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# Nonlinear dependencies in the Fama and French three-factor model.

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# Abstract

Praca dotyczy/obejmuje/podejmuje problematykę...\*) *<tekst wyjustowany, bez tabulacji, zawierający nie więcej niż 800 znaków ze spacjami. Nie podawać celu ani tematu pracy>*

# Key words

Fama and French model, Arbitrage Pricing Theory, nonlinear dependencies

**Area of study (codes according to Erasmus Subject Area Codes List))**

Economics (14300)

# The title of the thesis in Polish

*Nieliniowe zależności w trzyczynnikowym modelu Famy-Frencha.*

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**I. article**

**Introduction**

Since the creation of capital markets investors have been analyzing factors which influence returns on their investments. Therefore asset pricing theories are not only pure academic deliberations but rather useful optimization tools for capital market investors. Significant emphasis in the literature has been put on the stock market analysis, probably due to both relatively easy access to data and notable interest of the practitioners. The essential building block of any asset pricing theory is the generating process underlying returns on all assets, since it allows for estimation of the expected return on a given portfolio.

One of the first models describing the expected returns – the Capital Asset Pricing Model (CAPM) introduced by Sharpe (1964) and Lintner (1965) – establishes a linear relationship between the expected return and the portfolio’s sensitivity to systematic risk denoted by β. Later generalization of the model by Black (1972) and the popularity of the β measure have made the CAPM widely used by the investors on the stock markets. Further theoretical discussions resulted in creating the Arbitrage Pricing Theory (APT), which is based on no-arbitrage condition. The APT can be traced back to the paper by Ross (1976) where multifactor model is presented as an alternative to the aforementioned CAPM. This approach proved to be a productive field of study for plenty of scientists on both theoretical and empirical levels.

The three-factor model by Fama and French (1993) is one of the best known and most cited multifactor models in the literature. The three-factor became the flagship model of the APT in particular due to the novel approach to determining the factors on the basis of returns on specified portfolios. Work of Fama and French inspired many researchers to enhance the proposed model mainly by adding more factors as to boost its explanatory power. Among many attempts the most acknowledged are: four-factor model by Carhart (1997) and five-factor model by Fama and French (2015). The baseline three-factor model as well as its expanded versions were heavily tested among different markets and timespans. Griffin (2002) shows that the three-factor model is market specific and therefore it should be both estimated and analysed locally rather than globally. Another stream in the literature seeks for nonlinearities in multifactor models (e.g. Burmeister and McErloy 1988; Dubé *et al.* 2006; Wang 2018). This article fits into this strand of literature and investigate nonlinear dependencies in the Fama and French three-factor model.

The main aim of the study is to verify the existence and statistical significance of nonlinear dependencies in the Fama and French three-factor model. It is expected that the proposed nonlinear version of the model should statistically outperform its linear counterpart, which is the primary hypothesis of this article. Moreover, the article verifies the usefulness of the proposed nonlinear extensions with respect to practical applications. Bearing in mind the relatively high goodness-of-fit of the linear version of the three-factor model, it is expected that usefulness of its nonlinear versions may be limited to some particular cases only, which is the secondary hypothesis of this article.

The analysis is based on time series regressions for several portfolios in both US and European stock markets. The analysed timespan stretches from July of 1926 to January of 2022 for the US and from July of 1990 to January of 2022 for Europe which corresponds to the data available on the Kenneth R. French home page. The inference is based on information criteria and statistical significance of the estimates.

The structure of this article is as follows. Section I covers the literature overview regarding the Fama and French three-factor model as well as its formal definition. In Section II data and methodology of the analysis are presented. Finally, Section III compares results of estimations and creates basis for further conclusions.

1. **Literature overview**

***Origins of the multifactor models – the Arbitrage Pricing Theory (APT)***

The Arbitrage Pricing Theory (APT) introduced by Ross (1976) created a new type of asset pricing theory which, due to the higher number of degrees of freedom, is potent to outperform the standard CAPM. The line of reasoning is based on no-arbitrage arguments, hence the theory has strong economic and financial foundations. Although the set of assumptions behind the APT may seem questionable in some real world applications, the empirical success of the APT proves these assumptions to be useful in practice. The proposed return generating process for securities of assets is as follows,

(1)

where:

– return of the -th security,

­ – -th factor impacting return on -th security,

– constant,

– sensitivity of -th security to the -th factor,

– residual with the expected value of zero,

– number of factors.

In the case of portfolios of securities the return is the value-weighted average of returns of individual securities. The APT does not specify neither the number nor the characteristics of factors, therefore it may be treated as a very general model or theoretical framework for building specific models. Depending on the situation such generality may be perceived either as an advantage (elasticity) or a disadvantage (no clear baseline version of the model). It should be noted that although the return generating process is linear with respect to factors, the factors themselves can be nonlinear transformations of some variables.

In the literature there are three main approaches to the choice of factors. The first one is based on the factor analysis which differs from other methods by not choosing factors *a priori*. Roll and Ross (1980) show that under certain conditions the factor analysis yields positive results. The second one is following a natural intuition for selecting macroeconomic variables which may have fundamental impact on the expected returns (e.g. inflation, term structure, risk premium etc.). It has been shown that different sets of macroeconomic variables do have a significant impact on the expected return on portfolios (e.g. Chen, Roll and Ross 1986; Burmeister and Wall 1986; Clare and Thomas 1994). Third approach, which was introduced by Fama and French (1993), specifies *a priori* a set of portfolios which are considered to capture underlying return generating process. This method is sensitive for more specific movements in the analysed stock market. Furthermore, the portfolio approach was well received by other scholars and practitioners and became the main trend in the literature regarding the APT.

***Fama and French three-factor model***

The Fama and French (1993) three-factor model proposes the expected return on portfolio to be dependent on market portfolio excess return, size premium (historical tendency for stocks with smaller market capitalization to generate higher return than those with larger market capitalization) and value premium (spread between returns of value stocks with high book-to-market ratios and growth stocks with low book-to-market ratios). The return for the -thportfolio at time can be expressed with the following equation:

(2)

where:

– excess return on -th portfolio at time ,

– return on -th portfolio at time ,

– risk free rate at time ,

– return on value-weighted market portfolio of risky assets,

– return on diversified portfolio of stocks with small market capitalization minus the return on diversified portfolio of stocks with big market capitalization at time ,

– return on diversified portfolio of stocks with high Book to Market ratio minus the return on diversified portfolio of stocks with low Book to Market ratio at time ,

– constant,

– sensitivity of -th portfolio to the -th factor,

– residual with the expected value of zero.

In the following sections market capitalization is referred to as *Size* whereas book equity to market equity ratio as *B/M.* More specific definitions of the factors, in particular precise meaning of “low” and “high”, can be found in the Supplement.

