0)) = s(s^{n+m}(0)) \equiv s^{n+m+1}(0)$$

Step case "$\cdot_1$". $t \equiv t' \cdot 0$. IH: $t' = s^n(0)$.
Per (5), $t' + 0 = 0$.
Per Trans, $(t' \cdot 0 = 0) \wedge (t' = s^n(0)) \Rightarrow (t' \cdot 0 = 0)$.
Therefore, $t' \cdot 0 = 0$.

Step case "$\cdot_2$". $t \equiv t' \cdot s(r)$. IH: $t' = s^n(0)$ and $r = s^m(0)$ and $s^n(0) \cdot s^m(0) = s^{n \cdot m}(0)$ and $s^{n \cdot m}(0) + s^n(0) = s^{n \cdot (m+1)}(0)$.
Per (6), $t' \cdot s(r) = (t' \cdot r) + t$.

This case is basically analogous to $+_2$. We again apply Sym and Ext$_{\cdot,1}$, Ext$_{\cdot,2}$:

$$\begin{array}{lcl} t' = s^n(0) & \Rightarrow & s^n(0) = t' \\ r = s^m(0) & \Rightarrow & s^m(0) = r \\ s^n(0) = t'(0) & \Rightarrow & s^n(0) \cdot s^m(0) = t' \cdot s^m(0) \\ s^m(0) = r & \Rightarrow & s^n(0) \cdot s^m(0) = t' \cdot r \end{array}$$

Through Sym, we get $t' \cdot r = s^n(0) \cdot s^m(0)$ and, through the IH and Trans, $t' \cdot r = s^{n \cdot m}(0)$. We now apply Ext$_{+,1}$, Ext$_{+,2}$ to $(t' \cdot r) + t'$:

$$\begin{array}{lcl} t' \cdot r = s^{n \cdot m}(0) & \Rightarrow & (t' \cdot r) + t' = s^{n \cdot m}(0) + t' \\ t' = s^n(0) & \Rightarrow & (t' \cdot r) + t' = s^{n \cdot m}(0) + s^n(0) \end{array}$$

We apply the last part of the IH and Trans to get

$$(t' \cdot r) + t' = s^{n \cdot m}(0) + s^n(0) = s^{n \cdot (m+1)}(0)$$

The induction hypotheses (especially $s^n(0) + s^m(0) = s^{n+m}(0)$ and $s^n(0) \cdot s^m(0) = s^{n \cdot m}(0)$) might seem problematic, but these are always indeed always proven in the last lines of "$+_2$" and "$\cdot_2$". The hypothesis $s^{n \cdot m}(0) + s^n(0) = s^{n \cdot (m+1)}(0)$ can be derived from these two if we substitute suitable values for n and m.

The induction in this proof is not part of **Q**, but works on a metalinguistical level. Since t is a concrete (but arbitrary) term, this is not a problem, however: for any given t, we can unfold the definitions of **Q**'s formulas and obtain a finitely long proof which, *is constructed by induction*, but isn't inductive itself.

2 Exercise 19

Show that if

1. If s, t are ground terms, then either $\mathbf{Q} \vdash\ s = t$ or $\mathbf{Q} \vdash\ s \neq t$.

   Solution. In the previous example, we showed that all ground terms s, t are equal ($=$) to terms of the form $s^n(0), s^m(0)$. If $n = m$, then, through Refl, $\mathbf{Q} \vdash\ s = t$. Suppose, on the other hand, that $n \neq m$ and, w.l.o.g., $n < m$. We can give an indirect inductive proof:

Step case. If $s(s^{n-1}(0)) = s(s^{m-1}(0))$, then, per (2), $s^{n-1}(0) = s^{m-1}(0)$.

Base case. Since we assumed $n < m$, we must at some point come to the assertion that $0 = s^{m-k}(0)$ (for some k). However, this contradicts (1). Consequently, $s^n(0) = s^m(0)$ cannot hold if $n \neq m$ and thus, $s^n(0) \neq s^m(0)$.

