Spatial Orthogonal Geostatistical Model

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1 Full Dimensional Model Statement

1.1 Data Model

$$\boldsymbol{y}_t = \boldsymbol{H}_t \boldsymbol{X} \boldsymbol{\beta}_t + \boldsymbol{\eta}_t + \boldsymbol{\epsilon}_t$$

1.2 Process Model

$$\begin{aligned} \boldsymbol{\beta}_t &\sim N(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta) & \boldsymbol{\Sigma}_\beta &= \sigma^2_\beta \boldsymbol{I}_t & & \\ \boldsymbol{\eta}_t &\sim N(0, \boldsymbol{\Sigma}_\eta) & \boldsymbol{\Sigma}_\eta &= \sigma^2_\eta \boldsymbol{R}(\phi) & \boldsymbol{R}(\phi) &= \exp\left(-\boldsymbol{D}_t/\phi\right) \\ \boldsymbol{\epsilon}_t &\sim N(0, \boldsymbol{\Sigma}_\epsilon) & \boldsymbol{\Sigma}_\epsilon &= \sigma^2_\epsilon \boldsymbol{I}_t & & \end{aligned}$$

1.3 Parameter Model

$$\begin{aligned} \boldsymbol{\mu}_\beta &\sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) \\ \sigma^2_\beta &\sim IG(\alpha_\beta, \beta_\beta) \\ \sigma^2_\eta &\sim IG(\alpha_\eta, \beta_\eta) \\ \sigma^2_\epsilon &\sim IG(\alpha_\epsilon, \beta_\epsilon) \\ \phi &\sim IG(\alpha_\phi, \beta_\phi) \end{aligned}$$

where $\boldsymbol{I}_\beta$ is the identity matrix of size $\tau \times \tau$ where τ is the number of parameters in $\boldsymbol{\beta}_t$, $\boldsymbol{I}_t$ is the identity matrix of size $n_t \times n_t$ and n_t is the number of samples of y_t at time t and $\boldsymbol{D}_t$ is the distance matrix between locations observed at time t. Define $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_\eta + \boldsymbol{\Sigma}_\epsilon$

2 Orthogonalization

To preserve the fixed effects from being influence by the spatial process that should be explicitly second order, we construct the perpendicular projection operator (PPO)

$$\boldsymbol{P}^c_t = \boldsymbol{I}_{n_t} - \boldsymbol{H}_t \boldsymbol{X} \left((\boldsymbol{H}_t \boldsymbol{X})^T \boldsymbol{H}_t \boldsymbol{X} \right)^{-1} (\boldsymbol{H}_t \boldsymbol{X})^T.$$

The singular value decomposition of $\boldsymbol{P}^c_t$ is given by

$$\boldsymbol{P}^c_t = [\boldsymbol{L}|\boldsymbol{K}]^T \boldsymbol{\Phi} [\boldsymbol{L}|\boldsymbol{K}]$$

where $\boldsymbol{K}$ is a linear combination of the vectors of $\boldsymbol{L}$ (associated eignevalues of $\boldsymbol{L}$ are 0) and we can re-write our model as

$$\boldsymbol{y}_t = \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t + \boldsymbol{L}\boldsymbol{\eta}_t^* + \boldsymbol{\epsilon}_t$$

where $\boldsymbol{\eta}_t^* \sim N(0, \sigma_\eta^2 \boldsymbol{L}^T\boldsymbol{R}(\phi)\boldsymbol{L})$. Define $\boldsymbol{\Sigma}_t = \boldsymbol{\Sigma}_{\epsilon_t}$.

3 Posterior

$$\prod_{t=1}^T [\boldsymbol{\beta}_t, \boldsymbol{\eta}_t^*, \boldsymbol{\mu}_\beta, \sigma_\beta^2, \sigma_\eta^2, \sigma_\epsilon^2, \phi|\boldsymbol{y}_t] \propto \prod_{t=1}^T [\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\boldsymbol{\beta}_t|\boldsymbol{\mu}_\beta, \sigma_\beta^2][\boldsymbol{\mu}_\beta][\sigma_\beta^2][\sigma_\eta^2][\sigma_\epsilon^2][\phi]$$

4 Full Conditionals

4.1 Full Conditional for $\boldsymbol{\beta}_t$

For $t = 1, \ldots, T$,

$$\begin{aligned}
[\boldsymbol{\beta}_t|\cdot] &\propto [\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \sigma_\epsilon^2, \phi][\boldsymbol{\beta}_t|\boldsymbol{\mu}_\beta, \sigma_\beta^2] \\
&\propto e^{-\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t - \boldsymbol{L}\boldsymbol{\eta}_t^*)^T\boldsymbol{\Sigma}_\epsilon^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t - \boldsymbol{L}\boldsymbol{\eta}_t^*)} e^{-\frac{1}{2}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T\boldsymbol{\Sigma}_\beta^{-1}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)} \\
&\propto e^{-\frac{1}{2}\{\boldsymbol{\beta}_t^T(\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}_\epsilon^{-1}\boldsymbol{H}_t\boldsymbol{X} + \boldsymbol{\Sigma}_\beta^{-1})\boldsymbol{\beta}_t - 2\boldsymbol{\beta}_t^T(\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}_\epsilon^{-1}(\boldsymbol{y}_t - \boldsymbol{L}\boldsymbol{\eta}_t^*) + \boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\mu}_\beta)\}}
\end{aligned}$$

