Single recurrence: overview, open problems, and mysterious examples

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A probability measure preserving system (or MPS) is a triple (X, μ, T) where (X, μ) is a probability measure space and $T : X \to X$ is a transformation preserving μ:

$$\mu(T^{-1}A) = \mu(A) \qquad \text{for all measurable } A \subset X$$

In this talk we only consider invertible systems (T is invertible).

A topological dynamical system (X, T) is a compact metric space X together with a continuous map $T : X \to X$.

We say (X, T) is minimal if the only nonempty closed T-invariant subset of X is X.

Equivalently, (X, T) is minimal if for every nonempty open $U \subset X$, we have $\bigcup_{n=1}^{m} T^{-n}U = X$ for some $m \in \mathbb{N}$.

Group rotations

A group rotation (K, R_α) is a topological system where K is a compact abelian group, $\alpha \in K$, and $R_\alpha x := x + \alpha$.

Writing m for Haar probability measure on K, we get a MPS (K, m, R_α).

Example: $K = \mathbb{T} := \mathbb{R}/\mathbb{Z}$. Identifying $\mathbb{T}$ with the unit interval $[0, 1)$, fix an $\alpha \in \mathbb{T}$. Then (K, R_α) is a group rotation.

The Poincaré Recurrence Theorem

Theorem (PRT)

If (X, μ, T) *is a MPS and* $\mu(A) > 0$*, there exists* $n \in \mathbb{N}$ *such that* $\mu(A \cap T^{-n}A) > 0$*.*

The time n where positivity occurs can be specified more precisely.

Definition

$S \subset \mathbb{Z}$ is a set of measurable recurrence if $\forall$ MPS (X, μ, T) and all $A \subset X$ with $\mu(A) > 0$, we have $\mu(A \cap T^{-n}A) > 0$ for some $n \in S$.

PRT says that $\mathbb{N}$ is a set of measurable recurrence.

Definition

$E \subset \mathbb{Z}$ is a set of measurable recurrence if $\forall$ MPS (X, μ, T) and all $A \subset X$ with $\mu(A) > 0$, we have $\mu(A \cap T^{-n}A) > 0$ for some $n \in E$.

Examples:

- Sets of the form $\Delta(S)$, where S is infinite.
  $\Delta(S)$ means $\{s - s' : s \neq s' \in S\}$.
- Sets containing arbitrarily long intervals: $\bigcup_{n=1}^{\infty}[2^n, 2^n + n]$.
- $\{n^2 : n \in \mathbb{N}\}$ (Furstenberg [Fur77], Sárkőzy [S78]).
- $S_{5/2} := \{\lfloor n^{5/2} \rfloor : n \in \mathbb{N}\}$.
- Every translate of $S_{5/2}$ is a set of measurable recurrence.

Non-examples:

- $\{n^2 + 1 : n \in \mathbb{N}\}$ is not a set of measurable recurrence.
- The set $\{n! : n \in \mathbb{N}\}$ is not a set of measurable recurrence.
- Lacunary sets: if $S = \{s_1 < s_2 < \dots\}$ and $\inf s_{n+1}/s_n > 1$, S is not a set of measurable recurrence.

Theorem (PRT)

If $S \subset \mathbb{Z}$ is infinite, then $\Delta(S)$ is a set of mble recurrence.

Proof.

Fix infinite $S \subset \mathbb{Z}$, an MPS (X, μ, T), and $A \subset X$ with $\mu(A) > 0$.

We will find $s \neq s' \in S$ such that $\mu(A \cap T^{s-s'}A) > 0$.

Let $k \in \mathbb{N}, k > \frac{1}{\mu(A)}$, and let $s_1, \ldots, s_k$ be distinct elements of S.

Each $T^{-s_j}A$ has measure $\mu(A)$, so $\sum_{j=1}^{k} \mu(T^{-s_j}A) = k\mu(A) > 1$.

