Spatial Geostatistical Model

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1 Full Dimensional Model Statement

$$
\begin{aligned}
\boldsymbol{y}_t &= \boldsymbol{H}_t \boldsymbol{X}_t \boldsymbol{\beta}_t + \boldsymbol{\eta}_t + \boldsymbol{\epsilon}_t \\
\boldsymbol{\beta}_t &\sim N(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta) \\
\boldsymbol{\eta}_t &\sim N(0, \sigma^2_\eta \boldsymbol{R}(\phi)) \\
\boldsymbol{\epsilon}_t &\sim N(0, \sigma^2_\epsilon) \\
\boldsymbol{\mu}_\beta &\sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) \\
\boldsymbol{\Sigma}_\beta &\sim \sigma^2_\beta \boldsymbol{I} \\
\sigma^2_\beta &\sim IG(\alpha_\beta, \beta_\beta) \\
\sigma^2_\eta &\sim IG(\alpha_\eta, \beta_\eta) \\
\sigma^2_\epsilon &\sim IG(\alpha_\epsilon, \beta_\epsilon) \\
\phi &\sim IG(\alpha_\phi, \beta_\phi)
\end{aligned}
$$

2 Posterior

$$
\prod_{t=1}^T [\boldsymbol{\beta}_t, \boldsymbol{\mu}_t, \sigma^2_\beta, \sigma^2_\eta, \sigma^2_\epsilon, \phi | \boldsymbol{y}_t] \propto \prod_{t=1}^T [\boldsymbol{y}_t | \boldsymbol{\beta}_t, \sigma^2_\eta, \phi, \sigma^2_\epsilon][\boldsymbol{\beta}_t | \boldsymbol{\mu}_\beta, \sigma^2_\beta][\boldsymbol{\mu}_\beta][\sigma^2_\beta][\sigma^2_\eta][\sigma^2_\epsilon][\phi]
$$

3 Full Conditionals

3.1 Full Conditional for $\boldsymbol{\beta}_t$

For $t = 1, \ldots, T$,

$$
\begin{aligned}
[\boldsymbol{\beta}_t | \cdot] &\propto [\boldsymbol{y}_t | \boldsymbol{\beta}_t, \sigma^2_\eta, \sigma^2_\epsilon, \phi][\boldsymbol{\beta}_t | \boldsymbol{\mu}_\beta, \sigma^2_\beta] \\
&\propto e^{-\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{H}_t \boldsymbol{X} \boldsymbol{\beta}_t)^T (\sigma^2_\eta \boldsymbol{R}_t(\phi) + \sigma^2_\epsilon \boldsymbol{I})^{-1} (\boldsymbol{y}_t - \boldsymbol{H}_t \boldsymbol{X} \boldsymbol{\beta}_t)} e^{-\frac{1}{2}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T (\sigma^2_\beta \boldsymbol{\Lambda})^{-1} (\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)} \\
&\propto e^{-\frac{1}{2}\{\boldsymbol{\beta}_t^T (\boldsymbol{X}^T \boldsymbol{H}_t^T (\sigma^2_\eta \boldsymbol{R}_t(\phi) + \sigma^2_\epsilon \boldsymbol{I})^{-1} \boldsymbol{H}_t \boldsymbol{X} + (\sigma^2_\beta \boldsymbol{\Lambda})^{-1}) \boldsymbol{\beta}_t - 2 \boldsymbol{\beta}_t^T (\boldsymbol{X}^T \boldsymbol{H}_t^T (\sigma^2_\eta \boldsymbol{R}_t(\phi) + \sigma^2_\epsilon \boldsymbol{I})^{-1} \boldsymbol{y}_t + (\sigma^2_\beta \boldsymbol{\Lambda})^{-1} \boldsymbol{\mu}_\beta)\}}
\end{aligned}
$$

which is Normal with mean $\boldsymbol{A}^{-1}\boldsymbol{b}$ and variance $\boldsymbol{A}^{-1}$.

$$
\begin{aligned}
\boldsymbol{A}^{-1} &= (\boldsymbol{X}^T \boldsymbol{H}_t^T (\sigma^2_\eta \boldsymbol{R}_t(\phi) + \sigma^2_\epsilon \boldsymbol{I})^{-1} \boldsymbol{H}_t \boldsymbol{X} + (\sigma^2_\beta \boldsymbol{\Lambda})^{-1})^{-1} \\
&\approx (\boldsymbol{X}^T \boldsymbol{H}_t^T (\tilde{\boldsymbol{C}} + \sigma^2_\epsilon \boldsymbol{I})^{-1} \boldsymbol{H}_t \boldsymbol{X} + (\sigma^2_\beta \boldsymbol{\Lambda})^{-1})^{-1}
\end{aligned}
$$

$$
\begin{aligned}
\boldsymbol{b} &= (\boldsymbol{X}^T \boldsymbol{H}_t^T (\sigma^2_\eta \boldsymbol{R}_t(\phi) + \sigma^2_\epsilon \boldsymbol{I})^{-1} \boldsymbol{y}_t + (\sigma^2_\beta \boldsymbol{\Lambda})^{-1} \boldsymbol{\mu}_\beta) \\
&\approx (\boldsymbol{X}^T \boldsymbol{H}_t^T (\tilde{\boldsymbol{C}} + \sigma^2_\epsilon \boldsymbol{I})^{-1} \boldsymbol{y}_t + (\sigma^2_\beta \boldsymbol{\Lambda})^{-1} \boldsymbol{\mu}_\beta)
\end{aligned}
$$

