

**1.** If  $X$  and  $Z$  are locally compact Hausdorff, then for any space  $Y$ , the isomorphism of sets  $\text{Top}(Z \times X, Y) \rightarrow \text{Top}(Z, \text{Top}(X, Y))$  is a homeomorphsim of spaces.

**2.** Let  $X$  be compact and  $Y$  be Hausdorff. For a set map  $f : X \rightarrow Y$ ,  $f$  is continuous iff its graph  $G = \{(x, f(x)) \in X \times Y\}$  is closed.

**3.** It is a fact that you may prove or assume that

*Every  $n$ -dimensional Hausdorff topological vector space is homeomorphic to  $\mathbb{R}^n$  with its usual topology.*

Your problem: Let  $V$  be a topological vector space and  $f : V \rightarrow \mathbb{R}^n$  be linear. Prove that  $f$  is continuous iff  $\ker(f)$  is compact.

**4.** Let  $X$  be a topological space. There's a little logical universe contained in  $X$ . A *predicate* on  $X$  is an open set  $U \subseteq X$  or if you prefer, an open embedding of a topological space  $U \hookrightarrow X$ . A predicate  $U$  is *true* at a point  $x \in U$ . Entailment is containment  $U \vdash V$  means  $U \subseteq V$ . Conjunction is intersection  $U \wedge V := U \cap V$ , disjunction is union  $U \vee V = U \cup V$ , and implication is defined  $U \Rightarrow V$  is defined to be the largest open set  $W$  so that  $W \cap U \subset V$ . The universally true proposition is  $\top = X$ , the universally false is  $\perp = \emptyset$ . Negation is defined by  $\neg U = U \Rightarrow \emptyset$ .

- (a) Prove  $U \wedge V \vdash W$  if and only if  $U \vdash V \Rightarrow W$ .
- (b) Prove that  $U \wedge (U \Rightarrow V) \vdash V$ .
- (c) Here's a way to interpret the previous statement: if we make a category out of  $X$  where the objects are open sets and the morphisms are inclusions, then for each fixed  $U$ , the map  $V \mapsto V \wedge U$  has a right adjoint  $W \mapsto U \Rightarrow W$ .
- (d) Prove that  $U \vee \neg U = X \setminus \partial U$  and  $\neg\neg U = \text{int}(U)$ .
- (e) Prove that  $\neg\neg U = U$  and  $U \vee \neg U = \top$  iff  $X$  is discrete.

**5.** Let  $X$  be compact and  $Y$  be a metric space. We say a family of functions  $F \subseteq \text{Top}(X, Y)$  is equicontinuous iff for all  $x \in X$  and for all  $\epsilon > 0$  there exists a neighborhood  $U$  of  $x$  so that

$$d(f(x), f(x')) \leq \epsilon \text{ for all } x, x' \in U \text{ and for all } f \in F.$$

For this problem suppose  $F$  is an equicontinuous family of functions. For a compact set  $K \subset X$  and an open set  $U \subset Y$ , let  $S(K, U)$  denote the open set in the product topology given by

$$S(K, U) = \{f : X \rightarrow Y : f(K) \subseteq U\}.$$

- (a) Suppose  $F$  is equicontinuous. Prove that if  $K \subseteq X$  is compact and  $U \subseteq Y$  is open, then for any  $f \in F$  with  $f(K) \subseteq U$ , there exists a set  $V$  that is open in the product topology with  $f \in V \subseteq S(K, U) \cap F$ .
- (b) Let  $F$  be an equicontinuous family of functions from  $X$  to  $Y$ . Prove that  $F$  has compact closure in the product topology iff for each  $x \in X$ , the sets  $F_x = \{f(x) : x \in X\}$  all have compact closure in  $X$ .
- (c) Prove that a subset of functions  $F$  has compact closure in the topology induced by the sup norm if and only if it is equicontinuous and pointwise bounded.
- (d) Give a counterexample: Let  $X$  be compact,  $(Y, d)$  be a metric space and  $\{f_n\}$  be a sequence of functions in  $\text{Top}(X, Y)$ . If  $\{f_n\}$  is equicontinuous and if for each  $x \in X$  the set  $\{f_n(x)\}$  is bounded, then  $\{f_n(x)\}$  has a subsequence that converges uniformly.
- (e) Consider the family  $\mathcal{F} = \{f_a : 0 < a \leq 1\}$  where  $f_a(x) = 1 - \frac{x}{a}$ . Is  $\mathcal{F}$  a compact subspace of  $\text{Top}([0, 1], \mathbb{R})$ ?