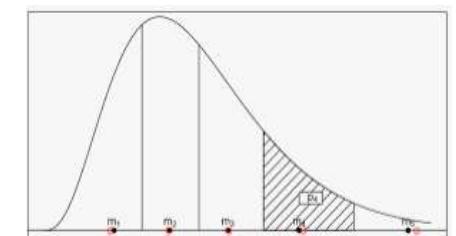


Statistics / Lecture1: Probability distributions, expected value, decision trees

1. Discrete probability distributions
2. Parameters: Expected Value, standard deviation
3. Continuous probability distributions
 - example: Normal distribution
4. Decision trees
 - example: Newsboy Problem



discrete distribution



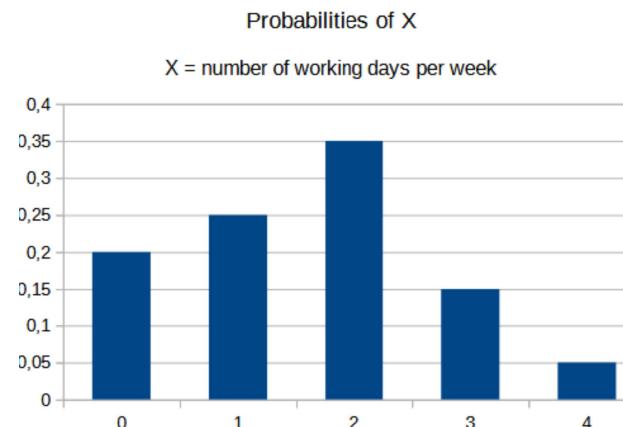
continuous distribution

1. Discrete probability distributions

- A discrete probability distribution shows all possible values of a discrete random variable x and their associated probabilities

Example1: Anne is a nursing student. She works occasionally as a sick leave substitute in a health centre. The number of weekly working days per week vary between 0 to 4. Following table shows the possible values of x and their probabilities. Visualization is bar chart.

X	P
0	0.2
1	0.25
2	0.35
3	0.15
4	0.05



Expected value $E(x)$

- Expected value describes the long-term average level of a random variable x based on its probability distribution.

$$\text{Expected value } E(x) = \sum p_i x_i$$

(that is: $E(x) = p_1 x_1 + p_2 x_2 + \dots + p_n x_n$)

X	P
0	0.2
1	0.25
2	0.35
3	0.15
4	0.05

What is the expected value of the example?

The expected value of Anne's working days per week
 $E(x) = 0.2*0 + 0.25*1 + 0.35*2 + 0.15*3 + 0.05*4 = 1.6$

In other words the long term average is 1.6 working days per week.

Variance $\text{Var}(x)$, Standard deviation

Variance and its square root are parameters, which describe how widely values of x are distributed around the average (expected return). The formulas for variance and standard deviation in case of discrete probability distribution are following.

Variance $\text{Var}(x) = \sum p_i (x_i - \mu)^2$, where $\mu = E(x)$ (expected value)

Standard deviation $\text{Std}(x) = \sqrt{(\text{Var}(x))}$ *)

x	$x-\mu$ $=x-1.6$	P
0	-1.6	0.2
1	-0.6	0.25
2	0.4	0.35
3	1.4	0.15
4	2.4	0.05

Calculate the variance and standard deviation of the example

$$\text{Var}(x) = 0.2*1.6^2+0.25*0.6^2+0.35*0.4^2+0.15*1.4^2+0.05*2.4^2= 1.24$$

$$\text{Std}(x) = \sqrt{1.24} = 1.11$$

*) $(-1.6)^2 = 1.6^2 \Rightarrow$ minus signs are not needed

Expected Return (= expected value of profit)

Ex. Midsummer music festival profitability analysis

The profit or loss of organizing the festival depends on weather conditions in the following way. Calculate the expected return. What are your conclusions.



weather	probability	Return (profit or loss)
Sunny weather	0.40	+12000
Cloudy, chilly, but no rain	0.35	+ 2000
rain	0.25	- 50000

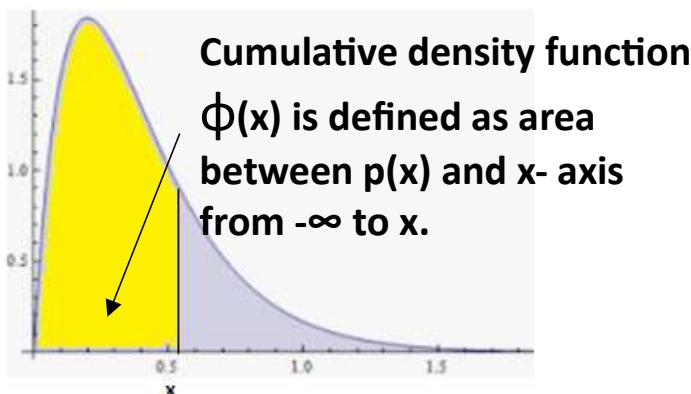
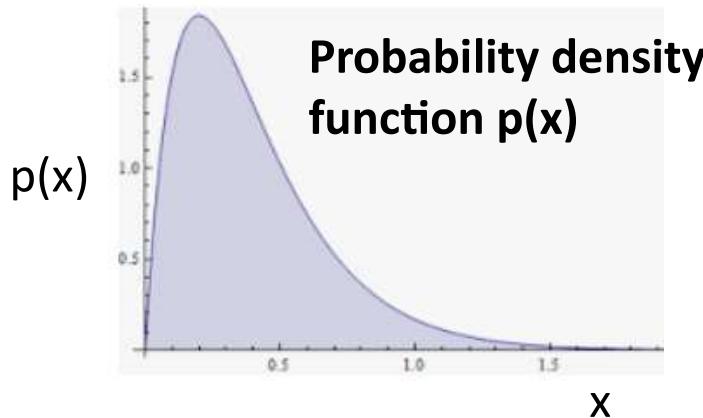
Expected return (long term average result) is following

$$ER = 0.4*12000 + 0.35*2000 + 0.25*(-50000) = -7000$$

Conclusion: Organizing the festival is not profitable.

2. Continuous probability distributions

A continuous probability distribution is one in which a continuous random variable X can take on any value within a given range of values (which can be infinite). Probabilities of values of x are given by a function $p(x)$, called **probability density function** or **probability mass function**.



Value $p(x)$ describes the probability of value of x

$p(x) \geq 0$ (function is always non-negative)

Area between $f(x)$ and x-axis equals 1

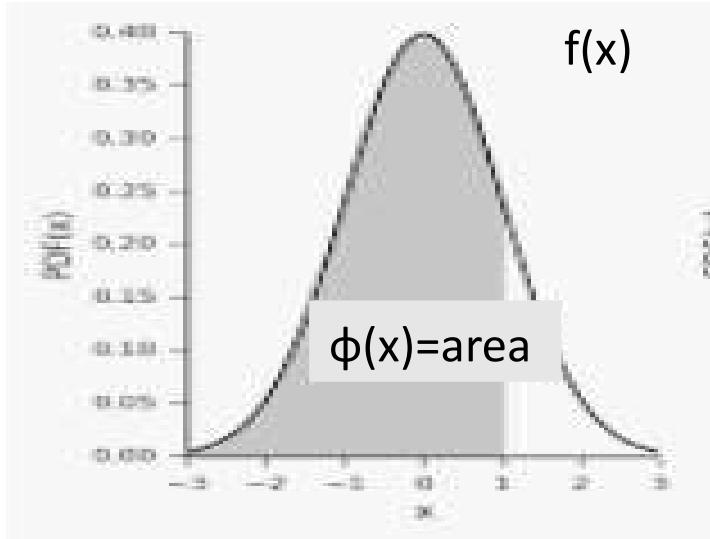
Interpretation: $\phi(x_0) = P(x \leq x_0)$

Value of cumulative density function at x_0 gives the probability that the variable value $x \leq x_0$

Normal distribution

The most common continuous distribution is "Gaussian" normal distribution.

It has two parameters: mean μ and standard deviation σ



$$p(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

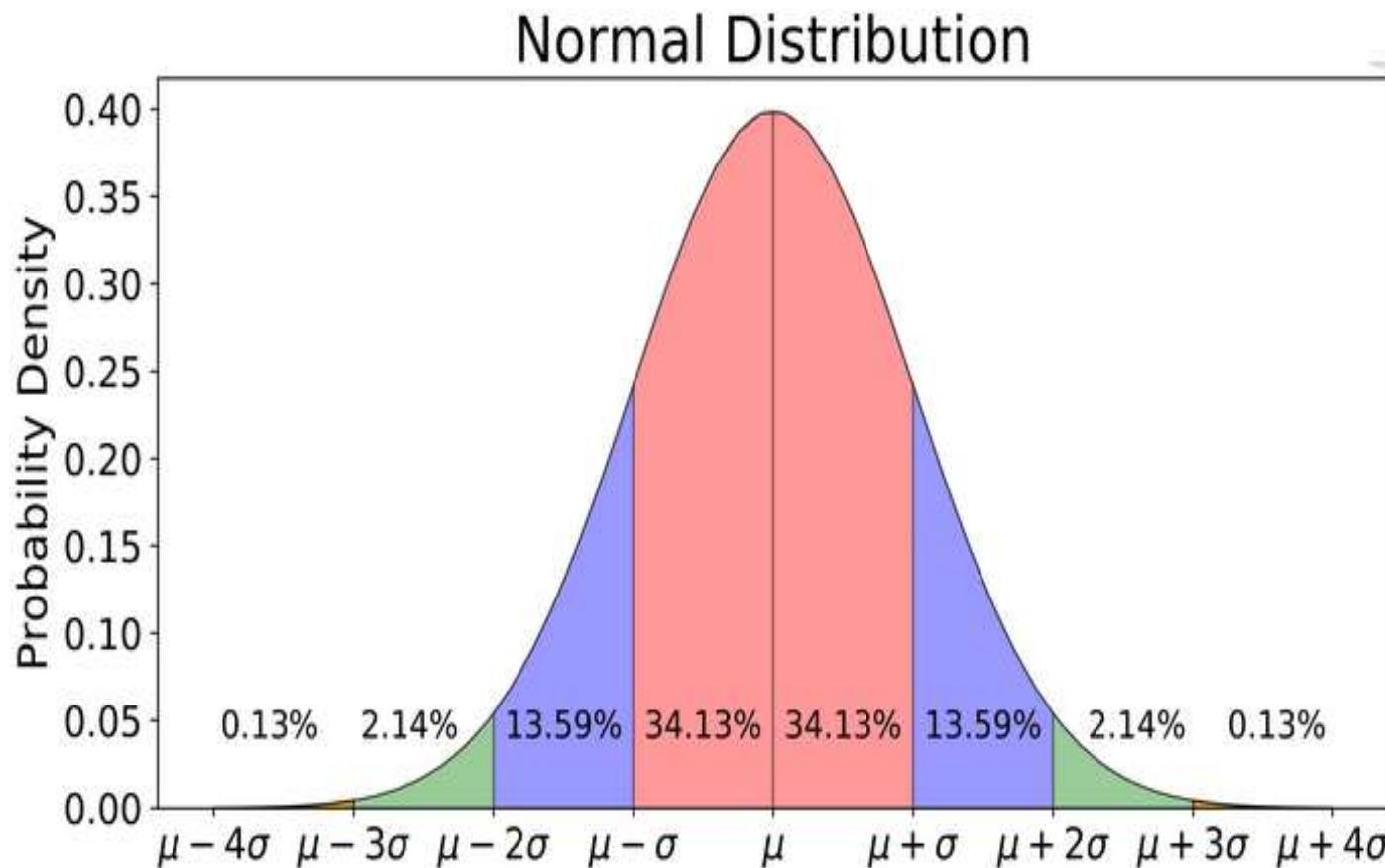
$$\Phi(x) = \frac{1}{2} \left[1 + \operatorname{erf}\left(\frac{x-\mu}{\sigma\sqrt{2}}\right) \right]$$

Both functions look complicated, but they are luckily found in calculators. In EXCEL function NORM.DIST can be used to calculate both $p(x)$ and $\Phi(x)$

Notation. $x \sim N(\mu, \sigma)$ means that variable x follows normal distribution with mean = μ and standard deviation σ .

For example if variable x is hemoglobin of male person, notation $x \sim N(153,9)$ would mean that x is normally distributed with mean of 154 and standard deviation of 9.

Picture of the probabilities of variable X in intervals limited by $\mu + k^*\sigma$, where $k=0,\pm 1,\dots$

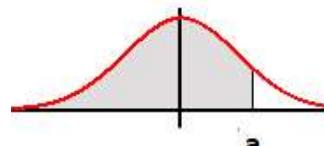


Around 68 % of values lie in range $[\mu-\sigma, \mu+\sigma]$

Around 95% of values lie in range $[\mu-2\sigma, \mu+2\sigma]$

Usage of cumulative density function $\Phi(x)$ and its inverse function $\Phi^{-1}(P)$

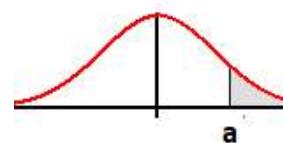
1. Calculation of probability that value of $x \leq a$



$$P(x \leq a) = \Phi(a)$$

Excel: =NORM.DIST (x;μ;σ;1)

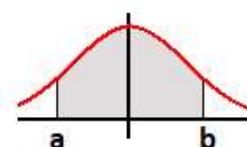
2. Calculation of probability that value of $x \geq a$



$$P(x \geq a) = 1 - \Phi(a)$$

Excel: = 1- NORM.DIST (x;μ;σ;1)

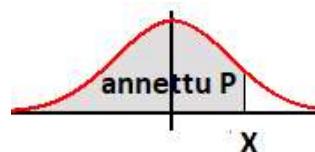
3. Calculation of probability that $a \leq x \leq b$



$$P(a \leq x \leq b) = \Phi(b) - \Phi(a)$$

Excel: = NORM.DIST (b,.) - NORM.DIST (a,.)

4. Calculation of value of x corresponding probability P



$$\Phi(x) = P \Rightarrow x = \Phi^{-1}(P)$$

Excel:

NORM.INV(P; μ; σ)

Ex3. Basket ball player. The mean height of Finnish adult men is 180.7 cm ja standard deviation is 7.4 cm. Assume that height is normally distributed.

- a) Calculate the probability that height \geq 213 cm (Lauri Markkanen's height is 213)
- b) Calculate the "P90 height". (a height that only 10% exceeds)
- c) Calculate the probability that height is between 190 cm – 200 cm?



a) $P(x \geq 213) = 1 - \phi(213) = 1 - 0.99999364 =$
0,00000636
6.4 miljoonasta

Excel $\phi(213)$:
=NORM.DIST(213;180,7;7,4;1)
which gives 0,99999517 =>
 $1 - \phi(213) = 0.00000636$

b) P90 height requires solving x from $\phi(x) = 0.90$
Using inverse function we get $x = \phi^{-1}(0.9) =$ 190.2 cm
=> Conclusion there are 90% with $h \leq 190.2$ cm and
10% with $h > 190.2$.

Excel's ϕ^{-1} is NORM.INV:
=NORM.INV(90%;180,7;7,4)
(arguments: percentage, mean,
st.dev). It gives 190,2 cm

c) $P(190 \leq x \leq 200) = \phi(200) - \phi(190) = 0.996 - 0.899 =$
0.097 = 9.7%

=NORM.DIST(200;180,7;7,4;1) -
NORM.DIST(190;180,7;7,4;1)

Ex4. A gravel is suitable for its purpose, if only 2.5% of its stones have diameter greater than 30 mm. In the table there are diameters of stones of a random sample of 80 stones. Assume that diameters are normally distributed. Does the gravel meet the requirements?



A	17,0	19,3	26,7	17,0	13,3	11,8	9,4	6,9	30,9	12,2
1	12,0	16,7	17,9	5,8	23,4	25,4	10,5	14,2	15,5	25,2
	13,0	11,0	19,9	21,1	25,6	25,5	25,4	31,1	19,2	21,8
	18,5	24,5	17,9	19,2	24,8	23,0	27,3	29,6	27,3	16,1
	29,3	24,2	16,2	19,9	10,4	16,8	17,9	19,8	22,1	20,8
	30,8	24,8	33,7	29,0	12,8	20,0	15,6	24,7	11,7	16,0
	15,6	16,7	20,0	28,0	20,4	11,8	19,1	13,7	17,4	25,6
	24,4	22,8	12,7	45,0	24,9	27,8	21,7	18,7	17,7	18,0

1. Calculate with Excel μ and σ

=average(A1:H10) gives $\mu = 20.1$

=stdev.s(A1:H10) gives $\sigma = 6.8$

2. Calculate using cumulative density function

$P(x < 30)$ (=probability that diameter < 30)

= NORM.DIST(30;20,1;6,8;1)

gives 0,927 = 92,7%

Conclusion: From gravel 92,7% has diameter < 30 mm

=> 7.3% has diameter > 30 mm, which is too much (maximum accepted share is 2.5%)

Another method would be to calculate the value of x corresponding probability $P = 97.5\%$

=NORM.INV(97,5%; 20,1 ; 6,8) gives 33,4 mm , which exceeds 30 mm limit.

3. Decision tree

- A decision tree is a decision support tree-like model of decisions and their possible consequences
- A decision tree consists of three types of nodes:
 - Decision nodes – typically represented by squares
 - Chance nodes – typically represented by circles
 - End nodes – typically represented by triangles

Decision trees are used in statistics, data mining and machine learning

"Newsboy problem" (classical decision tree example)

Ex5. How many papers the newsboy should buy from the publisher to maximize his profit? Probability distribution of number of sold papers are given.



