1 The Model

1.1 Data Model

$$\left[y_t|\boldsymbol{\beta},\boldsymbol{\gamma},\sigma^2_\epsilon\right] \sim \mathrm{N}\left(\boldsymbol{H}_t\boldsymbol{X}_t\boldsymbol{\beta}_{\boldsymbol{\gamma}},\sigma^2_\epsilon\right)$$

where $\boldsymbol{\beta}_{\boldsymbol{\gamma}} = \begin{bmatrix} \beta_0 \\ \beta_1\gamma_1 \\ \vdots \\ \beta_p\gamma_p \end{bmatrix}$ and $\boldsymbol{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_T \end{bmatrix}$.

1.2 Process Model - climate variable of interest is AR(1) in time

$$\left[y_t-\mu\middle|y_{t-1},\alpha,\sigma^2_\eta\right] \sim \mathrm{N}\left(\alpha\left(y_{t-1}-\mu\right),\sigma^2_\eta\right)$$

where μ is an intercept term, assumed to be known for the PDSI drought index ($\mu = 0$)

1.3 Parameter Model

$$\begin{aligned}
[\beta_0] &\propto 1 \\
\left[\beta_j\middle|\sigma^2_\epsilon,\gamma_j\right] &\overset{iid}{\sim} \mathrm{N}\left(0,\frac{\sigma^2_\epsilon\gamma_j}{\lambda_j}\right) && \text{for } j=1,\ldots,p \\
\left[\sigma^2_\epsilon\right] &\propto \frac{1}{\sigma^2_\epsilon} \\
\left[\sigma^2_\eta\right] &\propto \frac{1}{\sigma^2_\eta} \\
[\gamma_j] &\propto \mathrm{Bern}\left(\pi_j\right) && \text{for } j=1,\ldots,p \\
[\alpha] &\propto \mathrm{Unif}\left(0,1\right)
\end{aligned}$$

where π_j and λ_j are fixed hyperpriors for $j=1,\ldots,p$.

1.4 Posterior

$$\left[\boldsymbol{\beta}_{\boldsymbol{\gamma}},\boldsymbol{\gamma},\sigma^2_\epsilon,\sigma^2_\eta,\alpha\middle|\boldsymbol{y}\right] = \left(\prod_{t=1}^{T}\left[y_t,|\boldsymbol{\beta},\boldsymbol{\gamma},\sigma^2_\epsilon\right]\right)\left(\prod_{t=2}^{T}\left[y_t-\mu,\middle|y_{t-1},\alpha,\sigma^2_\eta\right]\right)\left[\boldsymbol{\beta}|\sigma^2_\epsilon,\boldsymbol{\gamma}\right]\left[\sigma^2_\epsilon\right]\left[\sigma^2_\eta\right]\left(\prod_{j=1}^{p}[\gamma_j]\right)[\alpha]$$

2 Full Conditionals

2.1 Full Conditional for unobserved y_t

For $t=1$,

$$\begin{aligned}
[y_1|\cdot] &\propto \left[y_1\middle|\boldsymbol{\beta},\boldsymbol{\gamma},\sigma^2_\epsilon\right]\left[y_2-\mu\middle|y_1,\alpha,\sigma^2_\eta\right] \\
&\propto \exp\left\{-\frac{1}{2\sigma^2_\epsilon}\left(y_1-\boldsymbol{H}_{1\boldsymbol{\gamma}}\boldsymbol{X}_{1\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}\right)^2\right\}\exp\left\{-\frac{1}{2\sigma^2_\eta}\left(\left(y_2-\mu\right)-\alpha\left(y_1-\mu\right)\right)^2\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(y_1^2\left(\frac{1}{\sigma^2_\epsilon}+\frac{\alpha^2}{\sigma^2_\eta}\right)-2y_1\left(\frac{\boldsymbol{H}_{1\boldsymbol{\gamma}}\boldsymbol{X}_{1\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}}{\sigma^2_\epsilon}+\frac{\alpha\left(y_2-\left(1-\alpha\right)\mu\right)}{\sigma^2_\eta}\right)\right)(\right\}
\end{aligned}$$

which is $\mathrm{N}\left(A^{-1}b, A^{-1}\right)$ where $A^{-1} = \left(\frac{1}{\sigma_\epsilon^2} + \frac{\alpha^2}{\sigma_\eta^2}\right)^{-1}$ and $b = \frac{\boldsymbol{H}_{1\boldsymbol{\gamma}}\boldsymbol{X}_{1\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}}{\sigma_\epsilon^2} + \frac{\alpha(y_2-(1-\alpha)\mu)}{\sigma_\eta^2}$

For $t = 2, \ldots, T-1$,

$$
\begin{aligned}
[y_t|\cdot] &\propto \left[y_t \middle| \boldsymbol{\beta}, \boldsymbol{\gamma}, \sigma_\epsilon^2\right] \left[y_t - \mu \middle| y_{t-1}, \alpha, \sigma_\eta^2\right] \left[y_{t+1} - \mu \middle| y_t, \alpha, \sigma_\eta^2\right] \\
&\propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\left(y_t - \boldsymbol{H}_{t\boldsymbol{\gamma}}\boldsymbol{X}_{t\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}\right)^2\right\} \exp\left\{-\frac{1}{2\sigma_\eta^2}\left((y_t-\mu) - \alpha\left(y_{t-1}-\mu\right)\right)^2\right\} \exp\left\{-\frac{1}{2\sigma_\eta^2}\left((y_{t+1}-\mu) - \alpha\left(y_t-\mu\right)\right)\right. \\
&\propto \exp\left\{-\frac{1}{2}\left(y_t^2\left(\frac{1}{\sigma_\epsilon^2} + \frac{\alpha^2+1}{\sigma_\eta^2}\right) - 2y_{t-1}\left(\frac{\boldsymbol{H}_{t\boldsymbol{\gamma}}\boldsymbol{X}_{t\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}}{\sigma_\epsilon^2} + \frac{\alpha y_{t-1} + (1-\alpha)\mu}{\sigma_\eta^2} + \frac{\alpha\left(y_{t+1} - (1-\alpha)\mu\right)}{\sigma_\eta^2}\right)\right)\right\}
\end{aligned}
$$

which is $\mathrm{N}\left(A^{-1}b, A^{-1}\right)$ where $A^{-1} = \left(\frac{1}{\sigma_\epsilon^2} + \frac{\alpha^2+1}{\sigma_\eta^2}\right)^{-1}$ and $b = \frac{\boldsymbol{H}_{t\boldsymbol{\gamma}}\boldsymbol{X}_{t\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}}{\sigma_\epsilon^2} + \frac{\alpha y_{t-1}+(1-\alpha)\mu}{\sigma_\eta^2} + \frac{\alpha(y_{t+1}-(1-\alpha)\mu)}{\sigma_\eta^2}$.

For $t = T$,

$$
\begin{aligned}
[y_T|\cdot] &\propto \left[y_T \middle| \boldsymbol{\beta}, \boldsymbol{\gamma}, \sigma_\epsilon^2\right] \left[y_T - \mu \middle| y_{T-1}, \alpha, \sigma_\eta^2\right] \\
&\propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\left(y_T - \boldsymbol{H}_{T\boldsymbol{\gamma}}\boldsymbol{X}_{T\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}\right)^2\right\} \exp\left\{-\frac{1}{2\sigma_\eta^2}\left((y_T-\mu) - \alpha\left(y_{T-1}-\mu\right)\right)^2\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(y_T^2\left(\frac{1}{\sigma_\epsilon^2} + \frac{1}{\sigma_\eta^2}\right) - 2y_T\left(\frac{\boldsymbol{H}_{T\boldsymbol{\gamma}}\boldsymbol{X}_{T\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}}{\sigma_\epsilon^2} + \frac{\alpha y_{T-1} + (1-\alpha)\mu}{\sigma_\eta^2}\right)\right)\right\}
\end{aligned}
$$

which is $\mathrm{N}\left(A^{-1}b, A^{-1}\right)$ where $A^{-1} = \left(\frac{1}{\sigma_\epsilon^2} + \frac{1}{\sigma_\eta^2}\right)^{-1}$ and $b = \frac{\boldsymbol{H}_{T\boldsymbol{\gamma}}\boldsymbol{X}_{T\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}}{\sigma_\epsilon^2} + \frac{\alpha y_{T-1}+(1-\alpha)\mu}{\sigma_\eta^2}$.

