Spatial Geostatistical Model

John Tipton

February 14, 2014

1 Full Dimensional Model Statement

1.1 Data Model

$$\boldsymbol{y}_t = \boldsymbol{H}_t \boldsymbol{X} \boldsymbol{\beta}_t + \boldsymbol{\eta}_t + \boldsymbol{\epsilon}_t$$

1.2 Process Model

$$\begin{aligned}
\boldsymbol{\beta}_t &\sim N(\boldsymbol{0}, \boldsymbol{\Sigma}_\beta) & \boldsymbol{\Sigma}_\beta &= \sigma^2_\epsilon \boldsymbol{D}_\gamma & \boldsymbol{D}_\gamma &= diag(\gamma^2_1, \ldots, \gamma^2_\tau) \\
\boldsymbol{\eta}_t &\sim N(0, \boldsymbol{\Sigma}_\eta) & \boldsymbol{\Sigma}_\eta &= \sigma^2_\eta \boldsymbol{R}(\phi) & \boldsymbol{R}(\phi) &= \exp\left(-\boldsymbol{D}_t/\phi\right) \\
\boldsymbol{\epsilon}_t &\sim N(0, \boldsymbol{\Sigma}_\epsilon) & \boldsymbol{\Sigma}_\epsilon &= \sigma^2_\epsilon \boldsymbol{I}_{n_t}
\end{aligned}$$

1.3 Parameter Model

$$\begin{aligned}
\sigma^2_\beta &\sim IG(\alpha_\beta, \beta_\beta) \\
\sigma^2_\eta &\sim IG(\alpha_\eta, \beta_\eta) \\
\sigma^2_\epsilon &\sim IG(\alpha_\epsilon, \beta_\epsilon) \\
\phi &\sim IG(\alpha_\phi, \beta_\phi)
\end{aligned}$$

where $\boldsymbol{I}_{n_t}$ is the identity matrix of size $n_t \times n_t$ where τ is the number of parameters in $\boldsymbol{\beta}_t$ and n_t is the number of samples of y_t at time t and $\boldsymbol{D}_t$ is the distance matrix between locations observed at time t. Define $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_\eta + \boldsymbol{\Sigma}_\epsilon$

2 Posterior

$$\prod_{t=1}^{T} [\boldsymbol{\beta}_t, \boldsymbol{\mu}_\beta, \sigma^2_\beta, \sigma^2_\eta, \sigma^2_\epsilon, \phi | \boldsymbol{y}_t] \propto \prod_{t=1}^{T} [\boldsymbol{y}_t | \boldsymbol{\beta}_t, \sigma^2_\eta, \phi, \sigma^2_\epsilon][\boldsymbol{\beta}_t | \boldsymbol{\mu}_\beta, \sigma^2_\beta][\boldsymbol{\mu}_\beta][\sigma^2_\beta][\sigma^2_\eta][\sigma^2_\epsilon][\phi]$$

3 Full Conditionals

3.1 Full Conditional for $\boldsymbol{\beta}_t$

For $t = 1, \ldots, T$,

$$\begin{aligned}
[\boldsymbol{\beta}_t | \cdot] &\propto [\boldsymbol{y}_t | \boldsymbol{\beta}_t, \sigma^2_\eta, \sigma^2_\epsilon, \phi][\boldsymbol{\beta}_t | \boldsymbol{\mu}_\beta, \sigma^2_\beta] \\
&\propto e^{-\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{H}_t \boldsymbol{X} \boldsymbol{\beta}_t)^T \boldsymbol{\Sigma}^{-1} (\boldsymbol{y}_t - \boldsymbol{H}_t \boldsymbol{X} \boldsymbol{\beta}_t)} e^{-\frac{1}{2}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T \boldsymbol{\Sigma}_\beta^{-1} (\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)} \\
&\propto e^{-\frac{1}{2}\{\boldsymbol{\beta}_t^T (\boldsymbol{X}^T \boldsymbol{H}_t^T \boldsymbol{\Sigma}^{-1} \boldsymbol{H}_t \boldsymbol{X} + \boldsymbol{\Sigma}_\beta^{-1}) \boldsymbol{\beta}_t - 2 \boldsymbol{\beta}_t^T (\boldsymbol{X}^T \boldsymbol{H}_t^T \boldsymbol{\Sigma}^{-1} \boldsymbol{y}_t + \boldsymbol{\Sigma}_\beta^{-1} \boldsymbol{\mu}_\beta)\}}
\end{aligned}$$

which is Normal with mean $\boldsymbol{A}^{-1}\boldsymbol{b}$ and variance $\boldsymbol{A}^{-1}$ where

$$\begin{aligned}\boldsymbol{A}^{-1} &= (\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}^{-1}\boldsymbol{H}_t\boldsymbol{X} + \boldsymbol{\Sigma}_\beta^{-1})^{-1}\\ \boldsymbol{b} &= (\boldsymbol{X}^T\boldsymbol{H}_t^T\boldsymbol{\Sigma}^{-1}\boldsymbol{y}_t + \boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\mu}_\beta)\end{aligned}$$

