Special cases and equivalent form of Katznelson's problem on recurrence

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Slides:
https://github.com/jtgriesmer/slides/blob/main/Crossroads.pdf

Let G be a countable abelian group.

A probability measure preserving G-system (or MPS) is a triple (X, μ, T) where (X, μ) is a probability measure space and T is an action of G on X by transformations T_g preserving μ:

$$\mu(T_g A) = \mu(A) \qquad \text{for all measurable } A \subset X, \text{ all } g \in G.$$

A topological G-system (X, T) is a compact metric space X together with an action of G on X by invertible homeomorphisms T_g.

We say (X, T) is minimal if the only nonempty closed T-invariant subset of X is X.

Group rotations

A group rotation (K, R_α) is a topological G-system where K is a compact abelian group, $\rho : G \to K$ is a homomorphism, and $R_g x := x + \rho(g)$.

Writing m for Haar probability measure on K, we get a MPS (K, m, R).

Ex: $G = \mathbb{Z}$. $K = \mathbb{T} := \mathbb{R}/\mathbb{Z}$. Fix $\alpha \in \mathbb{T}$, and let $R_n(x) = x + n\alpha$. Then (K, R_α) is a group rotation.

Definition

Let $S \subset G$. We say that S is a set of

1. measurable recurrence if $\forall$ (X, μ, T), $A \subset X$ with $\mu(A) > 0$, there exists $g \in S$ such that $\mu(A \cap T_g A) > 0$.
2. topological recurrence if $\forall$ minimal (X, T), nonempty open $U \subset X$ there exists $g \in S$ such that $U \cap T_g U \neq \varnothing$.
3. Bohr recurrence if for $\forall$ minimal group rotations (K, R), non-$\varnothing$ open $U \subset K$, $\exists$ $g \in S$ such that $U \cap R_g U \neq \varnothing$.

The definitions easily yield:

$$\begin{aligned} S \text{ is a set of measurable rec.} &\implies S \text{ is a set of topological rec.} \\ &\implies S \text{ is a set of Bohr rec.} \end{aligned}$$

Kriz [Kri87]: S is a set of topological rec. $\not\Longrightarrow$ S is a set of measurable rec. (in $\mathbb{Z}$). See also [Ruz85].
Actually: if $E \subset \mathbb{Z}$ is infinite, then there is a set $S \subset E - E$ which is a set of topological rec. but not measurable rec. [Gri21a]

Katznelson's and Veech's problem

Conjecture 1 (Katznelson [Kat01], Veech [Vee68])

Let G be a countable abelian group. Every subset of G which is a set of Bohr recurrence is also set of topological recurrence.

This is never stated as a **conjecture**; we do so for convenience. See [GKR21] for exposition.

Katznelson asked this question for $\mathbb{Z}$, but the answer is not known in any countable abelian group.

The natural intuition is no: topological systems can be much more complicated than group rotations.

Special cases: [HKM16] nilsystems, [GKR21] some distal systems.

Lemma ([GK03], Lemma 3.3, cf. [Giv03], [Dik01])

$S := \{7^{n+2d} + 7^{n+d} - 2 \cdot 7^n : n, d \in \mathbb{N}\}$ *is a set of Bohr recurrence.*

Is S a set of topological recurrence? Measurable recurrence?

For $x \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$, let $\tilde{x} \in [0,1)$ represent x.
$\|x\| := \min_{n\in\mathbb{Z}} |\tilde{x} - n|$.
For $\mathbf{x} = (x_1, \ldots, x_d) \in \mathbb{T}^d$, $\|\mathbf{x}\| := \max_{j\le d} \|x_j\|$.
Given a homomorphism $\psi : G \to \mathbb{T}^d$ and $\varepsilon > 0$, the basic Bohr neighborhood of 0 in G determined by ψ and ε is

$$\mathrm{Bohr}(\psi, \varepsilon) := \{g \in G : \|\psi(g)\| < \varepsilon\}.$$

Ex: $G = \mathbb{Z}$, $\psi(n) := n(1/2) \bmod 1$, $\varepsilon = 1/4$. $\mathrm{Bohr}(\psi, \varepsilon) = 2\mathbb{Z}$.

