

# Special cases and equivalent form of Katznelson's problem on recurrence

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Let  $G$  be a countable abelian group.

A **probability measure preserving  $G$ -system** (or MPS) is a triple  $(X, \mu, T)$  where  $(X, \mu)$  is a probability measure space and  $T$  is an action of  $G$  on  $X$  by transformations  $T_g$  preserving  $\mu$ :

$$\mu(T_g A) = \mu(A) \quad \text{for all measurable } A \subset X$$

A **topological  $G$ -system**  $(X, T)$  is a compact metric space  $X$  together with an action of  $G$  on  $X$  by invertible homeomorphisms  $T_g$ .

We say  $(X, T)$  is **minimal** if the only nonempty closed  $T$ -invariant subset of  $X$  is  $X$ .

# Group rotations

A **group rotation**  $(K, R_\alpha)$  is a topological  $G$ -system where  $K$  is a compact abelian group,  $\rho : G \rightarrow K$  is a homomorphism, and  $R_g x := x + \rho(g)$ .

Writing  $m$  for Haar probability measure on  $K$ , we get a MPS  $(K, m, R)$ .

Ex:  $G = \mathbb{Z}$ .  $K = \mathbb{T} := \mathbb{R}/\mathbb{Z}$ . Fix  $\alpha \in \mathbb{T}$ , and let  $R_n(x) = x + n\alpha$ . Then  $(K, R_\alpha)$  is a group rotation.

## Definition

Let  $S \subset G$ . We say that  $S$  is a set of

- 1 measurable recurrence if  $\forall (X, \mu, T), A \subset X$  with  $\mu(A) > 0$ , there exists  $g \in S$  such that  $\mu(A \cap T_g A) > 0$ .
- 2 **topological recurrence** if  $\forall$  minimal  $(X, T)$ , nonempty open  $U \subset X$  there exists  $g \in S$  such that  $U \cap T_g U \neq \emptyset$ .
- 3 **Bohr recurrence** if for  $\forall$  minimal group rotations  $(K, R)$ , non- $\emptyset$  open  $U \subset K$ ,  $\exists g \in S$  such that  $U \cap R_g U \neq \emptyset$ .

The definitions easily yield:

$$\begin{aligned} S \text{ is a set of measurable rec.} &\implies S \text{ is a set of topological rec.} \\ &\implies S \text{ is a set of Bohr rec.} \end{aligned}$$

Kriz [Kri87]:  $S$  is a set of topological rec.  $\not\Rightarrow$   $S$  is a set of measurable rec. (in  $\mathbb{Z}$ )

Actually: if  $E \subset \mathbb{Z}$  is infinite, then there is a set  $S \subset E - E$  which is a set of topological rec. but not measurable rec. [Gri21a]

# Katznelson's and Veech's problem

## Conjecture 1 (Katznelson [Kat01], Veech [Vee68])

*Let  $G$  be a countable abelian group. Every subset of  $G$  which is a set of Bohr recurrence is also set of topological recurrence.*

This is never stated as a **conjecture**; we do so for convenience. See [GKR21] for exposition.

Katznelson asked this question for  $\mathbb{Z}$ , but the answer is not known in any countable abelian group.

The natural intuition is **no**: topological systems can be much more complicated than group rotations.

## Lemma ([GK03], Lemma 3.3, cf [Giv03])

$S := \{7^{n+2d} + 7^{n+d} - 2 \cdot 7^n : n, d \in \mathbb{N}\}$  is a set of Bohr recurrence.

Is  $S$  a set of topological recurrence?

Is  $S$  a set of measurable recurrence?

For  $x \in \mathbb{T} := \mathbb{R}/\mathbb{Z}$ , let  $\tilde{x} \in [0, 1)$  represent  $x$ .

$$\|x\| := \min_{n \in \mathbb{Z}} |\tilde{x} - n|.$$

For  $\mathbf{x} = (x_1, \dots, x_d) \in \mathbb{T}^d$ ,  $\|\mathbf{x}\| := \max_{j \leq d} \|x_j\|$ .

Given a homomorphism  $\psi : G \rightarrow \mathbb{T}^d$  and  $\varepsilon > 0$ , the **basic Bohr neighborhood of 0 in  $G$**  determined by  $\psi$  and  $\varepsilon$  is

$$\text{Bohr}(\psi, \varepsilon) := \{g \in G : \|\psi(g)\| < \varepsilon\}.$$

Ex:  $G = \mathbb{Z}$ ,  $\psi(n) := n(1/2) \bmod 1$ ,  $\varepsilon = 1/4$ .  $\text{Bohr}(\psi, \varepsilon) = 2\mathbb{Z}$ .

