Spatial Geostatistical Model

John Tipton

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1 Full Dimensional Model Statement

1.1 Data Model

$$\boldsymbol{y}_t = \boldsymbol{H}_t \boldsymbol{X} \boldsymbol{\beta}_t + \boldsymbol{\eta}_t + \boldsymbol{\epsilon}_t$$

1.2 Process Model

$$\begin{aligned}
\boldsymbol{\beta}_t &\sim N(\boldsymbol{0}, \boldsymbol{\Sigma}_\beta) & \boldsymbol{\Sigma}_\beta &= \sigma^2_\epsilon \boldsymbol{D}_\gamma & \boldsymbol{D}_\gamma &= diag(\gamma^2_1, \ldots, \gamma^2_\tau)\\
\boldsymbol{\eta}_t &\sim N(0, \boldsymbol{\Sigma}_{\eta_t}) & \boldsymbol{\Sigma}_{\eta_t} &= \sigma^2_\eta \boldsymbol{R}_t(\phi) & \boldsymbol{R}_t(\phi) &= \exp\left(-\boldsymbol{D}_t/\phi\right)\\
\boldsymbol{\epsilon}_t &\sim N(0, \boldsymbol{\Sigma}_{\epsilon_t}) & \boldsymbol{\Sigma}_{\epsilon_t} &= \sigma^2_\epsilon \boldsymbol{I}_{n_t} & &\\
\boldsymbol{\gamma_i^2} &\sim Exp(\lambda^2/2) \text{ Equivalently} & \frac{1}{\gamma^2_i} &\sim IG(1, \lambda^2/2) & &\text{for } i = 1 \ldots, \tau
\end{aligned}$$

1.3 Parameter Model

$$\begin{aligned}
\sigma^2_\eta &\sim IG(\alpha_\eta, \beta_\eta)\\
\sigma^2_\epsilon &\sim IG(\alpha_\epsilon, \beta_\epsilon)\\
\phi &\sim IG(\alpha_\phi, \beta_\phi)\\
\lambda^2 &\sim \Gamma(\alpha_\lambda, \beta_\lambda)
\end{aligned}$$

where $\boldsymbol{I}_{n_t}$ is the identity matrix of size $n_t \times n_t$ where τ is the number of parameters in $\boldsymbol{\beta}_t$ and n_t is the number of samples of y_t at time t and $\boldsymbol{D}_t$ is the distance matrix between locations observed at time t. Define $\boldsymbol{\Sigma}_t = \boldsymbol{\Sigma}_{\eta_t} + \boldsymbol{\Sigma}_{\epsilon_t}$

2 Posterior

$$\prod_{t=1}^T [\boldsymbol{\beta}_t, \boldsymbol{\gamma}^2, \sigma^2_\eta, \sigma^2_\epsilon, \phi, \lambda^2 | \boldsymbol{y}_t] \propto \prod_{t=1}^T [\boldsymbol{y}_t | \boldsymbol{\beta}_t, \sigma^2_\eta, \phi, \sigma^2_\epsilon][\boldsymbol{\beta}_t | \sigma^2_\epsilon, \boldsymbol{\gamma^2}][\boldsymbol{\gamma} | \lambda^2][\sigma^2_\eta][\sigma^2_\epsilon][\phi][\lambda^2]$$

3 Full Conditionals

3.1 Full Conditional for $\boldsymbol{\beta}_t$

For $t = 1, \ldots, T$,

$$\begin{aligned}
[\boldsymbol{\beta}_t|\cdot] &\propto [\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \sigma_\epsilon^2, \phi][\boldsymbol{\beta}_t|\boldsymbol{\mu}_\beta, \sigma_\beta^2] \\
&\propto \exp\left(-\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right)\exp\left(-\frac{1}{2}\boldsymbol{\beta}_t^T\boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\beta}_t\right) \\
&\propto \exp\left(-\frac{1}{2}\{\boldsymbol{\beta}_t^T(\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}_t^{-1}\boldsymbol{H}_t\boldsymbol{X} + \boldsymbol{\Sigma}_\beta^{-1})\boldsymbol{\beta}_t - 2\boldsymbol{\beta}_t^T(\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}_t^{-1}\boldsymbol{y}_t)\}\right)
\end{aligned}$$

which is Normal with mean $\boldsymbol{A}^{-1}\boldsymbol{b}$ and variance $\boldsymbol{A}^{-1}$ where

$$\begin{aligned}
\boldsymbol{A}^{-1} &= (\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}_t^{-1}\boldsymbol{H}_t\boldsymbol{X} + \boldsymbol{\Sigma}_\beta^{-1})^{-1} \\
\boldsymbol{b} &= (\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}_t^{-1}\boldsymbol{y}_t)
\end{aligned}$$

