Fourier series: $x(t)=\sum_{k=-\infty}^{\infty} X[k]e^{jk\omega_0 t}$ ($x(t)$为periodic signal) ($\omega_0=\frac{2\pi}{T_0}$ 即为本身的角速度)

$X[k]=\frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)e^{-jk\omega_0 t}dt=\frac{1}{T_0}\int_{<T_0>} x(t)e^{-jk\omega_0 t}dt$ (注意,这个公式是根据上个公式得出的,不是指定$X[k]$的值)

这是exponential fourier series, $X[k]$为Fourier coefficients,可为复数

周期信号可用Fourier series表示(条件): $\int_{<T_0>}|x(t)|dt<\infty$,一个周期内不连续点、极大值、极小值有限

$|X[k]|$ vs ω为magnitude spectrum   spectrum频谱

$\arg[X[k]]$ vs ω为phase spectrum

ω_0为fundamental frequency, $k\omega_0$为kth harmonic

$X[0]=\frac{1}{T_0}\int_{<T_0>}x(t)dt$ 为$x(t)$的平均值,也叫DC component

若$x(t)$为实数,则$X[-k]=X^*[k]$, $|X[k]|=|X[-k]|$, $\arg[X[-k]]=-\arg[X[k]]$

因为虚数部分必须抵消

若为实数,则$x(t)=X[0]+\sum_{k=1}^{\infty}2|X[k]|\cos(k\omega_0 t+\arg[X[k]])$

$x(t)=X[0]+\sum_{k=1}^{\infty}[B[k]\cos(k\omega_0 t)+A[k]\sin(k\omega_0 t)]$

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其中: $B[k]=2\operatorname{Re}[X[k]]=\frac{2}{T_0}\int_{<T_0>}x(t)\cos(k\omega_0 t)dt$

$A[k]=-2\operatorname{Im}[X[k]]=\frac{2}{T_0}\int_{<T_0>}x(t)\sin(k\omega_0 t)dt$

$X[k]=\frac{B[k]}{2}-\frac{A[k]}{2}$

转换: $y(t)=Ex(t)+F$: $Y[0]=EX[0]+F$, $Y[k]=EX[k]$ ($k\neq 0$)

$y(t)=x^*(-t)$: $Y[k]=X^*[k]$ (若$x(t)$为实数,则$Y[k]=X[-k]$)

$y(t)=x(t-t_d)$: $Y[k]=X[k]e^{-jk\omega_0 t_d}$

$P_{avg}=\frac{1}{T_0}\int_{<T_0>}|x(t)|^2dt=\sum_{k=-\infty}^{\infty}|X[k]|^2$ (Parseval's Theorem,可以理解为各个频率功率之和)

$|X[k]|^2$ vs ω: Power spectrum

考虑在LTI system中(FRF为$H(j\omega)$): $x(t)=\sum_{k=-\infty}^{\infty}X[k]e^{jk\omega_0 t}\Rightarrow y(t)=\sum_{k=-\infty}^{\infty}X[k]\cdot H(jk\omega_0)e^{jk\omega_0 t}$

则Fourier cofficients: $Y[k]=X[k]H(jk\omega_0)$

考虑实数的三角形式: $x(t)=X[0]+\sum_{k=1}^{\infty}2|X[k]|\cos(k\omega_0 t+\arg[X[k]])$

$\Rightarrow y(t)=H(j0)X[0]+\sum_{k=1}^{\infty}2|X[k]||H(jk\omega_0)|\cos(k\omega_0 t+\arg[X[k]]+\arg[H(jk\omega_0)])$

$x(t)=X[0]+\sum_{k=1}^{\infty}[B[k]\cos(k\omega_0 t)+A[k]\sin(k\omega_0 t)]$

$\Rightarrow y(t)=H(j0)X[0]+\sum_{k=1}^{\infty}[B[k]|H(jk\omega_0)|\cos(k\omega_0 t+\arg[H(jk\omega_0)])+A[k]|H(jk\omega_0)|\sin(k\omega_0 t+\arg[H(jk\omega_0)])]$

