January 12, 2026

Continued Odyssey of the Figure Eight Puzzle

Justin Brown

Background

The infamous figure 8 puzzle was first published in a newsletter by Stewart Coffin in 1974 with a purposefully vague write-up. Coffin discusses it in his 1985 booklet *Puzzle Craft* and says “Many persons have been baffled by it” [2]. It seems simple enough and is pictured below. The goal is to remove the loop of string from the wire frame. In this paper, I will refer to the loop of string as the “ring”, but just keep in mind that it is stretchable and as long as needed. You can stretch, bend, push, pull as much as you want as long as you don’t do any cutting or gluing (i.e. only constrained topologically, not geometrically).

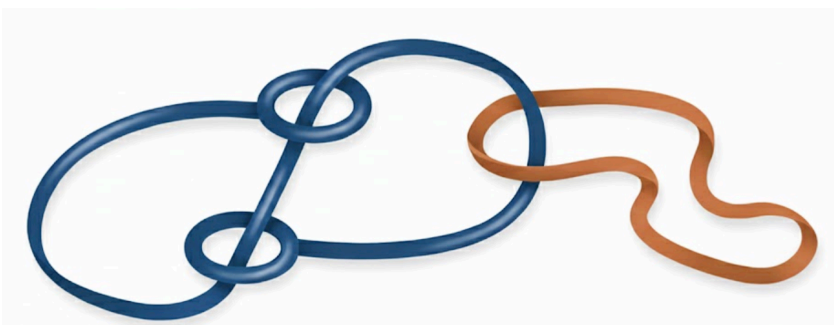

Figure 1: The Figure 8 Puzzle. The blue frame is made of wire and the orange “ring” is a loop of string.

This puzzle appeared in the September 1976 issue of *Games and Puzzles* and also the 1978 book *Creative Puzzles of the World*. Both sources claimed this puzzle was solvable, although there is a playful message in *Creative Puzzles* when describing the solution: “Now the cord should come free – or should it? After all, no one has proved it impossible” [1]. *Games and Puzzles* published 2 responses from readers claiming the puzzle was not possible, with one giving a somewhat convincing argument to its impossibility.

In an article by Coffin in the 1999 Martin Gardner tribute book, Coffin says he “had continued to receive numerous requests for the solution” and “was rather tired of the whole thing” [3]. He finally confesses he believed the puzzle was impossible and mentions a 7-page proof he received from Japan, although he did not examine it closely. There have been two published proofs of the impossibility of the figure 8 puzzle using somewhat sophisticated topological and knot theory techniques [4,5]. There was also a 2 part write up of the figure 8 puzzle in the *Cubism for Fun* magazine: issues 80 and 81 which also has a proof of the impossibility using knot theory and Jones polynomials [6,7].

With several proofs of the impossibility of disentangling the figure 8 puzzle, it seems that this subject should be put to rest. However the techniques used in the above proofs are fairly advanced and not easily followed by people unfamiliar with topology. I would like to propose (in my opinion) a more elementary proof. There are still some people out there that claim they have solved the figure 8 puzzle (although unfortunately unable to reproduce it). The aim of this paper is to provide a proof which can be followed with just some basic background in knot theory.

Crash Course in Knot Theory

Here is a quick overview of some of the concepts needed for the proof. There are lots of resources out there that go into more detail if needed, but I included some of the relevant information here to make this paper as self-contained as possible.

A **knot** is a loop that is tangled up and embedded in 3 dimensional space. When most people think of a knot, they think of a shoelace or the tangled up wires of their headphones - but a mathematical knot has one important difference - the ends are glued together to form a loop, so there is no way to unravel it.

