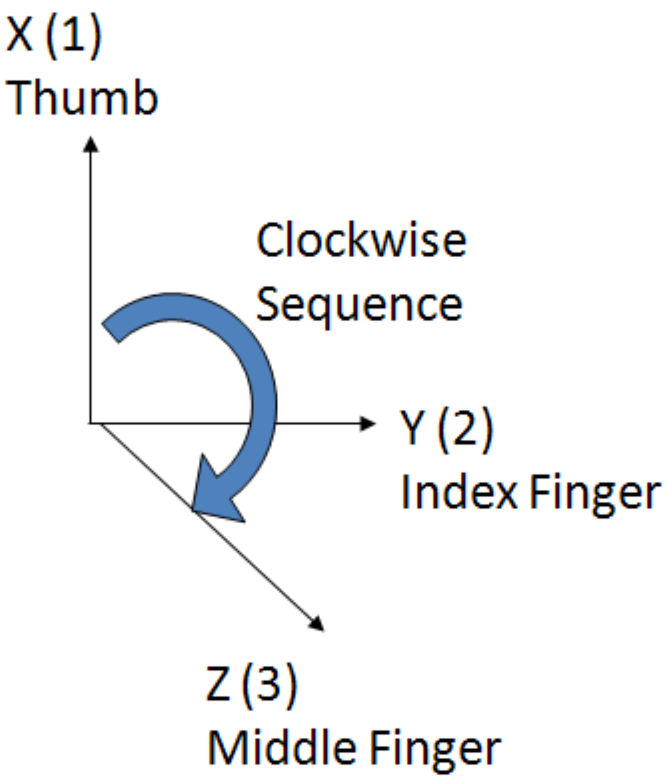
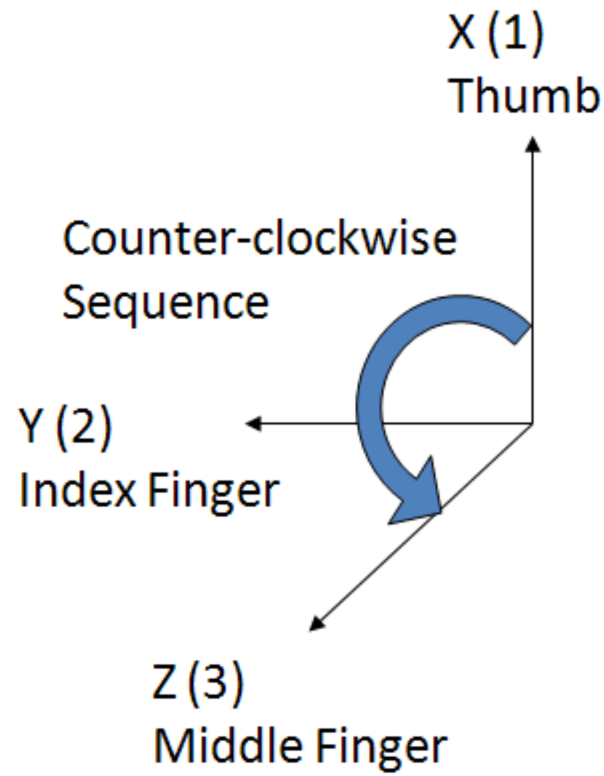
Co-ordinate Systems

- Left-handed



- Right-handed



Co-ordinate Systems (cont'd)

- Which are left-handed and which are right-handed systems?