Lemma (Equivalent definitions of Bohr recurrence)

Let $S \subset G$. The following are equivalent.

1. $\forall$ *minimal group rotations (K, R), non-$\varnothing$ open $U \subset K$, $\exists\, g \in S$ such that $U \cap R_g U \neq \varnothing$.*
2. *For all $d \in \mathbb{N}$, every homomorphism $\psi : G \to \mathbb{T}^d$, and all $\varepsilon > 0$, there is a $g \in S$ such that $\|\psi(g)\| < \varepsilon$.*
3. *For every Bohr neighborhood U of 0, $S \cap U \neq \varnothing$.*

In $\mathbb{Z}$: S is a set of Bohr recurrence iff for all $d \in \mathbb{N}$, $\boldsymbol{\alpha} \in \mathbb{T}^d$, and all $\varepsilon > 0$, $\exists m \in S$ such that $\|m\boldsymbol{\alpha}\| < \varepsilon$.

Lemma ([GK03], Lemma 3.3, cf [Giv03])

$S := \{7^{n+2d} + 7^{n+d} - 2 \cdot 7^n : n, d \in \mathbb{N}\}$ *is a set of Bohr recurrence.*

Proof.

Fix $d \in \mathbb{N}$ and $\boldsymbol{\alpha} \in \mathbb{T}^d$. We must find $m \in S$ such that $\|m\boldsymbol{\alpha}\| < \varepsilon$. Cover $\mathbb{T}^d$ with sets $U_1, \ldots, U_k$ of diameter $< \varepsilon/2$. For $n \in \mathbb{N}$, let

$$f(n) := \min\{j : 7^n\boldsymbol{\alpha} \in U_j\}$$

This is a finite coloring of $\mathbb{N}$. By van der Waerden's theorem on arithmetic progressions, there are $n, d \in \mathbb{N}$, $j \leq k$ so that $7^n\boldsymbol{\alpha}, 7^{n+d}\boldsymbol{\alpha}, 7^{n+2d}\boldsymbol{\alpha}$ all lie in U_j. Thus

$$\|7^{n+d}\boldsymbol{\alpha} - 7^n\boldsymbol{\alpha}\| < \varepsilon/2 \quad \text{and} \quad \|7^{n+2d}\boldsymbol{\alpha} - 7^n\boldsymbol{\alpha}\| < \varepsilon/2,$$

so $\|(7^{n+2d} + 7^{n+d} - 2 \cdot 7^n)\boldsymbol{\alpha}\| < \varepsilon$ (triangle inequality). □

A special case of Katznelson's problem in $\mathbb{Z}$

Proposition ([Gri21b], Proposition 2.2)

If $S \subset \{2^n - 2^m : n, m \in \mathbb{N}\}$ is a set of Bohr recurrence, then S is a set of topological recurrence.

This is not vacuous: the ambient set $\{2^n - 2^m\}$ has the form $E - E$, where E is infinite. So $E - E$ is a set of measurable recurrence (and therefore a set of topological recurrence and a set of Bohr recurrence), by the Poincaré recurrence theorem.

It is not trivial: $\exists\ S \subset E - E$ which is a set of topological recurrence and not a set of Bohr recurrence (Kriz).

It might be useless.

An equivalent form of Katznelson's problem

$\mathbb{Z}^\omega := \bigoplus_{j\in\mathbb{N}} \mathbb{Z} =$ the direct sum of countably many copies of $\mathbb{Z}$.
Write $\mathbf{n} \in \mathbb{Z}^\omega$ as $(n_1, n_2, \ldots,)$, only finitely many n_i nonzero.
$\mathbf{e}_1 = (1, 0, 0, \ldots)$, $\mathbf{e}_2 = (0, 1, 0, 0, \ldots)$, $\ldots$
$\mathcal{E}_1 := \{\mathbf{e}_i : i \in \mathbb{N}\}$.

$$\mathcal{E}_4 := \{(\mathbf{e}_i - \mathbf{e}_j) - (\mathbf{e}_k - \mathbf{e}_l) : i, j, k, l \text{ mutually distinct}\}$$

$(0, -1, 1, 0, 0, -1, 1, 0, \ldots) \in \mathcal{E}_4$.