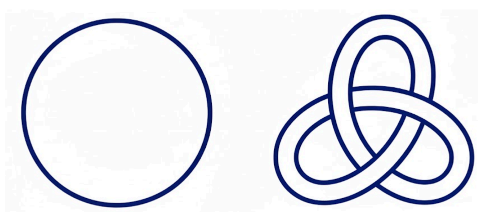

Figure 2: Examples of simple knots. The one on the left is the **Unknot**. The one on the right is a **Trefoil**

Two knots are **toplogically equivalent** if they can be arranged to look exactly the same, without cutting or gluing either one. It is convenient to represent knots as flat 2-d **knot diagrams**, where the over and under crossings are preserved (Figure 2 shows examples of knot diagrams). It turns out that if two knots are equivalent, you only need 3 types of moves to transform one knot diagram into the other. These are known as **Reidemeister moves**. Efficiently determining if two knots are equivalent is a hard and open problem. After all, maybe a few more Reidemeister moves might get you to the desired shape. How do you know for certain there is not some ingenious sequence of Reidemeister moves that will transform a trefoil knot into the unknot?

To show knots are not equivalent, mathematicians have developed several **knot invariants** that are not changed under Rediemeister moves. If two knots have different values of the invariant, then we know for certain they are not the same knot. However, just because two knots have the same value for an invariant, it doesn't mean they are equivalent! Also, a knot invariant may fail to show two knots are different even if they are.

One important and well-known family of invariants are the **Fox N-colorings**, named for Ralph Fox (although discovered independently by others). For a given N, assign to each segment in the knot diagram an integer from 0 to N-1, such that twice the over-strand equals the sum of the two under-strands (mod N) at every crossing. For each crossing, there will be 1 over-strand (a) and two under-strands (b and c), as shown in Figure 3.

Figure 3: An N-coloring assigns each segment a number 0 to N-1, such that the above equation is satisfied at all crossings.

The number of distinct, valid ways that integers can be assigned to a knot (also known as colorings) is invariant under Reidemeister moves. Therefore, no matter how you deform the knot, the number of colorings of any associated knot diagram remains the same.

You can always color a knot with just 1 color and satisfy the constraints since 2a = a+a. These are known as **trivial colorings**. Each knot has at least N colorings : namely the N trivial colorings that use just 1 of the N colors.

3-colorability is an important case of Fox N-coloring. In this case, the arithmetic constraints can be visualized with actual colors: each crossing should consist of all the same color or all different colors.

N-colorings are a useful way to show two knots are not equivalent. For example, the simplest knot, the unknot, only has the 3 trivial 3-colorings. The trefoil knot has at least one non-trivial 3-coloring (one shown below), and therefore is different from the unknot. This is a proof that no amount of moves will untangle it. Keep in mind, though, that just because two knots have the same number of colorings, it doesn't mean they are equivalent.

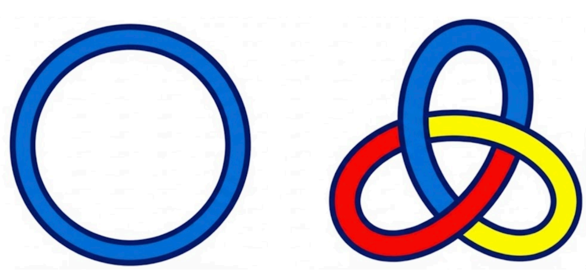

Figure 4: 3-colorings of the unknot and trefoil not. The unknot can only be trivially colored but the trefoil can be colored non-trivially. This proves they are different knots.

Finally, we define a **Link** to be a group of 2 or more disjoint knots. Fox N-colorings can be applied to links as well. Since tanglement puzzles typically involve two pieces that need to be separated from each other, the theory of links comes into play. The ring or string is (typically) the unknot, and the wire frame tends to be some more complicated looking knot (although might also be the unknot in disguise).

Figure 5: Racetrack by Karl Scherer. A beautiful example of a link with 2 knots. The puzzle is to split the link into two separated knots (without breaking or untying).

Useful Tools

In this section I described 2 useful tools for determining the colorability of a wire puzzle: Unzipping and Ring Vanishing.

Unzipping

One thing that you may have noticed is that the figure 8 frame is not a knot! It is a strand of wire with 2 loops at the end. A knot needs to be a simple loop (that can be tangled up in 3-space). To address this, we “unzip” the wire frame into a loop. This concept was first introduced by one of the readers of the 1976 *Games and Puzzle* article [2] and was also used in the proofs in [5] and [7].