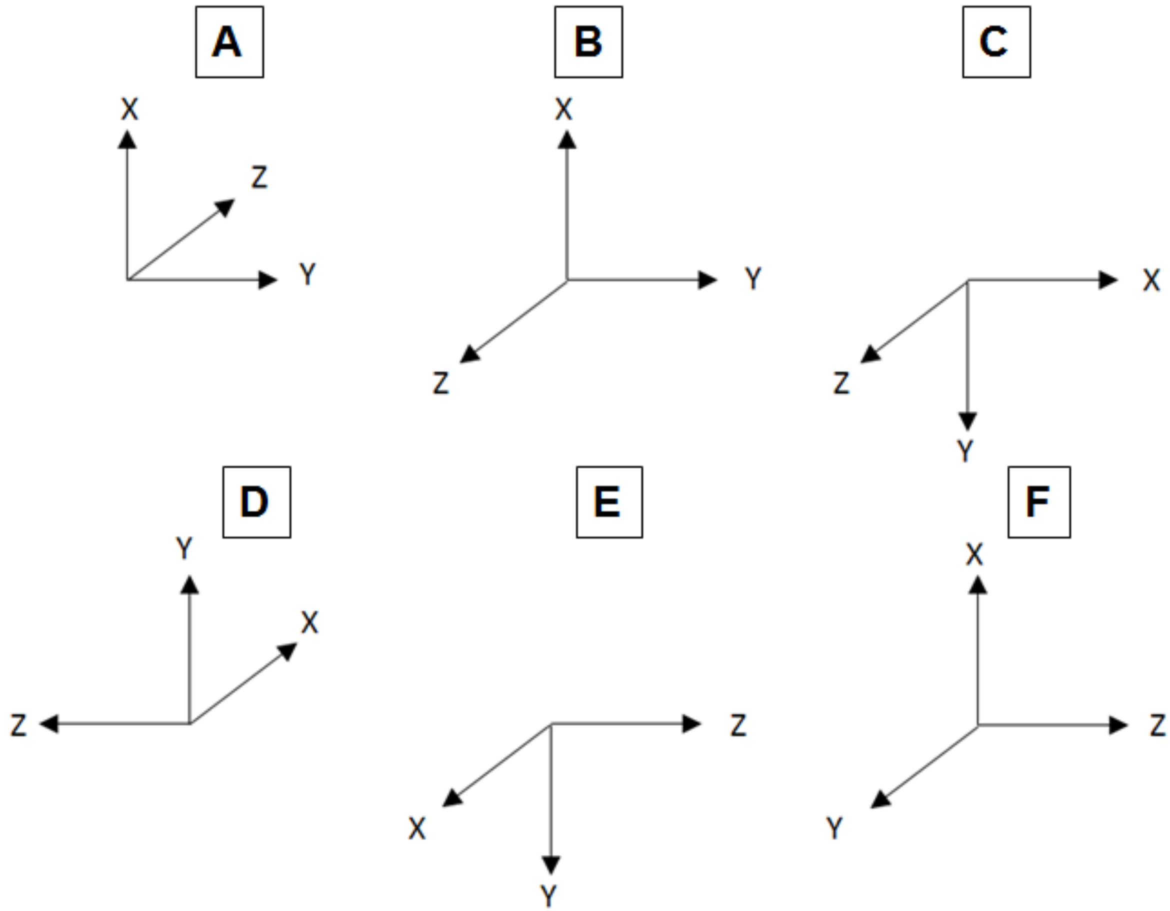
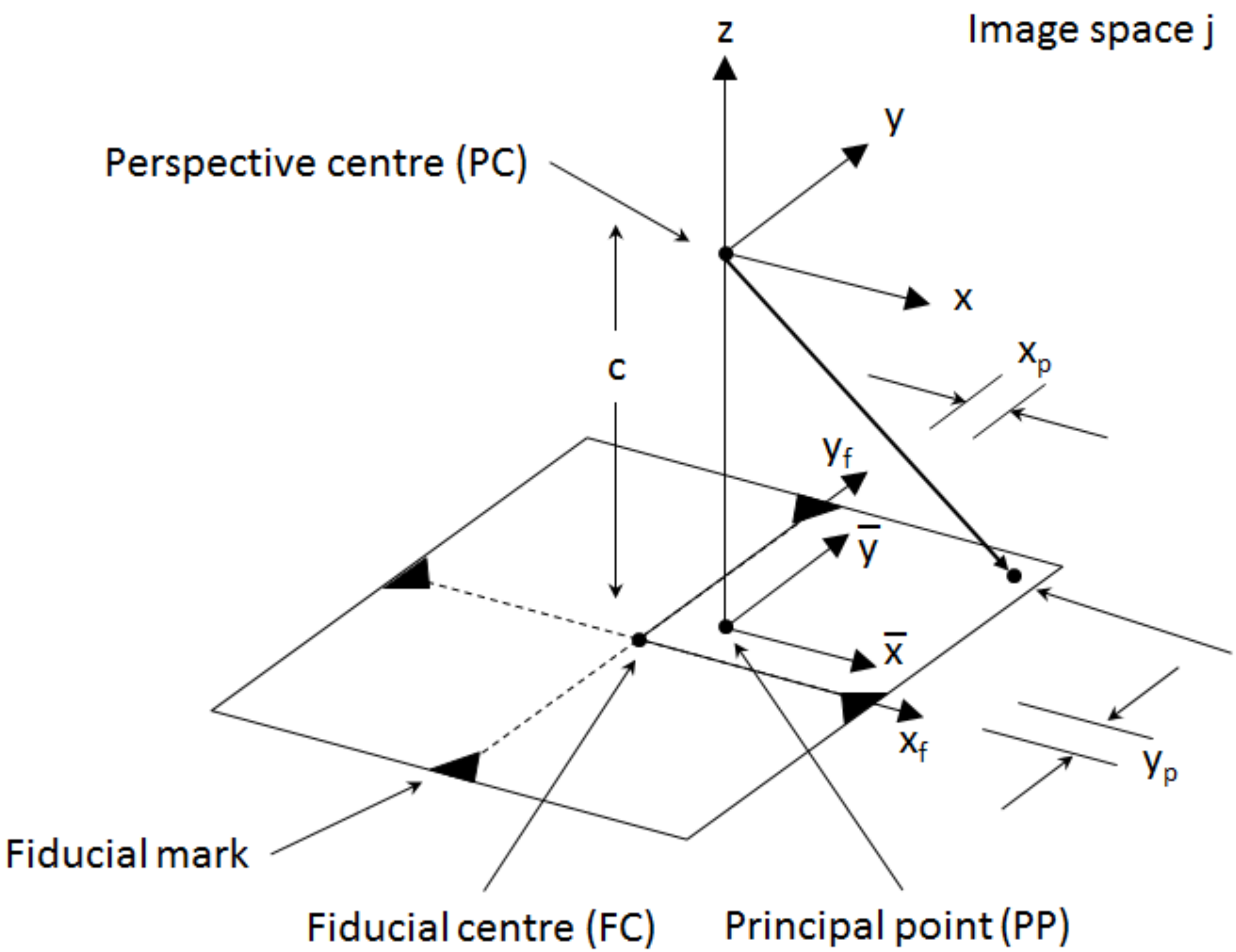
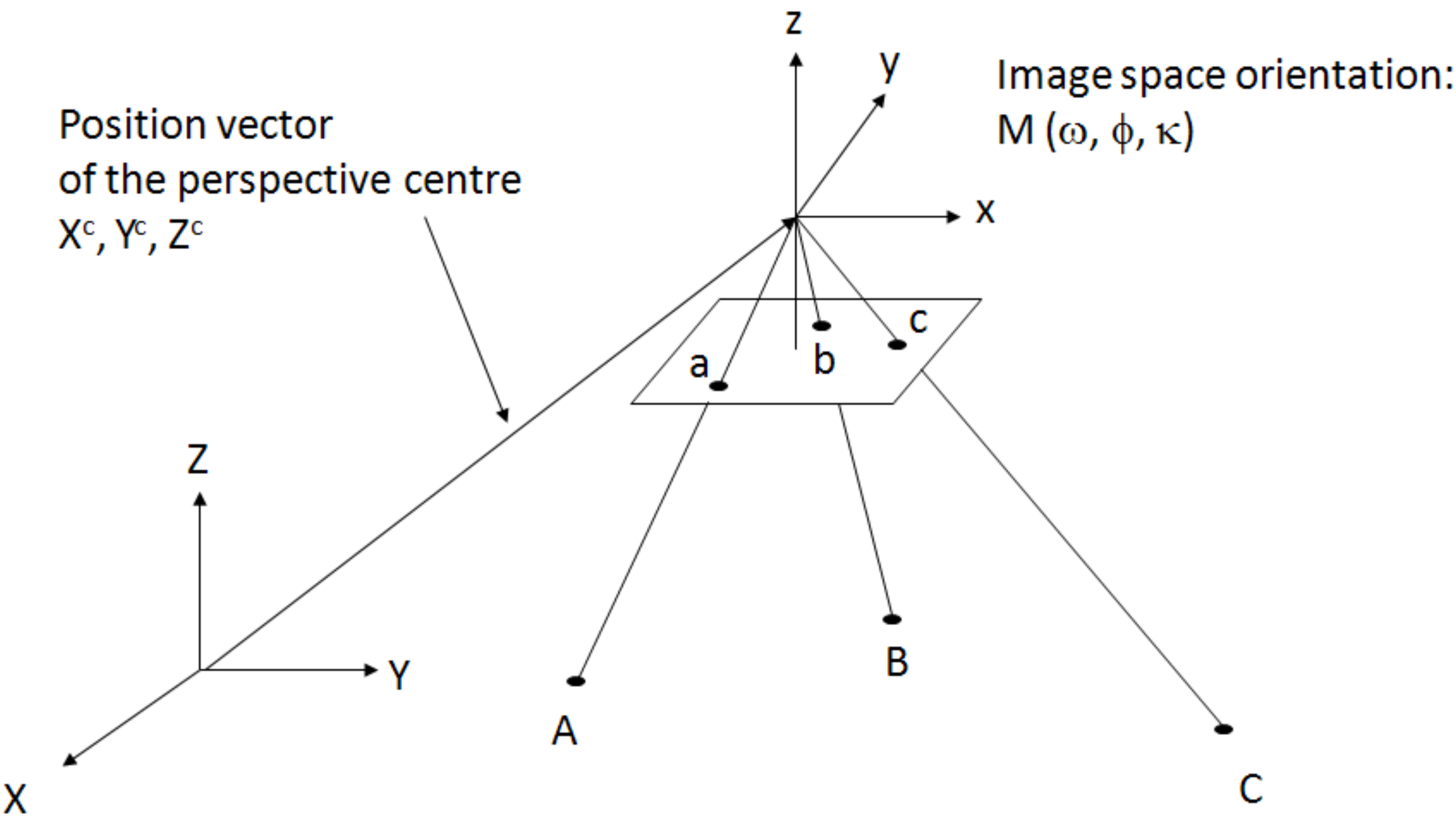


