Classification of postitive energy representations

James Tener
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Abstract

We will continue last week's discussion of Wassermann's paper *Operator Algebras and Conformal Field Theory III* by describing the classification of positive energy representations via the representation theory of $SU(n)$.

1 Introduction

Let $G = SU(n)$ and let $LG = C^\infty(S^1, G)$. We are looking at positive energy representations of LG - that is, projective representations of $LG \rtimes \mathbb{T}$ on a Hilbert space $\mathcal{H}$ such that

$$\mathcal{H} = \bigoplus_{n \geq 0} \mathcal{H}(n),$$

$\dim \mathcal{H}(n) < \infty$, and $e^{it} \cdot \zeta = e^{int}\zeta$ for $\zeta \in \mathcal{H}(n)$. We'll also usually insist on the normalization $\mathcal{H}(0) \neq \{0\}$. If this is not the case, we may obtain such a representation by tensoring with a character of $\mathbb{T}$. We obtained one such representation as follows.

Let $V = \mathbb{C}^n$, and for $f \in L^2(S^1, V)$ densely define $a(f)$ on $\Lambda\mathcal{H}$ by $a(f)\omega = f \wedge \omega$. Let $CAR(\mathcal{H})$ denote the C^*-algebra generated by $\{a(f)\}$ on $\Lambda\mathcal{H}$. For a projection p on $\mathcal{H}$, we discussed last week the existence of a pure state ϕ_p on $CAR(\mathcal{H})$ such that

$$\phi_p(a(f_1)^* \dots a(f_n)^* a(g_1) \dots a(g_m)) = \delta_{nm} \det\langle pg_i, f_j\rangle.$$

Lemma 1.1. *Quasi-free states exist.*

Proof. If $\dim \mathcal{H} < \infty$, choose ONB $\{\zeta_1, \dots, \zeta_k\}$ and $\{\eta_1, \dots, \eta_n\}$ for $p\mathcal{H}$ and $(1-p)\mathcal{H}$, respectively. Let $\eta = \eta_1 \wedge \cdots \wedge \eta_n$. Since $a(f)\eta = a(pf)\eta$, the vector state corresponding to η is quasi free of covariance p.

If $\dim \mathcal{H} = \infty$, choose ONB $\{\zeta_i\}$ and $\{\eta_j\}$ as before. Let V_k be the subspace spanned by ζ_i and η_j. We have such a state on V_k, and the inclusions are coherent, so we're done. □

If $u \in U(\mathcal{H})$ and α_u is the automorphism of $CAR(\mathcal{H})$ given by $\alpha(a(f)) = a(uf)$, then one can show that $\phi_p \circ \alpha = \phi_{upu^*}$. It turns out that $\|\phi_p - \phi_{upu^*}\|$ is controlled by tr $\left[(p - upu^*)^2\right]$. It can be shown that if $\|\phi_p - \phi_p \circ \alpha\| < 2$, then the GNS representations are equivalent and α is implemented on $CAR(\mathcal{H})_{\phi_p}$. Using these ideas, one shows that if $up - pu$ is Hilbert-Schmidt (which is equivalent to $p - u^*pu$ being Hilbert-Schmidt), then $a(f) \mapsto a(uf)$ is implemented on $CAR(\mathcal{H})_p$ (the GNS space for ϕ_p).

If it happens that $\phi_p \circ \alpha_u = \phi_p$ (that is, $[u, p] = 0$), then there is a unitary in $U_{\alpha_u} \in B(CAR(\mathcal{H})_p)$ characterized by $x\Omega \mapsto \alpha_u(x)\Omega$, where Ω is the cyclic vector in $CAR(\mathcal{H})_p$ implementing ϕ_p. It is easy to verify that $U_{\alpha_u}\pi_P(x)U_{\alpha_u}^* = \pi_P(\alpha_u(x))$ for $x \in CAR(\mathcal{H})$. In this case, we say that α is canonically quantized.

