1 The Model - starting with only one year...

1.1 Data Model

$$\left[\boldsymbol{y}_o \middle| \boldsymbol{\beta}_{\boldsymbol{\gamma}}, \boldsymbol{\gamma}, \sigma^2\right] \sim \mathrm{N}\left(\boldsymbol{H}_o \boldsymbol{X} \boldsymbol{\beta}_{\boldsymbol{\gamma}}, \sigma^2 \boldsymbol{I}_o\right)$$

1.2 Process Model - Unobserved data - needed for the model selection step

$$\left[\boldsymbol{y}_u \middle| \boldsymbol{\beta}_{\boldsymbol{\gamma}}, \boldsymbol{\gamma}, \sigma^2\right] \sim \mathrm{N}\left(\boldsymbol{H}_u \boldsymbol{X} \boldsymbol{\beta}_{\boldsymbol{\gamma}}, \sigma^2 \boldsymbol{I}_u\right)$$

where $\boldsymbol{y}_c \equiv (\boldsymbol{y}_o, \boldsymbol{y}_u)^T$.

1.3 Parameter Model

$$\begin{aligned}
\left[\beta_j | \sigma^2, \gamma_j\right] &\overset{iid}{\sim} \begin{cases} 0 & \text{if } \gamma_j = 0 \\ \mathrm{N}\left(0, \frac{\sigma^2}{\lambda_j}\right) & \text{if } \gamma_j = 1 \end{cases} && \text{for } j = 1, \ldots, p \\
\left[\sigma^2\right] &\propto \frac{1}{\sigma^2} \\
\left[\gamma_j\right] &\propto \mathrm{Bern}\left(\pi_j\right) && \text{for } j = 1, \ldots, p
\end{aligned}$$

where π_j and λ_j are fixed hyperpriors for $j = 1, \ldots, p$.

1.4 Posterior

$$\begin{aligned}
\left[\boldsymbol{\beta}_{\boldsymbol{\gamma}}, \boldsymbol{\gamma}, \sigma^2 \middle| \boldsymbol{y}_o\right] &= \int \left[\boldsymbol{y}_u, \boldsymbol{\beta}_{\boldsymbol{\gamma}}, \boldsymbol{\gamma}, \sigma^2 \middle| \boldsymbol{y}_o\right] \, d\boldsymbol{y}_u \\
&= \int \left[\boldsymbol{y}_o \middle| \boldsymbol{\beta}_{\boldsymbol{\gamma}}, \boldsymbol{\gamma}, \sigma^2\right] \left[\boldsymbol{y}_u \middle| \boldsymbol{\beta}_{\boldsymbol{\gamma}}, \boldsymbol{\gamma}, \sigma^2\right] \prod_{j=1}^{p} \left[\beta_j | \gamma_j, \sigma^2\right] \left[\gamma_j\right] \left[\sigma^2\right] \, d\boldsymbol{y}_u
\end{aligned}$$

2 Full Conditionals

2.1 Full Conditional for $\boldsymbol{y}_u$

$$\left[\boldsymbol{y}_u | \cdot\right] \propto \left[\boldsymbol{y}_u \middle| \boldsymbol{\beta}_{\boldsymbol{\gamma}}, \boldsymbol{\gamma}, \sigma^2\right]$$

which is $\mathrm{N}\left(\boldsymbol{H}_u \boldsymbol{X} \boldsymbol{\beta}_{\boldsymbol{\gamma}}, \sigma^2 \boldsymbol{I}_u\right)$