Image Space



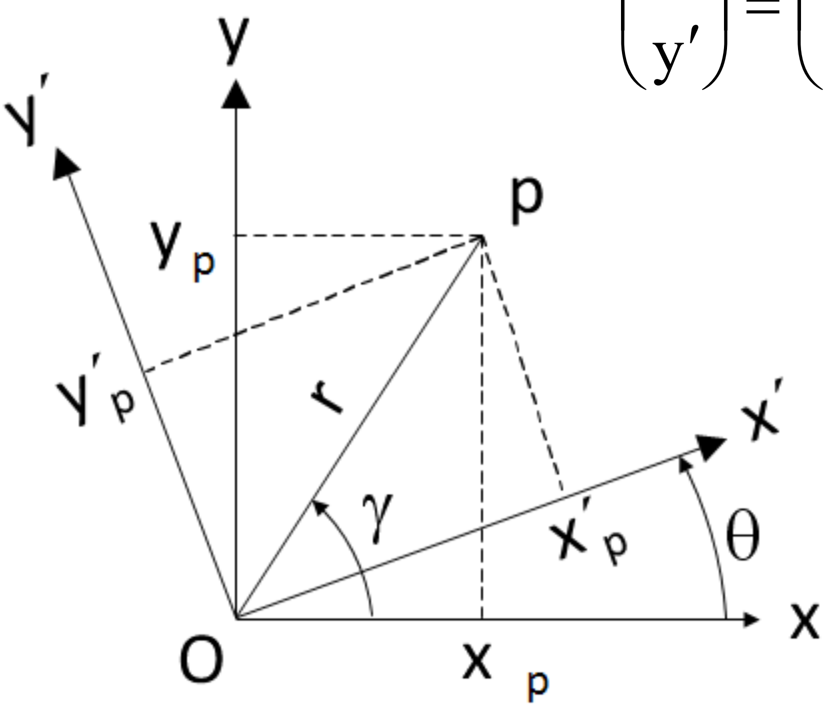
$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{ij} = \begin{bmatrix} \bar{x}_{ij} \\ \bar{y}_{ij} \\ -c_j \end{bmatrix}$$

Image Orientation

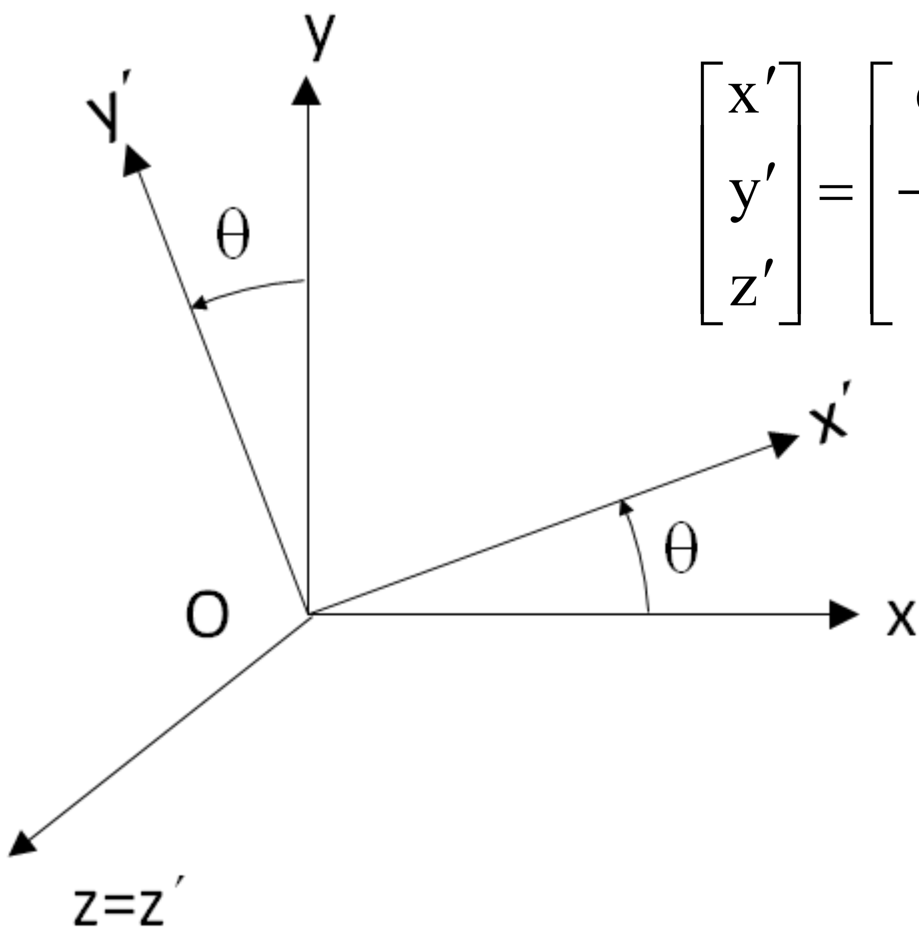


2D Rotation

$$\begin{pmatrix} x' \\ y' \end{pmatrix} = \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = R(\theta) \begin{pmatrix} x \\ y \end{pmatrix}$$