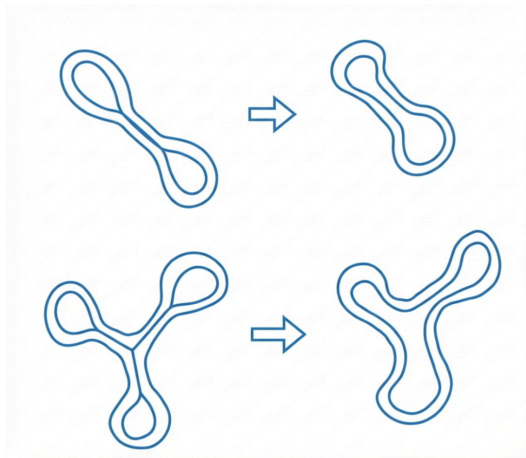

Figure 6: Two examples of Unzipping the wire puzzle frame. This technique can be applied to trees of wires with loops at the endpoints.

Unzipping the puzzle turns it into something topologically different. However, the thing to note here is that **if the original puzzle is solvable, then it will surely be solvable using the unzipped structure**. This is because the space that the ring can move within the original puzzle (also known as the complement of the knot) is a subset of the space of the unzipped puzzle. The contrapositive of this statement will be more useful to us: **If the unzipped puzzle is unsolvable, then the original puzzle is unsolvable.**

This allows us to apply the rich theory of knots to the disentanglement puzzle.

Ring Vanishing

Everything described up to this point is well known. This section describes a useful observation that makes it easier to calculate the number of Fox N-colorings for a disentanglement puzzle when N is odd, which I call Ring Vanishing.

If two strands go through a ring, we can apply a transformation where we cut the strands at the location of the ring, discard the ring, and glue the stands back together in a different way as shown in Figure 7. This move is of course illegal and changes the structure of the knot. However, one thing does not change: **the number of colorings remain the same!** Details of this are described in the next section.

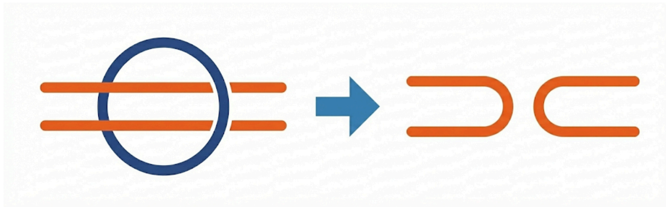

Figure 7: Ring Vanishing simplifies the topology but does not change the number of Fox N-Colorings.

Ring Vanishing Preserves Number of N-Colorings for Odd N

If N is odd, then this transformation preserves the number of colorings. Note that the two structures are not topologically equivalent but they have the same number of colorings. Typically, by applying this transformation, it is much easier to calculate the number of colorings of the original structure.

This is a useful technique to combine with Unzippping because in the original puzzle, the ring typically starts around just 1 wire. Once you unzip the puzzle, the ring will now have 2 wires passing through, and this technique can be used. Applying this technique usually greatly simplifies the knot diagram and makes it easy to compute the number of colorings.

Proof

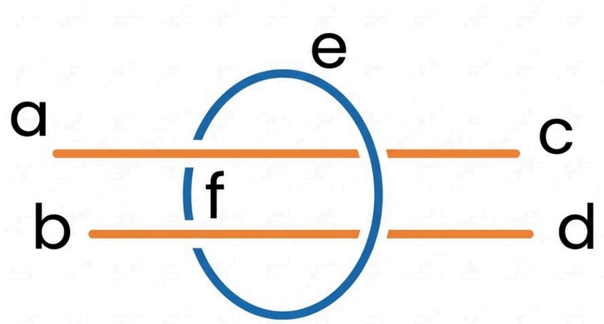

Figure 8: Diagram of two strands passing through a ring, with components labeled.

