

# 1 The Model

## 1.1 Data Model

$$[y_t | \boldsymbol{\beta}, \boldsymbol{\gamma}, \sigma_\epsilon^2] \sim N(\mathbf{H}_t \mathbf{X}_t \boldsymbol{\beta} \boldsymbol{\gamma}, \sigma_\epsilon^2)$$

$$\text{where } \boldsymbol{\beta} \boldsymbol{\gamma} = \begin{bmatrix} \beta_0 \\ \beta_1 \gamma_1 \\ \vdots \\ \beta_p \gamma_p \end{bmatrix} \text{ and } \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix}.$$

## 1.2 Process Model - climate variable of interest is AR(1) in time

$$[y_t - \mu | y_{t-1}, \alpha, \sigma_\eta^2] \sim N(\alpha(y_{t-1} - \mu), \sigma_\eta^2)$$

where  $\mu$  is an intercept term, assumed to be known for the PDSI drought index ( $\mu = 0$ )

## 1.3 Parameter Model

$$\begin{aligned} [\beta_0] &\propto 1 \\ [\beta_j | \sigma_\epsilon^2, \gamma_j] &\stackrel{iid}{\sim} N\left(0, \frac{\sigma_\epsilon^2 \gamma_j}{\lambda_j}\right) && \text{for } j = 1, \dots, p \\ [\sigma_\epsilon^2] &\propto \frac{1}{\sigma_\epsilon^2} \\ [\sigma_\eta^2] &\propto \frac{1}{\sigma_\eta^2} \\ [\gamma_j] &\propto \text{Bern}(\pi_j) && \text{for } j = 1, \dots, p \\ [\alpha] &\propto \text{Unif}(0, 1) \end{aligned}$$

where  $\pi_j$  and  $\lambda_j$  are fixed hyperpriors for  $j = 1, \dots, p$ .

## 1.4 Posterior

$$[\boldsymbol{\beta} \boldsymbol{\gamma}, \boldsymbol{\gamma}, \sigma_\epsilon^2, \sigma_\eta^2, \alpha | \mathbf{y}] = \left( \prod_{t=1}^T [y_t | \boldsymbol{\beta}, \boldsymbol{\gamma}, \sigma_\epsilon^2] \right) \left( \prod_{t=2}^T [y_t - \mu | y_{t-1}, \alpha, \sigma_\eta^2] \right) [\boldsymbol{\beta} | \sigma_\epsilon^2, \boldsymbol{\gamma}] [\sigma_\epsilon^2] [\sigma_\eta^2] \left( \prod_{j=1}^p [\gamma_j] \right) [\alpha]$$

# 2 Full Conditionals

## 2.1 Full Conditional for unobserved $y_t$

For  $t = 1$ ,

$$\begin{aligned} [y_1 | \cdot] &\propto [y_1 | \boldsymbol{\beta}, \boldsymbol{\gamma}, \sigma_\epsilon^2] [y_2 - \mu | y_1, \alpha, \sigma_\eta^2] \\ &\propto \exp \left\{ -\frac{1}{2\sigma_\epsilon^2} \left( y_1 - \mathbf{H}_1 \boldsymbol{\gamma} \mathbf{X}_1 \boldsymbol{\beta} \boldsymbol{\gamma} \right)^2 \right\} \exp \left\{ -\frac{1}{2\sigma_\eta^2} ((y_2 - \mu) - \alpha(y_1 - \mu))^2 \right\} \\ &\propto \exp \left\{ -\frac{1}{2} \left( y_1^2 \left( \frac{1}{\sigma_\epsilon^2} + \frac{\alpha^2}{\sigma_\eta^2} \right) - 2y_1 \left( \frac{\mathbf{H}_1 \boldsymbol{\gamma} \mathbf{X}_1 \boldsymbol{\beta} \boldsymbol{\gamma}}{\sigma_\epsilon^2} + \frac{\alpha(y_2 - (1 - \alpha)\mu)}{\sigma_\eta^2} \right) \right) \right\} \end{aligned}$$