Since $\mu(X) = 1$, at least two of the sets must intersect with positive measure, meaning

$$\mu(T^{-s_i}A \cap T^{-s_j}A) > 0$$

Since T preserves μ, we have

$$\mu(A \cap T^{s_i - s_j}A) = \mu(T^{-s_i}A \cap T^{-s_j}A) > 0. \qquad \square$$

Theorem (Mean Ergodic Theorem)

Let (X, μ, T) be a MPS and $f \in L^2(\mu)$. If $I_k = \{n_k, \ldots, m_k\}$ are intervals in $\mathbb{Z}$ with $|I_k| \to \infty$, then

$$\lim_{k\to\infty} \frac{1}{|I_k|} \sum_{n\in I_k} f \circ T^n = P_{inv} f \qquad \text{in } L^2(\mu)$$

where P_{inv} is orth. proj. onto the space of T-invariant functions.

Remark: $\int P_{inv} f \, d\mu = \int f \, d\mu$.

Corollary (Khintchine)

If $A \subset X$, then $\lim_{k\to\infty} \frac{1}{|I_k|} \sum_{n\in I_k} \mu(A \cap T^{-n}A) \geq \mu(A)^2$.

Consequently: for all $\varepsilon > 0$ there exists $n \in \bigcup_{k\in\mathbb{N}} I_k$ such that

$$\mu(A \cap T^{-n}A) > \mu(A)^2 - \varepsilon.$$

Khintchine: $\lim_{k\to\infty} \frac{1}{|I_k|} \sum_{n\in I_k} \mu(A \cap T^{-n}A) \geq \mu(A)^2$.

Proof.

Let $f = 1_A$, so $\mu(A \cap T^{-n}A) = \int f \cdot f \circ T^n \, d\mu$.

$$\begin{aligned}
\frac{1}{|I_k|} \sum_{n\in I_k} \mu(A \cap T^{-n}A) &= \frac{1}{|I_k|} \sum_{n\in I_k} \int f \cdot f \circ T^n \, d\mu \\
&= \int f \cdot \frac{1}{|I_k|} \sum_{n\in I_k} f \circ T^n \, d\mu \\
&\xrightarrow[k\to\infty]{} \int f \cdot P_{inv} f \, d\mu && \text{mean erg. thm.} \\
&= \int P_{inv} f \cdot P_{inv} f \, d\mu && \text{orth. proj.} \\
&\geq \left(\int P_{inv} f \, d\mu \right)^2 && \text{C-S, or Jensen} \\
&= \mu(A)^2. && \square
\end{aligned}$$

A sequence of integers s_n is Hartman uniformly distributed if

$$(1) \qquad \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} \exp(2\pi i s_n t) = 0 \qquad \text{for all } t \in (0,1)$$

Examples:

- $s_n = n$
- $s_n = \lfloor n^{5/2} \rfloor$

See [Nai98], [BKQW05], for more.

Theorem

If (s_n) is Hartman-u.d., (X, μ, T) is a MPS, and $f \in L^2(\mu)$, then

$$(2) \qquad \frac{1}{N} \sum_{n=1}^{N} f \circ T^{s_n} \underset{N\to\infty}{\longrightarrow} P_{inv} f.$$

Consequently, $\{s_n : n \in \mathbb{N}\}$ is a set of measurable recurrence.

(1) says s_n has the correct ergodic averages for torus rotations,
(2) says s_n has the correct ergodic averages for every MPS.

Definition

Let $S \subset \mathbb{Z}$. We say that S is a set of

1. measurable recurrence if $\forall$ (X, μ, T), $A \subset X$ with $\mu(A) > 0$, there exists $n \in S$ such that $\mu(A \cap T^{-n}A) > 0$.
2. topological recurrence if $\forall$ minimal (X, T), nonempty open $U \subset X$ there exists $n \in S$ such that $U \cap T^{-n}U \neq \varnothing$.
3. Bohr recurrence if for $\forall$ minimal group rotations (K, R_α), non-$\varnothing$ open $U \subset K$, $\exists$ $n \in S$ such that $U \cap R_\alpha^{-n}U \neq \varnothing$.

