Math 110, Fall 2012, Sections 109-110
Worksheet 9

1. Give an example of a matrix that is:

   (a) Diagonalizable and invertible.

   (b) Not diagonalizable, but invertible.

   (c) Not invertible, but diagonalizable.

   (d) Neither invertible nor diagonalizable.

2. Recall that a *nilpotent* linear operator $T : V \to V$ is one for which there exists a $k > 0$ with $T^k = 0$.

   (a) What can you say about the eigenvalues of a nilpotent linear operator?

   (b) What is the characteristic polynomial of a nilpotent linear operator?

   (c) When is a nilpotent linear operator diagonalizable?

   (d) Prove that if T is nilpotent, then $I + T$ is invertible.

3. Let
$$A = \begin{pmatrix} 1 & 1 \\ 3 & 4 \end{pmatrix} \begin{pmatrix} \pi & 0 \\ 0 & 47 \end{pmatrix} \begin{pmatrix} 4 & -1 \\ -3 & 1 \end{pmatrix}.$$

   (a) What are the eigenvalues of A? What are the corresponding eigenvectors?

   (b) Compute $([L_A]_\gamma)^3$ for whatever basis γ you want to pick.

4. Suppose that T is an operator on V , and that u and v are eigenvectors for T. If $u+v$ is an eigenvector for T with eigenvalue λ, what can you say about the eigenvalues of u and v?

5. Suppose $p(x) \in P_k(F)$ is given by
$$p(x) = a_k x^k + \cdots + a_1 x + a_0.$$
   Recall that if $A \in M_{n\times n}(F)$, we can define
$$p(A) = a_k A^k + \cdots + a_1 A + a_0 I_n.$$
   Prove that if A is diagonalizable with eigenvalues $\lambda_1, \ldots, \lambda_n$, then $p(A)$ is diagonalizable with eigenvalues $p(\lambda_1), \ldots, p(\lambda_n)$. (Hint: if D is diagonal, what is $p(D)$?)