① dual

dual: $\mathcal{L}(V,k)$, dual of V, V^*

dual basis: $B^* = \{f_1, \cdots, f_n\}$ $f_i : V \to K$, $f_i(b_j) = \begin{cases} 1, & i=j \\ 0, & i \neq j \end{cases}$

f_i: ith coordinate function with respect to B

$\forall \varphi \in V^*$, $\varphi = \sum_{i=1}^{n} \varphi(b_i) f_i$, $\forall x \in V$, $x = \sum_{i=1}^{n} f_i(x) b_i$

f^T, transpose of f: $f \in \mathcal{L}(V,W)$, $f^T : W^* \to V^*$, $f^T(w^*) = w^* \circ f$

$$f^T : (W \to k) \to (V \to k)$$

M: basis B' to B (B', B in same vectorspace), then $(M^{-1})^T$ is B'^* to B^*

predual basis: F basis of V^*, predual basis of F is B that $B^* = F$

annihilator: Let S be subset of V. Annihilator of S in V^* is

$$S^\perp = \{\varphi \in V^*, \forall s \in S, \varphi(s) = 0\}$$

Let L be subset of V^*, annihilator of L in V is

$$^\circ L = \{x \in V, \forall \varphi \in L, \varphi(x) = 0\}$$

$\dim V_0^\perp = \dim V - \dim V_0$, $\dim {}^\circ L = \dim V - \dim L$

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${}^\circ(V_0^\perp) = V_0$, $({}^\circ L)^\perp = L$, ${}^\circ(V^*) = \{0\}$

$f \in \mathcal{L}(V,W)$, then $\ker f^T = (\operatorname{im} f)^\perp$

codimension: V_0 subspace of V. $\operatorname{codim} V_0 = \dim V - \dim V_0$

② linear system

Elementary row operations: 交换两行，某行乘一个倍数，将另一行乘一个倍数加到自己身上

$\operatorname{rank}\left(\begin{array}{c|c} \alpha & \cdots \\ \hline 0 & A' \end{array}\right) = \operatorname{rank} A' + 1$ $(\alpha \neq 0)$

$GL_n(k)$ ($GL(V)$ means automorphism) matrix can be transformed into invertible matrix through elementary operations.

linear system: 一次方程组

affine space: A that $A=\Omega+V_0$ ($\Omega\in V$, V_0 subspace of V).
We say A is an affine space directed by V_0, passing through Ω.

③ permutation

transposition: 交换两个元素的位置

permutation: 多个 transposition 的组合（符号 σ_n，注意 permutation 集合的符号是 $\mathfrak{S}_n$，为 group）

signature: $\varepsilon(\sigma)=(-1)^{I(\sigma)}$（其中 $I(\sigma)=\sum_{i=1\sim n}^{j=1\sim n}(i<j)\ \underline{(\sigma(i)>\sigma(j))}$ true 1 false 0）

一次 transposition 必然改变 signature，奇数次则称排列为 odd，否则 even

关于符号：permutation σ，permutation 的集合 $\mathfrak{S}_n$，signature ε，transposition τ
（可用 $\sigma(i)$ 表示排列中的第 i 项，$I(\sigma)$ 表示 σ 中颠倒的数量）

σ,τ permutation，$\varepsilon(\sigma\tau)=\varepsilon(\sigma)\varepsilon(\tau)$

alternating group $\mathfrak{A}_n$: subgroup of $\mathfrak{S}$, all even

$\mathfrak{A}_n\tau=\{\sigma\tau,\sigma\in\mathfrak{A}_n\}$ 为 odd permutation 的集合

④ multi-linear map

multi-linear map: $f: V_1\times V_2\times\cdots\times V_p$, $f(v_1,v_2,\cdots v_p)=w$
且针对任意一项，满足线性关系（在其它项不变的情况下）

$f^\sigma(v)$: $f\in\mathcal{L}(V^p,W)$, $v=(v_1,\cdots,v_p)\in V^p$, $f^\sigma(v)=f(v_{\sigma(1)},\cdots,v_{\sigma(p)})$

f is antisymmetric if $f^\sigma=\varepsilon(\sigma)f$

f is alternating multi-linear map if 有两项相同，则 $f(v)=0$

f antisymmetric $\Leftrightarrow f^\tau=-f$, alternating $\cdots\Rightarrow$ antisymmetric
(if 2 isn't zero divider, antisymmetric $\Rightarrow$ alternating $\cdots$)

f alt$\cdots$, $v_1\sim v_p$ 线性不独立，则 $f(v_1,\cdots v_p)=0$

f alt$\cdots$，将某些项的线性组合添加到其它项，值不变

$f\in\mathcal{L}_p(V,W)$, $A(f)=\sum_{\sigma\in\mathfrak{S}_p}\varepsilon(\sigma)f^\sigma$ 为 alt$\cdots$

另外，若输入有 p 个，则一般称为 p-linear

⑤ determinant

广义的det：输入n个n维向量，输出一个数（与常规的det不同）

需满足n-linear alternating关系，且若输入为一组basis，输出必须为1

即 $\det_B(b_1,\dots,b_n)=1$ （$\{b_1,\dots,b_n\}$为basis）

$g(a_1,a_2,\dots,a_n)=g(b_1,b_2,\dots,b_n)\cdot\det_B(a_1,\dots,a_n)$

计算方法：$B=\{b_1,\dots,b_n\}$ $B^*=\{f_1,\dots,f_n\}$，$x=(x_1,\dots,x_n)$

$\det_B(x)=\det_B(x_1,\dots,x_n)=\sum_{\sigma\in S_n}\varepsilon(\sigma)\prod_{j=1}^{n}\langle x_{\sigma(j)},f_j\rangle=\sum_{\sigma\in S_n}\varepsilon(\sigma)\prod_{j=1}^{n}\langle x_j,f_{\sigma(j)}\rangle$