### Lemma (Equivalent definitions of Bohr recurrence)

*Let  $S \subset G$ . The following are equivalent.*

- 1**  $\forall$  minimal group rotations  $(K, R)$ , non- $\emptyset$  open  $U \subset K$ ,  
 $\exists g \in S$  such that  $U \cap R_g U \neq \emptyset$ .
- 2** For all  $d \in \mathbb{N}$ , every homomorphism  $\psi : G \rightarrow \mathbb{T}^d$ , and all  $\varepsilon > 0$ , there is a  $g \in S$  such that  $\|\psi(g)\| < \varepsilon$ .
- 3** For every Bohr neighborhood  $U$  of 0,  $S \cap U \neq \emptyset$ .

In  $\mathbb{Z}$ :  $S$  is a set of Bohr recurrence iff for all  $d \in \mathbb{N}$ ,  $\alpha \in \mathbb{T}^d$ , and all  $\varepsilon > 0$ ,  $\exists m \in S$  such that  $\|m\alpha\| < \varepsilon$ .

Lemma ([GK03], Lemma 3.3, cf [Giv03])

$S := \{7^{n+2d} + 7^{n+d} - 2 \cdot 7^n : n, d \in \mathbb{N}\}$  is a set of Bohr recurrence.

Proof.

Fix  $d \in \mathbb{N}$  and  $\alpha \in \mathbb{T}^d$ . We must find  $m \in S$  such that  $\|m\alpha\| < \varepsilon$ . Cover  $\mathbb{T}^d$  with sets  $U_1, \dots, U_k$  of diameter  $< \varepsilon/2$ . For  $n \in \mathbb{N}$ , let

$$f(n) := \min\{j : 7^n \alpha \in U_j\}$$

This is a finite coloring of  $\mathbb{N}$ . By van der Waerden's theorem on arithmetic progressions, there are  $n, d \in \mathbb{N}$ ,  $j \leq k$  so that  $7^n \alpha, 7^{n+d} \alpha, 7^{n+2d} \alpha$  all lie in  $U_j$ . Thus

$$\|7^{n+d} \alpha - 7^n \alpha\| < \varepsilon/2 \quad \text{and} \quad \|7^{n+2d} \alpha - 7^n \alpha\| < \varepsilon/2,$$

so  $\|(7^{n+2d} + 7^{n+d} - 2 \cdot 7^n) \alpha\| < \varepsilon$  (triangle inequality). □

# A special case of Katznelson's problem in $\mathbb{Z}$

## Proposition ([Gri21b], Proposition 2.2)

*If  $S \subset \{2^n - 2^m : n, m \in \mathbb{N}\}$  is a set of Bohr recurrence, then  $S$  is a set of topological recurrence.*

This is not vacuous: the ambient set  $\{2^n - 2^m\}$  has the form  $E - E$ , where  $E$  is infinite. So  $E - E$  is a set of measurable recurrence (and therefore a set of topological recurrence and a set of Bohr recurrence), by the Poincaré recurrence theorem.

It is not trivial:  $\exists S \subset E - E$  which is a set of topological recurrence and not a set of Bohr recurrence (Kriz).

It might be useless.



# An equivalent form of Katznelson's problem

$\mathbb{Z}^\omega$  := the countable direct sum of countably many copies of  $\mathbb{Z}$ .

Write  $\mathbf{n} \in \mathbb{Z}^\omega$  as  $(n_1, n_2, \dots)$ , only finitely many  $n_i$  nonzero.

$\mathbf{e}_1 = (1, 0, 0, \dots)$ ,  $\mathbf{e}_2 = (0, 1, 0, 0, \dots)$ ,  $\dots$

$\mathcal{E}_1 := \{\mathbf{e}_i : i \in \mathbb{N}\}$ .

$\mathcal{E}_4 := \{(\mathbf{e}_i - \mathbf{e}_j) - (\mathbf{e}_k - \mathbf{e}_l) : i, j, k, l \text{ mutually distinct}\}$

$(0, -1, 1, 0, 0, -1, 1, 0, \dots) \in \mathcal{E}_4$ .