Fourier transform：把之前的离散量变为连续量（不要求是周期函数）

$$F(\omega)=F(j\omega)=\int_{-\infty}^{\infty} f(t)\,e^{-j\omega t}\,dt \triangleq \mathcal{F}[f(t)]$$ （$F(\omega)$相当于$X[k]$）

$$f(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t}\,d\omega \triangleq \mathcal{F}^{-1}[F(\omega)]$$ （$f(t)$相当于$X(t)$）

$$\mathcal{F}[x(t)\cos(\omega_0 t)]=\frac{1}{2}X(\omega-\omega_0)+\frac{1}{2}X(\omega+\omega_0)$$ $$\mathcal{F}[x(t)\sin(\omega_0 t)]=\frac{jX(\omega+\omega_0)-jX(\omega-\omega_0)}{2}$$

可以看成是Fourier series的周期达到无穷大，所以ω变为无限小，从离散量变为连续量

任何信号可用Fourier transform表示的条件：$\int_{-\infty}^{\infty}|f(t)|\,dt<\infty$，$f(t)$在任意有限区间内的不连续点、极大值、极小值有限

Square-integrability：$\int_{-\infty}^{\infty}|f(t)|^2\,dt<\infty$

$f(t)$	$F(\omega)$
$e^{-at}u(t)$ $(Re[a]>0)$	$\frac{1}{a+j\omega}$
$te^{-at}u(t)$ $(Re[a]>0)$	$\frac{1}{(a+j\omega)^2}$
$t^{n-1}e^{-at}u(t)$ $(Re[a]>0)$	$\frac{(n-1)!}{(a+j\omega)^n}$
$e^{-a\|t\|}$ $(Re[a]>0)$	$\frac{2a}{a^2+\omega^2}$
$rect(\frac{t}{D})$	$\frac{2}{\omega}\sin(\frac{\omega D}{2})=D\,sinc(\frac{\omega D}{2\pi})$

$sinc\,x=\frac{\sin x}{x}$

$$\frac{1}{t\pi}\sin(tB)=\frac{B}{\pi}sinc(\frac{tB}{\pi})\quad rect(\frac{\omega}{2B})$$

$\delta(t)$：$\mathcal{F}[\delta(t)]=1$，$\mathcal{F}[\delta(t-t_0)]=e^{-j\omega t_0}$

$$\delta(t)=\frac{1}{2\pi}\int_{-\infty}^{\infty}e^{j\omega t}\,d\omega$$

$\mathcal{F}[\cdot]$操作为linear的

若$f(t)$为实函数，则$F^*(\omega)=F(-\omega)$

$|F(\omega)|=|F(-\omega)|$，$\arg[F(\omega)]=-\arg[F(-\omega)]$

Bode Plot：$20\log_{10}(|F(\omega)|)$ vs ω

单位：dB，相差20dB，强度相差10倍

RC电路：$H(\omega)=\frac{1}{1+(j\omega)\tau}$ $(\tau=RC)$（$V_C(t)$为输出）

$G(\omega)=\frac{(j\omega)\tau}{1+(j\omega)\tau}$（$V_R(t)$为输出）

RLC电路：
$$H(\omega)=\frac{\omega_n^2}{(j\omega)^2+2\zeta\omega_n(j\omega)+\omega_n^2}=\frac{1}{(j\frac{\omega}{\omega_n})^2+2\zeta j(\frac{\omega}{\omega_n})+1}$$ （$\omega_n=\frac{1}{\sqrt{LC}}$，$\zeta=\frac{R}{2}\sqrt{\frac{C}{L}}$）

ζ为damping factor，若$\zeta<\frac{1}{\sqrt{2}}$，会产生overshoot

$$\mathcal{F}[\cos(\omega_0 t)]=\pi\delta(\omega-\omega_0)+\pi\delta(\omega+\omega_0)\qquad \mathcal{F}[sgn(t)]=\frac{2}{j\omega}$$

$$\mathcal{F}[\sin(\omega_0 t)]=j\pi\delta(\omega+\omega_0)-j\pi\delta(\omega-\omega_0)$$

