RESEARCH STATEMENT

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1. Overview

The focus of my research is mathematical conformal field theory (CFT), particularly chiral, 1+1 dimensional CFT. I am especially interested in Graeme Segal's geometric proposal for the axioms of a conformal field theory ("Segal CFT"), and how Segal CFT relates to other axiomatizations of chiral CFT (i.e. conformal nets and vertex operator algebras). While it is believed that these different notions of CFT should have the same features, there are very few theorems that make their relationship precise. There are theoretical reasons, as well as preliminary results, which indicate that Segal CFT can be used to "interpolate" between the algebraic notion of a vertex operator algebra and the analytic notion of a conformal net. However, the development of Segal CFT has been slowed by the lack of examples, especially examples with analytic properties (i.e. built out of Hilbert spaces and trace class operators).

My results include (paper in preparation):

- The rigorous construction of the free fermion Segal CFT with analytic properties, the first such example.
- Obtaining the free fermion conformal net and vertex operator algebra as "boundary values" of Segal CFT, via bounded operators.
- The construction of genus zero Segal CFT for $SU(N)_\ell$ WZW models, and obtaining the $SU(N)_\ell$ VOA, its modules, and their intertwiner spaces as a boundary value.

Ongoing/future work:

- Construction of $SU(N)_\ell$ Segal CFT in arbitrary genus.
- Classes of constructions of Segal CFT. (e.g. lattice models, vertex operator algebras, cosets, orbifolds, etc.)
- Systematic relationship between fusion rules for conformal nets and vertex operator algebas, using Segal CFT and bounded operators.
- Extension of Segal CFT models to boundary with corners.

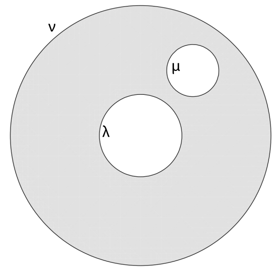
2. Background

The development of conformal field theory was inspired both by physical motivations (critical phenomena of lattice models, string theory) and mathematical motivations (subfactors, three manifold invariants). As a result, there have been several attempts to mathematically axiomatize the physical notion of a conformal field theory, with the best developed being vertex operator algebras (VOAs) and conformal nets.

In an unfinished manuscript from the 1980s (later published with small modifications [Seg04]), Graeme Segal gave a new geometric definition for a conformal field theory.

Roughly speaking, Segal proposed that a (rational, chiral) CFT should consist of a finite set of labels ("superselection sectors"), a Hilbert space for each label, and a finite-dimensional space of trace class maps assigned to complex cobordisms between labelled boundary circles. Moreover, this assignment should be compatible with the gluing of Riemann surfaces and the composition of linear maps.

$$\Sigma =$$



$$\rightsquigarrow E(\Sigma) \subseteq \mathcal{I}^1(H_\lambda \otimes H_\mu, H_\nu).$$

While Segal's definition provided new insight into the connections between CFT and other fields (e.g. topological modular forms [ST04], string theory [MS89], etc.) its development was slowed by an absence of examples. In fact, prior to my work it does not seem that there were any rigorously constructed examples of Segal CFTs using the original definition with Hilbert spaces and trace class maps (however, see [BK01, Pos03]). In Section 3, I will discuss my constructions, as well as future work aimed at constructing further examples.

One of the major benefits of Segal CFT is that it allows us to relate other axiomatizations of CFT, namely vertex operator alegbras and conformal nets. A VOA consists of a vector space of states, and an operator valued (formal) distribution $Y(a, z)$ assigned to each state a. The parameter z should be thought of as a formal variable (living, perhaps, on the complex unit circle S^1).

A conformal net is an assignment of von Neumann algebras to intervals $I \subset S^1$, subject to certain natural conditions. These algebras should be thought of as the algebras of local observables, and philosophically they can be obtained from a vertex operator algebra by integrating these distributions against functions whose support is localized in a given interval. There is ongoing work by Carpi, Kawahigashi, Longo, and Wiener that shows that a conformal net can indeed be obtained by this process.

One of the key features of both VOAs and conformal nets are the notions of modules and representations, respectively, which correspond to the physical notion of superselection sectors. In both cases, there is a notion of composition for sectors, which gives rise to a representation ring ("fusion ring") generated by the irreducible sectors. While one can construct examples of VOAs and conformal nets that should correspond to the same physical model, the problem of directly relating the fusion rules between settings has proven quite difficult. In Section 4, we will discuss a strategy for accomplishing this task using Segal CFT following a conjecture of Wassermann [Was98].

3. Construction of Segal CFTs

We will be particularly interested in constructing Segal CFTs where the operators assigned are manifestly trace class, and for which the analytic features come as a part of the construction. The focus on analytic properties is essential for understanding the behavior of the theory as it degenerates to a VOA or conformal net (see Section 4), or as it degenerates to allow corners on the boundary (see Section 5).