Fama and French (1993) argue that the proposed three-factor model exhibits significant explanatory power for the expected returns on the US market. The empirical success of the model on the US market resulted in intensified research for other markets. For example the work of Ajili (2002) shows that, in the case of French stock market, value-weighted returns on portfolios constructed on the basis of *Size* and *B/M* indeed follow the three-factor model. The estimates for Warsaw Stock Exchange reveal solid explanatory power of three-factor model for all test portfolios except for the abnormally high returns on portfolios of middle *Size* (Redlicki and Borowski 2017). Moreover, regressions on plenty of portfolios of German stocks proved the Fama and French three factor model right (Philipp and Franziska 2018). In the case of Japanese stock markets the all three factors were deemed significant in explain returns by Jagannathan *et al. (*1998) and later by Kubota and Takehara (2010). All above mentioned countries are developed, however the Fama and French three-factor model has significant explanatory power even for the case of the emerging markets such as Kenya or RPA (Achola and Muri 2016, Karp and van Vuuren 2017).

It is noteworthy to mention that the three-factor model is country specific and the estimations should be performed locally or for closely interconnected markets (Griffin 2002). Moreover, although adding foreign factors to the domestic model may improve its performance in terms of adjusted the economic importance of such an amendment is rather low.

Although the three-factor model by Fama and French (1993) was positively verified by a number of scholars, critical voices were present as well. For instance Daniel and Titman (1997) argue that in the case of stocks with small *Size* and high *B/M* the standard three-factor model does not provide sufficient explanatory power even on the US stock market. Michou, Mouselli and Stark (2012) find no satisfactory explanatory power of the three-factor model for the UK stock market despite testing many alternative models and definitions of *Size* and *B/M*. The critique resulted in attempts to expand the three-factor model with additional factors as to boost the explanatory power. One of the most important improvements was proposed by Carhart (1997), who introduced the *Momentum* factor i.e. difference between return on stocks with highest and lowest returns in previous period. The model assumes that short term behavior of a stock is stable, meaning that return from previous period may be informative about the present returns. The Carhart four-factor model induced discussion about the theory behind the momentum effect and was well received by practitioners. However, later results presented in Titman, Wei and Xie (2004) and Novy-Marx (2012) introduce two different factors ­– difference in return on portfolios with high and low profitability and difference in return on portfolios with high and low need for investment. The newly created five-factor model with two additional factors (i.e. profitability and investment) was later proposed by Fama and French (2015). This extension of the three-factor model not only substantially increased the explanatory power, but was also found satisfactory even for the emerging markets (Foye 2018).

All aforementioned multifactor models have been widely referred to in the literature and adapted by investors. To a great extent they are used interchangeably. This article acknowledges the existence of many extensions of the Fama and French thee-factor model, including the two best known i.e. four-factor model by Carhart (1997) and five-factor model by Fama and French (2015), but focuses on the three-factor model as a benchmark and still most widely cited multifactor model in the literature.

***Nonlinearities in multifactor APT models***

The APT, being a very general theory, does not precise any specific factors and leaves this task open for discussion. The aforementioned models and their applications assume a purely linear relationship in the equation for return generating process. Although the assumption of linearity is very intuitive and was shown to provide satisfactory performance of multifactor models on many markets, it may be questioned whether including possible nonlinearities has the potential to refine the results. This question catalyzed research in two main streams.

The first one represents return generating process as a nonlinear function of factors and securities’ sensitivities to these factors, and therefore is usually referred to as the nonlinear APT. The return generating process in the nonlinear APT may be represented with the following equation:

(3)

where:

– return of the -th security,

– nonlinear multivariable function,

­ – -th factor impacting return on -th security,

– sensitivity of -th security to the -th factor,

– residual with the expected value of zero,

– number of factors.

One of the first attempts of detecting nonlinear relationships as in (3) was proposed by Burmeister and McErloy (1988) who successfully used Iterated Nonlinear Seemingly Unrelated Regression. However, the theory of the nonlinear APT was established later in the papers by Bansal and Viswanathan (1993) and Bansal, Hsieh and Viswanathan (1993). Their proposition of a nonnegative nonlinear pricing kernel underlying the return generating process was consistent with the previously proposed multifactor models and simultaneously generalized them. Additionally, the nonlinear APT models have great advantage over their linear counterparts, since they do not restrict the return space and are potent to price derivatives with nonlinear payoffs (Bansal and Viswanathan 1993). On the one hand, the superiority of the nonlinear APT models over the linear ones in terms of higher explanatory power has been confirmed by e.g. Dittmar (2002). On the other hand Wang (2000) claims that the nonlinear APT models are no better in explaining the expected returns on stocks than conditional CAPM.

The second approach is based on linear equation for return generating process as in (1) but allows some factors to be nonlinear transformation (e.g. higher power terms or interactions) of some other “basic” factors. Therefore in this approach nonlinearity is modelled within the linear framework of the standard APT by taking advantage of its elasticity with respect to defining factors. The idea of considering higher moments in modelling portfolio returns can be traced back to the paper by Kraus and Litzenberger (1976) where the Three Moment CAPM is derived and proposed as a solution to weak performance of the original CAPM. Further work by Banz (1980) introduced second order term of market capitalization which explained larger returns on small firms than average and large ones. Likewise, Hung (2007) presents Four Moment CAPM which, in the case of US and UK data, has significant quadratic and cubic terms.

The multifactor APT models were modified in a similar fashion as CAPM. A study by Ang *et al.* (2014) uses Carhart four-factor model with additional second and third powers of market portfolio excess return and all additional factors are found to be significant on both the North American and the European markets. Racicot *et al.* (2014) and Rompotis (2016) show that augmented Fama and French three-factor model with additional quadratic and cubic terms of market portfolio excess return and *Size* premium proves to have significant nonlinear factors in case of respectively global, Asian and American markets. Dubé *et al.* (2006) modify the Fama and French three-factor model with supplementary quadratic terms of all three factors as well as cubic term of market portfolio excess return and find only some of the nonlinear factors to be significant. The study by Wang (2018) empirically tests a modified version of the Fama and French three-factor model on the Chinese Stock Market. The return generating process includes additional quadratic terms of size premium and value premium along with their interaction and all factors are found statistically significant with p-values close to zero.

The preceding literature overview demonstrates that the nonlinear augmentation of the Fama and French three-factor model is a productive field of research. Results of the aforementioned papers lead us to expect that some of the nonlinear transformations of the three factors should be statistically significant and provide more explanatory power. Furthermore, inclusion of said nonlinear factors should enlarge the explanatory power of the models. However, due to possible differences in characteristics of separate markets the analysis may show that the Fama and French three-factor model does not hold for aggregated European data.

