Weak forms for Bloch eigenproblems

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1 Notation

In this document, the following notation is observed:

ω temporal frequency

κ wave number (spatial frequency)

ι ("iota") the imaginary unit

i, j, k, l, m subscripts indicating vector/tensor components

Bold symbols indicate vectors, e.g. $\boldsymbol{x}$ is the position vector with components x_i.

2 Direct ($\boldsymbol{\kappa}(\omega)$)

In the direct formulation (see Collet et al., 2011), we prescribe the temporal frequency ω and wave direction $\boldsymbol{\phi}$ and compute the wavevector magnitudes as the eigenvalues. The periodic functions $\tilde{u}(\boldsymbol{x})$ are the eigenfunctions.
TODO: write me

3 Indirect ($\omega(\boldsymbol{\kappa})$), untransformed

In the untransformed indirect formulation, we prescribe the wavevector through Floquet-periodic boundary conditions and solve for the temporal frequencies ω as eigenvalues. The Floquet-periodic displacement fields $u(\boldsymbol{x})$ are the corresponding eigenfunctions.
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4 Indirect ($\omega(\boldsymbol{\kappa})$), with Bloch transformation

In the transformed indirect formulation, the Bloch ansatz $u(\boldsymbol{x}) = \tilde{u}(\boldsymbol{x})e^{-\iota\boldsymbol{\kappa}\cdot\boldsymbol{x}}$ is used to transform the equations of motion in terms of the periodic function $\tilde{u}(\boldsymbol{x})$. The wavevector $\boldsymbol{\kappa}$ is prescribed through additional terms that arise in the strong form of the problem, and the temporal frequency ω is solved as the eigenvalue. The periodic functions $\tilde{u}(\boldsymbol{x})$ are the corresponding eigenfunctions.

4.1 Scalar Helmholtz equation

Eigenproblem:

$$\begin{aligned}
\nabla \cdot (\boldsymbol{E}(\boldsymbol{x}) \nabla w(\boldsymbol{x},t)) =& \rho(\boldsymbol{x}) \ddot{w}(\boldsymbol{x},t) \\
\left(E_{ij}(\boldsymbol{x}) w_{,j}(\boldsymbol{x},t)\right)_{,i} =& \rho(\boldsymbol{x}) \ddot{w}(\boldsymbol{x},t) \\
w(\boldsymbol{x},t) =& u(\boldsymbol{x}) e^{-\iota\omega t} \\
\left(E_{ij}(\boldsymbol{x}) u_{,j}(\boldsymbol{x})\right)_{,i} e^{-\iota\omega t} =& -\omega^2 \rho(\boldsymbol{x}) u(\boldsymbol{x}) e^{-\iota\omega t} \\
\left(E_{ij}(\boldsymbol{x}) u_{,j}(\boldsymbol{x})\right)_{,i} =& -\omega^2 \rho(\boldsymbol{x}) u(\boldsymbol{x}) \qquad (1) \\
& \qquad (2)
\end{aligned}$$

Bloch transformation:

$$u(\boldsymbol{x}) = \tilde{u}(\boldsymbol{x}) e^{-\iota\boldsymbol{\kappa}\cdot\boldsymbol{x}} \qquad (3)$$