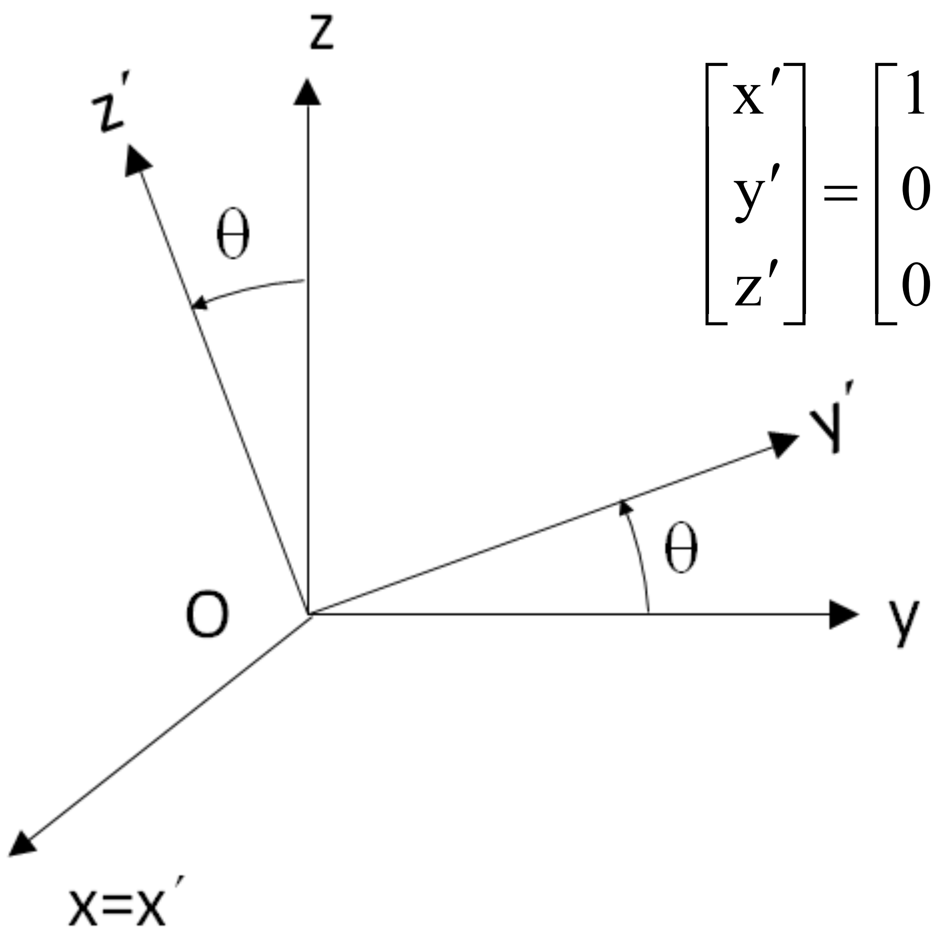


3D Rotation About z



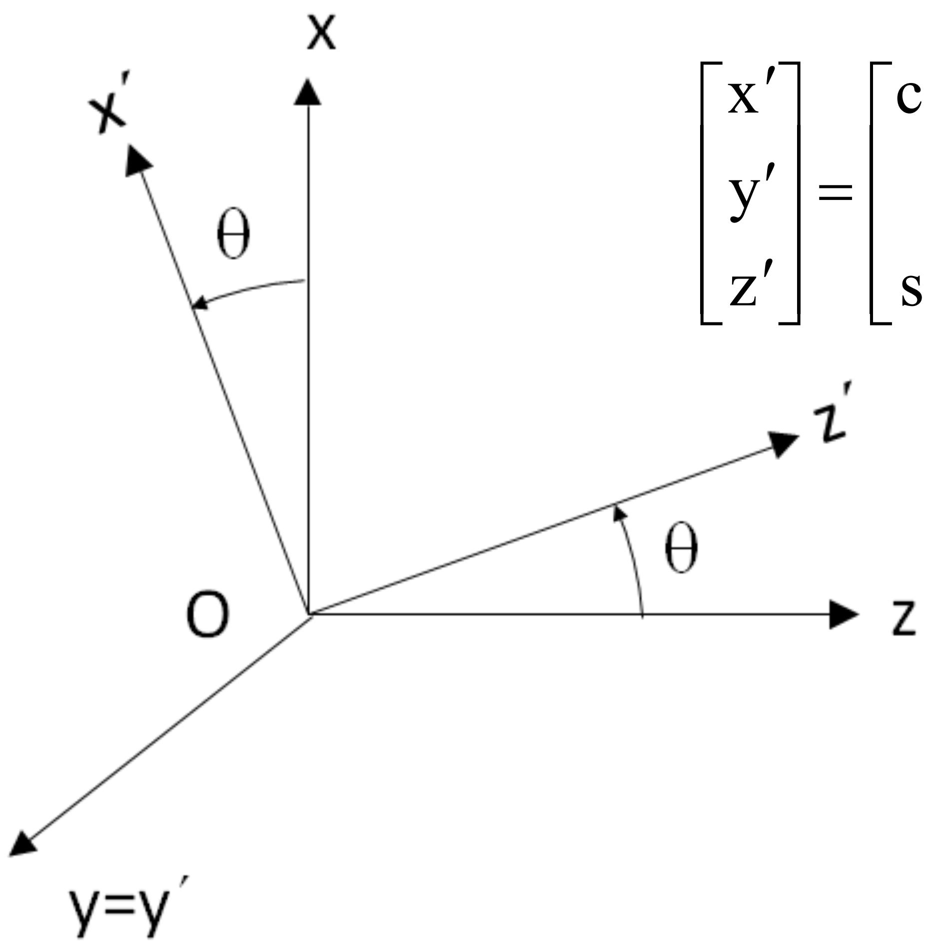
$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_3(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

3D Rotation About x



$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_1(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

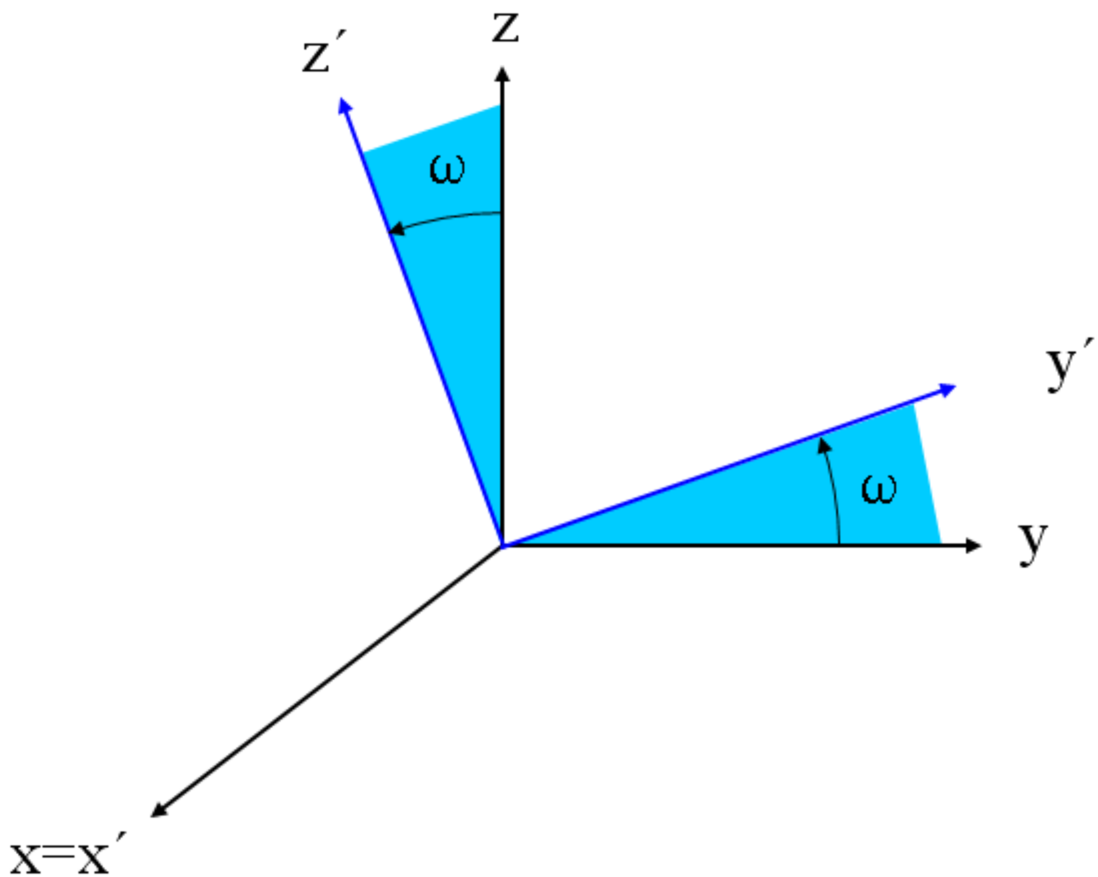
3D Rotation About y



$$\begin{bmatrix} x' \\ y' \\ z' \end{bmatrix} = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = R_2(\theta) \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