S is a set of mble rec. $\implies$ S is a set of top. rec. $\implies$ S is a set of Bohr rec.

S is a set of top. rec. $\not\Longrightarrow$ S is a set of mble rec. (Kriz, [Kri87]), answering a question of Bergelson.

Question (Katznelson [Kat01], Veech [Vee68])

Is every set of Bohr recurrence also a set of topological recurrence?

Equivalent form of Bohr recurrence; example

For $\alpha \in \mathbb{T}$, let $\|\alpha\| :=$ distance from α to the nearest integer (identifying α with $\tilde{\alpha} \in [0,1)$)

For $\alpha = (\alpha_1, \ldots, \alpha_d) \in \mathbb{T}^d$, let $\|\alpha\| = \max_{j \le d} \|\alpha_j\|$.

S is a set of Bohr recurrence iff for all $d \in \mathbb{N}$, $\alpha \in \mathbb{T}^d$, we have

$$\inf_{n \in S} \|n\alpha\| = 0.$$

Lemma ([GK03], Lemma 3.3, cf [Giv03])

$S := \{7^{n+2d} + 7^{n+d} - 2 \cdot 7^n : n, d \in \mathbb{N}\}$ *is a set of Bohr recurrence.*

Is S a set of topological recurrence? Measurable recurrence? This is an open problem.

Such sets arise in the study of Bohr topologies of certain abelian groups, like the direct sum of countably many copies of $\mathbb{Z}/2\mathbb{Z}$: [Kun98], [Dik01], [DW01], [dLD08].

Lemma ([GK03] Lemma 3.3, cf [Giv03])

$S := \{7^{n+2d} + 7^{n+d} - 2 \cdot 7^n : n, d \in \mathbb{N}\}$ is a set of Bohr recurrence.

Proof.

Fix $d \in \mathbb{N}$ and $\alpha \in \mathbb{T}^d$. We must find $m \in S$ such that $\|m\alpha\| < \varepsilon$. Cover $\mathbb{T}^d$ with sets $U_1, \ldots, U_k$ of diameter $< \varepsilon/2$. Let

$$A_j = \{n : 7^n\alpha \in U_j\},$$

so that $\mathbb{N} = A_1 \cup \cdots \cup A_k$.

By van der Waerden's theorem, at least one of the A_j contains a three-term arithmetic progression: there are $n, d \in \mathbb{N}$, $j \leq k$ so that $7^n\alpha, 7^{n+d}\alpha, 7^{n+2d}\alpha$ all lie in U_j. Thus

$$\|7^{n+d}\alpha - 7^n\alpha\| < \varepsilon/2 \quad \text{and} \quad \|7^{n+2d}\alpha - 7^n\alpha\| < \varepsilon/2,$$

and the triangle inequality yields

$$\|(7^{n+2d} + 7^{n+d} - 2 \cdot 7^n)\alpha\| < \varepsilon. \qquad \square$$

Katznelson's and Veech's problem

Question (Katznelson [Kat01], Veech [Vee68])

Is every set of Bohr recurrence also a set of topological recurrence?

The natural intuition is no: topological systems can be much more complicated than group rotations.

But recall: if (s_n) has correct ergodic averages for torus rotations, then it has correct ergodic averages for arbitrary MPSs.

Let $E = \{2^n : n \in \mathbb{N}\}$.

We will prove that if $S \subset \Delta(E)$ and S is a set of Bohr recurrence, then S is a set of topological recurrence.

This is not vacuous: $\Delta(E)$ is a set of measurable recurrence,by Poincaré's recurrence theorem.