3.2 Full Conditional for $\boldsymbol{\mu}_\beta$

$$
\begin{aligned}
[\boldsymbol{\mu}_\beta|\cdot] &\propto \prod_{t=1}^{T}[\boldsymbol{\beta}_t|\boldsymbol{\mu}_\beta, \sigma_\beta^2][\boldsymbol{\mu}_\beta] \\
&\propto e^{-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{\beta}_t-\boldsymbol{\mu}_\beta)^T(\sigma_\beta^2\boldsymbol{\Lambda})^{-1}(\boldsymbol{\beta}_t-\boldsymbol{\mu}_\beta)}e^{-\frac{1}{2}(\boldsymbol{\mu}_\beta-\boldsymbol{\mu}_0)^T\Sigma_0^{-1}(\boldsymbol{\mu}_\beta-\boldsymbol{\mu}_0)} \\
&\propto e^{-\frac{1}{2}(\boldsymbol{\mu}_\beta^T(T*(\sigma_\beta^2\boldsymbol{\Lambda})^{-1}+\boldsymbol{\Sigma}_0^{-1})\boldsymbol{\mu}_\beta-2\boldsymbol{\mu}_\beta^T(\sum_{t=1}^{T}(\sigma_\beta^2\boldsymbol{\Lambda})^{-1}\boldsymbol{\beta}_t+\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0))}
\end{aligned}
$$

which is multivariate normal with mean $(T*(\sigma_\beta^2\boldsymbol{\Lambda})^{-1}+\boldsymbol{\Sigma}_0^{-1})^{-1}(\sum_{t=1}^{T}(\sigma_\beta^2\boldsymbol{\Lambda})^{-1}\boldsymbol{\beta}_t+\boldsymbol{\Sigma}_0^{-1}\boldsymbol{\mu}_0)$ and variance $(T*(\sigma_\beta^2\boldsymbol{\Lambda})^{-1}+\boldsymbol{\Sigma}_0^{-1})^{-1}$

3.3 Full Conditional for σ_β^2

$$
\begin{aligned}
[\sigma_\beta^2|\cdot] &\propto \prod_{t=1}^{T}[\boldsymbol{\beta}_t|\boldsymbol{\mu}_\beta, \sigma_\beta^2][\sigma_\beta^2] \\
&\propto (\prod_{t=1}^{T}|\sigma_\beta^2\boldsymbol{\Lambda}|^{-\frac{1}{2}})e^{-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{\beta}_t-\boldsymbol{\mu}_t)^T(\sigma_\beta^2\boldsymbol{\Lambda})^{-1}(\boldsymbol{\beta}_t-\boldsymbol{\mu}_t)}(\sigma_\beta^2)^{-(\alpha_\beta+1)}e^{-\frac{\beta_\beta}{\sigma_\beta^2}} \\
&\propto (\sigma_\beta^2)^{-(\alpha_\beta+\frac{T*|\boldsymbol{\Lambda}|}{2}+1)}e^{-\frac{1}{\sigma_\beta^2}(\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{\beta}_t-\boldsymbol{\mu}_t)^T(\boldsymbol{\Lambda})^{-1}(\boldsymbol{\beta}_t-\boldsymbol{\mu}_t)+\beta_\beta)}
\end{aligned}
$$

which is IG$(\alpha_\beta+\frac{T*|\boldsymbol{\Lambda}|}{2}, \frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{\beta}_t-\boldsymbol{\mu}_t)^T(\boldsymbol{\Lambda})^{-1}(\boldsymbol{\beta}_t-\boldsymbol{\mu}_t)+\beta_\beta)$ where $|\boldsymbol{\Lambda}|$ is the determinant $\det(\boldsymbol{\Lambda}) = \prod_{i=1}^{J}\lambda_j^2$ where the λ_j are the singular values of $\boldsymbol{X}$.