产生overshoot的频率$\omega'=\omega_n\sqrt{1-2\zeta^2}$

若$f(t)=\sum_{k=-\infty}^{\infty}F[k]e^{jk\omega_0 t}$，则$F(\omega)=\sum_{k=-\infty}^{\infty}2\pi F[k]\delta(\omega-k\omega_0)$（只有在整数倍的时候才有定义，其余均为0）

怎么计算：$f(t)=\sum_{-\infty}^{\infty}\delta(t-kT_0)\Rightarrow f(t)=\sum_{-\infty}^{\infty}\frac{1}{T_0}e^{jk\omega_0 t}\Rightarrow F(\omega)=\sum_{k=-\infty}^{\infty}\omega_0\delta(\omega-k\omega_0)$

$f(t-t_0)\overset{\mathcal{F}}{\longleftrightarrow}e^{-j\omega t_0}F(\omega)$　$\mathcal{F}[f(t-t_0)]=e^{-j\omega t_0}F(\omega)$

信号时间变化会导致频谱上的相变化

You may have seen the expression "Linear Phase" in the description of audio components; this means that there is no distortion of the signal, only a time delay.

卷积：$x(t) * h(t) \xleftrightarrow{F} X(\omega)H(\omega)$ (convolution)

$e^{j\omega_0 t} f(t) \xleftrightarrow{F} F(\omega-\omega_0)$

Modulation: $x(t)m(t) \xleftrightarrow{F} \frac{1}{2\pi} X(\omega) * M(\omega)$

Time differentiation: $\frac{df(t)}{dt} \xleftrightarrow{F} j\omega F(\omega)$ (FRF为 $H(\omega)=j\omega$ 为 differentiator)

$$\frac{d^n f(t)}{dt^n} \xleftrightarrow{F} (j\omega)^n F(\omega)$$

Time integration: $\int_{-\infty}^{t} f(\tau)d\tau \xleftrightarrow{F} \frac{1}{j\omega}F(\omega)$ (FRF为 $H(\omega)=\frac{1}{j\omega}$，则为 integrator)

Frequency differentiation: $(-jt)^n f(t) \xleftrightarrow{F} \frac{d^n F(\omega)}{d\omega^n}$, $t^n f(t) \xleftrightarrow{F} \frac{d^n F(\omega)}{d\omega^n}$

Time scaling: $f(\alpha t) \xleftrightarrow{F} \frac{1}{|\alpha|} F(\frac{\omega}{\alpha})$

Parseval's Theorem: $\int_{-\infty}^{\infty} |f(t)|^2 dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2 d\omega$，和之前计算 Pavg 的相似

Energy Spectral Density (ESD): $\mathcal{E}_f(\omega) = \frac{1}{2\pi}|F(\omega)|^2$

若 $Y(\omega)=H(\omega)X(\omega)$ ($x(t)$ 输入，$y(t)$ 输出，$H(\omega)$ 为 FRF)，则 $\mathcal{E}_y(\omega) = |H(\omega)|^2 \mathcal{E}_x(\omega)$

$[\omega_1, \omega_2]$ 内的能量：$\int_{\omega_1}^{\omega_2} \mathcal{E}_f(\omega)d\omega$，严格来说应是 $f(t)$ 在此频率范围内对应的能量

Filtering: a filter is a FRF $H(j\omega) = H(\omega)$

Ideal filters: filters whose magnitude is either zero or one and whose phase is zero or, if non-zero, is a linear function of ω

Lowpass, highpass, bandpass, bandstop Ideal LPF: $H(j\omega) = \mathrm{rect}(\frac{\omega}{2W})$ 实际不存在

Linear phase 解释：考虑 $H(\omega) = 1e^{-j\omega t_d}$，只会带来时间平移的影响

Modulation: double-sided, suppressed carrier, sinusoidal amplitude modulation (DSB/SC-AM)