3.2 Full Conditional for σ_η^2

$$\begin{aligned}[\sigma_\eta^2|\cdot] &\propto \prod_{t=1}^T[\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\sigma_\eta^2]\\ &\propto (\prod_{t=1}^T|\boldsymbol{\Sigma}|^{-\frac{1}{2}})e^{-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)}(\sigma_\eta^2)^{-\alpha_\eta+1}e^{-\frac{\beta_\eta}{\sigma_\eta^2}}\\ &\propto |\boldsymbol{\Sigma}|^{-\frac{T}{2}}e^{-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)}(\sigma_\eta^2)^{-\alpha_\eta+1}e^{-\frac{\beta_\eta}{\sigma_\eta^2}}\end{aligned}$$

which can be sampled using a Metropolis-Hastings step

3.3 Full Conditional for σ_ϵ^2

$$\begin{aligned}[\sigma_\epsilon^2|\cdot] &\propto \prod_{t=1}^T[\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\boldsymbol{\beta}_t|\sigma_\eta^2, \boldsymbol{\gamma}][\sigma_\epsilon^2]\\ &\propto (\prod_{t=1}^T|\boldsymbol{\Sigma}|^{-\frac{1}{2}})e^{-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)}|\boldsymbol{\Sigma}_\beta|^{-\frac{1}{2}}e^{-\frac{1}{2}\sum_{t=1}^T\boldsymbol{\beta}_t^T\boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\beta}_t}(\sigma_\epsilon^2)^{-\alpha_\epsilon+1}e^{-\frac{\beta_\epsilon}{\sigma_\epsilon^2}}\\ &\propto |\boldsymbol{\Sigma}|^{-\frac{T}{2}}e^{-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)}|\boldsymbol{\Sigma}_\beta|^{-\frac{1}{2}}e^{-\frac{1}{2}\sum_{t=1}^T\boldsymbol{\beta}_t^T\boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\beta}_t}(\sigma_\epsilon^2)^{-\alpha_\epsilon+1}e^{-\frac{\beta_\epsilon}{\sigma_\epsilon^2}}\end{aligned}$$

which can be sampled using a Metropolis-Hastings step

3.4 Full Conditional for ϕ

$$\begin{aligned}[\phi|\cdot] &\propto \prod_{t=1}^T[y_t|\beta_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\phi]\\ &\propto \prod_{t=1}^T|\boldsymbol{\Sigma}|^{-\frac{1}{2}}e^{-\frac{1}{2}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)}\phi^{-\alpha_\phi+1}e^{-\frac{\beta_\phi}{\phi}}\\ &\propto |\boldsymbol{\Sigma}|^{-\frac{T}{2}}e^{-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}^{-1}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)}\phi^{-\alpha_\phi+1}e^{-\frac{\beta_\phi}{\phi}}\end{aligned}$$

which can be sampled using a Metropolis-Hastings step

Full Conditional for λ^2

$$\begin{aligned}[\lambda^2|\cdot] &\propto \prod_{t=1}^T\prod_{j=1}^\tau(\lambda^2)e^{-\frac{\lambda^2/2}{1/\gamma_j^2}}(\frac{\lambda^2}{2})^{\alpha_\lambda-1}e^{-\beta_\lambda(\lambda^2/2)}\\ &\propto (\frac{\lambda^2}{2})^{\alpha_\lambda+\tau-1}e^{-\lambda^2(\beta_\lambda+\sum_{j=1}^\tau\gamma_j^2/2)}\end{aligned}$$

which is Gamma$(\alpha_\lambda+\tau, \beta_\lambda+\sum_{j=1}^\tau\gamma_j^2/2)$

4 Posterior Predictive Distribution

The posterior predictive distribution for $\boldsymbol{y}_t$ is sampled a each MCMC iteration k by

$$\boldsymbol{y}_t^{(k)} \sim N(\boldsymbol{H_t}\boldsymbol{X}\boldsymbol{\beta}_t^{(k)}, \boldsymbol{\Sigma}^{(k)})$$