Conjecture 2

Every subset of $\mathcal{E}_4$ which is a set of Bohr recurrence is also a set of topological recurrence.

Conjecture 1 (Katznelson's problem)

Let G be a countable abelian group. If $S \subset G$ is a set of Bohr recurrence, then S is a set of topological recurrence.

Conjecture 2

Every subset of $\mathcal{E}_4$ which is a set of Bohr recurrence is also a set of topological recurrence.

Conjecture 1 (Katznelson's problem)

Let G be a countable abelian group. If $S \subset G$ is a set of Bohr recurrence, then S is a set of topological recurrence.

Proposition 3.3 of [Gri21b] says Conjecture 2 implies Conjecture 1.

Lemma 1

Let $\rho : \mathbb{Z}^\omega \to G$ be a homomorphism such that $\rho(\mathcal{E}_1) = G$.
If $S \subset G \setminus \{0\}$ is a set of Bohr recurrence in G, then $\rho^{-1}(S) \cap \mathcal{E}_4$ is a set of Bohr recurrence in $\mathbb{Z}^\omega$.

Lemma 2

Let $\rho : H \to G$ be a homomorphism. If $S \subset H$ is a set of topological rec., then $\rho(S)$ is a set of topological rec. in G.

Lemma 1

Let $\rho : \mathbb{Z}^\omega \to G$ be a homomorphism such that $\rho(\mathcal{E}_1) = G$. If $S \subset G$ is a set of Bohr recurrence in G, then $\rho^{-1}(S) \cap \mathcal{E}_4$ is a set of Bohr recurrence in $\mathbb{Z}^\omega$.

Such a ρ always exists. Every function $f : \mathcal{E}_1 \to G$ extends to a homomorphism:

$$\rho(n_1\mathbf{e}_1 + \cdots + n_k\mathbf{e}_k) := n_1 f(\mathbf{e}_1) + \cdots + n_k f(\mathbf{e}_k)$$

We need:

Theorem (Bogoliouboff, [Bog39])

Let G be a countable abelian group. If $k \in \mathbb{N}$ and $G = A_1 \cup A_2 \cup \cdots \cup A_k$, then for some i, the iterated difference set

$$\{(a-b)-(c-d) : a, b, c, d \in A_i \text{ mut. distinct}\} \cup \{0\}$$

contains a basic Bohr neighborhood of 0.

Proof of Lemma 1.

Let $\rho : \mathbb{Z}^\omega \to G$, $\rho(\mathcal{E}_1) = G$. Let $S \subset G \setminus \{0\}$ be Bohr rec. To prove $\rho^{-1}(S) \cap \mathcal{E}_4$ is Bohr rec. in $\mathbb{Z}^\omega$, we prove: If $\psi : \mathbb{Z}^\omega \to \mathbb{T}^d$ is a hom. and $\varepsilon > 0$, then $\exists$ mut. dist. $\mathbf{e}_p, \mathbf{e}_q, \mathbf{e}_r, \mathbf{e}_t$ such that

$$(1) \qquad \|\psi\big((\mathbf{e}_p - \mathbf{e}_q) - (\mathbf{e}_r - \mathbf{e}_t)\big)\| < \varepsilon \quad \text{and}$$
$$(2) \qquad \rho\big((\mathbf{e}_p - \mathbf{e}_q) - (\mathbf{e}_r - \mathbf{e}_t)\big) \in S.$$

Cover $\mathbb{T}^d$ by $U_i, i \leq k$, diam $< \frac{\varepsilon}{2}$. $B_i := \mathcal{E}_1 \cap \psi^{-1}(U_i)$ covers $\mathcal{E}_1$. Let $A_i = \rho(B_i)$. These cover G, since $\rho(\mathcal{E}_1) = G$. Bogoliouboff:

$$V := \{(a_p - a_q) - (a_r - a_t) : a_p, a_q, a_r, a_t \in A_i \text{ mut. dist.}\} \cup \{0\}$$

contains a Bohr nhood of 0 in G, some $i \leq k$. Since $S \subset G \setminus \{0\}$ is Bohr rec., $S \cap V \neq \varnothing, \neq \{0\}$. So $\exists \mathbf{e}_p, \mathbf{e}_q, \mathbf{e}_r, \mathbf{e}_t \in B_i$ satisfying (2). $\psi(B_i)$ has diam. $< \varepsilon/2$, so

$$\|\psi((\mathbf{e}_p - \mathbf{e}_q) - (\mathbf{e}_r - \mathbf{e}_t))\| \leq \|\psi(\mathbf{e}_p) - \psi(\mathbf{e}_q)\| + \|\psi(\mathbf{e}_r) - \psi(\mathbf{e}_t)\| < \varepsilon.$$