3.4 Full Conditional for σ_η^2

$$
\begin{aligned}
[\sigma_\eta^2|\cdot] &\propto \prod_{t=1}^{T}[\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\sigma_\eta^2] \\
&\propto (\prod_{t=1}^{T}|\sigma_\eta^2\boldsymbol{R}_t(\phi)+\sigma_\epsilon^2\boldsymbol{\Lambda}|^{-\frac{1}{2}})e^{-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T(\sigma_\eta^2\boldsymbol{R}_t(\phi)+\sigma_\epsilon^2\boldsymbol{\Lambda})^{-1}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)}(\sigma_\eta^2)^{-\alpha_\eta+1}e^{-\frac{\beta_\eta}{\sigma_\eta^2}} \\
&\propto |\sigma_\eta^2\boldsymbol{R}_t(\phi)+\sigma_\epsilon^2\boldsymbol{\Lambda}|^{-\frac{T}{2}}e^{-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T(\sigma_\eta^2\boldsymbol{R}_t(\phi)+\sigma_\epsilon^2\boldsymbol{\Lambda})^{-1}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)}(\sigma_\eta^2)^{-\alpha_\eta+1}e^{-\frac{\beta_\eta}{\sigma_\eta^2}} \\
&\approx |\tilde{\boldsymbol{C}}+\sigma_\epsilon^2\boldsymbol{\Lambda}|^{-\frac{T}{2}}e^{-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T(\tilde{\boldsymbol{C}}+\sigma_\epsilon^2\boldsymbol{\Lambda})^{-1}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)}(\sigma_\eta^2)^{-\alpha_\eta+1}e^{-\frac{\beta_\eta}{\sigma_\eta^2}}
\end{aligned}
$$

which can be sampled using a Metropolis-Hastings step

3.5 Full Conditional for σ_ϵ^2

$$
\begin{aligned}
[\sigma_\epsilon^2|\cdot] &\propto \prod_{t=1}^{T}[\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\sigma_\epsilon^2] \\
&\propto (\prod_{t=1}^{T}|\sigma_\eta^2\boldsymbol{R}_t(\phi)+\sigma_\epsilon^2\boldsymbol{\Lambda}|^{-\frac{1}{2}})e^{-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T(\sigma_\eta^2\boldsymbol{R}_t(\phi)+\sigma_\epsilon^2\boldsymbol{\Lambda})^{-1}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)}(\sigma_\epsilon^2)^{-\alpha_\epsilon+1}e^{-\frac{\beta_\epsilon}{\sigma_\epsilon^2}} \\
&\propto |\sigma_\eta^2\boldsymbol{R}_t(\phi)+\sigma_\epsilon^2\boldsymbol{\Lambda}|^{-\frac{T}{2}}e^{-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T(\sigma_\eta^2\boldsymbol{R}_t(\phi)+\sigma_\epsilon^2\boldsymbol{\Lambda})^{-1}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)}(\sigma_\epsilon^2)^{-\alpha_\epsilon+1}e^{-\frac{\beta_\epsilon}{\sigma_\epsilon^2}} \\
&\approx |\tilde{\boldsymbol{C}}+\sigma_\epsilon^2\boldsymbol{\Lambda}|^{-\frac{T}{2}}e^{-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T(\tilde{\boldsymbol{C}}+\sigma_\epsilon^2\boldsymbol{\Lambda})^{-1}(\boldsymbol{y}_t-\boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)}(\sigma_\epsilon^2)^{-\alpha_\epsilon+1}e^{-\frac{\beta_\epsilon}{\sigma_\epsilon^2}}
\end{aligned}
$$

which can be sampled using a Metropolis-Hastings step

3.6 Full Conditional for ϕ

$$
\begin{aligned}
[\phi|\cdot] &\propto \prod_{t=1}^{T}[y_t|\beta_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\phi] \\
&\propto \prod_{t=1}^{T} |\sigma_\eta^2 \boldsymbol{R}(\phi) + \sigma_\epsilon^2 \boldsymbol{\Lambda}|^{-\frac{1}{2}} e^{-\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{H}_t \boldsymbol{X} \boldsymbol{\beta}_t)^T (\sigma_\eta^2 \boldsymbol{R}(\phi) + \sigma_\epsilon^2 \boldsymbol{\Lambda})^{-1} (\boldsymbol{y}_t - \boldsymbol{H}_t \boldsymbol{X} \boldsymbol{\beta}_t)} \phi^{-\alpha_\phi + 1} e^{-\frac{\beta_\phi}{\phi}} \\
&\propto |\sigma_\eta^2 \boldsymbol{R}(\phi) + \sigma_\epsilon^2 \boldsymbol{\Lambda}]|^{-\frac{T}{2}} e^{-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t \boldsymbol{X} \boldsymbol{\beta}_t)^T (\sigma_\eta^2 \boldsymbol{R}(\phi) + \sigma_\epsilon^2 \boldsymbol{\Lambda})^{-1} (\boldsymbol{y}_t - \boldsymbol{H}_t \boldsymbol{X} \boldsymbol{\beta}_t)} \phi^{-\alpha_\phi + 1} e^{-\frac{\beta_\phi}{\phi}} \\
&\approx |\tilde{\boldsymbol{C}} + \sigma_\epsilon^2 \boldsymbol{\Lambda}|^{-\frac{T}{2}} e^{-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t \boldsymbol{X} \boldsymbol{\beta}_t)^T (\tilde{\boldsymbol{C}} + \sigma_\epsilon^2 \boldsymbol{\Lambda})^{-1} (\boldsymbol{y}_t - \boldsymbol{H}_t \boldsymbol{X} \boldsymbol{\beta}_t)} \phi^{-\alpha_\phi + 1} e^{-\frac{\beta_\phi}{\phi}}
\end{aligned}
$$

which can be sampled using a Metropolis-Hastings step