

# 561 HW 4

1a)  $p(x) = p_i$  for  $i = X: \{1, \dots, 6\}$

$q(y) = q_j$  for  $j = Y: \{1, \dots, 6\}$

|   | 1 | 2 | 3 | 4  | 5  | 6  |
|---|---|---|---|----|----|----|
| 1 | 2 | 3 | 4 | 5  | 6  | 7  |
| 2 | 3 | 4 | 5 | 6  | 7  | 8  |
| 3 | 4 | 5 | 6 | 7  | 8  | 9  |
| 4 | 5 | 6 | 7 | 8  | 9  | 10 |
| 5 | 6 | 7 | 8 | 9  | 10 | 11 |
| 6 | 7 | 8 | 9 | 10 | 11 | 12 |

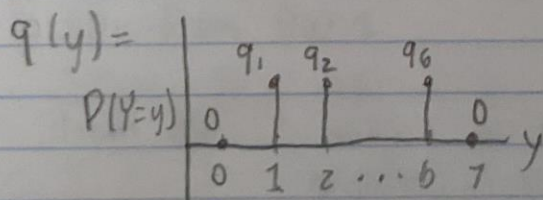
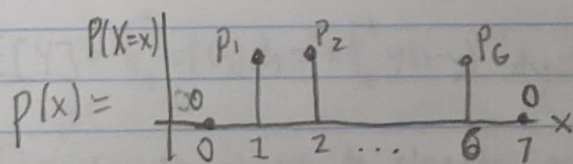
$$X + Y = Z$$

$$p(z) = \begin{cases} 2, & 1/36 \\ 3, & 2/36 \\ 4, & 3/36 \\ 5, & 4/36 \\ 6, & 5/36 \\ 7, & 6/36 \end{cases} \quad \begin{cases} 8, & 5/36 \\ 9, & 4/36 \\ 10, & 3/36 \\ 11, & 2/36 \\ 12, & 1/36 \end{cases}$$

1b)

$$X + Y = Z \rightarrow Y = Z - X$$

$$\text{so } \sum_i p(x_i) \cdot p(y_i) = p(z) \rightarrow p(z) = \sum_i p(x) \cdot p(z - x_i)$$



$$\text{If } z=3 \quad p(z) = p_x(x=1) \cdot p_y(y=z-1=2) + p_x(x=2) \cdot p_y(y=z-2=1) + p_x(x=3) \cdot p_y(y=z-3=0) + \dots + 0$$

this comes from us knowing  $p(x \leq 0), p(x \geq 7) = 0$   
 $p(y \leq 0), p(y \geq 7) = 0$

$$P(X \leq a | E(X)) \geq \frac{1}{a}$$

$$P(X \geq a) \leq \frac{E(X)}{a}$$

2a)  $X_i \sim U[0, 1]$

$$\text{pdf} = \begin{cases} 1 & \text{for } 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

$$Y = \min_{i=1 \dots n} X_i$$

trying to find pdf of Y

$$\text{cdf} = \begin{cases} 1 & x > 1 \\ x & 0 \leq x \leq 1 \\ 0 & \text{else} \end{cases}$$

$$F(y) = P(Y \leq y) = P(\min(X_1, \dots, X_n) \leq y)$$

if we take complement  $F(y) = 1 - P(\min(X_1, \dots, X_n) > y)$

$\min(X_1, \dots, X_n) > y$  will only happen when all  $X_i > y$   
and since each  $X_i$  is iid  $P(\min(X_1, \dots, X_n) > y) =$

$$P(X_1 > y) P(X_2 > y) P(X_3 > y) \dots$$

$$\text{so } P(\min(X_1, \dots, X_n) > y) = P(X_1 > y)^n$$

$$F(y) = 1 - P(X_1 > y)^n = 1 - \left( \int_y^1 1 dy \right)^n$$

$$= 1 - (1-y)^n$$

$$\boxed{\frac{dF(y)}{dy} = f(y) = n(1-y)^{n-1}}$$

2b)  $E[Y] = \int_0^1 y f(y) dy = n \int_0^1 y (1-y)^{n-1} dy = \boxed{\frac{1}{n+1}}$