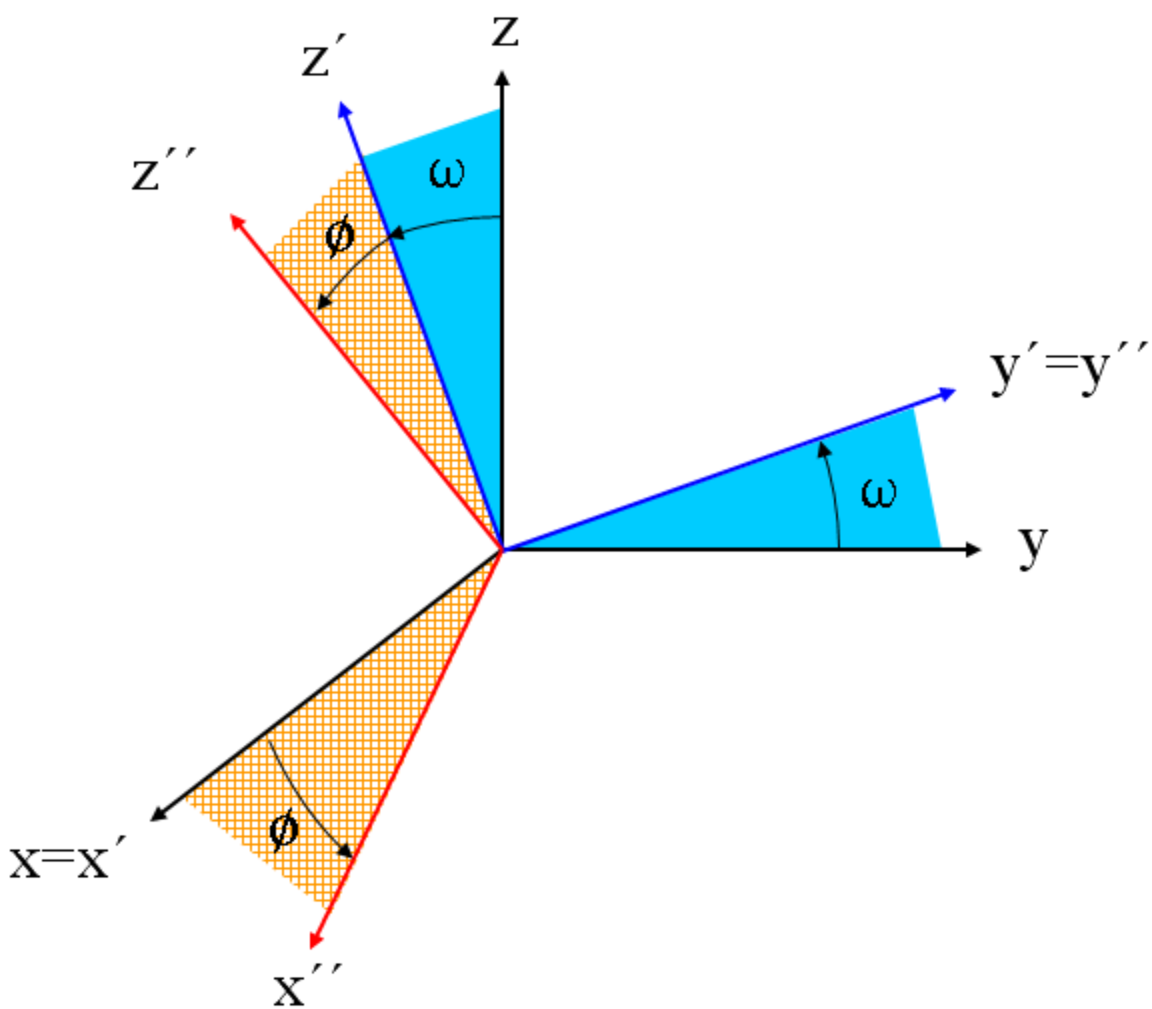
ωφκ Rotation Sequence

- First rotation by ω about x



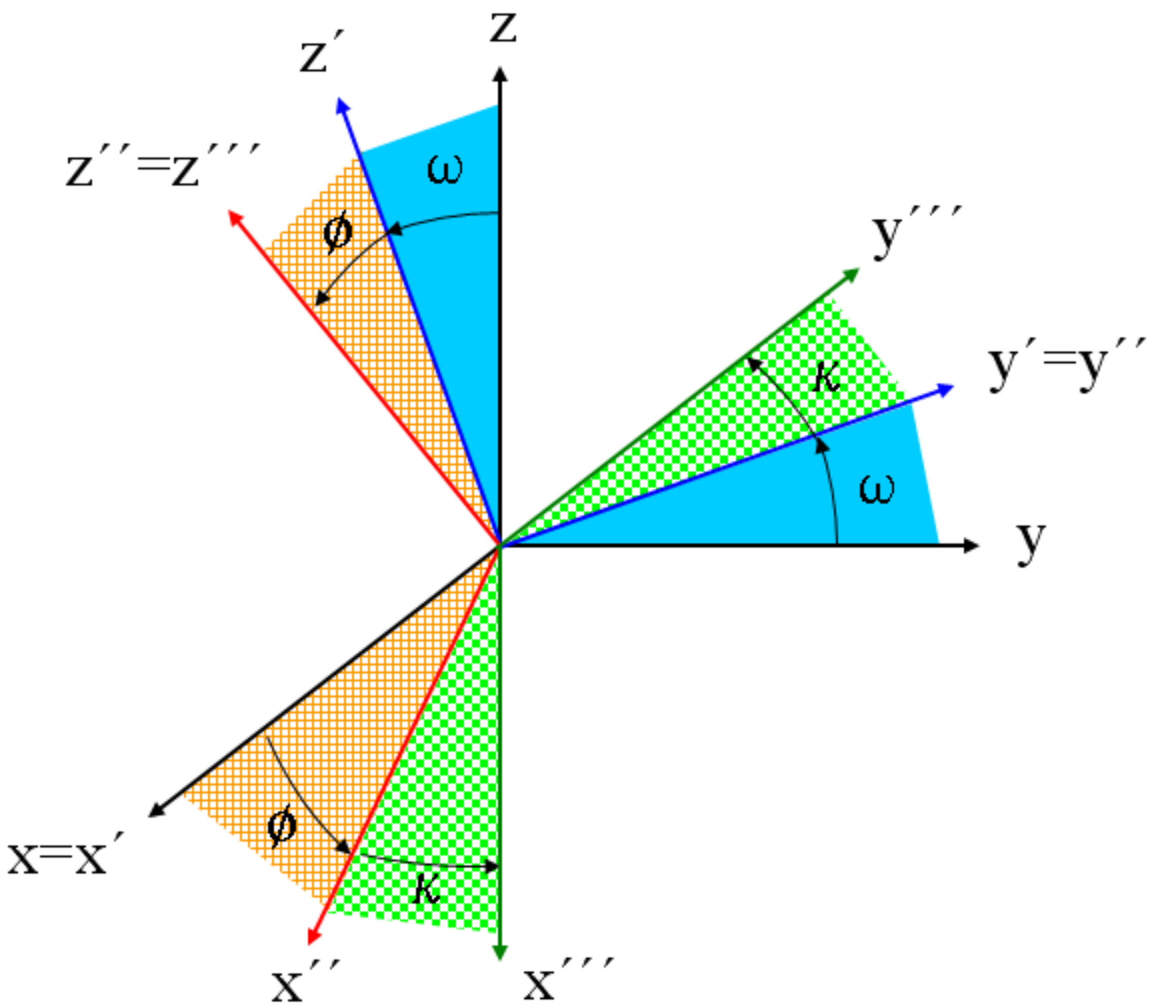
ωφκ Rotation Sequence (cont'd)

- Second rotation by φ about y′



ωφκ Rotation Sequence (cont'd)

- Third rotation by κ about z″



ωφκ Rotation Sequence (cont'd)

