Spatial Geostatistical Model

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1 Full Dimensional Model Statement

1.1 Data Model

$$\boldsymbol{y}_t = \boldsymbol{H}_t \boldsymbol{X} \boldsymbol{\beta}_t + \boldsymbol{\epsilon}_t$$

1.2 Process Model

$$\begin{aligned} \boldsymbol{\beta}_t &\sim N(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta) & \boldsymbol{\Sigma}_\beta &= \sigma^2_\beta \boldsymbol{\Lambda} \\ \boldsymbol{\epsilon}_t &\sim N(0, \boldsymbol{\Sigma}_{\epsilon_t}) & \boldsymbol{\Sigma}_{\epsilon_t} &= \sigma^2_\epsilon \boldsymbol{I}_{n_t} \end{aligned}$$

1.3 Parameter Model

$$\begin{aligned} \boldsymbol{\mu}_\beta &\sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) \\ \sigma^2_\beta &\sim IG(\alpha_\beta, \beta_\beta) \\ \sigma^2_\eta &\sim IG(\alpha_\eta, \beta_\eta) \\ \sigma^2_\epsilon &\sim IG(\alpha_\epsilon, \beta_\epsilon) \end{aligned}$$

where n_t is the number of samples of y_t at time t, $\boldsymbol{I}_{n_t}$ is the identity matrix of size $n_t \times n_t$ and $\boldsymbol{\Lambda}$ is the matrix of truncated eigenvalues of the original data $\boldsymbol{X}$ before the principal components regression.

2 Posterior

$$\prod_{t=1}^{T} [\boldsymbol{\beta}_t, \boldsymbol{\mu}_\beta, \sigma^2_\beta, \sigma^2_\epsilon | \boldsymbol{y}_t] \propto \prod_{t=1}^{T} [\boldsymbol{y}_t | \boldsymbol{\beta}_t, \sigma^2_\epsilon][\boldsymbol{\beta}_t | \boldsymbol{\mu}_\beta, \sigma^2_\beta][\boldsymbol{\mu}_\beta][\sigma^2_\beta][\sigma^2_\epsilon]$$

3 Full Conditionals

3.1 Full Conditional for $\boldsymbol{\beta}_t$

For $t = 1, \ldots, T$,

$$
\begin{aligned}
[\boldsymbol{\beta}_t|\cdot] &\propto [\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma^2_\epsilon][\boldsymbol{\beta}_t|\boldsymbol{\mu}_\beta, \sigma^2_\beta] \\
&\propto \exp\left\{-\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}_{\epsilon_t}^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right\}\exp\left\{-\frac{1}{2}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T\boldsymbol{\Sigma}_\beta^{-1}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)\right\} \\
&\propto \exp\left(-\frac{1}{2}\{\boldsymbol{\beta}_t^T(\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}_{\epsilon_t}^{-1}\boldsymbol{H}_t\boldsymbol{X} + \boldsymbol{\Sigma}_\beta^{-1})\boldsymbol{\beta}_t - 2\boldsymbol{\beta}_t^T(\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}_{\epsilon_t}^{-1}\boldsymbol{y}_t + \boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\mu}_\beta)\}\right)
\end{aligned}
$$

which is Normal with mean $\boldsymbol{A}^{-1}\boldsymbol{b}$ and variance $\boldsymbol{A}^{-1}$ where

$$
\begin{aligned}
\boldsymbol{A}^{-1} &= (\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}_{\epsilon_t}^{-1}\boldsymbol{H}_t\boldsymbol{X} + \boldsymbol{\Sigma}_\beta^{-1})^{-1} \\
\boldsymbol{b} &= (\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}_{\epsilon_t}^{-1}\boldsymbol{y}_t + \boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\mu}_\beta)
\end{aligned}
$$