DSB/SC-AM: $s(t) = m(t)c(t) = m(t)A_c\cos(\omega_c t)$ (实际上，$m(t)$ 为修改过的，为原信号 $m_0(t)$ 经过理想的 lowpass filter of $\omega_m < \omega_c$

($m(t)$ 为要被传输的信号，$c(t)$ 为正弦的 carrier，$A_c = 1$

$S(\omega) = \frac{1}{2}M(\omega+\omega_c) + \frac{1}{2}M(\omega-\omega_c)$

Demodulation: 从 $s(t)$ 恢复 $m(t)$ (不是 $m_0(t)$)

Synchronous (or coherent) demodulation: 乘上 $\cos(\omega_c t+\theta)$，再 lowpass

$V(\omega) = \frac{1}{2}S(\omega+\omega_c) + \frac{1}{2}S(\omega-\omega_c)$

$= \frac{1}{4}M(\omega+2\omega_c) + \frac{1}{2}M(\omega) + \frac{1}{4}M(\omega-2\omega_c)$ 实际 $\omega-2\omega_c<0$，但其绝对值大于 ω 也能被 lowpass 过滤，只要 $\omega_m<\omega_c$ 即可

FDM (frequency-division multiplexing): 在同一介质中传输不同频率信号

Realizable filters: ω_c (cut-off frequency of practical lowpass filter):

$|H(\omega_c)| = 0.707$, $20\log(|H(\omega_c)|) = -3\text{dB}$, ~~也叫 ω_{3dB}, 3-dB bandwidth~~

不是绝对的，而是连续变化的，即离得远的弱，$|H(\omega_c)|^2 = 0.5$ 为 cut-off frequency

half-energy half-power frequency

Baseband signals: $[-\omega_B, \omega_B]$, $BW_{abs} = \omega_B$

Bandpass signals: $[-\omega_2, -\omega_1] \cup [\omega_1, \omega_2]$, $BW_{abs} = \omega_2 - \omega_1$

$\dfrac{|F(\omega_{3dB})|}{|F(0)|} = \dfrac{1}{\sqrt{2}}$, 3-dB bandwidth, BW_{3dB} $(= \omega_2 - \omega_1)$

$[BW_{RMS}]^2 = \dfrac{\int \omega^2 |F(\omega)|^2 d\omega}{\int |F(\omega)|^2 d\omega}$ $\quad [TD_{RMS}]^2 = \dfrac{\int t^2 |f(t)|^2 dt}{\int |f(t)|^2 dt}$ (这里积分均为 $\int_{-\infty}^{\infty}$)

Time-Bandwidth Product: $BW_{RMS} \cdot TD_{RMS} \geq \frac{1}{2}$

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(3) SSB AM (Single Sideband Amplitude Modulation)

→ We transmit only one sideband instead of both sidebands in the method. The BW and power requirement is half that of DSB-SC AM.

(4) Quadrature AM:

→ We can send two message signals $x(t)$ and $y(t)$ in the same BW by transmitting the signal $z(t) = x(t)\cos(\omega_c t) + y(t)\sin(\omega_c t)$

→ At the receiver end, we perform synchronous demodulation as in the case of DSB-SC AM.

→ The BW required per signal is half of that of DSB-SC AM.

(5) Commercial AM Radio

→ The transmitted signal is $(x(t)+A)\cos(\omega_c t)$ ($\cos(\omega_c t)$ carrier, $x(t)$ message). The carrier is ALSO transmitted in this method. The demodulation will be done by envelope detection. (× synchronous de...)

→ The superheterodyne receiver consists of a local oscillator, an intermediate bandpass filter, an image rejection filter and the demodulator.

→ Two local oscillator frequencies are possible:

Case 1: $f_{LO} = f_c + f_{IF}$, Case 2: $f_{LO} = |f_c - f_{IF}|$

→ We run into an image frequency problem with commercial AM radio. The image frequencies for both cases are: Case 1: $f_I = f_{LO} + f_{IF}$

Case 2: This case needs to be handled carefully, depending on whether $f_{IF} > f_c$ or $f_{IF} < f_c$

→ This image rejection filter eliminates the image frequency problem.