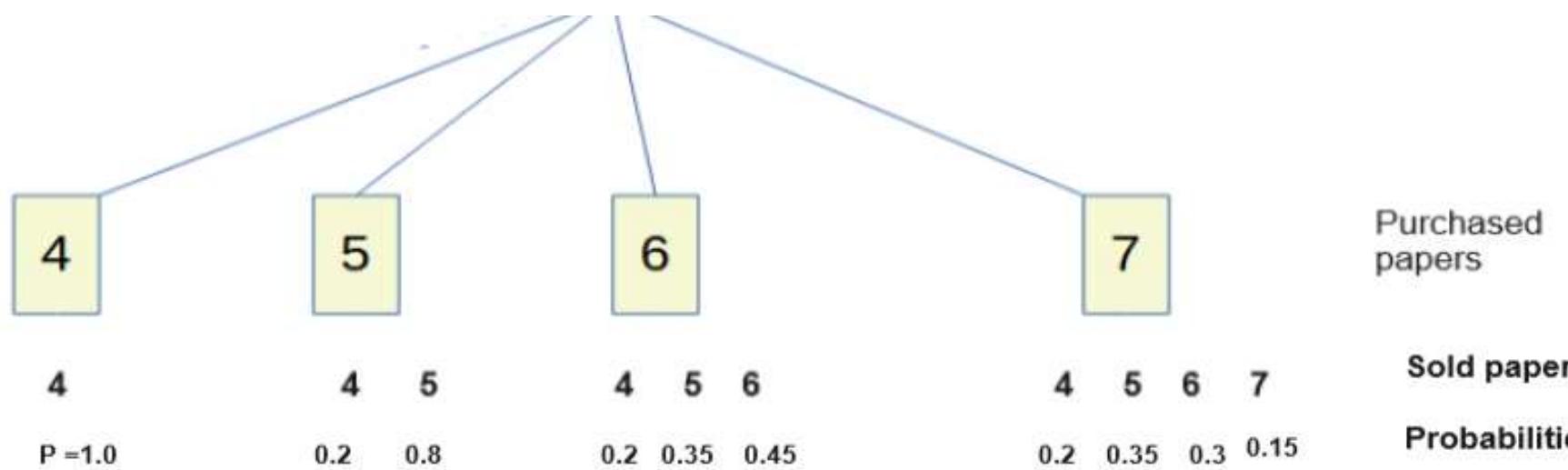
Demand of papers	Probability
4	0.2
5	0.35
6	0.3
7	0.15

Purchase price: 1 \$ per paper
Selling price: 2 \$ per paper

How many papers should the newsboy purchase daily to maximize his profit?

Newsboys were the main distributors of newspapers to the general public in USA 100-150 years ago. Youngest were 5 – 6 years old. They bought papers from the publisher at their own risk and sold them on the streets with a small profit. From unsold papers they got no compensation.

(Almost all shopkeepers have to solve problems of type: How many items he/she should order? Especially question is critical when we talk about products, which cannot be sold later (milk, fish, etc.)



Next we calculate expected returns (profit) in all four cases: Profit for sold paper = 1\$, loss for unsold is -1\$

Case1: Newsboy N buys 4 papers Everything is sold. Return (profit) = $4 \times 1 = 4\$$

Case2: N buys 5 papers. With $P = 0.2$ he sells 4 and with $P = 0.8$ he sells all 5:

$$ER = 0.2 \times (4-1) + 0.8 \times 5 = 4.6\$$$

Case3: N buys 6 papers: With $P = 0.2$ he sells 4. 2 remains unsold. Profit = $4-2 = 2\$$. With $P = 0.35$ he sells 5. 1 remains unsold. Profit = $5 - 1 = 4\$$. With $P = 0.45$ he sells all 6: Profit = $6\$$
 $ER = 0.2 \times 2 + 0.35 \times 4 + 0.45 \times 6 = 4.5\$$

Case4: N buys 7 papers: With $P = 0.2$ sells 4, unsold 3, profit=1. With $P=0.35$ sells 5 unsold 2, profit = 3. With $P=0.3$ sells 6 unsold 1, profit 5 and with $P = 0.15$ sells 7. profit = $7\$$. $ER = 0.2 \times 1 + 0.35 \times 3 + 0.3 \times 5 + 0.15 \times 7 = 3.8\$$

Best long term strategy is to buy 5 papers every day which gives 4.6\$ average income.

2. Lecture 20.3.24 topics

- Concepts and terminology
- Descriptive statistics
 - - Parameters
 - - Grouped data
 - - Graphical presentation

A. Basic concepts

1. Population

=the set of objects of interest, which we will to research

Example: Students of Lapland UAS

Example:

Customer feedback questionnaire

2. Sample

=proportion of population, which is researched

- usually randomly chosen subset of population

Example: Student, who were asked to fill a questionnaire

Population: All customers of the super market

Sample: Customers, who participated in the questionnaire

Variables:

- age
- gender
- household size
- total of purchases

3. Variables

The properties of objects of the population, which we collect information about.

Example: Degree programme, age, gender,...

4. Discrete and continuous variables

Variable is **discrete**, if the set of possible values is finite or enumerable
(example: number of children of a family)

Variable is **continuous**, if the set of possible values is continuous, infinite.
(Example: monthly salary)

5. Statistical parameters

= numerical measures which characterize the distribution of variable values
(Example: mean, variance)

Notation convention:

Latin letters are used for parameters of a sample, greek letters for parameters of the population.

(Example: population mean = μ

Sample mean = x

Discrete variables:

age in years
Gender
size of the household

satisfaction to service (scale 0 - 5)

Continuous variables:

total value of purchases

6. Levels of measurement

Statistical variables can be divided into four categories, which are called the levels of measurements

level of measurement	description	examples
nominal level variable or "class variable"	variable values are just names or labels, which define classes	-gender (male, female) -marital status (married, single, divorced, widow) -color (red, blue, ...)
ordinal level variable	values are labels, but an order can be added to the values	-quality class of potatoes -a players location on the ranking list
interval level variable	-a numerical variable, where it is meaningful to compare differences, but not ratios - absolute zero does not exist	- Celsius temperature
ratio level variable	a numerical variable where also ratios are comparable	-Kelvin temperature -monthly salary - age in years

Nominal level variables:

- gender

Ordinal level

- satisfaction to service (scale 0 - 5)

Ratio level:

- age
- Household size
- total of purchases

B. Descriptive statistics

- Statistical parameters:
 - Measures of central tendency
 - Measures of variation
- Grouping data
 - Calculation of parameters from grouped data
- Graphical presentations

1. Measures of central tendency

A

Mean value (or average)

Population Mean

$$\mu = \frac{\sum x}{N}$$

Sample Mean

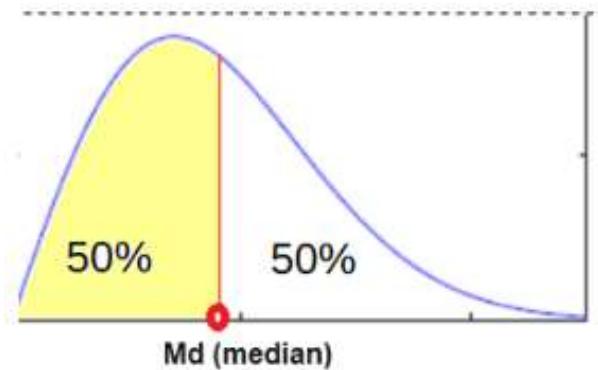
$$\bar{X} = \frac{\sum x}{n}$$

(Notice the convention of using greek letters for parameters of population. Formulas are the same:
mean = sum of values / count of values)

Excel- formulas

=average(A1:A28)

Median



Rule for manual calculation of median:

1. Sort data by magnitude
2. Median is the value in the middle of sorted data.
Half of the values are less than median, the other half is greater than median.

(If number of values is even, median is the average of the two values in the middle)

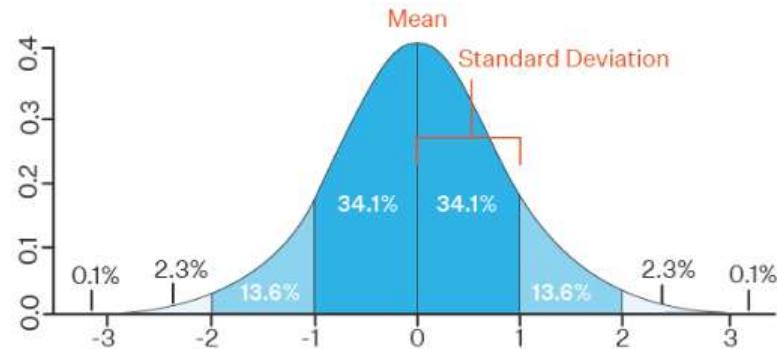
2. Measures of variation

Standard deviation

= the square root of the mean of squared distances from average: $(x_i - \mu)^2$.

Formulas are slightly different for whole population and sample.

Population	Sample
$\sigma = \sqrt{\frac{\sum(X - \mu)^2}{N}}$	$s = \sqrt{\frac{\sum(X - \bar{x})^2}{n - 1}}$
X - The Value in the data distribution μ - The population Mean N - Total Number of Observations	X - The Value in the data distribution \bar{x} - The Sample Mean n - Total Number of Observations



Example: Calculate the standard deviation of data in cells A1:A28 of the previous slide:

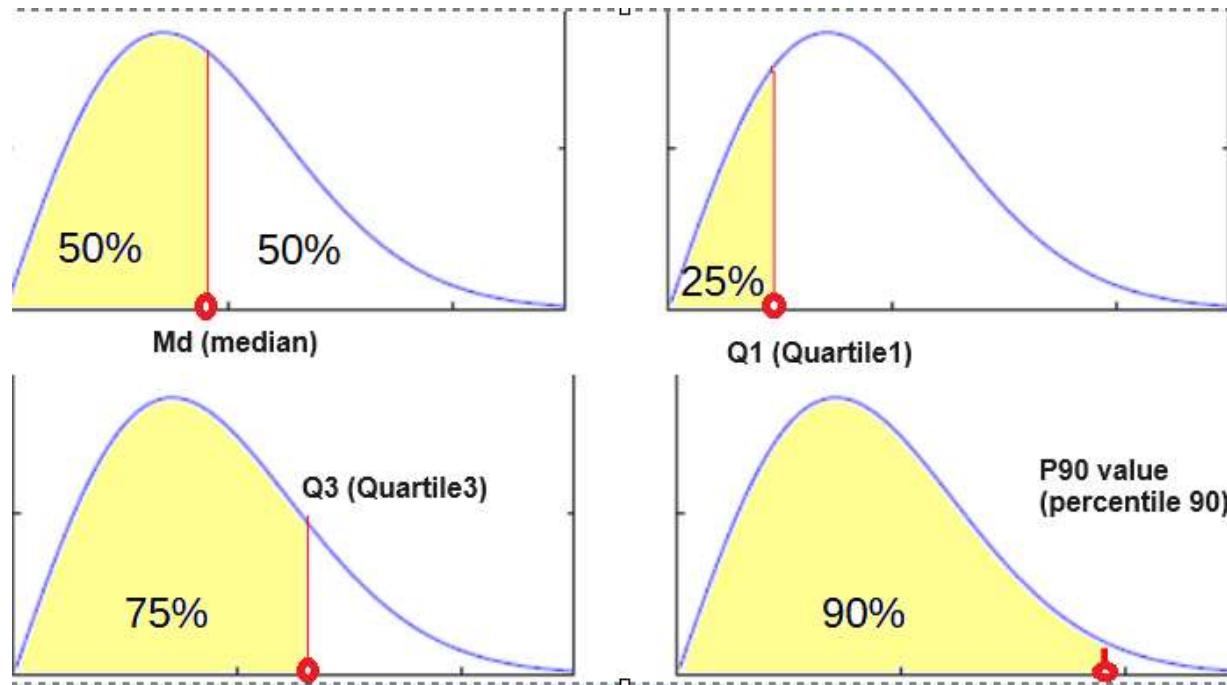
= STDEV.P(A1:A28)

Percentiles

= values of x which divide the variable data into two fractions according to the given percentages.

Most common percentiles are Quartiles 1 and 3 and P90 – value.

(P90- values are commonly used as reference values in medical measurements like blood pressure, cholesterol, etc.)



Usage of Excel functions

`Percentile(A1:A60; 90%)`
(or `Percentile(A1:A60; 0,9)`)

`Median(A1:A60)`
(or `Percentile(A1:A60; 50%)`)

`Quartile(A1:A60, 1)`
(or `Percentile(A1:A60; 25%)`)

`Quartile(A1:A60, 3)`
(or `Percentile(A1:A60; 75%)`)

Mean and standard deviation can be calculated also from a frequency table,
in which individual variable values are replaced by intervals.

Calculation of mean and standard deviation can be estimated using class centers as variable values.

Formulas: MEAN $\mu = \frac{\sum f_i x_i}{N}$

STDEV.P $\sigma = \sqrt{\frac{\sum f_i (x_i - \mu)^2}{N}}$ STDEV:S $s = \sqrt{\frac{\sum f_i (x_i - \bar{x})^2}{N - 1}}$

Example. In a psychological test the reaction times of 30 participants were measured. Results in form of a frequency table are on the right.
(unit is millisecond).

In our example the formulas give following results:

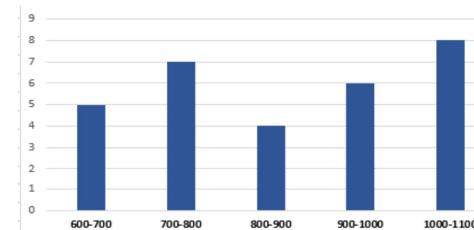
$$\mu = \frac{5 \cdot 650 + 4 \cdot 750 + 7 \cdot 850 + 6 \cdot 950 + 9 \cdot 1050}{30} = 867$$

$$\sigma = \sqrt{\frac{5 \cdot (650 - 867)^2 + 4 \cdot (750 - 867)^2 + 7 \cdot (850 - 867)^2 + 6 \cdot (950 - 867)^2 + 9 \cdot (1050 - 867)^2}{30}} = 146$$

Comment: The parameter values from grouped data are approximations. The exact mean and stdev calculated from original data were in this case 861 and 153.

reaction time	frequency
600 – 699	5
700 – 799	7
800 – 899	4
900 – 999	6
1000 – 1099	8

Frequency table



Bar char presentation

Common mathematical models:

Linear model

$$y = ax + b$$

Exponential model

$$y = ae^{bx}$$

Power model

$$y = ax^b$$

Polynomial model

$$y = ax^2 + bx + c$$

Excel:



Exponential



Linear



Polynomial



Power

Fitting a mathematical model
to observed data

Linear model

$$y = ax + b$$

Principle: Find parameters **slope a and constant b**, that the sum of squares of differences between observed y – values and those calculated from expression $a x + b$ is at minimum

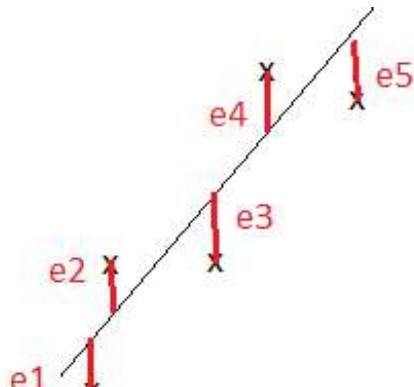
Example of observed data

Consists of x, y pairs

x	1.0	1.5	2.0	2.5	3.0	3.5
y	42.5	42.8	43.2	43.5	43.9	44.2

WolframAlpha function finds the minimum of the sum of squares $\sum (a x_i + b - y_i)^2$ with following command line.

minimize $(a*1.0+ b-42.5)^2 + (a*1.5+ b-42.8)^2 + (a*2+ b-43.2)^2 + (a*2.5+ b-43.8)^2 + (a*3+ b-43.9)^2 + (a*3.5+ b-44.3)^2$



(* "differences" are the vertical lines in the picture)

result: $(a, b) \approx (0.691429, 41.7943)$

Method is known as **linear regression analysis**
or the **method of least square sum**

Calculators have functions for regression analysis.