The conclusion cannot be improved to "then S is a set of measurable recurrence," by Kriz's construction ([Kri87], implicitly).

Lemma

Let $S \subset \mathbb{Z}$. The following are equivalent.

(i) *S is a set of topological recurrence.*

(ii) *For all $k \in \mathbb{N}$ and every $f : \mathbb{Z} \to \{1, \dots, k\}$, there exists $a, b \in \mathbb{Z}$ such that $b - a \in S$ and $f(b) = f(a)$.*

Proof of (ii) $\implies$ (i).

Let (X, T) be minimal, $U \subset X$ nonempty and open.

Fix $k \in \mathbb{N}$ such that $X = \bigcup_{m=1}^{k} T^{-m} U$ (minimality).

Fix $x \in X$, define $f(n) = \min\{m : T^n x \in T^{-m} U\}$, so $f(n) \leq k$.

By (ii), choose $a, b \in \mathbb{Z}$ such that $b - a \in S$ and $f(b) = f(a)$.

Writing m for the common value of $f(b)$, $f(a)$, we have

$$\begin{aligned} & T^b x \in T^{-m} U \quad \text{and} \quad T^a x \in T^{-m} U \\ \implies & T^m x \in T^{-b} U \cap T^{-a} U \\ \implies & U \cap T^{b-a} U \neq \varnothing. \end{aligned}$$

□

Lemma

Let $S \subset \mathbb{Z}$. The following are equivalent:

(i) *S is a set of Bohr recurrence.*

(ii) *For all trigonometric polynomials $p : \mathbb{Z} \to \mathbb{C}$, $\varepsilon > 0$, there exists $m \in S$ such that $|p(n+m) - p(n)| < \varepsilon$ for all $n \in \mathbb{Z}$.*

Proof of (i) $\implies$ (ii).

Fix $p(n) := \sum_{j=1}^{d} c_j e(n\alpha_j)$, $\varepsilon > 0$. Let $C = 1 + \sum |c_j|$.

Since S is a set of Bohr recurrence, we can find $m \in S$ such that $\|m\alpha_j\| < \varepsilon/4C$ for each j, meaning $|e(m\alpha_j) - 1| < \varepsilon/C$. Then

$$\begin{aligned} |p(n+m) - p(n)| &\leq \sum |c_j||e((n+m)\alpha_j) - e(n\alpha_j)| \\ &= \sum |c_j||e(m\alpha_j) - 1| \\ &< \sum |c_j|\varepsilon/C \\ &< \varepsilon \end{aligned}$$

□

I_0 sets

$\phi : \mathbb{Z} \to \mathbb{C}$ is Bohr almost periodic (Bohr-AP) if it is a uniform limit of trigonometric polynomials.

$E \subset \mathbb{Z}$ is an I_0-set if for all bounded $f : E \to \mathbb{C}$ there is a Bohr-AP ϕ such that $f(x) = \phi(x)$ for all $x \in E$.

$E = \{n_1 < n_2 < n_3 < \dots\}$ is lacunary if $\inf n_{k+1}/n_k > 1$.

Theorem (Strzelecki [Str63])

If $E \subset \mathbb{N}$ is lacunary, then E is an I_0 set.

For example, $\{3^n : n \in \mathbb{N}\}$ is an I_0 set.

cf. [GH13], [Le20], [KR99].

Lemma (Non-separation in differences of I_0 sets)

Let $E \subset \mathbb{Z}$ be an I_0 set. If $S \subset \Delta(E)$ and S is a set of Bohr recurrence, then S is a set of topological recurrence.

We recall the two preceding lemmas.

Lemma

Let $S \subset \mathbb{Z}$. The following are equivalent.