which is  $N(A^{-1}b, A^{-1})$  where  $A^{-1} = \left(\frac{1}{\sigma_\epsilon^2} + \frac{\alpha^2}{\sigma_\eta^2}\right)^{-1}$  and  $b = \frac{\mathbf{H}_1\boldsymbol{\gamma}\mathbf{X}_1\boldsymbol{\gamma}\boldsymbol{\beta}\boldsymbol{\gamma}}{\sigma_\epsilon^2} + \frac{\alpha(y_2 - (1-\alpha)\mu)}{\sigma_\eta^2}$

For  $t = 2, \dots, T-1$ ,

$$\begin{aligned} [y_t|\cdot] &\propto [y_t|\boldsymbol{\beta}, \boldsymbol{\gamma}, \sigma_\epsilon^2] [y_t - \mu|y_{t-1}, \alpha, \sigma_\eta^2] [y_{t+1} - \mu|y_t, \alpha, \sigma_\eta^2] \\ &\propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2} \left(y_t - \mathbf{H}_t\boldsymbol{\gamma}\mathbf{X}_t\boldsymbol{\gamma}\boldsymbol{\beta}\boldsymbol{\gamma}\right)^2\right\} \exp\left\{-\frac{1}{2\sigma_\eta^2} ((y_t - \mu) - \alpha(y_{t-1} - \mu))^2\right\} \exp\left\{-\frac{1}{2\sigma_\eta^2} ((y_{t+1} - \mu) - \alpha(y_t - \mu))^2\right\} \\ &\propto \exp\left\{-\frac{1}{2} \left(y_t^2 \left(\frac{1}{\sigma_\epsilon^2} + \frac{\alpha^2 + 1}{\sigma_\eta^2}\right) - 2y_{t-1} \left(\frac{\mathbf{H}_t\boldsymbol{\gamma}\mathbf{X}_t\boldsymbol{\gamma}\boldsymbol{\beta}\boldsymbol{\gamma}}{\sigma_\epsilon^2} + \frac{\alpha y_{t-1} + (1-\alpha)\mu}{\sigma_\eta^2} + \frac{\alpha(y_{t+1} - (1-\alpha)\mu)}{\sigma_\eta^2}\right)\right)\right\} \end{aligned}$$

which is  $N(A^{-1}b, A^{-1})$  where  $A^{-1} = \left(\frac{1}{\sigma_\epsilon^2} + \frac{\alpha^2 + 1}{\sigma_\eta^2}\right)^{-1}$  and  $b = \frac{\mathbf{H}_t\boldsymbol{\gamma}\mathbf{X}_t\boldsymbol{\gamma}\boldsymbol{\beta}\boldsymbol{\gamma}}{\sigma_\epsilon^2} + \frac{\alpha y_{t-1} + (1-\alpha)\mu}{\sigma_\eta^2} + \frac{\alpha(y_{t+1} - (1-\alpha)\mu)}{\sigma_\eta^2}$ .

For  $t = T$ ,

$$\begin{aligned} [y_T|\cdot] &\propto [y_T|\boldsymbol{\beta}, \boldsymbol{\gamma}, \sigma_\epsilon^2] [y_T - \mu|y_{T-1}, \alpha, \sigma_\eta^2] \\ &\propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2} \left(y_T - \mathbf{H}_T\boldsymbol{\gamma}\mathbf{X}_T\boldsymbol{\gamma}\boldsymbol{\beta}\boldsymbol{\gamma}\right)^2\right\} \exp\left\{-\frac{1}{2\sigma_\eta^2} ((y_T - \mu) - \alpha(y_{T-1} - \mu))^2\right\} \\ &\propto \exp\left\{-\frac{1}{2} \left(y_T^2 \left(\frac{1}{\sigma_\epsilon^2} + \frac{1}{\sigma_\eta^2}\right) - 2y_{T-1} \left(\frac{\mathbf{H}_T\boldsymbol{\gamma}\mathbf{X}_T\boldsymbol{\gamma}\boldsymbol{\beta}\boldsymbol{\gamma}}{\sigma_\epsilon^2} + \frac{\alpha y_{T-1} + (1-\alpha)\mu}{\sigma_\eta^2}\right)\right)\right\} \end{aligned}$$

