

# Spatial Geostatistical Model

John Tipton

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## 1 Full Dimensional Model Statement

$$\mathbf{y}_t = \mathbf{H}_t \mathbf{X}_t \boldsymbol{\beta}_t + \boldsymbol{\eta}_t + \boldsymbol{\epsilon}_t$$

$$\boldsymbol{\beta}_t \sim N(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta)$$

$$\boldsymbol{\eta}_t \sim N(0, \sigma_\eta^2 \mathbf{R}(\phi))$$

$$\boldsymbol{\epsilon}_t \sim N(0, \sigma_\epsilon^2 \mathbf{I})$$

$$\boldsymbol{\mu}_\beta \sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0)$$

$$\boldsymbol{\Sigma}_\beta \sim \sigma_\beta^2 \mathbf{I}$$

$$\sigma_\beta^2 \sim IG(\alpha_\beta, \beta_\beta)$$

$$\sigma_\eta^2 \sim IG(\alpha_\eta, \beta_\eta)$$

$$\sigma_\epsilon^2 \sim IG(\alpha_\epsilon, \beta_\epsilon)$$

$$\phi \sim IG(\alpha_\phi, \beta_\phi)$$

## 2 Posterior

$$\prod_{t=1}^T [\boldsymbol{\beta}_t, \boldsymbol{\mu}_\beta, \sigma_\beta^2, \sigma_\eta^2, \sigma_\epsilon^2, \phi | \mathbf{y}_t] \propto \prod_{t=1}^T [\mathbf{y}_t | \boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2] [\boldsymbol{\beta}_t | \boldsymbol{\mu}_\beta, \sigma_\beta^2] [\boldsymbol{\mu}_\beta] [\sigma_\beta^2] [\sigma_\eta^2] [\sigma_\epsilon^2] [\phi]$$

## 3 Full Conditionals

### 3.1 Full Conditional for $\boldsymbol{\beta}_t$

For  $t = 1, \dots, T$ ,

$$\begin{aligned} [\boldsymbol{\beta}_t | \cdot] &\propto [\mathbf{y}_t | \boldsymbol{\beta}_t, \sigma_\eta^2, \sigma_\epsilon^2, \phi] [\boldsymbol{\beta}_t | \boldsymbol{\mu}_\beta, \sigma_\beta^2] \\ &\propto e^{-\frac{1}{2}(\mathbf{y}_t - \mathbf{H}_t \mathbf{X}_t \boldsymbol{\beta}_t)^T (\sigma_\eta^2 \mathbf{R}_t(\phi) + \sigma_\epsilon^2 \mathbf{I})^{-1} (\mathbf{y}_t - \mathbf{H}_t \mathbf{X}_t \boldsymbol{\beta}_t)} e^{-\frac{1}{2}(\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)^T (\sigma_\beta^2 \mathbf{I})^{-1} (\boldsymbol{\beta}_t - \boldsymbol{\mu}_\beta)} \\ &\propto e^{-\frac{1}{2}\{\boldsymbol{\beta}_t^T (\mathbf{X}_t^T \mathbf{H}_t^T (\sigma_\eta^2 \mathbf{R}_t(\phi) + \sigma_\epsilon^2 \mathbf{I})^{-1} \mathbf{H}_t \mathbf{X}_t + (\sigma_\beta^2 \mathbf{I})^{-1}) \boldsymbol{\beta}_t - 2\boldsymbol{\beta}_t^T (\mathbf{X}_t^T \mathbf{H}_t^T (\sigma_\eta^2 \mathbf{R}_t(\phi) + \sigma_\epsilon^2 \mathbf{I})^{-1} \mathbf{y}_t + (\sigma_\beta^2 \mathbf{I})^{-1} \boldsymbol{\mu}_\beta)\}} \end{aligned}$$

which is Normal with mean  $\mathbf{A}^{-1} \mathbf{b}$  and variance  $\mathbf{A}^{-1}$ .

