GEOMETRIC STRUCTURES IN $\overline{\mathbb{R}}$-ENRICHED ADJUNCTIONS

JUAN LUIS GASTALDI, SAMANTHA JARVIS, THOMAS SEILLER, AND JOHN TERILLA

Abstract. A real $m \times n$ matrix M determines tropical row and column polytopes in tropical projective spaces $\mathbb{TP}^{m-1}$ and $\mathbb{TP}^{n-1}$, with canonical polyhedral cell structures that are naturally dual. We reinterpret this picture via Isbell duality: viewing M as an $\overline{\mathbb{R}}$-enriched profunctor $M : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \overline{\mathbb{R}}$, we study the associated order-reversing Isbell adjunction $M^* \dashv M_*$ and its fixed-point locus, the nucleus $\mathrm{Nuc}(M)$. After projectivization, $\mathbb{P}\mathrm{Nuc}(M)$ carries two interacting geometries.

On the metric side, $\overline{\mathbb{R}}$-enrichment induces a canonical Hilbert projective–type (max-spread) metric on projective (co)presheaves, and we show that the projective Isbell maps identify the presheaf and copresheaf realizations of $\mathbb{P}\mathrm{Nuc}(M)$ by mutually inverse isometries. On the polyhedral side, in the discrete real setting the Isbell inequalities cut out a canonical polyhedral decomposition of $\mathbb{P}\mathrm{Nuc}(M)$ recovering the usual tropical cell structure.

Our main new ingredient is a pointwise invariant of a nucleus point (f, g): the nonnegative *gap matrix* $\delta^{(f,g)}(c, d) = M(c, d) - f(c) - g(d)$. Its zero pattern determines the cell containing (f, g), while its positive entries compute exact metric distances to the boundary strata where additional inequalities become tight (Events Theorem). This distance-to-wall principle refines cells into order chambers and supports a constructible tower of complete lattices obtained by thresholding $\delta^{(f,g)}$.

Contents

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The authors contributed equally to this work.

1. Introduction

It is known that the tropical column span and tropical row span of a real $m \times n$ matrix determine polyhedral cell complexes in tropical projective spaces $\mathbb{TP}^{m-1}$ and $\mathbb{TP}^{n-1}$, and that these two complexes are related by a natural duality. The aim of this paper is to refine this polyhedral picture by endowing the underlying space with a canonical projective metric and by making the interaction between the metric geometry and the cell structure completely explicit. Our refinement is governed by a simple pointwise invariant: a *nonnegative* matrix attached to a point (a tropical analogue of a slack matrix) whose zero entries determine the cell containing the point, and whose positive entries compute the exact radii at which intrinsic metric balls first meet the walls of the polyhedral complex.

We formulate this refinement in the language of Isbell duality. Let $\overline{\mathbb{R}} = ([-\infty, \infty], \leq, +)$ be the monoidal poset of extended real numbers and let

$$M : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \longrightarrow \overline{\mathbb{R}}$$

be an $\overline{\mathbb{R}}$-enriched profunctor. The Isbell conjugates M^* and M_* form an order-reversing adjunction between presheaves on $\mathcal{C}$ and copresheaves on $\mathcal{D}$. Its nucleus $\mathrm{Nuc}(M)$ consists of the fixed points (f, g) with $g = M^*f$ and $f = M_*g$. In the discrete real setting, where $\mathcal{C}$ and $\mathcal{D}$ are finite sets and M is a matrix with entries in $\mathbb{R}$, these fixed points are precisely the simultaneous solutions of the inequalities

$$f(c) + g(d) \leq M(c, d),$$

together with the condition that each row and column attains equality somewhere. The construction is invariant under an $\mathbb{R}$ action $(f, g) \mapsto (f + \lambda, g - \lambda)$, so the geometry of interest lives on the projectivization $\mathbb{P}\mathrm{Nuc}(M)$.

Two complementary geometries meet on $\mathbb{P}\mathrm{Nuc}(M)$.

Metric geometry. The $\overline{\mathbb{R}}$-enrichment supplies a canonical "max-spread" (Hilbert projective–type) distance on projective (co)presheaves. Theorem 18 identifies the two realizations of $\mathbb{P}\mathrm{Nuc}(M)$—on the presheaf and copresheaf sides—by showing that the projective Isbell maps induced by M^* and M_* are inverse isometries. This statement is gauge-invariant and supplies the metric framework for what follows.

Polyhedral geometry. When $\mathcal{C}$ and $\mathcal{D}$ are finite and M has real entries, the projective nucleus $\mathbb{P}\mathrm{Nuc}(M)$ carries a canonical decomposition into polyhedral cells, governed by which inequalities are attained as equalities; this recovers the type decomposition of a tropical polytope.

The novelty of the paper is the interaction between these two structures. Given a nucleus point (f,g), we form its *gap matrix*

$$\delta^{(f,g)}(c,d) := M(c,d) - f(c) - g(d) \in [0,\infty],$$

which records the gaps of the inequalities $f(c)+g(d) \le M(c,d)$. The zeros of the gap matrix record the *witness pairs* at this point: $Z(f,g) = \{(c,d) \mid \delta^{(f,g)}(c,d) = 0\}$. Fixing this zero pattern determines the polyhedral cell containing the projective class $[(f,g)]$.

A main result of this paper is Theorem 49, which identifies each positive entry as an exact distance:

$$d_{\mathbb{P}\mathrm{Nuc}(M)}\left([(f,g)], \mathcal{E}_{c,d}\right) = \delta^{(f,g)}(c,d),$$

where $\mathcal{E}_{c,d}$ is the locus of points in $\mathbb{P}\mathrm{Nuc}(M)$ where (c,d) is a witness pair. Thus the gap matrix records both the combinatorics of the cell decomposition, via its zeros, and the metric distances to its bounding faces, via its positive entries. This leads naturally to a refinement of the polyhedral cells by *order chambers*, on which the weak order relations among the positive gap values are constant; ties correspond to simultaneous boundary events.

A reinterpretation of the same data gives a lattice-theoretic picture. Thresholding the gap matrix at $\varepsilon \ge 0$ defines a Boolean relation on $\mathcal{C} \times \mathcal{D}$ by declaring $c \sim d$ when $\delta^{(f,g)}(c,d) \le \varepsilon$. Taking nuclei of these relations yields a "tower" of complete lattices indexed by ε which can change only when ε crosses one of the event radii supplied by Theorem 49; the order-chamber stratification controls the relative order in which these changes occur across $\mathbb{P}\mathrm{Nuc}(M)$.

A running example. To keep the discussion concrete, we will repeatedly return to the following matrix. Let $C = \{c_0, c_1, c_2\}$ and $D = \{d_1, d_2, d_3, d_4\}$ and set

$$M = \begin{bmatrix} 0.7 & 1.5 & 1.7 & -1.3 \\ 1.2 & 2.6 & 0.1 & 2.2 \\ 2.0 & -1.6 & 2.0 & -2.9 \end{bmatrix}.$$

In this case $\mathbb{P}\mathrm{Nuc}(M)$ is a two-dimensional polyhedral complex and we work in the projective gauge $c_0 = 0$. We focus on the point $[(f,g)]$ where $f = (0,0,0)$ and $g = (0.7, -1.6, 0.1, -2.9)$. At this point, the gap matrix $\delta = \delta^{(f,g)}$ is given by

$$\delta = \begin{bmatrix} 0 & 3.1 & 1.6 & 1.6 \\ 0.5 & 4.2 & 0 & 5.1 \\ 1.3 & 0 & 1.9 & 0 \end{bmatrix}.$$

whose entries satisfy

$$0 < 0.5 < 1.3 < 1.6 = 1.6 < 1.9 < 3.1 < 4.2 < 5.1.$$

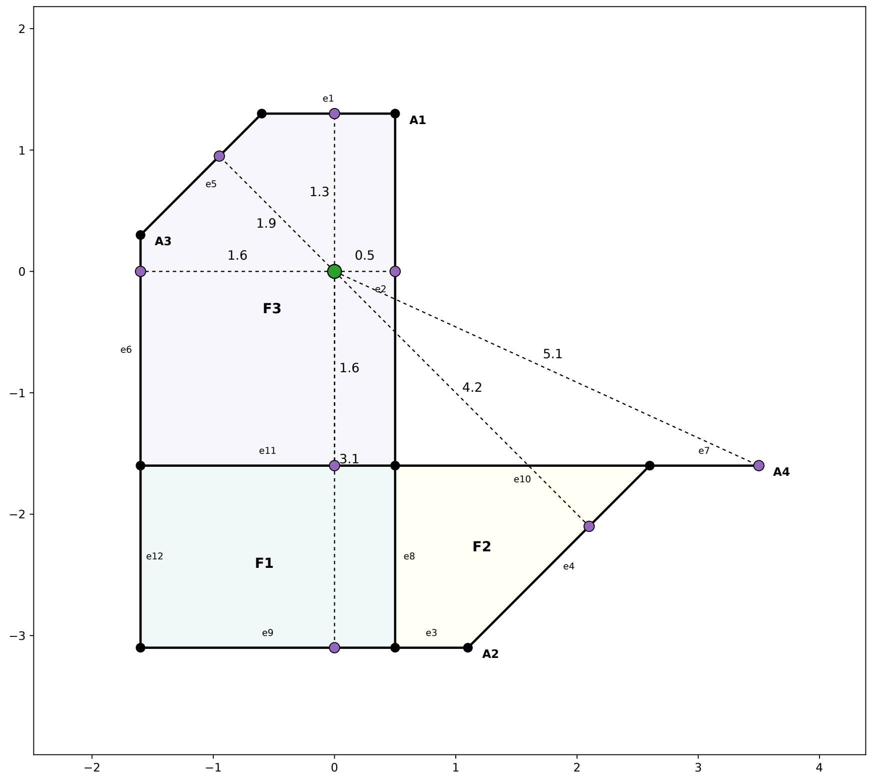

FIGURE 1. The projective nucleus $\mathbb{P}\mathrm{Nuc}(M)$ for the running example (in an affine chart), with its polyhedral decomposition. Here, the basepoint is marked in green. The other marked points illustrate Theorem 49: a positive gap value at a basepoint computes the distance to the locus where a wall is encountered. The shaded hexagons are metric balls in the max-spread metric.

The combinatorial type of the cell in which $[(f, g)]$ lies is given by the zero pattern $\{(c_0, d_1), (c_2, d_2), (c_1, d_3), (c_2, d_4)\}$ and the positive entries describe distances to walls. Figure 1 illustrates this "gap value as distance-to-wall" phenomenon.

1.1. **Relation to existing work.** The idea of viewing generalized metric spaces as categories enriched over an ordered monoid goes back to Lawvere [Law73]. The adjunction between covariant and contravariant presheaves that bears Isbell's name appears in Isbell's 1960 paper on adequate subcategories [Isb60], and Avery and Leinster [AL21] give a systematic treatment over an arbitrary base. In the concrete setting of Lawvere metric spaces, Willerton developed the enriched Isbell completion and its variants in detail, both expositively [Wil14] and in more systematic analyses [Wil13, Wil15], where Willerton connects Isbell duality with the Legendre-Fenchel transform. This in turn connects with the c-transform in optimal transport: for cost $c = M$ (up to sign conventions), the Isbell maps are c-transforms and fixed points correspond to the double c-transform, i.e. the c-concave envelope, see [AG13].

On the tropical side, Develin and Sturmfels introduced the polyhedral theory of tropical convexity and its canonical cell decompositions into combinatorial types [DS04], and later Elliott and Fujii observed that Isbell-type nuclei in quantale-enriched settings provide a natural categorical home for tropical polytopes [Ell17, Fuj19]. With particular applications in mind, Bradley, Terilla and Vlassopoulos use enrichment over $[0, 1] \cong [0, \infty]$ to organize structures in language [BTV22], and Gaubert and Vlassopoulos advance this circle of ideas in the language of directed metrics and tropical polyhedra [GV24]. Ingredients in our main Theorem 49 have precedents in the work of Gauber and Katz [GK06, GK11]. While we have applications in mind similar to those in [BTV22, GV24], background for which is reviewed in [BGT24], we do not discuss those applications here. Related structural developments of nuclei in enriched settings with additional compatible monoidal data appear in Jarvis's thesis [Jar25], with connections to linear logic in the spirit of Seiller's habilitation thesis [Sei24]; see also our companion work [GJST25]. In contrast, the present paper focuses on the projective metric geometry and its interaction with the witness polyhedral decomposition.

Our starting point is close in spirit to all these works, but we emphasize a two-sided base $\overline{\mathbb{R}}$ and the resulting projective metric geometry, and we isolate the gap-matrix as the mechanism linking metric balls to polyhedral boundaries. Small differences in conventions (like using $\overline{\mathbb{R}}$ instead of $[0, \infty]$ as a base) matter here, so we include a self-contained account of the enriched category theory tailored to the setting we use.

Organization of the paper. Section 2.2 fixes conventions for $\overline{\mathbb{R}}$ and $\overline{\mathbb{R}}$-enriched categories, functors, and profunctors. Section 2.5 reviews the Isbell adjunction and the nucleus. Section 3 develops the projective metric geometry and proves Theorem 18. Section 4 develops the witness polyhedral structure, proves Theorem 49, introduces order chambers, and explains the associated towers of concept lattices.

2. The nucleus of an $\overline{\mathbb{R}}$-profunctor

In this section we fix notation for enrichment over the symmetric monoidal closed poset $\overline{\mathbb{R}}$ and recall the Isbell adjunction associated to a profunctor $M : \mathcal{C} \nrightarrow \mathcal{D}$. The induced closure operators on presheaves and copresheaves have the fixed points which comprise the an $\overline{\mathbb{R}}$-category called the *nucleus* $\mathrm{Nuc}(M)$.

2.1. The extended real numbers $\overline{\mathbb{R}}$.

Let $\overline{\mathbb{R}}$ be the poset category on the extended real numbers

$$\overline{\mathbb{R}} = [-\infty, +\infty],$$

with a unique morphism $x \to y$ if and only if $x \leq y$. Finite products and coproducts are given by

$$x \times y = \min\{x, y\},$$
$$x \sqcup y = \max\{x, y\},$$

and arbitrary limits and colimits are infima and suprema. In particular, $\overline{\mathbb{R}}$ is complete and cocomplete. Its initial object is $-\infty$ and its terminal object is $+\infty$.

We use the symmetric monoidal structure on $\overline{\mathbb{R}}$ given by addition. We extend the usual addition on $\mathbb{R}$ to $\overline{\mathbb{R}}$ by declaring $-\infty$ to be absorbing:

$$-\infty + y = -\infty,$$

for all $y \in \overline{\mathbb{R}}$. This extension is equivalent to requiring that for each $y \in \overline{\mathbb{R}}$ the translation

$$\begin{aligned}(+y)\colon \overline{\mathbb{R}} &\to \overline{\mathbb{R}},\\ x &\mapsto x + y\end{aligned}$$

preserves arbitrary colimits (suprema) with the relevant detail being that, it preserves the initial object $-\infty$. For $y = +\infty$ this gives $-\infty + \infty = -\infty$, and hence also $\infty + (-\infty) = -\infty$ by commutativity.

Since each translation $+y$ preserves colimits, it has a right adjoint, denoted $[y, -]$. Thus $[y, z]$ is characterized by

$$x + y \leq z \iff x \leq [y, z].$$

We write $z - y$ for $[y, z]$. On finite reals this agrees with ordinary subtraction, but in general it is residuation:

$$z - y = \sup\{x \in \overline{\mathbb{R}} \mid x + y \leq z\}. \tag{1}$$

For example,

$$\begin{aligned}\infty - \infty &= \infty,\\ -\infty - (-\infty) &= \infty.\end{aligned}$$

In particular, subtraction of $-\infty$ is not the same as addition of $+\infty$, since $-\infty - (-\infty) = \infty$ but $-\infty + \infty = -\infty$.

We work with $\overline{\mathbb{R}}$ rather than Lawvere's base $([0, \infty], \geq, +, 0)$ for two reasons. First, although $[0, \infty]$ and $[-\infty, \infty]$ are isomorphic as ordered topological spaces, the monoidal unit in Lawvere's base is the top element. This leads to a conical geometry in which residuation is a truncated subtraction. In $\overline{\mathbb{R}}$, by contrast, residuation is not truncated and translation by any finite $\lambda \in \mathbb{R}$ is defined. This translation action on (co)presheaves will later yield an affine geometry and, after quotienting by finite shifts, a projective one. Moreover, the map $x \mapsto -x$ identifies $([0, \infty], \geq, +, 0)$ with the full sub-monoidal poset $[-\infty, 0] \subset \overline{\mathbb{R}}$.

Second, in applications to linear realizability [GJST25, Sei24, Jar25] one meets subsets $A \subseteq \mathcal{C} \times \mathbb{R}$ and the associated function

$$f_A(c) = \sup\{r \in \mathbb{R} \mid (c, r) \in A\}.$$

The values $-\infty$ (when the set of such r is empty) and $+\infty$ (when it is unbounded above) both occur naturally, so it is necessary to allow both infinite endpoints.