The proof is based on the crossing constraints (all congruences are modulo N), imposed by the Fox N-colorings:

(1) $2a \equiv e + f$
(2) $2b \equiv e + f$
(3) $2e \equiv a + c$
(4) $2e \equiv b + d$

We first show that $a \equiv b$ and $c \equiv d$

$$2a \equiv 2b \quad \text{(by 1 and 2)}$$
$$\Rightarrow 2^{-1}(2a) \equiv 2^{-1}(2b)$$
$$\Rightarrow a \equiv b$$

Note that we multiply both sides by the multiplicative inverse of 2. This only exists if N is odd, so this is why we are limited to odd N.[1]

Similarly,

$$a + c \equiv b + d \quad \text{(by 3 and 4)}$$
$$\Rightarrow c \equiv d$$

Next, we show that e and f can be written in terms of a and c.

$$2e \equiv a + c \text{ (by 3)}$$
$$\Rightarrow e \equiv 2^{-1}(a + c)$$

$$f \equiv 2a - e \text{ (by 1)}$$
$$\Rightarrow f \equiv 2a - 2^{-1}(a + c)$$

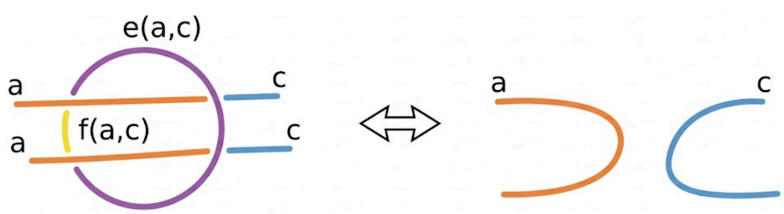

Figure 9: The number of colorings of the two diagrams are the same. Given colors for strands a and c, the rest of the strands are determined. A coloring of the left diagram gives a coloring of the right, and vice-versa.

Therefore, once a and c are determined, the ring has to be colored in a certain way. Also b must be the same color as a, and d must be the same color as c, so we can connect a and b and connect c and d as a different way of maintaining those constraints. So the number of colorings we get for a knot will remain unchanged if we substitute one of the structures in Figure 9 with the other. □

[1] The limitation to only odd N is not much of a limitation. A knot is n-colorable if and only if it's p-colorable for some prime factor p of n. Therefore, when searching for a non-trivial coloring, we only need to look at cases where n is prime, which will be odd for n >= 3.

Impossibility of the Figure 8 Puzzle

We now have all the background and tools to finally tackle the Figure 8 puzzle, which can mostly be shown in pictures.

Original Puzzle
Here is a picture again of the Figure 8 puzzle.

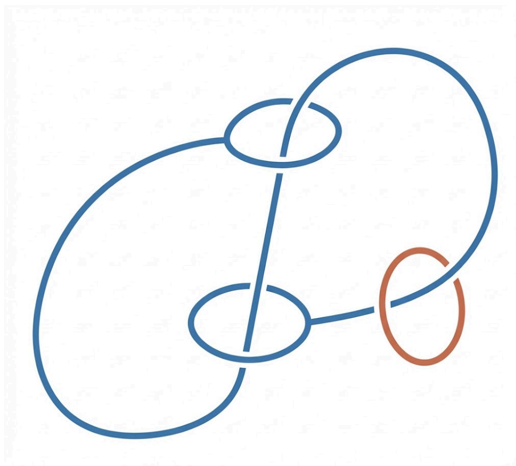

Figure 10: Original Figure 8 Puzzle

Step 1: Unzip
Unzip the Figure 8 frame so that it becomes a loop. Notice that the ring now has two wires going through it.

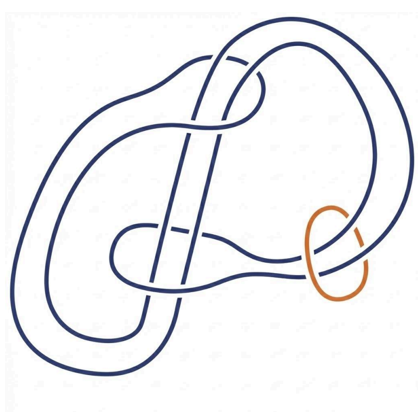

Figure 11: Unzipping the Figure 8 Puzzle

Step 2: Ring Vanish
We can apply the ring vanish to make a simpler structure. By shrinking this structure, we find that it is topologically equivalent to two unknots.