$\square$

Lemma 1

Let $\rho : \mathbb{Z}^\omega \to G$ be a homomorphism such that $\rho(\mathcal{E}_1) = G$. If $S \subset G \setminus \{0\}$ is a set of Bohr recurrence in G, then $\rho^{-1}(S) \cap \mathcal{E}_4$ is a set of Bohr recurrence in $\mathbb{Z}^\omega$.

Lemma 2

Let $\rho : H \to G$ be a homomorphism. If $S \subset H$ is a set of topological rec., then $\rho(S)$ is a set of topological rec. in G.

Let G be a ctbl. abel. group and assume that every Bohr recurrent subset of $\mathcal{E}_4 \subset \mathbb{Z}^\omega$ is a set of topological recurrence (Conjecture 2). We'll prove that every set of Bohr recurrence in G is a set of topological recurrence.

Let $\rho : \mathbb{Z}^\omega \to G$ such that $\rho(\mathcal{E}_1) = G$. Let $S \subset G$ be a set of Bohr recurrence. If $0 \in S$, then S is already a set of topological recurrence, so assume $0 \notin S$. Then $\rho^{-1}(S) \cap \mathcal{E}_4$ is a set of Bohr recurrence (Lemma 1). Assuming Conjecture 2, we have that $\rho^{-1}(S) \cap \mathcal{E}_4$ is a set of topological recurrence in $\mathbb{Z}^\omega$. By Lemma 2, $\rho(\rho^{-1}(S) \cap \mathcal{E}_4)$ is a set of topological recurrence in G. So S is, too.

Conjecture 2

Every subset of $\mathcal{E}_4$ which is a set of Bohr recurrence is also a set of topological recurrence.

Examples:

$$S_1 := \{\mathbf{e}_n - \mathbf{e}_{n+d} - \mathbf{e}_{n+2d} + \mathbf{e}_{n+3d} : n, d \in \mathbb{N}\} \subset \mathcal{E}_4$$

is a set of Bohr recurrence. So is

$$S_2 := \{\mathbf{e}_n - \mathbf{e}_{n+d^2} - \mathbf{e}_{n+2d^2} + \mathbf{e}_{n+3d^2} : n, d \in \mathbb{N}\}$$

Are these sets of topological recurrence?

I haven't proved any nontrivial special cases, beyond what can be done using the results on the next three slides.

I_0 sets

We say $S \subset \mathbb{Z}$ is an I_0-set if for all bounded $f : S \to \mathbb{C}$ and $\varepsilon > 0$ there is a trigonometric polynomial p such that $|f(s) - p(s)| < \varepsilon$ for all $s \in S$.

We say $S = \{s_1 < s_2 < s_3 < \dots\}$ is lacunary if inf $s_{n+1}/s_n > 1$.

Theorem (Strzelecki [Str63])

If $S \subset \mathbb{N}$ is lacunary, then S is an I_0 set.

For example, $\{2^n : n \in \mathbb{N}\}$ is an I_0 set.

cf. [Le20], [KR99].

Proposition ([Gri21b],Proposition 2.2)

Let $E \subset \mathbb{Z}$ be an I_0 set. If $S \subset E - E$ and S is a set of Bohr recurrence, then S is a set of topological recurrence.

We need two standard equivalences.

Lemma

Let $S \subset \mathbb{Z}$. The following are equivalent.