2.2. **$\overline{\mathbb{R}}$-categories.** Throughout, $\overline{\mathbb{R}}$ denotes the symmetric monoidal closed poset

$$(\overline{\mathbb{R}}, \leq, +, 0),$$

with internal hom $[x, y]$ characterized by $u + x \leq y \iff u \leq [x, y]$. We write $y - x$ for $[x, y]$; see §2.1 and (1).

Definition 1. A (small) $\overline{\mathbb{R}}$-*category* $\mathcal{C}$ consists of a set $\mathrm{Ob}(\mathcal{C})$ and, for each $c, c' \in \mathrm{Ob}(\mathcal{C})$, a hom-value $\mathcal{C}(c, c') \in \overline{\mathbb{R}}$ satisfying, for all c, c', c'',

$$0 \leq \mathcal{C}(c, c) \qquad \text{(identities)}, \tag{2a}$$

$$\mathcal{C}(c, c') + \mathcal{C}(c', c'') \leq \mathcal{C}(c, c'') \qquad \text{(composition)}. \tag{2b}$$

We call $\mathcal{C}$ *normalized* if $\mathcal{C}(c, c) = 0$ for all objects c.

The base quantale $\overline{\mathbb{R}}$ is itself an $\overline{\mathbb{R}}$-category, with $\overline{\mathbb{R}}(x, y) = y - x$. The *opposite* $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$ and

$$\mathcal{C}^{\mathrm{op}}(c, c') = \mathcal{C}(c', c).$$

Definition 2. An $\overline{\mathbb{R}}$-*functor* $F : \mathcal{C} \to \mathcal{D}$ is a function $\mathrm{Ob}(\mathcal{C}) \to \mathrm{Ob}(\mathcal{D})$ such that

$$\mathcal{C}(c, c') \leq \mathcal{D}(Fc, Fc')$$

for all objects c, c' of $\mathcal{C}$.

Given $\overline{\mathbb{R}}$-categories $\mathcal{C}, \mathcal{D}$, the set of $\overline{\mathbb{R}}$-functors $F, G : \mathcal{C} \to \mathcal{D}$ carries an $\overline{\mathbb{R}}$-enrichment

$$[\mathcal{C}, \mathcal{D}](F, G) = \inf_{c \in \mathcal{C}} \mathcal{D}(Fc, Gc), \tag{3}$$

making $[\mathcal{C}, \mathcal{D}]$ into an $\overline{\mathbb{R}}$-category. When the context is clear we write $[F, G]$ for $[\mathcal{C}, \mathcal{D}](F, G)$.

2.3. **Presheaves, copresheaves, and Yoneda.** A *presheaf* on $\mathcal{C}$ is an $\overline{\mathbb{R}}$-functor $f \colon \mathcal{C}^{\mathrm{op}} \to \overline{\mathbb{R}}$. A *copresheaf* on $\mathcal{D}$ is an $\overline{\mathbb{R}}$-functor $g \colon \mathcal{D} \to \overline{\mathbb{R}}$. We will often regard copresheaves as objects of the opposite $\overline{\mathbb{R}}$-category $[\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}$. Equivalently, we keep the same underlying functions $g \colon \mathcal{D} \to \overline{\mathbb{R}}$, but we reverse their pointwise order. This convention is convenient because the Isbell conjugates of a profunctor $M \colon \mathcal{C} \nrightarrow \mathcal{D}$ (defined in §2.5) then become $\overline{\mathbb{R}}$-functors.

For presheaves $f, f' \colon \mathcal{C}^{\mathrm{op}} \to \overline{\mathbb{R}}$ and copresheaves $g, g' \colon \mathcal{D} \to \overline{\mathbb{R}}$, the enriched homs in the functor categories are computed pointwise from (3):

$$[f, f'] = \inf_{c \in \mathcal{C}} \left(f'(c) - f(c) \right),$$

$$[g, g']_{[\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}} = [\mathcal{D}, \overline{\mathbb{R}}](g', g) = \inf_{d \in \mathcal{D}} \left(g(d) - g'(d) \right).$$

In particular, the underlying order in $[\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]$ is the pointwise order on functions $f \colon \mathcal{C} \to \overline{\mathbb{R}}$, while the underlying order in $[\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}$ is the opposite of the pointwise order on functions $g \colon \mathcal{D} \to \overline{\mathbb{R}}$.

The Yoneda embedding of $\mathcal{C}$ is the $\overline{\mathbb{R}}$-functor

$$\begin{aligned} y \colon \mathcal{C} &\to [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}], \\ c &\mapsto \mathcal{C}(-, c). \end{aligned}$$

Dually, copresheaves on $\mathcal{D}$ may be viewed as presheaves on $\mathcal{D}^{\mathrm{op}}$, and the Yoneda embedding of $\mathcal{D}^{\mathrm{op}}$ is

$$\begin{aligned} \mathcal{D}^{\mathrm{op}} &\to [\mathcal{D}, \overline{\mathbb{R}}], \\ d &\mapsto \mathcal{D}(d, -). \end{aligned}$$

We will use the same assignment, but write it covariantly as a functor

$$\begin{aligned} y^{\mathrm{co}} \colon \mathcal{D} &\to [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}, \\ d &\mapsto \mathcal{D}(d, -). \end{aligned}$$

The enriched Yoneda lemma gives natural equalities

$$\begin{aligned} [\mathcal{C}(-, c), f] &= f(c), \\ [g, \mathcal{D}(d, -)]_{[\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}} &= g(d), \end{aligned}$$

and in particular

$$\begin{aligned} [\mathcal{C}(-, c), \mathcal{C}(-, c')] &= \mathcal{C}(c, c'), \\ [\mathcal{D}(d, -), \mathcal{D}(d', -)]_{[\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}} &= \mathcal{D}(d, d'). \end{aligned}$$

Thus y and y^{co} are fully faithful in the enriched sense.

We will also use the enriched density theorem in its concrete, pointwise form. We recall the relevant colimit notions and then record the formulas.

Let $\mathcal{X}$ be a cocomplete $\overline{\mathbb{R}}$-category. For $r \in \overline{\mathbb{R}}$ and $x \in \mathcal{X}$, write $r \odot x$ for the copower of x by r, characterized by

$$\mathcal{X}\left(r \odot x, x'\right) = \left[r, \mathcal{X}\left(x, x'\right)\right] = \mathcal{X}\left(x, x'\right) - r \tag{4}$$

for all $x' \in \mathcal{X}$. For $\mathcal{X} = \overline{\mathbb{R}}$ this recovers $r \odot x = r + x$.

More generally, let $w \colon \mathcal{C}^{\mathrm{op}} \to \overline{\mathbb{R}}$ be a presheaf (a weight) and let $F \colon \mathcal{C} \to \mathcal{X}$ be an $\overline{\mathbb{R}}$-functor. A w-weighted colimit of F is an object $w \star F \in \mathcal{X}$ characterized by

$$\mathcal{X}\left(w \star F, x'\right) = \left[w, \mathcal{X}\left(F-, x'\right)\right] = \inf_{c \in \mathcal{C}} \left(\mathcal{X}\left(F\left(c\right), x'\right) - w\left(c\right)\right) \tag{5}$$

for all $x' \in \mathcal{X}$, where $\mathcal{X}\left(F-, x'\right) : \mathcal{C}^{\mathrm{op}} \to \overline{\mathbb{R}}$ denotes the presheaf $c \mapsto \mathcal{X}\left(F(c), x'\right)$.

In presheaf categories, copowers and colimits are computed pointwise. Since colimits in $\overline{\mathbb{R}}$ are suprema, the density theorem gives, for every presheaf $f \colon \mathcal{C}^{\mathrm{op}} \to \overline{\mathbb{R}}$,

$$\begin{aligned} f &\cong \bigvee_{c \in \mathcal{C}} \left(f(c) + \mathcal{C}(-, c)\right), \\ f(x) &= \sup_{c \in \mathcal{C}} \left(f(c) + \mathcal{C}(x, c)\right). \end{aligned} \tag{6}$$

Likewise, viewing copresheaves on $\mathcal{D}$ as presheaves on $\mathcal{D}^{\mathrm{op}}$, every copresheaf $g \colon \mathcal{D} \to \overline{\mathbb{R}}$ satisfies

$$g(x) = \sup_{d \in \mathcal{D}} \left(g(d) + \mathcal{D}(d, x)\right). \tag{7}$$

The density theorem is equivalent to the statement that $[\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]$ is the free cocompletion of $\mathcal{C}$. In particular, every $\overline{\mathbb{R}}$-functor $F \colon \mathcal{C} \to \mathcal{X}$ admits a unique cocontinuous extension $\widehat{F} \colon [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}] \to \mathcal{X}$ together with a canonical natural isomorphism $\widehat{F} \circ y \cong F$. Equivalently, $\widehat{F}$ is the enriched left Kan extension of F along y. It is computed by the weighted colimit formula

$$\widehat{F}(f) \cong \bigvee_{c \in \mathcal{C}} \left(f(c) \odot F(c)\right), \tag{8}$$

where the supremum denotes the colimit in $\mathcal{X}$. Taking $\mathcal{X} = [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]$ and $F = y$ recovers (6).

For reference, the data $\widehat{F}$ and $\widehat{F} \circ y \cong F$ may be summarized by the triangle

$$\begin{tikzcd} \mathcal{C} \arrow[r, "F"] \arrow[d, "y"'] & \mathcal{X} \\ {[\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]} \arrow[ur, "\widehat{F}"'] & \end{tikzcd}$$

together with the specified natural isomorphism on the composite.

2.4. **Profunctors.** The tensor product of $\overline{\mathbb{R}}$-categories $\mathcal{C}$ and $\mathcal{D}$ is the $\overline{\mathbb{R}}$-category $\mathcal{C} \otimes \mathcal{D}$ with objects $\mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D})$ and hom-values

$$(\mathcal{C} \otimes \mathcal{D})\left((c,d),(c',d')\right) = \mathcal{C}(c,c') + \mathcal{D}(d,d').$$

Definition 3. A *profunctor* $M : \mathcal{C} \nrightarrow \mathcal{D}$ is an $\overline{\mathbb{R}}$-functor

$$M : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \overline{\mathbb{R}}.$$

Equivalently, M is a function $M : \mathrm{Ob}(\mathcal{C}) \times \mathrm{Ob}(\mathcal{D}) \to \overline{\mathbb{R}}$ such that for all $c, c' \in \mathcal{C}$ and $d, d' \in \mathcal{D}$,

$$\mathcal{C}(c',c) + \mathcal{D}(d,d') \leq [M(c,d), M(c',d')] = M(c',d') - M(c,d). \tag{9}$$

Any set S determines a discrete $\overline{\mathbb{R}}$-category (also denoted S) with hom-values

$$S(s,s') = \begin{cases} 0 & s = s', \\ -\infty & s \neq s'. \end{cases}$$

For discrete $\mathcal{C}$ and $\mathcal{D}$ the profunctor condition (9) is void, so any function $M : \mathcal{C} \times \mathcal{D} \to \overline{\mathbb{R}}$ defines a profunctor.

2.5. **Isbell duality and the nucleus.** Let $M : \mathcal{C} \nrightarrow \mathcal{D}$ be a profunctor. For each $d \in \mathcal{D}$ the function $c \mapsto M(c,d)$ is a presheaf $M(-,d) : \mathcal{C}^{\mathrm{op}} \to \overline{\mathbb{R}}$, and for each $c \in \mathcal{C}$ the function $d \mapsto M(c,d)$ is a copresheaf $M(c,-) : \mathcal{D} \to \overline{\mathbb{R}}$. These assemble into $\overline{\mathbb{R}}$-functors

$$\mathcal{D} \to [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}], \quad d \mapsto M(-,d),$$
$$\mathcal{C} \to [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}, \quad c \mapsto M(c,-).$$

We can equivalently regard M as a family of presheaves $M(-,d)$ indexed by $d \in \mathcal{D}$ and as a family of copresheaves $M(c,-)$ indexed by $c \in \mathcal{C}$. The Isbell conjugates extend these assignments to arbitrary (co)presheaves. The resulting transforms are the *Isbell conjugates*.

Definition 4. Define maps

$$M^*\colon [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}] \to [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}},$$
$$M_*\colon [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}} \to [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]$$

by

$$(M^* f)(d) := \inf_{c \in \mathcal{C}} \left(M(c,d) - f(c)\right), \tag{10}$$

$$(M_* g)(c) := \inf_{d \in \mathcal{D}} \left(M(c,d) - g(d)\right). \tag{11}$$

Proposition 5. *The assignments* (10)–(11) *define* $\overline{\mathbb{R}}$*-functors and satisfy*
$$M^* \dashv M_*,$$
meaning that
$$[\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(M^* f, g) = [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}](f, M_* g)$$
for all $f \in [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]$ *and* $g \in [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}$.

Proof. We first check functoriality of M^*. Let $f, f' \in [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]$ and put $\delta = [f, f'] = \inf_c (f'(c) - f(c))$. Then $\delta + f(c) \le f'(c)$ for all c. Fix $d \in \mathcal{D}$ and set $x_c = M(c, d) - f'(c)$, so that $(M^* f')(d) = \inf_c x_c$. From $\delta + f(c) \le f'(c)$ we obtain $x_c + \delta \le M(c, d) - f(c)$. Taking $\inf_c$ and using monotonicity gives
$$(M^* f')(d) + \delta = \inf_c x_c + \delta \le \inf_c (x_c + \delta) \le \inf_c (M(c, d) - f(c)) = (M^* f)(d).$$
Thus $\delta \le (M^* f)(d) - (M^* f')(d)$ for all d, hence $[f, f'] \le [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(M^* f, M^* f')$. The verification for M_* is analogous.

For the adjunction, compute using the definition of the hom in the opposite functor category:
$$\begin{aligned}[\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(M^* f, g) &= [\mathcal{D}, \overline{\mathbb{R}}](g, M^* f) \\ &= \inf_{d \in \mathcal{D}} ((M^* f)(d) - g(d)) = \inf_{d \in \mathcal{D}} \left(\inf_{c \in \mathcal{C}} (M(c, d) - f(c)) - g(d) \right).\end{aligned}$$
For each $r \in \overline{\mathbb{R}}$ the map $(-) - r$ is right adjoint to $+r$, hence preserves limits, which are infima. We may therefore pass $(-) - g(d)$ through the inner infimum, obtaining
$$\inf_c (M(c, d) - f(c)) - g(d) = \inf_c (M(c, d) - f(c) - g(d)).$$
It follows that
$$\begin{aligned}[\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(M^* f, g) &= \inf_d \inf_c (M(c, d) - f(c) - g(d)) \\ &= \inf_c \left(\inf_d (M(c, d) - g(d)) - f(c) \right) = [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}](f, M_* g),\end{aligned}$$
as claimed. □

From this point on, inequalities between presheaves and copresheaves refer to the pointwise order. For presheaves this coincides with the underlying order of $[\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]$:
$$f \le f' \iff 0 \le [f, f'].$$
For copresheaves one has
$$g \le g' \iff 0 \le [\mathcal{D}, \overline{\mathbb{R}}](g, g') \iff 0 \le [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(g', g).$$

Lemma 6. *The maps* M^* *and* M_* *are order-reversing for the pointwise order:* $f \le f' \Rightarrow M^* f' \le M^* f$ *and* $g \le g' \Rightarrow M_* g' \le M_* g$.

Proof. If $f \le f'$ then $M(c, d) - f'(c) \le M(c, d) - f(c)$ for each c, d because residuation is antitone in its first argument. Taking $\inf_c$ gives $(M^* f')(d) \le (M^* f)(d)$ for all d. The proof for M_* is the same. □

Since M^* and M_* are antitone, the composites

$$\mathrm{cl}_{\mathcal{C}} := M_* M^* \colon [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}] \to [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}],$$
$$\mathrm{cl}_{\mathcal{D}} := M^* M_* \colon [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}} \to [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}.$$

are monotone. The adjunction $M^* \dashv M_*$ implies that they are closure operators:

$$f \le \mathrm{cl}_{\mathcal{C}}(f), \quad \mathrm{cl}_{\mathcal{C}}^2 = \mathrm{cl}_{\mathcal{C}}, \qquad g \le \mathrm{cl}_{\mathcal{D}}(g), \quad \mathrm{cl}_{\mathcal{D}}^2 = \mathrm{cl}_{\mathcal{D}}. \tag{12}$$

Definition 7. The *nucleus* of M is the $\overline{\mathbb{R}}$-category $\mathrm{Nuc}(M)$ whose objects are pairs

$$\mathrm{Nuc}(M) = \left\{(f, g) \mid f \in [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}],\ g \in [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}},\ g = M^* f,\ f = M_* g\right\},$$

with hom-values inherited from either side:

$$\mathrm{Nuc}(M)\left((f, g), (f', g')\right) = [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}](f, f') = [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(g, g').$$

The equality holds because $M^* \dashv M_*$.