**2. Data and Methodology**

The literature covers two mainstream approaches to testing return generating processes within the APT. The first one is based on cross-section regressions while the second one on time series regressions. The former approach is more accessible as it requires less data but it provides less insight into the overall return generating process than the latter. The original paper by Fama and French (1993) performed calculations on NYSE, NASDQ and Amex monthly data ranging from July 1963 to December 1991. Authors applied time series regression method which has been described in detail by Black, Jensen and Scholes (1972), who used it in order to test the CAPM. The procedure can be easily adjusted for the case of a multifactor model. Tests were conducted on portfolios of stocks rather than individual securities. Portfolios were constructed according to the bivariate sort of securities on *Size* and *B/M* values. Since the main aim of this article is to verify nonlinear dependencies in the Fama and French three-factor model it is suitable to use the same methods i.e. time series regression as in Black, Jensen and Scholes (1972). Analysis makes use of popular econometric method used in regression tasks – general to specific procedure (GETS), which allows for statistically valid elimination of jointly insignificant variables. In order to alleviate the effects of potential heteroscedasticity and autocorrelation within time series models the Newey-West estimator (HAC) of variance-covariance matrix is used.

Data used for the purposes of this article consists of monthly values of the Fama and French three-factor model factors as well as excess returns on six portfolios formed on *Size* and *B/M*. Analysis is performed on two distinct datasets: US stock market and European stock markets of sixteen developed European countries. In the case of the US data spans from July of 1926 to January of 2022 which translates to 1147 observations, whereas European data spans from July of 1990 to January of 2022 which amounts to 379 observations. The data has been downloaded from Kenneth R. French’s Data Library[[1]](#footnote-1) where the values of factors are calculated monthly on the basis of CRSP stock market database[[2]](#footnote-2). The Supplement covers detailed statistics of both datasets that are used in the analysis.

Time series regressions are performed for five types of nonlinear models along with the linear Fama and French three-factor model as benchmark. The analysed types of models are listed in order of complexity: linear with interactions, quadratic, quadratic with interactions, cubic and cubic with interactions. The last one can be represented by the following equation:

(4)

where:

– excess return on -th portfolio at time ,

– return on -th portfolio at time ,

– risk free rate at time ,

– return on value-weighted market portfolio of risky assets,

– return on diversified portfolio of stocks with small market capitalization minus the return on diversified portfolio of stocks with big market capitalization at time ,

– return on diversified portfolio of stocks with high Book to Market ratio minus the return on diversified portfolio of stocks with low Book to Market ratio at time ,

– constant,

– sensitivity of -th portfolio to the -th factor,

– residual with the expected value of zero.

The factors in the rest of the estimated models are subsets of the factors present in the cubic model with interactions. Regressions are performed for each type of model and each portfolio separately. Higher order terms and interactions in subsequent models corresponds with the gradual increase in nonlinearity.

**3. Results and Interpretations**

Throughout the whole thesis the significance level α is set to 5%. Implementation of the Newey-West estimator of variance-covariance matrix ensures that estimations are robust to possible heteroscedasticity and autocorrelation. In the first place the benchmark Fama and French three-factor model given by the Equation (2) is assessed with use of the time series regression. The results for the benchmark for the US data are presented in Table 1. Calculated p-values of the RESET tests show that models for four portfolios do not fulfil assumption of *correct specification* of the Ordinary Least Squares method (OLS). Meaning that the Fama and French three-factor model explains the return generating process only in the case of Big – Low BM and Big – High BM portfolios. All factors of the former model proved to be significant both in statistical and economic sense, whereas the SMB of the latter model turned out to be insignificant. Noticeably the market premium is the most influential factor for all portfolios, which is in line with the aim of expanding the CAPM by the three-factor model. The coefficient estimations tend to be similar in value to the ones presented by Fama and French (1993) despite the timespan being more than three times larger. The sensitivity to the market premium is around 1.0 for all portfolios. The size premium coefficient for the three Small portfolios is close to 1.0 whereas for the two Big portfolios values are negative alike in the original paper. Similarly, the value premium coefficients show the same pattern as in the cited article i.e. negative for the Low BM portfolio, change sign at the Medium BM portfolio and are close to 0.8 for High BM portfolio. This presents a time based robustness of the Fama and French three-factor model. the data set is much larger the coefficient estimations and patterns do not change dramatically.

The results show that some of the quadratic and cubic terms as well as interactions are indeed both individually and jointly significant. Increasing nonlinearity in models leads to fulfilling the *correct specification* assumption for more portfolios with each step ending on cubic version with interactions where all portfolios meet the OLS requirements. Moreover, including more nonlinear factors produces an upward trend that can be seen in the values of adjusted throughout all portfolios. It means that higher order terms along with interactions allow for greater explanatory power of the underlying return generating process. Further, the cubic models with interactions have minimum values of both the Akaike information criterion (AIC) and the Schwarz information criterion (SIC). The situation when both the AIC and the SIC choose the same model as the best fit to data is rather uncanny, usually the SIC tends to point to models with fewer parameters than the AIC. The aforementioned statements can be seen in Table 2. which presents coefficient estimations and appropriate p-values for all considered models for the Small – Low BM portfolio. Considering additional nonlinear factors neither change the estimations drastically nor alter the statistical significance.

Table 3. presents time series regression results of the cubic model with interactions given by the Equation (4) for all portfolios for the US data. As mentioned before the nonlinear factors are both individually and jointly significant, only the SMB factor turned out to be insignificant in the case of the Big – High BM portfolio. The values of the sensitivity to market premium range from 0.94 to 1.095, the size premium sensitivity starts at 1.039 for the Small – Low BM portfolio and ends up as -0.123 for the Big – Medium BM portfolio, finally the value premium coefficient rises from -0.22 for Small – Low BM portfolio to 0.734 for Small – High BM portfolio and again from -0.252 for Big – Low BM portfolio to 0.813 for Big – High BM portfolio. The results above coincide with the benchmark model and with the results by Fama and French (1993) both in values and patterns. Models for all of the portfolios have been successfully augmented i.e. at least one nonlinear factor is statistically significant. The interactions between base factors are significant with exception of the Big – Medium BM portfolio and have more influence on the excess return than the quadratic and cubic terms. The higher order terms of the three factors complement the benchmark model and allow for more flexibility in fitting to the data. Notice that the values of coefficients for all of the nonlinear factors is two or three orders of magnitude smaller than those of the linear factors. Thus the nonlinear factors being statistically significant may not prove to be economically significant, especially for smaller portfolios.