- Final rotation matrix

$$M = R_3(\kappa)R_2(\phi)R_1(\omega)$$

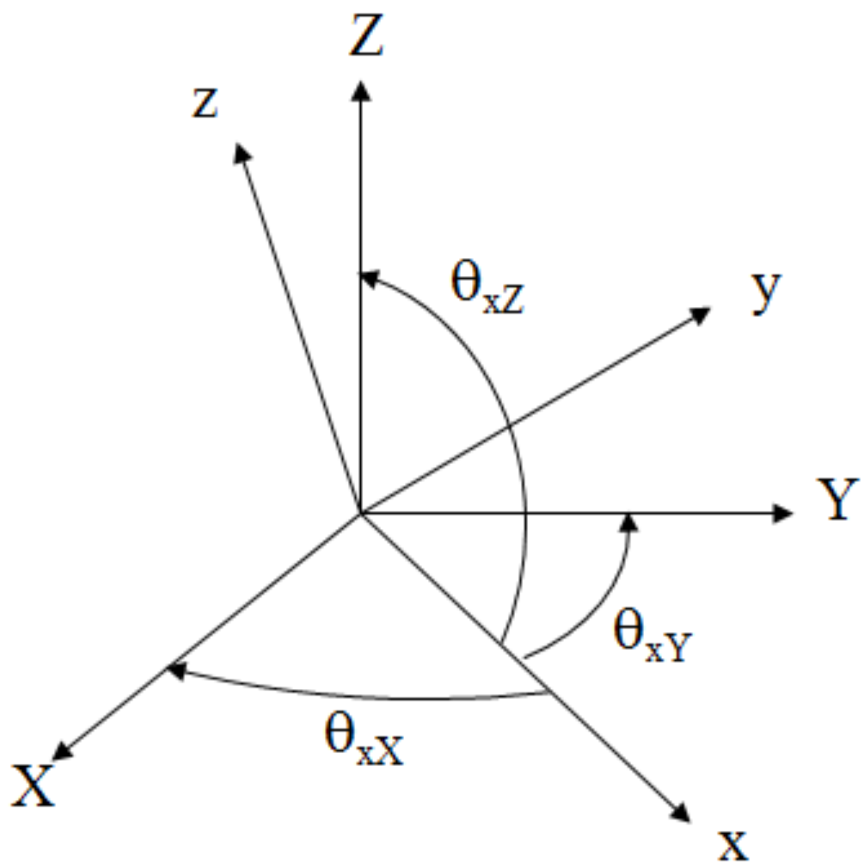
$$= \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & 0 & -\sin\phi \\ 0 & 1 & 0 \\ \sin\phi & 0 & \cos\phi \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\omega & \sin\omega \\ 0 & -\sin\omega & \cos\omega \end{bmatrix}$$

$$= \begin{bmatrix} \cos\kappa & \sin\kappa & 0 \\ -\sin\kappa & \cos\kappa & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\phi & \sin\omega\sin\phi & -\cos\omega\sin\phi \\ 0 & \cos\omega & \sin\omega \\ \sin\phi & -\sin\omega\cos\phi & \cos\omega\cos\phi \end{bmatrix}$$

$$= \begin{bmatrix} \cos\phi\cos\kappa & \cos\omega\sin\kappa + \sin\omega\sin\phi\cos\kappa & \sin\omega\sin\kappa - \cos\omega\sin\phi\cos\kappa \\ -\cos\phi\sin\kappa & \cos\omega\cos\kappa - \sin\omega\sin\phi\sin\kappa & \sin\omega\cos\kappa + \cos\omega\sin\phi\sin\kappa \\ \sin\phi & -\sin\omega\cos\phi & \cos\omega\cos\phi \end{bmatrix}$$

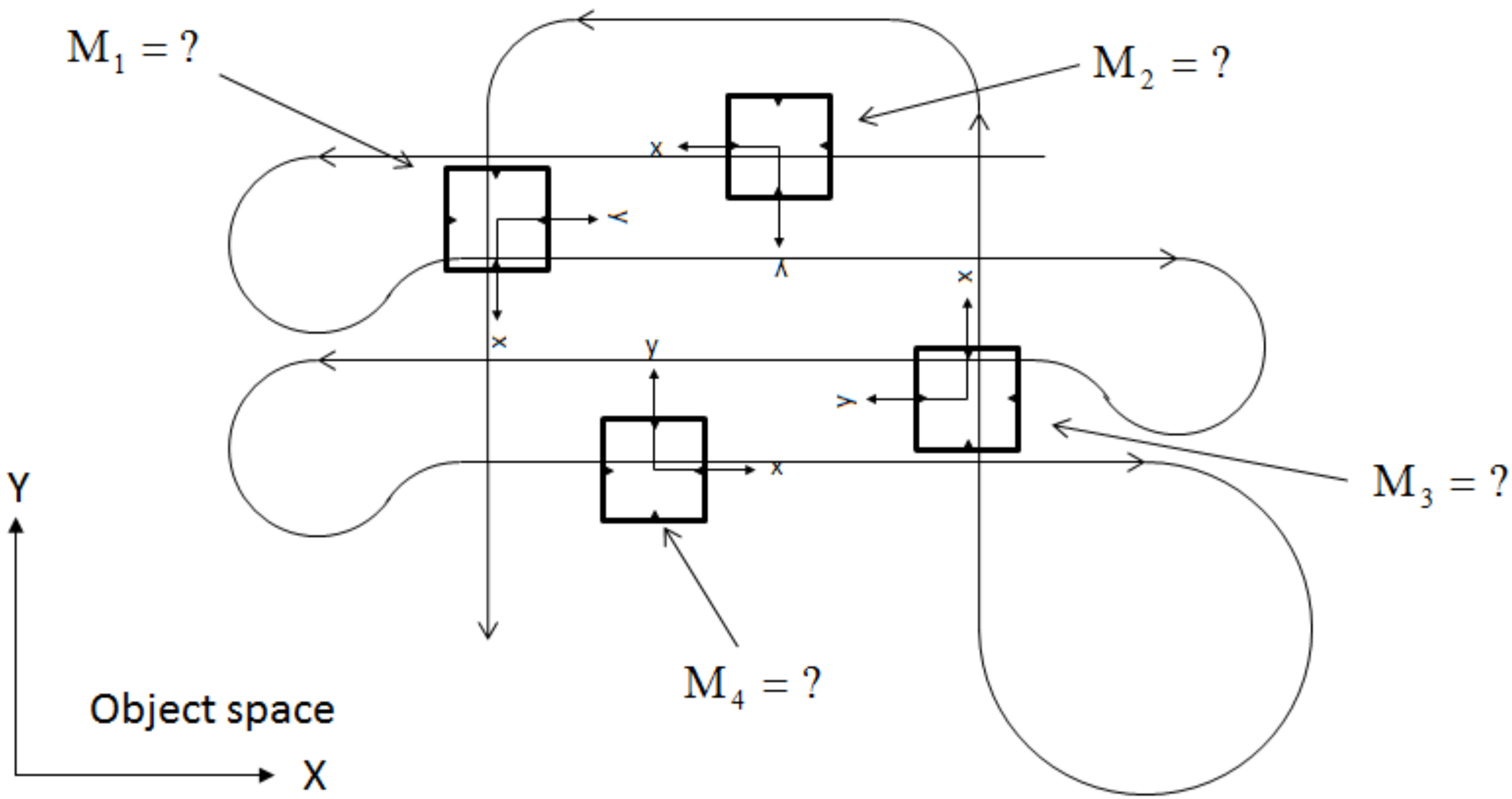
$$= \begin{bmatrix} m_{11} & m_{12} & m_{13} \\ m_{21} & m_{22} & m_{23} \\ m_{31} & m_{32} & m_{33} \end{bmatrix} = \begin{bmatrix} \cos(\theta_{xX}) & \cos(\theta_{xY}) & \cos(\theta_{xZ}) \\ \cos(\theta_{yX}) & \cos(\theta_{yY}) & \cos(\theta_{yZ}) \\ \cos(\theta_{zX}) & \cos(\theta_{zY}) & \cos(\theta_{zZ}) \end{bmatrix}$$