Proposition 8. *There are canonical isomorphisms of* $\overline{\mathbb{R}}$*-categories*

$$\mathrm{Nuc}(M) \cong \mathrm{Fix}(\mathrm{cl}_{\mathcal{C}}) \cong \mathrm{Fix}(\mathrm{cl}_{\mathcal{D}}) \cong \mathrm{im}(M^*) \cong \mathrm{im}(M_*),$$

where $\mathrm{Fix}(\mathrm{cl}_{\mathcal{C}}) = \{f \mid \mathrm{cl}_{\mathcal{C}}(f) = f\}$ *and similarly for* $\mathcal{D}$*, and* $\mathrm{im}(M^*)$ *(resp.* $\mathrm{im}(M_*)$*) denotes the full subcategory spanned by objects of the form* $M^* f$ *(resp.* $M_* g$*).*

Proof. The projections $(f, g) \mapsto f$ and $(f, g) \mapsto g$ identify $\mathrm{Nuc}(M)$ with the fixed-point subcategories of $\mathrm{cl}_{\mathcal{C}}$ and $\mathrm{cl}_{\mathcal{D}}$ because $(f, g) \in \mathrm{Nuc}(M)$ if and only if $f = M_* M^* f$ and $g = M^* M_* g$. The remaining identifications follow from the idempotence identities $M^* M_* M^* = M^*$ and $M_* M^* M_* = M_*$ which are immediate from (12). $\square$

Remark 9. Proposition 8 yields an explicit way to produce objects of $\mathrm{Nuc}(M)$. For any presheaf f the pair $(\mathrm{cl}_{\mathcal{C}}(f), M^* f)$ lies in $\mathrm{Nuc}(M)$, and for any copresheaf g the pair $(M_* g, \mathrm{cl}_{\mathcal{D}}(g))$ lies in $\mathrm{Nuc}(M)$.

Corollary 10. *A presheaf* f *is* $\mathrm{cl}_{\mathcal{C}}$*-closed if and only if it is the largest presheaf (for the pointwise order) among those with the same* M^**-image:*

$$M^* h = M^* f \Longrightarrow h \le f.$$

Dually, a copresheaf g *is* $\mathrm{cl}_{\mathcal{D}}$*-closed if and only if it is the largest copresheaf among those with the same* M_**-image.*

Proof. If $f = \mathrm{cl}_{\mathcal{C}}(f)$ and $M^* h = M^* f$, then

$$h \le M_* M^* h = M_* M^* f = f$$

by (12). Conversely, applying the displayed implication to $h = \mathrm{cl}_{\mathcal{C}}(f)$ shows $\mathrm{cl}_{\mathcal{C}}(f) \le f$, and (12) gives $f \le \mathrm{cl}_{\mathcal{C}}(f)$, hence equality. The dual statement is analogous. $\square$

3. The geometry of the nucleus

The nucleus constructed in §2.5 is an $\overline{\mathbb{R}}$-enriched category. In this section we extract from the enrichment a metric on its projectivization. The idea is straightforward. The enriched hom encodes a directed distance, symmetrization produces a translation-invariant seminorm, and passing to translation classes removes its kernel giving a metric. This is also the natural setting for the Isbell transforms, which are equivariant for the action of $\mathbb{R}$ by constant translation.

More precisely, for presheaves f, f' the residuation formulation for the enriched hom

$$[f, f'] = \sup \{\lambda \in \mathbb{R} : f + \lambda \leq f'\} .$$

suggests the interpretation of $-[f, f']$ as the least constant λ such that $f \leq f' + \lambda$. Define

$$d_{\to}(f, f') := -[f, f'].$$

Then the enriched composition law

$$[f, f'] + [f', f''] \leq [f, f'']$$

becomes, after negation, the triangle inequality

$$d_{\to}(f, f'') \leq d_{\to}(f, f') + d_{\to}(f', f'').$$

Its symmetrization

$$d_H(f, f') := d_{\to}(f, f') + d_{\to}(f', f) = -[f, f'] - [f', f]$$

is the max-spread, or oscillation, of $f - f'$. In particular, it is symmetric and nonnegative, and it vanishes exactly when $f - f'$ is constant. Therefore d_H descends to a genuine metric on the quotient by constant translation, which is the projectivization we use throughout.

3.1. **Finite index sets and the Isbell transforms.**

For the geometric constructions below it is convenient to work with finite, discrete $\overline{\mathbb{R}}$-categories. Thus, for the remainder of this section, $\mathcal{C}$ and $\mathcal{D}$ are finite sets (regarded as discrete $\overline{\mathbb{R}}$-categories), and a profunctor $M : \mathcal{C} \nrightarrow \mathcal{D}$ is simply a function

$$M : \mathcal{C} \times \mathcal{D} \to \overline{\mathbb{R}}.$$

In this setting, presheaves on $\mathcal{C}$ and copresheaves on $\mathcal{D}$ are just functions $\mathcal{C} \to \overline{\mathbb{R}}$ and $\mathcal{D} \to \overline{\mathbb{R}}$, and the infima in Definition 4 are minima. Accordingly,

$$\begin{aligned} (M^* f)(d) &= \min_{c \in \mathcal{C}} \left(M(c, d) - f(c) \right), \\ (M_* g)(c) &= \min_{d \in \mathcal{D}} \left(M(c, d) - g(d) \right), \end{aligned} \tag{13}$$

where $z - y$ denotes residuation in $\overline{\mathbb{R}}$ (cf. (1)).

Lemma 11. *For any finite constant $\lambda \in \mathbb{R}$ one has*

$$\begin{aligned} M^*(f + \lambda) &= M^* f - \lambda, \\ M_*(g - \lambda) &= M_* g + \lambda, \end{aligned} \tag{14}$$

where λ denotes the constant function on $\mathcal{C}$ or $\mathcal{D}$.

Proof. For the first identity, subtracting λ from each term inside the minimum gives

$$\begin{aligned}(M^*(f+\lambda))(d) &= \min_c \left(M(c,d) - f(c) - \lambda\right) \\ &= \Big(\min_c \left(M(c,d) - f(c)\right)\Big) - \lambda \\ &= (M^* f)(d) - \lambda.\end{aligned}$$

The second identity is analogous. □

The equivariance (14) is the algebraic shadow of a projective symmetry: if (f, g) satisfies $g = M^* f$ and $f = M_* g$, then so does $(f + \lambda, g - \lambda)$.

3.2. **Projective (co)presheaves and the Hilbert–oscillation metric.** We now isolate the locus on which translation by constants acts freely.

Definition 12. Define the *finite somewhere* presheaves $[\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]_{\mathrm{fs}}$ to be the full subcategory of presheaves $f\colon \mathcal{C} \to \overline{\mathbb{R}}$ for which $f(c) \in \mathbb{R}$ for at least one $c \in \mathcal{C}$. Define $[\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}_{\mathrm{fs}}$ similarly for copresheaves on $\mathcal{D}$.

On these full subcategories, $(\mathbb{R}, +)$ acts freely by constant translation $f \mapsto f + \lambda$.

Definition 13. The *projective presheaf space* of $\mathcal{C}$ is the quotient

$$\mathbb{P}\mathcal{C} := [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]_{\mathrm{fs}}/\mathbb{R}.$$

Let $[f] \in \mathbb{P}\mathcal{C}$ denote the translation class of f. Similarly, the *projective copresheaf space* of $\mathcal{D}$ is

$$\mathbb{P}\mathcal{D} := [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}_{\mathrm{fs}}/\mathbb{R}.$$

For $\lambda \in \mathbb{R}$ and presheaves f, f', the enriched hom satisfies $[f + \lambda, f'] = [f, f'] - \lambda$ and $[f, f' + \lambda] = [f, f'] + \lambda$. In particular, the symmetrized quantity $-[f, f'] - [f', f]$ is $\mathbb{R}$-invariant. This is the tropical analogue of Hilbert's projective metric: it measures only the oscillation of the difference.

Definition 14. Let $f, f' \in [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]_{\mathrm{fs}}$. Set

$$S(f, f') := \{c \in \mathcal{C} \mid (f(c), f'(c)) = (+\infty, +\infty) \text{ or } (f(c), f'(c)) = (-\infty, -\infty)\}.$$

On $\mathcal{C} \setminus S(f, f')$ the anti-symmetry $(x - y) = -(y - x)$ holds, so the values $f(c) - f'(c)$ behave as ordinary extended differences. Define the *projective distance* between $[f], [f'] \in \mathbb{P}\mathcal{C}$ by

(15)
$$d_{\mathcal{C}}([f], [f']) := \begin{cases} \sup\limits_{c \notin S(f,f')} (f(c) - f'(c)) - \inf\limits_{c \notin S(f,f')} (f(c) - f'(c)) & \text{if both extrema lie in } \mathbb{R}, \\ +\infty & \text{otherwise.} \end{cases}$$

Define $d_{\mathcal{D}}$ on $\mathbb{P}\mathcal{D}$ analogously.

Proposition 15. *The function $d_{\mathcal{C}}$ is an extended metric on $\mathbb{P}\mathcal{C}$. Moreover, whenever $d_{\mathcal{C}}([f], [f']) < \infty$ one has the identity*

$$d_{\mathcal{C}}([f], [f']) = -[f, f'] - [f', f], \tag{16}$$

where $[f, f']$ is the enriched hom in $[\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]$.

Proof. First note that for any finite constant $\lambda \in \mathbb{R}$ we have $S(f+\lambda, f') = S(f, f') = S(f, f' + \lambda)$, since translating by a finite constant does not change whether a value is $\pm\infty$. Moreover, on $\mathcal{C} \setminus S(f, f')$ we have $(f+\lambda)(c) - f'(c) = (f(c) - f'(c)) + \lambda$ and $f(c) - (f'(c) + \lambda) = (f(c) - f'(c)) - \lambda$. Hence both $\sup(f - f')$ and $\inf(f - f')$ shift by the same constant, so their difference is unchanged. Therefore $d_{\mathcal{C}}$ is well-defined on $\mathbb{P}\mathcal{C}$. Symmetry is immediate since on $\mathcal{C} \setminus S(f, f')$ one has $f'(c) - f(c) = -(f(c) - f'(c))$, and nonnegativity is clear from $\sup - \inf \geq 0$ (or $+\infty$).

If $d_{\mathcal{C}}([f], [f']) = 0$, then both extrema in (15) lie in $\mathbb{R}$ and are equal. Thus $f(c) - f'(c)$ is a single real constant on $\mathcal{C} \setminus S(f, f')$, say $f'(c) - f(c) = \lambda$ for all such c. For $c \in S(f, f')$ we have $f(c) = f'(c) \in \{\pm\infty\}$, and adding λ does not change $\pm\infty$. Hence $f' = f + \lambda$ pointwise, so $[f] = [f']$.

Now assume $d_{\mathcal{C}}([f], [f']) < \infty$. Then for every $c \notin S(f, f')$ the value $f(c) - f'(c)$ lies in $\mathbb{R}$, and the enriched hom in $[\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]$ satisfies

$$[f, f'] \;=\; \inf_{c \in \mathcal{C}} \big(f'(c) - f(c)\big) \;=\; \inf_{c \notin S(f, f')} \big(f'(c) - f(c)\big),$$

since for $c \in S(f, f')$ one has $f'(c) - f(c) = +\infty$ and so such indices cannot affect the infimum in the finite case. Using anti-symmetry on $\mathcal{C} \setminus S(f, f')$ we obtain

$$\begin{aligned} -[f, f'] &= \sup_{c \notin S(f, f')} \big(f(c) - f'(c)\big) \text{ and} \\ -[f', f] &= \sup_{c \notin S(f, f')} \big(f'(c) - f(c)\big) \\ &= - \inf_{c \notin S(f, f')} \big(f(c) - f'(c)\big). \end{aligned}$$

Therefore

$$\begin{aligned} -[f, f'] - [f', f] &= \sup_{c \notin S(f, f')} (f - f') - \inf_{c \notin S(f, f')} (f - f') \\ &= d_{\mathcal{C}}([f], [f']), \end{aligned}$$

which is (16).

Finally, for the triangle inequality: if $d_{\mathcal{C}}([f], [f']) = +\infty$ or $d_{\mathcal{C}}([f'], [f'']) = +\infty$ there is nothing to prove. Otherwise all three distances are finite and we may use enriched composition twice:

$$[f, f'] + [f', f''] \leq [f, f''] \text{ and } [f'', f'] + [f', f] \leq [f'', f].$$

Negating and adding these inequalities gives

$$-[f, f''] - [f'', f] \leq \big(-[f, f'] - [f', f]\big) + \big(-[f', f''] - [f'', f']\big),$$

i.e. $d_{\mathcal{C}}([f], [f'']) \leq d_{\mathcal{C}}([f], [f']) + d_{\mathcal{C}}([f'], [f''])$. □

3.3. **Projective nuclei and isometries.** We now impose the mild hypothesis needed to pass M^* and M_* to projective spaces.

Definition 16 (Nondegeneracy)**.** We call the profunctor M *nondegenerate* if the Isbell transforms preserve the finite-somewhere condition:

$$\begin{aligned} M^* \left([\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]_{\mathrm{fs}}\right) &\subseteq [\mathcal{D}, \overline{\mathbb{R}}]_{\mathrm{fs}}^{\mathrm{op}}, \\ M_* \left([\mathcal{D}, \overline{\mathbb{R}}]_{\mathrm{fs}}^{\mathrm{op}}\right) &\subseteq [\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]_{\mathrm{fs}}. \end{aligned}$$

Under this hypothesis, Lemma 11 implies that M^* and M_* descend to well-defined maps

$$M^*\colon \mathbb{P}\mathcal{C} \to \mathbb{P}\mathcal{D},$$
$$M_*\colon \mathbb{P}\mathcal{D} \to \mathbb{P}\mathcal{C}.$$

Definition 17 (Projective nucleus)**.** Let $\mathrm{Nuc}(M)$ be the nucleus of M (Definition 7) and set

$$\mathrm{Nuc}(M)_{\mathrm{fs}} := \mathrm{Nuc}(M) \cap \left([\mathcal{C}^{\mathrm{op}}, \overline{\mathbb{R}}]_{\mathrm{fs}} \times [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}_{\mathrm{fs}}\right).$$

The group $\mathbb{R}$ acts on $\mathrm{Nuc}(M)_{\mathrm{fs}}$ by

$$\lambda \cdot (f, g) = (f + \lambda, g - \lambda).$$

The *projective nucleus* is the quotient

$$\mathbb{P}\mathrm{Nuc}(M) := \mathrm{Nuc}(M)_{\mathrm{fs}}/\mathbb{R}.$$

We metrize $\mathbb{P}\mathrm{Nuc}(M)$ by

$$d_{\mathbb{P}\mathrm{Nuc}}\left([(f,g)], [(f',g')]\right) := \max\left\{d_{\mathcal{C}}([f],[f']), d_{\mathcal{D}}([g],[g'])\right\}.$$

Write $\mathrm{Fix}_{Proj}(M_*M^*) \subseteq \mathbb{P}\mathcal{C}$ and $\mathrm{Fix}_{Proj}(M^*M_*) \subseteq \mathbb{P}\mathcal{D}$ for the images in projective space of the fixed-point sets of the closure operators M_*M^* and M^*M_*.

Theorem 18. *Let $M\colon \mathcal{C} \nrightarrow \mathcal{D}$ be a nondegenerate profunctor. Then the maps $M^*\colon \mathbb{P}\mathcal{C} \to \mathbb{P}\mathcal{D}$ and $M_*\colon \mathbb{P}\mathcal{D} \to \mathbb{P}\mathcal{C}$ are 1-Lipschitz for the metrics $d_{\mathcal{C}}$ and $d_{\mathcal{D}}$. Moreover, they restrict to mutually inverse isometries*

$$M^*\colon\ \mathrm{Fix}_{Proj}(M_*M^*) \xrightarrow{\cong} \mathrm{Fix}_{Proj}(M^*M_*),$$
$$M_*\colon\ \mathrm{Fix}_{Proj}(M^*M_*) \xrightarrow{\cong} \mathrm{Fix}_{Proj}(M_*M^*).$$

and hence identify $\mathbb{P}\mathrm{Nuc}(M)$ isometrically with either projective fixed-point set.

Proof. Functoriality of M^* in the enriched sense gives, for presheaves f, f',

$$[f, f'] \le [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}(M^*f, M^*f') = [\mathcal{D}, \overline{\mathbb{R}}](M^*f', M^*f),$$

and the same inequality with f and f' exchanged. Negating and adding yields

$$-[M^*f, M^*f'] - [M^*f', M^*f] \le -[f, f'] - [f', f].$$

If $d_{\mathcal{C}}([f],[f']) < \infty$, Proposition 15 identifies both sides with the corresponding projective metrics, giving

$$d_{\mathcal{D}}\left(M^*[f], M^*[f']\right) \le d_{\mathcal{C}}\left([f],[f']\right).$$

If $d_{\mathcal{C}}([f],[f']) = +\infty$, the inequality is tautological. Thus M^* is 1-Lipschitz, and similarly M_*.

On the projective fixed-point sets, M^* and M_* are inverse bijections (Proposition 8). Since each is 1-Lipschitz, the two inequalities

$$d_{\mathcal{D}}\left(M^*[f], M^*[f']\right) \le d_{\mathcal{C}}\left([f],[f']\right),$$
$$d_{\mathcal{C}}\left(M_*[g], M_*[g']\right) \le d_{\mathcal{D}}\left([g],[g']\right).$$

apply to inverse pairs and force equality. Hence both restrictions are isometries.