My first construction was the following.

Theorem 3.1 (Construction of the free fermion)**.** *There exists a Segal CFT for the free fermion that assigns Hilbert spaces to boundary circles and trace class maps to Riemann surfaces. The Hilbert space is fermionic Fock space, and the maps correspond to the determinant line of the Hardy space of the surface.*

In [Seg04], Segal remarks that the most important examples to construct are the WZW models for compact Lie groups, as all other models are conjectured to be related to these via operations natural to conformal field theory. At present, I am working to construct Segal CFTs for the $SU(N)_\ell$ WZW models.

Theorem 3.2 (Construction of genus zero $SU(N)_\ell$ model)**.** *There exists a Segal CFT for the $SU(N)_\ell$ model in genus zero that assigns Hilbert spaces to boundary circles and trace class maps to Riemann surfaces. The Hilbert spaces and maps are obtained by restricting the free fermion to irreducible representations of the loop group $LSU(N)$.*

The techniques for proving this theorem follow the philosophy of A. Wassermann that analytic properties for the $SU(N)_\ell$ models can be obtained via the inclusion $SU(N)_\ell \subset U(N\ell)_1$ (i.e. via the study of the free fermion). Wassermann used this strategy to great effect in proving the rationality of the $SU(N)_\ell$ conformal nets [Was98]. This leads to the following natural directions for future work.

Problem 3.3. Use the free fermion Segal CFT to construct the Segal CFT for $SU(N)_\ell$ models in higher genus.

Project 3.4. Develop constructions for producing new Segal CFTs from old Segal CFTs. Constructions of particular interest are coset constructions, orbifolds, and conformal inclusions.

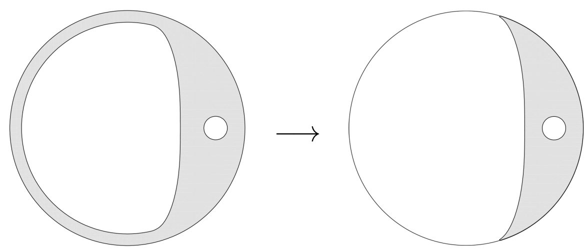
Project 3.5. Develop constructions for producing Segal CFTs from different kinds of objects. Objects of particular interest are lattices and vertex operator algebras.

Similar projects have been undertaken successfully for conformal nets. Coset and orbifold constructions for conformal nets were developed by Xu [Xu00, Xu01], and conformal nets associated to lattices were constructed by Dong and Xu [DX06]. Carpi, Kawahigashi, Longo, and Wiener have constructed a conformal net from a vertex operator algebra, but the problem of transporting the fusion rules between conformal nets and vertex operator algebras remains quite difficult. In the next section, I will discuss how Segal CFT could be used to solve this problem.

4. Fusion rules: Segal CFT at the boundary

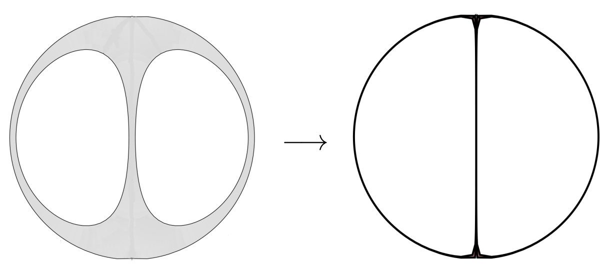
While the theory of VOAs and conformal nets are expected to coincide (modulo technical hypotheses), the development of these two fields has been almost entirely independent. Landmark results like the modularity of the tensor category of representations of rational models [KLM01, Hua05] were obtained separately. The classification of rational models with central charge less than 1 has been accomplished in the context of conformal nets [KL04], but remains open for VOAs (although there has been some progress [DZ08]. It would be extremely beneficial to the development of CFT to have a mechanism for transporting results between these two contexts, and Segal CFT fills this role. The relationship between VOAs and genus zero Segal CFTs has been studied by Huang [Hua98], but in a purely algebraic context that is incompatible with conformal nets. It is for this reason that we have insisted on constructing Segal CFT with nice analytic properties (i.e. using Hilbert spaces and trace class maps).

One should be able to recover a conformal net from a Segal CFT by considering limits of the form

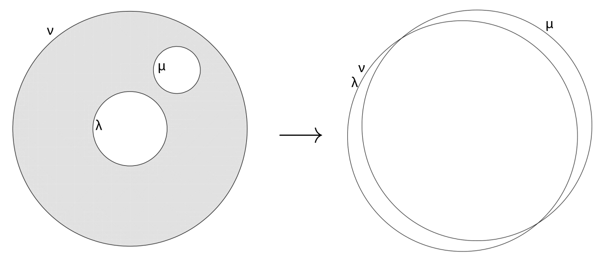
(1) 

with convergence in the strong operator topology on maps $H \otimes H \to H$. That is, we wish to assign a bounded operator T to the limiting "surface" above, where the incoming and outgoing boundary overlap on an interval I. Now one assigns to the complementary interval I^c the von Neumann algebra generated by operators of the form $T(\cdot \otimes \xi)$ with $\xi \in H$. This procedure should produce a conformal net as we vary the interval I.