3.2 Full Conditional for γ_i^2

For $i = 1, \ldots, \tau$

$$\begin{aligned}
[\gamma_i^2|\cdot] &\propto \prod_{t=1}^T[\boldsymbol{\beta}_t|\sigma_\epsilon^2, \gamma_i^2][\gamma_i^2|\lambda^2] \\
&\propto \prod_{t=1}^T|\sigma_\epsilon^2\gamma_i^2|^{-\frac{1}{2}}\exp\left(-\frac{1}{2}\boldsymbol{\beta}_t^T(\frac{\sigma_\epsilon^2}{\gamma_i^2})^{-1}\boldsymbol{\beta}_t\right)\exp\left(-\frac{\lambda^2}{2}\gamma_i^2\right) \\
&\propto (\gamma_i^2)^{-\frac{T}{2}}\exp\left(-\gamma_i^2(\frac{\sum_{t=1}^T\boldsymbol{\beta}_t^T\boldsymbol{\beta}_t}{2\sigma_\epsilon^2} + \frac{\lambda^2}{2})\right)
\end{aligned}$$

3.3 Full Conditional for σ_η^2

$$\begin{aligned}
[\sigma_\eta^2|\cdot] &\propto \prod_{t=1}^T[\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\sigma_\eta^2] \\
&\propto (\prod_{t=1}^T|\boldsymbol{\Sigma}_t|^{-\frac{1}{2}})\exp\left(-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right)(\sigma_\eta^2)^{-\alpha_\eta-1}\exp\left(-\frac{\beta_\eta}{\sigma_\eta^2}\right) \\
&\propto |\boldsymbol{\Sigma}_t|^{-\frac{T}{2}}\exp\left(-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right)(\sigma_\eta^2)^{-\alpha_\eta-1}\exp\left(-\frac{\beta_\eta}{\sigma_\eta^2}\right)
\end{aligned}$$

which can be sampled using a Metropolis-Hastings step

3.4 Full Conditional for σ_ϵ^2

$$
\begin{aligned}
[\sigma_\epsilon^2|\cdot] &\propto \prod_{t=1}^{T}[\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\boldsymbol{\beta}_t|\sigma_\eta^2, \boldsymbol{\gamma}][\sigma_\epsilon^2] \\
&\propto (\prod_{t=1}^{T}|\boldsymbol{\Sigma}_t|^{-\frac{1}{2}}) \exp\left(-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right)|\boldsymbol{\Sigma}_\beta|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\sum_{t=1}^{T}\boldsymbol{\beta}_t^T\boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\beta}_t\right) \\
&\quad \times (\sigma_\epsilon^2)^{-\alpha_\epsilon - 1}\exp\left(-\frac{\beta_\epsilon}{\sigma_\epsilon^2}\right) \\
&\propto |\boldsymbol{\Sigma}_t|^{-\frac{T}{2}} \exp\left(-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right)|\boldsymbol{\Sigma}_\beta|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}\sum_{t=1}^{T}\boldsymbol{\beta}_t^T\boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\beta}_t\right) \\
&\quad \times (\sigma_\epsilon^2)^{-\alpha_\epsilon - 1}\exp\left(-\frac{\beta_\epsilon}{\sigma_\epsilon^2}\right)
\end{aligned}
$$

which can be sampled using a Metropolis-Hastings step

3.5 Full Conditional for ϕ

$$
\begin{aligned}
[\phi|\cdot] &\propto \prod_{t=1}^{T}[y_t|\beta_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\phi] \\
&\propto \prod_{t=1}^{T}|\boldsymbol{\Sigma}_t|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right)\phi^{-\alpha_\phi - 1}\exp\left(-\frac{\beta_\phi}{\phi}\right) \\
&\propto |\boldsymbol{\Sigma}_t|^{-\frac{T}{2}} \exp\left(-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right)\phi^{-\alpha_\phi - 1}\exp\left(-\frac{\beta_\phi}{\phi}\right)
\end{aligned}
$$

which can be sampled using a Metropolis-Hastings step

Full Conditional for λ^2

$$
\begin{aligned}
[\lambda^2|\cdot] &\propto \prod_{t=1}^{T}\prod_{j=1}^{\tau}(\lambda^2)\exp\left(-\frac{\lambda^2/2}{1/\gamma_j^2}\right)(\frac{\lambda^2}{2})^{\alpha_\lambda - 1}\exp\left(-\beta_\lambda(\lambda^2/2)\right) \\
&\propto (\frac{\lambda^2}{2})^{\alpha_\lambda + \tau - 1}\exp\left(-\lambda^2(\beta_\lambda + \sum_{j=1}^{\tau}\gamma_j^2/2)\right)
\end{aligned}
$$

which is Gamma$(\alpha_\lambda + \tau, \beta_\lambda + \sum_{j=1}^{\tau}\gamma_j^2/2)$

4 Posterior Predictive Distribution

The posterior predictive distribution for $\boldsymbol{y}_t$ is sampled a each MCMC iteration k by

$$\boldsymbol{y}_t^{(k)} \sim N(\boldsymbol{H_t}\boldsymbol{X}\boldsymbol{\beta}_t^{(k)}, \boldsymbol{\Sigma}^{(k)})$$