1 Scalar operations

2 Vector operations

1. Norm $\alpha = ||\mathbf{x}||_2$
   - First Order:
     - Forward: $\dot{\alpha} = \dfrac{\mathbf{x}^T\dot{\mathbf{x}}}{||\mathbf{x}||_2}$
     - Reverse: $\bar{\mathbf{x}} = \dfrac{\bar{\alpha}}{||\mathbf{x}||_2}\mathbf{x}$
   - Second Order:
     - H-Forward: $\dot{\alpha}_i = \dfrac{\mathbf{x}^T\dot{\mathbf{x}}_i}{||\mathbf{x}||_2}$
     - H-Reverse: $\hat{\mathbf{x}}_i = \dfrac{\hat{\alpha}_i}{||\mathbf{x}||_2}\mathbf{x} + \bar{\alpha}\left(\dfrac{\dot{\mathbf{x}}_i}{||\mathbf{x}||_2} - \bar{\mathbf{x}}\dfrac{\mathbf{x}\cdot\dot{\mathbf{x}}_i}{||\mathbf{x}||_2^3}\right)$
2. Scale $\mathbf{v} = \alpha\mathbf{x}$
   - First Order:
     - Forward: $\dot{\mathbf{v}} = \dot{\alpha}\mathbf{x} + \alpha\dot{\mathbf{x}}$
     - Reverse: $\bar{\alpha} = \bar{\mathbf{v}}^T\mathbf{x}$, $\bar{\mathbf{x}} = \alpha\bar{\mathbf{v}}$
   - Second Order:
     - H-Forward: $\dot{\mathbf{v}}_i = \dot{\alpha}_i\mathbf{x} + \alpha\dot{\mathbf{x}}_i$
     - H-Reverse: $\hat{\alpha}_i = \hat{\mathbf{v}}_i \cdot \mathbf{x} + \bar{\mathbf{v}} \cdot \dot{\mathbf{x}}_i$, $\hat{\mathbf{x}}_i = \alpha\hat{\mathbf{v}}_i + \dot{\alpha}_i\bar{\mathbf{v}}$
3. AXPY $\mathbf{v} = \alpha\mathbf{x} + \mathbf{y}$
   - First Order:
     - Forward: $\dot{\mathbf{v}} = \dot{\alpha}\mathbf{x} + \alpha\dot{\mathbf{x}} + \dot{\mathbf{y}}$
     - Reverse: $\bar{\alpha} = \bar{\mathbf{v}}^T\mathbf{x}$, $\bar{\mathbf{x}} = \alpha\bar{\mathbf{v}}$, $\bar{\mathbf{y}} = \bar{\mathbf{v}}$
   - Second Order:
     - H-Forward: $\dot{\mathbf{v}}_i = \dot{\alpha}_i\mathbf{x} + \alpha\dot{\mathbf{x}}_i + \dot{\mathbf{y}}_i$
     - H-Reverse: $\hat{\alpha}_i = \hat{\mathbf{v}}_i \cdot \mathbf{x} + \bar{\mathbf{v}} \cdot \dot{\mathbf{x}}_i$, $\hat{\mathbf{x}}_i = \alpha\hat{\mathbf{v}}_i + \dot{\alpha}_i\bar{\mathbf{v}}$, $\hat{\mathbf{y}}_i = \hat{\mathbf{v}}_i$
4. Dot-product $\alpha = \mathbf{x}^T\mathbf{y}$
   - First Order:
     - Forward: $\dot{\alpha} = \dot{\mathbf{x}}^T\mathbf{y} + \mathbf{x}^T\dot{\mathbf{y}}$
     - Reverse: $\bar{\mathbf{x}} = \bar{\alpha}\mathbf{y}$, $\bar{\mathbf{y}} = \bar{\alpha}\mathbf{x}$
   - Second Order:
     - H-Forward: $\dot{\alpha}_i = \dot{\mathbf{x}}_i \cdot \mathbf{y} + \mathbf{x} \cdot \dot{\mathbf{y}}_i$
     - H-Reverse: $\hat{\mathbf{x}}_i = \hat{\alpha}_i\mathbf{y} + \bar{\alpha}\dot{\mathbf{y}}_i$, $\hat{\mathbf{y}}_i = \hat{\alpha}_i\mathbf{x} + \bar{\alpha}\dot{\mathbf{x}}_i$
5. Cross-product $\mathbf{v} = \mathbf{x} \times \mathbf{y}$,
   - First Order:
     - Forward: $\dot{\mathbf{v}} = \dot{\mathbf{x}} \times \mathbf{y} + \mathbf{x} \times \dot{\mathbf{y}}$
     - Reverse: $\bar{\mathbf{x}} = \mathbf{y} \times \bar{\mathbf{v}}$, $\bar{\mathbf{y}} = \bar{\mathbf{v}} \times \mathbf{x}$,
   - Second Order:
     - H-Forward: $\dot{\mathbf{v}}_i = \dot{\mathbf{x}}_i \times \mathbf{y} + \mathbf{x} \times \dot{\mathbf{y}}_i$