which is  $N(A^{-1}b, A^{-1})$  where  $A^{-1} = \left(\frac{1}{\sigma_\epsilon^2} + \frac{1}{\sigma_\eta^2}\right)^{-1}$  and  $b = \frac{\mathbf{H}_T\boldsymbol{\gamma}\mathbf{X}_T\boldsymbol{\gamma}\boldsymbol{\beta}\boldsymbol{\gamma}}{\sigma_\epsilon^2} + \frac{\alpha y_{T-1} + (1-\alpha)\mu}{\sigma_\eta^2}$ .

## 2.2 Full Conditional for $\boldsymbol{\beta}$

$$\begin{aligned} [\boldsymbol{\beta}|\cdot] &\propto \prod_{t=1}^T [y_t|\boldsymbol{\beta}, \boldsymbol{\gamma}, \sigma_\epsilon^2] [\boldsymbol{\beta}|\boldsymbol{\gamma}, \sigma_\eta^2] \\ &\propto \prod_{t=1}^T \exp\left\{-\frac{1}{2\sigma_\epsilon^2} \left(y_t - \mathbf{H}_t\boldsymbol{\gamma}\mathbf{X}_t\boldsymbol{\gamma}\boldsymbol{\beta}\boldsymbol{\gamma}\right)^2\right\} \exp\left\{-\frac{1}{2\sigma_\epsilon^2} \boldsymbol{\beta}^T \boldsymbol{\Lambda} \boldsymbol{\gamma} \boldsymbol{\beta} \boldsymbol{\gamma}\right\} \\ &\propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2} \left[ \boldsymbol{\beta}^T \boldsymbol{\gamma} \left( \sum_{t=1}^T \mathbf{X}_t^T \mathbf{H}_t \boldsymbol{\gamma} \mathbf{H}_t \boldsymbol{\gamma} \mathbf{X}_t \boldsymbol{\gamma} + \boldsymbol{\Lambda} \boldsymbol{\gamma} \right) \boldsymbol{\beta} \boldsymbol{\gamma} - 2 \boldsymbol{\beta}^T \boldsymbol{\gamma} \left( \sum_{t=1}^T \mathbf{X}_t^T \mathbf{H}_t \boldsymbol{\gamma} y_t \right) \right] \right\} \end{aligned}$$

which is MVN  $(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1})$  where  $\mathbf{A}^{-1} = \left(\frac{\mathbf{X}_t^T \mathbf{H}_t \boldsymbol{\gamma} \mathbf{H}_t \boldsymbol{\gamma} \mathbf{X}_t \boldsymbol{\gamma} + \boldsymbol{\Lambda} \boldsymbol{\gamma}}{\sigma_\epsilon^2}\right)^{-1}$  and  $\mathbf{b} = \frac{\mathbf{X}_t^T \mathbf{H}_t \boldsymbol{\gamma} y_t}{\sigma_\epsilon^2}$