Excel graphics trendline

Easiest way to find parameters of
a mathematical model is TRENDLINE

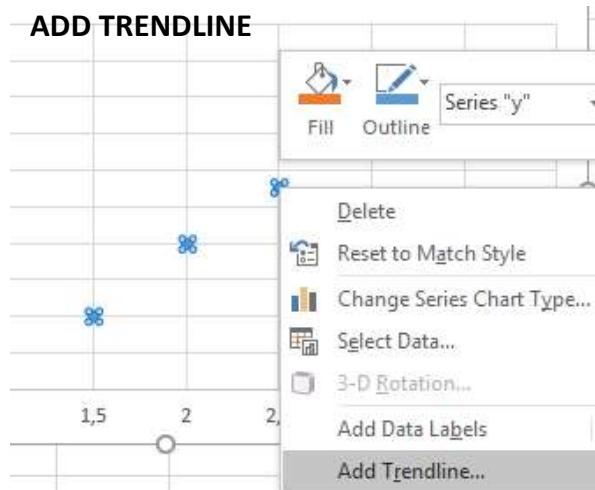
Function of Excel Graphics

Linear model using trendline property

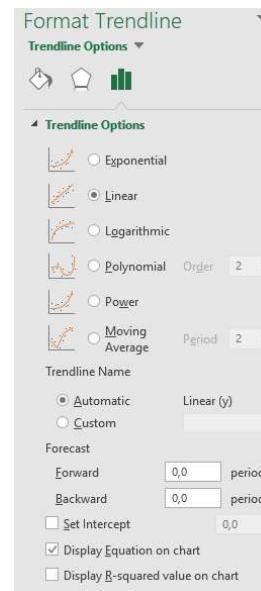
x	y
1	42,5
1,5	42,8
2	43,2
2,5	43,5
3	43,9
3,5	44,2

Data

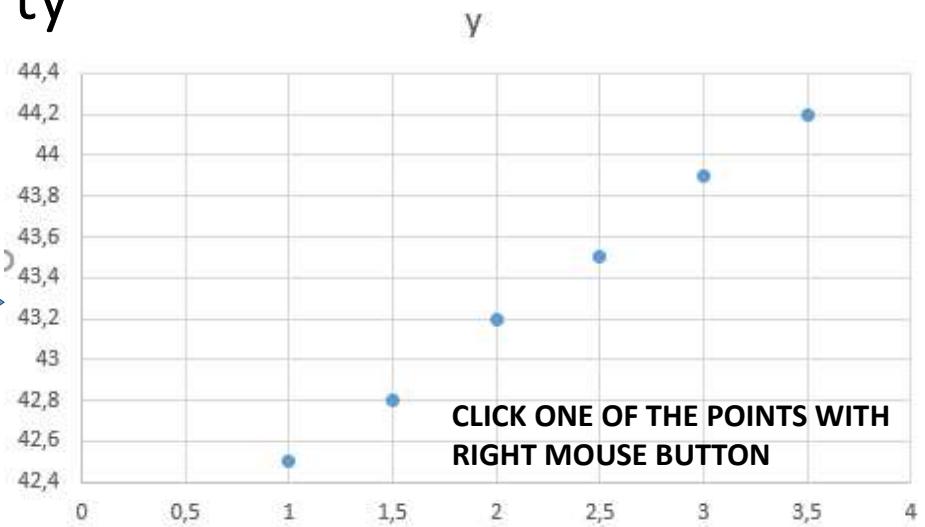
ADD TRENDLINE



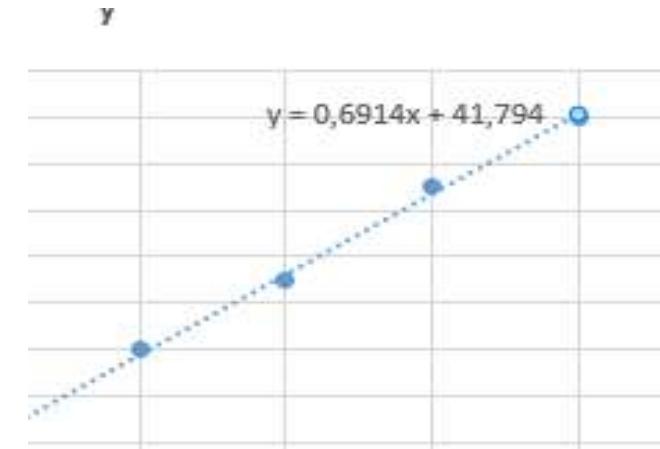
**TYPE =
LINEAR**



**CLICK ONE OF THE POINTS WITH
RIGHT MOUSE BUTTON**

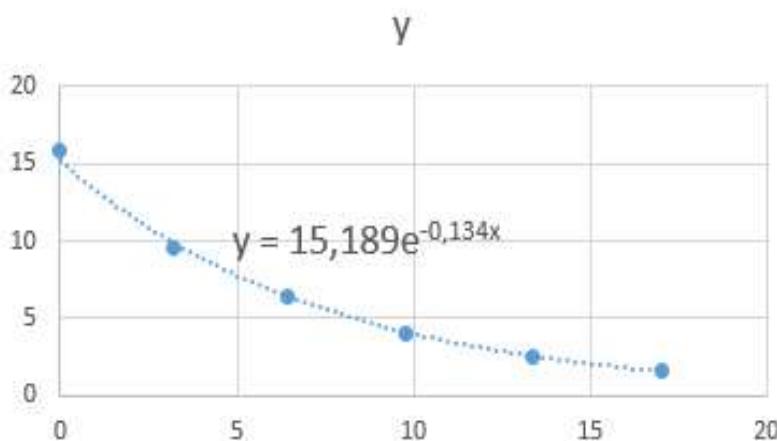


RESULT: EQUATION OF THE MODEL



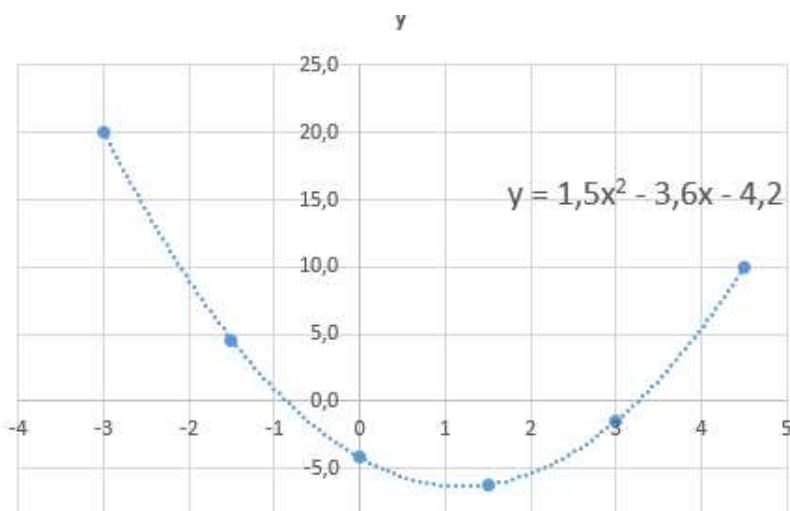
Other mathematical models (in Excel graphics trendline)

X	y
0	15,8
3,2	9,6
6,4	6,4
9,8	4
13,4	2,5
17	1,6



Exponent model
 $y = a e^{-bx}$

X	y
-3	20,1
-1,5	4,6
0	-4,2
1,5	-6,2
3	-1,5
4,5	10,0



Polynomial model
 $y = ax^2 + bx + c$

Topics of 28.3.24 lesson

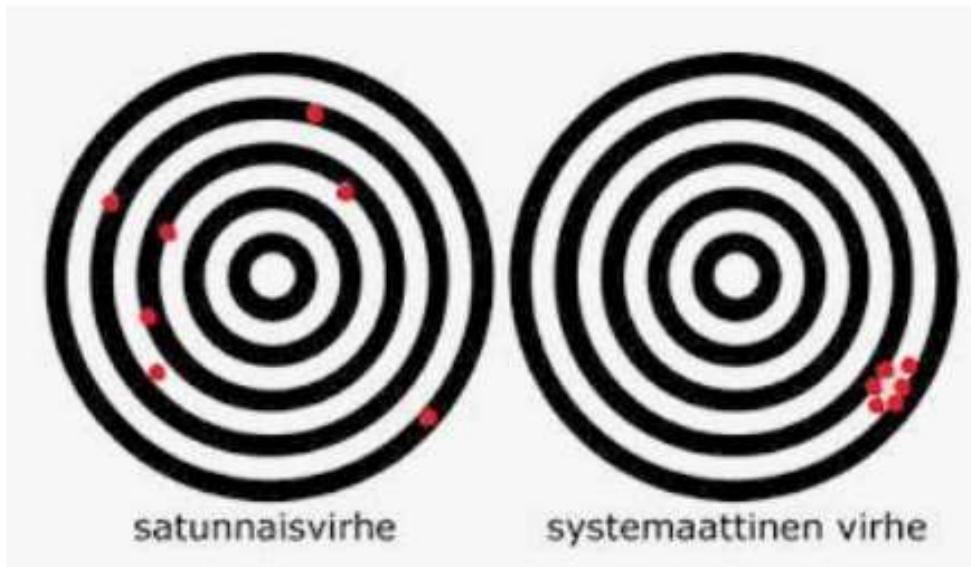
- 1. Estimation of population mean using sample**
 - error margin of population mean**
 - confidence levels and confidence intervals**
- 2. Error margins of regression line parameters**

Error types: Random error and systematic error

Type A: Random error

is an error caused by random factors in measurement. It appears in the way that measured values are randomly distributed around the mean.

By increasing the number of measurements, error of type A can be eliminated.



Type B: Systematic error

Errors appears always in the same direction.

Statistical methods cannot be used to eliminate systematic error.

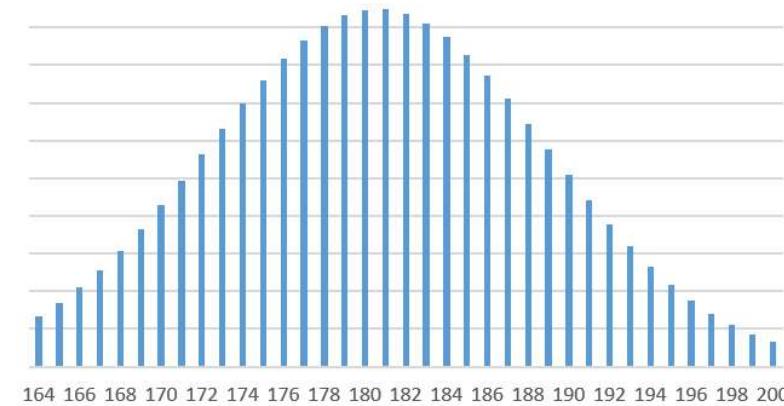
Error margin of
population mean

Sampling errors

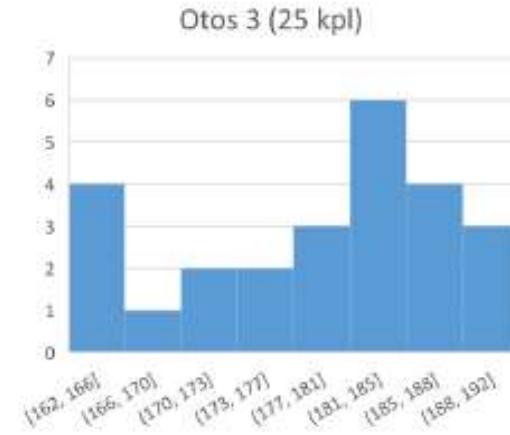
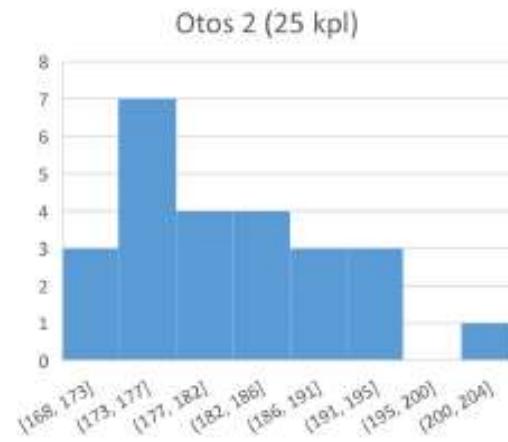
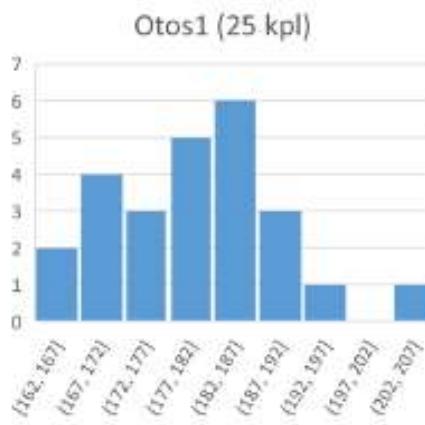
Confidence intervals

SAMPLING ERRORS

Height of 18 year old men follows normal distribution with mean = 180 cm and standard deviation = 8 cm.

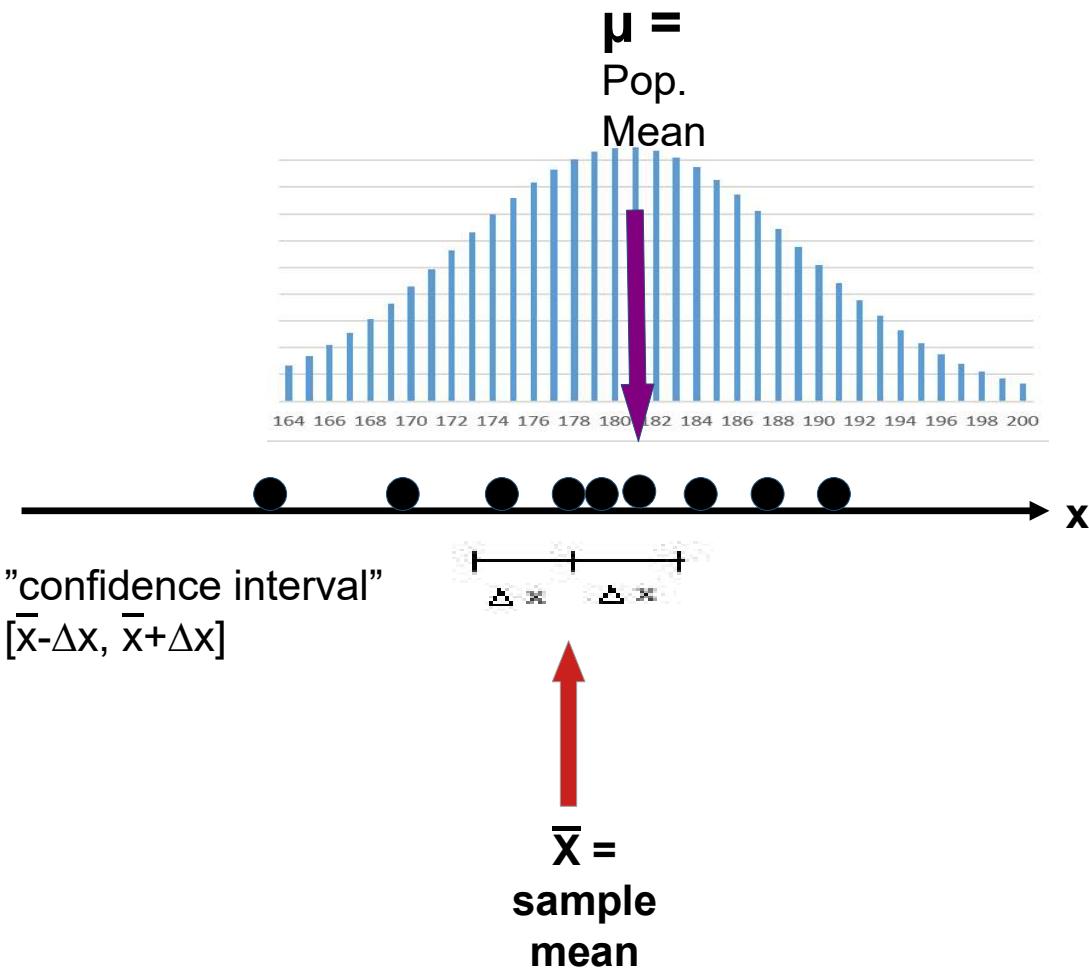


When we measure heights of men of samples of size 25, all samples look different. However sample means 180.3 cm, 180.7 cm and 181.3 cm are not very far from the population mean.



By increasing sample size sample distribution would get closer to the normal distribution.

Goal is to determine the population mean μ of variable x using sample.



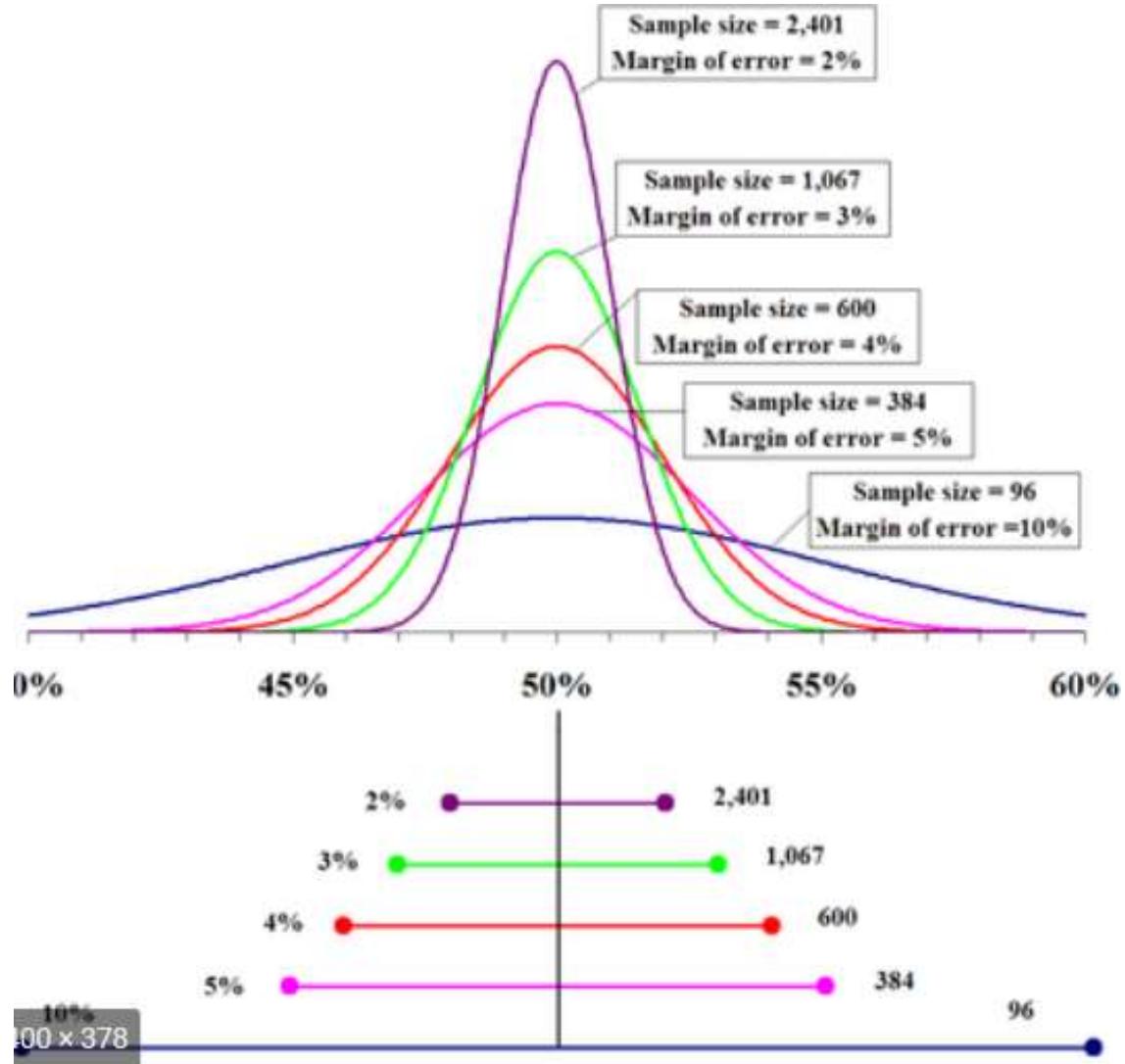
How far from the sample mean \bar{x} can be from the population mean μ ?

