Topology HW 5 **Fall 2025**

1. Let μ be a finitely additive, nonnegative measure on the integers $\mathbb{Z}$.

(a) Prove that the collection of full-measure sets $\{A \subset \mathbb{Z} : \mu(A) = \mu(\mathbb{Z})\}$ is a filter.

(b) Prove that if μ is $\{0, 1\}$ valued then the collection of full-measure sets is an ultrafilter.

2. Let $X = [0, 1]^{[0,1]}$ and consider two topologies on X: the product topology and the topology induced by the sup norm $\|f\|_\infty = \sup_{x\in[0,1]} |f(x)|$. Prove that topology induced by the sup norm is strictly finer than the product topology.

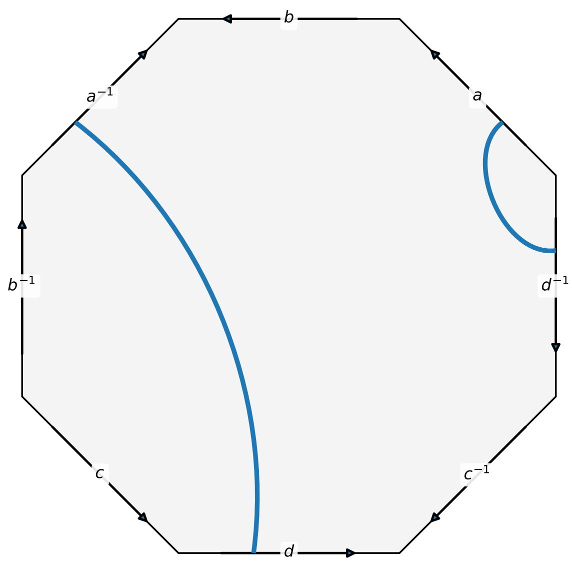
3. Let X be the pushout in **Top** of the following diagram

$$\begin{tikzcd}
\coprod_{(x,y)\in\mathbb{R}^2} \{*\} \arrow[r, "f"] \arrow[d, "g"'] & \coprod_{(x,y)\in\mathbb{R}^2} \mathbb{R} \arrow[d] \\
\mathbb{R}^2 \arrow[r] & X \arrow[ul, phantom, "\ulcorner", very near start]
\end{tikzcd}$$

where g sends $*$ in the (x, y) summand to $(x, y) \in \mathbb{R}^2$ and the map g, in each summand, sends $*$ to $0 \in \mathbb{R}$. Prove that X is homeomorphic to $\mathbb{R}^3$ with the lumberjack metric.

4. Let CH be the category of compact Hausdorff spaces (objects are compact Hausdorff spaces morphisms are continuous maps). Prove that every morphism $X \to Y$ factors $X \to Z \to Y$ where the first map is a quotient map and the second map is a closed embedding.

5. Consider the surface obtained as a quotient of an octogon where edges are identified according to the following diagram. Cutting along the blue curve results in a surface with two boundary circles. If you cap the two new boundary circles with disks, what new surface do you get?



6. There are only finitely many surfaces that admit an embedding of a 3 regular graph in which every face is a hexagon.

(a) Which are they?

(b) Pick one and draw an embedding of a 3 regular graph in which every face is a hexagon.

7. A compactification of a space X is an embedding of X as a dense subset of a compact Hausdorff space. You can make a category out of compactifications of X by letting the objects be compactifications with morphisms defined as follows: a morphism from a compactifaction $f_1 : X \to K_1$ to a compactification $f_2 : X \to K_2$ is a map $g : K_1 \to K_2$ so that $gf_1 = f_2$

$$\begin{tikzcd} & K_1 \arrow[dd, "g"] \\ X \arrow[ur, "f_1"] \arrow[dr, "f_2"'] & \\ & K_2 \end{tikzcd}$$

Prove that the category of compactifications of X is *thin* meaning that there is at most one morphism between any two compactifications.