3.2 Full Conditional for $\boldsymbol{\mu}_\beta$

$$
\begin{aligned}
[\boldsymbol{\mu}_\beta|\cdot] &\propto \prod_{t=1}^T[\boldsymbol{\beta}_t|\boldsymbol{\mu}_\beta, \sigma^2_\beta][\boldsymbol{\mu}_\beta] \\
&\propto \exp\left(-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T\boldsymbol{\Sigma}_\beta^{-1}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)\right)\exp\left(-\frac{1}{2}(\boldsymbol{\mu}_\beta - \boldsymbol{\mu}_0)^T\Sigma_0^{-1}(\boldsymbol{\mu}_\beta - \boldsymbol{\mu}_0)\right) \\
&\propto \exp\left(-\frac{1}{2}(\boldsymbol{\mu}_\beta^T(T\boldsymbol{\Sigma}_\beta^{-1} + \boldsymbol{\Sigma}_0^{-1})\boldsymbol{\mu}_\beta - 2\boldsymbol{\mu}_\beta^T(\sum_{t=1}^T\boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\beta}_t + \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0))\right)
\end{aligned}
$$

which is Normal with mean $\boldsymbol{A}^{-1}\boldsymbol{b}$ and variance $\boldsymbol{A}^{-1}$ where

$$
\begin{aligned}
\boldsymbol{A}^{-1} &= (T\boldsymbol{\Sigma}_\beta^{-1} + \boldsymbol{\Sigma}_0^{-1})^{-1} \\
\boldsymbol{b} &= \sum_{t=1}^T\boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\beta}_t + \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0
\end{aligned}
$$

3.3 Full Conditional for σ^2_β

$$
\begin{aligned}
[\sigma^2_\beta|\cdot] &\propto \prod_{t=1}^T[\boldsymbol{\beta}_t|\boldsymbol{\mu}_\beta, \sigma^2_\beta][\sigma^2_\beta] \\
&\propto (\prod_{t=1}^T|\boldsymbol{\Sigma}_\beta|^{-\frac{1}{2}})\exp\left(-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T\boldsymbol{\Sigma}_\beta^{-1}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)\right)(\sigma^2_\beta)^{-(\alpha_\beta+1)}\exp\left(-\frac{\beta_\beta}{\sigma^2_\beta}\right) \\
&\propto (\sigma^2_\beta)^{-(\alpha_\beta + \frac{T\tau}{2}+1)}\exp\left(-\frac{1}{\sigma^2_\beta}(\frac{1}{2}\sum_{t=1}^T(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T\boldsymbol{\Lambda}^{-1}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta) + \beta_\beta)\right)
\end{aligned}
$$

which is $\text{IG}(\alpha_\beta + \frac{T\tau}{2}, \frac{1}{2}\sum_{t=1}^T(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T\boldsymbol{\Lambda}^{-1}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta) + \beta_\beta)$ since the determinant $|\boldsymbol{\Sigma}_\beta| = (\sigma^2_\beta)^\tau|\boldsymbol{\Lambda}|$ and $\boldsymbol{\Sigma}_\beta^{-1} = \frac{1}{\sigma^2_\beta}\boldsymbol{\Lambda}^{-1}$

3.4 Full Conditional for σ_ϵ^2

$$
\begin{aligned}
[\sigma_\epsilon^2|\cdot] &\propto \prod_{t=1}^{T}[\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\sigma_\epsilon^2] \\
&\propto (\prod_{t=1}^{T}|\boldsymbol{\Sigma}_{\epsilon_t}|^{-\frac{1}{2}}) \exp\left(-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}_{\epsilon_t}^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right)(\sigma_\epsilon^2)^{-(\alpha_\epsilon+1)}\exp\left(-\frac{\beta_\epsilon}{\sigma_\epsilon^2}\right) \\
&\propto (\sigma_\epsilon^2)^{(-\frac{\sum_{t=1}^{T}n_t}{2})}\exp\left(-\frac{1}{\sigma_\epsilon^2}\left(\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right)\right)(\sigma_\epsilon^2)^{-(\alpha_\epsilon+1)}\exp\left(-\frac{\beta_\epsilon}{\sigma_\epsilon^2}\right) \\
&\propto (\sigma_\epsilon^2)^{-\left(\alpha_\epsilon+\frac{\sum_{t=1}^{T}n_t}{2}+1\right)}\exp\left(-\frac{1}{\sigma_\epsilon^2}\left(\beta_\epsilon + \frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right)\right)
\end{aligned}
$$

which is $\text{IG}\Big(\alpha_\epsilon + \frac{\sum_{t=1}^{T}n_t}{2}, \beta_\epsilon + \frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\Big)$.

4 Posterior Predictive Distribution

The posterior predictive distribution for $\boldsymbol{y}_t$ is sampled a each MCMC iteration k by