Direction Cosines



Direction Cosines Examples

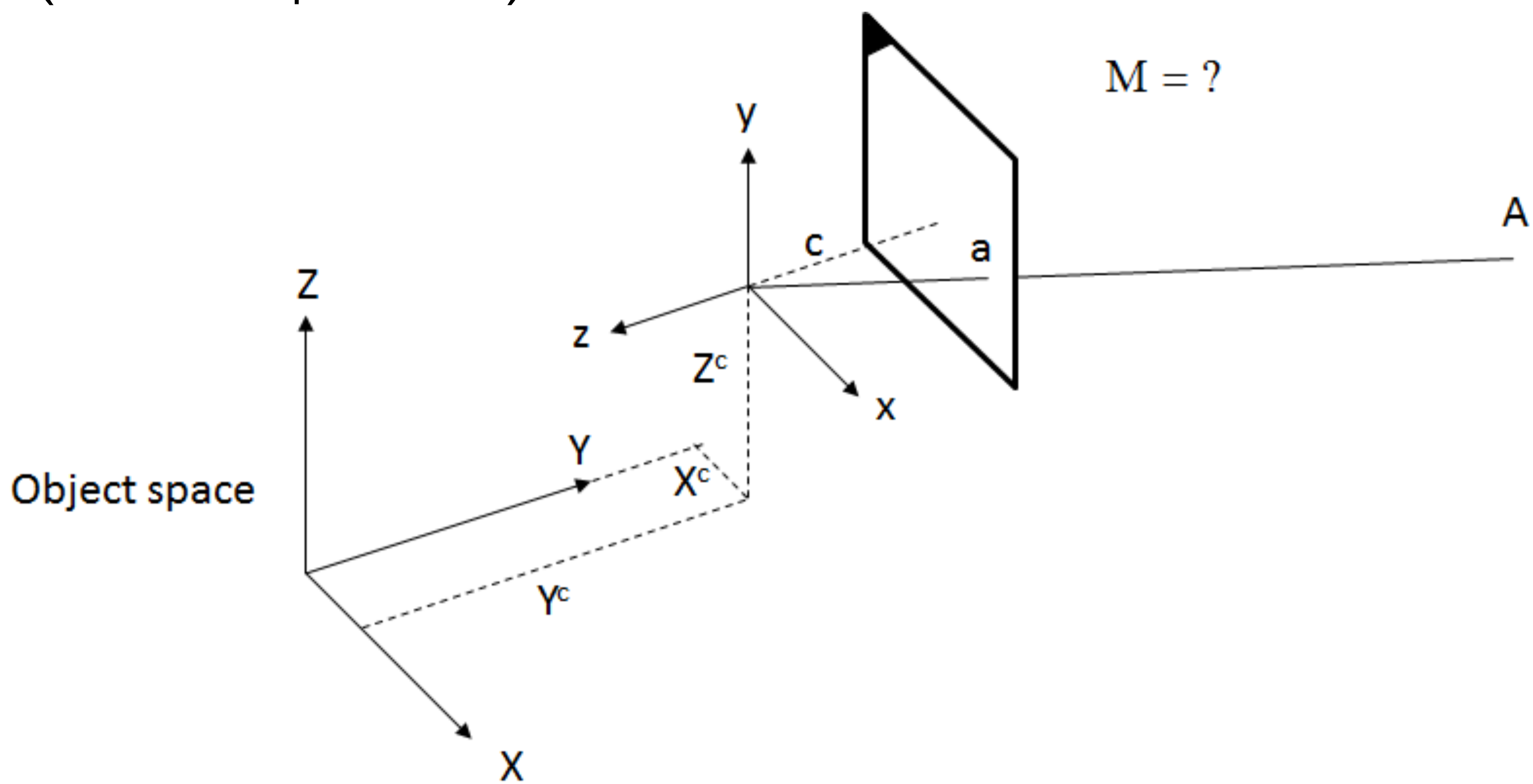
- Vertical photography with EW and NS flight lines ($\omega = \phi = 0°$)



Direction Cosines Examples (cont'd)

- Nominally level terrestrial imagery

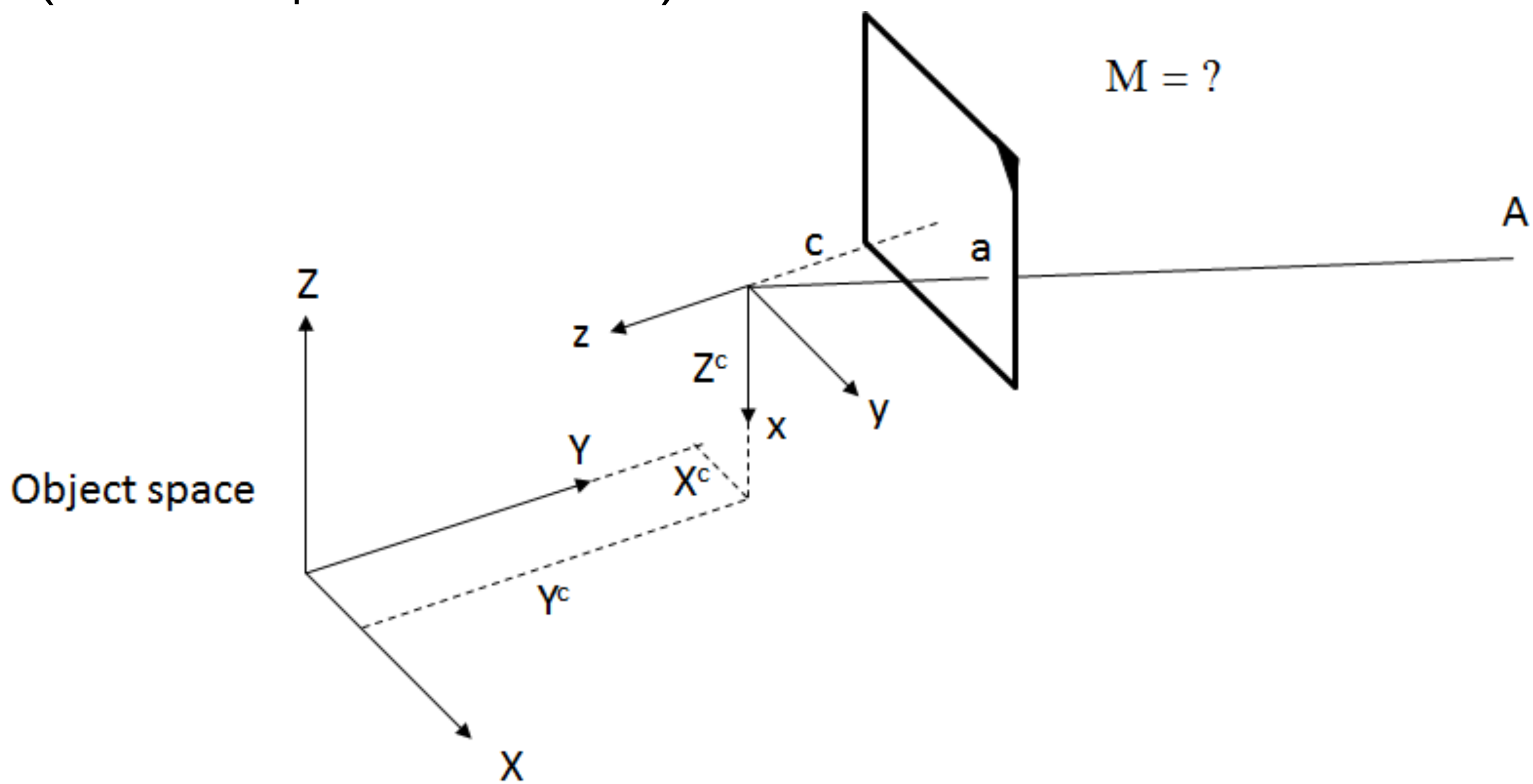
  (ω=90°, ϕ=κ=0°)