$$\mathbf{A}^{-1} = (\mathbf{X}_t^T \mathbf{H}_t^T (\sigma_\eta^2 \mathbf{R}_t(\phi) + \sigma_\epsilon^2 \mathbf{I})^{-1} \mathbf{H}_t \mathbf{X}_t + (\sigma_\beta^2 \mathbf{I})^{-1})^{-1}$$

$$\mathbf{b} = (\mathbf{X}_t^T \mathbf{H}_t^T (\sigma_\eta^2 \mathbf{R}_t(\phi) + \sigma_\epsilon^2 \mathbf{I})^{-1} \mathbf{y}_t + (\sigma_\beta^2 \mathbf{I})^{-1} \boldsymbol{\mu}_\beta)$$

### 3.2 Full Conditional for $\mu_\beta$

$$\begin{aligned}
[\mu_\beta | \cdot] &\propto \prod_{t=1}^T [\beta_t | \mu_\beta, \sigma_\beta^2] [\mu_\beta] \\
&\propto e^{-\frac{1}{2} \sum_{t=1}^T (\beta_t - \mu_\beta)^T (\sigma_\beta^2 \mathbf{I})^{-1} (\beta_t - \mu_\beta)} e^{-\frac{1}{2} (\mu_\beta - \mu_0)^T \Sigma_0^{-1} (\mu_\beta - \mu_0)} \\
&\propto e^{-\frac{1}{2} (\mu_\beta^T (T * (\sigma_\beta^2 \mathbf{I})^{-1} + \Sigma_0^{-1}) \mu_\beta - 2 \mu_\beta^T (\sum_{t=1}^T (\sigma_\beta^2 \mathbf{I})^{-1} \beta_t + \Sigma_0^{-1} \mu_0))}
\end{aligned}$$

which is multivariate normal with mean  $(T * (\sigma_\beta^2 \mathbf{I})^{-1} + \Sigma_0^{-1})^{-1} (\sum_{t=1}^T (\sigma_\beta^2 \mathbf{I})^{-1} \beta_t + \Sigma_0^{-1} \mu_0)$  and variance  $(T * (\sigma_\beta^2 \mathbf{I})^{-1} + \Sigma_0^{-1})^{-1}$

### 3.3 Full Conditional for $\sigma_\beta^2$

$$\begin{aligned}
[\sigma_\beta^2 | \cdot] &\propto \prod_{t=1}^T [\beta_t | \mu_\beta, \sigma_\beta^2] [\sigma_\beta^2] \\
&\propto \left( \prod_{t=1}^T |\sigma_\beta^2 \mathbf{I}|^{-\frac{1}{2}} \right) e^{-\frac{1}{2} \sum_{t=1}^T (\beta_t - \mu_t)^T (\sigma_\beta^2 \mathbf{I})^{-1} (\beta_t - \mu_t)} (\sigma_\beta^2)^{-(\alpha_\beta + 1)} e^{-\frac{\beta_\beta}{\sigma_\beta^2}} \\
&\propto (\sigma_\beta^2)^{-(\alpha_\beta + \frac{T * \tau}{2} + 1)} e^{-\frac{1}{\sigma_\beta^2} (\frac{1}{2} \sum_{t=1}^T (\beta_t - \mu_t)^T (\mathbf{I})^{-1} (\beta_t - \mu_t) + \beta_\beta)}
\end{aligned}$$

which is  $\text{IG}(\alpha_\beta + \frac{T * \tau}{2}, \frac{1}{2} \sum_{t=1}^T (\beta_t - \mu_t)^T (\mathbf{I})^{-1} (\beta_t - \mu_t) + \beta_\beta)$