If x is normally distributed, sampling error could be in theory very large, but probability for large error is close to zero.

Next slides show formulas of estimation of sampling error. Formulas for error margin Δx are based on desired certainty levels, called confidence levels.

Standard confidence level is 95%
(Other, like 99%, are possible)

Confidence level means, that in 95% of random samples the population mean μ lies in interval $[\bar{x}-\Delta x, \bar{x}+\Delta x]$, which is called the confidence interval.



The best estimate for population mean μ is sample mean \bar{x} .

Picture shows how variation of sample means drops, when sample size n gets higher

=> The bigger is the number of measurement, the more accurate is the result.

Picture shows the error margins with different sample sizes in candidate support polls of presidential election.
To reach 2% error margin, sample size should be about 2500.

How far can the population mean μ be from the sample mean \bar{x} ?

1) Theoretical result: If the standard deviation of variable X of population = σ , then

the standard deviation of sample means = σ/\sqrt{n} .

(If σ is not known, we use sample st.deviation s , which gives s/\sqrt{n})

2) To calculate the error margin we need to multiply s/\sqrt{n} with a "coverage parameter" t , which depends on the required confidence level.

$$\Delta x = t \frac{s}{\sqrt{n}}$$

s = sample standard deviation

n = sample size

t = "coverage parameter" called **critical value**

3) If sample size ≥ 500 , we can use critical value t based on normal distribution:

- at confidence level of 95%, coefficient $t = 1.96$,
- at confidence level of 99%, coefficient $t = 2.57$

4) If sample size is smaller than 500, values of t are based on Student's distribution, in which t values are larger. (table on the next slide)

6) Interpretation: population mean μ is at 95% probability in interval

$$\bar{x} - t \frac{s}{\sqrt{n}} < \mu < \bar{x} + t \frac{s}{\sqrt{n}}$$

Student's distribution: Table of t – values

called "critical values" or "coverage factors"

"Degrees of freedom"
= n - 1

95 % confidence

n = sample size

6	2.447	3.707
7	2.365	3.499
8	2.306	3.355
9	2.262	3.250
10	2.228	3.169
11	2.201	3.106
12	2.179	3.055
13	2.160	3.012
14	2.145	2.977
15	2.131	2.947
16	2.120	2.921
17	2.110	2.898
18	2.101	2.878
19	2.093	2.861
20	2.086	2.845
21	2.080	2.831
30	2.042	2.750
40	2.021	2.704
50	2.010	2.679
60	2.000	2.660
∞	1.960	2.576

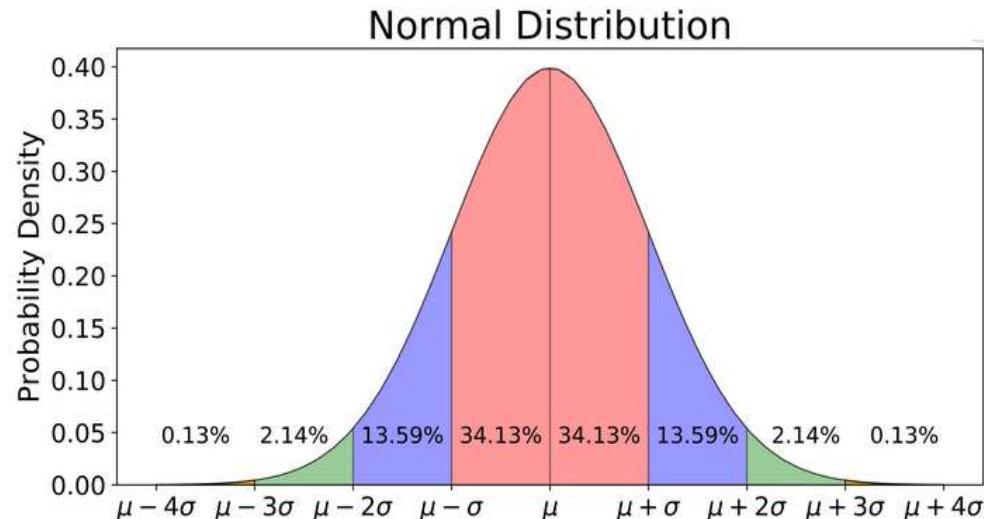
Usage of the table:

***If sample size is large (> 500), use
t = 1.96 at 95% confidence level***

***If sample size is smaller, look the t- value from
row, closest to the sample size n.***

***For example: If sample size is about 30,
use t = 2.042***

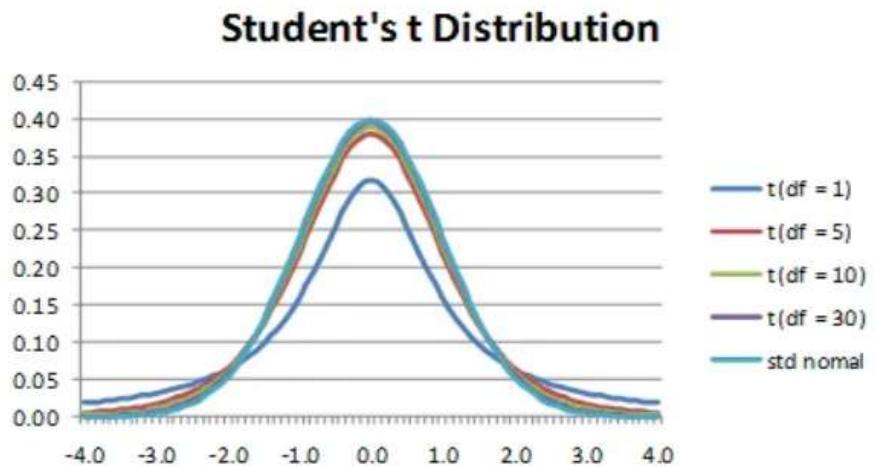
Excel functions confidence.norm and confidence.t



For large sample size ($n > 500$), sample means follow normal distribution

If n is large, we could use Excel function **confidence.norm** to evaluate Δx using fixed $t = 1.96$ in formula

$$\Delta x = t \frac{s}{\sqrt{n}}$$



For ($n < 500$), sample means follow t – distribution, which depends on n (see picture)
For $n > 500$, Student's distribution and normal distributions are nearly equal.

If $n < 500$, use Excel function **confidence.t**
Notice: **confidence.t** can be used in all cases. Its result are more accurate and better.

Example. Health institute wanted to determine the average blood sugar level of 70 year old citizens. In their research sample size was 200.

6,61	6,26	7,13	3,31	5,91	4,86	5,15	6,37	6,45	6,14	4,95	7,30	5,92	5,64	4,55	6,30	6,12	7,75	5,91	5,69
5,64	7,04	6,35	7,06	3,31	6,26	7,13	6,10	7,97	5,33	6,03	4,95	6,88	7,67	6,49	6,82	6,55	5,09	4,56	6,01
5,58	8,36	7,83	6,41	7,59	4,41	7,23	5,92	6,98	4,73	6,07	5,47	7,02	6,27	6,04	6,95	6,25	4,38	5,61	6,36
4,61	5,45	6,49	5,82	5,02	7,58	6,45	7,42	5,37	6,51	6,92	6,35	5,65	5,78	5,17	5,11	6,10	5,66	6,52	5,34
5,33	5,85	6,15	5,28	5,75	5,68	5,90	5,77	7,44	6,36	6,05	4,88	7,46	7,40	5,64	5,36	7,03	5,80	6,79	4,55
4,98	6,69	6,14	5,32	7,21	5,16	7,26	5,94	4,83	6,73	5,68	5,79	6,46	5,92	6,41	4,50	5,98	7,63	8,09	6,10
4,64	5,91	5,99	5,32	5,18	6,40	5,92	5,35	5,93	5,90	5,69	5,12	6,35	6,59	6,00	7,31	6,37	4,95	5,74	6,82
5,68	6,93	6,54	6,50	6,10	6,51	4,76	5,66	7,33	6,04	4,67	6,70	4,69	5,73	4,56	7,37	5,47	6,64	6,10	4,84
6,06	5,68	3,76	7,42	6,13	4,98	7,00	5,59	5,47	6,51	5,01	5,88	5,80	5,29	7,11	6,50	6,57	5,02	7,44	5,20
5,04	6,82	6,60	6,89	6,42	5,16	8,09	5,50	3,82	4,58	3,91	6,13	6,65	5,32	5,49	6,34	7,27	6,39	4,11	6,18
average																			6,01
standard deviation																			0,94

What is a) the average sugar level of population and b) its error margin at 95% confidence level?

a) Average blood sugar = 6.01

b) error margin = $t^*s/\sqrt{n} = 1.96*0.94/\sqrt{200} = 0.13$

=> average blood sugar level = 6.01 ± 0.13

Using confidence interval, this can be written

$$5.88 \leq \mu \leq 6.14$$

Error margin using Excel function.

=confidence.t(5%;0,94;200) gives 0.13

Arguments:

"Significance level" = "risk of being wrong" =

100%- confidence level = 100%-95% = 5%

Sample mean = 0.94

Sample size = 200

Confidence margin of proportions

In opinion surveys the error margin depends on the sample size according to the following formula.

$$\Delta p = t \sqrt{\frac{p(1-p)}{n}}$$

p = measured proportion in decimal form

n = sample size

t = coverage factor = 1.96 at confidence level of 95% (most usual)

A newspaper The Times made a research of support of Brexit in 2016. Brexit got 58.0% support among 1000 participants of the survey. Calculate the error margin of the measured value.
(Brexit = leaving EU)

$$\Delta p = t \sqrt{\frac{p(1-p)}{n}} = 1.96 \sqrt{\frac{0.58(1-0.58)}{1000}} = 0.031 = 3.1\%$$

Answer: Error margin was 3.1%

Interpretation: "Support percent of Brexit lies in interval [58.0 – 3.1, 58.0 + 3.1] at 95% probability"

Formula for calculation of the sample size for a given error margin

Solving n from $\Delta p = t \sqrt{\frac{p(1-p)}{n}}$ gives

Sample size formula:

$$n = \frac{t^2 p(1-p)}{\Delta p^2}$$

How big sample size would give 2.0 % error margin in the previous example, where p = 58% ?

$$n = \frac{t^2 p(1-p)}{\Delta p^2} = \frac{1.96^2 \cdot 0.58(1-0.58)}{0.02^2} = 2340$$

Sample size should be close to 2500.

Solved examples

200 randomly chosen customers of CityMarket were asked about their age.

Sample mean was 41.3 years and sample standard deviation was 23.5 years.

Estimate the mean age of all customers. Give also the error margin

Excel- function = **CONFIDENCE.T(5% ; 23.5 ; 200)** gives 3.3

Result: Mean age = 41.3 ± 3.3 years

500 randomly chosen individuals were asked, whether they use the bank card's contactless payment feature at the store's checkout. 47.5 % answered YES. **Estimate how many percents of all customers use that feature. Give also the error margin**

$$\Delta p = t \sqrt{\frac{p(1-p)}{n}} = 1.96 \sqrt{\frac{0.475(1-0.475)}{500}} = 0.044 = 4.4\%$$

Answer: 47.5% \pm 4.4 %

Error margins of linear
models $y = a x + b$
parameters a and b

Find a linear model $y = a x + b$ based on the table below. Variables: x = temperature, y = heating costs. Determine the confidence margins of parameters a and b . Calculate the coefficient of determination R^2 , which is a statistical measure of how well regression line approximates the observed data.

Mean	
temperature in Rovaniemi	Households heating costs (€)
-17,3	338
-12,2	294
-5,2	260
-1,2	203
9,7	128
13,4	99
17,4	58
18,2	65
9,3	134
1,7	180
-3,5	234
-9,2	256

1. Paint 3 x 3 cell area for results

-7,73	201,00
0,24	2,77
99,0 %	9,49

2. Insert function LINEST

3. Fill in the argument list

Known y's	<input type="button" value="fx"/>	D\$4:D\$15
Known x's	<input type="button" value="fx"/>	C\$4:C\$15
Type	<input type="button" value="fx"/>	1
Stats	<input type="button" value="fx"/>	1

paint y values

paint x values

4. Finally PRESS SHIFT-CTRL-ENTER combination to get all the answers:

1st row: parameters a and b

2nd row: standard errors of a and b

3rd row: R -squared value and standard error of predicted y

Summary of formulas of estimation the error of mean

$$\Delta x = t \frac{s}{\sqrt{n}}$$

Excel:
confidence.t

Error margin of mean of numeric variable

n = sample size, s = sample standard deviation
t depends on sample size and significance level

$$\Delta p = t \sqrt{\frac{p(1 - p)}{n}}$$

Error margin of proportion

n = sample size, p = proportion (measured of sample)
t depends on significance level, for 95% level t = 1.96

$$n = \frac{t^2 p(1 - p)}{\Delta p^2}$$

Formula for calculation of adequate sample size

p = proportion (measured), Δp = error of proportion
t depends on significance level, for 95% level t = 1.96

Estimation of error margins of parameters **a** and **b** of linear model $y = a x + b$
is performed using Excel function **LINEST**

Explanation of the results given by LINEST function

Linear model: $y = -7.73*x + 201.0$

-7,73	201,00
0,24	2,77
99,0 %	9,49

(If mean temperature of some month = 10°C, then heating costs are $-7.73*10+201 = 193.27$)

Standard errors of a and b are: $s_a = 0.24$ and $s_b = 2.77$.

The sample size = 12 => we need to use coverage factor value $t = 2.2$ to get the confidence margins for a and b:

See slide 7 table of t - values.
N - 1 t

10	2.228
11	2.201
12	2.179

Confidence margins of a and b :

$$\Delta a = t * s_a = 2.2 * 0.24 = 0.52 \quad \text{and} \quad \Delta b = t * s_b = 2.2 * 2.77 = 6.1$$

Coefficient $R^2 = 99\% \Rightarrow$ Model explains 99% of the variation of y values. (The observed points lie almost perfectly on the regression line. If all points would lie exactly on the line, R^2 would be 100%). \Rightarrow Model is excellent.

Standard error for the y -estimate = 9.5

The standard deviation of differences between observed y -values and y – values calculated from the model $y = a x + b$

Optimization / lec1

1. Limit values of a function $f(x)$
2. Derivative at a given point $f'(x_0)$
3. Derivative function $f'(x)$
4. Derivation rules of simple functions
5. Derivative in calculators

1. Limit values of a function

Consider a rational function $f(x) = \frac{2x^2 + x}{3x}$

1. Function $f(x)$ is not defined at point $x = 0$

2. If $x \neq 0$, $f(x)$ can be simplified as $\frac{2x^2 + x}{3x} = \frac{x(2x + 1)}{3x} = \frac{(2x + 1)}{3}$

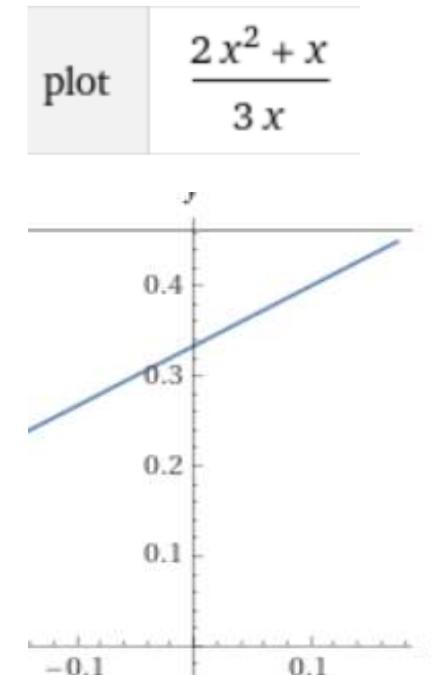
**3. Graph of $f(x)$ is the same as graph of $(2x+1)/3$,
with the exception that at one point at $x = 0$ is missing**

**4. The graph shows, that the value of $f(x)$ approaches
a specific value, as x approaches 0.**

We call this value the limit value of $f(x)$ as x tends to 0.