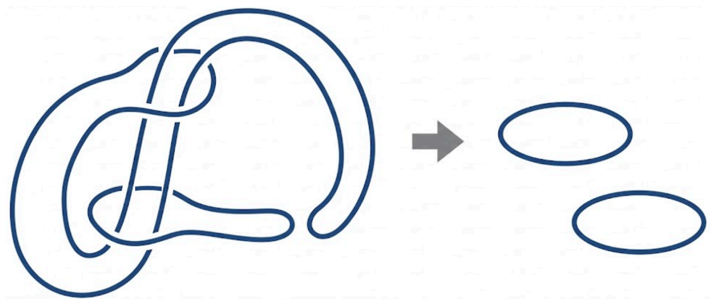

Figure 12: After applying the Ring Vanish, we see that the number of colorings of the structure in Figure 11 is the same as two separate unknots.

From this we can easily see that the number of colorings of Figure 12 is N^2. Therefore Figure 11 also has N^2 colorings. We could have directly counted the number of colorings in Figure 11, but applying the ring vanish makes it much easier to see.

If we can show that the unzipped puzzle with a separate ring (Figure 13) has more than N^2 colorings, we have proven the ring cannot be separated! To do this, we only need to show 1 non-trivial coloring for some odd N.

It turns out that N=3 suffices. Here is a nontrivial 3-coloring of the unzipped figure 8, alongside the separated ring.

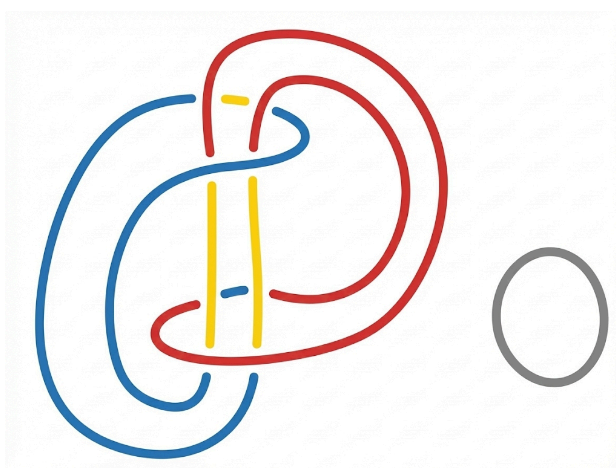

Figure 13: A non-trivial 3-coloring of Unzipped Figure 8. This proves the Figure 8 puzzle is impossible

The ring can only be colored in the N trivial ways. The unzipped Figure 8 can be colored in all the trivial ways, plus some non–trivial colorings, like shown above. Therefore, the number of colorings of Figure 13 is greater than N^2. But when the ring is attached (Figure 11), there are only N^2 colorings.

Because there are a different number of colorings between the unzipped figure 8 with the ring attached and with the ring detached, we know that there are no moves that can get one to the other. Therefore we also know the original puzzle is unsolvable!

More Examples

Now that we've added the above techniques to our tool chest, we might as well try applying it to other puzzles. Here are a few more examples to show that it can be useful for more than just the Figure 8 puzzle (with the last example illustrating a case where it doesn't help).

Example 1

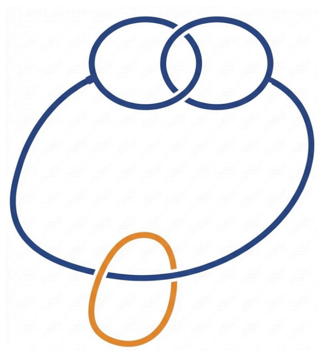

Figure 14: Another impossible puzzle

This is another simple impossible puzzle that can be easily proved with the above techniques. Applying the unzip and the ring vanish lead to 2 rings linked together (known as the Hopf link).