Secondly, the time series regression of the benchmark model which follows the Equation (2) is performed on the European data. The results are presented in Table 4. On the contrary to the US case all of the linear models, except the Big – High BM portfolio, fulfill the OLS assumptions, in particular fitted RESET tests indicate that *correct specifications* are met. All of the factors are individually and jointly significant besides the HML in the case of the Big – Medium BM portfolio, thus the Fama and French three-factor model explains the return generating process in the case of aggregated sixteen European capital markets. Once more the market premium is the most influential factor for all portfolios which corresponds with the goal of the three-factor model - expanding the CAPM. The sensitivities to the market premium are in general similar to what the US calculations show – the largest difference in values is present in the Big – Low BM portfolio with difference of 0.079 and the coefficient value of 0.941. The size premium is a significant factor for all portfolios and its sensitivities differ from the US ones by no more than 0.102. The key difference can be seen in the sensitivities to the vale premium. The overall pattern is similar meaning that the coefficients are negative for the Low BM portfolio, change sign at the Medium BM portfolio and are positive for High BM portfolio. However, all of the coefficient values are substantially lower – on average by 0.247, showing that the HML factor has a different characteristics than in the US.

In comparison with the US, the European models have fewer nonlinear factors which are in fact statistically significant. For instance the linear models with interactions shrink to the linear models due to the joint insignificance of the interactions. Adding nonlinear transformations of the factors to the models does not change the fulfillment of the OLS assumptions, apart from the Big – High BM portfolio that meets the OLS requirements only in the cubic form with interactions. Simultaneously the adjusted has a tendency to grow along with adding nonlinear factors, hence proving the increase in explanatory power. Regarding the information criteria the Small – Low BM, Small – Medium BM, Big – Low BM and Big –Medium BM portfolios have minimal values of both the AIC and the SIC for the cubic model with interactions, whereas the Small – High BM and Big – High BM portfolios have the lowest value of the AIC for the cubic model with interactions and the lowest value of the SIC for the benchmark linear model. Observations noted in this paragraph can be seen in Table 5. which contains estimations of all models for the Big – Medium BM portfolio. This particular portfolio shows step by step the nonlinear factors being added to the benchmark model alongside increasing adjusted and decreasing values of the information criteria.

The calculations of the time series regression of the cubic model with interactions following the Equation (4) for the European data are presented in Table 6. The nonlinear factors are individually and jointly significant, however the HML factor is insignificant for the Small – Medium BM and the Big – Medium BM portfolios. The values of the coefficient of the market premium range from 0.937 to 1.076, the size premium sensitivity lies within -0.089 and 0.910, the value premium sensitivity rises from -0.408 for the Small – Low BM portfolio to 0.490 for the Small – High BM portfolio and again from -0.467 for the Big – Low BM portfolio to 0.610 for the Big – High BM portfolio. The estimations listed above overlap with corresponding values for the US capital market. Models for all of the portfolios have been successfully modified, meaning that one or more nonlinear factors are statistically significant. The European models consist of fewer nonlinear factor than the US ones, in particular only one interaction between market premium and value premium is significant which might be caused by the insignificance of the linear term of the HML. Analogous to the US case, the sensitivities to all of the nonlinear factors are two or three orders of magnitude smaller than those of the linear factors. Hence the nonlinear factors being statistically significant may not prove to be economically significant, in particular in the case of smaller portfolios.

**CONCLUSIONS**

This thesis is devoted to the topic of nonlinear dependencies in one of the most cited models of the APT – Fama and French three-factor model. The main hypothesis is that some of the nonlinear transformations of the three factors are statistically significant and contribute to achieving greater explanatory power. The secondary hypothesis states that the standard Fama and French three-factor model as well as its augmented versions do not hold for the aggregated European capital markets.

The thesis begins with a historical overview of the evolution of the pricing theory from first return generating processes of the Capital Asset Pricing Model to the modern Arbitrage Pricing Theory with examples of the most cited models – three-factor model by Fama and French (1993), four-factor model by Carhart (1997) and five-factor model by Fama and French (2015). Following, two approaches existing in the literature regarding the nonlinear models of the Arbitrage Pricing Theory are discussed, in particular the modifications of the Fama and French three-factor model. The empirical part of the thesis covers the data specification as well as the methodology and the results interpretation.

Seventy-two time series regressions have been performed on two datasets, representing the US and Europe, for six portfolios with bivariate sort by *Size* and *B/M* with six different models from linear to cubic with interactions. Estimations of the coefficients of the three linear factors market premium and size premium are similar for both the US and Europe, however the sensitivities to the value premium vary substantially. Additionally, there are two key differences in the behaviour of the return generating processes. Firstly, most of the linear models for the US data do not comply with the OLS assumptions, but the augmented versions with nonlinear factors do. Whereas, in the case of Europe almost all of the linear models fulfil requirements of the OLS. Secondly, the European models tend to have fewer statistically significant nonlinear factors than the US models. Overall some of the nonlinear factors prove to be both individually and jointly statistically significant which varies substantially between portfolios.

The obtained results allow for positive verification of the main hypothesis of this thesis. In fact the additional nonlinear factors to the original Fama and French three-factor model are statistically significant and provide greater explanatory power. However, relatively small values of the sensitivities to the nonlinear factors question the economic significance of these variables. The sixteen European capital markets that are aggregated in the dataset prove to be intertwined enough so that the benchmark model and modified models hold. Hence, the secondary hypothesis can be rejected.

This thesis complements the existing literature concerning the less popular approach to augmenting the Fama and French three-factor model. The nonlinear modifications have a potential usage for the capital markets investors by giving them more insight into the return generating process without extensive complication of the previously used methods.

There are numerous aspects of the topic open to further research. The analysed models take into considerations quadratic and cubic form of the three factors, however there are no objections to using higher order terms – this would allow for more accurate approximation of the Taylor expansion of the underlying function. Moreover, the influence of the outliers on the final results could be investigated. An important matter in the Arbitrage Pricing Theory is the time robustness of the models, which should be examined in future for larger datasets. Another issue worth researching is to check whether the significance of the nonlinear factors are market specific. Abovementioned matters motivate further exploration of this topic.

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Table 1. Estimation of parameters of the benchmark model for the US data.