$$\begin{aligned}
\left(E_{ij}(\boldsymbol{x}) \left[\tilde{u}(\boldsymbol{x}) e^{-\iota\kappa_k x_k}\right]_{,j}\right)_{,i} &= -\omega^2 \rho(\boldsymbol{x}) \tilde{u}(\boldsymbol{x}) e^{-\iota\kappa_k x_k} \\
\left(E_{ij}(\boldsymbol{x}) \left[\tilde{u}_{,j}(\boldsymbol{x}) e^{-\iota\kappa_k x_k} + \tilde{u}(\boldsymbol{x}) (e^{-\iota\kappa_k x_k})_{,j}\right]\right)_{,i} &= -\omega^2 \rho(\boldsymbol{x}) \tilde{u}(\boldsymbol{x}) e^{-\iota\kappa_k x_k} \\
\left(E_{ij}(\boldsymbol{x}) \left[\tilde{u}_{,j}(\boldsymbol{x}) e^{-\iota\kappa_k x_k} + \tilde{u}(\boldsymbol{x}) e^{-\iota\kappa_m x_m} (-\iota\kappa_k \delta_{kj})\right]\right)_{,i} &= -\omega^2 \rho(\boldsymbol{x}) \tilde{u}(\boldsymbol{x}) e^{-\iota\kappa_k x_k} \\
\left(E_{ij}(\boldsymbol{x}) \left[\tilde{u}_{,j}(\boldsymbol{x}) e^{-\iota\kappa_k x_k} - \iota\kappa_j \tilde{u}(\boldsymbol{x}) e^{-\iota\kappa_k x_k}\right]\right)_{,i} &= -\omega^2 \rho(\boldsymbol{x}) \tilde{u}(\boldsymbol{x}) e^{-\iota\kappa_k x_k} \\
\left(E_{ij}(\boldsymbol{x}) \tilde{u}_{,j}(\boldsymbol{x}) e^{-\iota\kappa_k x_k} - \iota E_{ij}(\boldsymbol{x}) \kappa_j \tilde{u}(\boldsymbol{x}) e^{-\iota\kappa_k x_k}\right)_{,i} &= -\omega^2 \rho(\boldsymbol{x}) \tilde{u}(\boldsymbol{x}) e^{-\iota\kappa_k x_k} \\
\left(E_{ij}(\boldsymbol{x}) \tilde{u}_{,j}(\boldsymbol{x}) e^{-\iota\kappa_k x_k}\right)_{,i} - \iota \left(E_{ij}(\boldsymbol{x}) \kappa_j \tilde{u}(\boldsymbol{x}) e^{-\iota\kappa_k x_k}\right)_{,i} &= -\omega^2 \rho(\boldsymbol{x}) \tilde{u}(\boldsymbol{x}) e^{-\iota\kappa_k x_k} \\
\left(\left[E_{ij}(\boldsymbol{x}) \tilde{u}_{,j}(\boldsymbol{x})\right]_{,i} e^{-\iota\kappa_k x_k} - \iota\kappa_i E_{ij}(\boldsymbol{x}) \tilde{u}_{,j}(\boldsymbol{x}) e^{-\iota\kappa_k x_k}\right) & \\
-\iota \left(\left[E_{ij}(\boldsymbol{x}) \kappa_j \tilde{u}(\boldsymbol{x})\right]_{,i} e^{-\iota\kappa_k x_k} - \iota\kappa_i E_{ij}(\boldsymbol{x}) \kappa_j \tilde{u}(\boldsymbol{x}) e^{-\iota\kappa_k x_k}\right) &= -\omega^2 \rho(\boldsymbol{x}) \tilde{u}(\boldsymbol{x}) e^{-\iota\kappa_k x_k} \\
\left(\left[E_{ij}(\boldsymbol{x}) \tilde{u}_{,j}(\boldsymbol{x})\right]_{,i} - \iota\kappa_i E_{ij}(\boldsymbol{x}) \tilde{u}_{,j}(\boldsymbol{x})\right) & \\
-\iota \left(\left[E_{ij}(\boldsymbol{x}) \kappa_j \tilde{u}(\boldsymbol{x})\right]_{,i} - \iota\kappa_i E_{ij}(\boldsymbol{x}) \kappa_j \tilde{u}(\boldsymbol{x})\right) &= -\omega^2 \rho(\boldsymbol{x}) \tilde{u}(\boldsymbol{x}) \qquad (4)
\end{aligned}$$

4.1.1 Transformed strong forms

General case:

$$\left(\left[E_{ij}(\boldsymbol{x}) \tilde{u}_{,j}(\boldsymbol{x})\right]_{,i} - \kappa_i E_{ij}(\boldsymbol{x}) \kappa_j \tilde{u}(\boldsymbol{x})\right) - \iota \left(\left[E_{ij}(\boldsymbol{x}) \kappa_j \tilde{u}(\boldsymbol{x})\right]_{,i} + \kappa_i E_{ij}(\boldsymbol{x}) \tilde{u}_{,j}(\boldsymbol{x})\right) = -\omega^2 \rho(\boldsymbol{x}) \tilde{u}(\boldsymbol{x}) \qquad (5)$$

Isotropic modulus, i.e. $E_{ij}(\boldsymbol{x}) = E(\boldsymbol{x}) \delta_{ij}$:

$$\left(\left[E(\boldsymbol{x}) \tilde{u}_{,i}(\boldsymbol{x})\right]_{,i} - \kappa_i E(\boldsymbol{x}) \kappa_i \tilde{u}(\boldsymbol{x})\right) - \iota \left(\left[E(\boldsymbol{x}) \kappa_i \tilde{u}(\boldsymbol{x})\right]_{,i} + \kappa_i E(\boldsymbol{x}) \tilde{u}_{,i}(\boldsymbol{x})\right) = -\omega^2 \rho(\boldsymbol{x}) \tilde{u}(\boldsymbol{x}) \qquad (6)$$

Piecewise constant modulus, i.e. $E_{ij,k}(\boldsymbol{x}) = 0$:

$$\left(E_{ij}(\boldsymbol{x}) \tilde{u}_{,ji}(\boldsymbol{x}) - \kappa_i E_{ij}(\boldsymbol{x}) \kappa_j \tilde{u}(\boldsymbol{x})\right) - \iota \left(E_{ij}(\boldsymbol{x}) \kappa_j \tilde{u}_{,i}(\boldsymbol{x}) + \kappa_i E_{ij}(\boldsymbol{x}) \tilde{u}_{,j}(\boldsymbol{x})\right) = -\omega^2 \rho(\boldsymbol{x}) \tilde{u}(\boldsymbol{x}) \qquad (7)$$

Piecewise constant isotropic modulus:

$$E(\boldsymbol{x}) \left(\tilde{u}_{,ii}(\boldsymbol{x}) - \kappa_i \kappa_i \tilde{u}(\boldsymbol{x})\right) - \iota E(\boldsymbol{x}) \left(\kappa_i \tilde{u}_{,i}(\boldsymbol{x}) + \kappa_i \tilde{u}_{,i}(\boldsymbol{x})\right) = -\omega^2 \rho(\boldsymbol{x}) \tilde{u}(\boldsymbol{x}) \qquad (8)$$