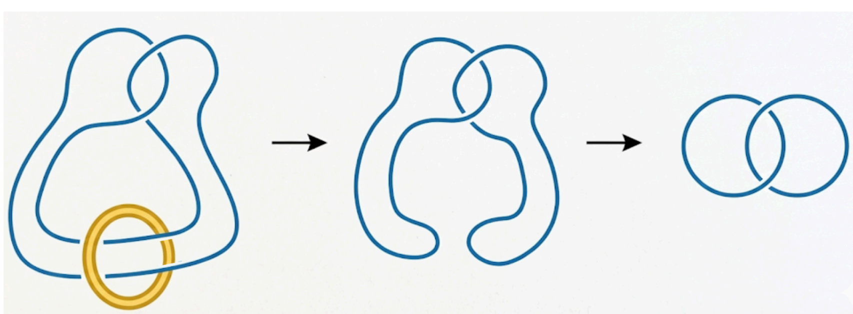

Figure 15: Unzip and Ring Vanish, showing that the left most diagram is only trivially colorable.

It is easy to show that this link can only be trivially colored, so only has N colorings.

If the ring is separate from the frame, then we have two separate knots, and at least N^2 colorings. Therefore, we already know this has a different number of colorings, so we can immediately conclude the puzzle is impossible.

Example 2

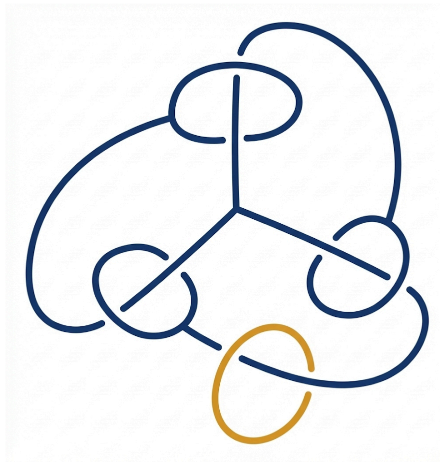

Figure 16: Impossible puzzle where the wire frame has branches like a tree.

This example uses a more complex wire frame, where the wire is a tree with a loop at the end of each of the 3 endpoints, rather than a single strand of 2 loops. The same techniques apply.

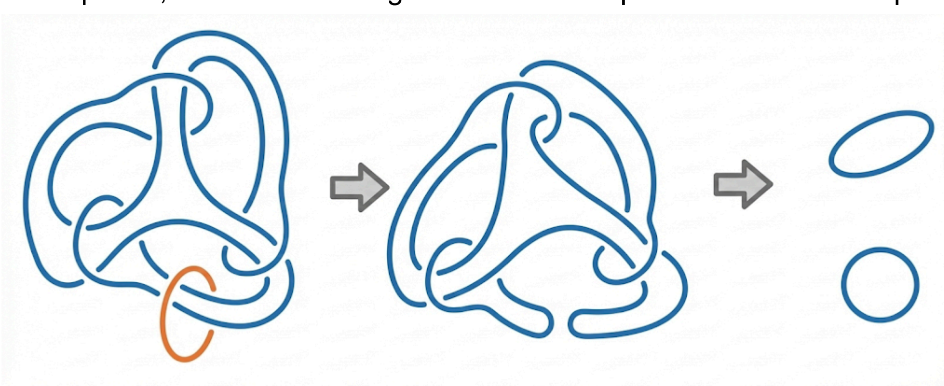

Figure 17: Unzip and Ring Vanish. The diagram on the left has N^2 colorings because it has the same number of colorings as two separate rings.

After the transformations, we are left with two unknots, so we know that the diagram on the left of Figure 17 can be colored with N^2 colors.

What about the number of colorings of the unzipped structure with the ring detached? It turns out that the unzipped structure is only trivially 3-colorable so that doesn't help us. Likewise it only has trivial 5-colorings. However, it turns out that it is non-trivially colorable for N=7, one of which is shown in Figure 18. Admittedly, I used a computer to find a non-trivial coloring, but once it is found, it is easily checked by hand to verify it satisfies the colorability constraints.