2.2 Full Conditional for $\boldsymbol{\beta}_{\boldsymbol{\gamma}}$

$$\begin{aligned}
\left[\boldsymbol{\beta}_{\boldsymbol{\gamma}} \middle| \cdot\right] &\propto \left[\boldsymbol{y}_o \middle| \boldsymbol{\beta}_{\boldsymbol{\gamma}}, \boldsymbol{\gamma}, \sigma^2\right] \left[\boldsymbol{\beta}_{\boldsymbol{\gamma}} \middle| \boldsymbol{\gamma}, \sigma^2\right] \\
&\propto \exp\left\{-\frac{1}{2\sigma^2} \left(\boldsymbol{y}_o - \boldsymbol{H}_o \boldsymbol{X}_{o\gamma} \boldsymbol{\beta}_{\boldsymbol{\gamma}}\right)^T \left(\boldsymbol{y}_o - \boldsymbol{H}_o \boldsymbol{X}_{o\gamma} \boldsymbol{\beta}_{\boldsymbol{\gamma}}\right)\right\} \exp\left\{-\frac{1}{2\sigma^2} \boldsymbol{\beta}_{\boldsymbol{\gamma}}^T \boldsymbol{\Delta}_{\boldsymbol{\gamma}} \boldsymbol{\beta}_{\boldsymbol{\gamma}}\right\} \\
&\propto \exp\left\{-\frac{1}{2\sigma^2} \left[\boldsymbol{\beta}_{\boldsymbol{\gamma}}^T \left(\boldsymbol{X}_{o\gamma}^T \boldsymbol{H}_o^T \boldsymbol{H}_o \boldsymbol{X}_{o\gamma} + \boldsymbol{\Delta}_{\boldsymbol{\gamma}}\right) \boldsymbol{\beta}_{\boldsymbol{\gamma}} - 2 \boldsymbol{\beta}_{\boldsymbol{\gamma}}^T \left(\boldsymbol{X}_{o\gamma}^T \boldsymbol{H}_o^T \boldsymbol{y}_o\right)\right]\right\}
\end{aligned}$$

which is MVN $\left(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1}\right)$ where $\boldsymbol{A}^{-1} = \left(\boldsymbol{X}_{o\boldsymbol{\gamma}}^T \boldsymbol{H}_o^T \boldsymbol{H}_o \boldsymbol{X}_{o\boldsymbol{\gamma}} + \boldsymbol{\Delta}_{\boldsymbol{\gamma}}\right)^{-1}$ and $\boldsymbol{b} = \boldsymbol{X}_{o\boldsymbol{\gamma}}^T \boldsymbol{H}_o^T \boldsymbol{y}_o$

2.3 Full Conditional for σ^2

$$
\begin{aligned}
\left[\sigma^2 \middle| \cdot\right] &\propto \left[\boldsymbol{y}_o, \middle| \boldsymbol{\beta}_{\boldsymbol{\gamma}}, \boldsymbol{\gamma}, \sigma^2\right] \left[\boldsymbol{\beta}_{\boldsymbol{\gamma}} \middle| \gamma, \sigma^2\right] \left[\sigma^2\right] \\
&\propto \left(\sigma^2\right)^{-\frac{n_o}{2}} \exp\left\{-\frac{1}{\sigma^2} \frac{\left(\boldsymbol{y}_o - \boldsymbol{H}_o \boldsymbol{X}_{o\boldsymbol{\gamma}} \boldsymbol{\beta}_{\boldsymbol{\gamma}}\right)^T \left(\boldsymbol{y}_o - \boldsymbol{H}_o \boldsymbol{X}_{o\boldsymbol{\gamma}} \boldsymbol{\beta}_{\boldsymbol{\gamma}}\right)}{2}\right\} \left(\sigma^2\right)^{-\frac{p_{\boldsymbol{\gamma}}}{2}} \exp\left\{-\frac{1}{\sigma^2} \frac{\boldsymbol{\beta}_{\boldsymbol{\gamma}}^T \boldsymbol{\Lambda}_{\boldsymbol{\gamma}} \boldsymbol{\beta}_{\boldsymbol{\gamma}}}{2}\right\} \left(\sigma^2\right)^{-1} \\
&\propto \left(\sigma^2\right)^{-\frac{n_o + p_{\boldsymbol{\gamma}}}{2} - 1} \exp\left\{-\frac{1}{\sigma^2} \frac{\left(\boldsymbol{y}_o - \boldsymbol{H}_o \boldsymbol{X}_{o\boldsymbol{\gamma}} \boldsymbol{\beta}_{\boldsymbol{\gamma}}\right)^T \left(\boldsymbol{y}_o - \boldsymbol{H}_o \boldsymbol{X}_{o\boldsymbol{\gamma}} \boldsymbol{\beta}_{\boldsymbol{\gamma}}\right) + \boldsymbol{\beta}_{\boldsymbol{\gamma}}^T \boldsymbol{\Lambda}_{\boldsymbol{\gamma}} \boldsymbol{\beta}_{\boldsymbol{\gamma}}}{2}\right\}
\end{aligned}
$$

which is IG $\left(\frac{n_o + p_{\boldsymbol{\gamma}}}{2}, \frac{\left(\boldsymbol{y}_o - \boldsymbol{H}_o \boldsymbol{X}_{o\boldsymbol{\gamma}} \boldsymbol{\beta}_{\boldsymbol{\gamma}}\right)^T \left(\boldsymbol{y}_o - \boldsymbol{H}_o \boldsymbol{X}_{o\boldsymbol{\gamma}} \boldsymbol{\beta}_{\boldsymbol{\gamma}}\right) + \boldsymbol{\beta}_{\boldsymbol{\gamma}}^T \boldsymbol{\Lambda}_{\boldsymbol{\gamma}} \boldsymbol{\beta}_{\boldsymbol{\gamma}}}{2}\right)$