(i) *S is a set of topological recurrence.*

(ii) *For all $k \in \mathbb{N}$ and every $f : \mathbb{Z} \to \{1, \ldots, k\}$, there exists $a, b \in \mathbb{Z}$ such that $b - a \in S$ and $f(b) = f(a)$.*

The following are equivalent:

(iii) *S is a set of Bohr recurrence.*

(iv) *For all trigonometric polynomials $p : \mathbb{Z} \to \mathbb{C}$, $\varepsilon > 0$, there exists $m \in S$ such that $|p(n + m) - p(n)| < \varepsilon$ for all $n \in \mathbb{Z}$.*

Lemma

Let $E \subset \mathbb{Z}$ be an I_0 set. If $S \subset E - E$ and S is a set of Bohr recurrence, then S is a set of topological recurrence.

Proof.

Let $S \subset E - E$ be Bohr recurrent. To prove S is topologically recurrent, we fix $k \in \mathbb{N}$ and an arbitrary $f : \mathbb{Z} \to \{1, \dots, k\}$, and will prove $\exists\ a, b \in \mathbb{Z}$ such that $f(b) = f(a)$ and $b - a \in S$.

E is I_0, so let p be a trig. poly. with $|f(n) - p(n)| < \frac{1}{3}\ \forall n \in E$.

S is Bohr recurrent, so fix $m \in S$ with
$|p(n+m) - p(n)| < \frac{1}{3}\ \forall n \in \mathbb{Z}$.

Since $S \subset E - E$, write $m = b - a$, where $a, b \in E$.

Then $|p(b) - p(a)| = |p(a+m) - p(a)| < \frac{1}{3}$, so

$$\begin{aligned} |f(b) - f(a)| &= |f(b) - p(b) + p(b) - p(a) + p(a) - f(a)| \\ &\leq |f(b) - p(b)| + |p(b) - p(a)| + |p(a) - f(a)| < 1, \end{aligned}$$

so $f(b) = f(a)$. Since $b - a \in S$, we are done. □

Separating recurrence properties

Question (Katznelson [Kat01], Veech [Vee68])

Is every set of Bohr recurrence also a set of topological recurrence?

If there is a set of Bohr recurrence which is not a set of topological recurrence, it cannot be a subset of a difference set of an I_0 set.

Lemma (Non-separation)

If $E \subset \mathbb{Z}$ is an I_0 set and $S \subset E - E$ is a set of Bohr recurrence, then S is a set of topological recurrence.

Question

Can the hypothesis "E is an I_0 set" be weakened?

Theorem (Kriz, [Kri87])

If $S \subset \mathbb{Z}$ is infinite, then there is a set $E \subset S - S$ which is a set of topological recurrence but not of measurable recurrence.

Conjecture

If $S \subset \mathbb{Z}$ is a set of measurable recurrence, then $\exists\ S' \subset S$ which is a set of topological recurrence and not a set of measurable recurrence.

No such result is known for "local" recurrence properties.
We say S is dense in the Bohr topology on $\mathbb{Z}$ if for all $m \in \mathbb{Z}$, $S + m$ is a set of Bohr recurrence.

Theorem ([Gri20])

If $S \subset \mathbb{Z}$ is dense in the Bohr topology then there is a subset $S' \subset S$ such that S' is dense in the Bohr topology and S' is not a set of measurable recurrence.

This answers (negatively) a question by Ruzsa [GR09], Bergelson and Ruzsa [BR09], Hegyvári and Ruzsa [HR16], and G. [Gri12]:

Question

Is every set which is dense in the Bohr topology also a set of measurable recurrence?

Sets with unknown recurrence properties

The following are known to be sets of Bohr recurrence, but not known to be sets of topological recurrence.

1. $\{7^{n+2d} + 7^{n+d} - 2 \cdot 7^n : n, d \in \mathbb{N}\}$
   B. N. Givens PhD thesis [Giv03], Givens and Kunen [GK03].
2. Grivaux and Roginskaya's examples [GR13]
3. Translates of the IP set generated by the Erdős-Taylor sequence [Kat73], [GM79]
4. $\{n!2^m 3^k : n, m, k \in \mathbb{N}\}$ (said to be Bohr recurrent in [FM12])
5. The sets constructed in [Gri20]: S is dense in the Bohr topology, not a set of measurable recurrence.

The first four could possibly be sets of measurable recurrence.

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