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Small - Low BM | | Small - Medium BM | | Small - High BM | | Big - Low BM | | Big - Medium BM | | Big - High BM | |
|  | coef | p-v | coef | p-v | coef | p-v | coef | p-v | coef | p-v | coef | p-v |
| Intercept | 0,120 | 1E-03 | 0,310 | 1E-16 | 0,292 | 2E-23 | 0,346 | 4E-35 | 0,203 | 7E-05 | 0,173 | 2E-05 |
| MKT\_RF | 1,079 | 0 | 0,982 | 0 | 1,018 | 0 | 1,020 | 0 | 0,979 | 8E-279 | 1,080 | 0 |
| SMB | 1,040 | 1E-295 | 0,823 | 1E-199 | 0,926 | 0 | -0,099 | 2E-09 | -0,126 | 1E-06 | 0,015 | 0,475 |
| HML | -0,188 | 5E-22 | 0,301 | 4E-22 | 0,788 | 1E-149 | -0,221 | 5E-63 | 0,320 | 4E-21 | 0,802 | 9E-254 |
| p-v fitted RESET test | 1E-07 | | 2E-04 | | 2E-28 | | 0,133 | | 2E-17 | | 0,130 | |
| p-v rhs RESET test | 7 E-07 | | 5E-23 | | 1E-37 | | 5E-09 | | 3E-16 | | 1E-03 | |
| p-v Breusch-Godfrey | 0,158 | | 0,300 | | 0,080 | | 3E-07 | | 4E-10 | | 0,251 | |
| R2 | 0,9724 | | 0,9764 | | 0,9915 | | 0,9793 | | 0,9500 | | 0,9664 | |
| Adj R2 | 0,9723 | | 0,9764 | | 0,9915 | | 0,9792 | | 0,9498 | | 0,9663 | |
| AIC | 3761 | | 3420 | | 2593 | | 2635 | | 3793 | | 3884 | |
| SIC | 3786 | | 3445 | | 2618 | | 2660 | | 3818 | | 3909 | |
| N | 1147 | | 1147 | | 1147 | | 1147 | | 1147 | | 1147 | |

MKT\_RF denotes the market premium.

Table 2. Estimation of parameters of all models for Small – Low BM portfolio on the US data.

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Linear | | Linear with interactions | | Quadratic | | Quadratic with interactions | | Cubic | | Cubic with interactions | |
|  | coef | p-v | coef | p-v | coef | p-v | coef | p-v | coef | p-v | coef | p-v |
| Intercept | 0,120 | 2E-03 | 0,076 | 0,055 | 0,064 | 0,117 | 0,008 | 0,851 | 0,113 | 0,004 | 0,055 | 0,187 |
| MKT\_RF | 1,079 | 0 | 1,086 | 0 | 1,077 | 0 | 1,084 | 0 | 1,084 | 0 | 1,081 | 0 |
| SMB | 1,040 | 2E-295 | 1,031 | 0 | 1,031 | 0 | 1,035 | 0 | 1,027 | 0 | 1,039 | 0 |
| HML | -0,188 | 6E-22 | -0,189 | 2E-18 | -0,204 | 8E-30 | -0,209 | 2E-31 | -0,217 | 2E-33 | -0,220 | 1E-36 |
| MKT\_RF^2 | - | - | - | - | 0,00171 | 0,006 | - | - | - | - | 0,00160 | 0,094 |
| SMB^2 | - | - | - | - | 0,00146 | 0,028 | - | - | - | - | - | - |
| HML^2 | - | - | - | - | - | - | 0,00560 | 0,008 | - | - | - | - |
| MKT\_RF^3 | - | - | - | - | - | - | - | - | -0,00006 | 0,074 | - | - |
| SMB^3 | - | - | - | - | - | - | - | - | 0,00005 | 0,003 | - | - |
| HML^3 | - | - | - | - | - | - | - | - | 0,000161 | 1E-05 | 0,00022 | 6E-05 |
| MKT\_RF\*SMB | - | - | 0,01004 | 7E-05 | - | - | 0,00729 | 0,019 | - | - | 0,00760 | 0,009 |
| MKT\_RF\*HML | - | - | - | - | - | - | -0,00441 | 0,039 | - | - | -0,00587 | 0,005 |
| SMB\*HML | - | - | -0,00996 | 0,003 | - | - | -0,00835 | 0,015 | - | - | -0,00736 | 0,030 |
| p-v ^2 = 0 | - | | - | | 4E-05 | | 4E-05 | | - | | 0,016 | |
| p-v ^3 = 0 | - | | - | | - | | - | | 6E-07 | | 5E-07 | |
| p-v inter = 0 | - | | 3E-09 | | - | | 5E-09 | | - | | 2E-10 | |
| p-v ^2 & ^3 = 0 | - | | - | | - | | - | | - | | 4E-07 | |
| p-v ^2 & ^3 & inter = 0 | - | | - | | - | | - | | - | | 4E-14 | |
| p-v fitted RESET test | 1E-07 | | 8E-03 | | 1E-03 | | 0,756 | | 4E-06 | | 0,409 | |
| p-v rhs RESET test | 7E-07 | | 2E-06 | | 2E-03 | | 3E-06 | | 1E-04 | | 6E-05 | |
| p-v Breusch-Godfrey | 0,158 | | 0,203 | | 0,503 | | 0,472 | | 0,437 | | 0,692 | |
| R2 | 0,9724 | | 0,9733 | | 0,9729 | | 0,9739 | | 0,9731 | | 0,9741 | |
| Adj R2 | 0,9723 | | 0,9732 | | 0,9727 | | 0,9737 | | 0,9730 | | 0,9739 | |
| AIC | 3761 | | 3725 | | 3744 | | 3708 | | 3735 | | 3698 | |
| SIC | 3786 | | 3761 | | 3779 | | 3758 | | 3775 | | 3749 | |
| N | 1147 | | 1147 | | 1147 | | 1147 | | 1147 | | 1147 | |

Rows from number 14 to 18 contain p-values of joint significance F-tests. The “-“ sign indicates that this factor has been removed during GETS procedure or was not considered in the model.

Table 3. Estimation of parameters of the cubic model with interactions for the US data.

