Definition. A morphism $f : X \to Y$ in a category is a *monomorphism* (mono) if $f \circ g = f \circ h$ implies $g = h$ for all $g, h : Z \to X$. A morphism $f : X \to Y$ in a category is *split* mono if there exists $g : Y \to X$ with $gf = \mathrm{id}_X$.

Definition. A morphism $f : X \to Y$ in a category is an *epimorphism* (epi) if $g \circ f = h \circ f \Rightarrow g = h$ for all $g, h : Y \to Z$. A morphism $f : X \to Y$ in a category is *split* epi if there exists $g : Y \to X$ with $fg = \mathrm{id}_Y$.

1. A subset A of a topological space X is *dense* if and only if the smallest closed set containing A is all of X.

(a) Prove that A is dense iff for every nonempty open set U in X, $U \cap A \neq \emptyset$.

(b) Prove that having a countable dense subset is a topological property (it's sometimes called *second countable*.)

2. On $\mathbb{R}^3$ define the *Lumberjack metric* d_L by

$$d_L\big((x,y,z),(x',y',z')\big) = \begin{cases} |z - z'|, & \text{if } (x,y) = (x',y'), \\ |z| + \sqrt{(x-x')^2 + (y-y')^2} + |z'|, & \text{otherwise.} \end{cases}$$

(a) Prove or disprove: There is a countable dense subset of $\mathbb{R}^3$ with the lumberjack topology.

(b) Prove that the d_L–topology is strictly finer than the Euclidean topology.

(c) Prove or disprove: The inclusion of $i : \mathbb{R}^2 \to \mathbb{R}^3$ defined by $i(x,y) = (x,y,0)$ defines an embedding of $\mathbb{R}^2$ with the usual topology into $\mathbb{R}^3$ with the lumberjack topology.

3. Monomorphisms in **Top**.

(a) Prove that monomorphisms in **Top** are exactly the injective continuous maps.

(b) Split monos in **Top** are given many names: *embeddings, sections, inclusions that have retractions...* Just give a couple of example of monomorpisms in **Top** that are not split.

4. Define the *Sierpiński space* $\mathbb{S} = \{0,1\}$ with topology $\{\varnothing, \{1\}, \{0,1\}\}$.

(a) Prove that continuous maps $Y \to \mathbb{S}$ are exactly characteristic functions of open sets of Y. That is, a function $\chi : Y \to \mathbb{S}$ is continuous iff $\chi^{-1}(\{1\})$ is open in Y.

(b) Let $f : X \to Y$ be a set function between spaces X and Y. Show that f is continuous iff for every continuous $\chi : Y \to \mathbb{S}$, the composite $\chi \circ f : X \to \mathbb{S}$ is continuous.

5. Epimorphisms in **Top**.

(a) Prove that if $f : X \to Y$ is an epi in **Top**, then $f(X)$ is dense in Y. *Hint:* Suppose $f(X)$ is not dense and construct distinct continuous maps $g, h : Y \to \mathbb{S}$ with $g \circ f = h \circ f$.

(b) Give an explicit example showing that there exists a continuous $f : X \to Y$ with dense image that is *not* epi in **Top**.

6. More reasons the category **Top** is different than **Set**.

(a) In the category of sets, if there exists a monomorphism $f : X \to Y$ and a monomorphism $g : Y \to X$ then there exists an isomorphism $h : X \to Y$ (this is called the Canter-Schroder-Bernstein theorem). Prove, by example, that there is no such theorem in topology.

(b) In the category of sets, if $f : X \to Y$ is both a monomorphism and an epimorphism then f is an isomorphism. Prove, by example, that there are continuous functions $f : X \to Y$ that are both monic and epic but are not isomorphisms.

7. Split epimorphisms are called retracts or retractions. Prove that split epimorphisms in **Top** are precisely surjective quotient maps.

8. Let X and Y be topological spaces and let $A \subseteq X$ and $B \subseteq Y$ be subsets. There are two ostensibly different ways to put a topology on the set $A \times B$.

(a) First give A and B the *subspace* topologies from X and Y and then put the product topology on $A \times B$.

(b) Second, give the subset $A \times B$ of $X \times Y$ the subspace topology inherited from the product topology on $X \times Y$.

Prove or disprove: these two constructions yield the same topology.

9. Let $q : X \to Y$ be a surjection and let Z any space. There are two ostensibly different ways to get a topology on $Y \times Z$.

(a) First take the quotient topology on Y defined by q, then take the product with Z.

(b) Give $X \times Z$ the product topology and then give $Y \times Z$ the quotient topology defined by $q \times \mathrm{id}_Z : X \times Z \to Y \times Z$.

Do you think these two constructions are the same?