PSYCH-GA.2211/NEURL-GA.2201 – Fall 2018
Mathematical Tools for Neural and Cognitive Science

Homework 3

Due: 26 Oct 2018
(late homeworks penalized 10% per day)

See the course web site for submission details. Do yourself a favor, and don't wait until the day before the due date... *start now*!

1. **LSI system characterization.** You are trying to experimentally characterize three auditory neurons, in terms of their responses to sounds. For purposes of this problem, the responses of these neurons are embodied in compiled matlab functions `unknownSystemX.p`, (with X=1, 2, 3), each of which takes an input column vector of length $N = 64$ whose elements represent sound pressure over time. The response of each is a column vector (of the same length) representing the mean spike count over time. Your task is to examine them to see if they behave like they're linear and/or shift-invariant with circular (periodic) boundary-handling. For each neuron,

   (a) "Kick the tires" by measuring the response to an impulse in the first position of an input vector. Check that this impulse response is shift-invariant by comparing to the response to an impulse at positions $n = 2, 4, 8$. Also check that the response to a sum of any two of these impulses is equal to the sum of their individual responses.

   (b) If the previous tests succeeded, examine the response of the system to sinusoids with frequencies $\{2\pi/N, 4\pi/N, 8\pi/N, 16\pi/N\}$, and random phases, and check whether the outputs are sinusoids of the same frequency (i.e., verify that the output vector lies completely in the subspace containing all the sinusoids of that frequency). [note: Make the input stimuli positive, by adding one to each sinusoid, and the responses should then be positive (mean spike counts)].

   (c) If the previous tests succeeded, verify that the change in amplitude and phase from input to output is predicted by the amplitude (`abs`) and phase (`angle`) of the corresponding terms of the Fourier transform of the impulse response. If not, explain which property (linearity, or shift-invariance, or both) seems to be missing from the system. If so, do you think that the combination of all tests *guarantees* that the system is linear and shift-invariant? What combination of tests would provide such a guarantee?

2. **Fourier transform of periodic signals.**

   (a) Generate and plot a signal (vector) of length 2048 containing the function $f(n) = \mathrm{mod}(n, 32)/32$. This waveform is known as a "sawtooth". Note that it's periodic, with a period of 32 samples. Play it through your computer speakers or headphones, using the function `sound`. Assuming you use the default playback rate of 8192 samples/sec, what is the duration of your signal (in sec), what is the duration of one cycle of the sawtooth, and what is the frequency of the repetitions (in cycles/sec)? What note on the piano is closest to this (consult Google, or a piano!)?

(b) Compute and plot the Fourier amplitude spectrum, centered at zero. Label the x-axis using units of cycles/sec. What do you see? What about the plot indicates that the signal is periodic, and how can you determine the period? Test your assertion by generating another sawtooth signal, with a period of 24 samples and noting what changes to the plot of the fourier amplitude spectrum.

(c) Generate and plot another periodic signal, with function $g(n) = (1+cos(2\pi\, 64\, n/2048))^2$. Again, compute and plot the Fourier amplitude spectrum, centered at zero. How does this differ from the plot of the Fourier spectrum of $f(n)$? Is the periodicity the same or different? Compare the shape of the function by plotting a period of this function on top of one period of the previous function. What in the Fourier spectrum indicates that the waveform shape is different? Play this signal using the `sound` function. Does it sound different to you from the previous one? [you might also want to compare to the sound of functions that are the same shape, but different period].

3. **Gabor filter.** The response properties of neurons in primary visual cortex (area V1) are often described using linear filters. We'll examine a one-dimensional cross-section of the most common choice, known as a "Gabor filter" (named after Electrical Engineer/Physicist Denis Gabor, who developed it in 1946 for use in signal processing).

   (a) Create a one-dimensional linear filter that is a product of a Gaussian and a sinusoid, $\exp\left(-\frac{n^2}{2\sigma^2}\right)\cos(\omega n)$, with parameters $\sigma = 3.5$ and $\omega = 2\pi * 10/64$ samples. The filter should contain 25 samples, and the Gaussian should be centered on the middle (13th) sample. Plot the filter to verify that it looks like what you'd expect. Plot the amplitude of the Fourier transform of this filter, sampled at 64 locations (MATLAB's `fft` function takes an optional additional argument). What kind of filter is this? Why does it have this shape, and how is the shape related to the choice of parameters (σ, ω)? Specifically, how does the Fourier amplitude change if you alter each of these parameters?