2.2 Full Conditional for $\boldsymbol{\beta}$

$$
\begin{aligned}
[\boldsymbol{\beta}|\cdot] &\propto \prod_{t=1}^{T}\left[y_t \middle| \boldsymbol{\beta}, \boldsymbol{\gamma}, \sigma_\epsilon^2\right] \left[\boldsymbol{\beta} \middle| \boldsymbol{\gamma}, \sigma_\eta^2\right] \\
&\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\left(y_t - \boldsymbol{H}_{t\boldsymbol{\gamma}}\boldsymbol{X}_{t\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}\right)^2\right\} \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\boldsymbol{\beta}_{\boldsymbol{\gamma}}^T\boldsymbol{\Lambda}_{\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}\right\} \\
&\propto \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\left[\boldsymbol{\beta}_{\boldsymbol{\gamma}}^T\left(\sum_{t=1}^{T}\boldsymbol{X}_{t\boldsymbol{\gamma}}^T\boldsymbol{H}_{t\boldsymbol{\gamma}}^T\boldsymbol{H}_{t\boldsymbol{\gamma}}\boldsymbol{X}_{t\boldsymbol{\gamma}} + \boldsymbol{\Lambda}_{\boldsymbol{\gamma}}\right)\boldsymbol{\beta}_{\boldsymbol{\gamma}} - 2\boldsymbol{\beta}_{\boldsymbol{\gamma}}^T\left(\sum_{t=1}^{T}\boldsymbol{X}_{t\boldsymbol{\gamma}}^T\boldsymbol{H}_{t\boldsymbol{\gamma}}^T y_t\right)\right]\right\}
\end{aligned}
$$

which is MVN $\left(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1}\right)$ where $\boldsymbol{A}^{-1} = \left(\frac{\boldsymbol{X}_{t\boldsymbol{\gamma}}^T\boldsymbol{H}_{t\boldsymbol{\gamma}}^T\boldsymbol{H}_{t\boldsymbol{\gamma}}\boldsymbol{X}_{t\boldsymbol{\gamma}}+\boldsymbol{\Lambda}_{\boldsymbol{\gamma}}}{\sigma_\epsilon^2}\right)^{-1}$ and $\boldsymbol{b} = \frac{\boldsymbol{X}_{t\boldsymbol{\gamma}}^T\boldsymbol{H}_{t\boldsymbol{\gamma}}^T\boldsymbol{y}_t}{\sigma_\epsilon^2}$

2.3 Full Conditional for σ_ϵ^2

$$
\begin{aligned}
\left[\sigma_\epsilon^2|\cdot\right] &\propto \prod_{t=1}^{T}\left[y_t|\boldsymbol{\beta},\boldsymbol{\gamma},\sigma_\epsilon^2\right]\left[\boldsymbol{\beta}|\boldsymbol{\gamma},\sigma_\epsilon^2\right]\left[\sigma_\epsilon^2\right] \\
&\propto \prod_{t=1}^{T}\left(\sigma_\epsilon^2\right)^{-\frac{1}{2}}\exp\left\{-\frac{1}{\sigma_\epsilon^2}\frac{\left(y_t-\boldsymbol{H}_{t\boldsymbol{\gamma}}\boldsymbol{X}_{t\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}\right)^2}{2}\right\}\left|\sigma_\epsilon^2\boldsymbol{\Lambda}_{\boldsymbol{\gamma}}\right|^{-\frac{1}{2}}\exp\left\{-\frac{1}{\sigma_\epsilon^2}\frac{\boldsymbol{\beta}_{\boldsymbol{\gamma}}^T\boldsymbol{\Lambda}_{\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}}{2}\right\}\left(\sigma_\epsilon^2\right)^{-1} \\
&\propto \left(\sigma_\epsilon^2\right)^{-\frac{T+p_{\boldsymbol{\gamma}}}{2}-1}\exp\left\{-\frac{1}{\sigma_\epsilon^2}\frac{\sum_{t=1}^{T}\left(y_t-\boldsymbol{H}_{t\boldsymbol{\gamma}}\boldsymbol{X}_{t\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}\right)^2+\boldsymbol{\beta}_{\boldsymbol{\gamma}}^T\boldsymbol{\Lambda}_{\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}}{2}\right\}
\end{aligned}
$$