### 2.3 Full Conditional for $\sigma_\epsilon^2$

$$\begin{aligned}
[\sigma_\epsilon^2 | \cdot] &\propto \prod_{t=1}^T [y_t | \beta, \gamma, \sigma_\epsilon^2] [\beta | \gamma, \sigma_\epsilon^2] [\sigma_\epsilon^2] \\
&\propto \prod_{t=1}^T (\sigma_\epsilon^2)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{\sigma_\epsilon^2} \frac{(y_t - \mathbf{H}_t \gamma \mathbf{X}_t \gamma \beta_\gamma)^2}{2} \right\} |\sigma_\epsilon^2 \mathbf{\Lambda} \gamma|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{\sigma_\epsilon^2} \frac{\beta_\gamma^T \mathbf{\Lambda} \gamma \beta_\gamma}{2} \right\} (\sigma_\epsilon^2)^{-1} \\
&\propto (\sigma_\epsilon^2)^{-\frac{T+p\gamma}{2}-1} \exp \left\{ -\frac{1}{\sigma_\epsilon^2} \frac{\sum_{t=1}^T (y_t - \mathbf{H}_t \gamma \mathbf{X}_t \gamma \beta_\gamma)^2 + \beta_\gamma^T \mathbf{\Lambda} \gamma \beta_\gamma}{2} \right\}
\end{aligned}$$

which is IG  $\left( \frac{T+p\gamma}{2}, \frac{\sum_{t=1}^T (y_t - \mathbf{H}_t \gamma \mathbf{X}_t \gamma \beta_\gamma)^2 + \beta_\gamma^T \mathbf{\Lambda} \gamma \beta_\gamma}{2} \right)$

### 2.4 Full Conditional for $\sigma_\eta^2$

$$\begin{aligned}
[\sigma_\eta^2 | \cdot] &\propto \prod_{t=2}^T [y_t | y_{t-1}, \alpha, \sigma_\eta^2] [\sigma_\eta^2] \\
&\propto \prod_{t=2}^T (\sigma_\eta^2)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{\sigma_\eta^2} \frac{((y_t - \mu) - \alpha(y_{t-1} - \mu))^2}{2} \right\} (\sigma_\eta^2)^{-1} \\
&\propto (\sigma_\eta^2)^{-\frac{T-1}{2}-1} \exp \left\{ -\frac{1}{\sigma_\eta^2} \frac{\sum_{t=2}^T ((y_t - \mu) - \alpha(y_{t-1} - \mu))^2}{2} \right\}
\end{aligned}$$

which is IG  $\left( \frac{T-1}{2}, \frac{\sum_{t=2}^T ((y_t - \mu) - \alpha(y_{t-1} - \mu))^2}{2} \right)$

### 2.5 Full Conditional for $\alpha$

$$\begin{aligned}
[\alpha | \cdot] &\propto \prod_{t=2}^T [y_t | y_{t-1}, \alpha, \sigma_\eta^2] [\alpha] \\
&\propto \prod_{t=2}^T \exp \left\{ -\frac{1}{2\sigma_\eta^2} ((y_t - \mu) - \alpha(y_{t-1} - \mu))^2 \right\} \mathbf{I} \{ \alpha \in (0, 1) \} \\
&\propto \exp \left\{ -\frac{1}{2} \left( \alpha^2 \frac{\sum_{t=2}^T (y_{t-1} - \mu)^2}{\sigma_\eta^2} - 2\alpha \frac{\sum_{t=2}^T (y_{t-1} - \mu)(y_t - \mu)}{\sigma_\eta^2} \right) \right\} \mathbf{I} \{ \alpha \in (0, 1) \}
\end{aligned}$$

which is Truncated Normal  $(A^{-1}b, A^{-1})$  where  $A^{-1} = \left( \frac{\sum_{t=2}^T (y_{t-1} - \mu)^2}{\sigma_\eta^2} \right)^{-1}$  and  $b = \frac{\sum_{t=2}^T (y_{t-1} - \mu)(y_t - \mu)}{\sigma_\eta^2}$  restricted to the interval  $(0, 1)$ .