– H-Reverse: $\hat{\mathbf{x}}_i = \mathbf{y} \times \hat{\mathbf{v}}_i + \bar{\mathbf{v}} \times \dot{\mathbf{y}}_i$, $\hat{\mathbf{y}}_i = \hat{\mathbf{v}}_i \times \mathbf{x} + \dot{\mathbf{x}}_i \times \bar{\mathbf{v}}$

6. Normalize $\mathbf{v} = \frac{\mathbf{x}}{||\mathbf{x}||_2}$

- First Order:

– Forward: $\dot{\mathbf{v}} = \frac{\dot{\mathbf{x}}}{||\mathbf{x}||_2} - \frac{(\dot{\mathbf{x}}^T\mathbf{x})\mathbf{x}}{||\mathbf{x}||_2^3}$

– Reverse: $\bar{\mathbf{x}} = \frac{\bar{\mathbf{v}}}{||\mathbf{x}||_2} - \frac{(\bar{\mathbf{v}}^T\mathbf{x})\mathbf{x}}{||\mathbf{x}||_2^3}$

- Second Order:

– H-Forward: $\dot{\mathbf{v}}_i = \frac{\dot{\mathbf{x}}_i}{||\mathbf{x}||_2} - \mathbf{x}\frac{\dot{\mathbf{x}}_i \cdot \mathbf{x}}{||\mathbf{x}||_2^3}$

– H-Reverse: $\hat{\mathbf{x}}_i = \frac{\hat{\mathbf{v}}_i}{||\mathbf{x}||_2} - \mathbf{x}\frac{\hat{\mathbf{v}}_i \cdot \mathbf{x}}{||\mathbf{x}||_2^3} - \mathbf{x}\frac{\bar{\mathbf{v}} \cdot \dot{\mathbf{x}}_i}{||\mathbf{x}||_2^3} - \mathbf{v}\frac{\dot{\mathbf{x}}_i \cdot \mathbf{x}}{||\mathbf{x}||_2^3} - \dot{\mathbf{x}}_i\frac{\bar{\mathbf{v}} \cdot \mathbf{x}}{||\mathbf{x}||_2^3} + \mathbf{x}\frac{3(\dot{\mathbf{x}}_i \cdot \mathbf{x})(\bar{\mathbf{v}} \cdot \mathbf{x})}{||\mathbf{x}||_2^5}$

7. Symmetric outer-product $\mathbf{S} = \alpha\mathbf{x}\mathbf{x}^T$

- First Order:

– Forward: $\dot{\mathbf{S}} = \dot{\alpha}\mathbf{x}\mathbf{x}^T + \alpha(\dot{\mathbf{x}}\mathbf{x}^T + \mathbf{x}\dot{\mathbf{x}}^T)$

– Reverse: $\bar{\alpha} = \mathbf{x}^T\bar{\mathbf{S}}\mathbf{x}$, $\bar{\mathbf{x}} = 2\alpha\bar{\mathbf{S}}\mathbf{x}$

- Second Order:

– H-Forward: $\dot{\mathbf{S}}_i = \dot{\alpha}_i\mathbf{x}\mathbf{x}^T + \alpha\left(\dot{\mathbf{x}}_i\mathbf{x}^T + \mathbf{x}\dot{\mathbf{x}}_i^T\right)$

– H-Reverse: $\hat{\alpha}_i = \mathbf{x} \cdot \left(\hat{\mathbf{S}}_i \cdot \mathbf{x} + 2\bar{\mathbf{S}} \cdot \dot{\mathbf{x}}_i\right)$, $\hat{\mathbf{x}}_i = 2\alpha\left(\hat{\mathbf{S}}_i \cdot \mathbf{x} + \bar{\mathbf{S}} \cdot \dot{\mathbf{x}}_i\right) + 2\dot{\alpha}_i\bar{\mathbf{S}} \cdot \mathbf{x}$

3 Matrix-vector operations

1. Matrix-vector product $\mathbf{v} = \mathbf{A}\mathbf{x}$

- Forward: $\dot{\mathbf{v}} = \dot{\mathbf{A}}\mathbf{x} + \mathbf{A}\dot{\mathbf{x}}$
- Reverse: $\bar{\mathbf{x}} = \mathbf{A}^T\bar{\mathbf{y}}$, $\bar{\mathbf{A}} = \bar{\mathbf{v}}\mathbf{x}^T$

2. Matrix-vector product $\mathbf{v} = \mathbf{A}^T\mathbf{x}$

- Forward: $\dot{\mathbf{v}} = \dot{\mathbf{A}}^T\mathbf{x} + \mathbf{A}^T\dot{\mathbf{x}}$
- Reverse: $\bar{\mathbf{x}} = \mathbf{A}\bar{\mathbf{y}}$, $\bar{\mathbf{A}} = \mathbf{x}\bar{\mathbf{v}}^T$

3. Matrix inner product $\alpha = \mathbf{x}^T\mathbf{A}\mathbf{y}$

- Forward: $\dot{\alpha} = \dot{\mathbf{x}}^T\mathbf{A}\mathbf{y} + \mathbf{x}^T\dot{\mathbf{A}}\mathbf{y} + \mathbf{x}^T\mathbf{A}\dot{\mathbf{y}}$
- Reverse: $\bar{\mathbf{x}} = \bar{\alpha}\mathbf{A}\mathbf{y}$, $\bar{\mathbf{y}} = \bar{\alpha}\mathbf{A}^T\mathbf{x}$, $\bar{\mathbf{A}} = \bar{\alpha}\mathbf{x}\mathbf{y}^T$

4 Matrix-matrix operations

1. Trace $\alpha = \mathrm{tr}(\mathbf{A})$

- Forward $\dot{\alpha} = \mathrm{tr}(\dot{\mathbf{A}})$
- Reverse $\bar{\mathbf{A}}_{ii} = \bar{\alpha}$

2. Symmetric trace $\alpha = \mathrm{tr}(\mathbf{S})$

- Forward $\dot{\alpha} = \mathrm{tr}(\dot{\mathbf{S}})$
- Reverse $\bar{\mathbf{S}}_{ii} = \bar{\alpha}$