which is IG $\left(\frac{T+p_{\boldsymbol{\gamma}}}{2},\frac{\sum_{t=1}^{T}\left(y_t-\boldsymbol{H}_{t\boldsymbol{\gamma}}\boldsymbol{X}_{t\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}\right)^2+\boldsymbol{\beta}_{\boldsymbol{\gamma}}^T\boldsymbol{\Lambda}_{\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}}{2}\right)$

2.4 Full Conditional for σ_η^2

$$
\begin{aligned}
\left[\sigma_\eta^2|\cdot\right] &\propto \prod_{t=2}^{T}\left[y_t|y_{t-1},\alpha,\sigma_\eta^2\right]\left[\sigma_\eta^2\right] \\
&\propto \prod_{t=2}^{T}\left(\sigma_\eta^2\right)^{-\frac{1}{2}}\exp\left\{-\frac{1}{\sigma_\eta^2}\frac{\left((y_t-\mu)-\alpha\left(y_{t-1}-\mu\right)\right)^2}{2}\right\}\left(\sigma_\eta^2\right)^{-1} \\
&\propto \left(\sigma_\eta^2\right)^{-\frac{T-1}{2}-1}\exp\left\{-\frac{1}{\sigma_\eta^2}\frac{\sum_{t=2}^{T}\left((y_t-\mu)-\alpha\left(y_{t-1}-\mu\right)\right)^2}{2}\right\}
\end{aligned}
$$

which is IG $\left(\frac{T-1}{2},\frac{\sum_{t=2}^{T}\left((y_t-\mu)-\alpha(y_{t-1}-\mu)\right)^2}{2}\right)$

2.5 Full Conditional for α

$$
\begin{aligned}
[\alpha|\cdot] &\propto \prod_{t=2}^{T}\left[y_t|y_{t-1},\alpha,\sigma_\eta^2\right][\alpha] \\
&\propto \prod_{t=2}^{T}\exp\left\{-\frac{1}{2\sigma_\eta^2}\left((y_t-\mu)-\alpha\left(y_{t-1}-\mu\right)\right)^2\right\}\boldsymbol{I}\left\{\alpha\in(0,1)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left(\alpha^2\frac{\sum_{t=2}^{T}\left(y_{t-1}-\mu\right)^2}{\sigma_\eta^2}-2\alpha\frac{\sum_{t=2}^{T}\left(y_{t-1}-\mu\right)\left(y_t-\mu\right)}{\sigma_\eta^2}\right)\right\}\boldsymbol{I}\left\{\alpha\in(0,1)\right\}
\end{aligned}
$$

which is Truncated Normal $\left(A^{-1}b,A^{-1}\right)$ where $A^{-1}=\left(\frac{\sum_{t=2}^{T}(y_{t-1}-\mu)^2}{\sigma_\eta^2}\right)^{-1}$ and $b=\frac{\sum_{t=2}^{T}(y_{t-1}-\mu)(y_t-\mu)}{\sigma_\eta^2}$ restricted to the interval $(0,1)$.