## Conjecture 2

*Every subset of  $\mathcal{E}_4$  which is a set of Bohr recurrence is also a set of topological recurrence.*

## Conjecture 1 (Katznelson's problem)

*Let  $G$  be a countable abelian group. If  $S \subset G$  is a set of Bohr recurrence, then  $S$  is a set of topological recurrence.*

## Conjecture 2

*Every subset of  $\mathcal{E}_4$  which is a set of Bohr recurrence is also a set of topological recurrence.*

## Conjecture 1 (Katznelson's problem)

*Let  $G$  be a countable abelian group. If  $S \subset G$  is a set of Bohr recurrence, then  $S$  is a set of topological recurrence.*

We will prove Conjecture 2 implies Conjecture 1.

## Lemma 1

*Let  $\rho : \mathbb{Z}^\omega \rightarrow G$  be a homomorphism such that  $\rho(\mathcal{E}_1) = G$ . If  $S \subset G \setminus \{0\}$  is a set of Bohr recurrence in  $G$ , then  $\rho^{-1}(S) \cap \mathcal{E}_4$  is a set of Bohr recurrence in  $\mathbb{Z}^\omega$ .*

## Lemma 2

*Let  $\rho : H \rightarrow G$  be a homomorphism. If  $S \subset H$  is a set of topological rec., then  $\rho(S)$  is a set of topological rec. in  $G$ .*

## Lemma 1

*Let  $\rho : \mathbb{Z}^\omega \rightarrow G$  be a homomorphism such that  $\rho(\mathcal{E}_1) = G$ .  
If  $S \subset G$  is a set of Bohr recurrence in  $G$ , then  $\rho^{-1}(S) \cap \mathcal{E}_4$  is a set of Bohr recurrence in  $\mathbb{Z}^\omega$ .*

We need:

## Theorem (Bogoliouboff, [Bog39])

*Let  $G$  be a countable abelian group. If  $k \in \mathbb{N}$  and  $G = A_1 \cup A_2 \cup \cdots \cup A_k$ , then for some  $i$ , the iterated difference set*

$$\{(a - b) - (c - d) : a, b, c, d \in A_i \text{ mut. distinct}\} \cup \{0\}$$

*contains a basic Bohr neighborhood of 0.*

## Proof of Lemma 1.

Let  $\rho : \mathbb{Z}^\omega \rightarrow G$ ,  $\rho(\mathcal{E}_1) = G$ . Let  $S \subset G \setminus \{0\}$  be Bohr rec. To prove  $\rho^{-1}(S) \cap \mathcal{E}_4$  is Bohr rec. in  $\mathbb{Z}^\omega$ , we prove: If  $\psi : \mathbb{Z}^\omega \rightarrow \mathbb{T}^d$  is a hom. and  $\varepsilon > 0$ , then  $\exists$  mut. dist.  $\mathbf{e}_p, \mathbf{e}_q, \mathbf{e}_r, \mathbf{e}_t$  such that

- (1)  $\|\psi((\mathbf{e}_p - \mathbf{e}_q) - (\mathbf{e}_r - \mathbf{e}_t))\| < \varepsilon$  and
- (2)  $\rho((\mathbf{e}_p - \mathbf{e}_q) - (\mathbf{e}_r - \mathbf{e}_t)) \in S$ .

Cover  $\mathbb{T}^d$  by  $U_i, i \leq k$ ,  $\text{diam} < \frac{\varepsilon}{2}$ .  $B_i := \mathcal{E}_1 \cap \psi^{-1}(U_i)$  covers  $\mathcal{E}_1$ . Let  $A_i = \rho(B_i)$ . These cover  $G$ , since  $\rho(\mathcal{E}_1) = G$ . Bogoliouboff:

$$V := \{(\mathbf{a}_p - \mathbf{a}_q) - (\mathbf{a}_r - \mathbf{a}_t) : \mathbf{a}_p, \mathbf{a}_q, \mathbf{a}_r, \mathbf{a}_t \in A_i \text{ mut. dist.}\} \cup \{0\}$$

contains a Bohr nhood of 0 in  $G$ . Since  $S \subset G \setminus \{0\}$  is Bohr rec.,  $S \cap V \neq \emptyset, \neq \{0\}$ . So  $\exists \mathbf{e}_p, \mathbf{e}_q, \mathbf{e}_r, \mathbf{e}_t \in B_i$  satisfying (2).