   We can encode this proof in $\mathbf{Q}$ through the following formula:

$$s^n(0) = s^m(0) \Rightarrow s^{n-1}(0) = s^{m-1}(0) \Rightarrow \cdots \Rightarrow 0 = s^{m-k}(0)$$

   By using $0 \neq s(x)$, we show $\neg(0 = s^{m-1}(0))$ and therefrom "roll up" the chain of implications until we get $\neg(s^n(0) = s^m(0))$.

2. If s, t are ground terms, then either $\mathbf{Q} \vdash\ s > t$ or $\mathbf{Q} \vdash\ s = t$ or $\mathbf{Q} \vdash\ s < t$.

   Solution. This follows immediately from (9). The semantics of $Q \vdash\ s < t \vee s = t \vee s > t$ are precisely "$\mathbf{Q} \vdash\ s > t$ or $\mathbf{Q} \vdash\ s = t$ or $\mathbf{Q} \vdash\ s < t$".

3 Exercise 20

Prove Proposition 10: for all $k \in \mathbb{N}$ we have $\mathbf{Q} \vdash\ [\forall x]\, x < k \Leftrightarrow (x = 0 \vee x = 1 \vee \cdots \vee x = (k-1))$.

Solution. We can unroll (8) by repeatedly instantiating y to attain this formula. We can construct the proof inductively, in a sense: we construct a proof for $k = 1$ and, having a proof of $k = n$, we can construct a proof for $k = n+1$. Merely the *construction* of the proof is inductive, the proof itself won't be.

Optional base case. It's not clear whether the formula is defined for $k = 0$, but we can do so if we assume the empty disjunction to be $\perp$ (the neural element of $\vee$).

$$
\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\mathbf{Q}', x_0 < 0 \vdash\ \perp, x_0 < 0}{\mathbf{Q}', \neg(x_0 < 0), x_0 < 0 \vdash\ \perp}\,\neg l}{\mathbf{Q}, x_0 < 0 \vdash\ \perp}\,\forall l}{\mathbf{Q} \vdash\ x_0 < 0 \Rightarrow \perp}\,\Rightarrow r \qquad \dfrac{\dfrac{\dfrac{\text{(Note: } \vdash\ A \equiv \top \vdash\ A)}{\vdash\ \top, x_0 < 0}}{\perp \vdash\ x_0 < 0}\,\neg l}{\vdash\ \perp \Rightarrow x_0 < 0}\,\Rightarrow r}{\mathbf{Q} \vdash\ x_0 < 0 \Rightarrow \perp \wedge \perp \Rightarrow x_0 < 0}\,\wedge r}{\mathbf{Q} \vdash\ x_0 < 0 \Leftrightarrow \perp}\,\text{def.} \Leftrightarrow}{\mathbf{Q} \vdash\ [\forall x]\, x < 0 \Leftrightarrow \perp}\,\forall r
$$

As we can see, even this case is quite cumbersome; I will therefore sketch the other two somewhat more informally.

Base case. Let $k = 1 = s(0)$. We have to construct a proof s.t.

$$\mathbf{Q} \vdash\ [\forall x]\, x < s(0) \Leftrightarrow x = 0$$

We can instantiate (8) with $y \to s(0)$. It becomes:

$$[\forall x]\, x < s(0) \Leftrightarrow x < 0 \vee x = 0$$

From (7), we know that $x < 0$ is false and thus, if we appropriately unpack and re-pack the formula above, we get, $[\forall x]\, x < s(0) \Leftrightarrow x = 0$ remains, which is what we wanted.