(i) *S is a set of topological recurrence.*

(ii) *For all $k \in \mathbb{N}$ and every $f : \mathbb{Z} \to \{1, \ldots, k\}$, there exists $a, b \in \mathbb{Z}$ such that $b - a \in S$ and $f(b) = f(a)$.*

The following are equivalent:

(iii) *S is a set of Bohr recurrence.*

(iv) *For all Bohr-AP $\phi : \mathbb{Z} \to \mathbb{C}$, $\varepsilon > 0$, there exists $m \in S$ such that $|\phi(n + m) - \phi(n)| < \varepsilon$ for all $n \in \mathbb{Z}$.*

Lemma

Let $E \subset \mathbb{Z}$ be an I_0 set. If $S \subset \Delta(E)$ and S is a set of Bohr recurrence, then S is a set of topological recurrence.

Proof.

Let $S \subset \Delta(E)$ be Bohr recurrent. To prove S is topologically recurrent, we fix $k \in \mathbb{N}$ and an arbitrary $f : \mathbb{Z} \to \{1, \dots, k\}$, and will prove $\exists\ a, b \in \mathbb{Z}$ such that $f(b) = f(a)$ and $b - a \in S$.

S is I_0, so let ϕ be Bohr-AP with $\phi(x) = f(x)\ \forall x \in E$.

S is Bohr recurrent, so fix $m \in S$ with $|\phi(n+m) - \phi(n)| < 1\ \forall n$.

Since $S \subset \Delta(E)$, write $m = b - a$, where $a, b \in E$. In particular $|\phi(b) - \phi(a)| = |\phi(a+m) - \phi(a)| < 1$, so

$$|f(b) - f(a)| = |\phi(b) - \phi(a)| < 1.$$

so $f(b) = f(a)$. Since $b - a \in S$, we are done. □

Lemma

Let $E \subset \mathbb{Z}$ be an I_0 set. If $S \subset \Delta(E)$ and S is a set of Bohr recurrence, then S is a set of topological recurrence.

If E is an I_0 set, then $\mathbb{N} \not\subseteq \Delta(E)$: 0 is an isolated point of the closure of $\Delta(E)$ in the Bohr topology.

Let $\mathbb{Z}^\omega = \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \ldots$, with elements written as $(n_1, n_2, \ldots)$, and standard basis $\mathrm{e}_1 = (1, 0, 0, \ldots), \mathrm{e}_2 = (0, 1, 0, \ldots)$. Let $\mathcal{E} = \{\mathrm{e}_i : i \in \mathbb{N}\}$.

Conjecture

If $S \subset (\mathcal{E} - \mathcal{E}) - (\mathcal{E} - \mathcal{E})$ is a set of Bohr recurrence, then S is a set of topological recurrence.

This conjecture implies an affirmative answer to Katznelson and Veech's question.

It is superficially similar to the lemma above.

Separating recurrence properties

Question (Katznelson [Kat01], Veech [Vee68])

Is every set of Bohr recurrence also a set of topological recurrence?

If there is a set of Bohr recurrence which is not a set of topological recurrence, it cannot be a subset of a difference set of an I_0 set.

Lemma (Non-separation)

If $E \subset \mathbb{Z}$ is an I_0 set and $S \subset \Delta(E)$ is a set of Bohr recurrence, then S is a set of topological recurrence.

Question

Can the hypothesis "E is an I_0 set" be weakened?

Theorem (Kriz, [Kri87])

If $E \subset \mathbb{Z}$ is infinite, then there is a set $S \subset \Delta(E)$ which is a set of topological recurrence but not of measurable recurrence.

Conjecture

If $S \subset \mathbb{Z}$ is a set of measurable recurrence, then there is a set $S' \subset S$ such that S' is a set of topological recurrence and not a set of measurable recurrence.

No such result is known for "local" recurrence properties. For some translation-invariant recurrence properties, separation is known.

Theorem ([Gri20])

If every translate of S is a set of measurable recurrence, $\exists S' \subset S$ such that every translate of S' is a set of Bohr recurrence and S' is not a set of measurable recurrence.