$\psi(B_i)$  has diam.  $< \varepsilon/2$ , so

$$\begin{aligned} \|\psi((\mathbf{e}_p - \mathbf{e}_q) - (\mathbf{e}_r - \mathbf{e}_t))\| &\leq \|\psi(\mathbf{e}_p) - \psi(\mathbf{e}_q)\| + \|\psi(\mathbf{e}_r) - \psi(\mathbf{e}_t)\| \\ &< \varepsilon. \end{aligned}$$



## Lemma 1

*Let  $\rho : \mathbb{Z}^\omega \rightarrow G$  be a homomorphism such that  $\rho(\mathcal{E}_1) = G$ . If  $S \subset G \setminus \{0\}$  is a set of Bohr recurrence in  $G$ , then  $\rho^{-1}(S) \cap \mathcal{E}_4$  is a set of Bohr recurrence in  $\mathbb{Z}^\omega$ .*

## Lemma 2

*Let  $\rho : H \rightarrow G$  be a homomorphism. If  $S \subset H$  is a set of topological rec., then  $\rho(S)$  is a set of topological rec. in  $G$ .*

Let  $G$  be a ctbl. abel. group and assume that every Bohr recurrent subset of  $\mathcal{E}_4 \subset \mathbb{Z}^\omega$  is a set of topological recurrence (Conjecture 2). We'll prove that every set of Bohr recurrence in  $G$  is a set of topological recurrence.

Let  $\rho : \mathbb{Z}^\omega \rightarrow G$  such that  $\rho(\mathcal{E}_1) = G$ . Let  $S \subset G$  be a set of Bohr recurrence. If  $0 \in S$ , then  $S$  is already a set of topological recurrence, so assume  $0 \notin S$ . Then  $\rho^{-1}(S) \cap \mathcal{E}_4$  is a set of Bohr recurrence (Lemma 1). Assuming Conjecture 2, we have that  $\rho^{-1}(S) \cap \mathcal{E}_4$  is a set of topological recurrence in  $\mathbb{Z}^\omega$ . By Lemma 2,  $\rho(\rho^{-1}(S) \cap \mathcal{E}_4)$  is a set of topological recurrence in  $G$ . So  $S$  is, too.

## Conjecture 2

*Every subset of  $\mathcal{E}_4$  which is a set of Bohr recurrence is also a set of topological recurrence.*

Examples:

$$S_1 := \{\mathbf{e}_n - \mathbf{e}_{n+d} - \mathbf{e}_{n+2d} + \mathbf{e}_{n+3d} : n, d \in \mathbb{N}\} \subset \mathcal{E}_4$$

is a set of Bohr recurrence. So is

$$S_2 := \{\mathbf{e}_n - \mathbf{e}_{n+d^2} - \mathbf{e}_{n+2d^2} + \mathbf{e}_{n+3d^2} : n, d \in \mathbb{N}\}$$

Are these sets of topological recurrence?

I haven't proved any nontrivial special cases, beyond what can be done using the results on the next three slides.

We say  $S \subset \mathbb{Z}$  is an  $I_0$ -set if for all bounded  $f : S \rightarrow \mathbb{C}$  and  $\varepsilon > 0$  there is a trigonometric polynomial  $p$  such that  $|f(s) - p(s)| < \varepsilon$  for all  $s \in S$ .

We say  $S = \{s_1 < s_2 < s_3 < \dots\}$  is lacunary if  $\inf s_{n+1}/s_n > 1$ .

## Theorem (Strzelecki [Str63])

*If  $S \subset \mathbb{N}$  is lacunary, then  $S$  is an  $I_0$  set.*

For example,  $\{2^n : n \in \mathbb{N}\}$  is an  $I_0$  set.

cf. [Le20], [KR99].

## Proposition ([Gri21b], Proposition 2.2)

*Let  $E \subset \mathbb{Z}$  be an  $I_0$  set. If  $S \subset E - E$  and  $S$  is a set of Bohr recurrence, then  $S$  is a set of topological recurrence.*

We need two standard equivalences.