1. Let $k = n + 1 = s^{n+1}(0)$ and let us assume the existence of a proof P_n for $k = n$ as the IH — that is:

$$\dfrac{P_n}{\mathbf{Q} \vdash\ [\forall x]\, x < s^n(0) \Leftrightarrow (x = 0 \vee \cdots \vee x = s^{n-1}(0))}$$

From this, we construct a proof P_{n+1} by instantiating (8) with $y \rightarrow s^{n+1}(0)$, getting

$$[\forall x]\, x < s^{n+1}(0) \Leftrightarrow x < s^n(0) \vee x = s^n(0)$$

Now we use the IH and replace $s^n(0)$ with $(x = 0 \vee \cdots \vee x = s^{n-1}(0))$, again by unpacking and re-packing the formula according to the rules of $\Leftarrow$ and $\forall r$. We get:

$$[\forall x]\, x < s^{n+1}(0) \Leftrightarrow x = 0 \vee \cdots \vee x = s^{n-1}(0) \vee x = s^n(0)$$

If we write this procedure down as an LK proof, we get P_{n+1} s.t.

$$\frac{P_{n+1} \text{ (containing } P_n)}{[\forall x]\, x < s^{n+1}(0) \Leftrightarrow x = 0 \vee \cdots \vee x = s^{n-1}(0) \vee x = s^n(0)}$$

4 Exercise 21

Prove that if F is a ground formula, then either $\mathbf{Q} \vdash F$ or $\mathbf{Q} \vdash \neg F$.

Solution. We can proceed via structural induction. The base cases consists of atoms of the form $s = t$ or $s < t$, since $=$ and $<$ are the only two predicates in L. The step cases are formed via logical connectives.

Base case "$=$". Let F be an atom of the form $s = t$. In Exercise 19, showed that, if s, t are ground terms, then $\mathbf{Q} \vdash s = t$ or $\mathbf{Q} \vdash s \neq t$.

Base case "$<$". Let F be an atom of the form $s < t$. Also in Exercise 19, we showed that $\mathbf{Q} \vdash s < t$ or $\mathbf{Q} \vdash s = t$ or $\mathbf{Q} \vdash t < s$. Two sub-cases:

- If $\mathbf{Q} \vdash s < t$, then $\mathbf{Q} \vdash F$.
- If $\mathbf{Q} \vdash s = t$ or $\mathbf{Q} \vdash t < s$, then, $s \neq 0 \wedge \cdots \wedge s \neq t - 1$[1]. Per Exercise 20, this is a direct negation of $s < t$. Thereby, we can prove $s \nless t$.

Step case. Let F be $\neg F_1$, $F_1 \vee F_2$ or $F_1 \wedge F_2$. Without quantifiers, any complete, propositional calculus (like LK) suffices to show $\mathbf{Q} \vdash F$ or $\mathbf{Q} \vdash \neg F$.

Prove Proposition 11: If $F(x)$ is a formula with x being the only free variable, then $\mathbb{N} \models [\exists x]\, F(x)$ iff $\mathbf{Q} \vdash [\exists x]\, F(x)$.

Solution.

$\Rightarrow$-direction. Suppose that $\mathbb{N} \models [\exists x]\, F(x)$. Then there exists a witness n s.t. $F(n)$ is true. Since $F(n)$ is ground, there exists a proof P_F for $F(n)$ with the theory $\mathbf{Q}$, as we showed above. That proof can be transformed into one of $[\exists x]\, F(x)$ thus:

$$\cfrac{\cfrac{P_F}{\mathbf{Q} \vdash F(n)}}{\mathbf{Q} \vdash F(x)}\ \exists r$$

$\Leftarrow$-direction. Suppose that $\mathbf{Q}$ is consistent. Since we know that LK is sound and complete, it follows that LK with theory $\mathbf{Q}$ is also sound — that is, if $\mathbf{Q} \vdash [\exists x]\, F(x)$, then $\mathbb{N} \models [\exists x]\, F(x)$. $\mathbf{Q}$ is consistent if it has a model; we assume $\mathbb{N}$ to be such a model, although no proof of that exists in $\mathbf{Q}$ itself.

[1] This is so because otherwise, there would be two distinct numbers n_1, n_2 s.t. $n_1 \neq n_2$ and $s = n_1$ and $s = n_2$. Applying Trans would then lead to a contradiction.