2c) The result above no closely matches the results from HW 1

| from HW | n         | min                         |
|---------|-----------|-----------------------------|
|         | n=10      | 0.11                        |
|         | n=1000    | 0.0015                      |
|         | n=100,000 | $\approx 1.1 \cdot 10^{-5}$ |



$$3a) E[X] = \sum_{x \in X} x \cdot p(x) = 1\left(\frac{1}{2}\right) + 2\left(\frac{1}{4}\right) + 3\left(\frac{1}{4}\right) \\ = \frac{1}{2} + \frac{1}{2} + \frac{3}{4} = \frac{7}{4} = \boxed{1.75}$$

$$b) E[g(x)]; g(x) = x^2 = \sum_{x \in X} g(x) \cdot p(x) \\ = 1\left(\frac{1}{2}\right) + 4\left(\frac{1}{4}\right) + 9\left(\frac{1}{4}\right) \\ = \frac{1}{2} + 1 + \frac{9}{4} = \frac{15}{4} = \boxed{3.75}$$

$$c) \text{Var}(X) = E[X^2] - E[X]^2 \\ = 3.75 - (1.75)^2 = \boxed{0.6875}$$

$$d) E[-\log(g(x))] = \sum_{x \in X} -\log(g(x)) \cdot p(x)$$

$$e) E[-\log_2(P(x))] = -\log_2(P(x=1)) \cdot P(x=1) + -\log_2(P(x=2)) \cdot P(x=2) \\ + -\log_2(P(x=2)) \cdot P(x=2) \\ = 1.5$$

4a) They are not independent.

~~xx~~  $P(x=2)$  varies as the value of  $Y$  changes so the two must depend on each other.

$$4b) P(x=1) = \frac{1}{4} \quad P(x=2) = \frac{3}{4} \quad P(y=1) = \frac{5}{8} \quad P(y=2) = \frac{3}{8}$$

$$E[X] = \frac{1}{4} + \frac{6}{4} = \frac{7}{4} \quad E[Y] = \frac{5}{8} + \frac{6}{8} = \frac{11}{8}$$

$$E\left[\left(x - \frac{7}{4}\right)\left(y - \frac{11}{8}\right)\right]$$

$$\frac{1}{8} \left( \frac{-3}{4} \cdot \frac{-3}{8} \right) + \frac{1}{8} \left( \frac{-3}{4} \cdot \frac{5}{8} \right) + \frac{1}{2} \left( \frac{1}{4} \cdot \frac{-3}{8} \right) + \frac{1}{4} \left( \frac{1}{4} \cdot \frac{5}{8} \right)$$

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4c)  $Z = X + Y$

|              |   |
|--------------|---|
| $Z = 2, 1/8$ | $p(z) = \begin{cases} z < 2, & 0 \\ z = 2, & 1/8 \\ z = 3, & 5/8 \\ z = 4, & 1/4 \\ z > 4, & 0 \end{cases}$ |
| $Z = 3, 1/8$ |   |
| $Z = 3, 1/2$ |   |
| $Z = 4, 1/4$ |   |
|              |   |

4d) see attached code

5) ~~W~~ let's define  $E[X|Y]$  as  $g(x)$

$$\begin{aligned} \text{so } E_y[E[X|Y]] &= E_y[g(x)] \\ &= \sum_{y \in Y} g(x) p(y) \\ &= \sum_{y \in Y} E[X|Y] p(y) \end{aligned}$$

where  $E[X|Y] = \sum_{x \in X} x \cdot p(x|y)$

putting it all together

$$E_y[E_x[X|Y]] = \sum_{y \in Y} \sum_{x \in X} x \cdot p(x|y) \cdot p(y)$$

For all values of  $y$   $\sum_{x \in X} p(x|y) p(y) = p(y)$

$$E_y[E_x[X|Y]] = \sum_{x \in X} x \cdot p(x) = E[X]$$

## Question 4d

In [1]: `import random`

In [9]: `def getXY():  
 x = 1 if random.uniform(0,1) <= 0.25 else 2  
 if x==1:  
 y = 1 if random.uniform(0,1) <= 0.5 else 2  
 else:  
 y = 1 if random.uniform(0,1) <= 0.666 else 2  
 return x,y`

In [10]: `x1y1 = 0  
x1y2 = 0  
x2y1 = 0  
x2y2 = 0  
num = 100000  
for i in range(num):  
 x,y = getXY()  
 if x==1 and y==1:  
 x1y1+=1  
 elif x==1 and y==2:  
 x1y2+=1  
 elif x==2 and y==1:  
 x2y1+=1  
 elif x==2 and y==2:  
 x2y2+=1  
print("P(X=1,Y=1) = "+str(x1y1/num))  
print("P(X=1,Y=2) = "+str(x1y2/num))  
print("P(X=2,Y=1) = "+str(x2y1/num))  
print("P(X=2,Y=2) = "+str(x2y2/num))`

P(X=1,Y=1) = 0.12524  
P(X=1,Y=2) = 0.1237  
P(X=2,Y=1) = 0.49916  
P(X=2,Y=2) = 0.2519

In [ ]: