

# Spatial Geostatistical Model

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## 1 Full Dimensional Model Statement

### 1.1 Data Model

$$\mathbf{y}_t = \mathbf{H}_t \mathbf{X} \boldsymbol{\beta}_t + \boldsymbol{\eta}_t + \boldsymbol{\epsilon}_t$$

### 1.2 Process Model

$$\begin{aligned} \boldsymbol{\beta}_t &\sim N(\boldsymbol{\mu}_\beta, \boldsymbol{\Sigma}_\beta) & \boldsymbol{\Sigma}_\beta &= \sigma_\beta^2 \mathbf{I}_{n_t} \\ \boldsymbol{\eta}_t &\sim N(0, \boldsymbol{\Sigma}_\eta) & \boldsymbol{\Sigma}_\eta &= \sigma_\eta^2 \mathbf{R}(\phi) & \mathbf{R}(\phi) &= \exp(-\mathbf{D}_t/\phi) \\ \boldsymbol{\epsilon}_t &\sim N(0, \boldsymbol{\Sigma}_\epsilon) & \boldsymbol{\Sigma}_\epsilon &= \sigma_\epsilon^2 \mathbf{I}_{n_t} \end{aligned}$$

### 1.3 Parameter Model

$$\begin{aligned} \boldsymbol{\mu}_\beta &\sim N(\boldsymbol{\mu}_0, \boldsymbol{\Sigma}_0) \\ \sigma_\beta^2 &\sim IG(\alpha_\beta, \beta_\beta) \\ \sigma_\eta^2 &\sim IG(\alpha_\eta, \beta_\eta) \\ \sigma_\epsilon^2 &\sim IG(\alpha_\epsilon, \beta_\epsilon) \\ \phi &\sim IG(\alpha_\phi, \beta_\phi) \end{aligned}$$

where  $\mathbf{I}_{n_t}$  is the identity matrix of size  $n_t \times n_t$  where  $\tau$  is the number of parameters in  $\boldsymbol{\beta}_t$  and  $n_t$  is the number of samples of  $\mathbf{y}_t$  at time  $t$  and  $\mathbf{D}_t$  is the distance matrix between locations observed at time  $t$ . Define  $\boldsymbol{\Sigma} = \boldsymbol{\Sigma}_\eta + \boldsymbol{\Sigma}_\epsilon$

## 2 Posterior

$$\prod_{t=1}^T [\boldsymbol{\beta}_t, \boldsymbol{\mu}_\beta, \sigma_\beta^2, \sigma_\eta^2, \sigma_\epsilon^2, \phi | \mathbf{y}_t] \propto \prod_{t=1}^T [\mathbf{y}_t | \boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2] [\boldsymbol{\beta}_t | \boldsymbol{\mu}_\beta, \sigma_\beta^2] [\boldsymbol{\mu}_\beta] [\sigma_\beta^2] [\sigma_\eta^2] [\sigma_\epsilon^2] [\phi]$$

### 3 Full Conditionals

#### 3.1 Full Conditional for $\beta_t$

For  $t = 1, \dots, T$ ,

$$\begin{aligned} [\beta_t | \cdot] &\propto [\mathbf{y}_t | \beta_t, \sigma_\eta^2, \sigma_\epsilon^2, \phi] [\beta_t | \boldsymbol{\mu}_\beta, \sigma_\beta^2] \\ &\propto e^{-\frac{1}{2}(\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)^T \boldsymbol{\Sigma}^{-1}(\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)} e^{-\frac{1}{2}(\beta_t - \boldsymbol{\mu}_\beta)^T \boldsymbol{\Sigma}_\beta^{-1}(\beta_t - \boldsymbol{\mu}_\beta)} \\ &\propto e^{-\frac{1}{2}\{\beta_t^T (\mathbf{X}^T \mathbf{H}_t^T \boldsymbol{\Sigma}^{-1} \mathbf{H}_t \mathbf{X} + \boldsymbol{\Sigma}_\beta^{-1}) \beta_t - 2\beta_t^T (\mathbf{X}^T \mathbf{H}_t^T \boldsymbol{\Sigma}^{-1} \mathbf{y}_t + \boldsymbol{\Sigma}_\beta^{-1} \boldsymbol{\mu}_\beta)\}} \end{aligned}$$