Finally, the identification with $\mathbb{P}\mathrm{Nuc}(M)$ is obtained by projecting $[(f,g)] \mapsto [f]$ or $[(f,g)] \mapsto [g]$. $\square$

Consequently we obtain a diagram of metric spaces in which every arrow is an isometry:

$$\begin{tikzcd}
 & \mathbb{P}\mathrm{Nuc}(M) \arrow[dl, bend right, "\pi_1"'] \arrow[dl, bend left, "i_1"] \arrow[dr, bend right, "i_2"'] \arrow[dr, bend left, "\pi_2"] & \\
\mathrm{Fix}_{Proj}(M_* M^*) \arrow[rr, shift left, "M^*"] & & \mathrm{Fix}_{Proj}(M^* M_*) \arrow[ll, shift left, "M_*"]
\end{tikzcd}$$

where $\pi_1([(f,g)]) = [f]$, $\pi_2([(f,g)]) = [g]$, $i_1([f]) = [(f, M^*f)]$, and $i_2([g]) = [(M_*g, g)]$.

3.4. **External gauge transformations.** Beyond translation by constants there is a larger symmetry, familiar from tropical linear algebra: one may reweight rows and columns by arbitrary potentials. Let $\mathbb{R}^{\mathcal{C}}$ and $\mathbb{R}^{\mathcal{D}}$ denote the additive groups of real-valued functions on $\mathcal{C}$ and $\mathcal{D}$.

Definition 19 (Gauge action and gauge transform)**.** For $u \in \mathbb{R}^{\mathcal{C}}$ define $L_u : \mathbb{P}\mathcal{C} \to \mathbb{P}\mathcal{C}$ by $[f] \mapsto [f-u]$. For $v \in \mathbb{R}^{\mathcal{D}}$ define $R_v : \mathbb{P}\mathcal{D} \to \mathbb{P}\mathcal{D}$ by $[g] \mapsto [g-v]$. Given (u,v), define the *gauge transform* of M by

$$M^{(u,v)}(c,d) := M(c,d) - u(c) - v(d).$$

Lemma 20. *On projective spaces, the Isbell transforms of $M^{(u,v)}$ are conjugate to those of M:*

$$\left(M^{(u,v)}\right)^* = R_v \circ M^* \circ L_u^{-1},$$
$$\left(M^{(u,v)}\right)_* = L_u \circ M_* \circ R_v^{-1}.$$

Proof. For a representative f and any $d \in \mathcal{D}$,

$$\begin{aligned}
\left(M^{(u,v)}\right)^* (f-u)(d) &= \min_c \left(M(c,d) - u(c) - v(d) - (f(c) - u(c))\right) \\
&= \min_c \left(M(c,d) - f(c)\right) - v(d) \\
&= M^* f(d) - v(d).
\end{aligned}$$

This is precisely $(R_v \circ M^*)(f)(d)$. The statement for $\left(M^{(u,v)}\right)_*$ is analogous. $\square$

Proposition 21. *For any $(u,v) \in \mathbb{R}^{\mathcal{C}} \times \mathbb{R}^{\mathcal{D}}$, the map*

$$\begin{aligned}
\Phi_{u,v} \colon \mathbb{P}\mathrm{Nuc}(M) &\to \mathbb{P}\mathrm{Nuc}\left(M^{(u,v)}\right), \\
[(f,g)] &\mapsto [(f-u, g-v)].
\end{aligned}$$

is a well-defined isometry with inverse $[(f', g')] \mapsto [(f'+u, g'+v)]$. Consequently the projective fixed-point sets and projective nuclei of M and $M^{(u,v)}$ are canonically isometric.

Proof. The maps L_u and R_v are isometries by Definition 14, since subtracting the same potential from both arguments leaves the difference, hence its oscillation, unchanged. Lemma 20 identifies the Isbell equations for M with those for $M^{(u,v)}$ under these isometries. Thus $(f,g) \in \mathrm{Nuc}(M)$ if and only if $(f-u, g-v) \in \mathrm{Nuc}\left(M^{(u,v)}\right)$, and the induced map on projective quotients is an isometry. $\square$

3.5. **Witness cells.** The projective metric geometry of $\mathbb{P}\mathrm{Nuc}(M)$ is intrinsic and invariant under external gauge transformations (Proposition 21). When the indexing categories are finite, the Isbell inequalities also endow $\mathbb{P}\mathrm{Nuc}(M)$ with a canonical polyhedral stratification. The bridge between these two viewpoints is provided by the *gap matrix* $\delta^{(f,g)}$: its zero entries record the witness relation and hence the combinatorial cell of (f, g), while its positive entries measure slack.

Throughout this subsection, $\mathcal{C}$ and $\mathcal{D}$ are finite sets, regarded as discrete $\overline{\mathbb{R}}$-categories. Thus a profunctor

$$M : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \overline{\mathbb{R}}$$

is simply a matrix $M : \mathcal{C} \times \mathcal{D} \to \overline{\mathbb{R}}$. In this case, (co)presheaves are functions $f : \mathcal{C} \to \overline{\mathbb{R}}$ and $g : \mathcal{D} \to \overline{\mathbb{R}}$, and the infima in (10)–(11) are minima.

We emphasize that the definitions below make sense for general small $\overline{\mathbb{R}}$-categories $\mathcal{C}$ and $\mathcal{D}$ (with "min" replaced by "inf"). In later applications, $\mathcal{C}$ and $\mathcal{D}$ will not be discrete; then witness sets need not be nonempty, but the slack formalism remains available and will still organize the geometry.

Fix a nondegenerate profunctor M. For a presheaf $f : \mathcal{C} \to \overline{\mathbb{R}}$ write $g := M^* f$, so that

$$g(d) = \min_{c \in \mathcal{C}} \left(M(c, d) - f(c) \right), \ d \in \mathcal{D},$$

where subtraction is residuation in $\overline{\mathbb{R}}$ (cf. (1)). Similarly, for a copresheaf $g : \mathcal{D} \to \overline{\mathbb{R}}$ write $f := M_* g$, so that

$$f(c) = \min_{d \in \mathcal{D}} \left(M(c, d) - g(d) \right), \ c \in \mathcal{C}.$$

Definition 22. Let $f : \mathcal{C} \to \overline{\mathbb{R}}$ and let $g := M^* f$. An element $c \in \mathcal{C}$ is a *witness for f at $d \in \mathcal{D}$* if

$$g(d) = M(c, d) - f(c),$$

that is, if c realizes the minimum defining $M^* f(d)$. Dually, if $g : \mathcal{D} \to \overline{\mathbb{R}}$ and $f := M_* g$, we call $d \in \mathcal{D}$ a *witness for g at $c \in \mathcal{C}$* if

$$f(c) = M(c, d) - g(d),$$

that is, if d realizes the minimum defining $M_* g(c)$.

Because $M^*(f + \lambda) = M^* f - \lambda$ and $M_*(g - \lambda) = M_* g + \lambda$ for every $\lambda \in \mathbb{R}$, the witness relation depends only on the projective classes $[f] \in \mathbb{P}\mathcal{C}$ and $[g] \in \mathbb{P}\mathcal{D}$.

By definition of $g = M^* f$, the inequalities

$$f(c) + g(d) \leq M(c, d), \ c \in \mathcal{C}, \ d \in \mathcal{D}$$

always hold. The gap matrix measures how far they are from equality.

Definition 23. Let $[f] \in \mathbb{P}\mathcal{C}$ and choose a representative $f : \mathcal{C} \to \overline{\mathbb{R}}$. Set $g := M^* f$. The *gap matrix* of $[f]$ is the function

$$\delta^f(c, d) := M(c, d) - \left(f(c) + g(d) \right). \tag{17}$$

Dually, for $[g] \in \mathbb{P}\mathcal{D}$ with representative g and $f := M_* g$, we set

$$\delta^g(c, d) := M(c, d) - \left(f(c) + g(d) \right).$$

If $(f, g) \in \mathrm{Nuc}(M)$, then $\delta^f = \delta^g$, and we write $\delta^{(f,g)}$.

Lemma 24. *(a) For every $\lambda \in \mathbb{R}$, one has $\delta^{f+\lambda} = \delta^f$.*

(b) For $u \in \mathbb{R}^{\mathcal{C}}$ and $v \in \mathbb{R}^{\mathcal{D}}$, let $M^{(u,v)}$ be the gauge transform

$$M^{(u,v)}(c,d) := M(c,d) - u(c) - v(d).$$

If $g = M^* f$, then $\left(M^{(u,v)}\right)^* (f - u) = g - v$, and the corresponding gap matrices agree:

$$\delta^f = \delta^{f-u},$$

where δ^f is computed with M and δ^{f-u} with $M^{(u,v)}$.

Proof. (a) follows from $M^*(f + \lambda) = M^* f - \lambda$ and the identity $(f(c) + \lambda) + (g(d) - \lambda) = f(c) + g(d)$. For (b), compute

$$M^{(u,v)}(c,d) - ((f-u)(c) + (g-v)(d)) = M(c,d) - (f(c) + g(d)). \qquad \square$$

On the finite locus, the zeros of δ are exactly the witness pairs. The next lemma isolates the only subtlety: in $\overline{\mathbb{R}}$, a zero gap forces finiteness.

Lemma 25. *If $\delta^f(c,d) = 0$, then $f(c)$, $g(d)$, and $M(c,d)$ are all finite real numbers.*

Proof. By definition, $\delta^f(c,d) = M(c,d) - (f(c) + g(d))$ is a residuation. If $f(c) + g(d) = +\infty$ then $\delta^f(c,d) = M(c,d) - \infty$ lies in $\{-\infty, \infty\}$, never 0. If $f(c) + g(d) = -\infty$ then $\delta^f(c,d) = M(c,d) - (-\infty) = \infty$, again not 0. Thus $f(c) + g(d) \in \mathbb{R}$. Since $-\infty$ is absorbing for $+$, a finite sum forces $f(c), g(d) \in \mathbb{R}$. Finally, $\delta^f(c,d) = 0$ forces $M(c,d) \in \mathbb{R}$ as well: if $M(c,d) = \pm\infty$ then $M(c,d) - (f(c) + g(d)) = \pm\infty$. $\square$

Proposition 26. *Let $[f] \in \mathbb{P}\mathcal{C}$ with representative f, let $g := M^* f$, and let $\delta = \delta^f$. Then:*

(a) $\delta(c,d) \geq 0$ for all $(c,d) \in \mathcal{C} \times \mathcal{D}$.
(b) If $\delta(c,d) = 0$, then c is a witness for f at d.
(c) If f and g are finite-valued, then $\delta(c,d) = 0$ if and only if c is a witness for f at d. In particular, every column contains at least one zero.
(d) If moreover $(f,g) \in \mathrm{Nuc}(M)$ and f, g are finite-valued, then $\delta(c,d) = 0$ if and only if d is a witness for g at c. In particular, every row contains at least one zero.
(e) If f is finite-valued and every row of δ^f contains a zero, then f is a fixed point of $M_* M^*$:

$$f = M_* M^* f.$$

Proof. (a) Since $g(d) = \min_{c'} (M(c',d) - f(c'))$, we have $g(d) \leq M(c,d) - f(c)$ for every c. By residuation, this is equivalent to $f(c) + g(d) \leq M(c,d)$, hence $\delta(c,d) = M(c,d) - (f(c) + g(d)) \geq 0$.

(b) If $\delta(c,d) = 0$, then Lemma 25 shows that the relevant entries are finite, so subtraction is ordinary: $0 = M(c,d) - f(c) - g(d)$, hence $g(d) = M(c,d) - f(c)$ and c realizes the minimum in $M^* f(d)$.

(c) If f, g are finite-valued and c is a witness for f at d, then $g(d) = M(c,d) - f(c)$ and $\delta(c,d) = M(c,d) - f(c) - g(d) = 0$. Conversely, $\delta(c,d) = 0$ implies (b). Since $\mathcal{C}$ is finite, every minimum defining $g(d)$ is attained, so every column contains a witness and hence a zero.

(d) Apply (c) to the dual description $f = M_* g$.

(e) Assume f is finite-valued and every row contains a zero. Fix c and choose d with $\delta(c,d)=0$. Then $f(c)=M(c,d)-g(d)$, hence

$$(M_*g)(c)=\min_{d'}\left(M(c,d')-g(d')\right)\le M(c,d)-g(d)=f(c).$$

On the other hand, $f\le M_*M^*f=M_*g$ holds for all f by (12), so equality holds coordinatewise. $\square$

Corollary 27. *If $(f,g)\in\mathbb{P}\mathrm{Nuc}(M)$ and f,g are finite-valued, then c is a witness for f at d if and only if d is a witness for g at c.*

Proof. This is Proposition 26(c) and (d). $\square$

Corollary 28. *A finite-valued presheaf f is a fixed point of M_*M^* if and only if every row of δ^f contains a zero. Equivalently, for every $c\in\mathcal{C}$ there exists $d\in\mathcal{D}$ such that d is a witness for M^*f at c.*

Proof. If f is finite-valued and fixed by M_*M^*, write $g:=M^*f$, so $f=M_*g$. Since $\mathcal{D}$ is finite, for each $c\in\mathcal{C}$ the minimum defining $f(c)$ is attained at some d, and then $\delta^f(c,d)=0$. Conversely, if every row contains a zero then Proposition 26(e) gives $f=M_*M^*f$. $\square$

For a finite-valued nucleus point, the witness pattern is exactly the zero set of its gap matrix.

Definition 29. For $(f,g)\in\mathbb{P}\mathrm{Nuc}(M)$ with f,g finite-valued, define the *witness relation*

$$Z(f,g):=\left\{(c,d)\in\mathcal{C}\times\mathcal{D}\mid\delta^{(f,g)}(c,d)=0\right\}. \tag{18}$$

By Proposition 26(d), the relation $Z(f,g)$ meets every row and every column. These relations partition the finite part of $\mathbb{P}\mathrm{Nuc}(M)$: we declare $(f,g)\sim(f',g')$ if $Z(f,g)=Z(f',g')$.

Definition 30. For a relation $Z\subseteq\mathcal{C}\times\mathcal{D}$ meeting every row and column, define the *open witness cell*

$$\mathrm{Cell}^\circ(Z):=\{(f,g)\in\mathbb{P}\mathrm{Nuc}(M)\mid f,g\text{ finite-valued and }Z(f,g)=Z\}.$$

In Section 4 we show that, after choosing an affine chart for the projective quotient (for example $\min f=0$), the closure of $\mathrm{Cell}^\circ(Z)$ is a classical polytope. It is cut out by the equalities $M(c,d)=f(c)+g(d)$ for $(c,d)\in Z$ together with the inequalities $M(c,d)\ge f(c)+g(d)$ for all (c,d), and $\mathrm{Cell}^\circ(Z)$ is its relative interior.

Because $\mathcal{D}$ is discrete, the copresheaf represented by $d\in\mathcal{D}$ is the delta function $g_d(d)=0$ and $g_d(d')=-\infty$ for $d'\ne d$. A direct computation gives $M_*g_d=M(-,d)$; the columns of the matrix M are the images of the representables.

Definition 31. For $d\in\mathcal{D}$ define the dth *anchor presheaf* $A_d:\mathcal{C}\to\overline{\mathbb{R}}$ by $A_d=M(-,d)$.

Proposition 32. *Each anchor A_d lies in $\mathrm{Fix}(M_*M^*)=\mathrm{Im}(M_*)$. Moreover, for any $f\in\mathrm{Fix}(M_*M^*)$ there exist weights $\lambda_d\in\overline{\mathbb{R}}$ such that*

$$f(c)=\min_{d\in\mathcal{D}}\left(A_d(c)-\lambda_d\right),\ \ c\in\mathcal{C}. \tag{19}$$

Proof. Since $A_d = M_* g_d$, we have $A_d \in \mathrm{Im}(M_*) = \mathrm{Fix}(M_* M^*)$. Conversely, if $f \in \mathrm{Fix}(M_* M^*) = \mathrm{Im}(M_*)$ then $f = M_* g$ for some copresheaf g, hence

$$f(c) = \min_{d \in \mathcal{D}} (M(c,d) - g(d)) = \min_{d \in \mathcal{D}} (A_d(c) - \lambda_d)$$

with $\lambda_d := g(d)$. □

Equivalently, $\mathrm{Fix}(M_* M^*)$ is the closureof the anchors A_d under weighted coproducts: every fixed point f can be written as the pointwise minimum of translates $A_d - \lambda_d$, and conversely every such minimum lies in $\mathrm{Fix}(M_* M^*)$.

Let $d_{\mathcal{C}}$ denote the metric on $\mathbb{P}\mathcal{C}$ defined in (15). We measure the size of M by the maximal distance from the origin attained by its projective image.