which is Normal with mean $\boldsymbol{A}^{-1}\boldsymbol{b}$ and variance $\boldsymbol{A}^{-1}$ where

$$\begin{aligned}
\boldsymbol{A}^{-1} &= (\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}_\epsilon^{-1}\boldsymbol{H}_t\boldsymbol{X} + \boldsymbol{\Sigma}_\beta^{-1})^{-1} \\
\boldsymbol{b} &= (\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}_\epsilon^{-1}(\boldsymbol{y}_t - \boldsymbol{L}\boldsymbol{\eta}_t^*) + \boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\mu}_\beta)
\end{aligned}$$

4.2 Full Conditional for $\boldsymbol{\mu}_\beta$

$$\begin{aligned}
[\boldsymbol{\mu}_\beta|\cdot] &\propto \prod_{t=1}^T [\boldsymbol{\beta}_t|\boldsymbol{\mu}_\beta, \sigma_\beta^2][\boldsymbol{\mu}_\beta] \\
&\propto e^{-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T\boldsymbol{\Sigma}_\beta^{-1}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)} e^{-\frac{1}{2}(\boldsymbol{\mu}_\beta - \boldsymbol{\mu}_0)^T\Sigma_0^{-1}(\boldsymbol{\mu}_\beta - \boldsymbol{\mu}_0)} \\
&\propto e^{-\frac{1}{2}(\boldsymbol{\mu}_\beta^T(T\boldsymbol{\Sigma}_\beta^{-1} + \boldsymbol{\Sigma}_0^{-1})\boldsymbol{\mu}_\beta - 2\boldsymbol{\mu}_\beta^T(\sum_{t=1}^T \boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\beta}_t + \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0))}
\end{aligned}$$

which is multivariate normal with mean $(T\boldsymbol{\Sigma}_\beta^{-1} + \boldsymbol{\Sigma}_0^{-1})^{-1}(\sum_{t=1}^T \boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\beta}_t + \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0)$ and variance $(T\boldsymbol{\Sigma}_\beta^{-1} + \boldsymbol{\Sigma}_0^{-1})^{-1}$

4.3 Full Conditional for σ_β^2

$$\begin{aligned}
[\sigma_\beta^2|\cdot] &\propto \prod_{t=1}^T [\boldsymbol{\beta}_t|\boldsymbol{\mu}_\beta, \sigma_\beta^2][\sigma_\beta^2] \\
&\propto (\prod_{t=1}^T |\boldsymbol{\Sigma}_\beta|^{-\frac{1}{2}}) e^{-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T\boldsymbol{\Sigma}_\beta^{-1}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)} (\sigma_\beta^2)^{-(\alpha_\beta + 1)} e^{-\frac{\beta_\beta}{\sigma_\beta^2}} \\
&\propto (\sigma_\beta^2)^{-(\alpha_\beta + \frac{\sum_{t=1}^T n_t}{2} + 1)} e^{-\frac{1}{\sigma_\beta^2}(\frac{1}{2}\sum_{t=1}^T(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T\boldsymbol{\Lambda}^{-1}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta) + \beta_\beta)}
\end{aligned}$$

which is IG($\alpha_\beta + \frac{\sum_{t=1}^T n_t}{2}, \beta_\beta + \frac{1}{2}\sum_{t=1}^T (\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T \boldsymbol{\Lambda}^{-1}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)$) since the determinant $|\boldsymbol{\Sigma}_\beta| = (\sigma_\beta^2)^{n_t}|\boldsymbol{\Lambda}|$ and $\boldsymbol{\Sigma}_\beta^{-1} = \frac{1}{\sigma_\beta^2}\boldsymbol{\Lambda}^{-1}$

4.4 Full Conditional for $\boldsymbol{\eta}_t^*$

For $t = 1, \ldots, T$,

$$
\begin{aligned}
[\boldsymbol{\beta}_t|\cdot] &\propto [\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \sigma_\epsilon^2, \phi][\boldsymbol{\eta}_t^*|\sigma_\eta^2] \\
&\propto e^{-\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t - \boldsymbol{L}\boldsymbol{\eta}_t^*)^T\boldsymbol{\Sigma}_\epsilon^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t - \boldsymbol{L}\boldsymbol{\eta}_t^*)} e^{-\frac{1}{2}\boldsymbol{\eta}_t^{*T}\boldsymbol{\Sigma}_{\eta^*}^{-1}\boldsymbol{\eta}_t^*} \\
&\propto e^{-\frac{1}{2}\{\boldsymbol{\eta}_t^{*T}(\boldsymbol{L}^T\boldsymbol{\Sigma}_\epsilon^{-1}\boldsymbol{L} + \boldsymbol{\Sigma}_{\eta^*}^{-1})\boldsymbol{\eta}_t^* - 2\boldsymbol{\eta}_t^{*T}(\boldsymbol{L}^T\boldsymbol{\Sigma}_\epsilon^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\}}
\end{aligned}
$$