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Small – Low BM | | Small – Medium BM | | Small – High BM | | Big – Low BM | | Big – Medium BM | | Big – High BM | |
|  | coef | p-v | coef | p-v | 0,302 | 1E-33 | coef | p-v | coef | p-v | coef | p-v |
| Intercept | 0,055 | 0,187 | 0,373 | 3E-31 | 1,006 | 0 | 0,341 | 6E-35 | 0,230 | 6E-07 | 0,085 | 0,061 |
| MKT\_RF | 1,081 | 0 | 0,960 | 0 | 0,916 | 0 | 1,010 | 0 | 0,940 | 0 | 1,095 | 0 |
| SMB | 1,039 | 0 | 0,808 | 4E-278 | 0,734 | 0 | -0,117 | 9E-12 | -0,123 | 2E-06 | 0,008 | 0,623 |
| HML | -0,220 | 1E-36 | 0,307 | 3E-52 | - | - | -0,252 | 3E-86 | 0,285 | 5E-28 | 0,813 | 4E-277 |
| MKT\_RF^2 | 0,00160 | 0,094 | - | - | - | - | - | - | - | - | 0,00189 | 0,034 |
| SMB^2 | - | - | - | - | - | - | 0,00230 | 0,011 | - | - | - | - |
| HML^2 | - | - | -0,00736 | 1E-07 | - | - | - | - | - | - | - | - |
| MKT\_RF^3 | - | - | 0,00014 | 7E-04 | - | - | 3E-05 | 0,015 | 0,00015 | 3E-10 | - | - |
| SMB^3 | - | - | - | - | 0,00025 | 1E-11 | - | - | - | - | - | - |
| HML^3 | 0,00022 | 5E-05 | - | - | - | - | 0,00017 | 1E-07 | - | - | - | - |
| MKT\_RF\*SMB | 0,00760 | 0,009 | 0,00646 | 0,002 | - | - | - | - | - | - | 0,00964 | 0,002 |
| MKT\_RF\*HML | -0,00587 | 0,005 | -0,00587 | 0,005 | -0,00272 | 0,013 | -0,00516 | 4E-05 | - | - | -0,00416 | 0,007 |
| SMB\*HML | -0,00736 | 0,030 | -0,00736 | 0,030 | 0,00373 | 4E-04 | 0,00312 | 0,010 | - | - | -0,00892 | 0,024 |
| p-v ^2 = 0 | 0,016 | | 3E-19 | | - | | 3E-04 | | - | | 0,007 | |
| p-v ^3 = 0 | 5E-07 | | 9E-15 | | 4E-22 | | 4E-11 | | 5E-18 | | - | |
| p-v inter = 0 | 2E-10 | | 2E-10 | | 1E-06 | | 1E-06 | | - | | 2E-12 | |
| p-v ^2 & ^3 = 0 | 4E-07 | | 5E-20 | | - | | 7E-13 | | - | | - | |
| p-v ^2 & ^3 & inter = 0 | 4E-14 | | 5E-24 | | 1E-40 | | 9E-16 | | - | | - | |
| p-v fitted RESET test | 0,409 | | 0,258 | | 0,315 | | 0,250 | | 0,108 | | 0,216 | |
| p-v rhs RESET test | 6E-05 | | 0,001 | | 0,002 | | 1E-04 | | 0,001 | | 0,001 | |
| p-v Breusch-Godfrey | 0,692 | | 0,259 | | 0,003 | | 9E-08 | | 5E-07 | | 0,145 | |
| R2 | 0,9741 | | 0,9786 | | 0,9928 | | 0,9807 | | 0,9532 | | 0,9681 | |
| Adj R2 | 0,9739 | | 0,9785 | | 0,9928 | | 0,9805 | | 0,9530 | | 0,9679 | |
| AIC | 3698 | | 3314 | | 2410 | | 2565 | | 3720 | | 3831 | |
| SIC | 3749 | | 3354 | | 2450 | | 2615 | | 3750 | | 3870 | |
| N | 1147 | | 1147 | | 1147 | | 1147 | | 1147 | | 1147 | |

Rows from number 14 to 18 contain p-values of joint significance F-tests. The “-“ sign indicates that this factor has been removed during GETS procedure or was not considered in the model.

Table 4. Estimation of parameters of benchmark model for the EU data.

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Small – Low BM | | Small – Medium BM | | Small – High BM | | Big – Low BM | | Big – Medium BM | | Big – High BM | |
|  | coef | p-v | coef | p-v | coef | p-v | coef | p-v | coef | p-v | coef | p-v |
| Intercept | 0,092 | 0,066 | 0,210 | 5E-10 | 0,317 | 1E-18 | 0,293 | 3E-14 | 0,257 | 8E-09 | 0,069 | 0,201 |
| MKT\_RF | 1,075 | 2E-239 | 0,986 | 0 | 0,945 | 2E-248 | 0,941 | 3E-216 | 0,994 | 5E-250 | 1,072 | 4E-232 |
| SMB | 0,953 | 4E-137 | 0,848 | 5E-206 | 0,824 | 2E-133 | -0,194 | 8E-12 | -0,116 | 2E-07 | -0,065 | 0,011 |
| HML | -0,412 | 1E-34 | 0,079 | 6E-05 | 0,493 | 2E-69 | -0,481 | 1E-44 | 0,025 | 0,402 | 0,614 | 3E-55 |
| p-v fitted RESET test | 0,298 | | 0,055 | | 0,537 | | 0,190 | | 0,081 | | 0,039 | |
| p-v rhs RESET test | 0,007 | | 4E-06 | | 0,007 | | 0,001 | | 0,001 | | 0,037 | |
| p-v Breusch-Godfrey | 5E-06 | | 0 | | 0,144 | | 0,935 | | 0 | | 4E-05 | |
| R2 | 0,979 | | 0,990 | | 0,986 | | 0,977 | | 0,978 | | 0,977 | |
| Adj R2 | 0,979 | | 0,989 | | 0,986 | | 0,977 | | 0,978 | | 0,977 | |
| AIC | 900 | | 587 | | 718 | | 818 | | 865 | | 1001 | |
| SIC | 920 | | 607 | | 737 | | 838 | | 884 | | 1021 | |
| N | 379 | | 379 | | 379 | | 379 | | 379 | | 379 | |

MKT\_RF denotes the market premium.

Table 5. Estimation of parameters of all models for Small – Low BM portfolio on the European data.