For $\mathcal{H} = L^2(S^1, V)$ and $p\mathcal{H} = H^2(S^1, V)$, we have $\|[M_f, p]\|_2 = \|f\|_{H^{1/2}}$ for $f \in LG$ and $[r_\theta, p] = 0$ for $r_\theta \in \mathbb{T}$. Thus we get a projective unitary representation π of $LG \rtimes \mathbb{T}$ on $CAR(\mathcal{H})_p$ which restricts to an ordinary representation of $\mathbb{T}$. This representation satisfies $\pi(M_g)\pi_p(a(f))\pi(M_g)^* = \pi_p(a(gf))$ and $\pi(r_\theta)\pi_p(a(f))\pi(M_g)^* = \pi_p(a(r_\theta f))$ for $g \in LG$, $f \in \mathcal{H}$ and $\theta \in [0, 2\pi]$. We'll denote $\pi(r_\theta) = R_\theta$.

An irreducible positive energy representation of level ℓ is an irreducible subrepresentation of $\pi^{\otimes \ell}$. We've glossed over the (true statement) that the irreducible LG-subrepresentations of $\pi^{\otimes \ell}$ are positive energy representations in the obvious way. In this talk, we will prove that all irreducible positive energy representations are characterized by the isomorphism class of $\mathcal{H}(0)$ (the subspace fixed by the action of $\mathbb{T}$) as an $SU(n)$-module (where $SU(n)$ has been identified with constant loops in LG).

2 Representation theory of $SU(n)$

Let $G = SU(n)$ and let V be the defining representation on $\mathbb{C}^n$. Let $\mathfrak{g}$ be the corresponding Lie algebra consisting of traceless skew-adjoint matrices and let $\mathfrak{g}_\mathbb{C}$ be its complexification, the traceless matrices. We have representations of G and $\mathfrak{g}$ on $V^{\otimes m}$ via $g \cdot (v_1 \otimes \cdots \otimes v_m) = gv_1 \otimes \cdots gv_m$ for $g \in G$ and for $x \in \mathfrak{g}_\mathbb{C}$

$$x \cdot (v_1 \otimes \cdots \otimes v_m) = \sum_{j=1}^{m} v_1 \otimes \cdots \otimes xv_j \otimes \cdots \otimes v_m.$$

Let $\pi : G \to U(W)$ be a subrepresentation of $V^{\otimes m}$, irreducible for G (and hence for $\mathfrak{g}$ and $\mathfrak{g}_\mathbb{C}$). From Lie group theory, we may write

$$W = \bigoplus_{g \in \mathbb{Z}^n} W_g$$

where for z a diagonal matrix in G with entries $z_1, \ldots, z_n$ and $w \in W_g$ we have $\pi(z)w = (z_1^{g_1} \cdots z_n^{g_n})w$. Since $z_1 \cdots z_n = 1$, we can only define g up to addition of a vector $(a, \ldots, a) \in \mathbb{Z}^n$. On the Lie algebra side, if D is a traceless diagonal matrix with entries $d_1, \ldots, d_n$, then

$$\pi(D)w = (d_1 g_1 + \cdots d_n g_n)w.$$

Using the relation $DE_{ij} = E_{ij}D + (d_i - d_j)E_{ij}$, we see that $\pi(E_{ij})$ takes W_g into the weight space for $g + e_i - e_j$ (where $\{e_k\}$ is the standard basis for $\mathbb{Z}^n$). If we order the weights lexicographically, then E_{ij} takes W_g into a higher weight space if $i < j$ and into a lower weight space if $i > j$. These operators are called raising and lowering operators, respectively. Since monomial matrices in $SU(n)$ permute weight spaces and W is finite dimensional, there is a highest weight space W_f with $f_1 \geq f_2 \geq \cdots \geq f_N$. We call this the signature of the representation.

We now prove that a finite-dimensional $SU(n)$ representation is characterized up to unitary equivalence by its signature. Let v and v' be non-zero highest weight vectors. Since W is irreducible for G and hence $\mathfrak{g}_\mathbb{C}$, there is some T in the associative algebra generated by $\pi(\mathfrak{g}_\mathbb{C})$ such that $Tv = v'$. Every element of $\mathfrak{g}_\mathbb{C}$ is of the form

$$D + \sum L_i + R_i$$

Where D is a traceless diagonal operators, L_i is a linear combination of operators of the form $\pi(E_{ij})$ for $i > j$ and R_i is a linear combination of operators of the form $\pi(E_{ij})$ for $i < j$. Using the commutation relations