We also expect to obtain the fusion of sectors of conformal nets in a similar manner. In [Was98], Wassermann defined and computed the fusion of representations of the $SU(N)_\ell$ nets using the Connes-Sauvageot relative tensor product ("Connes' fusion"). He conjectured that this definition of fusion should be a unitary boundary value of Segal's notion, as one takes the limit

(2)  .

On the other hand, a Segal CFT should produce a vertex operator algebra, its modules, and the intertwiners for fusion of modules as a limit of pairs of pants

(3) 

as all radii tend to 1.

This leads to the following natural problem.

Project 4.1. Construct (families of) examples of Segal CFTs where one can recover both a vertex operator algebra and a conformal net, and use the Segal CFT to identify their fusion rules.

The theory of the free fermion is the first example where I have taken this approach.

Theorem 4.2. *The limits* (1) *and* (3) *converge for the free fermion Segal CFT constructed in Theorem 3.1, and one recovers the free fermion conformal net, the free fermion VOA and the VOA fusion rules.*

The same techniques should show that (2) converges to Connes' fusion of representations of conformal nets in this model.

The analysis of the free fermion was greatly simplified by the presence of only a single representation. In order to construct a modular functor for more complicated models, one must incorporate their fusion rules.

Theorem 4.3. *The limits* (1) *and* (3) *converge for the* $SU(N)_\ell$ *Segal CFT from Theorem 3.2, and one recovers the vacuum sector of the* $SU(N)_\ell$ *conformal net, the* $SU(N)_\ell$ *VOA, its modules, and the intertwiner spaces for their fusion.*

The limit (2) is more involved in this case, and establishing its convergence would affirmatively answer Wassermann's conjecture. I propose to answer the following question.

Problem 4.4. Obtain Connes' fusion for the free fermion and $SU(N)_\ell$ conformal nets as limits of operators from the corresponding Segal CFTs, and use this to identify the fusion rules with the ones from the corresponding VOAs.

While it is already known that these models have the same fusion rules in the VOA and conformal net settings, this is only known *a posteriori*. Interpolation via Segal CFTs provide an *a priori* mechanism for identifying the fusion rules, and the techniques involved would be applicable to other models.

5. Extended Segal CFT

In [ST04], Stolz and Teichner propose an extension of Segal's definition of a conformal theory that "extends down to points." More precisely, one allows the boundary circle to decompose into incoming and outgoing line segments, to which one associates von Neumann algebras (in practice, the hyperfinite factor of type III_1). Now instead of a Hilbert space assigned to the boundary circle, one has a bimodule over these von Neumann algebras. In the case when the boundary circle is entirely incoming or entirely outgoing, this reduces to the type I case of ${}_{B(\mathcal{H})}\mathcal{H}$ or $(\mathcal{H}^*)_{B(\mathcal{H})}$.

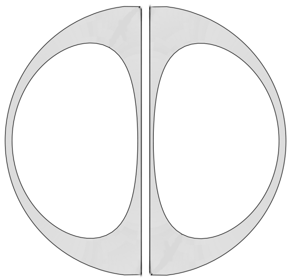
The principal technical obstruction to extending Segal CFT in this manner is that one must deal with Riemann surfaces whose boundary has corners. As this is an analytic problem, one expects that extending a Segal CFT in this manner should be possible if the theory possesses sufficiently nice analytic properties.

Project 5.1. Show that the Segal CFTs constructed in Section 3 can be extended to Riemann surfaces whose boundary has corners.

As a first step, I propose to address the following problem.

Problem 5.2. Extend the Segal CFTs for the free fermion and $SU(N)_\ell$ to planar domains with polygonal boundaries.

Already, Problem 5.2 encodes a great deal of the complexity of the Segal CFT, as one can obtain the field operators by gluing annuli along part of their boundaries.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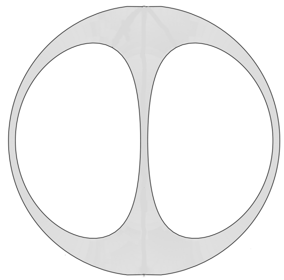
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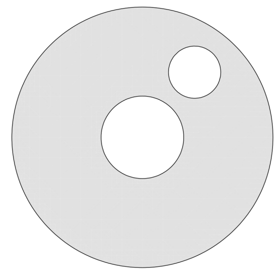
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On the other hand, it should be possible to solve this problem via a limiting process like in Section 4, or via the machinery of fermionic second quantization that was used to prove Theorem 3.1.

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