Notation for limit value is

$$\lim_{x \rightarrow 0} \frac{2x^2 + x}{3x}$$



Calculation of the limit value of a rational function at a zero of its denominator.

”Rational function” means a function $f(x) = \frac{P(x)}{Q(x)}$, where $P(x)$ and $Q(x)$ are polynomials

At a point $x = x_0$, where both $P(x_0)$ and $Q(x_0)$ are zero, $f(x)$ can be simplified by reducing by common factor $x - x_0$. After reduction limit value at $x = x_0$ can be calculated by substitution.

Ex. Calculate:

$$\lim_{x \rightarrow 0} \frac{2x^2 + x}{3x}$$

Simplification:

$$\frac{2x^2 + x}{3x} = \frac{x(2x + 1)}{3x} = \frac{(2x + 1)}{3}$$

Substitution $x = 0$
gives:

$$\frac{2*0+1}{3} = \frac{1}{3}$$

Ex2. Calculate limit value of $\frac{x^2 - 4}{x - 2}$, as $x \rightarrow 2$

Direct substitution of $x = 2$ gives 0/0 .

Simplification :

$$\frac{x^2 - 4}{x - 2} = \frac{(x - 2)(x + 2)}{x - 2} = x + 2$$

Formula

$$(a^2 - b^2) = (a - b)(a + b)$$

Substitution $x = 2$ gives $2 + 2 = 4$

Limits of other functions can be calculated with calculator

Ex3. Calculate limit value of $\frac{e^x - 1}{x}$ as $x \rightarrow 0$

Direct substitution of $x = 0$ gives $(e^0 - 1)/0 = (1-1)/0 = 0/0$

Simplification by simple methods is not possible =>

The limit value is calculated using WolframAlpha calculator

$$\text{limit } \frac{e^x - 1}{x} \text{ as } x \rightarrow 0$$

returns $\lim_{x \rightarrow 0} \frac{e^x - 1}{x} = 1$

Notation convention

$$\lim_{x \rightarrow 0} \frac{3x^2 + 2x}{2x} = \lim_{x \rightarrow 0} \frac{x(3x + 2)}{2x} = \lim_{x \rightarrow 0} \frac{(3x + 2)}{2} = \frac{0 + 2}{2} = 1$$

Read: " $(3x^2+2x)/(2x)$ tends to 1 as x tends to 0 "

$$\lim_{x \rightarrow 2} \frac{x^2 - 4}{x - 2} = \lim_{x \rightarrow 2} \frac{(x - 2)(x + 2)}{x - 2} = \lim_{x \rightarrow 2} (x + 2) = 2 + 2 = 4$$

Read: " $(x^2-4)/(x-2)$ tends to 4 as x tends to 2 "

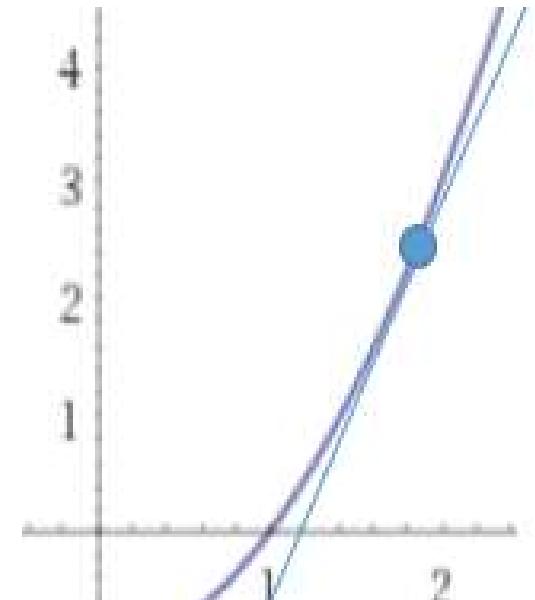
Definition of the derivative

The value of the derivative of $f(x)$ at $x = x_0$ is a measure of rate of change of $f(x)$ at that point

Graphical definition of derivative

The derivative of function $f(x)$ at $x = x_0$ is the slope of the tangent line at $x = x_0$.

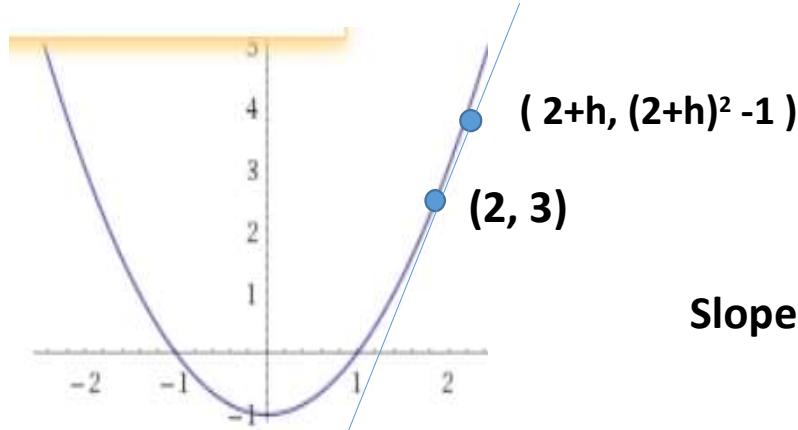
The notation of the derivative is $f'(x_0)$



Calculation of derivative at a given point

Example. Determine the value of derivative of function $y = x^2 - 1$ at $x = 2$

Consider the "secant line" between points of the curve



x	$y = x^2 - 1$
2	3
$2 + h$	$(2+h)^2 - 1$

$$\text{Slope } k = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(2+h)^2 - 1 - 3}{(2+h) - 2} = \frac{(2+h)^2 - 4}{h}$$

The limit value of k as $h \rightarrow 0$ is the derivative (=the slope of the tangent)

$$\lim_{h \rightarrow 0} \frac{(2+h)^2 - 4}{h} = 4$$

Answer: $f'(2) = 4$

W.A limit ((2+h)^2-1-3)/h as h>0

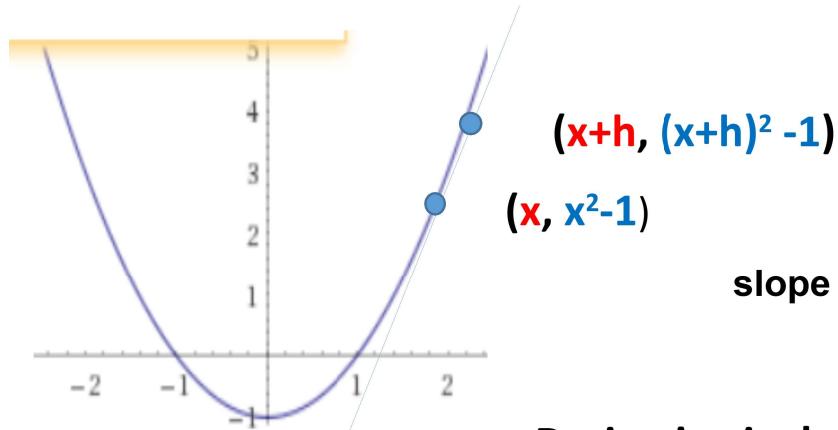
manually

$$\frac{(2+h)^2 - 4}{h} = \frac{4+4h+h^2 - 4}{h} = \frac{4h+h^2}{h} = \frac{h(4+h)}{h} = 4+h \rightarrow 4, \text{ as } h \rightarrow 0$$

Derivative function

Example: Determine the derivative function of $y = x^2 - 1$ = general expression for calculation of the derivative at any point x .

Consider points at x and $x + h$



x	$y = x^2 - 1$
x	$x^2 - 1$
$x + h$	$(x+h)^2 - 1$

slope $k = \frac{y_2 - y_1}{x_2 - x_1} = \frac{(x+h)^2 - 1 - (x^2 - 1)}{h} = \frac{(x+h)^2 - x^2}{h}$

Derivative is the limit value of slope:

$$\lim_{h \rightarrow 0} \frac{(x+h)^2 - 1 - (x^2 - 1)}{h} = 2x$$

Answer: $f'(x) = 2x$

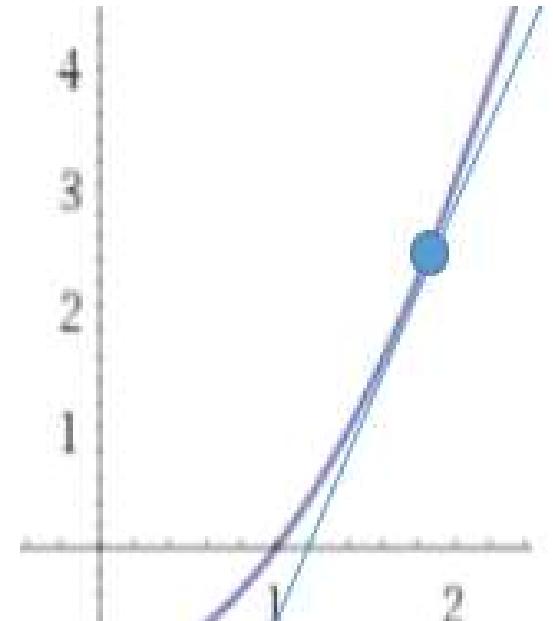
manually $\frac{(x+h)^2 - x^2}{h} = \frac{x^2 + 2xh + h^2 - x^2}{h} = \frac{2xh + h^2}{h} = \frac{h(2x+h)}{h} = 2x + h \rightarrow 2x, \text{ as } h \rightarrow 0$

Algebraic definition of derivative function

$$f'(x) = \lim_{h \rightarrow 0} \frac{f(x+h) - f(x)}{h}$$

All following notations mean the derivative of function $y = f(x)$

$$f'(x) \quad y' \quad \frac{df(x)}{dx} \quad \frac{dy}{dx} \quad Df(x)$$



From now on we use mostly notation $Df(x)$

$$D(x^2 - 1) = 2x$$

Derivatives are mostly calculated either using formulas of derivation or calculators

Derivation formulas for some basic functions:

1. Derivative of a constant function $y = c$:

$$Dc = 0$$

2. Derivative of power function

$$Dx = 1$$

$$Dx^n = n \cdot x^{n-1}$$

3. Power function with negative exponent.

$$D \frac{1}{x^n} = -\frac{n}{x^{n+1}}$$

Solved examples

- 1 a) Determine the derivate function of $f(x) = x^5$ a) $Dx^5 = 5 x^4$
a) b) Calculate the slope of the tangent of $f(x)$ at $x = 2$ b) $f'(2) = 5 \cdot 2^4 = 80$
b) c) Calculate the slope of the tangent of $f(x)$ at $x = -1$ c) $f'(-1) = 5 \cdot (-1)^4 = 5$

2. Calculate following derivates:

a) $D x^{2015}$

a) $2015 x^{2014}$

b) $D \frac{1}{x}$

b) $-\frac{1}{x^2}$

c) $D \frac{1}{x^5}$

c) $-\frac{5}{x^6}$

Derivative of a polynomial function

Laws for derivation of linear combinations of basic functions:

$$1) D a f(x) = a Df(x)$$

$$2) D (f(x) + g(x)) = D f(x) + D g(x)$$

Example calculate the derivative

$$D (- 3 x^3 + 5 x^2 - 4 x + 7)$$

Check result with W.A

Applying above rules we get

$$= -3*Dx^3 + 5*Dx^2 - 4*Dx$$

$$= -3*3x^2 + 5*2x - 4*1 + 0$$

$$= -9x^2 + 10x - 4$$

$$D (- 3 x^3 + 5 x^2 - 4 x + 7)$$

$$= \underline{-9x^2 + 10x - 4}$$

Derivation rules of special functions

$$D e^x = e^x$$

$$D \ln(x) = 1/x$$

$$D \sin(x) = \cos(x)$$

$$D \cos(x) = -\sin(x)$$

Derivation rules of combined functions

$$D e^{ax} = a e^{ax}$$

$$D \sin(ax) = a \cos(ax)$$

$$D \cos(ax) = -a \sin(ax)$$

Solved problems

$$D(3x^4 - 2x^3 + 5x^2 - x - 3) =$$

$$3 \cdot 4x^3 - 2 \cdot 3x^2 + 5 \cdot 2x - 1 = 12x^3 - 6x^2 + 10x - 1$$

$$D\left(\frac{2}{3}x^3 - 5/x^2\right) = \frac{2}{3} \cdot 3x^2 - 5 \cdot (-2)/x^3 = 2x^2 + 10/x^3$$

$$D(5 - x + 2e^x) = -1 + 2e^x$$

Derivation rules:

1. $Dc = 0$
2. $Dx = 1$
3. $Dx^n = nx^{n-1}$
4. $D(1/x^n) = -n/x^{n+1}$
5. $De^x = e^x$
6. $Da^*f(x) = a^*Df(x)$
7. $D(f(x) + g(x)) = Df(x) + Dg(x)$

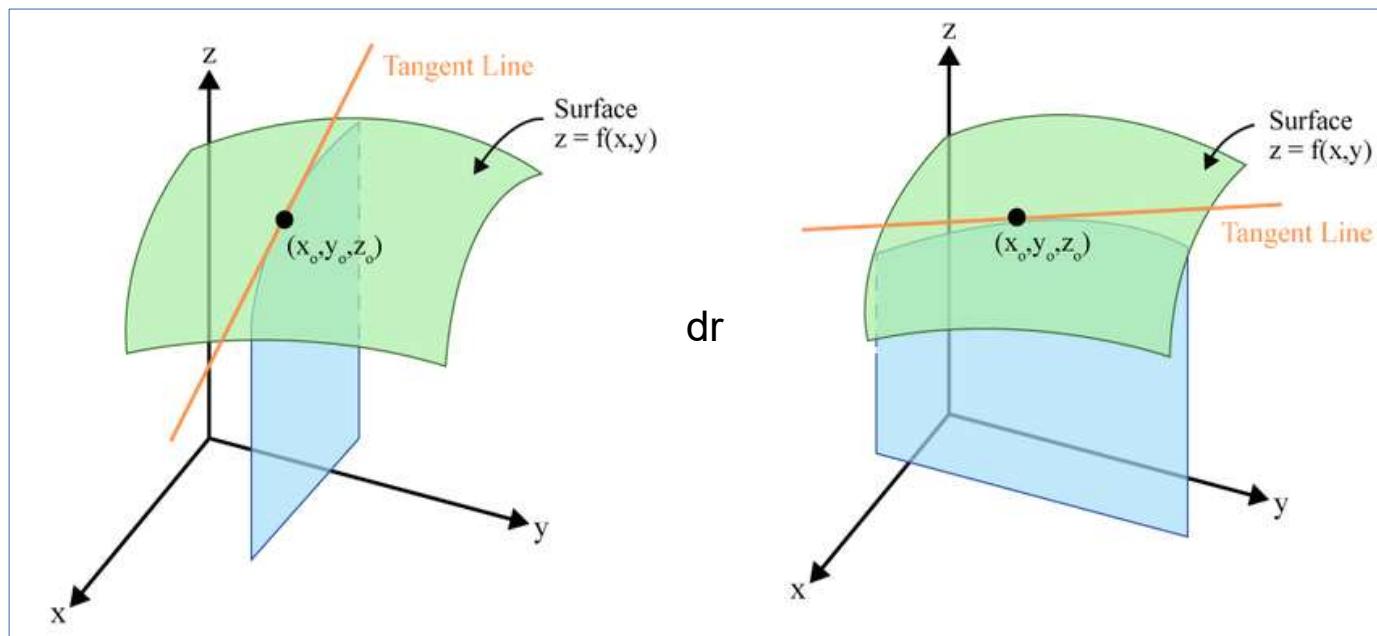
Functions of several variables

Partial derivatives

Graphical interpretation

Gradient vector

A function of two variables $z = f(x,y)$ presents a surface in 3D space.



At any point of the surface we can draw tangent lines to directions of both coordinate axes.

The slopes of these tangent lines are called **partial derivatives** of $f(x,y)$.