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Linear | | Linear with interactions | | Quadratic | | Quadratic with interactions | | Cubic | | Cubic with interactions | |
|  | coef | p-v | coef | p-v | coef | p-v | coef | p-v | coef | p-v | coef | p-v |
| Intercept | 0,257 | 8E-09 | 0,257 | 8E-09 | 0,271 | 5E-09 | 0,274 | 4E-10 | 0,257 | 2E-09 | 0,226 | 7E-08 |
| MKT\_RF | 0,994 | 5E-250 | 0,994 | 5E-250 | 0,993 | 5E-262 | 0,995 | 2,1E-262 | 0,994 | 1,9E-254 | 0,998 | 3E-269 |
| SMB | -0,116 | 2E-07 | -0,116 | 2E-07 | -0,121 | 5E-08 | -0,115 | 6,95E-08 | -0,089 | 2E-04 | -0,089 | 2E-04 |
| HML | 0,025 | 0,402 | 0,025 | 0,402 | 0,012 | 0,636 | 0,022 | 0,394 | 0,023 | 0,430 | 0,006 | 0,805 |
| MKT\_RF^2 | - | - | - | - | -0,00257 | 0,024 | - | - | - | - | - | - |
| SMB^2 | - | - | - | - | - | - | - | - | - | - | - | - |
| HML^2 | - | - | - | - | 0,00779 | 0,039 | - | - | - | - | 0,00758 | 0,048 |
| MKT\_RF^3 | - | - | - | - | - | - | - | - | - | - | - | - |
| SMB^3 | - | - | - | - | - | - | - | - | -0,00143 | 0,010 | -0,00134 | 0,017 |
| HML^3 | - | - | - | - | - | - | - | - | - | - | - | - |
| MKT\_RF\*SMB | - | - | - | - | - | - | - | - | - | - | - | - |
| MKT\_RF\*HML | - | - | - | - | - | - | -0,00583 | 0,073 | - | - | -0,00671 | 0,031 |
| SMB\*HML | - | - | - | - | - | - | - | - | - | - | - | - |
| p-v ^2 = 0 | - | | - | | 0,001 | | - | | - | | 0,002 | |
| p-v ^3 = 0 | - | | - | | - | | - | | 0,113 | | 0,132 | |
| p-v inter = 0 | - | | - | | - | | 0,004 | | - | | 0,001 | |
| p-v ^2 & ^3 = 0 | - | | - | | - | | - | | - | | 0,003 | |
| p-v ^2 & ^3 & inter = 0 | - | | - | | - | | - | | - | | 2E-04 | |
| p-v fitted RESET test | 0,081 | | 0,081 | | 0,088 | | 0,602 | | 0,092 | | 0,147 | |
| p-v rhs RESET test | 0,001 | | 0,001 | | 0,026 | | 0,012 | | 0,009 | | 0,384 | |
| p-v Breusch-Godfrey | 1E-04 | | 1E-04 | | 3E-04 | | 0,001 | | 2E-04 | | 0,001 | |
| R2 | 0,9777 | | 0,9777 | | 0,9785 | | 0,9782 | | 0,9779 | | 0,9789 | |
| Adj R2 | 0,9775 | | 0,9775 | | 0,9782 | | 0,9780 | | 0,9776 | | 0,9785 | |
| AIC | 865 | | 865 | | 854 | | 858 | | 864 | | 850 | |
| SIC | 884 | | 884 | | 882 | | 882 | | 888 | | 882 | |
| N | 379 | | 379 | | 379 | | 379 | | 379 | | 379 | |

Rows from number 14 to 18 contain p-values of joint significance F-tests. The “-“ sign indicates that this factor has been removed during GETS procedure or was not considered in the model.

Table 6. Estimation of parameters of the cubic model with interactions for the European data.

|  |  |  |  |  |  |  |  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- | --- |
|  | Small – Low BM | | Small – Medium BM | | Small – High BM | | Big – Low BM | | Big – Medium BM | | Big – High BM | |
|  | 0,093 | 0,045 | 0,257 | 8E-15 | 0,267 | 4E-12 | 0,346 | 2E-17 | 0,226 | 7E-08 | 0,069 | 0,168 |
| Intercept | 1,076 | 0 | 0,985 | 0 | 0,946 | 0 | 0,937 | 0 | 0,998 | 0 | 1,072 | 0 |
| MKT\_RF | 0,910 | 3E-112 | 0,874 | 1E-173 | 0,856 | 1E-137 | -0,195 | 2E-13 | -0,089 | 2E-04 | -0,107 | 0,001 |
| SMB | -0,408 | 2E-42 | 0,035 | 0,055 | 0,490 | 2E-79 | -0,467 | 4E-57 | 0,006 | 0,805 | 0,618 | 9E-65 |
| HML | - | - | -0,00191 | 0,001 | - | - | - | - | - | - | - | - |
| MKT\_RF^2 | - | - | - | - | 0,01045 | 0,024 | - | - | - | - | - | - |
| SMB^2 | - | - | - | - | - | - | -0,00798 | 0,024 | 0,00758 | 0,04808 | - | - |
| HML^2 | - | - | - | - | - | - | - | - | - | - | - | - |
| MKT\_RF^3 | 0,00231 | 1E-04 | -0,00169 | 0,003 | -0,00161 | 0,006 | - | - | -0,00134 | 0,01689 | 0,00224 | 0,001 |
| SMB^3 | - | - | 0,00083 | 0,009 | - | - | - | - | - | - | - | - |
| HML^3 | - | - | - | - | - | - | - | - | - | - | - | - |
| MKT\_RF\*SMB | - | - | - | - | - | - | - | - | -0,00671 | 0,03096 | - | - |
| MKT\_RF\*HML | - | - | - | - | - | - | - | - | - | - | - | - |
| SMB\*HML | 0,093 | 0,045 | 0,257 | 8E-15 | 0,267 | 4E-12 | 0,346 | 2E-17 | 0,226 | 7E-08 | 0,069 | 0,168 |
| p-v ^2 = 0 | - | | 0,001 | | 0,009 | | 4E-04 | | 0,002 | | - | |
| p-v ^3 = 0 | 0,014 | | 1E-05 | | 0,035 | | - | | 0,132 | | 0,038 | |
| p-v inter = 0 | - | | - | | - | | - | | 0,001 | | - | |
| p-v ^2 & ^3 = 0 | - | | 8E-07 | | - | | - | | 0,003 | | - | |
| p-v ^2 & ^3 & inter = 0 | - | | - | | 0,012 | | - | | 2E-04 | | - | |
| p-v fitted RESET test | 0,209 | | 0,348 | | 0,259 | | 0,471 | | 0,147 | | 0,052 | |
| p-v rhs RESET test | 0,162 | | 0,054 | | 0,111 | | 0,045 | | 0,384 | | 0,332 | |
| p-v Breusch-Godfrey | 4E-05 | | 0,002 | | 0,205 | | 0,693 | | 0,001 | | 2E-04 | |
| R2 | 0,9798 | | 0,9904 | | 0,9864 | | 0,9780 | | 0,9789 | | 0,9774 | |
| Adj R2 | 0,9796 | | 0,9902 | | 0,9862 | | 0,9778 | | 0,9785 | | 0,9771 | |
| AIC | 896 | | 562 | | 713 | | 808 | | 850 | | 999 | |
| SIC | 920 | | 593 | | 740 | | 831 | | 882 | | 1022 | |
| N | 379 | | 379 | | 379 | | 379 | | 379 | | 379 | |

Rows from number 14 to 18 contain p-values of joint significance F-tests. The “-“ sign indicates that this factor has been removed during GETS procedure or was not considered in the model.

