Appendix 2. Cyclic groups, group generators

The bonus problem 2 (1p) in Moodle is based on this appendix.

Diffie Hellman key exchange and ECDHE key exchange are both based on cyclic groups. In this appendix, we review the concepts of group and cyclic group and use their properties to find a group generator of multiplicative group Z_p.

Basic concepts

Definition 1: A set G with operation * defined in G is called a group, if it has the following properties.

G1) a*b ϵ G   for all a,b ϵ G

G2) (a*b)*c = a*(b*c)   for a,b,c ϵ G

G3) G has a neutral element e for which a*e = e*a = a for all a ϵ G

G4) Every element a ϵ G has an inverse element a^{-1} for which $a^{-1} * a = a * a^{-1} = e$

Example : The number set Z_p* of integers {1,2,….,p-1} is a group where the group operation is multiplication a*b (mod p).

The neutral element of Z_p* is number 1. All elements a ϵ Z_p* have an inverse element a^{-1} ϵ Z_p*. Indeed, modulus p is a prime and therefore gcd(a,p) = 1 for all a ϵ Z_p*. Inverse can be calculated using Euclid's extendedGDC algorithm.

Definition 2: Let G be a finite group of n elements.

If there is such an element g ϵ G that the set of its powers {g, g^2, … , g^n } includes all elements of G, we say that group G is **cyclic** and the element g is a **generator** of group G.

If p is prime, the number set Z_p* of integers {1, 2,…., p -1} is a cyclic group. Diffie Hellman key exchange uses this group.

Example: Group Z_{13}* is cyclic. For example number 7 is a generator of Z_{13}*, because the set of powers {7^1 mod 13, 7^2 mod 13, …, 7^{12} mod 13} = {7, 10, 5, 9, 11, 12, 6, 3, 8, 4, 2, 1} contains all the elements of Z_{13}*.

Definition 3: If H is a group and H is a subset of of group G, we say that H is a **subgroup** of group G.

Example: Set H = {1,3,9} is a subgroup of Z_{13}*.
The multiplication table, where operation is a*b (mod 13) shows that all group properties G1,…,G4 hold.

	1	**3**	**9**
1	*1*	*3*	*9*
3	*3*	*9*	*1*
9	*9*	*1*	*3*

This group is cyclic, because 3 generate all its elements: {$3,3^2,3^3$} mod 13 = {3,9,1}

Properties of multiplicative groups Z_p*

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1. All elements a of Zp* = {1,2,…., p -1) generate a cyclic subgroup of Zp*

2. The size of the subgroup generated by element a is called **the order of element a** and denoted **Ord(a)**.

3. Lagrange's theorem: Ord(a) is a divisor of p - 1 for all a ϵ Zp*.

4. If Ord(g) is p -1, *element g is called* a generator of Zp* or primitive root of Zp*

The next property follows from properties 1 – 4.

5. Let d_1, d_2, …, d_n be the list of divisors of p – 1 in ascending order (where d_1=1 and d_n= p -1)

Then, the generators are those elements g of Zp*, for which only the last power of g in the sequence g^{d1}, g^{d2}, …, g^{dn} equals 1 (mod p).

Example. Below is a table of powers (a^k mod 11) of elements of Z_{11}*.

a	a^1	a^2	a^3	a^4	a^5	a^6	a^7	a^8	a^9	a^{10}	Subgroup size
1	1	1	1	1	1	1	1	1	1	1	1
2	2	4	8	5	10	9	7	3	6	1	10 (generator)
3	3	9	5	4	1	3	9	5	4	1	5
4	4	5	9	3	1	4	5	9	3	1	5
5	5	3	4	9	1	5	3	4	9	1	5
6	6	3	7	9	10	5	8	4	2	1	10 (generator)
7	7	5	2	3	10	4	6	9	8	1	10 (generator)
8	8	9	6	4	10	3	2	5	7	1	10 (generator)
9	9	4	3	5	1	9	4	3	5	1	5
10	10	1	10	1	10	1	10	1	10	1	2

There are 4 generators: {2,6,7,8}
All the subgroup sizes 1,2,5, and 10 are divisors of 10 (in general p – 1) as Lagrange's theorem predicts.

The previous theory provides a tool for finding a group generator of Zp*

Case1: Generator test, when p is a relatively small prime

Let d_1, d_2, …, d_n be the divisors of p – 1 in ascending order.
An integer g ϵ Z_p* is a generator if and only if the last integer in the list of powers g^{d1}, g^{d2}, …., g^{dn} equals 1.

Example: a) Test if number 5 a generator of Z_{29}* or not?

Divisors of p – 1 = 28 are {1,2,4, 7,14,28}. Raising 5 to all the powers in the divisor list gives (using WolframAlpha)
5^{1,2,4,7,14,28} mod 29 = {5, 25, 16, 28, 1, 1}. Number 5 is not a generator, because the last two numbers are ones.
(WolframAlpha allows to calculate all powers with only one command line)

2^{1,2,4,7,14,28} mod 29 = {2, 4, 16, 12, 28, 1}. Number 2 is a generator of Z_{29}*

Case2: Finding a generator, when p is very large prime

When prime p is very large, for example 1000 bit integer, it is often impossible to factor p – 1. Some divisors are trivial: 1, 2 and p – 1 (2 is a divisor, because p – 1 is even). However, if p – 1 has very large factors, some of the divisors may be not be found.

Example. If p = 265738830135992486377941683556469254997964098756853, then p – 1 = 265738830135992486377941683556469254997964098756852 Attempts to factor p – 1 further fail.

Strong primes

Definition: A prime p is called a "**strong prime**", if also (p-1)/2 is a prime.

In other words: If p is a strong prime, then p - 1 has only four divisors: {1,2, (p-1)/2, p - 1}

It is recommended that Diffie-Hellman key exchange protocol should use strong primes as modulus p. To find these primes, one can write a Python program or utilize the help of chatGPT. Such a program generated 1033 primes, out of which 75 were strong primes within the range of 10000-20000. In practise, the system parameter g can also be any element of Z_p* that generates an adequately large subgroup.

Generator test is easy to do for Z_p*, if p is a strong prime.

Example. Integer p = 200087 is a strong prime. Divisors of p – 1 are {1,2,100043, 200086}. Test if number 5 is a generator of Z_{200087}*.

Calculation of powers **5^{1,2,100043,200086} mod 200087** gives {5,25, 200086, 1}, which shows that number 5 is a generator of Z_{200087}*.

Rule: Number of generators of Zp* = φ(p-1), where φ is Euler's totient function.

(More general rule is that for any divisor d of p - 1 the number of elements of order d is φ(d))

If p is a strong prime, then p – 1 = 2*r, where r = (p-1)/2 is also a prime.
Now φ(2*r) = (2-1)(r-1) = (p-1)/2 – 1 ≈ (p-1)/2.

Hence: If p is a large prime, nearly 50% of elements of Z_p* are generators.