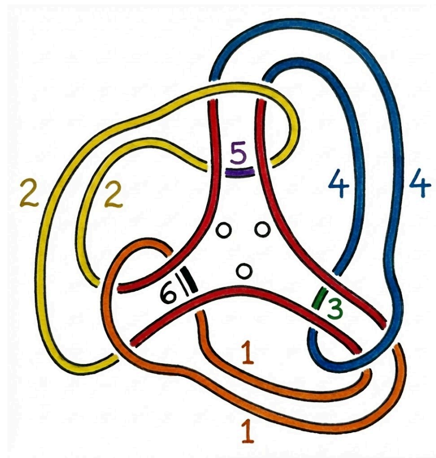

Figure 18: A nontrivial coloring of the unzip wire frame from Figure 16. We needed to look at N=7 to find a non-trivial coloring.

Example 3: The Un-Example

Sometimes a puzzle may not be solvable but the above techniques will not be able to prove it. This is an important thing to keep in mind - you can't assume the puzzle is solvable just because the two diagrams have the same number of colorings.

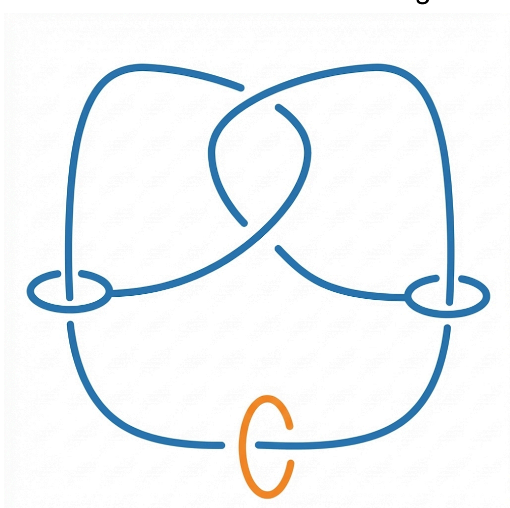

Figure 19: Example of an impossible puzzle where the above techniques fail to prove it.

The puzzle in Figure 19 is the same puzzle pointed out in [5] (figure 6) that is impossible but the techniques cannot identify it as such. By applying the Unzip and Ring Vanish, we get two separate links, so this link has N^2 colorings. However, the unzipped wire frame by itself only has trivial colorings. Therefore we cannot make any conclusion about the solvability of this puzzle using this technique.

One way to show the puzzle is impossible is to look at the Jones polynomial, which is the invariant used in [7]. So this is another method that can be used, albeit less intuitive and a little more computational. It is an important consideration to keep in mind - it is useful to know multiple knot invariants because knot invariants cannot always distinguish different knots.

Conclusion

Topological wire puzzles are closely related to links and knots. By "unzipping" a wire frame shaped like a tree with loops at the ends, we can turn the problem into one about knots, and use the rich theory of knots and links to try to help determine if the puzzle is solvable. The colorability invariant is one such technique that works on several examples, including the Figure 8 puzzle, the most famous unsolvable wire puzzle. By applying the Ring Vanish, one can often quickly compute the number of N-colorings of the unzipped wire puzzle. Other invariants, such as the Jones polynomial can also be used to determine if a puzzle is impossible to solve.

Note about Diagrams

The diagrams in this document were first created by hand and then processed through Google's Nano Banana Pro to make them look more visually appealing.

References

1. Van Delft, Pieter, and Jack Botermans. Creative Puzzles of the World. H.N. Abrams (1978): 154.
2. Coffin, Stewart. Puzzle Craft. Coffin (1985): 16.
3. Coffin, Stewart. The Odyssey of the Figure Eight Puzzle, in: Elwyn Berlekamp. Tom Rodgers (ed.), The Mathemagician and Pied Puzzler, a Collection in Tribute to Martin Gardner. A.K. Peters LTD, Natick (1999): 127-129.
4. Bertuccioni, Inta. "A topological puzzle." The American mathematical monthly 110.10 (2003): 937-939.
5. Melvin, Paul. "A topological menagerie." The American Mathematical Monthly 113.4 (2006): 348-351.
6. Kiviharju, Mikko. "Later Odysseys of the Figure Eight Puzzle." Cubism for Fun 80 (2009): 4-8.
7. Kiviharju, Mikko. "Later Odysseys of the Figure Eight Puzzle, Part 2." Cubism for Fun 81 (2010): 4-9.