Spatial Geostatistical Model

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1 Full Dimensional Model Statement - Cressie and Wikle Statistics for Spatio-Temporal Data p. 139

1.1 Data Model

$$\boldsymbol{y}_t = \boldsymbol{z}_t + \boldsymbol{\epsilon}_t$$

1.2 Process Model

$$\boldsymbol{z}_t = \boldsymbol{H}_t \boldsymbol{X} \boldsymbol{\beta}_t + \boldsymbol{\eta}_t$$

1.3 Parameter Model

$$\begin{aligned}
\boldsymbol{\beta}_t &\sim N(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta) & \boldsymbol{\Sigma}_\beta &= \sigma^2_\beta \boldsymbol{I}_{n_t} & & \\
\boldsymbol{\eta}_t &\sim N(0, \boldsymbol{\Sigma}_{\eta_t}) & \boldsymbol{\Sigma}_{\eta_t} &= \sigma^2_\eta \boldsymbol{R}(\phi) & \boldsymbol{R}(\phi) &= \exp\left(-\boldsymbol{D}_t/\phi\right) \\
\boldsymbol{\epsilon}_t &\sim N(0, \boldsymbol{\Sigma}_\epsilon) & \boldsymbol{\Sigma}_\epsilon &= \sigma^2_\epsilon \boldsymbol{I}_{n_t} & & \\
\boldsymbol{\mu}_\beta &\sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) & & & & \\
\sigma^2_\beta &\sim IG(\alpha_\beta, \beta_\beta) & & & & \\
\sigma^2_\eta &\sim IG(\alpha_\eta, \beta_\eta) & & & & \\
\sigma^2_\epsilon &\sim IG(\alpha_\epsilon, \beta_\epsilon) & & & & \\
\phi &\sim IG(\alpha_\phi, \beta_\phi) & & & &
\end{aligned}$$

where $\boldsymbol{I}_{n_t}$ is the identity matrix of size $n_t \times n_t$ where τ is the number of parameters in $\boldsymbol{\beta}_t$ and n_t is the number of samples of y_t at time t and $\boldsymbol{D}_t$ is the distance matrix between locations observed at time t. Define $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_\eta + \boldsymbol{\Sigma}_\epsilon$

2 Posterior

$$\prod_{t=1}^T [\boldsymbol{\beta}_t, \boldsymbol{\mu}_\beta, \sigma^2_\beta, \sigma^2_\eta, \sigma^2_\epsilon, \phi | \boldsymbol{y}_t] \propto \prod_{t=1}^T [\boldsymbol{y}_t | \boldsymbol{\beta}_t, \sigma^2_\eta, \phi, \sigma^2_\epsilon][\boldsymbol{\beta}_t | \boldsymbol{\mu}_\beta, \sigma^2_\beta][\boldsymbol{\mu}_\beta][\sigma^2_\beta][\sigma^2_\eta][\sigma^2_\epsilon][\phi]$$

3 Full Conditionals

3.1 Full Conditional for $\boldsymbol{\beta}_t$

For $t = 1, \ldots, T$,

$$
\begin{aligned}
[\boldsymbol{\beta}_t|\cdot] &\propto [\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma^2_\eta, \sigma^2_\epsilon, \phi][\boldsymbol{\beta}_t|\boldsymbol{\mu}_\beta, \sigma^2_\beta] \\
&\propto \exp\left\{-\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right\}\exp\left\{-\frac{1}{2}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T\boldsymbol{\Sigma}_\beta^{-1}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)\right\} \\
&\propto \exp\left(-\frac{1}{2}\{\boldsymbol{\beta}_t^T(\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}^{-1}\boldsymbol{H}_t\boldsymbol{X} + \boldsymbol{\Sigma}_\beta^{-1})\boldsymbol{\beta}_t - 2\boldsymbol{\beta}_t^T(\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}^{-1}\boldsymbol{y}_t + \boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\mu}_\beta)\}\right)
\end{aligned}
$$

which is Normal with mean $\boldsymbol{A}^{-1}\boldsymbol{b}$ and variance $\boldsymbol{A}^{-1}$ where

$$
\begin{aligned}
\boldsymbol{A}^{-1} &= (\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}^{-1}\boldsymbol{H}_t\boldsymbol{X} + \boldsymbol{\Sigma}_\beta^{-1})^{-1} \\
\boldsymbol{b} &= (\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}^{-1}\boldsymbol{y}_t + \boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\mu}_\beta)
\end{aligned}
$$