## Lemma

*Let  $S \subset \mathbb{Z}$ . The following are equivalent.*

- (i)  $S$  is a set of topological recurrence.*
- (ii) For all  $k \in \mathbb{N}$  and every  $f : \mathbb{Z} \rightarrow \{1, \dots, k\}$ , there exists  $a, b \in \mathbb{Z}$  such that  $b - a \in S$  and  $f(b) = f(a)$ .*

*The following are equivalent:*

- (iii)  $S$  is a set of Bohr recurrence.*
- (iv) For all trigonometric polynomials  $p : \mathbb{Z} \rightarrow \mathbb{C}$ ,  $\varepsilon > 0$ , there exists  $m \in S$  such that  $|p(n + m) - p(n)| < \varepsilon$  for all  $n \in \mathbb{Z}$ .*



## Lemma

*Let  $E \subset \mathbb{Z}$  be an  $I_0$  set. If  $S \subset E - E$  and  $S$  is a set of Bohr recurrence, then  $S$  is a set of topological recurrence.*

## Proof.

Let  $S \subset E - E$  be Bohr recurrent. To prove  $S$  is topologically recurrent, we fix  $k \in \mathbb{N}$  and an arbitrary  $f : \mathbb{Z} \rightarrow \{1, \dots, k\}$ , and will prove  $\exists a, b \in \mathbb{Z}$  such that  $f(b) = f(a)$  and  $b - a \in S$ .

$E$  is  $I_0$ , so let  $p$  be a trig. poly. with  $|f(n) - p(n)| < \frac{1}{3} \forall n \in E$ .

$S$  is Bohr recurrent, so fix  $m \in S$  with

$$|p(n+m) - p(n)| < \frac{1}{3} \forall n \in \mathbb{Z}.$$

Since  $S \subset E - E$ , write  $m = b - a$ , where  $a, b \in E$ .

Then  $|p(b) - p(a)| = |p(a+m) - p(a)| < \frac{1}{3}$ , so

$$\begin{aligned} |f(b) - f(a)| &= |f(b) - p(b) + p(b) - p(a) + p(a) - f(a)| \\ &\leq |f(b) - p(b)| + |p(b) - p(a)| + |p(a) - f(a)| < 1, \end{aligned}$$

so  $f(b) = f(a)$ . Since  $b - a \in S$ , we are done.



# Separating recurrence properties

Question (Katznelson [Kat01], Veech [Vee68])

*Is every set of Bohr recurrence also a set of topological recurrence?*

If there is a set of Bohr recurrence which is not a set of topological recurrence, it **cannot** be a subset of a difference set of an  $I_0$  set.

Lemma (Non-separation)

*If  $E \subset \mathbb{Z}$  is an  $I_0$  set and  $S \subset E - E$  is a set of Bohr recurrence, then  $S$  is a set of topological recurrence.*

Question

*Can the hypothesis “ $E$  is an  $I_0$  set” be weakened?*

Theorem (Kriz, [Kri87])

*If  $S \subset \mathbb{Z}$  is infinite, then there is a set  $E \subset S - S$  which is a set of topological recurrence but not of measurable recurrence.*

## Conjecture

*If  $S \subset \mathbb{Z}$  is a set of measurable recurrence, then  $\exists S' \subset S$  which is a set of topological recurrence and not a set of measurable recurrence.*

No such result is known for “local” recurrence properties.  
We say  $S$  is **dense in the Bohr topology** on  $\mathbb{Z}$  if for all  $m \in \mathbb{Z}$ ,  $S + m$  is a set of Bohr recurrence.

## Theorem ([Gri20])

*If  $S \subset \mathbb{Z}$  is dense in the Bohr topology then there is a subset  $S' \subset S$  such that  $S'$  is dense in the Bohr topology and  $S'$  is not a set of measurable recurrence.*

This answers (negatively) a question by Ruzsa [GR09], Bergelson and Ruzsa [BR09], Hegyvári and Ruzsa [HR16], and G. [Gri12]:

## Question

Is every set which is dense in the Bohr topology also a set of measurable recurrence?






# Sets with unknown recurrence properties

The following are known to be sets of Bohr recurrence, but not known to be sets of topological recurrence.


- 1  $\{7^{n+2d} + 7^{n+d} - 2 \cdot 7^n : n, d \in \mathbb{N}\}$   
B. N. Givens PhD thesis [Giv03], Givens and Kunen [GK03].
- 2 Grivaux and Roginskaya's examples [GR13]
- 3 Translates of the IP set generated by the Erdős-Taylor sequence [Kat73], [GM79]
- 4  $\{n!2^m3^k : n, m, k \in \mathbb{N}\}$  (said to be Bohr recurrent in [FM12])
- 5 The sets constructed in [Gri20]:  $S$  is dense in the Bohr topology, not a set of measurable recurrence.

The first four could possibly be sets of measurable recurrence.