Definition 33. Define

$$\|M\| := \sup \left\{ d_{\mathcal{C}} \left([M_* g], [0] \right) \mid g \in [\mathcal{D}, \overline{\mathbb{R}}]^{\mathrm{op}}_{\mathrm{fs}} \right\} \in [0, \infty].$$

Lemma 34. *If $\|M\| < \infty$, then every anchor A_d has finite distance from $[0]$, and*

$$\|M\| = \max_{d \in \mathcal{D}} d_{\mathcal{C}} \left([A_d], [0] \right).$$

Proof. For each d, the represented copresheaf g_d is finite somewhere, hence

$$d_{\mathcal{C}} \left([A_d], [0] \right) = d_{\mathcal{C}} \left([M_* g_d], [0] \right) \le \|M\|.$$

Thus $\|M\| \ge \max_d d_{\mathcal{C}} \left([A_d], [0] \right)$.

For the reverse inequality, let g be any finite-somewhere copresheaf and put $f := M_* g$. By Proposition 32 we can write $f = \min_d (A_d - \lambda_d)$ for suitable λ_d. Choose a representative of $[f]$ normalized by $\min_c f(c) = 0$. Pick c_0 with $f(c_0) = 0$ and choose d_0 attaining the minimum at c_0, so $0 = f(c_0) = A_{d_0}(c_0) - \lambda_{d_0}$. Then for any c,

$$f(c) = \min_d (A_d(c) - \lambda_d) \le A_{d_0}(c) - \lambda_{d_0} = A_{d_0}(c) - A_{d_0}(c_0).$$

Hence $0 \le f(c) \le \sup_c A_{d_0}(c) - \inf_c A_{d_0}(c) = d_{\mathcal{C}} \left([A_{d_0}], [0] \right)$ for all c, so $d_{\mathcal{C}} \left([f], [0] \right) \le \max_d d_{\mathcal{C}} \left([A_d], [0] \right)$. Taking the supremum over g gives the desired inequality. □

Corollary 35. *If $\|M\| < \infty$, then $\mathbb{P}\mathrm{Nuc}(M)$ is compact.*

Proof. Lemma 34 implies that all anchors A_d lie at finite distance from $[0]$. In particular, their projective classes admit real-valued representatives. Then every point of $\mathbb{P}\mathrm{Fix}(M_* M^*) = \mathrm{Im}(M_*)/\mathbb{R}$ is represented by a real-valued function as in (19), and hence lies in the finite-dimensional space $\mathbb{R}^{\mathcal{C}}/\mathbb{R}$.

On an affine slice, for example $\min f = 0$, the closed ball

$$\{[f] \mid d_{\mathcal{C}} \left([f], [0] \right) \le \|M\|\}$$

is a compact cube. It therefore suffices to know that $\mathbb{P}\mathrm{Fix}(M_* M^*)$ is closed. But $M_* M^* \colon \mathbb{P}\mathcal{C} \to \mathbb{P}\mathcal{C}$ is continuous (indeed M^* and M_* are nonexpansive for the projective metric), hence its fixed-point set is closed. Therefore $\mathbb{P}\mathrm{Fix}(M_* M^*)$ is compact. Since $\mathbb{P}\mathrm{Nuc}(M)$ is canonically isometric to $\mathbb{P}\mathrm{Fix}(M_* M^*)$ by Theorem 18, it follows that $\mathbb{P}\mathrm{Nuc}(M)$ is compact. □

4. Polyhedral structures of $\mathbb{P}\mathrm{Nuc}(M)$

In §3.5 we associated to each finite-valued nucleus point (f, g) its *gap matrix*

$$\delta^{(f,g)}(c, d) := M(c, d) - f(c) - g(d)$$

and its zero set

$$Z(f, g) := \left\{(c, d) \in \mathcal{C} \times \mathcal{D} \mid \delta^{(f,g)}(c, d) = 0\right\}.$$

The witness cell decomposition of $\mathbb{P}\mathrm{Nuc}(M)$ is obtained by fixing this zero pattern. In this section we pass from the combinatorics of $Z(f, g)$ to an explicit polyhedral model. Under mild finiteness hypotheses, the projective nucleus is a finite polytopal complex whose closed cells are *witness polyhedra* cut out by linear equalities indexed by subsets $Y \subseteq \mathcal{C} \times \mathcal{D}$, and whose open cells are their relative interiors.

4.1. **Hypotheses for this section.** As in the previous subsection, we continue with the standing hypotheses that $\mathcal{C}$ and $\mathcal{D}$ are finite sets, viewed as discrete $\overline{\mathbb{R}}$-categories. We also assume that the profunctor

$$M : \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \mathbb{R}$$

has finite real entries. By Corollary 35, $\mathbb{P}\mathrm{Nuc}(M)$ is compact; in particular, each projective class admits a representative (f, g) with $f : \mathcal{C} \to \mathbb{R}$ and $g : \mathcal{D} \to \mathbb{R}$.

For polyhedral arguments it is convenient to choose an affine chart for the projective quotient. Fix $c_0 \in \mathcal{C}$ and set

$$\mathrm{Nuc}(M)_0 := \{(f, g) \in \mathrm{Nuc}(M) \mid f(c_0) = 0\}.$$

Since the $\mathbb{R}$-action is given by $(f, g) \mapsto (f + \lambda, g - \lambda)$, each class in $\mathbb{P}\mathrm{Nuc}(M)$ has a unique representative in $\mathrm{Nuc}(M)_0$, so $\mathrm{Nuc}(M)_0 \cong \mathbb{P}\mathrm{Nuc}(M)$. We use this identification throughout.

4.2. **Witness polyhedra and admissibility.** Working in the gauge slice $\mathrm{Nuc}(M)_0$, the Isbell inequalities

$$f(c) + g(d) \leq M(c, d) \text{ for all } c \in \mathcal{C}, d \in \mathcal{D}$$

cut out a feasibility polyhedron in $\mathbb{R}^{\mathcal{C}} \times \mathbb{R}^{\mathcal{D}}$. The Isbell fixed-point conditions for the nucleus are equivalent to the requirement that every row and every column attains equality. Given $Y \subseteq \mathcal{C} \times \mathcal{D}$, we obtain a smaller closed polyhedron by forcing equality along Y.

Definition 36. Let $Y \subseteq \mathcal{C} \times \mathcal{D}$. We say that Y *covers* $\mathcal{C}$ if for every $c \in \mathcal{C}$ there exists $d \in \mathcal{D}$ with $(c, d) \in Y$. We say that Y *covers* $\mathcal{D}$ if for every $d \in \mathcal{D}$ there exists $c \in \mathcal{C}$ with $(c, d) \in Y$.

Definition 37. For $Y \subseteq \mathcal{C} \times \mathcal{D}$, define $\mathrm{Cell}(Y)$ to be the set of pairs of real-valued functions $(f, g) \in \mathbb{R}^{\mathcal{C}} \times \mathbb{R}^{\mathcal{D}}$ satisfying the gauge condition $f(c_0) = 0$ and the constraints

$$f(c) + g(d) \leq M(c, d) \text{ for all } (c, d) \in \mathcal{C} \times \mathcal{D}, \tag{20}$$

$$f(c) + g(d) = M(c, d) \text{ for all } (c, d) \in Y. \tag{21}$$

Thus $\mathrm{Cell}(Y)$ is a (possibly empty) closed polyhedron. For general Y, feasibility of (20)–(21) does not imply that (f, g) is a nucleus point. The next lemma isolates the combinatorial condition that forces the Isbell equalities in every row or column.

Lemma 38. *Let $Y \subseteq \mathcal{C} \times \mathcal{D}$ and let (f, g) satisfy* (20)–(21)*.*

(a) If Y covers $\mathcal{D}$, then $g = M^* f$.

(b) If Y covers $\mathcal{C}$, then $f = M_* g$.

Consequently, if Y covers both $\mathcal{C}$ and $\mathcal{D}$, then every $(f, g) \in \mathrm{Cell}(Y)$ lies in $\mathrm{Nuc}(M)_0$.

Proof. (a) Fix $d \in \mathcal{D}$. From (20) we obtain $g(d) \leq M(c, d) - f(c)$ for all c, hence

$$g(d) \leq \min_{c \in \mathcal{C}} (M(c, d) - f(c)) = (M^* f)(d).$$

Since Y covers $\mathcal{D}$, there exists c with $(c, d) \in Y$. Then (21) gives $g(d) = M(c, d) - f(c)$, hence $g(d) \geq (M^* f)(d)$. Thus $g(d) = (M^* f)(d)$.

(b) The proof is symmetric. If Y covers both sides, (a) and (b) give $g = M^* f$ and $f = M_* g$, so $(f, g) \in \mathrm{Nuc}(M)$, and the gauge condition places it in $\mathrm{Nuc}(M)_0$. □

Covering is a necessary condition for $\mathrm{Cell}(Y)$ to lie in the nucleus, but it is far from sufficient: for a generic set Y the system (20)–(21) is infeasible. In other words, Lemma 38 says that if Y covers both $\mathcal{C}$ and $\mathcal{D}$, then every $(f, g) \in \mathrm{Cell}(Y)$ lies in $\mathrm{Nuc}(M)_0$, but usually $\mathrm{Cell}(Y)$ is empty.

Definition 39. A subset $Y \subseteq \mathcal{C} \times \mathcal{D}$ is *admissible* if it covers both $\mathcal{C}$ and $\mathcal{D}$ and $\mathrm{Cell}(Y) \neq \varnothing$.

When Y is admissible, $\mathrm{Cell}(Y)$ is a nonempty polyhedron consisting entirely of nucleus points by Lemma 38.

Lemma 40. *Let $(f, g) \in \mathrm{Nuc}(M)_0$, and set*

$$Z(f, g) = \{(c, d) \in \mathcal{C} \times \mathcal{D} \mid f(c) + g(d) = M(c, d)\}.$$

Then $(f, g) \in \mathrm{Cell}(Z(f, g))$. In particular, $Z(f, g)$ is admissible.

Proof. The inequalities (20) are exactly the Isbell inequalities for (f, g). The definition of $Z(f, g)$ is the equality condition (21). Since $\mathcal{C}$ and $\mathcal{D}$ are finite and (f, g) is a nucleus point, each row and each column attains equality, so $Z(f, g)$ covers both $\mathcal{C}$ and $\mathcal{D}$. □

Corollary 41. *The set $\mathrm{Nuc}(M)_0 \cong \mathbb{P}\mathrm{Nuc}(M)$ is a union of finitely many polytopes:*

$$\mathrm{Nuc}(M)_0 = \bigcup_{Y \text{ admissible}} \mathrm{Cell}(Y).$$

Proof. By Lemma 40, each $(f, g) \in \mathrm{Nuc}(M)_0$ lies in $\mathrm{Cell}(Z(f, g))$, and $Z(f, g)$ is admissible. Since $\mathcal{C} \times \mathcal{D}$ is finite, there are only finitely many admissible subsets Y.

Each $\mathrm{Cell}(Y)$ is a closed convex polyhedron by construction. When $\|M\| < \infty$, Corollary 35 implies that $\mathrm{Nuc}(M)_0$ is compact, hence each nonempty $\mathrm{Cell}(Y) \subseteq \mathrm{Nuc}(M)_0$ is bounded. Thus every admissible $\mathrm{Cell}(Y)$ is a polytope. □

Lemma 42. *For any $Y, Y' \subseteq \mathcal{C} \times \mathcal{D}$ one has*

$$\mathrm{Cell}(Y) \cap \mathrm{Cell}(Y') = \mathrm{Cell}(Y \cup Y').$$

Proof. The inequalities (20) are common to all $\mathrm{Cell}(\cdot)$, and the equalities (21) imposed by Y and by Y' together are exactly those imposed by $Y \cup Y'$. $\square$

Corollary 43. *The admissible polytopes* $\{\mathrm{Cell}(Y)\}$ *form a finite polytopal complex inside* $\mathrm{Nuc}(M)_0 \cong \mathbb{P}\mathrm{Nuc}(M)$*: if* Y *and* Y' *are admissible, then* $\mathrm{Cell}(Y) \cap \mathrm{Cell}(Y')$ *is either empty or a common face of both.*

Proof. By Lemma 42, the intersection is $\mathrm{Cell}(Y \cup Y')$. If it is nonempty, then $Y \cup Y'$ covers both $\mathcal{C}$ and $\mathcal{D}$, hence is admissible, and $\mathrm{Cell}(Y \cup Y')$ is obtained from $\mathrm{Cell}(Y)$ and $\mathrm{Cell}(Y')$ by imposing additional linear equalities. Therefore it is a face of each. $\square$

4.3. **Special case $\mathcal{D} = \mathcal{C}$.** One case that we are particularly interested in is when M is a profunctor from $\mathcal{C}$ to itself which we study elsewhere [GJST25]. Here, we use this case to illustrate how restrictive admissibility can be.

Assume $\mathcal{C} = \mathcal{D}$ and $|\mathcal{C}| = n$, and M is an arbitrary $n \times n$ real matrix. The projective space $\mathbb{P}\mathcal{C}$ has dimension $n-1$, so every witness polyhedron has dimension at most $n-1$. We call $\mathrm{Cell}(Y)$ *full-dimensional* if it has dimension $n-1$.

Any subset $Y \subseteq \mathcal{C} \times \mathcal{C}$ that covers both sides has cardinality at least n. Moreover, $|Y| = n$ if and only if Y is the graph of a permutation $\sigma \in S_n$:

$$\Gamma_\sigma := \{(c, \sigma(c)) \mid c \in \mathcal{C}\}.$$

Thus only permutation graphs can support full-dimensional cells.

Proposition 44. *If* $\mathrm{Cell}(Y)$ *is full-dimensional, then* $Y = \Gamma_\sigma$ *for a permutation* $\sigma \in S_n$*.*

Proof. Suppose Y contains two pairs (c, d) and (c', d) with the same second coordinate d. For any $(f, g) \in \mathrm{Cell}(Y)$, the equalities (21) give

$$f(c) + g(d) = M(c, d),$$
$$f(c') + g(d) = M(c', d),$$

hence

$$f(c') - f(c) = M(c', d) - M(c, d)$$

throughout $\mathrm{Cell}(Y)$. In the gauge $f(c_0) = 0$, this is a nontrivial affine relation among the $n-1$ free coordinates of f, so $\mathrm{Cell}(Y)$ cannot have dimension $n-1$.

Therefore, if $\mathrm{Cell}(Y)$ is full-dimensional, each $d \in \mathcal{C}$ occurs in at most one pair of Y. Since Y covers $\mathcal{D} = \mathcal{C}$, each d occurs in exactly one pair. By symmetry, each $c \in \mathcal{C}$ occurs in exactly one pair, so Y is the graph of a permutation. The permutation is uniquely determined by Y. $\square$

The full-dimensional part of $\mathbb{P}\mathrm{Nuc}(M)$ is therefore controlled by permutation patterns. Admissibility is considerably stronger than being a permutation: among the $n!$ permutation graphs, typically only one is admissible. While this result follows from Corollary 25 of [DS04] which gives a combinatorial formula for the number of faces of every dimension in a tropical complex, here we give a constructive argument that finds the admissible permutation. Proposition 45 below is a min-plus version of an optimality criterion for a linear assignment problem, and the proof using shortest-path potential is related to the Hungarian method [Kuh55, BDM09]. For any permutation $\sigma \in S_n$ define its *M-cost* to be $\sum_{c\in\mathcal{C}} M\left(c, \sigma(c)\right)$ and the *value* of M is defined to be the minimum cost over all permutations:

$$\mathrm{val}(M) := \min_{\sigma\in S_n} \sum_{c\in\mathcal{C}} M\left(c, \sigma(c)\right).$$

The value of M is also known as the tropical determinant, which is known to solve the assignment problem [MS15].

Proposition 45. *For $\sigma \in S_n$, the permutation graph Γ_σ is admissible if and only if*

$$\sum_{c\in\mathcal{C}} M\left(c, \sigma(c)\right) = \mathrm{val}(M).$$

Proof. Suppose Γ_σ is admissible and choose $(f, g) \in \mathrm{Cell}(\Gamma_\sigma)$. Summing the equalities $f(c) + g(\sigma(c)) = M(c, \sigma(c))$ over c gives

$$\sum_{c\in\mathcal{C}} f(c) + \sum_{d\in\mathcal{C}} g(d) = \sum_{c\in\mathcal{C}} M\left(c, \sigma(c)\right), \tag{22}$$

since σ is a permutation. For any $\tau \in S_n$, summing the inequalities $f(c)+g(\tau(c)) \leq M(c, \tau(c))$ yields

$$\sum_{c\in\mathcal{C}} f(c) + \sum_{d\in\mathcal{C}} g(d) \leq \sum_{c\in\mathcal{C}} M\left(c, \tau(c)\right).$$

Combining with (22) shows that σ attains the minimum $\mathrm{val}(M)$.