3.2 Full Conditional for $\boldsymbol{\mu}_\beta$

$$
\begin{aligned}
[\boldsymbol{\mu}_\beta|\cdot] &\propto \prod_{t=1}^T[\boldsymbol{\beta}_t|\boldsymbol{\mu}_\beta, \sigma^2_\beta][\boldsymbol{\mu}_\beta] \\
&\propto \exp\left(-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T\boldsymbol{\Sigma}_\beta^{-1}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)\right)\exp\left(-\frac{1}{2}(\boldsymbol{\mu}_\beta - \boldsymbol{\mu}_0)^T\Sigma_0^{-1}(\boldsymbol{\mu}_\beta - \boldsymbol{\mu}_0)\right) \\
&\propto \exp\left(-\frac{1}{2}(\boldsymbol{\mu}_\beta^T(T\boldsymbol{\Sigma}_\beta^{-1} + \boldsymbol{\Sigma}_0^{-1})\boldsymbol{\mu}_\beta - 2\boldsymbol{\mu}_\beta^T(\sum_{t=1}^T\boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\beta}_t + \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0))\right)
\end{aligned}
$$

which is multivariate normal with mean $(T\boldsymbol{\Sigma}_\beta^{-1} + \boldsymbol{\Sigma}_0^{-1})^{-1}(\sum_{t=1}^T\boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\beta}_t + \boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0)$ and variance $(T\boldsymbol{\Sigma}_\beta^{-1} + \boldsymbol{\Sigma}_0^{-1})^{-1}$

3.3 Full Conditional for σ^2_β

$$
\begin{aligned}
[\sigma^2_\beta|\cdot] &\propto \prod_{t=1}^T[\boldsymbol{\beta}_t|\boldsymbol{\mu}_\beta, \sigma^2_\beta][\sigma^2_\beta] \\
&\propto (\prod_{t=1}^T|\boldsymbol{\Sigma}_\beta|^{-\frac{1}{2}})\exp\left(-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T\boldsymbol{\Sigma}_\beta^{-1}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)\right)(\sigma^2_\beta)^{-(\alpha_\beta+1)}\exp\left(-\frac{\beta_\beta}{\sigma^2_\beta}\right) \\
&\propto (\sigma^2_\beta)^{-(\alpha_\beta+\frac{T\tau}{2}+1)}\exp\left(-\frac{1}{\sigma^2_\beta}(\frac{1}{2}\sum_{t=1}^T(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta) + \beta_\beta)\right)
\end{aligned}
$$

which is IG$(\alpha_\beta + \frac{t\tau}{2}, \frac{1}{2}\sum_{t=1}^T(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta) + \beta_\beta)$ since the determinant $|\boldsymbol{\Sigma}_\beta| = (\sigma^2_\beta)^\tau$ and $\boldsymbol{\Sigma}_\beta^{-1} = \frac{1}{\sigma^2_\beta}\boldsymbol{I}_\tau$

3.4 Full Conditional for σ_η^2

$$
\begin{aligned}
[\sigma_\eta^2|\cdot] &\propto \prod_{t=1}^{T}[\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\sigma_\eta^2] \\
&\propto (\prod_{t=1}^{T}|\boldsymbol{\Sigma}|^{-\frac{1}{2}}) \exp\left(-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right)(\sigma_\eta^2)^{-\alpha_\eta+1}\exp\left(-\frac{\beta_\eta}{\sigma_\eta^2}\right) \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{T}{2}} \exp\left(-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right)(\sigma_\eta^2)^{-\alpha_\eta+1}\exp\left(-\frac{\beta_\eta}{\sigma_\eta^2}\right)
\end{aligned}
$$

which can be sampled using a Metropolis-Hastings step

3.5 Full Conditional for σ_ϵ^2

$$
\begin{aligned}
[\sigma_\epsilon^2|\cdot] &\propto \prod_{t=1}^{T}[\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\sigma_\epsilon^2] \\
&\propto (\prod_{t=1}^{T}|\boldsymbol{\Sigma}|^{-\frac{1}{2}}) \exp\left(-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right)(\sigma_\epsilon^2)^{-\alpha_\epsilon+1}\exp\left(-\frac{\beta_\epsilon}{\sigma_\epsilon^2}\right) \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{T}{2}} \exp\left(-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right)(\sigma_\epsilon^2)^{-\alpha_\epsilon+1}\exp\left(-\frac{\beta_\epsilon}{\sigma_\epsilon^2}\right)
\end{aligned}
$$

which can be sampled using a Metropolis-Hastings step

3.6 Full Conditional for ϕ

$$
\begin{aligned}
[\phi|\cdot] &\propto \prod_{t=1}^{T}[y_t|\beta_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\phi] \\
&\propto \prod_{t=1}^{T}|\boldsymbol{\Sigma}|^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right)\phi^{-\alpha_\phi+1}\exp\left(-\frac{\beta_\phi}{\phi}\right) \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{T}{2}} \exp\left(-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)\right)\phi^{-\alpha_\phi+1}\exp\left(-\frac{\beta_\phi}{\phi}\right)
\end{aligned}
$$

which can be sampled using a Metropolis-Hastings step

4 Posterior Predictive Distribution

The posterior predictive distribution for $\boldsymbol{y}_t$ is sampled a each MCMC iteration k by

$$
\left(\boldsymbol{z}_t^{(k)}\left(\boldsymbol{s}_0\right)\middle|\boldsymbol{y}\right) = \boldsymbol{X}\left(\boldsymbol{s}_0\right)^T\boldsymbol{\beta}_t^{(k)} + \boldsymbol{c}_Y\left(\boldsymbol{s}_0\right)^T\boldsymbol{\Sigma}^{-1}
$$