561 HW 4

1a) $p(x) = p_i$ for $i = X : \{1, \dots, 6\}$

$q(y) = q_j$ for $j = Y : \{1, \dots, 6\}$

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

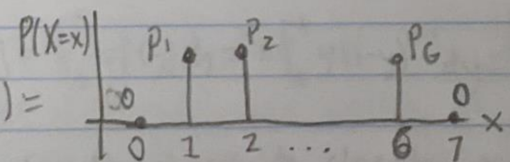
$X + Y = Z$

$$p(z) = \begin{cases} 2, \ 1/36 & 8, \ 5/36 \\ 3, \ 2/36 & 9, \ 4/36 \\ 4, \ 3/36 & 10, \ 3/36 \\ 5, \ 4/36 & 11, \ 2/36 \\ 6, \ 5/36 & 12, \ 1/36 \\ 7, \ 6/36 & \end{cases}$$

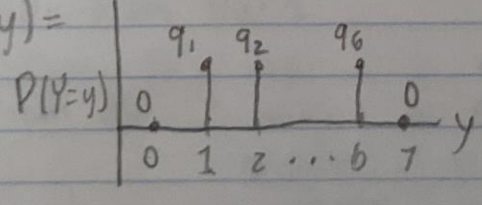
1b) $X + Y = Z \rightarrow Y = Z - X$

so $\sum_i p(x_i) \cdot p(y_i) = p(z) \rightarrow \quad p(z) = \sum_i p(x) \cdot p(z - x_i)$

$p(x) =$



$q(y) =$



If $z = 3$ $\quad p(z) = p_x(x=1) \cdot p_y(y = z-1 = 2) + p_x(x=2) p_y(y = z-2 = 1)$

$+ \underbrace{p_x(x=3) \cdot p_y(y = z-3 = 0)}_{= 0} \dots + 0$

this comes from us knowing $p(x \le 0), p(x \ge 7) = 0$

$p(y \le 0), p(y \ge 7) = 0$

$$P(X \le \alpha E(X)) \ge \frac{1}{\alpha} \qquad P(X \ge a) \le \frac{E(X)}{a}$$

2a) $X_i \sim U[0,1]$

$$\text{pdf} = \begin{cases} 1 & \text{for } 0 \le x \le 1 \\ 0, & \text{else} \end{cases}$$

$Y = \min_{i=1 \dots n} X_i$

$$\text{cdf} = \begin{cases} 1 ; & x > 1 \\ x ; & 0 \le x \le 1 \\ 0 , & \text{else} \end{cases}$$

trying to find pdf of Y

$F(y) = P(Y \le y) = P(\min(X_1, \dots X_n) \le y)$

if we take complement $F(y) = 1 - P(\min(X_1, \dots X_n) > y)$

$\min(X_1 \dots X_n) > y$ will only happen when all $X_i > y$ and since each X_i is iid $P(\min(X_1, \dots X_n) > y) = P(X_1 > y) P(X_2 > y) P(X_3 > y) \dots$

so $P(\min(X_1 \dots X_n) > y) = P(X_1 > y)^n$

$F(y) = 1 - P(X_1 > y)^n = 1 - \left(\int_y^1 1 \, dy\right)^n$

$= 1 - (1-y)^n$

$$\boxed{\frac{dF(y)}{dy} = f(y) = n(1-y)^{n-1}}$$

2b) $E[Y] = \int_0^1 y f(y)\, dy = n \int_0^1 y(1-y)^{n-1} dy = \boxed{\frac{1}{n+1}}$

2c) The result above closely matches the results from HW 1

from HW		min
	$n = 10$	0.111
	$n = 1000$	0.0015
	$n = 100{,}000$	$1.1 \cdot 10^{-5}$

3a) $E[X] = \sum_{x \in X} x \cdot p(x) = 1(1/2) + 2(1/4) + 3(1/4)$

$= \frac{1}{2} + \frac{1}{2} + \frac{3}{4} = 7/4 = \boxed{1.75}$

b) $E[g(X)]; g(x) = x^2 = \sum_{x \in X} g(x) \cdot p(x)$

$= 1(1/2) + 4(1/4) + 9(1/4)$

$= \frac{1}{2} + 1 + \frac{9}{4} = 15/4 = \boxed{3.75}$

c) $Var(X) = E[X^2] - E[X]^2$

$= 3.75 - (1.75)^2 = \boxed{0.6875}$

d) $E[-\log(g(X))] = \sum_{x \in X} -\log(g(x)) \cdot p(x)$

e) $E[-\log_2(P(X))] = -\log_2(P(X\ 1)) \cdot P(X=1) + -\log_2(P(x=22)) \cdot P(x=2)$

$+ -\log_2(P(x=2)) \cdot P(x=2)$

$= 1.5$

4a) They are not independent.

$P(X=2)$ ~~varis~~ varies as the value of Y changes so the two must depend on each other.

4b) $P(X=1) = 1/4 \quad P(X=2) = 3/4 \qquad P(y=1) = 5/8 \quad P(y=2) = 3/8$

$E[X] = \frac{1}{4} + \frac{6}{4} = \frac{7}{4} \qquad E[Y] = \frac{5}{8} + \frac{6}{8} = \frac{11}{8}$

$E[(X - \frac{7}{4})(Y - \frac{11}{7})]$

$\frac{1}{8}\left(\frac{-3}{4} \cdot \frac{-3}{8}\right) + \frac{1}{8}\left(\frac{-3}{4} \cdot \frac{5}{8}\right) + \frac{1}{2}\left(\frac{1}{4} \cdot \frac{-3}{8}\right) + \frac{1}{4}\left(\frac{1}{4} \cdot \frac{5}{8}\right)$

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4c) $Z = X + Y$

$Z = 2, \ 1/8$
$Z = 3, \ 1/8$
$Z = 3, \ 1/2$
$Z = 4, \ 1/4$

$$p(z) = \begin{cases} z < 2, & 0 \\ z = 2, & 1/8 \\ z = 3, & 5/8 \\ z = 4, & 1/4 \\ z > 0, & 0 \end{cases}$$

4d) see attatched code

5) lets define $E[X|Y]$ as $g(x)$

$$\text{so } E_Y[E_X[X|Y]] = E_Y[g(x)]$$
$$= \sum_{y \in Y} g(x) p(y)$$
$$= \sum_{y \in Y} E[X|Y] \, p(y)$$

where $E[X|Y] = \sum_{x \in X} x \cdot p(x|y)$

putting it all together

$$E_Y[E_X[X|Y]] = \sum_{y \in Y} \sum_{x \in X} x \cdot p(x|y) \cdot p(y)$$

for all values of y $\sum_{y \in Y} p(x|y) p(y) = p(x)$

$$E_Y[E_X[X|Y]] = \sum_{x \in X} x \cdot p(x) = E[X]$$