Conversely, assume that σ attains $\mathrm{val}(M)$. We will construct a point $(f, g) \in \mathrm{Cell}(\Gamma_\sigma)$. Consider the complete directed graph on $\mathcal{C}$ with edge weights

$$w(c \to c') := M\left(c', \sigma(c)\right) - M\left(c, \sigma(c)\right).$$

Define $f: \mathcal{C} \to \mathbb{R}$ by shortest-path distances from the base vertex c_0, so $f(c_0) = 0$ and

$$f(c') \leq f(c) + w(c \to c')$$

for all edges. Define $g: \mathcal{C} \to \mathbb{R}$ by

$$g\left(\sigma(c)\right) := M\left(c, \sigma(c)\right) - f(c).$$

Then for any $c, c' \in \mathcal{C}$ we have

$$f(c') + g\left(\sigma(c)\right) = f(c') + M\left(c, \sigma(c)\right) - f(c) \leq M\left(c', \sigma(c)\right),$$

which is (20) for pairs of the form $(c', \sigma(c))$. Since σ is bijective, this is (20) for all pairs (c', d). Moreover, $f(c) + g(\sigma(c)) = M(c, \sigma(c))$ for all c, so (21) holds on Γ_σ. Thus $(f, g) \in \mathrm{Cell}(\Gamma_\sigma)$, and Γ_σ is admissible. Finally, to gaurantee that there is a shortest-path distance, one needs to know that there are no directed cycles with negative total weight. If $c_0 \to c_1 \to \cdots \to c_k = c_0$ had negative total weight, then

$$\sum_{i=0}^{k-1} M\left(c_i, \sigma(c_i)\right) > \sum_{i=0}^{k-1} M\left(c_{i+1}, \sigma(c_i)\right).$$

The right-hand side is the cost of the permutation obtained from σ by cycling the images along this cycle, contradicting optimality. $\square$

In particular, if M is tropically nonsingular in the standard sense that the minimum in $\mathrm{val}(M)$ is achieved by a unique permutation, then there is exactly one admissible permutation graph, hence exactly one full-dimensional cell in $\mathbb{P}\mathrm{Nuc}(M)$.

Remark 46. If σ and τ are permutations with $\sigma \neq \tau$, then $\Gamma_\sigma \cup \Gamma_\tau$ has at least $n+2$ elements. Indeed, if $\sigma \neq \tau$ then the set of c with $\sigma(c) \neq \tau(c)$ has cardinality at least 2, and

$$|\Gamma_\sigma \cup \Gamma_\tau| = n + |\{c \mid \sigma(c) \neq \tau(c)\}| \geq n + 2.$$

By Lemma 42 one has

$$\mathrm{Cell}(\Gamma_\sigma) \cap \mathrm{Cell}(\Gamma_\tau) = \mathrm{Cell}(\Gamma_\sigma \cup \Gamma_\tau),$$

so any intersection forces at least two new equalities beyond those of a permutation graph. In particular, two distinct full-dimensional permutation cells cannot meet along a codimension-one face in $\mathbb{P}\mathrm{Nuc}(M)$.

4.4. **From witness polyhedra to witness cells.** The witness polyhedra $\mathrm{Cell}(Y)$ are best regarded as closures of the open witness cells from §3.5. For $(f,g) \in \mathrm{Nuc}(M)_0$, write $\delta = \delta^{(f,g)}$ and $Z = Z(f,g) = \delta^{-1}(0)$. Then $\mathrm{Cell}(Z)$ is the smallest witness polyhedron containing (f,g), and its relative interior consists of those points for which no additional inequalities become equalities:

$$\mathrm{Cell}^\circ(Z) = \left\{(f',g') \in \mathrm{Cell}(Z) \mid \delta^{(f',g')}(c,d) > 0 \text{ for all } (c,d) \notin Z\right\}.$$

In particular, for every admissible Z the witness cell $\mathrm{Cell}^\circ(Z)$ is the relative interior of the polytope $\mathrm{Cell}(Z)$.

In the classical min-plus setting this recovers the type decomposition of a tropical polytope, for instance via tropical hyperplane arrangements [DS04]. What is new in the present framework is that the same gap matrix controls the metric geometry of the decomposition: nonzero entries of $\delta^{(f,g)}$ appear as critical radii at which balls about (f,g) meet the boundary of $\mathrm{Cell}(Z)$.

4.5. **The events theorem.** We now analyze the nonzero entries of the gap matrix as critical radii, measured in the projective metric, that define the distances to cellular event loci. This is summarized in Theorem 49 and illustrated in an example. Related formulas for the distance to a given halfspace appear computed in Hilbert's projective metric appear in idempotent semimodule theory [NS07]. Our contribution is to show that, for nucleus points, each gap entry $\delta^{(f,g)}(c,d)$ in the gap matrix is exactly the distance to the event locus where (c,d) becomes a witness — these loci correspond to simultaneous half-space events — and yielding a metric refinement of the witness-cell decomposition.

Fix a point $(f,g) \in \mathbb{P}\mathrm{Nuc}(M)$ and let $\delta = \delta^{(f,g)}$ be its gap matrix.

Definition 47. For each pair $(c,d) \in \mathcal{C} \times \mathcal{D}$, we define the corresponding *event locus*

$$\mathcal{E}_{c,d} := \left\{(f',g') \in \mathbb{P}\mathrm{Nuc}(M) \mid \delta^{(f',g')}(c,d) = 0\right\}.$$

Thus $\mathcal{E}_{c,d}$ is the locus in $\mathbb{P}\mathrm{Nuc}(M)$ where (c,d) is a witness pair.

Lemma 48. *Let $(f,g) \in \mathbb{P}\mathrm{Nuc}(M)$ and let $\lambda = \delta^{(f,g)}(c_i,d_j) > 0$. Then there exists $(f',g') \in \mathcal{E}_{c_i,d_j}$ such that*

$$d_{\mathbb{P}\mathrm{Nuc}}\left((f,g),(f',g')\right) = \lambda.$$

Proof. Work in the gauge slice $\mathrm{Nuc}(M)_0$ and choose representatives with $f(c_0) = 0$. Define $f'' : \mathcal{C} \to \mathbb{R}$ by

$$f''(c) = \begin{cases} f(c) & c \neq c_i, \\ f(c_i) + \lambda & c = c_i. \end{cases}$$

Set $g' := M^* f''$ and $f' := M_* g' = M_* M^* f''$. By the identities $M^* M_* M^* = M^*$ and $M_* M^* M_* = M_*$, we have $g' = M^* f'$ and $f' = M_* g'$, hence $(f', g') \in \mathrm{Nuc}(M)_0$.

For each $d \in \mathcal{D}$ one has

$$\begin{aligned} g'(d) &= \min\left(\min_{c \neq c_i} (M(c,d) - f(c)),\, M(c_i,d) - f(c_i) - \lambda \right) \\ &= \min\left(g(d),\, g(d) + \delta^{(f,g)}(c_i,d) - \lambda \right) \\ &= g(d) - \max\left(\lambda - \delta^{(f,g)}(c_i,d), 0 \right). \end{aligned}$$

In particular,

$$g(d) - g'(d) = \max\left(\lambda - \delta^{(f,g)}(c_i,d), 0 \right)$$

takes values in $[0, \lambda]$. Moreover, $g(d_j) = g'(d_j)$ because $\delta^{(f,g)}(c_i, d_j) = \lambda$, and if d_k satisfies $\delta^{(f,g)}(c_i, d_k) = 0$ then $g(d_k) - g'(d_k) = \lambda$. Therefore $d_{\mathcal{D}}([g], [g']) = \lambda$.

Since (f, g) and (f', g') are nucleus points, $f = M_* g$ and $f' = M_* g'$. The map M_* is 1-Lipschitz for the projective metrics by Theorem 18, so

$$d_{\mathcal{C}}([f], [f']) \leq d_{\mathcal{D}}([g], [g']) = \lambda.$$

By definition of $d_{\mathbb{P}\mathrm{Nuc}}$ it follows that

$$d_{\mathbb{P}\mathrm{Nuc}}((f,g), (f',g')) = \lambda.$$

Finally, at the distinguished pair (c_i, d_j) we have $g'(d_j) = g(d_j)$ and

$$f'(c_i) = \min_{d \in \mathcal{D}} (M(c_i,d) - g'(d)) \leq M(c_i,d_j) - g'(d_j) = M(c_i,d_j) - g(d_j) = f(c_i) + \lambda.$$

On the other hand, $f'' \leq f' = M_* M^* f''$ by extensivity of the closure operator $M_* M^*$, so $f'(c_i) \geq f''(c_i) = f(c_i) + \lambda$. Thus $f'(c_i) = f(c_i) + \lambda$, and

$$\delta^{(f',g')}(c_i,d_j) = M(c_i,d_j) - f'(c_i) - g'(d_j) = \delta^{(f,g)}(c_i,d_j) - \lambda = 0.$$

Hence $(f', g') \in \mathcal{E}_{c_i,d_j}$, as claimed. $\square$

It is sometimes useful to express f' directly in terms of the original gap matrix. Writing $\delta = \delta^{(f,g)}$, the formula for g' above gives

$$f'(c) = \min_{d \in \mathcal{D}} (M(c,d) - g'(d)) = f(c) + \min_{d \in \mathcal{D}} (\delta(c,d) + \max(\lambda - \delta(c_i,d), 0)).$$

Given any positive value $\lambda = \delta^{(f,g)}(c,d)$ of the gap matrix $\delta^{(f,g)}$, Lemma 48 gives a constructive way to find a point $(f', g') \in \mathbb{P}\mathrm{Nuc}(M)$ in a different cell from (f, g) with $d((f,g),(f',g')) = \lambda$. The points (f, g) and (f', g') lie in different witness cells since $\delta^{(f',g')}(c,d) = 0$ and $\delta^{(f,g)}(c,d) = \lambda > 0$. The "events theorem" says that this value λ is sharp, meaning that λ is precisely the distance from (f, g) to the event locus $\mathcal{E}_{c,d}$ where (c, d) becomes a witness. As usual, for any subset $S \subseteq \mathbb{P}\mathrm{Nuc}(M)$, let

$$d_{\mathbb{P}\mathrm{Nuc}}((f,g), S) = \inf_{(f',g') \in S} d_{\mathbb{P}\mathrm{Nuc}}((f,g), (f',g')).$$

Theorem 49 (The Events Theorem)**.** *Let $(f,g) \in \mathbb{P}\mathrm{Nuc}(M)$ and let $(c,d) \in \mathcal{C} \times \mathcal{D}$. Then*

$$d_{\mathbb{P}\mathrm{Nuc}}\left((f,g), \mathcal{E}_{c,d}\right) = \delta^{(f,g)}(c,d).$$

Proof. If $\delta^{(f,g)}(c,d) = 0$, then $(f,g) \in \mathcal{E}_{c,d}$ and both sides are 0. Assume $\lambda = \delta^{(f,g)}(c,d) > 0$. By Theorem 49 there exists $(f',g') \in \mathcal{E}_{c,d}$ with $d_{\mathbb{P}\mathrm{Nuc}}\left((f,g),(f',g')\right) = \lambda$, so $d_{\mathbb{P}\mathrm{Nuc}}\left((f,g), \mathcal{E}_{c,d}\right) \le \lambda$.

For the reverse inequality, work in the gauge slice $\mathrm{Nuc}(M)_0$ and fix $(f_1,g_1) \in \mathcal{E}_{c,d}$. Set $a(c) = f_1(c) - f(c)$ and let

$$\alpha = \min_{c \in \mathcal{C}} a(c), \qquad \beta = \max_{c \in \mathcal{C}} a(c).$$

By definition of the projective metric on $\mathbb{P}\mathcal{C}$ one has

$$d_{\mathcal{C}}\left([f],[f_1]\right) = \beta - \alpha.$$

Since $g = M^* f$ and $g_1 = M^* f_1$, for each $d' \in \mathcal{D}$,

$$g_1(d') = \min_{c \in \mathcal{C}}\left(M(c,d') - f_1(c)\right) = \min_{c \in \mathcal{C}}\left(M(c,d') - f(c) - a(c)\right).$$

Therefore

$$g(d') - \beta \le g_1(d') \le g(d') - \alpha,$$

so $g_1(d') - g(d') \in [-\beta, -\alpha]$. It follows that for all $c' \in \mathcal{C}$ and $d' \in \mathcal{D}$,

$$\left(f_1(c') + g_1(d')\right) - \left(f(c') + g(d')\right) = a(c') + \left(g_1(d') - g(d')\right) \in [\alpha - \beta, \beta - \alpha],$$

hence

$$\left|\left(f_1(c') + g_1(d')\right) - \left(f(c') + g(d')\right)\right| \le \beta - \alpha.$$

Evaluating at the distinguished pair (c,d) and using $f_1(c) + g_1(d) = M(c,d)$ gives

$$\lambda = M(c,d) - f(c) - g(d) = \left(f_1(c) + g_1(d)\right) - \left(f(c) + g(d)\right) \le \beta - \alpha.$$

Finally, by definition of $d_{\mathbb{P}\mathrm{Nuc}}$ one has

$$d_{\mathcal{C}}\left([f],[f_1]\right) \le d_{\mathbb{P}\mathrm{Nuc}}\left((f,g),(f_1,g_1)\right),$$

so $\lambda \le d_{\mathbb{P}\mathrm{Nuc}}\left((f,g),(f_1,g_1)\right)$ for every $(f_1,g_1) \in \mathcal{E}_{c,d}$. Taking the infimum over $\mathcal{E}_{c,d}$ yields $d_{\mathbb{P}\mathrm{Nuc}}\left((f,g),\mathcal{E}_{c,d}\right) \ge \lambda$, hence equality. □

We illustrate Theorem 49 on the 3×4 example from §3.5. Let $f = (0,0,0)$ and $g = M^* f = (0.7, -1.6, 0.1, -2.9)$, viewed in the gauge slice $\mathrm{Nuc}(M)_0$. The gap matrix is

$$\delta^{(f,g)} = \begin{bmatrix} 0 & 3.1 & 1.6 & 1.6 \\ 0.5 & 4.2 & 0 & 5.1 \\ 1.3 & 0 & 1.9 & 0 \end{bmatrix}.$$

Figure 2 shows the base point in green and the projective ball of radius $\lambda = 1.9 = \delta^{(f,g)}(c_2,d_2)$ in $\mathbb{P}\mathcal{C} \cong \mathbb{R}^2$. Define $f'' : \mathcal{C} \to \mathbb{R}$ by $f''(c) = f(c)$ for $c \ne c_2$ and $f''(c_2) = f(c_2) + \lambda$. This presheaf need not be $\mathrm{cl}_{\mathcal{C}}$-closed, and in fact is not. It is pictured as by a red point, which lies outside the nucleus. Applying the closure produces $f' = M_* M^* f''$ and hence a nucleus point (f',g') with $\delta^{(f',g')}(c_2,d_2) = 0$. In this example one finds $f' = (0.6, 0, 1.9)$, which is projectively equivalent to $(0, -0.6, 1.3)$ in the gauge $f(c_0) = 0$ and is pictured as a blue point.

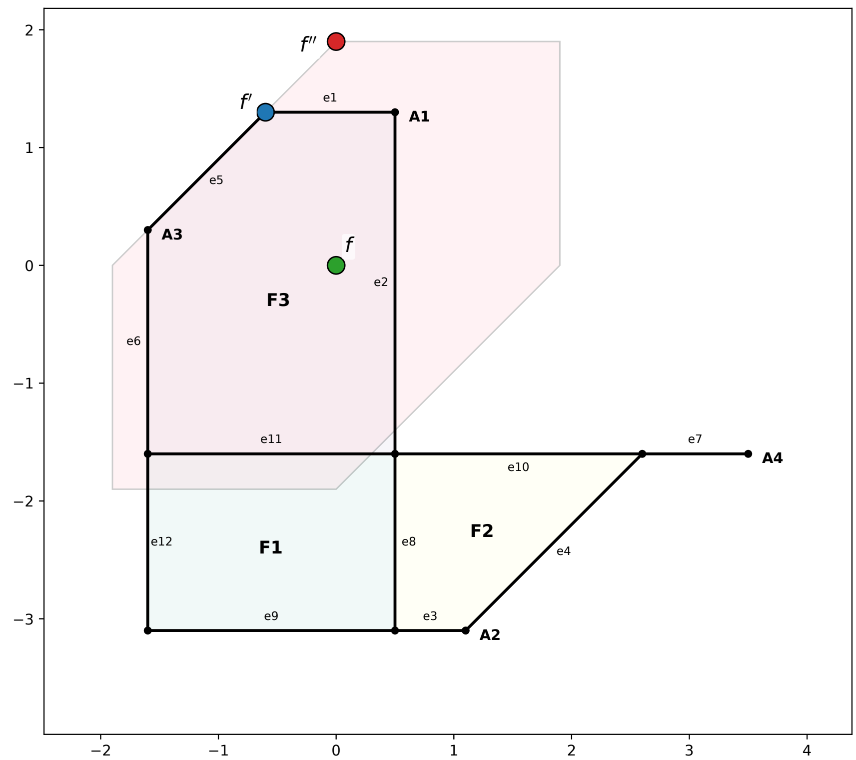

FIGURE 2. Projective ball and event at radius 1.9 in $\mathbb{P}\mathcal{C}$ illustrating how Lemma 48 and Theorem 49 work.

4.6. **Order chambers.** A witness cell is determined by the zero pattern $Z = \delta^{-1}(0)$ of the gap matrix. Theorem 49 shows that the remaining entries carry metric information: for each $(c, d) \notin Z$ the gap value $\delta(c, d)$ is the distance from (f, g) to the event locus $\mathcal{E}_{c,d}$. Keeping track only of the relative order of the positive gaps refines the witness decomposition.

Fix $(f, g) \in \mathrm{Nuc}(M)_0$ and write $\delta = \delta^{(f,g)}$. Define a total preorder $\preceq_{f,g}$ on $\mathcal{C} \times \mathcal{D}$ by

$$(c, d) \preceq_{f,g} (c', d') \iff \delta(c, d) \leq \delta(c', d').$$

Two points lie in the same *order chamber* if they induce the same preorder.

Conversely, let $\preceq$ be a total preorder on $\mathcal{C} \times \mathcal{D}$ and let $Y \subseteq \mathcal{C} \times \mathcal{D}$ be its set of minimal elements. Assume that Y covers $\mathcal{C}$ and $\mathcal{D}$, so that $\mathrm{Cell}(Y) \subseteq \mathrm{Nuc}(M)_0$. The closure of the corresponding order chamber is the subset of $\mathrm{Cell}(Y)$ cut out by the weak inequalities

$$(c, d) \preceq (c', d') \Rightarrow \delta^{(f,g)}(c, d) \leq \delta^{(f,g)}(c', d').$$

The order chamber itself is the relative interior obtained by requiring strict inequality between distinct equivalence classes. These conditions refine the witness polyhedra and descend to $\mathbb{P}\mathrm{Nuc}(M)$ by the gap invariances from §3.5.

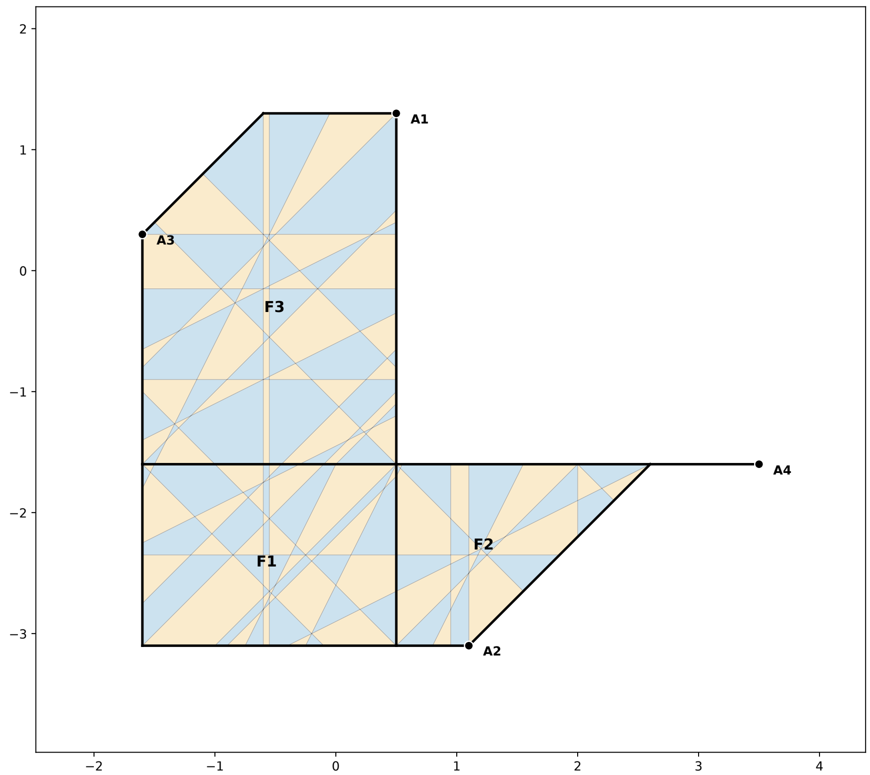

FIGURE 3. The order chambers refining the witness cells.

The reader will note that the order chambers in Figure 3 are two-colored. Since order chambers are regions cut out by a hyperplane arrangement, and the the chamber graph of any arrangement is bipartite [AM17], the adjacency graph of order chambers is always bipartite.

4.7. **Base change to Booleans — formal concept lattices.** We close this section by indicating a further base-change construction that assigns to each point of the nucleus a family of Boolean nuclei. By Booleans, we mean the two element monoidal poset $\{0,1\}$ with order $0 \leq 1$ and monoidal product $\vee$.

A Boolean-valued profunctors, pre- and copresheaves on discrete sets $\mathcal{C}, \mathcal{D}$ are just Boolean valued functions and are identified as subsets. A profunctor is a relation $R \subseteq \mathcal{C} \times \mathcal{D}$, where $R(c,d) = 1$. The nucleus of a Boolean-valued profunctor R is precisely the classical *concept lattice* of R in the sense of formal concept analysis [Wil82, GW99], which we briefly review.

The Isbell conjugates of R are a pair of order-reversing maps between power sets

$$R^*\colon \mathcal{P}(\mathcal{C}) \to \mathcal{P}(\mathcal{D}), \quad R^*(F) = \{d \in \mathcal{D} \mid (c,d) \in R \text{ for all } c \in F\}\,,$$
$$R_*\colon \mathcal{P}(\mathcal{D}) \to \mathcal{P}(\mathcal{C}), \quad R_*(G) = \{c \in \mathcal{C} \mid (c,d) \in R \text{ for all } d \in G\}\,.$$

Equivalently, $R^*(F)$ is the subset of $\mathcal{D}$ related to every $c \in F$, and $R_*(G)$ is the subset of $\mathcal{C}$ related to every $d \in G$. These maps form a Galois connection in the sense that for all $F \subseteq \mathcal{C}$ and $G \subseteq \mathcal{D}$,

$$F \subseteq R_*(G) \iff G \subseteq R^*(F).$$

A *formal concept* of R is a point in the Boolean nucleus of R; that is, a pair (F, G) of subsets of $\mathcal{C}$ and $\mathcal{D}$ with $G = R^*(F)$ and $F = R_*(G)$. Following [Wil82], F is called the *extent* and G the *intent* of the pair (F, G). The set of all formal concepts is a complete lattice, ordered by inclusion of extents; equivalently, by containment of intents:

$$(F, G) \le (F', G') \iff F \subseteq F' \iff G' \subseteq G.$$

Meets and joins are computed by intersecting extents and intents, which are the specialization of Isbell duality

$$\bigwedge_i (F_i, G_i) = \left(\bigcap_i F_i, R^* \left(\bigcap_i F_i \right) \right), \quad \bigvee_i (F_i, G_i) = \left(R_* \left(\bigcap_i G_i \right), \bigcap_i G_i \right).$$

When $R \subseteq R' \subseteq C \times D$ are relations, there are two canonical ways to transport a formal concept of R to a formal concept of R': one may re-close the intent, or re-close the extent.

Proposition 50. *Let $R \subseteq R' \subseteq C \times D$ be relations on sets of objects C and attributes D, with extensions to power sets $R^* \colon \mathcal{P}(C) \to \mathcal{P}(D)$ and $R_* \colon \mathcal{P}(D) \to \mathcal{P}(C)$, and similarly for R'. Write $L = \mathrm{Nuc}(R)$ and $L' = \mathrm{Nuc}(R')$ for their concept lattices. The maps*

$$T^{\mathrm{ext}}_{R \to R'}, T^{\mathrm{int}}_{R \to R'} : L \to L'$$

defined by

$$T^{\mathrm{ext}}_{R \to R'}(F, G) := (R'_*(R'^* F),\, R'^* F)\,,$$
$$T^{\mathrm{int}}_{R \to R'}(F, G) := (R'_* G,\, R'^*(R'_* G))\,.$$

are monotone. Moreover:

(a) $T^{\mathrm{ext}}_{R \to R'}(F, G)$ is the least *concept of $L(R')$ whose extent contains F.*
(b) $T^{\mathrm{int}}_{R \to R'}(F, G)$ is the greatest *concept of $L(R')$ whose intent contains G.*
(c) One has the following inequality in L':

$$T^{\mathrm{ext}}_{R \to R'}(F, G) \;\le\; T^{\mathrm{int}}_{R \to R'}(F, G)$$

4.8. **Chamber-indexed towers and specialization to faces.** Now, let us return to our $\overline{\mathbb{R}}$-profunctor M on discrete $\overline{\mathbb{R}}$ categories $\mathcal{C}$ and $\mathcal{D}$ and fix a point $(f, g) \in \mathrm{Nuc}(M)$. For $\varepsilon \ge 0$, we relax the witness condition "up to ε" by defining a relation on $\mathcal{C} \times \mathcal{D}$.

Definition 51. For $\varepsilon \ge 0$, define a Boolean profunctor $R^{(f,g)}_\varepsilon \colon \mathcal{C}^{\mathrm{op}} \otimes \mathcal{D} \to \{0, 1\}$ by

$$R^{(f,g)}_\varepsilon(c, d) = \begin{cases} 1 & \text{if } \delta^{(f,g)}(c, d) \le \varepsilon, \\ 0 & \text{if } \delta^{(f,g)}(c, d) > \varepsilon. \end{cases}$$

Let $L_\varepsilon(f, g)$ denote the nucleus of $R^{(f,g)}_\varepsilon$, that is, its lattice of formal concepts.

So, an element of $L_\varepsilon(f,g) = \mathrm{Nuc}(R_\varepsilon^{(f,g)})$ is a pair (F,G) with

$$F = (R_\varepsilon)_* G \text{ and } G = (R_\varepsilon)^* F.$$

For brevity, write $R_\varepsilon := R_\varepsilon^{(f,g)}$ and $L_\varepsilon := L_\varepsilon(f,g) = \mathrm{Nuc}(R_\varepsilon)$. If $\varepsilon \le \varepsilon'$ then $R_\varepsilon \subseteq R_{\varepsilon'}$. We have the two monotone maps

$$T^{\mathrm{int}}_{\varepsilon,\varepsilon'},\, T^{\mathrm{ext}}_{\varepsilon,\varepsilon'} \colon L_\varepsilon \to L_{\varepsilon'}$$

given by

$$T^{\mathrm{int}}_{\varepsilon,\varepsilon'}(F,G) = \left((R_{\varepsilon'})_* G, (R_{\varepsilon'})^*(R_{\varepsilon'})_* G\right),$$
$$T^{\mathrm{ext}}_{\varepsilon,\varepsilon'}(F,G) = \left((R_{\varepsilon'})_*(R_{\varepsilon'})^* F, (R_{\varepsilon'})^* F\right).$$

For a fixed numerical value of ε, the relation $R_\varepsilon^{(f,g)}$ need not be locally constant as (f,g) varies in an order chamber, since the gap values $\delta^{(f,g)}(c,d)$ move. What is locally constant on an order chamber is the order in which incidences enter as ε increases. It is therefore convenient to reindex the construction by the equivalence classes of the chamber preorder.

Fix an order chamber Q with total preorder $\preceq_Q$ on $\mathcal{C} \times \mathcal{D}$. Let

$$E_0 \prec_Q E_1 \prec_Q \cdots \prec_Q E_m$$

be the equivalence classes of $\preceq_Q$, ordered from smallest to largest. For $0 \le k \le m$ define

$$R_k^Q := \bigcup_{i=0}^{k} E_i \subseteq \mathcal{C} \times \mathcal{D}$$

to create a finite chain of relations

$$R_0^Q \subseteq R_1^Q \subseteq \cdots \subseteq R_m^Q.$$

By setting $L_k^Q := \mathrm{Nuc}(R_k^Q)$ and setting

$$T_{k,k'} : L_k^Q \to L_{k'}^Q$$

to be the monotone map induced by the inclusions $R_k^Q \subseteq R_{k'}^Q$, for instance by re-closing intents (or re-closing extents) as in the maps T^{int} and T^{ext} above, we obtain a finite tower of concept lattices

$$L_0^Q \to L_1^Q \to \cdots \to L_m^Q.$$

For any $(f,g) \in Q$ and any $\varepsilon \ge 0$, the sublevel relation $R_\varepsilon^{(f,g)}$ is an initial segment of this filtration: there is a unique k such that $R_\varepsilon^{(f,g)} = R_k^Q$. Equivalently, the real-parameter family $\varepsilon \mapsto L_\varepsilon(f,g)$ factors through the chamber tower $\left\{L_k^Q\right\}_{k=0}^{m}$ by a reparameterization of ε that depends on the point (f,g).

4.9. **The global structure over the order chamber complex.** Now we explain how the lattice towers over different order chambers fit together and illustrate using our running example. Let Q' be a face of the closure of Q. Then $\preceq_{Q'}$ is obtained from $\preceq_Q$ by allowing additional ties; each equivalence class for $\preceq_{Q'}$ is a union of consecutive classes for $\preceq_Q$. Consequently, the chain $\left\{R_\ell^{Q'}\right\}$ is obtained from $\left\{R_k^Q\right\}$ by deleting the intermediate relations corresponding to the merged classes, and the

$$\begin{tikzcd}
L_0^Q \arrow[r] \arrow[d, equal] & L_1^Q \arrow[r] & L_2^Q \arrow[r] \arrow[d, equal] & L_3^Q \arrow[r] & L_4^Q \arrow[r] \arrow[d, equal] & L_5^Q \arrow[d, equal] \\
L_0^{Q'} \arrow[rr, "T_{0,2}"] & & L_1^{Q'} \arrow[rr, "T_{2,4}"] & & L_2^{Q'} \arrow[r, "T_{4,5}"] & L_3^{Q'}
\end{tikzcd}$$

FIGURE 4. Specialization to a face: if $Q' \leq \overline{Q}$ merges consecutive preorder blocks, then the chamber tower over Q' is obtained from the tower over Q by deleting the intermediate floors. Here $T_{a,b}$ denotes the chosen structure map $L_a^Q \to L_b^Q$ induced by $R_a^Q \subseteq R_b^Q$ (e.g. $T_{a,b}^{\mathrm{ext}}$ or $T_{a,b}^{\mathrm{int}}$), and on a face these maps compose when intermediate relations are deleted. The bottom structure maps are composites of the skipped maps (e.g. $T_{0,2} = T_{1,2} \circ T_{0,1}$).

lattice tower for Q' is obtained from that for Q by composing the corresponding structure maps.

This provides canonical specialization maps from towers over chambers to towers over their faces, assembling the chamberwise towers into a constructible family over the order-chamber complex.

Now we return to our running example. Let $\mathcal{C} = \{c_0, c_1, c_2\}$, $\mathcal{D} = \{d_1, d_2, d_3, d_4\}$ and let

$$M = \begin{bmatrix} 0.7 & 1.5 & 1.7 & -1.3 \\ 1.2 & 2.6 & 0.1 & 2.2 \\ 2.0 & -1.6 & 2.0 & -2.9 \end{bmatrix}.$$

Work in the gauge slice $f(c_0) = 0$, so f may be written as $f = (0, x, y) \in \mathbb{R}^3$.

Consider the three nucleus points

$$f_1 = (0,0,0), \qquad f_2 = (0,-0.1,0), \qquad f_3 = (0,0.1,0),$$

and set $g_i := M^* f_i$ and $\delta_i(c,d) := M(c,d) - f_i(c) - g_i(d)$. One computes

$$g_1 = (0.7,-1.6,0.1,-2.9), \qquad g_2 = (0.7,-1.6,0.2,-2.9), \qquad g_3 = (0.7,-1.6,0.0,-2.9)$$

and gap matrices

$$\delta_1 = \begin{bmatrix} 0.0 & 3.1 & 1.6 & 1.6 \\ 0.5 & 4.2 & 0.0 & 5.1 \\ 1.3 & 0.0 & 1.9 & 0.0 \end{bmatrix}, \qquad \delta_2 = \begin{bmatrix} 0.0 & 3.1 & 1.5 & 1.6 \\ 0.6 & 4.3 & 0.0 & 5.2 \\ 1.3 & 0.0 & 1.8 & 0.0 \end{bmatrix}, \qquad \delta_3 = \begin{bmatrix} 0.0 & 3.1 & 1.7 & 1.6 \\ 0.4 & 4.1 & 0.0 & 5.0 \\ 1.3 & 0.0 & 2.0 & 0.0 \end{bmatrix}.$$

In particular,

$$\delta_2(c_0,d_3) < \delta_2(c_0,d_4), \qquad \delta_1(c_0,d_3) = \delta_1(c_0,d_4), \qquad \delta_3(c_0,d_4) < \delta_3(c_0,d_3),$$

so f_2 and f_3 lie in adjacent order chambers Q_2 and Q_3 separated by the wall $Q_1 = \overline{Q_2} \cap \overline{Q_3}$ through f_1.