**II. SUPPLEMENT**

**1. In depth factors overview**

The main part of this thesis describes three factors from the Fama and French three-factor model introduced by Fama and French (1993) without getting into calculation details. This section of the Supplement specifies the construction of these factors. The market premium factor is the most straightforward one. It is calculated by subtracting the risk free rate of return (usually the one-month treasury bill rate) from the whole market portfolio return measured by an index. In order to define the SMB and HML factors certain portfolios have to be defined. Firstly, for all of the firms participating in the capital market *Size* (market capitalization) and *B/M* (book equity to market equity ratio) are calculated. Secondly, a bivariate sort on *Size* and *B/M* is performed resulting in creation of six portfolios with breakpoints being median of *Size* and 30th and 70th percentiles of *B/M*. This definition resolves the ambiguity of “low” and “high” values form the literature overview section. Figure 1. presents the portfolios and assigns names to each one.

|  |  |  |
| --- | --- | --- |
| 70th *B/M* percentile  30th *B/M* percentile | Median *Size* | |
| Small Value | Big Value |
| Small Neutral | Big Neutral |
| Small Growth | Big Growth |

Figure 1. – Diagram of portfolios creation.

After dividing stocks into portfolios the values of the factors for each time period can be calculated according to Equation (5) and Equation (6).

(5)

where:

– the *Size* factor for period

names of the portfolios represent their value-weighted returns for period .

(6)

where:

– the *B/M* factor for period

names of the portfolios represent their value-weighted returns for period .

More details can be found in the paper by Fama and French (2015). The process of computing values of the three factors is universal throughout the capital markets.

**Further data analysis**

Data used in this thesis consists of monthly values of the Fama and French three factors calculated in line with the method detailed in the previous section as well as excess returns on six portfolios formed on *Size* and *B/M*  with use of breakpoints mentioned in preceding section. Two separate datasets are used. The US monthly data covers the CRSP[[3]](#footnote-3) firms incorporated in the US and listed on the NYSE, AMEX and NASDAQ from July of 1926 to January of 2022. The European monthly data covers firms from capital markets of the following sixteen developed countries: Austria, Belgium, Switzerland, Germany, Denmark, Spain, Finland, France, Great Britain, Greece, Ireland, Italy, the Netherlands, Norway, Portugal and Sweden. The time span begins on July of 1990 and ends on January of 2022. The Fama and French three-factor model has become a standard for the capital markets investors. The popularity and need for standardisation has been answered by one of the authors – Kenneth French who posts monthly values of the factors on their website[[4]](#footnote-4). The database covers factors used by the three most cited and most popular models: three-factor by Fama and French (1993), four-factor by Carhart (1997) and five-factor by Fama and French (2015).

In general in the time series regression high correlation of the independent variables is undesirable. In order to verify if the datasets comply with this rule the correlation matrices are calculated for both the US (Table 7.) and the European case (Table 8.). The p-values of the Pearson correlation tests for all pairs of factors for the US are below the significance level α of 5%, therefore conclusions can be drawn from the correlation coefficients. Pearson correlation coefficients for the US data do not exceed 0.317 in absolute value so that no significant correlation between any pair of the factors can be identified. In the case of Europe the Pearson correlation tests for all pairs but the SMB – HML one have p-values lower than the significance level α of 5%. Additionally, the absolute values of the Pearson correlation coefficients are lower than 0.204, thus there are no significant correlations in the European case, apart from the SMB-HML pairing where the data does not allow for a clear answear..

Table 7. Pearson correlation coefficients for market premium, size premium and value premium for the US.

|  |  |  |  |
| --- | --- | --- | --- |
|  | Mkt.RF | SMB | HML |
| Mkt.RF | 1 | 0,317 | 0,236 |
| SMB |  | 1 | 0,118 |
| HML |  |  | 1 |

Table 8. Pearson correlation coefficients for market premium, size premium and value premium for the Europe.

|  |  |  |  |
| --- | --- | --- | --- |
|  | Mkt.RF | SMB | HML |
| Mkt.RF | 1 | -0,108 | 0,204 |
| SMB |  | 1 | -0,088 |
| HML |  |  | 1 |

The dependent variables in the considered return generating processes are excess returns on six portfolios formed as shown in the previous section. They are calculated by subtracting the risk free rate from the value-weighted return of stocks contained in a certain portfolio. The summary statistics of the portfolios’ returns are shown in Table 9. and Table 10.

Table 9. Distribution statistics for the six portfolios – the US data.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | mean | sd | min | median | max |
| Small - Low BM | 1,00 | 7,47 | -32,48 | 1,21 | 59,57 |
| Small - Medium BM | 1,25 | 6,97 | -30,05 | 1,52 | 62,20 |
| Small - High BM | 1,44 | 8,11 | -33,87 | 1,65 | 83,50 |
| Big - Low BM | 0,95 | 5,28 | -28,87 | 1,27 | 33,77 |
| Big - Medium BM | 0,96 | 5,63 | -28,18 | 1,22 | 51,94 |
| Big - High BM | 1,20 | 7,14 | -34,90 | 1,38 | 67,75 |

Table 10. Distribution statistics for the six portfolios – the European data.

|  |  |  |  |  |  |
| --- | --- | --- | --- | --- | --- |
|  | mean | sd | min | median | max |
| Small - Low BM | 0,57 | 5,47 | -25,82 | 0,92 | 16,84 |
| Small - Medium BM | 0,76 | 5,08 | -26,47 | 1,08 | 18,33 |
| Small - High BM | 0,94 | 5,22 | -27,04 | 1,17 | 21,01 |
| Big - Low BM | 0,66 | 4,67 | -19,10 | 0,87 | 13,33 |
| Big - Medium BM | 0,78 | 5,01 | -20,56 | 1,20 | 16,23 |
| Big - High BM | 0,78 | 5,92 | -24,93 | 1,28 | 25,53 |

Firstly, there is a clear difference in mean and median returns between the two datasets – the US portfolios on average have higher returns than the European ones. This observation may be caused by the aggregation of sixteen diverse capital markets. Secondly, all of the respective standard deviations of the US portfolios ale larger than the European ones. This fact along with higher average returns of the US portfolios are consistent with the conventional assumption of finance that the more volatile assets should yield higher on average return. Lastly, both of the Small – High BM portfolios happen to have the highest average return in each dataset.

**SUPPLEMENTARY REFRENCES**

Carhart, M. “On Persistence in Mutual Fund Performance.” The Journal of Finance, 52, (1997): 57-82.

Fama, E., and French K., “Common risk factors in the returns on stocks and bonds” Journal of Financial Economics, 33.1, (1993): 3-56.

Fama, E. and French, K., “A Five-Factor Asset Pricing Model.” Journal of Financial Economics, 116.1 (2015): 1-22.

1. <https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html> [↑](#footnote-ref-1)
2. <https://www.crsp.org/> [↑](#footnote-ref-2)
3. <https://www.crsp.org/> [↑](#footnote-ref-3)
4. <https://mba.tuck.dartmouth.edu/pages/faculty/ken.french/data_library.html> [↑](#footnote-ref-4)