Calculation of partial derivatives

Derivation is applied to one variable at the time, while the other variables are considered as constants

A function of many variables has derivatives with respect to all variables

$$D(y^*x^2 + 4x + 5y, x) = y^*2x + 4*1 + 0 = 2xy + 4$$

$$D(y^*x^2 + 4x + 5y, y) = 1*x^2 + 0 + 5*1 = x^2 + 5$$

Another notation of
partial derivatives: $\frac{\partial f}{\partial x}$ $\frac{\partial f}{\partial y}$ or simply f_x, f_y

In these slides we use mostly notation $D(f,x), D(f,y)$

Also on WolframAlpha you can write [D\(y^*x^2 + 4x + 5y, x\)](#), which gives [2xy + 4](#)

Example. Linear function $f(x,y) = 2x + 3y - 1$ represents a plane

Calculate the partial derivatives of $f(x,y)$

$$f_x = D(2x + 3y - 1, x) = 2$$

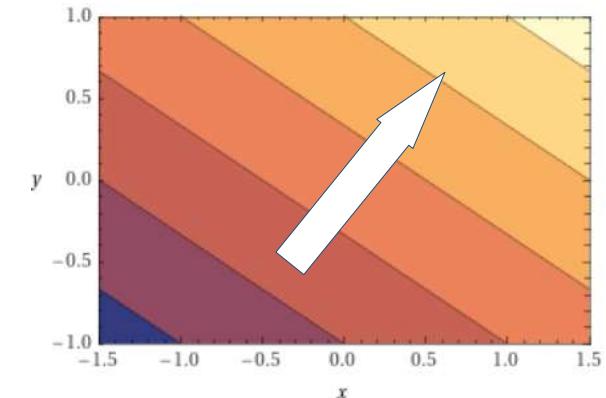
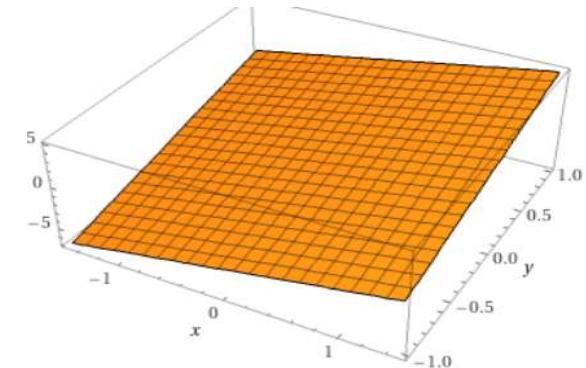
$$f_y = D(2x + 3y - 1, y) = 3$$

The vector of partial derivatives (f_x, f_y) is called the **gradient of function $f(x,y)$**

Gradient $(f_x, f_y) = (2, 3)$ gives the direction in which the function grows fastest

Function decreases the fastest in the opposite direction $(-2, -3)$

plot $2x + 3y - 1$



Example. Function $f(x,y) = x^2 + 2x + y^2 - y + 3$ represents a paraboloid surface

Calculate the partial derivatives of $f(x,y)$

$$f_x = D(x^2 + 2x + y^2 - y + 3, x) = 2x + 2$$

$$f_y = D(x^2 + 2x + y^2 - y + 3, y) = 2y - 1$$

Calculate the values of partial derivatives at point where $x = 1, y = 0$

$$f_x(1, 0) = 2*1 + 2 = 4$$

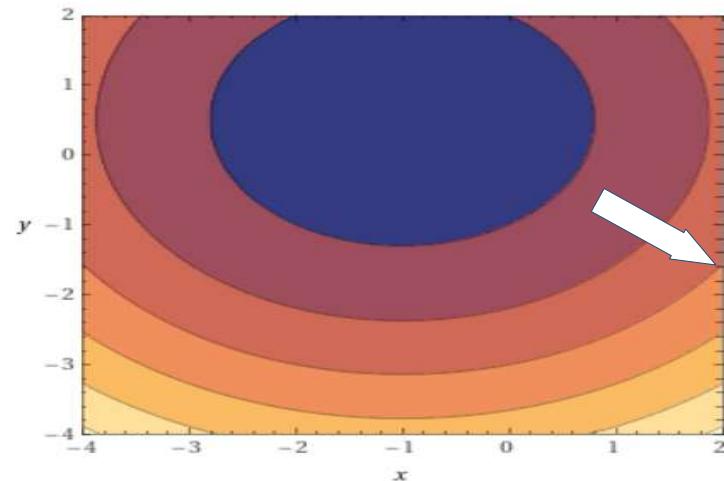
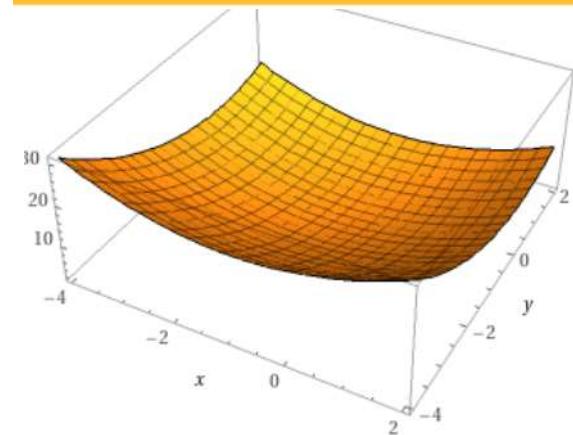
$$f_y(1, 0) = 2*0 - 1 = -1$$

At point $(1,0)$ gradient vector $(f_x, f_y) = (4, -1)$

=> Function grows fastest in direction $(4, -1)$

=> Minimum of the function is in direction $(-4, 1)$

plot $x^2 + 2x + y^2 - y + 3$



Gradient vector of a multivariate function

Grad $f(x,y) = (f_x, f_y)$ is the vector of partial derivatives

It is denoted often in mathematics as

$$\nabla f(x, y)$$

Operator ∇

is called "nabla"

WolframAlpha command

gradient of $x^2 + 2x + y^2 - y + 3$

gives $(2(1+x), -1+2y)$, which includes both partial derivatives

It can be simplified to normal form $(2x+2, 2y-1)$

When we want to calculate value of gradient at a specific point, we can use where statement

gradient of $x^2 + 2x + y^2 - y + 3$ where $x = 1, y = 0$

calculates the gradient vector at point $(1,0)$. Answer is $(4, -1)$

Functions of more than 2 variables cannot be visualized ("graph" would be a surface of 4 dimensional space). We can still calculate the gradient easily.

Example: $f(x,y,z) = x^2 + y^2 + z^2$ is a function of 3 variables.

The partial derivatives $f_x = 2x$, $f_y = 2y$ and $f_z = 2z$ measure the rate of change of function f in the directions of coordinate axes.

The gradient vector is $(2x, 2y, 2z)$

Gradient vector at point $(3, 2, -1) = (2 \cdot 3, 2 \cdot 2, 2 \cdot -1) = (3, 4, -1)$

=> At point $x=3, y=2, z=-1$ vector $(6, 4, -2)$ gives the direction in 3D space, in which function value increases most rapidly.

Result in 3D Cartesian coordinates

$$\text{grad}(x^2 + y^2 + z^2) = (2x, 2y, 2z)$$

Wolframalpha function grad (or gradient) works only, when variable names are x, y and z .
f.e gradient of $a^2 + b^2$ will not work
In this case we have to use partial derivates:
 $D(a^2+b^2, a) = 2a$ and $D(a^2+b^2, b) = 2b$

Exercises: Calculate partial derivatives with respect to all variables

1) $f(x,y) = x^2 + 5x - y^2$

$f_x =$

$f_y =$

2) $f(x,y) = e^x - 5x^2 - 3y^2 + 2xy + 1$

$f_x =$

$f_y =$

3) $g(a,b) = (2a + 3b)^2$

$g_a =$

$g_b =$

4) $f(x,y) = x/y$

$f_x =$

$f_y =$

Derivation rules:

$Dc = 0$

$Dx = 1$

$Dx^n = n x^{n-1}$

$D(1/x^n) = -n/x^{n+1}$

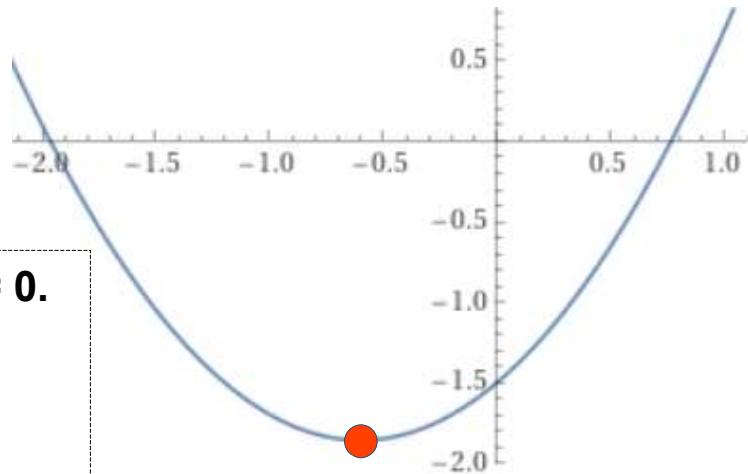
$De^x = e^x$

$D(a*f(x)+b*g(x)) =$
 $a*Df(x) + b*Dg(x)$

Other useful rules

$(x+y)^2 = x^2 + 2xy + y^2$

Example1. Find the minimum value of $f(x) = x^2 + 1.2x - 1.5$



Solution: At the minimum point, the slope of the tangent line of $f(x) = 0$.

In other words: at the minimum point the derivative $f'(x) = 0$.

Step1: Find the general expression of derivative:

$$f'(x) = 2x + 1.2$$

Step2: Solve equation $f'(x) = 0$ for x

$$2x + 1.2 = 0 \Rightarrow 2x = -1.2 \Rightarrow x = -0.6$$

Step3: Calculate the y – coordinate value at minumum point

$$y_{\min} = f(-0.6) = (-0.6)^2 + 1.2 * (-0.6) - 1.5 = -1.86$$

Next time more about finding extreme values of function of one or several variables.

Extreme value problems

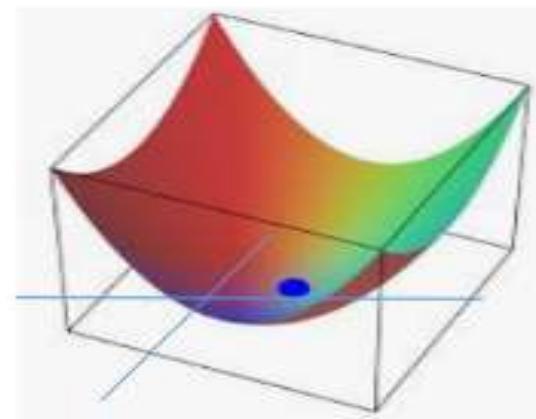
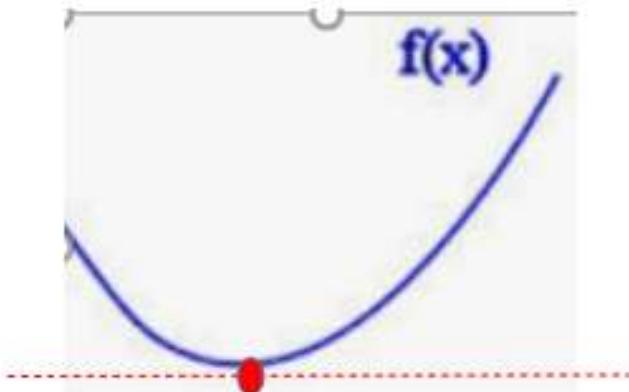
The goal is to find the minimum or maximum value of a function.

Applications:

- A) optimization of parameters of a model to find the best fit
(examples: regression analysis, machine learning)
- B) maximize profit
- C) minimize costs, time or use of materials

Terminology: "The objective function"

is the function of which we wish to find minimum or maximum)



Extreme values of one variable functions are often found **at points, where the derivative is zero.** (called critical points)

Extreme values of multivariable functions are mostly found **at points, where both partial derivatives are equal to zero.** (derivatives to directions of both axes).

Methods of optimization

1. ANALYTICAL METHOD

Extreme values of a single variable function $f(x)$ are found by solving roots of $f'(x) = 0$

Extreme value of a multivariable function $f(x,y)$ are found by solving the group of equations

$$D(f,x) = 0, D(f,y) = 0$$

2. ITERATION BASED METHODS

Extreme values are found by moving in steps in direction opposite to gradient vector until minimum is found (or **in direction of gradient vector for maximum**).

Algorithm "Gradient Descent Method" for finding minimum is used in Machine Learning

Extreme values of one variable function $f(x)$

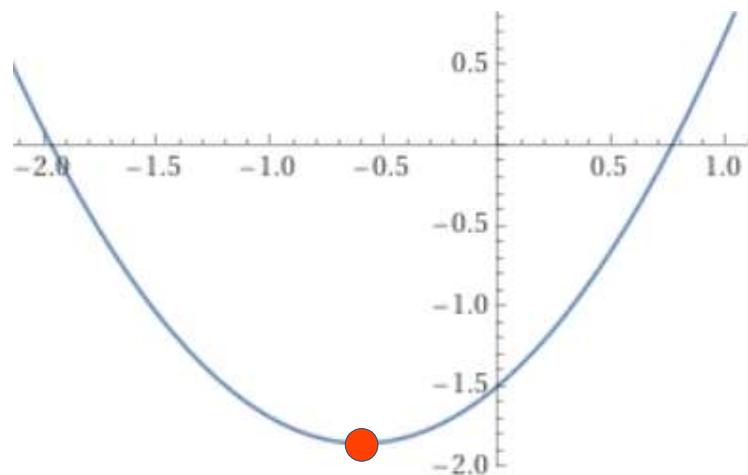
1A. Analytical method : manual solution

1B and C. Analytical method: using calculator functions

2. Gradient Descent method
- Python , Excel

Example1. Find the minimum value of $f(x) = x^2 + 1.2x - 1.5$

Next slides show different methods to solve the problem



Example1 cont... Find the minimum value of $f(x) = x^2 + 1.2x - 1.5$

plot $x^2+1.2x-1.5$

Method 1A: Analytical method (manual version)

1. Calculate the derivative function

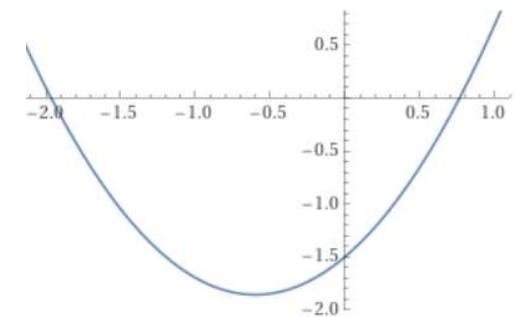
$$f'(x) = 2x + 1.2$$

2. Solve the zero of derivative

$$2x + 1.2 = 0 \Rightarrow 2x = -1.2 \Rightarrow x = -0.6$$

3. Calculate $f(-0.6)$

$$f(-0.6) = (-0.6)^2 + 1.2 * (-0.6) - 1.5 = -1.86$$



Answer: $f(x)$ has a minimum value -1.86 at $x = -0.6$

Example1 cont... Find the minimum value of $f(x) = x^2 + 1.2x - 1.5$

Method 1B. Analytical method using W.A calculator

Calculate zero of derivative (solve $f'(x) = 0$)

solve $D(x^2 + 1.2*x - 1.5) = 0$

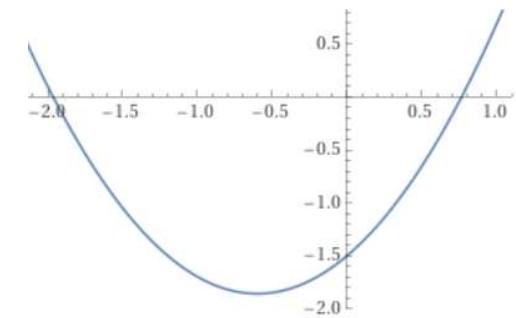
Result: $x = -0.6$

Calculate $f(x)$ at point $x = -0.6$

$x^2 + 1.2*x - 1.5$ where $x = -0.6$

Result: $x = -1.86$

plot $x^2 + 1.2*x - 1.5$



Method 1C. Direct solution using WolframAlpha's minimize function

minimize $x^2 + 1.2*x - 1.5$

Global minimum

$\min\{x^2 + 1.2x - 1.5\} = -1.86$ at $x = -0.6$

Example1 cont.... Find the minimum value of $f(x) = x^2 + 1.2x - 1.5$

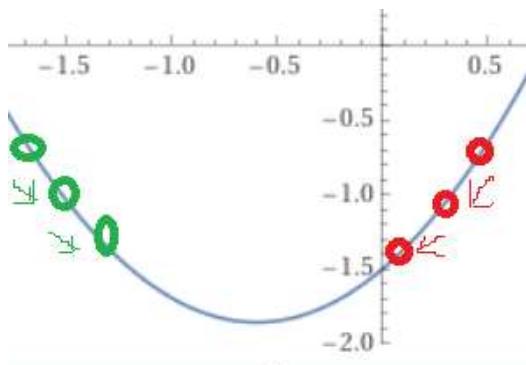
Method 2. Iteration Gradient descent method

1. Input start value for x

2. If $f'(x) > 0$, step dx to the left
If $f'(x) < 0$, step dx to the right

step size = $dx = -f'(x)*t$
where t is a coefficient (for ex. 0.2)

Near minimum $|dx|$ decreases and tends to zero.



```
# Python code
# 1. Define function f(x)
def f(x):
    return x**2 + 1.2*x - 1.5

# Define derivative function fx(x)
def fx(x):
    return 2*x + 1.2

# 2. Set an arbitrary initial value for variable x
x = 2.0

# 3. Define step coefficient t and step dx=-fx(x)*t
t = 0.2
dx=-fx(x)*t

#4. Create a loop to update x until |dx| < 0.01
while abs(dx)>0.01:
    dx=-fx(x)*t
    x=x+dx
    print(f"x= {x:5.2f}, y = {f(x):7.4f}")

#5. Print coordinates of minimum using Python f-string
print(f"\nMinimum : x= {x:5.2f}, f(x) = {f(x):5.2f}")
```

Example2: Find the minimum value of $f(x) = e^x - 2.5 x + 1$

Method 1A: Analytical method (manual version)