Direction Cosines Examples (cont'd)

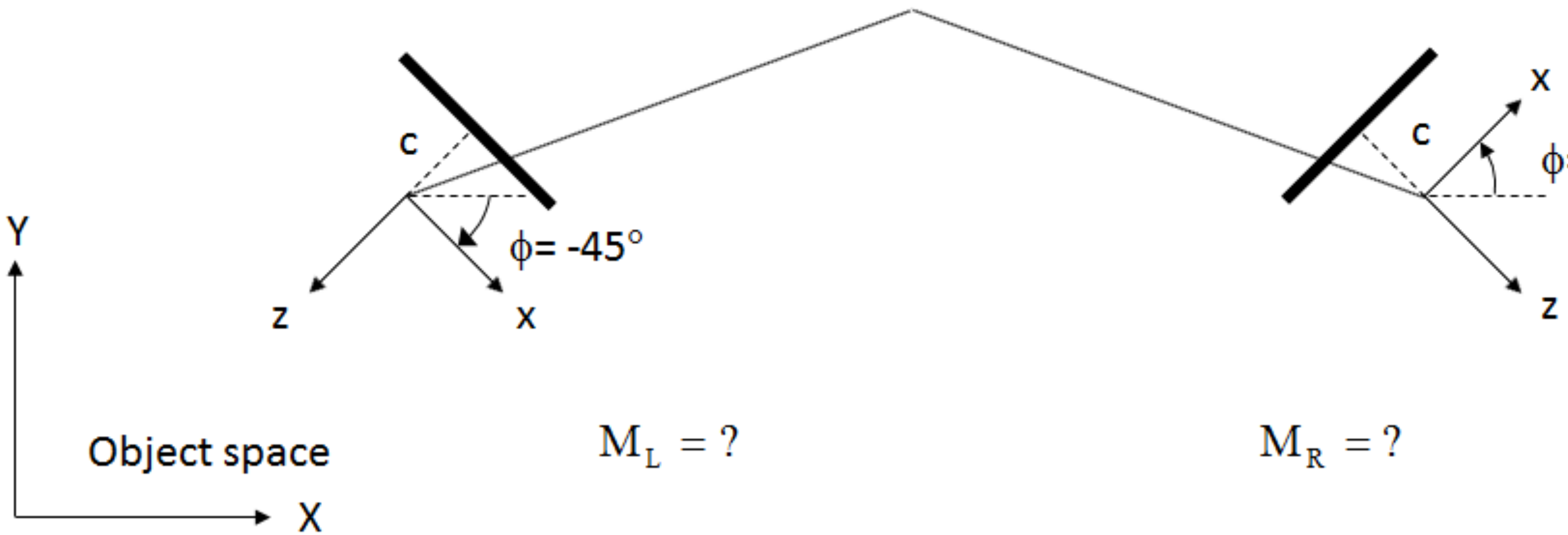
- Nominally level terrestrial imagery

  (ω=90°, ϕ=0°, κ=-90°)



Direction Cosines Examples (cont'd)

- Nominally level terrestrial imagery (ω=90°, ϕ=±45°, κ=0°)



M_L = ?

M_R = ?

Example 1

- Given the Cardan angles

$$\omega = 101.6595^\circ$$
$$\phi = -32.4075^\circ$$
$$\kappa = 3.2442^\circ$$

- Calculate the rotation matrix M

Example 2

- Given the rotation matrix

$$M = \begin{vmatrix} -0.6153 & -0.7883 & -0.0050 \\ 0.7883 & -0.6153 & -0.0060 \\ 0.0017 & -0.0076 & 1.0000 \end{vmatrix}$$

- Calculate the Cardan angles ω, ϕ, κ

Example 3

- The following rotation matrix

$$M = \begin{vmatrix} -0.1801 & 0.1888 & -0.9654 \\ -0.8471 & 0.4691 & 0.2497 \\ 0.5000 & 0.8627 & 0.0755 \end{vmatrix}$$

was constructed with the $\phi\omega\kappa$ sequence:

$$M = R_3(\kappa)R_1(\omega)R_2(\phi)$$

1. Derive the analytical form of the rotation matrix and calculate the angles ω, ϕ, κ for this sequence
2. Extract the ω, ϕ, κ from M assuming the <u>Cardan</u> sequence, i.e.

$$M = R_3(\kappa)R_2(\phi)R_1(\omega)$$

- Do the two sets of angles differ?

Summary of Rotation Matrix Equations

$$R_1(\theta)=\begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta & \sin\theta\\ 0 & -\sin\theta & \cos\theta\end{bmatrix}\quad R_2(\theta)=\begin{bmatrix}\cos\theta & 0 & -\sin\theta\\ 0 & 1 & 0\\ \sin\theta & 0 & \cos\theta\end{bmatrix}\quad R_3(\theta)=\begin{bmatrix}\cos\theta & \sin\theta & 0\\ -\sin\theta & \cos\theta & 0\\ 0 & 0 & 1\end{bmatrix}$$

$$\begin{aligned}M &= R_3(\kappa)R_2(\phi)R_1(\omega)\\ &= \begin{bmatrix}\cos\phi\cos\kappa & \cos\omega\sin\kappa+\sin\omega\sin\phi\cos\kappa & \sin\omega\sin\kappa-\cos\omega\sin\phi\cos\kappa\\ -\cos\phi\sin\kappa & \cos\omega\cos\kappa-\sin\omega\sin\phi\sin\kappa & \sin\omega\cos\kappa+\cos\omega\sin\phi\sin\kappa\\ \sin\phi & -\sin\omega\cos\phi & \cos\omega\cos\phi\end{bmatrix}\\ &= \begin{bmatrix}m_{11} & m_{12} & m_{13}\\ m_{21} & m_{22} & m_{23}\\ m_{31} & m_{32} & m_{33}\end{bmatrix}=\begin{bmatrix}\cos(\theta_{xX}) & \cos(\theta_{xY}) & \cos(\theta_{xZ})\\ \cos(\theta_{yX}) & \cos(\theta_{yY}) & \cos(\theta_{yZ})\\ \cos(\theta_{zX}) & \cos(\theta_{zY}) & \cos(\theta_{zZ})\end{bmatrix}\end{aligned}$$

$$\omega=\arctan\left(\frac{-m_{32}}{m_{33}}\right)\qquad \phi=\arcsin(m_{31})\qquad \kappa=\arctan\left(\frac{-m_{21}}{m_{11}}\right)$$