Conjecture

If every translate of S is a set of measurable recurrence, then $\exists\, S' \subset S$ such that every translate of S' is a set of topological recurrence and not a set of measurable recurrence.

Sets with unknown recurrence properties

The following are known to be sets of Bohr recurrence, but not known to be sets of topological recurrence.

1. $\{7^{n+2d} + 7^{n+d} - 2 \cdot 7^n : n, d \in \mathbb{N}\}$
   B. N. Givens PhD thesis [Giv03], Givens and Kunen [GK03].
2. Grivaux and Roginskaya's examples [GR13]
3. Translates of the IP set generated by the Erdős-Taylor sequence [Kat73], [GM79]
4. $\{n!2^m3^k : n, m, k \in \mathbb{N}\}$ (said to be Bohr recurrent in [FM12])
5. The sets constructed in [Gri20]: S is dense in the Bohr topology, not a set of measurable recurrence.

The first four could possibly be sets of measurable recurrence.

Grivaux and Roginskaya's examples

The following two facts suggest an appealing strategy for constructing sets of Bohr recurrence lacking stronger recurrence properties.

- Bohr recurrence properties of long arithmetic progressions $\{a, a+d, \ldots, a+(k-1)d\}$ can be understood using diophantine approximation arguments. In partiuclar, the dichotomy between ε-denseness of $\{\alpha, 2\alpha, \ldots, N\alpha\}$ in $\mathbb{T}$ and quantitative approximation of α by rationals.
- Lacunary sets are not Bohr recurrent.

[GR13] inductively constructs a set S as a union of increasingly sparse arithmetic progressions and uses diophantine approximation arguments to prove it is a set of Bohr recurrence.

There are several parameters in the construction. Can they be chosen to produce a set of Bohr recurrence which is not a set of topological recurrence?

The IP set generated by the Erdős-Taylor sequence

The Erdős-Taylor sequence is defined by $n_1 = 1$ and $n_{k+1} = kn_k + 1$.

Let S be the IP set generated by $(n_k)_{k\in\mathbb{N}}$:

$$S := \{n_{i_1} + \cdots + n_{i_r} : i_1 < \cdots < i_r, r \in \mathbb{N}\}$$

Katznelson proved that S is dense in the Bohr topology, but is not Hartman-u.d. [Kat73], [GM79].
Used to construct rigidity sequences in [BGM19], [Gri13].

In particular, the translate $S + 1$ is a set of Bohr recurrence.

Is $S + 1$ a set of topological recurrence? Measurable recurrence? Strong recurrence?

What about other translates?

The set $\{n!2^m3^k : n, m, k \in \mathbb{N}\}$

$$S := \{n!2^m3^k : n, m, k \in \mathbb{N}\}$$

[FM12] asks: what are its recurrence properties?

No explicit proof or disproof of Bohr recurrence, topological recurrence, or measurable recurrence appears in print.

Hamming Balls and Bohr-Hamming Balls

[Gri20] constructs sets $S \subset \mathbb{Z}$ which are dense in the Bohr topology and are not sets of measurable recurrence.

It builds on the standard method for constructing dense sets A where $A - A$ lacks structure (niveau sets, Kriz's example)

Question

Are the sets constructed in [Gri20] also sets of topological recurrence?

Is the Kriz example the only way to separate toplogical recurrence from measurable recurrence?

[Kri87] constructs a set of topological recurrence which is not a set of measurable recurrence.

Topological recurrence is obtained by locating Kneser graphs of high chromatic number (Lovász's theorem) in a Cayley graph associated to a set $S \subset \mathbb{Z}$.

This was simplified by Ruzsa [McC99], [McC95], [Wei00], and extended: [For90], [Gri20].

Variations on Kriz's construction are the only known examples of dense sets A where $A - A$ lacks some prescribed structure.

Question

Is there a fundamentally different construction of a set of topological recurrence which is not a set of measurable recurrence?

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