## 2.6 Full Conditional for $\gamma_j$

$$\begin{aligned}
[\gamma|\cdot] &\propto \prod_{t=1}^T [y_t|\beta, \gamma, \sigma_\epsilon^2] [\gamma] \\
&\propto \prod_{t=1}^T (\sigma^2)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2\sigma_\epsilon^2} \left( y_t - \mathbf{H}_{t\gamma} \mathbf{X}_{t\gamma} \beta_\gamma \right)^2 \right\} |\sigma_\epsilon^2 \mathbf{\Lambda}_\gamma|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2\sigma_\epsilon^2} \beta_\gamma^T \mathbf{\Lambda}_\gamma \beta_\gamma \right\} \left( \prod_{j=1}^p \pi_j^{\gamma_j} (1 - \pi_j)^{(1-\gamma_j)} \right) \\
&\propto (\sigma^2)^{-\frac{T+p}{2}} |\mathbf{\Lambda}_\gamma|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2\sigma_\epsilon^2} \left( y_t - \mathbf{H}_{t\gamma} \mathbf{X}_{t\gamma} \beta_\gamma \right)^2 \right\} \exp \left\{ -\frac{1}{2\sigma_\epsilon^2} \beta_\gamma^T \mathbf{\Lambda}_\gamma \beta_\gamma \right\} \left( \prod_{j=1}^p \pi_j^{\gamma_j} (1 - \pi_j)^{(1-\gamma_j)} \right)
\end{aligned}$$

which can be sampled using a Metropolis Hastings proposal. This requires a smart choice of proposal distribution.

If  $H_t X_t$  has orthogonal columns (e.g  $X_t^T H_t^T H_t X_t$  is diagonal) then **ADD IN A SUM OVER ALL  $\mathbf{T}$**

$$\begin{aligned}
[\gamma_j|\cdot] &\propto \prod_{t=1}^T [y_t, \beta_j, \gamma_j, \sigma_\epsilon^2] [\beta_j|\gamma_j, \sigma^2] [\gamma_j] \\
&\propto \exp \left\{ -\frac{1}{2\sigma_\epsilon^2} (y_t - \mathbf{H}_{tj} \mathbf{X}_{tj} \gamma_j \beta_j)^2 \right\} \left( \frac{\sigma_\epsilon^2}{\lambda_j} \right)^{-\frac{\gamma_j}{2}} \exp \left\{ -\frac{\gamma_j \lambda_j \beta_j^2}{2\sigma^2} \right\} \pi^{\gamma_j} (1 - \pi)^{1-\gamma_j} \\
&\propto \left( \frac{\sigma_\epsilon^2}{\lambda_j} \right)^{-\frac{\gamma_j}{2}} \exp \left\{ -\frac{1}{2\sigma_\epsilon^2} \gamma_j \left( \beta_j^2 \left( \mathbf{X}_{tj}^T \mathbf{H}_{tj}^T \mathbf{H}_{tj} \mathbf{X}_{tj} \lambda_j \right) - 2y_t \mathbf{H}_{tj} \mathbf{X}_{tj} \beta_j \right) \right\} \pi^{\gamma_j} (1 - \pi)^{1-\gamma_j} \\
&\propto \left( \left( \frac{\sigma_\epsilon^2}{\lambda_j} \right)^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2\sigma_\epsilon^2} \left( \beta_j^2 \left( \mathbf{X}_{tj}^T \mathbf{H}_{tj}^T \mathbf{H}_{tj} \mathbf{X}_{tj} \lambda_j \right) - 2y_t \mathbf{H}_{tj} \mathbf{X}_{tj} \beta_j \right) \right\} \frac{\pi_j}{1 - \pi_j} \right)^{\gamma_j} \\
&\propto \Psi^{\gamma_j}
\end{aligned}$$

so  $\gamma_j$  is Bern $\left(\frac{\Psi_j}{1+\Psi_j}\right)$  where  $\Psi_j = \left(\frac{\lambda_j}{\sigma^2}\right)^{\frac{1}{2}} \exp \left\{ -\frac{1}{2\sigma^2} [\beta_j^2 (\delta_j + \lambda_j) - 2\delta_j \beta_j^2] \right\} \frac{\pi}{1-\pi}$