### 3.4 Full Conditional for $\sigma_\eta^2$

$$\begin{aligned}
[\sigma_\eta^2 | \cdot] &\propto \prod_{t=1}^T [\mathbf{y}_t | \beta_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2] [\sigma_\eta^2] \\
&\propto \left( \prod_{t=1}^T |\sigma_\eta^2 \mathbf{R}_t(\phi) + \sigma_\epsilon^2 \mathbf{I}|^{-\frac{1}{2}} \right) e^{-\frac{1}{2} \sum_{t=1}^T (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)^T (\sigma_\eta^2 \mathbf{R}_t(\phi) + \sigma_\epsilon^2 \mathbf{I})^{-1} (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)} (\sigma_\eta^2)^{-\alpha_\eta + 1} e^{-\frac{\beta_\eta}{\sigma_\eta^2}} \\
&\propto |\sigma_\eta^2 \mathbf{R}_t(\phi) + \sigma_\epsilon^2 \mathbf{I}|^{-\frac{T}{2}} e^{-\frac{1}{2} \sum_{t=1}^T (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)^T (\sigma_\eta^2 \mathbf{R}_t(\phi) + \sigma_\epsilon^2 \mathbf{I})^{-1} (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)} (\sigma_\eta^2)^{-\alpha_\eta + 1} e^{-\frac{\beta_\eta}{\sigma_\eta^2}}
\end{aligned}$$

which can be sampled using a Metropolis-Hastings step

### 3.5 Full Conditional for $\sigma_\epsilon^2$

$$\begin{aligned}
[\sigma_\epsilon^2 | \cdot] &\propto \prod_{t=1}^T [\mathbf{y}_t | \beta_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2] [\sigma_\epsilon^2] \\
&\propto \left( \prod_{t=1}^T |\sigma_\eta^2 \mathbf{R}_t(\phi) + \sigma_\epsilon^2 \mathbf{I}|^{-\frac{1}{2}} \right) e^{-\frac{1}{2} \sum_{t=1}^T (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)^T (\sigma_\eta^2 \mathbf{R}_t(\phi) + \sigma_\epsilon^2 \mathbf{I})^{-1} (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)} (\sigma_\epsilon^2)^{-\alpha_\epsilon + 1} e^{-\frac{\beta_\epsilon}{\sigma_\epsilon^2}} \\
&\propto |\sigma_\eta^2 \mathbf{R}_t(\phi) + \sigma_\epsilon^2 \mathbf{I}|^{-\frac{T}{2}} e^{-\frac{1}{2} \sum_{t=1}^T (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)^T (\sigma_\eta^2 \mathbf{R}_t(\phi) + \sigma_\epsilon^2 \mathbf{I})^{-1} (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)} (\sigma_\epsilon^2)^{-\alpha_\epsilon + 1} e^{-\frac{\beta_\epsilon}{\sigma_\epsilon^2}}
\end{aligned}$$

which can be sampled using a Metropolis-Hastings step

### 3.6 Full Conditional for $\phi$

$$\begin{aligned}
[\phi|\cdot] &\propto \prod_{t=1}^T [y_t|\beta_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\phi] \\
&\propto \prod_{t=1}^T |\sigma_\eta^2 \mathbf{R}(\phi) + \sigma_\epsilon^2 \mathbf{I}|^{-\frac{1}{2}} e^{-\frac{1}{2}(\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)^T (\sigma_\eta^2 \mathbf{R}(\phi) + \sigma_\epsilon^2 \mathbf{I})^{-1} (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)} \phi^{-\alpha_\phi + 1} e^{-\frac{\beta \phi}{\phi}} \\
&\propto |\sigma_\eta^2 \mathbf{R}(\phi) + \sigma_\epsilon^2 \mathbf{I}|^{-\frac{T}{2}} e^{-\frac{1}{2} \sum_{t=1}^T (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)^T (\sigma_\eta^2 \mathbf{R}(\phi) + \sigma_\epsilon^2 \mathbf{I})^{-1} (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)} \phi^{-\alpha_\phi + 1} e^{-\frac{\beta \phi}{\phi}}
\end{aligned}$$

which can be sampled using a Metropolis-Hastings step