1. Calculate the derivative function

$$f'(x) = e^x - 2.5$$

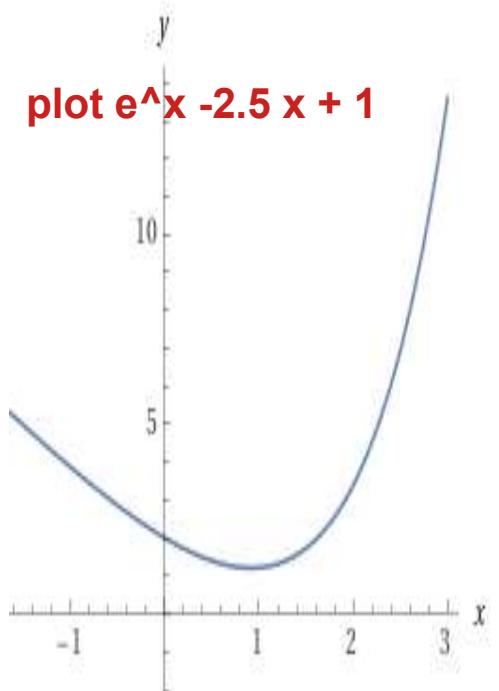
2. Solve the zero of derivative

$$e^x - 2.5 = 0 \Rightarrow e^x = 2.5 \Rightarrow x = \ln(2.5) = 0.916$$

3. Calculate $f(0.916)$

$$f(0.916) = e^{0.916} - 2.5 \cdot 0.916 + 1 = 1.21$$

Answer: Minimum 1.21 at $x = 0.916$



Method2. Gradient Descent Method (Excel version)

A	B	C
1 step coefficient t	0,2	
2	X	
3 initial value	0	
4	0,300	iteration formula = B4 - (exp(B4)-2,5)*\$B\$1
5	0,530	
6	0,690	
7	0,791	
8	0,850	
9	0,882	
10	0,899	
11	0,908	
12	0,912	
13	0,914	
14	0,915	
15	0,916	
16	0,916	minimum value at 0,916
17	0,916	
18	0,916	

1. Write step coefficient t and initial value for x in cells marked yellow

2. Iteration formula

New x value = $x - f'(x) \cdot t$
 $= x - (e^x - 2,5) \cdot t$

Excel formula uses cell addresses
 $=B4 - (exp(B4)-2,5)*$B1

3. Copy down iteration formula.

The values stabilize to the x value, which corresponds the minimum

Extreme values of one two variable function $f(x,y)$

1A. Analytical method : manual solution

1B and C. Analytical method: using calculator functions

2. Gradient Descent method
- Python , Excel

Function of two variables

Example3: Find the minimum of $f(x,y) = x^2 + y^2 + 2x - y + 1$

Method1A: Analytical method (manual version)

1. Calculate both partial derivatives

$$f_x(x,y) = 2x + 2$$

$$f_y(x,y) = 2y - 1$$

2. Solve point where both partial derivatives are zero.

$$2x + 2 = 0 \Rightarrow x = -1$$

$$2y - 1 = 0 \Rightarrow y = \frac{1}{2}$$

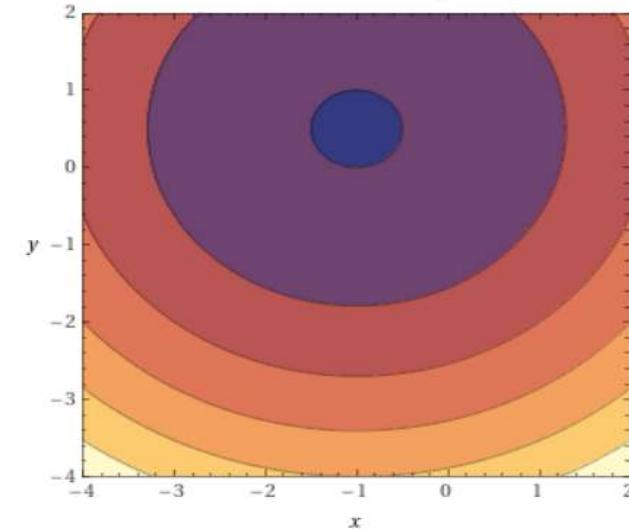
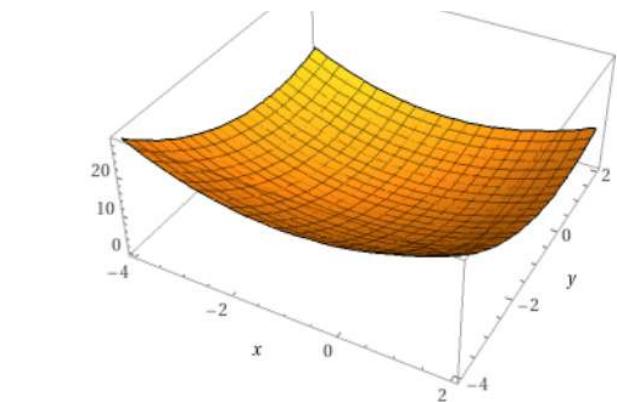
3. Calculate $f(-1, 1/2)$

$$f(-1, 1/2) = (-1)^2 + (1/2)^2 + 2*(-1) - 1/2 + 1 = -1/4$$

Answer: Function has a minimum value

-1/4 at point (-1, 1/2)

plot $x^2 + y^2 + 2x - y + 1$ from -4 to 2



Example3 cont.... find the minimum of $f(x,y) = x^2 + y^2 + 2x - y + 1$

Method 1B. Analytical method using WolframAlpha calculator

solve $D(x^2 + y^2 + 2x - y + 1, x) = 0, D(x^2 + y^2 + 2x - y + 1, y) = 0$

Result: $x = -1, y = 1/2$

$x^2 + y^2 + 2x - y + 1$ where $x=-1, y=1/2$

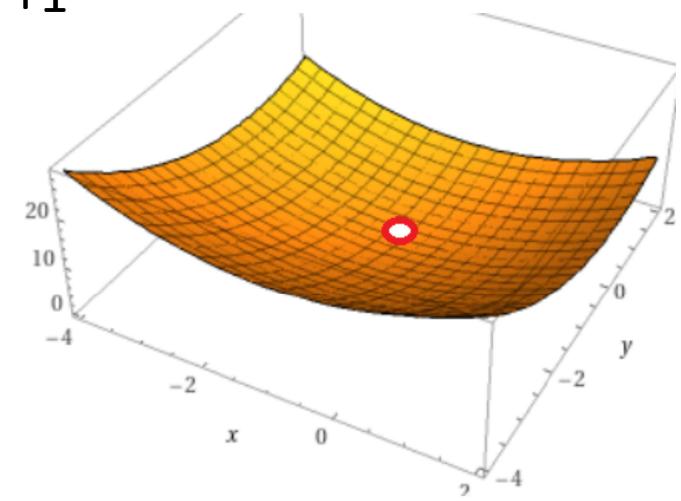
Result: $-1/4$

Method 1C. Direct solution using WolframAlpha's minimize command

minimize $x^2 + y^2 + 2x - y + 1$

$$\min\{x^2 + y^2 + 2x - y + 1\} = -\frac{1}{4} \text{ at } (x, y) = \left(-1, \frac{1}{2}\right)$$

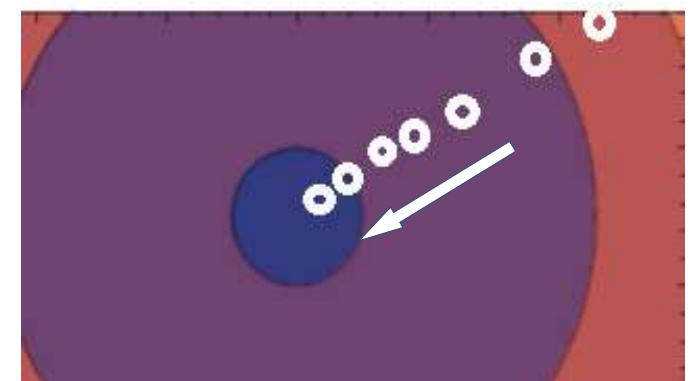
Example3 cont.... find the minimum of $f(x,y) = x^2 + y^2 + 2x - y + 1$



Method 2. Iteration using Gradient Descent Method

Calculate the gradient (=vector of partial derivatives)

Step in the opposite direction of gradient
until minimum is reached



Example3 cont... finding minimum
of $f(x,y) = x^2 + y^2 + 2x - y + 1$

Gradient descent method

Iteration formula:

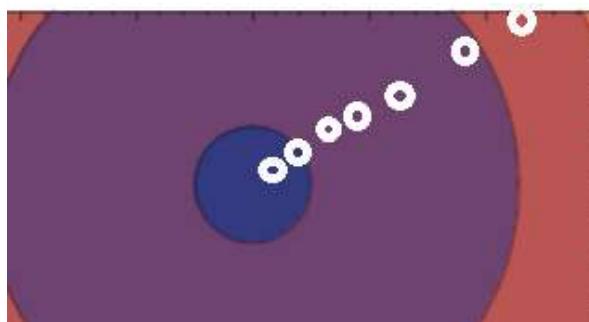
$$(x,y) = (x,y) - \text{grad}(f)*t$$

Iteration can be written also:

$$x = x - f_x * t$$

$$y = y - f_y * t$$

(f_x and f_y are partial derivatives,
 t = coefficient defining the step)



```
# Python code. Define f(x) and partial derivatives fx(x,y) and fy(x,y)
from math import sqrt
def f(x,y):
    return x**2 + y**2 + 2*x - y + 1
def fx(x,y):
    return 2*x + 2
def fy(x,y):
    return 2*y - 1
# 2. Give an arbitrary initial value for variable x
x = 1.5
y = 2.0
# 3. Define step_scale coeff. t and initialize steps dx and dy
t = 0.2
dx=-fx(x,y)*t
dy=-fy(x,y)*t
nr=0 #counter
#4. run the loop to update x as long as step >0.005
while sqrt(dx**2+dy**2)>0.005:
    dx=-fx(x,y)*t
    dy=-fy(x,y)*t
    print(f"nr {nr:2d} x = {x:5.2f} y ={y:5.2f} f(x,y)={z:9.5f} ")
    x=x+dx # new x
    y=y+dy # new y
    z=f(x,y) # function value
    nr+=1 # increase round nr
#5. Print minimum value and coordinates of minimum point
print(f"\nMinimum found by iteration at (x,y) =[{x:7.4f},{y:7.4f}] ")
print(f"\nMinimum value by iteration = {z:7.4f}")
print("\nCompare: True minimum at zero of gradient ",[-1,0.5,f(-1,0.5)])
```

Output of the Python code

```
1  x =  1.50  y = 2.00  f(x,y)=  8.25000
2  x =  0.50  y = 1.40  f(x,y)=  2.81000
3  x = -0.10  y = 1.04  f(x,y)=  0.85160
4  x = -0.46  y = 0.82  f(x,y)=  0.14658
5  x = -0.68  y = 0.69  f(x,y)= -0.10723
6  x = -0.81  y = 0.62  f(x,y)= -0.19860
7  x = -0.88  y = 0.57  f(x,y)= -0.23150
8  x = -0.93  y = 0.54  f(x,y)= -0.24334
9  x = -0.96  y = 0.53  f(x,y)= -0.24760
10 x = -0.97  y = 0.52  f(x,y)= -0.24914
11 x = -0.98  y = 0.51  f(x,y)= -0.24969
12 x = -0.99  y = 0.51  f(x,y)= -0.24989
13 x = -0.99  y = 0.50  f(x,y)= -0.24996
14 x = -1.00  y = 0.50  f(x,y)= -0.24999
15 x = -1.00  y = 0.50  f(x,y)= -0.24999
16 x = -1.00  y = 0.50  f(x,y)= -0.25000
```

Minimum found by iteration at (x,y) =[-0.9993, 0.5004]

Minimum value by iteration = -0.2500

Compare: True minimum at zero of gradient [-1, 0.5, -0.25]

Excel version of minimizing $f(x,y) = x^2 + y^2 + 2x - y + 1$

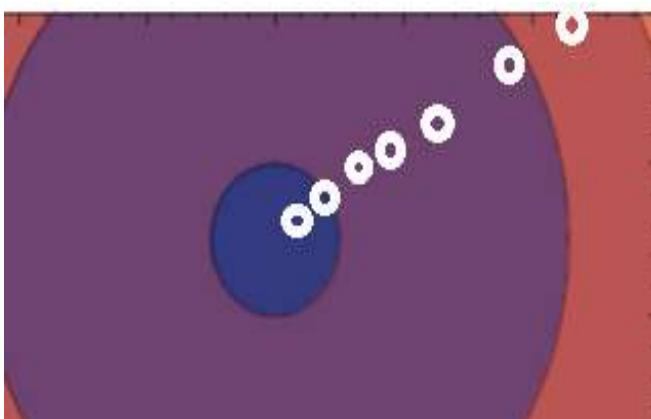
Gradient descent method

Iteration for x and y

$$x = x - fx*t = x - (2x+2)*t$$

$$y = y - fy*t = y - (2y-1)*t$$

(fx and fy are partial derivatives,
t = coefficient defining the step)



	A	B	C	
1	step coefficient t	0,25		
2				
3	X	Y		
4	initial value	2	1	
5	iteration formulas	0,500	0,750	=B4-(2*B4+2)*\$B\$1
6		-0,250	0,625	=C4-(2*C4-1)*\$B\$1
7		-0,625	0,563	
8		-0,813	0,531	
9		-0,906	0,516	
10		-0,953	0,508	
11		-0,977	0,504	
12		-0,988	0,502	
13		-0,994	0,501	
14		-0,997	0,500	
15		-0,999	0,500	
16		-0,999	0,500	
17		-1,000	0,500	
18		-1,000	0,500	

Found minimum at (-1, 0.5)

Linear regression as an optimization problem of two variables

1. Analytical method

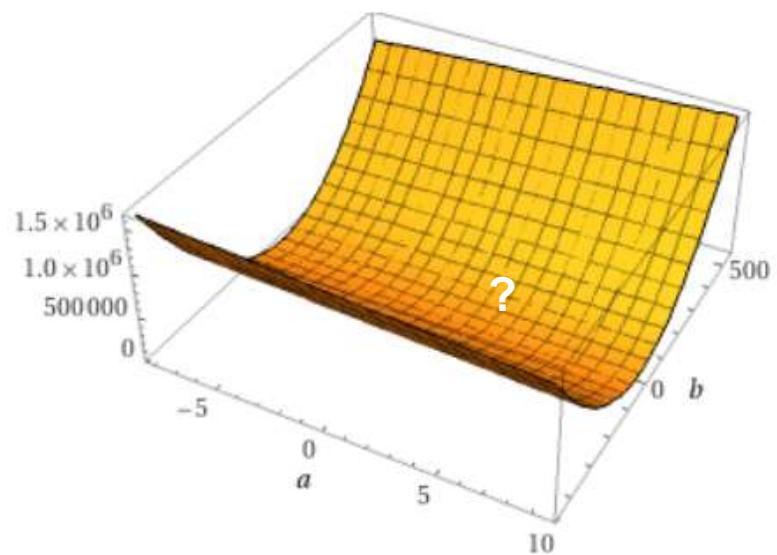
2. Gradient Descent method
- Python , Excel

Example4. Fit a linear model $y = a x + b$ to following observed data

x	1.0	1.5	2.0	2.5	3.0	3.5
y	42.5	42.8	43.2	43.5	43.9	44.2

Best fitting model is obtained by minimizing the square sum ("least square sum method")

$$(a*1.0+ b -42.5)^2 + (a*1.5+ b -42.8)^2 + (a*2+ b -43.2)^2 + (a*2.5+ b -43.8)^2 + (a*3+ b -43.9)^2 + (a*3.5+ b -44.3)^2$$



From plot it is impossible to locate the minimum.

Analytical method finds a point (a,b) where both partial derivatives of the square sum are zero.

WolframAlpha command minimize automatizes this procedure.

minimize $(a*1.0 + b - 42.5)^2 + (a*1.5 + b - 42.8)^2 + (a*2 + b - 43.2)^2 + (a*2.5 + b - 43.8)^2 + (a*3 + b - 43.9)^2 + (a*3.5 + b - 44.3)^2$

Result: 0.0510476 at $(a, b) \approx (0.737143, 41.7581)$

On the next slide we try Gradient Descent Method

Gradient descend method for Linear Regression

```
from math import sqrt
def f(a,b): #define the square sum function, which we want to find the minimum
    return (a*1.0+b-42.5)**2 +(a*1.5+b-42.8)**2+(a*2+b-43.2)**2+(a*2.5+b-43.8)**2+(a*3+b-43.9)**2+(a*3.5+b-44.3)**2

def fa(a,b): #partial derivative fa in simplified form, calculated with WolframAlpha
    return 69.5*a + 27.0*b -1178.7

def fb(a,b): #partial derivative fb in simplified form, calculated with WolframAlpha
    return 27*a + 12*b -521

# initial values of a and b
a=1.0
b=40.0
z=f(a,b)
nr=1 #iteration round counter

# step_scaling coefficient t and iteration step vector (da,db)
t = 0.02;
da=-fa(a,b)*t
db=-fb(a,b)*t
LOOPLIMIT=0.0003 # while loop stops when iteration steps get lower than this constant

while sqrt(da**2+db**2)>LOOPLIMIT:
    da=-fa(a,b)*t # iteration step in x-direction
    db=-fb(a,b)*t # iteration step in y-direction
    print(f"\n{nr:3d}. a={a:4.2f} b={b:4.2f} goal={z:4.2f}")
    a=a+da # new x
    b=b+db # new y
    z=f(a,b) # goal function value
    nr+=1 # increase counter

#formatted f-string prints the results
print(f"\nMinimum found by iteration at a ={a:4.2f} b ={b:4.2f}")
```

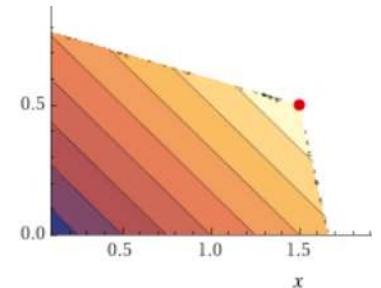
Comments:

1. If there is a lot of data, the function $f(a,b)$ is quite long.
2. Finding suitable value for coefficient t needs several trials
3. Finding suitable value for loop conditions
 $\text{abs}(da)>0.0003$
needs several trials

(Wrong values lead to deadlock)

```
Minimum found by iteration at a = 0.748   b =41.730
Compare: True minimum found with W.A minimize at [0.74, 41.8]
```

Linear optimization : 2D case



Analytical method

Linear multivariable functions, gradient, extreme values

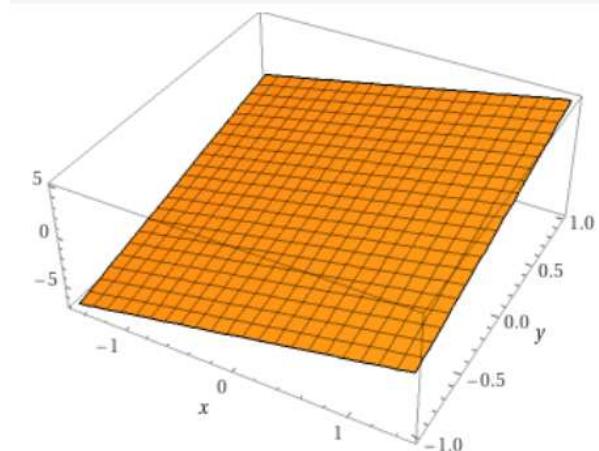
plot

$2x + 3y - 1$

Linear multivariable functions, for example $f(x,y) = 2x + 3y - 1$ have constant gradients. In our example $df/dx = 2$ and $df/dy = 3 \Rightarrow$ Gradient vector = (2,3).

