1. Suppose X and Z are locally compact Hausdorff, and Y is any space. Prove that the isomorphism of sets $\mathsf{Top}(Z \times X, Y) \to \mathsf{Top}(Z, \mathsf{Top}(X, Y))$ is a homeomorphsim of spaces (where all the spaces of functions are given the compact-open topology).

2. Let X be any topological space and let Y be compact Hausdorff. For a set map $f : X \to Y$, f is continuous iff its graph $G = \{(x, f(x) \in X \times Y)\}$ is closed.

3. The following is a fact that you may assume (or prove!)

> *Every n-dimensional Hausdorff topological vector space over $\mathbb{R}$ is homeomorphic to $\mathbb{R}^n$ with its usual topology.*

Your problem: Let V be a topological vector space and $f : V \to \mathbb{R}^n$ be linear. Prove that f is continuous iff $\ker(f)$ is compact.

4. Let X be a topological space. There's a little logical universe contained in X. A *predicate* on X is an open set $U \subseteq X$ or if you prefer, an open embedding of a topological space $U \hookrightarrow X$. A predicate U is *true* at a point $x \in U$. Entailment is containment $U \vdash V$ means $U \subseteq V$. Conjunction is intersection $U \wedge V := U \cap V$, disjunction is union $U \vee V = U \cup V$, and implication $U \Rightarrow V$ is defined to be the largest open set W so that $W \cap U \subset V$. The universally true proposition is $\top := X$, the universally false is $\bot := \varnothing$. Negation is defined by $\neg U = U \Rightarrow \varnothing$.

(a) Prove $U \wedge V \vdash W$ if and only if $U \vdash V \Rightarrow W$.

(b) Prove that $U \wedge (U \Rightarrow V) \vdash V$.

(c) Here's a way to interpret the previous statement: if we make a category out of X where the objects are open sets and the morphisms are inclusions, then for each fixed U, the map $V \mapsto V \wedge U$ has a right adjoint $W \mapsto U \Rightarrow W$.

(d) Prove that $U \vee \neg U = X \setminus \partial U$ and $\neg\neg U = \operatorname{int}(\overline{U})$.

(e) Prove that $\neg\neg U = U$ and $U \vee \neg U = \top$ iff X is discrete.

5. Let X be compact and Y be a metric space. We say a family of functions $F \subseteq \mathsf{Top}(X, Y)$ is equicontinuous iff for all $x \in X$ and for all $\epsilon > 0$ there exists a neighborhood U of x so that

$$d(f(x), f(x')) \leq \epsilon \text{ for all } x, x' \in U \text{ and for all } f \in F.$$

For a compact set $K \subset X$ and an open set $U \subset Y$, let $S(K, U)$ denote the open set in the compact-open topology given by

$$S(K, U) = \{f : X \to Y : f(K) \subseteq U\}.$$

(a) (This problem requires an ϵ argument). Suppose F is equicontinuous. Prove that if $K \subseteq X$ is compact and $U \subseteq Y$ is open, then for any $f \in F$ with $f(K) \subseteq U$, there exists a set V that is open in the product topology with $f \in V \subseteq S(K, U) \cap F$.

(b) Let F be any family of functions from X to Y. Prove that F has compact closure in the product topology iff for each $x \in X$, the sets $F_x = \{f(x) : f \in F\}$ all have compact closure in X.

(c) Prove that a subset of functions F has compact closure in the topology induced by the sup norm if and only if it is equicontinuous and pointwise bounded.

(d) Give a counterexample: Let X be compact, (Y, d) be a metric space and $\{f_n\}$ be a sequence of functions in $\mathsf{Top}(X, Y)$. If $\{f_n\}$ is equicontinuous and if for each $x \in X$ the set $\{f_n(x)\}$ is bounded, then $\{f_n(x)\}$ has a subsequence that converges uniformly.

(e) Consider the family $\mathcal{F} = \{f_a : 0 < a \leq 1\}$ where $f_a(x) = 1 - \frac{x}{a}$. Is $\mathcal{F}$ a compact subspace of $\mathsf{Top}([0, 1], \mathbb{R})$?

(f) (If you know Cauchy's integral formula.) Let $0 < r < R$ and suppose F is a family of uniformly bounded holomorphic functions on the disc $D(0, R) = \{z \in \mathbb{C} : |z| \leq R\}$. Prove that any sequence $\{f_n\}$ in F has a subsequence whose restrictions to the smaller disc $\overline{D(0, r)}$ converges to a holomorphic function.