2.6 Full Conditional for γ_j

$$
\begin{aligned}
[\boldsymbol{\gamma}|\cdot] &\propto \prod_{t=1}^{T} \left[y_t \middle| \boldsymbol{\beta}, \boldsymbol{\gamma}, \sigma_\epsilon^2\right] [\boldsymbol{\gamma}] \\
&\propto \prod_{t=1}^{T} \left(\sigma^2\right)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\left(y_t - \boldsymbol{H}_{t\boldsymbol{\gamma}}\boldsymbol{X}_{t\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}\right)^2\right\} \left|\sigma_\epsilon^2 \boldsymbol{\Lambda}_{\boldsymbol{\gamma}}\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\boldsymbol{\beta}_{\boldsymbol{\gamma}}^T \boldsymbol{\Lambda}_{\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}\right\} \left(\prod_{j=1}^{p} \pi_j^{\gamma_j}\left(1-\pi_j\right)^{(1-\gamma_j)}\right) \\
&\propto \left(\sigma^2\right)^{-\frac{T+p_{\boldsymbol{\gamma}}}{2}} \left|\boldsymbol{\Lambda}_{\boldsymbol{\gamma}}\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\left(y_t - \boldsymbol{H}_{t\boldsymbol{\gamma}}\boldsymbol{X}_{t\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}\right)^2\right\} \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\boldsymbol{\beta}_{\boldsymbol{\gamma}}^T \boldsymbol{\Lambda}_{\boldsymbol{\gamma}}\boldsymbol{\beta}_{\boldsymbol{\gamma}}\right\} \left(\prod_{j=1}^{p} \pi_j^{\gamma_j}\left(1-\pi_j\right)^{(1-\gamma_j)}\right)
\end{aligned}
$$

which can be sampled using a Metropolis Hastings proposal. This requires a smart choice of proposal distribution.

If $H_t X_t$ has orthogonal columns (e.g $X_t^T H_t^T H_t X_t$ is diagonal) then **ADD IN A SUM OVER ALL T**

$$
\begin{aligned}
[\gamma_j|\cdot] &\propto \prod_{t=1}^{T} \left[y_t, \middle| \beta_j, \gamma_j, \sigma_\epsilon^2\right] \left[\beta_j \middle| \gamma_j, \sigma^2\right] [\gamma_j] \\
&\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\left(y_t - \boldsymbol{H}_{tj}\boldsymbol{X}_{tj}\gamma_j\beta_j\right)^2\right\} \left(\frac{\sigma_\epsilon^2}{\lambda_j}\right)^{-\frac{\gamma_j}{2}} \exp\left\{-\frac{\gamma_j\lambda_j\beta_j^2}{2\sigma^2}\right\} \pi^{\gamma_j}\left(1-\pi\right)^{1-\gamma_j} \\
&\propto \left(\frac{\sigma_\epsilon^2}{\lambda_j}\right)^{-\frac{\gamma_j}{2}} \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\gamma_j\left(\beta_j^2\left(\sum_{t=1}^{T}\boldsymbol{X}_{tj}^T\boldsymbol{H}_{tj}^T\boldsymbol{H}_{tj}\boldsymbol{X}_{tj}\lambda_j\right) - 2\sum_{t=1}^{T} y_t \boldsymbol{H}_{tj}\boldsymbol{X}_{tj}\beta_j\right)\right\} \pi^{\gamma_j}\left(1-\pi\right)^{1-\gamma_j} \\
&\propto \left(\left(\frac{\sigma_\epsilon^2}{\lambda_j}\right)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\left(\beta_j^2\left(\sum_{t=1}^{T}\boldsymbol{X}_{tj}^T\boldsymbol{H}_{tj}^T\boldsymbol{H}_{tj}\boldsymbol{X}_{tj}\lambda_j\right) - 2\sum_{t=1}^{T} y_t \boldsymbol{H}_{tj}\boldsymbol{X}_{tj}\beta_j\right)\right\} \frac{\pi_j}{1-\pi_j}\right)^{\gamma_j} \\
&\propto \Psi^{\gamma_j}
\end{aligned}
$$

so γ_j is $\text{Bern}\left(\frac{\Psi_j}{1+\Psi_j}\right)$ where $\Psi_j = \left(\frac{\sigma_\epsilon^2}{\lambda_j}\right)^{-\frac{1}{2}} \exp\left\{-\frac{1}{2\sigma_\epsilon^2}\left(\beta_j^2\left(\sum_{t=1}^{T}\boldsymbol{X}_{tj}^T\boldsymbol{H}_{tj}^T\boldsymbol{H}_{tj}\boldsymbol{X}_{tj}\lambda_j\right) - 2\sum_{t=1}^{T} y_t\boldsymbol{H}_{tj}\boldsymbol{X}_{tj}\beta_j\right)\right\}\frac{\pi_j}{1-\pi_j}$