Delft University of Technology

Special Topics for Computational Science and Engineering
WI4475

Assignment: Parallel Time Integration

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1 Time Integration

Let's consider the heat equation of the form,

$$\frac{d\boldsymbol{u}}{dt} = -\alpha\nabla^2\boldsymbol{u}, \text{for } \boldsymbol{u} \in \mathcal{R}^3. \tag{1}$$

On the domain $\Omega = [0,1]^3$. Discretizing Ω (in space) with an equidistant grid ($N = N_x = N_y = N_z$) of $\Delta x = h = \frac{1-0}{N}$ using a finite difference method gives,

$$-\nabla^2 u_i = \frac{1}{h^2}(-u_{i-n^2} - u_{i-n} - u_{i-1} + 6u_i - u_{i+1} - u_{i+n} - u_{i+n^2}) + \mathcal{O}(h^2), \quad i = 1, .., N^3 \tag{2}$$

Now we define,

$$L = \frac{1}{h^2}(-u_{i-n^2} - u_{i-n} - u_{i-1} + 6u_i - u_{i+1} - u_{i+n} - u_{i+n^2}), \tag{3}$$

gives,

$$\frac{d\mathbf{u}}{dt} = L\mathbf{u} \tag{4}$$

When using the Euler Forward method to get the solution over time, we get

$$\mathbf{u}^{k+1} = \mathbf{u}^k - \Delta t L\mathbf{u}^k \tag{5}$$

$$\mathbf{u}^{k+1} + (-I + \Delta t L)\mathbf{u}^k = 0 \tag{6}$$

When we put this equation for all k together in a system, we have

$$A = \begin{pmatrix} I & 0 & \cdots & & & & 0 \\ -I+\Delta tL & I & 0 & \cdots & & & 0 \\ 0 & -I+\Delta tL & I & 0 & \cdots & & 0 \\ 0 & 0 & -I+\Delta tL & I & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \vdots \\ 0 & \cdots & & 0 & -I+\Delta tL & I & 0 \\ 0 & \cdots & & & 0 & -I+\Delta tL & I \end{pmatrix}.$$

This results in the following system,

$$A\boldsymbol{x} = b, \text{ with } \boldsymbol{x} = \begin{pmatrix} \boldsymbol{u}^0 \\ \boldsymbol{u}^1 \\ \vdots \\ \boldsymbol{u}^k \\ \vdots \\ \boldsymbol{u}^T \end{pmatrix} \text{ and } \boldsymbol{b} = \begin{pmatrix} \boldsymbol{u}_b \\ \mathbf{0} \\ \vdots \\ \mathbf{0} \end{pmatrix}, \tag{7}$$

where T is the number of steps in time.

1.1 Sequential Methods: Jacobi and Gauss-Seidel

For the Jacobi method, we use $M_{JAC} = D = I$. Therefore the iterative method is for iteration i

$$\mathbf{r}^i = \mathbf{b} - A\mathbf{x}^i$$
$$\mathbf{x}^{i+1} = \mathbf{x}^i + \mathbf{r}^i$$

The calculation of $A\mathbf{x}^i$ can be done in parallel.

For the Gauss-Seidel method, we use $M_{GS} = D - E$, however this is the whole matrix A, so we will do Block Gauss-Seidel.