

1. Let X be locally compact and Hausdorff. Prove that $(X \times \mathbb{R})^*$, the one-point compactification of $X \times \mathbb{R}$ is homeomorphic to the suspension ΣX . What does this say for $X = \mathbb{R}^n$?

2. Consider the suspension-loop adjunction $\Sigma : \mathbf{Top} \leftrightarrows \mathbf{Top} : \Omega$. The unit and counit of this adjunction are maps

$$\eta_X : X \rightarrow \Omega \Sigma X \text{ and } \epsilon_Y : \Sigma \Omega Y \rightarrow Y$$

Describe these maps and verify the two “triangle identities”

$$\epsilon_{\Sigma X} \circ \Sigma(\eta_X) = \text{id}_{\Sigma X} \text{ and } \Omega(\epsilon_Y) \circ \eta_{\Omega Y} = \text{id}_{\Omega Y}.$$

3. Consider the points $e = 1$, $n = i$, $w = -1$, and $s = -i$ on S^1 and let $U = S^1 \setminus \{e, w\}$ and $V = S^1 \setminus \{n, s\}$. Then S^1 is the pushout

$$\begin{array}{ccc} U \cap V & \xrightarrow{j_U} & U \\ j_V \downarrow & \lrcorner & \downarrow \\ V & \longrightarrow & S^1 \end{array}$$

Use the groupoid version of SVK to compute $\Pi_1(S^1)$ and $\pi_1(S^1, 1)$.

4. The map $\exp : \mathbb{C} \rightarrow \mathbb{C}^*$ is a covering map. Prove that there cannot exist a function $g : \mathbb{C}^* \rightarrow \mathbb{C}$ with $\exp \circ g = \text{id}_{\mathbb{C}^*}$.

5. Consider the diagram $S^1 \leftarrow S^1 \leftarrow S^1 \leftarrow \dots$ where every arrow is the map $z \mapsto z^p$ for a prime p . The limit of this diagram, call it X_p , comes with maps to each S^1 in the picture. Decide whether these maps $X_p \rightarrow S^1$ are covering maps and (if so) analyze the relevant groups.

6. Let $X = S^1 \vee S^1$.

(a) Classify all two-sheeted covers of X up to isomorphism.

(b) Now take two of these two-sheeted covers $E \rightarrow X$ and $E' \rightarrow X$ and draw their pullback, which is a new cover of X .

$$\begin{array}{ccc} E'' & \longrightarrow & E' \\ \downarrow & \lrcorner & \downarrow p' \\ E & \xrightarrow{p} & X \end{array}$$

(c) The covers in your pullback correspond to subgroups H and K of G , the free group on two generators. What is the index of $H \cap K$ in G ?

7. The fundamental group of the Klein bottle K is given by $\langle r, s : rsr = s \rangle$. If we add the relations r^n and s^2 we get the Dihedral group D_n . So, K admits a cover with Galois group D_n . Describe it and describe the cover when we only add the relation s^2 .