   (b) If you were to convolve this filter with sinusoids of different frequencies, which of them would produce a response with the largest amplitude? Obtain this answer by reasoning about the equation defining the filter (above), and also by finding the maximum of the computed Fourier amplitudes (using the `max` function), and verify that the answers are the same. Compute the *period* of this sinusoid, measured in units of sample spacing (hint: this is the inverse of its frequency, in cycles/sample), and verify by eye that this is roughly matched to the oscillations in the graph of the filter itself. Which two sinusoids would produce responses with about 25% of this maximal amplitude?

   (c) Create three unit-amplitude 64-sample sinusoidal signals at the three frequencies (low, mid, high) that you found in part (b). Convolve the filter with each, and verify that the amplitude of the response is approximately consistent with the answers you gave in part (b). (hint: to estimate amplitude, you'll either need to project the response onto sine and cosine of the appropriate frequency, or compute the DFT of the response and measure the ampitude at the appropriate frequency).

4. **Deconvolution of the Haemodynamic Response.** Neuronal activity causes local changes in deoxyhemoglobin concentration in the blood, which can be measured using magnetic resonance imaging (MRI). One drawback of this is that the haemodynamic response is both delayed and slower than the underlying neural responses. We can model the delay and spread of the measurements relative to the neural signals using a linear shift-invariant system:

$$r(n) = \sum_k x(n-k)h(k), \qquad (1)$$

where $x(n)$ is an input signal delivered over time (for example, a sequence of light intensities), $h(k)$ is the haemodynamic response to a single light flash at time $k = 0$ (i.e., the impulse response of the MRI measurement), and $r(n)$ is the MRI response to the full input signal.

In the file `hrfDeconv.mat`, you will find a response vector r and an input vector x containing a sequence of impulses (indicating flashes of light). Your goal is to estimate the HRF, h, from the data. Each of these signals are sampled at 1 Hz. Plot vectors r and x versus time to get a sense for the data. (Use the `stem` command for x, and label the x-axis).

(a) Convolution is linear, and thus we can re-write the equation above as a matrix multiplication, $r = Xh$, where h is a vector of length M, and X is an $N + M - 1 \times M$ matrix (N is the length of the input x). Write a matlab function `createConvMat`, that takes as arguments an input vector x and M (the dimensionality of h) and generates a matrix X such that the response $r = Xh$ is as defined in Eq. 1 for any h. Verify that the matrix generated by your function produces the same response as MatLab's `conv` function when applied to a few random h vectors of length $M = 15$. Visualize the matrix X as an image (call `imagesc(X)`, and describe its structure.

(b) Now, given the X generated by your function for $M = 15$, solve for h by formulating a least-squares regression problem:

$$h_{\text{opt}} = \arg\min_h ||r - Xh||^2$$

Plot h_{opt} as a function of time (label your x-axis, including units). How would you describe it? How long does it last?

(c) Its often easier to understand an LSI system by viewing it in the frequency domain. Plot the power-spectrum of the HRF (i.e. $|\mathcal{F}(\langle)|^{\in}$, where $\mathcal{F}(\langle)$ is the Fourier transform of the HRF). Re-center around zero frequency, and label the x axis, in Hz. Based on this plot, what kind of filter is the HRF? Specifically, which frequencies does it allow to pass, and which does it block?

5. **Sampling and aliasing.** Load the file `myMeasurements.mat` into matlab. It contains a vector, `sig`, containing voltage values measured from an EEG electrode, sampled at 100Hz. Plot `sig` as a function of vector `time` (time, in seconds), using the flag 'ko-' in matlab's plot command so you can see the samples.

(a) This signal is only a small portion of the full data, and you don't want to store all those values. Create a subsampled version of the signal, which contains every *fourth* value. Plot this, against the corresponding entries of the `time` vector, on top of the original data (use matlab's `hold` function, and plot with flag 'r*-'). How does this reduced version of the data look, compared to the original? Does it provide a good summary of the original measurements? Is the subsampling operation linear? Shift-invariant? Explain.

(b) Let's examine the signal in the frequency domain. First plot the magnitude (amplitude) of the Fourier transform of the original signal, over the range $[-N/2, (N/2) - 1]$ (use `fftshift`). Now plot the same thing for the subsampled signal, after upsampling it back up to full size (i.e., make a full-size signal filled with zeros, and set every fourth sample equal to one of the subsampled values). Explain the relationship between the two plots. In particular, what has happened to the frequency components of the original signal? How does this transformation explain the appearance of the subsampled signal? [Hints: remember that the DFT is periodic. It might help to plot four periods of the DFT of the subsampled signal, fftshift'ed which will be the same size as the DFT of the original signal.]