2.4 Full Conditional for γ_j

For $j = 1, \ldots, p$ and using the fact that $\boldsymbol{X}_i^T \boldsymbol{X}_j = 0$ (by orthogonality of principal components), $\hat{\beta}_j = \left(\left(\boldsymbol{X}^T \boldsymbol{X}\right)^{-1} \boldsymbol{X}^T \boldsymbol{y}_c\right)_j$ is the j^{th} element of the vector $\hat{\boldsymbol{\beta}}$ and $\left(\boldsymbol{X}^T \boldsymbol{X}\right)_j = \delta_j$is the j^{th} element of the diagonal matrix.

$$
\begin{aligned}
\left[\gamma_j \middle| \cdot\right] &\propto \left[\boldsymbol{y}_c, \middle| \boldsymbol{\beta}_{\boldsymbol{\gamma}}, \boldsymbol{\gamma}, \sigma^2\right] \left[\beta_j \middle| \gamma_j, \sigma^2\right] \left[\gamma_j\right] \\
&\propto \exp\left\{-\frac{1}{2\sigma^2} \left(\boldsymbol{y}_c - \boldsymbol{X}\boldsymbol{\gamma}\boldsymbol{\beta}\right)^T \left(\boldsymbol{y}_c - \boldsymbol{X}\boldsymbol{\gamma}\boldsymbol{\beta}\right)\right\} \left(\frac{\sigma^2}{\lambda_j}\right)^{-\frac{\gamma_j}{2}} \exp\left\{-\frac{\gamma_j \lambda_j \beta_j^2}{2\sigma^2}\right\} \pi^{\gamma_j} \left(1 - \pi\right)^{1-\gamma_j} \\
&\propto \exp\left\{-\frac{1}{2\sigma^2} \left(\boldsymbol{y}_c - \boldsymbol{X}_j \gamma_j \beta_j\right)^T \left(\boldsymbol{y}_c - \boldsymbol{X}_j \gamma_j \beta_j\right)\right\} \left(\frac{\sigma^2}{\lambda_j}\right)^{-\frac{\gamma_j}{2}} \exp\left\{-\frac{\gamma_j \lambda_j \beta_j^2}{2\sigma^2}\right\} \pi^{\gamma_j} \left(1 - \pi\right)^{1-\gamma_j} \\
&\propto \left[\left(\frac{\lambda_j}{\sigma^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2} \left[\beta_j^2 \left(\boldsymbol{X}_j^T \boldsymbol{X}_j + \lambda_j\right) - 2\beta_j \boldsymbol{X}_j^T \boldsymbol{y}_c\right]\right\} \frac{\pi}{1 - \pi}\right]^{\gamma_j} \\
&\propto \left[\left(\frac{\lambda_j}{\sigma^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2} \left[\beta_j^2 \left(\boldsymbol{X}_j^T \boldsymbol{X}_j + \lambda_j\right) - 2\beta_j \delta_j \left(\boldsymbol{X}^T \boldsymbol{X}\right)_j^{-1} \boldsymbol{X}_j^T \boldsymbol{y}_c\right]\right\} \frac{\pi}{1 - \pi}\right]^{\gamma_j} \\
&\propto \left[\left(\frac{\lambda_j}{\sigma^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2} \left[\beta_j^2 \left(\delta_j + \lambda_j\right) - 2\delta_j \beta_j^2\right]\right\} \frac{\pi}{1 - \pi}\right]^{\gamma_j} \\
&\propto \Psi^{\gamma_j}
\end{aligned}
$$

which is Bern$\left(\frac{\Psi_j}{1+\Psi_j}\right)$ where $\Psi_j = \left(\frac{\lambda_j}{\sigma^2}\right)^{\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma^2} \left[\beta_j^2 \left(\delta_j + \lambda_j\right) - 2\delta_j \beta_j^2\right]\right\} \frac{\pi}{1-\pi}$