which is Normal with mean $\boldsymbol{A}^{-1}\boldsymbol{b}$ and variance $\boldsymbol{A}^{-1}$ where

$$
\begin{aligned}
\boldsymbol{A}^{-1} &= (\boldsymbol{L}^T\boldsymbol{\Sigma}_\epsilon^{-1}\boldsymbol{L} + \boldsymbol{\Sigma}_{\eta^*}^{-1})^{-1} \\
\boldsymbol{b} &= (\boldsymbol{L}^T\boldsymbol{\Sigma}_\epsilon^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)
\end{aligned}
$$

4.5 Full Conditional for σ_η^2

$$
\begin{aligned}
[\sigma_\eta^2|\cdot] &\propto \prod_{t=1}^T [\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\sigma_\eta^2] \\
&\propto (\prod_{t=1}^T |\boldsymbol{\Sigma}_\epsilon|^{-\frac{1}{2}}) e^{-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t - \boldsymbol{L}\boldsymbol{\eta}_t^*)^T\boldsymbol{\Sigma}_\epsilon^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t - \boldsymbol{L}\boldsymbol{\eta}_t^*)} (\sigma_\eta^2)^{-\alpha_\eta+1} e^{-\frac{\beta_\eta}{\sigma_\eta^2}} \\
&\propto |\boldsymbol{\Sigma}_\epsilon|^{-\frac{T}{2}} e^{-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t - \boldsymbol{L}\boldsymbol{\eta}_t^*)^T\boldsymbol{\Sigma}_\epsilon^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t - \boldsymbol{L}\boldsymbol{\eta}_t^*)} (\sigma_\eta^2)^{-\alpha_\eta+1} e^{-\frac{\beta_\eta}{\sigma_\eta^2}}
\end{aligned}
$$

which can be sampled using a Metropolis-Hastings step

4.6 Full Conditional for σ_ϵ^2

$$
\begin{aligned}
[\sigma_\epsilon^2|\cdot] &\propto \prod_{t=1}^T [\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\sigma_\epsilon^2] \\
&\propto (\prod_{t=1}^T |\boldsymbol{\Sigma}|^{-\frac{1}{2}}) e^{-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t - \boldsymbol{L}\boldsymbol{\eta}_t^*)^T\boldsymbol{\Sigma}_\epsilon^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t - \boldsymbol{L}\boldsymbol{\eta}_t^*)} (\sigma_\epsilon^2)^{-\alpha_\epsilon+1} e^{-\frac{\beta_\epsilon}{\sigma_\epsilon^2}} \\
&\propto |\boldsymbol{\Sigma}_\epsilon|^{-\frac{T}{2}} e^{-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t - \boldsymbol{L}\boldsymbol{\eta}_t^*)^T\boldsymbol{\Sigma}_\epsilon^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t) - \boldsymbol{L}\boldsymbol{\eta}_t^*} (\sigma_\epsilon^2)^{-\alpha_\epsilon+1} e^{-\frac{\beta_\epsilon}{\sigma_\epsilon^2}}
\end{aligned}
$$

which can be sampled using a Metropolis-Hastings step

4.7 Full Conditional for ϕ

$$
\begin{aligned}
[\phi|\cdot] &\propto \prod_{t=1}^T [y_t|\beta_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\phi] \\
&\propto \prod_{t=1}^T |\boldsymbol{\Sigma}_\epsilon|^{-\frac{1}{2}} e^{-\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t - \boldsymbol{L}\boldsymbol{\eta}_t^*)^T\boldsymbol{\Sigma}_\epsilon^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t - \boldsymbol{\eta}_t^*)} \phi^{-\alpha_\phi+1} e^{-\frac{\beta_\phi}{\phi}} \\
&\propto |\boldsymbol{\Sigma}_\epsilon|^{-\frac{T}{2}} e^{-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t - \boldsymbol{L}\boldsymbol{\eta}_t^*)^T\boldsymbol{\Sigma}_\epsilon^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t - \boldsymbol{L}\boldsymbol{\eta}_t^*)} \phi^{-\alpha_\phi+1} e^{-\frac{\beta_\phi}{\phi}}
\end{aligned}
$$

which can be sampled using a Metropolis-Hastings step

5 Posterior Predictive Distribution

The posterior predictive distribution for $\boldsymbol{y}_t$ is sampled a each MCMC iteration k by

$$\boldsymbol{y}_t^{(k)} \sim N(\boldsymbol{H_t} \boldsymbol{X} \boldsymbol{\beta}_t^{(k)}, \boldsymbol{\Sigma}^{(k)})$$