$$DE_{ij} = E_{ij}D', \qquad E_{ij}E_{k\ell} = E_{k\ell}E_{ij} + \delta jkE_{i\ell} - \delta_{\ell i}E_{ki}$$

we can write any element of the algebra generated by $\pi(\mathfrak{g}_{\mathbb{C}})$ as a sum of operators of the form LDR where L is a product of lowering operators, D is traceless diagonal, and R is a product of raising operators. Since v is highest weight, either $LDRv$ is a multiple of v or lower weight. Thus $v' = Tv = \lambda v$, and the high weight space is one-dimensional. Similarly, $\langle LDRv, v\rangle$ only depends on the weight of v, so $\langle A_1 v, A_2 v\rangle$ only depends on the weight of v (writing $A_2^* A_1$ as a sum of terms LDR). Thus if W_1 and W_2 are two irreducible representations of $SU(n)$ with the same signature, the map $Av \mapsto Av'$ is a unitary intertwining the action of $\mathfrak{g}$ and hence G.

Every signature occurs: if $f_1 \geq f_2 \geq \cdots \geq f_n \geq 0$, then the vector

$$e_f = e_1^{\otimes(f_1 - f_2)} \otimes (e_1 \wedge e_2)^{\otimes(f_2 - f_3)} \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_n)^{\otimes f_n}$$

is the unique highest weight vector in $\Lambda^1 V^{\otimes(f_1 - f_2)} \otimes \cdots \otimes \Lambda^n V^{\otimes f_n}$. By uniqueness, e_f generates an irreducible submodule.

3 Existence of positive energy representations

If H is an irreducible positive energy represntation of $LSU(n)$, we can think of its low weight space $H(0)$ as an $SU(n)$-module, where $SU(n)$ has been identified with the constant loops in $LSU(n)$. It turns out, the isomorphism class of this module determines the representation.

Theorem 3.1 (Classification of representations)**.** *If H is an irreducible positive energy representation of $LSU(n)$, then $H(0)$ is an irreducible $SU(n)$-module. The H is an irreducible subrepresentation of $\pi^{\otimes \ell}$, then its signature satisfies $f_1 - f_n \leq \ell$, and all such signatures occur at level ℓ. If H and H' are two positive energy representations at level ℓ with the same signature, then they are unitarily equivalent.*

We'll prove selected parts of this theorem today. First, we'll show that $H(0)$ is an $SU(n)$-module and outline the proof that every signature occurs. Note that if g is a constant loop in $LSU(n)$ and v is fixed by rotation, then

$$R_\theta \pi_i(g) v = \pi_i(r_\theta g) R_\theta^* v = \pi_i(g) v.$$

Thus $H(0)$ is an $SU(n)$-module.

We now try to see which signatures occur. To do this, we have to look a little more closely at the action of of $\mathbb{T} = \{r_\theta\}$ on $CAR(\mathcal{H})_p$. It will be useful to think of $CAR(\mathcal{H})_p$ as $\Lambda \mathcal{H}_p$, where $\mathcal{H}_p$ is $\mathcal{H}$ with a new complex structure. Hence $CAR(\mathcal{H})_p \cong \Lambda p\mathcal{H} \otimes \Lambda \overline{(1-p)\mathcal{H}}$. ΛV sits inside $\Lambda p\mathcal{H}$, so ΛV is inside $CAR(\mathcal{H})_p$ and is fixed by rotation. Since $\dim V = n$, $\Lambda V = \Lambda^1 V \oplus \cdots \oplus \Lambda^n V$. Note that ΛV contains irreducible subrepresentations of $SU(n)$ for every signature such that $f_1 - f_n \leq 1$. For each signature, we'd like for there to be some irreducible subrepresentation of $LSU(n) \rtimes \mathbb{T}$ that has that signature for its low weight space.