Equation $f(x,y) = 2x + 3y - 1$ represents a plane.
Function has neither minima nor maxima.

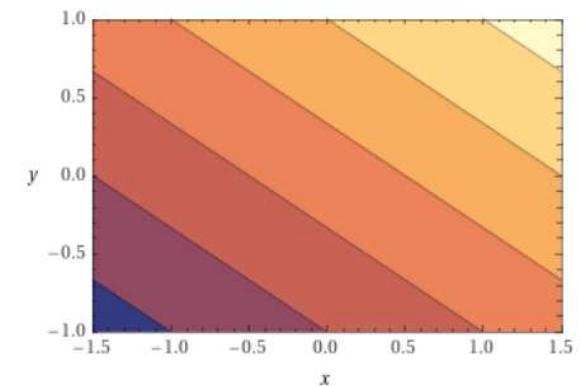
3D plot



Generally: Gradient of function $f(x,y) = a x + b y + c$ is a constant vector (a,b).

=> Function values increase most rapidly in the direction of (a,b)

Contour plot



"Linear optimization" = finding a minimum (or maximum) of a linear function inside a closed domain

If we restrict the domain of $f(x,y) = a x + b y + c$ with inequalities to a closed area, maximum is found at the corner vertex of the domain that is farthest in the direction of the gradient

Example: Find the maximum of function $f(x,y) = 2x + 3y - 1$ inside the polygon, which is defined with conditions

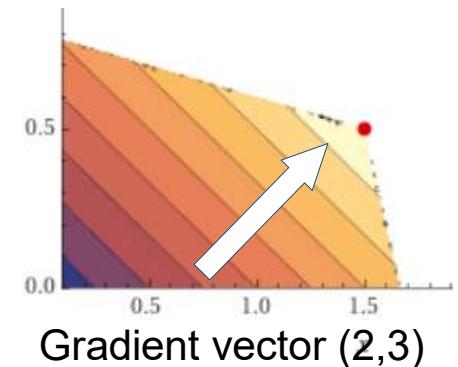
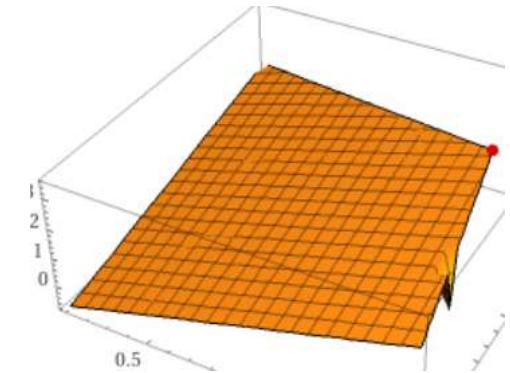
$$x \geq 0$$

$$y \geq 0$$

$$3x+y \leq 5$$

$$x+5y \leq 4$$

maximize $2x+3y-1$ where $x \geq 0, y \geq 0, 3x+y \leq 5, x+5y \leq 4$



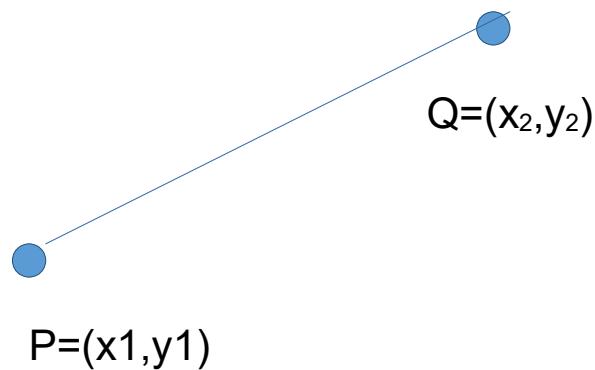
$$\max\{2x + 3y - 1 \mid x \geq 0 \wedge y \geq 0 \wedge 3x + y \leq 5 \wedge x + 5y \leq 4\} = \frac{7}{2} \text{ at } (x, y) = \left(\frac{3}{2}, \frac{1}{2}\right)$$

ADDITIONAL OPTIMIZATION EXAMPLES

Example1. Finding an optimal location for a logistic centre

Example2. Minimizing material of a 1 liter can

Example3. Minimizing time in orienteering competition



Example 1 needs formula:

Distance between P and Q

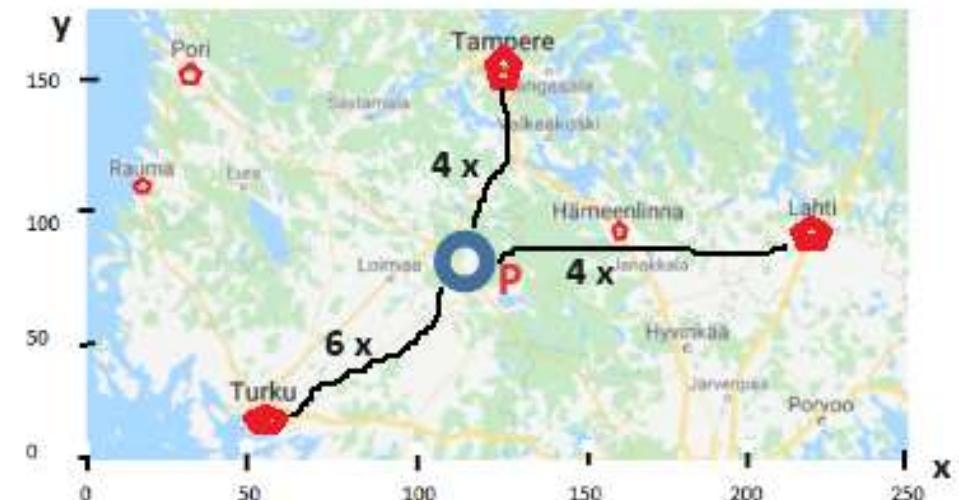
$$|PQ| = \sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$$

Example 1. Optimizing location of a logistic centre

A company has 3 apartments stores in three cities

City	X	Y	nr of lorries per week
Turku	52	15	6
Tampere	125	150	4
Lahti	220	95	4

Determine the optimal location for the logistic centre
(total number of driving km per week is minimized)



WolframAlpha command:

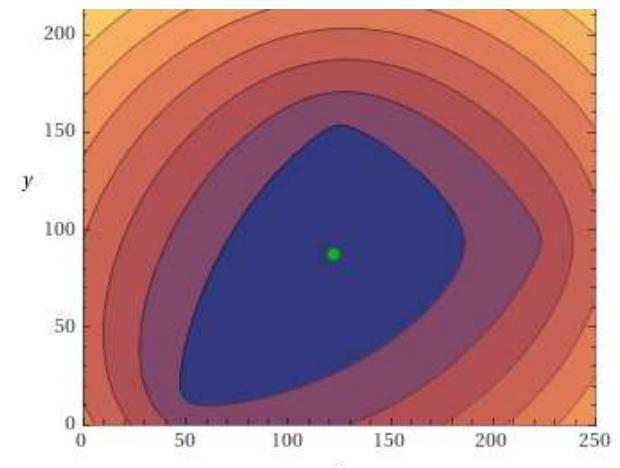
$$\min 6\sqrt{(x-52)^2 + (y-15)^2} + 4\sqrt{(x-125)^2 + (y-150)^2} + 4\sqrt{(x-220)^2 + (y-95)^2} \text{ where } 0 < x < 250, 0 < y < 250$$

Result: $\min \left\{ 6\sqrt{(x-52)^2 + (y-15)^2} + 4\sqrt{(x-125)^2 + (y-150)^2} + 4\sqrt{(x-220)^2 + (y-95)^2} \mid 0 < x < 250 \wedge 0 < y < 250 \right\} \approx 1248.1 \text{ at } (x, y) \approx (122.058, 87.3024)$

Answer: Optimal logistic center location is (122, 87)

See also Jupyter Notebook version in the Moodle

Part of output: contourplot



```

# PYTHON CODE (Standard IDLE Python 3.11)
# Find a point P=(x,y) for logistic centre using Gradient descend method of two variable function
# minimize 6*|PA|+4*|PB|+4*|PC|, where A=(52,15), B=(220,95), C=(125,150)

from math import sqrt # sqrt function from math package

def f(x,y): # define function f
    return 6*sqrt((x-52)**2+(y-15)**2)+ 4*sqrt((x-220)**2+(y-95)**2)+ 4*sqrt((x-125)**2+(y-150)**2)

def f_x(x,y): # define partial derivative fx
    return 6*(x-52)/sqrt((x-52)**2+(y-15)**2)+ 4*(x-220)/sqrt((x-220)**2+(y-95)**2)+ 4*(x-125)/sqrt((x-125)**2+(y-150)**2)

def f_y(x,y): # define partial derivative fy
    return 6*(y-15)/sqrt((x-52)**2+(y-15)**2)+ 4*(y-95)/sqrt((x-220)**2+(y-95)**2)+ 4*(y-150)/sqrt((x-125)**2+(y-150)**2)

# give initial values for iteration
x=100.0
y=100.0
z=f(x,y)
nr=1 # counter for number of iterations

# iteration coefficient and iteration step
t = 0.7;
dx=-f_x(x,y)*t
dy=-f_y(x,y)*t

print("Iteration steps: \n")

# loop which prints all the steps of iteration
while sqrt(dx**2+dy**2)>0.01:
    dx=-f_x(x,y)*t # iteration step in x-direction
    dy=-f_y(x,y)*t # iteration step in y-direction
    print(f"nr {nr:2d} x = {x:5.1f} y = {y:5.1f} f(x,y)={z:9.0f} ")
    x=x+dx # new x value
    y=y+dy # new y value
    z=f(x,y) # function value
    nr+=1 # increase round nr

# print results
print(f"\nMinimum {z:4.0f} km per week found at location (x,y)=[{x:3.0f},{y:3.0f}] ")
print("\n\n")

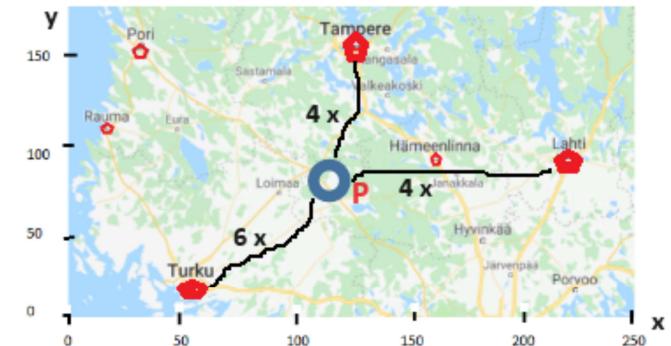
```

OUTPUT:

Iteration steps:

nr 1	x = 100.0	y = 100.0	f(x,y)=	1290
nr 2	x = 102.0	y = 98.7	f(x,y)=	1282
nr 3	x = 103.8	y = 97.6	f(x,y)=	1276
...				
nr 63	x = 121.8	y = 87.1	f(x,y)=	1248
nr 64	x = 121.8	y = 87.1	f(x,y)=	1248
nr 65	x = 121.8	y = 87.1	f(x,y)=	1248
nr 66	x = 121.8	y = 87.1	f(x,y)=	1248
nr 67	x = 121.8	y = 87.1	f(x,y)=	1248

**Minimum 1248 km per week found at location
(x,y)=[122, 87]**



Example2. Minimizing material of a 1 liter can

Function to be minimized is the total area of
Bottom, lid and jacket of the can:

$$A = 2 \cdot \pi r^2 + 2\pi r h$$

Because volume $V = \pi r^2 h = 1$ (unit dm³), we
can substitute $h = 1/(\pi r^2)$ which gives a one
variable function

$$A = 2 \cdot \pi r^2 + \frac{2\pi r}{\pi r^2} = 2\pi r^2 + \frac{2}{r}$$

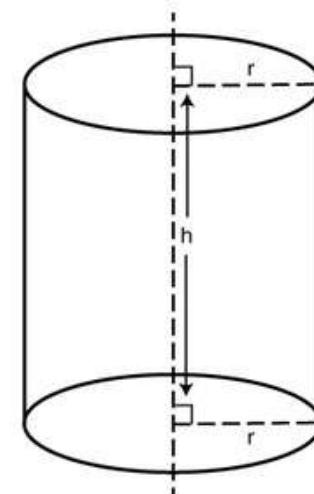
Minimum is found at zero of derivative:

$$\text{Derivative } A'(r) = 4\pi r - 2/r^2$$

$$\text{Zero of derivative: } 4\pi r = 2/r^2 \Rightarrow r^3 = 1/(2\pi) \Rightarrow$$

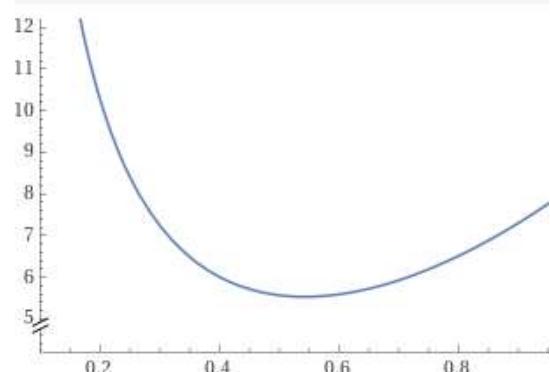
$$r = \frac{1}{\sqrt[3]{2\pi}} = 0.54 \text{ dm}$$

$$h = \frac{1}{\pi \cdot 0.54^2} = 1.09 \text{ dm}$$



$$\text{Volume (V)} = \pi r^2 h$$

$$\text{plot } 2\pi r^2 + 2/r \text{ from 0.1 to 1}$$



Result: Optimal form:
Diameter d of bottom equals height h



Example3. Finding a fastest path from A to B in orienteering competition

An athlete runs from point A in forest to point B in swamp (wetland). Her speed in the forest is 3m/s and in the swamp 1.5m/s. Calculate the optimal crossing point X from forest to swamp to minimize running time.

Function to be minimized:

$$t(x) = \frac{AX}{v_1} + \frac{XB}{v_2} = \frac{\sqrt{x^2 + 200^2}}{3} + \frac{\sqrt{(800-x)^2 + 150^2}}{1.5}$$

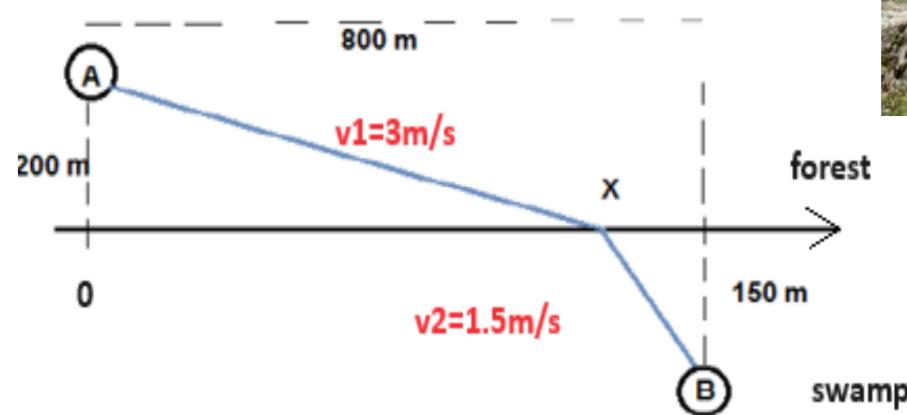
Minimum is at zero of derivative

$$\text{roots } D\left(\frac{\sqrt{x^2+200^2}}{3} + \frac{\sqrt{(800-x)^2+150^2}}{1.5}\right) = 0$$

Result

$$x \approx 717.56$$

Answer : Optimal crossing point $x = 718$ m



$$\text{plot } \frac{\sqrt{x^2+200^2}}{3} + \frac{\sqrt{(800-x)^2+150^2}}{1.5} \text{ from } 0 \text{ to } 800$$

