

1 The Model - starting with only one year...

1.1 Data Model

$$[\mathbf{y}_o | \boldsymbol{\beta}_\gamma, \gamma, \sigma^2] \sim \text{N}(\mathbf{H}_o \mathbf{X} \boldsymbol{\beta}_\gamma, \sigma^2 \mathbf{I}_o)$$

1.2 Process Model - Unobserved data - needed for the model selection step

$$[\mathbf{y}_u | \boldsymbol{\beta}_\gamma, \gamma, \sigma^2] \sim \text{N}(\mathbf{H}_u \mathbf{X} \boldsymbol{\beta}_\gamma, \sigma^2 \mathbf{I}_u)$$

where $\mathbf{y}_c \equiv (\mathbf{y}_o, \mathbf{y}_u)^T$.

1.3 Parameter Model

$$\begin{aligned} [\beta_j | \sigma^2, \gamma_j] &\stackrel{iid}{\sim} \begin{cases} 0 & \text{if } \gamma_j = 0 \\ \text{N}\left(0, \frac{\sigma^2}{\lambda_j}\right) & \text{if } \gamma_j = 1 \end{cases} & \text{for } j = 1, \dots, p \\ [\sigma^2] &\propto \frac{1}{\sigma^2} \\ [\gamma_j] &\propto \text{Bern}(\pi_j) & \text{for } j = 1, \dots, p \end{aligned}$$

where π_j and λ_j are fixed hyperpriors for $j = 1, \dots, p$.

1.4 Posterior

$$\begin{aligned} [\boldsymbol{\beta}_\gamma, \gamma, \sigma^2 | \mathbf{y}_o] &= \int [\mathbf{y}_u, \boldsymbol{\beta}_\gamma, \gamma, \sigma^2 | \mathbf{y}_o] d\mathbf{y}_u \\ &= \int [\mathbf{y}_o | \boldsymbol{\beta}_\gamma, \gamma, \sigma^2] [\mathbf{y}_u | \boldsymbol{\beta}_\gamma, \gamma, \sigma^2] \prod_{j=1}^p [\beta_j | \gamma_j, \sigma^2] [\gamma_j] [\sigma^2] d\mathbf{y}_u \end{aligned}$$

2 Full Conditionals

2.1 Full Conditional for \mathbf{y}_u

$$[\mathbf{y}_u | \cdot] \propto [\mathbf{y}_u | \boldsymbol{\beta}_\gamma, \gamma, \sigma^2]$$

which is $\text{N}(\mathbf{H}_u \mathbf{X} \boldsymbol{\beta}_\gamma, \sigma^2 \mathbf{I}_u)$

2.2 Full Conditional for $\boldsymbol{\beta}_\gamma$

$$\begin{aligned} [\boldsymbol{\beta}_\gamma | \cdot] &\propto [\mathbf{y}_o | \boldsymbol{\beta}_\gamma, \gamma, \sigma^2] [\boldsymbol{\beta}_\gamma | \gamma, \sigma^2] \\ &\propto \exp \left\{ -\frac{1}{2\sigma^2} (\mathbf{y}_o - \mathbf{H}_o \mathbf{X}_{o\gamma} \boldsymbol{\beta}_\gamma)^T (\mathbf{y}_o - \mathbf{H}_o \mathbf{X}_{o\gamma} \boldsymbol{\beta}_\gamma) \right\} \exp \left\{ -\frac{1}{2\sigma^2} \boldsymbol{\beta}_\gamma^T \boldsymbol{\Delta}_\gamma \boldsymbol{\beta}_\gamma \right\} \\ &\propto \exp \left\{ -\frac{1}{2\sigma^2} \left[\boldsymbol{\beta}_\gamma^T (\mathbf{X}_{o\gamma}^T \mathbf{H}_o^T \mathbf{H}_o \mathbf{X}_{o\gamma} + \boldsymbol{\Delta}_\gamma) \boldsymbol{\beta}_\gamma - 2\boldsymbol{\beta}_\gamma^T (\mathbf{X}_{o\gamma}^T \mathbf{H}_o^T \mathbf{y}_o) \right] \right\} \end{aligned}$$

which is MVN $(\mathbf{A}^{-1}\mathbf{b}, \mathbf{A}^{-1})$ where $\mathbf{A}^{-1} = (\mathbf{X}_{o\gamma}^T \mathbf{H}_o^T \mathbf{H}_o \mathbf{X}_{o\gamma} + \Delta\gamma)^{-1}$ and $\mathbf{b} = \mathbf{X}_{o\gamma}^T \mathbf{H}_o^T \mathbf{y}_o$