Proposition 3.1. *If H is the cyclic $LSU(n) \rtimes \mathbb{T}$-module generated by a lowest energy vector, it contains an irreducible $LSU(n) \rtimes \mathbb{T}$-module generated by some lowest energy vector. The weights of the low energy space of H and its irreducible submodule are the same.*

Proof. Let $\Gamma = LSU(n) \rtimes \mathbb{T}$. Let $H(0)$ be the lowest energy subspace of H, and let K be any Γ-invariant subspace of H with corresponding projection $p \in \Gamma'$. Since H is the closed linear space of $\Gamma H(0)$, $K = pH$ is the closed linear span of $\Gamma pH(0)$. But $pH(0) \subseteq H(0)$, since p commutes with $\mathbb{T}$. Choosing $pH(0)$ in $H(0)$ of smallest dimension, $K = pH$ must be irreducible as a Γ-module. Thus H contains an irreducible submodule K generated by any non-zero pv with $v \in H(0)$. Since p commutes with the action of $SU(n)$, $pH(0)$ has the same weights as $H(0)$. □

If we believe that the low weight space of the generated $LSU(n)$-module has not gotten any bigger (i.e. that if $v \in H(0)$ then $\pi(g)v \in H(0)$ if and only if $g \in SU(n)$), then this shows that there is an irreducible positive energy representation whose low weight space has the given signature, so long as $f_1 - f_n \leq 1$.

Since the low weight space of π contained ΛV, the low weight space of $\pi^{\otimes \ell}$ contains $(\Lambda V)^{\otimes \ell}$. Since we can construct a representation with a given signature f as a subrepresentation of

$$e_f = e_1^{\otimes(f_1-f_2)} \otimes (e_1 \wedge e_2)^{\otimes(f_2-f_3)} \otimes \cdots \otimes (e_1 \wedge \cdots \wedge e_n)^{\otimes f_n},$$

we have V_f as the low weight space of an irreducible subrepresentation of $V^{\otimes \ell}$ if $f_1 - f_n \leq \ell$ (since $f_1 - f_n$ says how many tensor products are required in the above expression if we normalize to $f_n = 0$).

The assumption that the low weight space could not have gotten any larger will be justified when we show that the low weight space of a positive energy representation is irreducible as an $SU(n)$ module.

4 Summary of the rest (Part II)

We do the rest by understanding the action of $L\mathfrak{g}$. If $X \in \mathfrak{g}$, let $X_n = Xe^{-int}$. We can write down explicitly an action of X_n on $CAR(\mathcal{H})_p$ such that the action exponentiates to give the action of e^{X_n}. Let's start with the action of $\mathbb{T}$. Everything is easies to discuss on a basis. By Stone's theorem, we can write $R_\theta = e^{iD\theta}$ for self-adjoint D. Let's look at the action of D on a basis. In the interest of time, we assert that an orthonormal basis for $CAR(\mathcal{H})_p$ is given by

$$e_{i_1}(n_1) \cdots e_{i_p}(n_p) e_{j_1}(m_1)^* \cdots e_{j_q}(m_q)^* \Omega$$

where $n_i \leq 0$ and $m_j > 0$. A reason to believe this is to think of $e_i(n)$ as creation if $n \leq 0$ and annihilation if $n > 0$. As an example, we'll show $e_i(n)\Omega = 0$ for $n > 0$. Compute

$$\langle e_i(n)\Omega, e_i(n)\Omega \rangle = \langle e_i(n)^* e_i(n)\Omega, \Omega \rangle = \phi_p(e_i(n)^* e_i(n)) = \langle pe_i(n), e_i(n) \rangle.$$

This is 1 if $n \leq 0$ and 0 otherwise.

If $X = \sum a_{ij} E_{ij}$, then let $X(n)$ act on $CAR(\mathcal{H})$ by linear extending the action

$$E_{ij}(n) = \sum_{m \geq 0} e_i(n-m) e_j(-m)^* - \sum_{m \geq 0} e_j(m)^* e_i(m+n).$$

Note that $E_{ij}(n)$ raises energy by $-n$. One shows, in analogy with the rep theory of $SU(n)$, that everything in $L^0\mathfrak{g}$ is products of RDL, where D, the energy preserving ones, are precisely the $E_{ij}(0)$, or the constant loops. This shows that the low energy space cannot grow even when used to generate an $LSU(n)$ module, as before.

A key fact is that $L^0\mathfrak{g} \rtimes \mathbb{R}$ is algebraically irreducible on H^0. This shows that the low energy space is, in fact, always an irreducible $SU(n)$ module. It can also be used to prove unitary equivalence of irreps with the same signature.