# References I

-  N. Bogoliouboff, *Sur quelques propriétés arithmétiques des presque-périodes*, Ann. Chaire Phys. Math. Kiev **4** (1939), 185–205. MR 20164
-  Vitaly Bergelson and Imre Z. Ruzsa, *Sumsets in difference sets*, Israel J. Math. **174** (2009), 1–18. MR 2581205
-  Dikran Dikranjan, *Continuous maps in the Bohr topology*, Appl. Gen. Topol. **2** (2001), no. 2, 237–270. MR 1890040
-  Lorenzo de Leo and Dikran Dikranjan, *Straightening theorem for bounded abelian groups*, Topology Appl. **155** (2008), no. 17-18, 2158–2176. MR 2458001
-  Dikran Dikranjan and Stephen Watson, *A solution to van Douwen's problem on Bohr topologies*, J. Pure Appl. Algebra **163** (2001), no. 2, 147–158. MR 1846658

# References II

-  Nikos Frantzikinakis and Randall McCutcheon, *Ergodic theory: recurrence*, Mathematics of complexity and dynamical systems. Vols. 1–3, Springer, New York, 2012, pp. 357–368. MR 3220681
-  Berit Nilsen Givens, *Hypergraphs and chromatic numbers, with applications to the Bohr topology*, ProQuest LLC, Ann Arbor, MI, 2003, Thesis (Ph.D.)—The University of Wisconsin - Madison. MR 2704616
-  Berit Nilsen Givens and Kenneth Kunen, *Chromatic numbers and Bohr topologies*, Topology Appl. **131** (2003), no. 2, 189–202. MR 1981873
-  Daniel Glasscock, Andreas Koutsogiannis, and Florian K. Richter, *On Katznelson's Question for skew product systems*, arXiv:2106.11393, to appear in Bulletin AMS (2021).

# References III



Colin C. Graham and O. Carruth McGehee, *Essays in commutative harmonic analysis*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Science], vol. 238, Springer-Verlag, New York-Berlin, 1979. MR 550606








Alfred Geroldinger and Imre Z. Ruzsa, *Combinatorial number theory and additive group theory*, Advanced Courses in Mathematics. CRM Barcelona, Birkhäuser Verlag, Basel, 2009, Courses and seminars from the DocCourse in Combinatorics and Geometry held in Barcelona, 2008.








Sophie Grivaux and Maria Roginskaya, *Some new examples of recurrence and non-recurrence sets for products of rotations on the unit circle*, Czechoslovak Math. J. **63(138)** (2013), no. 3, 603–627. MR 3125645




## References IV

-  John T. Griesmer, *Sumsets of dense sets and sparse sets*, Israel J. Math. **190** (2012), 229–252.
-  John T. Griesmer, *Separating bohr denseness from measurable recurrence*, 2020, arxiv:2002.06994.
-  John T. Griesmer, *Separating topological recurrence from measurable recurrence: exposition and extension of Kriz's example*, arxiv:2108.01642 (2021).
-  ———, *Special cases and equivalent forms of Katznelson's problem on recurrence*, arXiv e-prints (2021), arXiv:2108.02190.
-  Norbert Hegyvári and Imre Z. Ruzsa, *Additive structure of difference sets and a theorem of Følner*, Australas. J. Combin. **64** (2016), 437–443.



# References V

-  Y. Katznelson, *Sequence of integers dense in the bohr group*, Proc. Royal. Inst. Techn. Stockholm (1973), 79–86.
-  ———, *Chromatic numbers of Cayley graphs on  $\mathbb{Z}$  and recurrence*, vol. 21, 2001, Paul Erdős and his mathematics (Budapest, 1999), pp. 211–219. MR 1832446
-  Kenneth Kunen and Walter Rudin, *Lacunarity and the Bohr topology*, Math. Proc. Cambridge Philos. Soc. **126** (1999), no. 1, 117–137. MR 1681658
-  Igor Kriz, *Large independent sets in shift-invariant graphs: solution of Bergelson's problem*, Graphs Combin. **3** (1987), no. 2, 145–158. MR 932131
-  Kenneth Kunen, *Bohr topologies and partition theorems for vector spaces*, Topology Appl. **90** (1998), no. 1-3, 97–107. MR 1648302

-  Anh N. Le, *Interpolation sets and nilsequences*, Colloq. Math. **162** (2020), no. 2, 181–199. MR 4128459
-  E. Strzelecki, *On a problem of interpolation by periodic and almost periodic functions*, Colloq. Math. **11** (1963), 91–99. MR 160080
-  William A. Veech, *The equicontinuous structure relation for minimal Abelian transformation groups*, Amer. J. Math. **90** (1968), 723–732. MR 232377