which is Normal with mean  $\mathbf{A}^{-1} \mathbf{b}$  and variance  $\mathbf{A}^{-1}$  where

$$\begin{aligned} \mathbf{A}^{-1} &= (\mathbf{X}^T \mathbf{H}_t^T \boldsymbol{\Sigma}^{-1} \mathbf{H}_t \mathbf{X} + \boldsymbol{\Sigma}_\beta^{-1})^{-1} \\ \mathbf{b} &= (\mathbf{X}^T \mathbf{H}_t^T \boldsymbol{\Sigma}^{-1} \mathbf{y}_t + \boldsymbol{\Sigma}_\beta^{-1} \boldsymbol{\mu}_\beta) \end{aligned}$$

#### 3.2 Full Conditional for $\boldsymbol{\mu}_\beta$

$$\begin{aligned} [\boldsymbol{\mu}_\beta | \cdot] &\propto \prod_{t=1}^T [\beta_t | \boldsymbol{\mu}_\beta, \sigma_\beta^2] [\boldsymbol{\mu}_\beta] \\ &\propto e^{-\frac{1}{2} \sum_{t=1}^T (\beta_t - \boldsymbol{\mu}_\beta)^T \boldsymbol{\Sigma}_\beta^{-1} (\beta_t - \boldsymbol{\mu}_\beta)} e^{-\frac{1}{2} (\boldsymbol{\mu}_\beta - \boldsymbol{\mu}_0)^T \boldsymbol{\Sigma}_0^{-1} (\boldsymbol{\mu}_\beta - \boldsymbol{\mu}_0)} \\ &\propto e^{-\frac{1}{2} (\boldsymbol{\mu}_\beta^T (T \boldsymbol{\Sigma}_\beta^{-1} + \boldsymbol{\Sigma}_0^{-1}) \boldsymbol{\mu}_\beta - 2 \boldsymbol{\mu}_\beta^T (\sum_{t=1}^T \boldsymbol{\Sigma}_\beta^{-1} \beta_t + \boldsymbol{\Sigma}_0^{-1} \boldsymbol{\mu}_0))} \end{aligned}$$

which is multivariate normal with mean  $(T \boldsymbol{\Sigma}_\beta^{-1} + \boldsymbol{\Sigma}_0^{-1})^{-1} (\sum_{t=1}^T \boldsymbol{\Sigma}_\beta^{-1} \beta_t + \boldsymbol{\Sigma}_0^{-1} \boldsymbol{\mu}_0)$  and variance  $(T \boldsymbol{\Sigma}_\beta^{-1} + \boldsymbol{\Sigma}_0^{-1})^{-1}$

#### 3.3 Full Conditional for $\sigma_\beta^2$

$$\begin{aligned} [\sigma_\beta^2 | \cdot] &\propto \prod_{t=1}^T [\beta_t | \boldsymbol{\mu}_\beta, \sigma_\beta^2] [\sigma_\beta^2] \\ &\propto \left( \prod_{t=1}^T |\boldsymbol{\Sigma}_\beta|^{-\frac{1}{2}} \right) e^{-\frac{1}{2} \sum_{t=1}^T (\beta_t - \boldsymbol{\mu}_\beta)^T \boldsymbol{\Sigma}_\beta^{-1} (\beta_t - \boldsymbol{\mu}_\beta)} (\sigma_\beta^2)^{-(\alpha_\beta + 1)} e^{-\frac{\beta_\beta}{\sigma_\beta^2}} \\ &\propto (\sigma_\beta^2)^{-(\alpha_\beta + \frac{\sum_{t=1}^T n_t}{2} + 1)} e^{-\frac{1}{\sigma_\beta^2} (\frac{1}{2} \sum_{t=1}^T (\beta_t - \boldsymbol{\mu}_\beta)^T (\beta_t - \boldsymbol{\mu}_\beta) + \beta_\beta)} \end{aligned}$$

which is  $\text{IG}(\alpha_\beta + \frac{\sum_{t=1}^T n_t}{2}, \frac{1}{2} \sum_{t=1}^T (\beta_t - \boldsymbol{\mu}_\beta)^T (\beta_t - \boldsymbol{\mu}_\beta) + \beta_\beta)$  since the determinant  $|\boldsymbol{\Sigma}_\beta| = (\sigma_\beta^2)^{n_t}$  and  $\boldsymbol{\Sigma}_\beta^{-1} = \frac{1}{\sigma_\beta^2} \mathbf{I}_t$