3. Determinant $\alpha = \det(\mathbf{A})$
4. Symmetric determinant $\alpha = \det(\mathbf{A})$
5. Inverse $\mathbf{B} = \mathbf{A}^{-1}$
   - Forward: $\dot{\mathbf{B}} = -\mathbf{A}^{-1}\dot{\mathbf{A}}\mathbf{A}^{-1}$
   - Reverse $\bar{\mathbf{A}} = -\mathbf{A}^{-T}\bar{\mathbf{B}}\mathbf{A}^{-T}$
6. Symmetric Inverse $\mathbf{T} = \mathbf{S}^{-1}$
   - Forward: $\dot{\mathbf{T}} = -\mathbf{S}^{-1}\dot{\mathbf{S}}\mathbf{S}^{-1}$
   - Reverse $\bar{\mathbf{S}} = -\mathbf{S}^{-T}\bar{\mathbf{T}}\mathbf{S}^{-T}$
7. Symmetric part $\mathbf{S} = \alpha(\mathbf{A} + \mathbf{A}^T)$
8. Anti-symmetric part $\mathbf{B} = \alpha(\mathbf{A} - \mathbf{A}^T)$
9. Matrix addition $\mathbf{C} = \mathbf{A} + \mathbf{B}$
   - Forward: $\dot{\mathbf{C}} = \dot{\mathbf{A}} + \dot{\mathbf{B}}$
   - Reverse: $\bar{\mathbf{A}} = \bar{\mathbf{C}}$, $\bar{\mathbf{B}} = \bar{\mathbf{C}}$
10. Matrix-matrix product $\mathbf{C} = \mathbf{A}\mathbf{B}$
    - Forward $\dot{\mathbf{C}} = \dot{\mathbf{A}}\mathbf{B} + \mathbf{A}\dot{\mathbf{B}}$
    - Reverse $\bar{\mathbf{A}} = \bar{\mathbf{C}}\mathbf{B}^T$, $\bar{\mathbf{B}} = \mathbf{A}^T\bar{\mathbf{C}}$,
11. Matrix-matrix product $\mathbf{C} = \mathbf{A}^T\mathbf{B}$
    - Forward $\dot{\mathbf{C}} = \dot{\mathbf{A}}^T\mathbf{B} + \mathbf{A}^T\dot{\mathbf{B}}$
    - Reverse $\bar{\mathbf{A}} = \mathbf{B}\bar{\mathbf{C}}^T$, $\bar{\mathbf{B}} = \mathbf{A}\bar{\mathbf{C}}$,
12. Matrix-matrix product $\mathbf{C} = \mathbf{A}\mathbf{B}^T$
    - Forward $\dot{\mathbf{C}} = \dot{\mathbf{A}}\mathbf{B}^T + \mathbf{A}\dot{\mathbf{B}}^T$
    - Reverse $\bar{\mathbf{A}} = \bar{\mathbf{C}}\mathbf{B}$, $\bar{\mathbf{B}} = \bar{\mathbf{C}}^T\mathbf{A}$
13. Matrix-matrix product $\mathbf{C} = \mathbf{A}^T\mathbf{B}^T$
    - Forward $\dot{\mathbf{C}} = \dot{\mathbf{A}}^T\mathbf{B}^T + \mathbf{A}^T\dot{\mathbf{B}}^T$
    - Reverse $\bar{\mathbf{A}} = \mathbf{B}^T\bar{\mathbf{C}}^T$, $\bar{\mathbf{B}} = \bar{\mathbf{C}}^T\mathbf{A}^T$
14. Linear Lagrange-Green strain $\mathbf{E} = \frac{1}{2}\left(\mathbf{U} + \mathbf{U}^T\right)$
    - Forward $\dot{\mathbf{E}} = \frac{1}{2}\left(\dot{\mathbf{U}} + \dot{\mathbf{U}}^T\right)$
    - Reverse $\bar{\mathbf{U}} = \bar{\mathbf{E}}$
15. Lagrange-Green strain $\mathbf{E} = \frac{1}{2}\left(\mathbf{U} + \mathbf{U}^T + \mathbf{U}^T\mathbf{U}\right)$
    - Forward $\dot{\mathbf{E}} = \frac{1}{2}\left(\dot{\mathbf{U}} + \dot{\mathbf{U}}^T + \dot{\mathbf{U}}^T\mathbf{U} + \mathbf{U}^T\dot{\mathbf{U}}\right)$
    - Reverse $\bar{\mathbf{U}} = (\mathbf{I} + \mathbf{U})\bar{\mathbf{E}}$
16. Isotropic constitutive $\mathbf{S} = 2\mu\mathbf{E} + \lambda\mathrm{tr}(\mathbf{E})\mathbf{I}$
    - Forward $\dot{\mathbf{S}} = 2\mu\dot{\mathbf{E}} + \lambda\mathrm{tr}(\dot{\mathbf{E}})\mathbf{I}$
    - Reverse $\bar{\mathbf{E}} = 2\mu\bar{\mathbf{S}} + \lambda\mathrm{tr}(\bar{\mathbf{S}})\mathbf{I}$