2.3 Full Conditional for σ^2

$$\begin{aligned} [\sigma^2 | \cdot] &\propto [\mathbf{y}_o, \beta\gamma, \gamma, \sigma^2] [\beta\gamma | \gamma, \sigma^2] [\sigma^2] \\ &\propto (\sigma^2)^{-\frac{n_o}{2}} \exp \left\{ -\frac{1}{\sigma^2} \frac{(\mathbf{y}_o - \mathbf{H}_o \mathbf{X}_{o\gamma} \beta\gamma)^T (\mathbf{y}_o - \mathbf{H}_o \mathbf{X}_{o\gamma} \beta\gamma)}{2} \right\} (\sigma^2)^{-\frac{p\gamma}{2}} \exp \left\{ -\frac{1}{\sigma^2} \frac{\beta\gamma^T \Lambda \gamma \beta\gamma}{2} \right\} (\sigma^2)^{-1} \\ &\propto (\sigma^2)^{-\frac{n_o+p\gamma}{2}-1} \exp \left\{ -\frac{1}{\sigma^2} \frac{(\mathbf{y}_o - \mathbf{H}_o \mathbf{X}_{o\gamma} \beta\gamma)^T (\mathbf{y}_o - \mathbf{H}_o \mathbf{X}_{o\gamma} \beta\gamma) + \beta\gamma^T \Lambda \gamma \beta\gamma}{2} \right\} \end{aligned}$$

which is IG $\left(\frac{n_o+p\gamma}{2}, \frac{(\mathbf{y}_o - \mathbf{H}_o \mathbf{X}_{o\gamma} \beta\gamma)^T (\mathbf{y}_o - \mathbf{H}_o \mathbf{X}_{o\gamma} \beta\gamma) + \beta\gamma^T \Lambda \gamma \beta\gamma}{2} \right)$

2.4 Full Conditional for γ_j

For $j = 1, \dots, p$ and using the fact that $\mathbf{X}_i^T \mathbf{X}_j = 0$ (by orthogonality of principal components),

$$\begin{aligned} [\gamma_j | \cdot] &\propto [\mathbf{y}_c, \beta\gamma, \gamma, \sigma^2] [\beta_j | \gamma_j, \sigma^2] [\gamma_j] \\ &\propto \exp \left\{ -\frac{1}{2\sigma^2} (\mathbf{y}_c - \mathbf{X}\gamma\beta)^T (\mathbf{y}_c - \mathbf{X}\gamma\beta) \right\} \left(\frac{\sigma^2}{\lambda_j} \right)^{-\frac{\gamma_j}{2}} \exp \left\{ -\frac{\gamma_j \lambda_j \beta_j^2}{2\sigma^2} \right\} \pi^{\gamma_j} (1-\pi)^{1-\gamma_j} \\ &\propto \exp \left\{ -\frac{1}{2\sigma^2} (\mathbf{y}_c - \mathbf{X}_j \gamma_j \beta_j)^T (\mathbf{y}_c - \mathbf{X}_j \gamma_j \beta_j) \right\} \left(\frac{\sigma^2}{\lambda_j} \right)^{-\frac{\gamma_j}{2}} \exp \left\{ -\frac{\gamma_j \lambda_j \beta_j^2}{2\sigma^2} \right\} \pi^{\gamma_j} (1-\pi)^{1-\gamma_j} \\ &\propto \left[\left(\frac{\lambda_j}{\sigma^2} \right)^{\frac{1}{2}} \exp \left\{ -\frac{1}{2\sigma^2} [\beta_j^2 (\mathbf{X}_j^T \mathbf{X}_j + \lambda_j) - 2\beta_j \mathbf{X}_j^T \mathbf{y}_c] \right\} \frac{\pi}{1-\pi} \right]^{\gamma_j} \\ &\propto \left[\left(\frac{\lambda_j}{\sigma^2} \right)^{\frac{1}{2}} \exp \left\{ -\frac{1}{2\sigma^2} [\beta_j^2 (\delta_j + \lambda_j) - 2\delta_j \beta_j^2] \right\} \frac{\pi}{1-\pi} \right]^{\gamma_j} \\ &\propto \Psi^{\gamma_j} \end{aligned}$$

which is Bern $\left(\frac{\Psi_j}{1+\Psi_j} \right)$ where $\Psi_j = \left(\frac{\lambda_j}{\sigma^2} \right)^{\frac{1}{2}} \exp \left\{ -\frac{1}{2\sigma^2} [\beta_j^2 (\delta_j + \lambda_j) - 2\delta_j \beta_j^2] \right\} \frac{\pi}{1-\pi}$