$u\in\mathcal{L}(V)$，$f(u(x_1),\dots,u(x_n))=f(x_1,\dots,x_n)\cdot\lambda$ （$\lambda=\det u$）

怎样理解：$x_1\sim x_n$看作basis（$b_1\sim b_n$），$u(x)$看作（$a_1\sim a_n$），det u当作a~b的转换方式，（若输入$x_1\sim x_n$确实为1）

$\det(\lambda u)=\lambda^n\det u$，$\det(vu)=\det v\cdot\det u$，$\det u^{-1}=(\det u)^{-1}$，$\det A=\det A^T$

cofactor：代数余子式，$(-1)^{i+j}\Delta_{i,j}$

$\det\begin{pmatrix}A & B\\ 0 & C\end{pmatrix}=\det A\det C$

comatrix：每个位置都是其对应的cofactor

若B为A的comatrix，则 $AB^T=B^TA=(\det A)\cdot I_n$

⑥ eigenvalue, eigenvector

trace：对角线数字之和

$\mathrm{tr}\,AB=\mathrm{tr}\,BA$，$\mathrm{tr}\,A=\mathrm{tr}(P^{-1}AP)$，$\mathrm{tr}: M_n(K)\to K$ 为linear form

eigen value, eigenvector

λ is eigenvalue $\Leftrightarrow \ker(f-\lambda I_n)\neq\{0\}\Leftrightarrow f-\lambda I_n$ is not injective

eigenspace：λ对应eigenvector组成的space，记作$E_\lambda(f)$　spectrum：eigenvalue的集合（σ_f）

~~若n×n矩阵~~有n个不同eigenvalue，则其对应的eigenspace线性独立

χ_A：$\chi_A(\lambda)=\det(A-\lambda I_n)$ (characteristic polynomial)

$\chi_A(\lambda)=(-\lambda)^n+(-\lambda)^{n-1}\mathrm{tr}(A)+\cdots+\det A$

若A与B similar（$A=P^{-1}BP$），则 $\chi_A(\lambda)=\chi_B(\lambda)$

algebraic multiplicity：degree of λ_0 in χ_f（重根）

geometric multiplicity：dimension of eigenspace of λ_0（代入后的实际情况）

$1 \le$ geometric $\le$ algebraic

diagonalizable：存在与其similar的对角矩阵

diagonalizable $\Leftrightarrow$ eigenvectors 为basis $\Leftrightarrow$ eigenspaces 形成 V $\Leftrightarrow$ eigenspaces dimension 之和 $= V$

diagonalizable $\Leftrightarrow$ 有n个实数解(含重根)，且每个根 $\dim(E_\lambda(f)) =$ algebraic ...

若n维，且有n个不同eigenvalue，则 diagonalizable

⑦ polynomial

$P(f) = \sum_{i=0}^{N} a_i f^i$，$P(A) = \sum_{i=0}^{N} a_i A^i$

$(\alpha P + Q)(f) = \alpha P(f) + Q(f)$，$(PQ)f = P(f) \circ Q(f)$

若 $f \circ g = g \circ f$，则 $P(f) \circ Q(g) = Q(g) \circ P(f)$

annihilating polynomial：P that $P(f) = 0$ 或 $P(A) = 0$

若λ为f的eigenvalue，则 $(P(f))(x) = P(\lambda)x$

若P为f的annihilating polynomial，则 $\sigma_f \subset P^{-1}(\{0\})$

diagonalizable $\Leftrightarrow$ 存在P使 $P(f) = 0$ 且P在K上仅有实根

Cayley-Hamilton theorem：$\chi_f(f) = 0$

Let V be a vectorspace, $f \in \mathcal{L}(V)$, $N \in \mathbb{N}^*$, $P_i \in K[X]$, $i \in [1, N]$, P_i are pairwise prime polynomials. The vector subspaces $\ker P_i(f)$ are in direct sum and $\bigoplus_{i=1}^{N} \ker P_i(f) = \ker\left(\left(\prod_{i=1}^{N} P_i\right)(f)\right)$

Let V be a finite dimensional vector space, $f \in \mathcal{L}(V)$, $N \in \mathbb{N}^*$ and $P_i \in K[X]$ $(i \in [1, N])$ be N pairwise coprime polynomials such that $\chi_f = \prod_{i=1}^{N} P_i$. Then there exits a basis B of V such that the matrix of f in B is block diagonal.

⑧ 理解 $M' = P^{-1} M P$

这里应该认为M、M'是函数，而不是一个普通的矩阵，M'：旧坐标→旧坐标 M：新坐标→新坐标，P的作用是把点从旧坐标变换为新坐标，然后在新坐标中运算(M)，然后再通过 P^{-1} 变换回去。怎样对角化：计算 eigenvalue 和 eigenvector，所有 eigenvector 作为列组成矩阵，即P为P，eigenvalue 按对应顺序放在对角线上，即为M。

⑨ 一些符号

T_n：upper triangular

D_n：diagonal

S_n：symmetric

A_n：antisymmetric $(A^T = -A)$

Group：存在0和相反数，交换率结合率