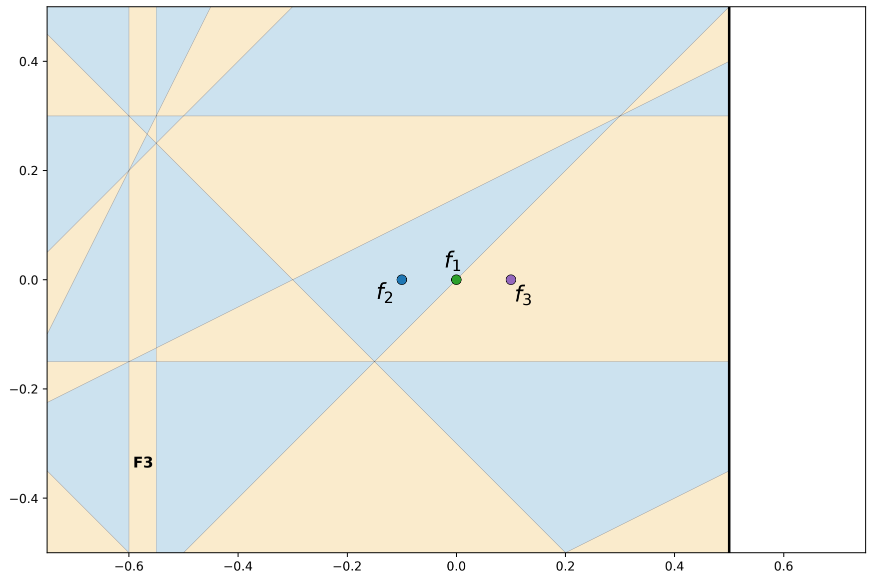

FIGURE 5. The order chambers refining the witness cells. The point f_1 lies directly on an order chamber wall reflecting the tie at 1.6. Nudging off the wall to the left or right by 0.1 gives points f_2 and f_3.

Let $R_\varepsilon^{(f_i,g_i)} := \{(c,d) \mid \delta_i(c,d) \le \varepsilon\}$ be the threshold relations and write $L(R) := \mathrm{Nuc}(R)$ for the concept lattice of a relation $R \subseteq \mathcal{C} \times \mathcal{D}$. Define the relations

$$\begin{aligned} R_0 &:= \{(c_0,d_1),(c_1,d_3),(c_2,d_2),(c_2,d_4)\}, \\ R_1 &:= R_0 \cup \{(c_1,d_1)\}, \qquad R_2 := R_1 \cup \{(c_2,d_1)\}, \\ R_{3a} &:= R_2 \cup \{(c_0,d_3)\}, \qquad R_{3b} := R_2 \cup \{(c_0,d_4)\}, \\ R_4 &:= R_2 \cup \{(c_0,d_3),(c_0,d_4)\}, \\ R_5 &:= R_4 \cup \{(c_2,d_3)\}, \qquad R_6 := R_5 \cup \{(c_0,d_2)\}, \\ R_7 &:= R_6 \cup \{(c_1,d_2)\}, \qquad R_8 := R_7 \cup \{(c_1,d_4)\}. \end{aligned}$$

Then the chamberwise towers are:

$$\begin{aligned} &\text{at } f_2 \in Q_2: && R_0 \subset R_1 \subset R_2 \subset R_{3a} \subset R_4 \subset R_5 \subset R_6 \subset R_7 \subset R_8 \\ &\text{at } f_1 \in Q_1: && R_0 \subset R_1 \subset R_2 \subset R_4 \subset R_5 \subset R_6 \subset R_7 \subset R_8 \\ &\text{at } f_3 \in Q_3: && R_0 \subset R_1 \subset R_2 \subset R_{3b} \subset R_4 \subset R_5 \subset R_6 \subset R_7 \subset R_8. \end{aligned}$$

The only combinatorial difference between the two chambers is the order in which (c_0,d_3) and (c_0,d_4) enter; on the wall they enter simultaneously.

The corresponding concept lattices $L(R) = \mathrm{Nuc}(R)$ are:

$$\begin{aligned} L(R_2) = \{&(\varnothing \mid \{d_1,d_2,d_3,d_4\}),\ (\{c_1\} \mid \{d_1,d_3\}), \\ &(\{c_2\} \mid \{d_1,d_2,d_4\}),\ (\{c_0,c_1,c_2\} \mid \{d_1\})\}, \\ L(R_{3a}) = \{&(\varnothing \mid \{d_1,d_2,d_3,d_4\}),\ (\{c_2\} \mid \{d_1,d_2,d_4\}), \\ &(\{c_0,c_1\} \mid \{d_1,d_3\}),\ (\{c_0,c_1,c_2\} \mid \{d_1\})\}, \end{aligned}$$

$$\begin{tikzcd}
& L(R_{3a}) \arrow[dr, "T_{3a,4}"] & \\
L(R_2) \arrow[ur, "T_{2,3a}"] \arrow[rr, "T_{2,4}"] \arrow[dr, "T_{2,3b}"'] & & L(R_4) \\
& L(R_{3b}) \arrow[ur, "T_{3b,4}"'] &
\end{tikzcd}$$

FIGURE 6. Wall specialization at the 1.6 tie in the running example. The adjacent chambers Q_2 and Q_3 correspond to the two strict refinements of the wall preorder: in Q_2 the incidence (c_0, d_3) enters before (c_0, d_4) (so the intermediate relation is R_{3a}), while in Q_3 the order is reversed (intermediate relation R_{3b}). On the wall Q_1, the tie merges these floors, yielding a direct inclusion $R_2 \subset R_4$. The two composites agree with the wall map $T_{2,4}$.

$$\begin{aligned}
L(R_{3b}) = \{&(\varnothing \mid \{d_1, d_2, d_3, d_4\}),\ (\{c_1\} \mid \{d_1, d_3\}),\ (\{c_2\} \mid \{d_1, d_2, d_4\}),\\
&(\{c_0, c_2\} \mid \{d_1, d_4\}),\ (\{c_0, c_1, c_2\} \mid \{d_1\})\},\\
L(R_4) = \{&(\varnothing \mid \{d_1, d_2, d_3, d_4\}),\ (\{c_0\} \mid \{d_1, d_3, d_4\}),\ (\{c_2\} \mid \{d_1, d_2, d_4\}),\\
&(\{c_0, c_1\} \mid \{d_1, d_3\}),\ (\{c_0, c_2\} \mid \{d_1, d_4\}),\ (\{c_0, c_1, c_2\} \mid \{d_1\})\}.
\end{aligned}$$

Moreover, for $k \geq 4$ the lattices are the same in all three towers: $L(R_5)$ is a 3-element chain, $L(R_6)$ and $L(R_7)$ are 2-element chains, and $L(R_8)$ is the one-point lattice.

Let T denote either transport map T^{ext} or T^{int}, applied to the inclusions $R \subseteq R'$. Then the wall map $T_{2,4} : L(R_2) \to L(R_4)$ is independent of the choice of chamber refinement: one has

$$T_{2,4} = T_{3a,4} \circ T_{2,3a} \qquad \text{and} \qquad T_{2,4} = T_{3b,4} \circ T_{2,3b}.$$

Equivalently, both chamber towers specialize to the wall tower by deleting the intermediate floor and composing the corresponding structure maps.

5. Conclusion and outlook

The guiding theme of this paper is that the projective nucleus $\mathbb{P}\mathrm{Nuc}(M)$ of an $\overline{\mathbb{R}}$ enriched profunctor carries not only categorical structure, but *two* canonical geometries—a projective metric coming from $\mathbb{R}$-enrichment, and a polyhedral cell structure coming from Isbell inequalities—and that these two geometries interact in a concrete, computable way.

5.1. **Why the gap matrix is clarifying.**

A nucleus point (f, g) comes equipped with a nonnegative profunctor

$$\delta^{(f,g)}(c, d) = M(c, d) - f(c) - g(d),$$

which we call the *gap matrix.* The matrix $\delta^{(f,g)}$ is precisely the profunctor that is externally gauged from M so that f and g are translated to the zero pre and co-presheaves. Conceptually, this single object plays a role in each geometry.

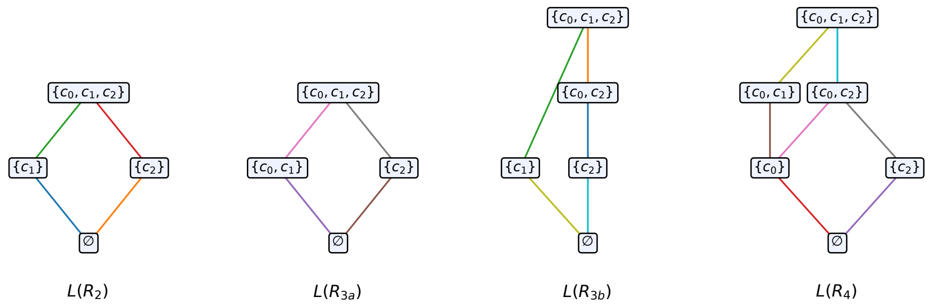

FIGURE 7. Hasse snapshots of the concept lattices at the wall and its two adjacent chambers (extents shown). The wall tower jumps from $L(R_2)$ to $L(R_4)$, while each adjacent chamber inserts a different intermediate lattice.

The vanishing pattern $Z(f,g) = \delta^{-1}(0)$ records exactly which inequalities are tight, and hence determines the witness cell containing (f,g). In this sense, the gap matrix packages the polyhedral cell structure, the usual "type data" of tropical convexity into a pointwise invariant that is intrinsic to the Isbell fixed-point locus.

The Events Theorem upgrades this combinatorial data to a quantitative statement involving the projective geometry: for each pair (c,d), the value $\delta^{(f,g)}(c,d)$ is *exactly* the distance in the projective nucleus metric from (f,g) to the event locus $\mathcal{E}_{c,d}$ where (c,d) becomes a witness. Each entry of the gap matrix is a *critical radius* at which the metric ball about (f,g) first meets a specific boundary stratum of the polyhedral complex.

This distance-to-wall principle has two immediate consequences. First, it provides a local "radial profile" of the cell structure around a point: sorting the positive gaps produces an ordered list of the radii at which *some* new inequality becomes an equality, and identifies which inequality appears at each radius. Second, the proof is constructive: given a desired event (c,d), one can explicitly produce a nucleus point at the corresponding distance by a controlled perturbation followed by Isbell closure. This makes the metric geometry usable for navigation on $\mathbb{P}\mathrm{Nuc}(M)$ and not only a background structure.

5.2. **What becomes canonical.** Several pieces of structure that could conceivably be presentation-dependent in tropical geometry become canonical in the Isbell picture.

- **A projective metric on the underlying space.** The max-spread (Hilbert–oscillation) metric on projective (co)presheaves is induced directly from the enriched hom. Moreover, the two standard realizations of $\mathbb{P}\mathrm{Nuc}(M)$ (on the presheaf and copresheaf sides) are identified by inverse isometries given by the projective Isbell maps.
- **Gauge-invariant local geometry.** The gap matrix is unchanged by the internal projective scaling action and is compatible with external gauge transformations of the matrix M. As a result, the distance data encoded

by δ is an invariant of the underlying profunctor geometry rather than an artifact of a chosen chart or normalization.

- **A canonical refinement of the type decomposition.** Beyond the zero pattern, the *relative order* of the positive gap values is locally constant on regions (order chambers), and ties correspond to simultaneous boundary events. This refines the witness-cell decomposition by recording not only *which* walls exist, but the *order* in which they appear in the projective metric around a point.

5.3. **Boolean shadows and constructible concept-lattice data.** A natural first impulse is to study an $\overline{\mathbb{R}}$-profunctor M by "Booleanizing" it, for instance by thresholding its values to obtain relations

$$R_\varepsilon^M := \{(c,d) \in \mathcal{C} \times \mathcal{D} \mid M(c,d) \leq \varepsilon\},$$

and then take their Boolean nuclei (concept lattices). However, this procedure is not well adapted to the geometry of $\mathbb{P}\mathrm{Nuc}(M)$, and among the difficulties with this approach is the dependence on the choice of external gauge for M. We have found the correct thresholding is *pointed*, leading to a global way to integrate the combinatorics of Boolean relations with the geometry of the $\overline{\mathbb{R}}$ nucleus.

To extract Boolean combinatorics that is intrinsic to $\mathbb{P}\mathrm{Nuc}(M)$, first fix a nucleus point $[(f,g)] \in \mathbb{P}\mathrm{Nuc}(M)$ and pass to the *gap matrix*

$$\delta^{(f,g)}(c,d) := M(c,d) - f(c) - g(d) \in [0,\infty].$$

Equivalently, $\delta^{(f,g)}$ is the external gauge transform of M that translates (f,g) to the zero pre- and copresheaves; by construction it is invariant under projective scaling and compatible with external gauge transformations of M.

Then, thresholding the gap matrix yields a tower of relations For $\varepsilon \geq 0$ define

$$R_\varepsilon^{(f,g)} := \{(c,d) \in \mathcal{C} \times \mathcal{D} \mid \delta^{(f,g)}(c,d) \leq \varepsilon\}.$$

The Boolean nucleus $L_\varepsilon(f,g) := \mathrm{Nuc}\big(R_\varepsilon^{(f,g)}\big)$ of each relation is a complete lattice of formal concepts. The point of Step 1 is that $R_\varepsilon^{(f,g)}$ is not an arbitrary truncation: by the Events Theorem, $\delta^{(f,g)}(c,d)$ is the sharp distance from $[(f,g)]$ to the event locus where (c,d) becomes a witness. Thus thresholding $\delta^{(f,g)}$ records exactly which witness events occur within radius ε of the basepoint.

This is the means by which we answer the question **Where are the Boolean concepts in the projective geometry?** Our answer is that the lattices $L_\varepsilon(f,g)$ organize into a constructible family over the order-chamber refinement: on an order chamber the relative order of the critical radii $\{\delta^{(f,g)}(c,d)\}$ is constant, so the one-parameter family $\varepsilon \mapsto L_\varepsilon(f,g)$ factors through a finite tower indexed by preorder blocks, and these towers specialize functorially to faces by merging consecutive blocks. Viewed as a family stratified over the polyhedral order chamber complex, the stalk captures how concepts appear as one relaxes the witness condition, and the specialization maps record exactly how these concept structures merge when one moves to walls where event radii tie. In this sense, the lattice towers behave like stratified Morse data: as ε increases, combinatorial changes occur only at the discrete critical radii recorded by the gap matrix, and crossing a wall corresponds to a functorial merging of consecutive stages. Global information about $\mathbb{P}\mathrm{Nuc}(M)$

and its polyhedral and metric geometries can be reconstructed from the discrete combinatorial input of a single stalk of locally constant lattice stalks. We expect this viewpoint to yield computable invariants and practical navigation schemes for $\overline{\mathbb{R}}$-enriched nuclei and tropical polytopes.

5.4. **Outlook: reconstruction and persistence-type invariants.** The gap matrix viewpoint does more than relate two structures abstractly: it gives a concrete way to pass between polyhedral data (which inequalities are tight) and metric data (how far one is from making further inequalities tight). This suggests two directions that we expect will be useful both conceptually and computationally.

5.4.1. *Reconstruction from a single pointed stalk.* Fix a basepoint $x = [(f, g)] \in \mathbb{P}\mathrm{Nuc}(M)$ and consider the associated *pointed* threshold relations $R_\varepsilon^{(f,g)}$ and their Boolean nuclei $L_\varepsilon(f, g)$. The Events Theorem identifies each entry $\delta^{(f,g)}(c, d)$ as the *exact* distance from x to the event locus $\mathcal{E}_{c,d}$, so the multiset of gap values, together with the labels (c, d), specifies which walls are encountered at which radii. Meanwhile, on an order chamber the relative order of these radii is fixed, and the chamber-indexed tower records how witness data (and hence cells) appear as the radius increases, with wall-crossing encoded by the specialization maps when radii tie. In particular, this single pointed tower carries enough discrete information to recover the witness-cell complex of $\mathbb{P}\mathrm{Nuc}(M)$ together with the distance-from-x function to its faces in a way that is reminiscent of Morse theory.

5.4.2. *Birth–death data and barcode-type summaries from lattice towers.* Because $L_\varepsilon(f, g)$ can change only when ε crosses a gap value, each point x determines a finite sequence of lattice changes at a finite set of critical radii. One can therefore attach *birth* and *death* radii to lattice features tracked through the tower by the canonical transport maps (e.g. re-closing intents or extents): for instance, when a new join-irreducible concept first appears, and when it merges into a previously existing feature. Collecting these intervals produces barcode-like summaries, in the spirit of persistence, attached to points of $\mathbb{P}\mathrm{Nuc}(M)$; a natural question is how these barcodes vary across chambers and how they behave under wall-crossing.

5.4.3. *Beyond the discrete setting and further structures.* Although the polyhedral arguments here use finite discrete indexing categories, witnesses, gaps, and the projective metric make sense for general small $\overline{\mathbb{R}}$-categories, and it would be interesting to understand what replaces the polytopal stratification in that generality. It is also natural to ask how the metric–polyhedral picture interacts with additional algebraic structure on M. This is especially relevant for example monoidal compatibility, as in our companion work [GJST25].

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ETH Zurich, Switzerland
Email address: juan.luis.gastaldi@inf.ethz.ch

CUNY Queens College, New York, USA
Email address: Samantha.jarvis@qc.cuny.edu

CNRS Paris, France
Email address: thomas.seiller@cnrs.fr

CUNY Queens College and CUNY Graduate Center, New York, USA
Email address: jterilla@gc.cuny.edu