#### 3.4 Full Conditional for $\sigma_\eta^2$

$$\begin{aligned} [\sigma_\eta^2 | \cdot] &\propto \prod_{t=1}^T [\mathbf{y}_t | \beta_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2] [\sigma_\eta^2] \\ &\propto \left( \prod_{t=1}^T |\boldsymbol{\Sigma}|^{-\frac{1}{2}} \right) e^{-\frac{1}{2} \sum_{t=1}^T (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)^T \boldsymbol{\Sigma}^{-1} (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)} (\sigma_\eta^2)^{-\alpha_\eta + 1} e^{-\frac{\beta_\eta}{\sigma_\eta^2}} \\ &\propto |\boldsymbol{\Sigma}|^{-\frac{T}{2}} e^{-\frac{1}{2} \sum_{t=1}^T (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)^T \boldsymbol{\Sigma}^{-1} (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \beta_t)} (\sigma_\eta^2)^{-\alpha_\eta + 1} e^{-\frac{\beta_\eta}{\sigma_\eta^2}} \end{aligned}$$

which can be sampled using a Metropolis-Hastings step

### 3.5 Full Conditional for $\sigma_\epsilon^2$

$$\begin{aligned}
[\sigma_\epsilon^2 | \cdot] &\propto \prod_{t=1}^T [\mathbf{y}_t | \boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2] [\sigma_\epsilon^2] \\
&\propto \left( \prod_{t=1}^T |\boldsymbol{\Sigma}|^{-\frac{1}{2}} \right) e^{-\frac{1}{2} \sum_{t=1}^T (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \boldsymbol{\beta}_t)^T \boldsymbol{\Sigma}^{-1} (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \boldsymbol{\beta}_t)} (\sigma_\epsilon^2)^{-\alpha_\epsilon + 1} e^{-\frac{\beta_\epsilon}{\sigma_\epsilon^2}} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{T}{2}} e^{-\frac{1}{2} \sum_{t=1}^T (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \boldsymbol{\beta}_t)^T \boldsymbol{\Sigma}^{-1} (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \boldsymbol{\beta}_t)} (\sigma_\epsilon^2)^{-\alpha_\epsilon + 1} e^{-\frac{\beta_\epsilon}{\sigma_\epsilon^2}}
\end{aligned}$$

which can be sampled using a Metropolis-Hastings step

### 3.6 Full Conditional for $\phi$

$$\begin{aligned}
[\phi | \cdot] &\propto \prod_{t=1}^T [y_t | \beta_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2] [\phi] \\
&\propto \prod_{t=1}^T |\boldsymbol{\Sigma}|^{-\frac{1}{2}} e^{-\frac{1}{2} (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \boldsymbol{\beta}_t)^T \boldsymbol{\Sigma}^{-1} (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \boldsymbol{\beta}_t)} \phi^{-\alpha_\phi + 1} e^{-\frac{\beta_\phi}{\phi}} \\
&\propto |\boldsymbol{\Sigma}|^{-\frac{T}{2}} e^{-\frac{1}{2} \sum_{t=1}^T (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \boldsymbol{\beta}_t)^T \boldsymbol{\Sigma}^{-1} (\mathbf{y}_t - \mathbf{H}_t \mathbf{X} \boldsymbol{\beta}_t)} \phi^{-\alpha_\phi + 1} e^{-\frac{\beta_\phi}{\phi}}
\end{aligned}$$

which can be sampled using a Metropolis-Hastings step

## 4 Posterior Predictive Distribution

The posterior predictive distribution for  $\mathbf{y}_t$  is sampled at each MCMC iteration  $k$  by

$$\mathbf{y}_t^{(k)} \sim N(\mathbf{H}_t \mathbf{X} \boldsymbol{\beta}_t^{(k)}, \boldsymbol{\Sigma}^{(k)})$$