A little algebraic topology

1. A *retraction* (of X onto A) is a continuous map $r : X \to A$ with $ri = \mathrm{id}_A$ where $i : A \hookrightarrow X$ is the inclusion:

$$\begin{tikzcd} A \arrow[r, hook, "i"] \arrow[rr, bend right, "\mathrm{id}_A"'] & X \arrow[r, "r"] & A \end{tikzcd}$$

Let $D^{n+1} = \{x \in \mathbb{R}^{n+1} : |x| \leq 1\}$ and $S^n = \{x \in \mathbb{R}^{n+1} : |x| = 1\}$. Prove that there is no retraction of D^{n+1} onto S^n.

A little set theory

2. Let X and Y be sets and $f : X \to Y$. Some people define f to be surjective iff

$$\text{for all } y \in Y \text{ there exists } x \in X \text{ with } fx = y$$

and define f to be injective iff

$$\text{for all } x, x' \in X \text{ if } x \neq x' \text{ then } fx \neq fx'.$$

Use these definitions of surjective and injective to prove:

(a) f is surjective iff f is right-cancellable.

(b) f is injective iff f is left-cancellable.

(c) f is surjective iff $\mathsf{Set}(Y, Z) \xrightarrow{f^*} \mathsf{Set}(X, Z)$ is injective for all Z.

(d) f is injective iff $\mathsf{Set}(Z, X) \xrightarrow{f_*} \mathsf{Set}(Z, Y)$ is injective for all Z.

(danger below this line)

(e) f is surjective iff f is right-invertible.

(f) f is injective iff f is left-invertible.

(g) f is injective iff $\mathsf{Set}(Y, Z) \xrightarrow{f^*} \mathsf{Set}(X, Z)$ is surjective for all Z.

(h) f is surjective iff $\mathsf{Set}(Z, X) \xrightarrow{f_*} \mathsf{Set}(Z, Y)$ is surjective for all Z.

(i) If f has a unique left inverse then it is (two-sided) invertible.

(j) If f has a unique right inverse then it is (two-sided) invertible.

A little point-set topology

3. Let (X, d) be a metric space. Define a set $U \subset X$ to be *open* iff for all $x \in U$ there exists $\epsilon > 0$ so that for all $x' \in X$ if $d(x, x') < \epsilon$ then $x' \in U$.

(a) Prove that the arbitrary union of open sets is open.

(b) Prove that the intersection of finitely many open sets is open.

(c) Prove $\emptyset$ and X are both open.

4. A function $f : X \to Y$ between two metric space is continuous iff for all $x \in X$ and for all $\epsilon > 0$ there exists a $\delta > 0$ so that if $d_X(x, x') < \delta$ then $d_Y(fx, fx') < \epsilon$.. Prove that f is continuous iff $f^{-1}(U)$ is open in X whenever U is open in Y.