Spatial Geostatistical Model

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1 Full Dimensional Model Statement

1.1 Data Model

$$\boldsymbol{y}_t = \boldsymbol{H}_t \boldsymbol{X} \boldsymbol{\beta}_t + \boldsymbol{\eta}_t + \boldsymbol{\epsilon}_t$$

1.2 Process Model

$$\begin{aligned}
\boldsymbol{\beta}_t &\sim N(\boldsymbol{0}, \boldsymbol{\Sigma}_\beta) & \boldsymbol{\Sigma}_\beta &= \sigma^2_\epsilon \boldsymbol{D}_\gamma & \boldsymbol{D}_\gamma &= diag(\gamma_1^2, \ldots, \gamma_\tau^2) \\
\boldsymbol{\eta}_t &\sim N(0, \boldsymbol{\Sigma}_{\eta_t}) & \boldsymbol{\Sigma}_{\eta_t} &= \sigma^2_\eta \boldsymbol{R}_t(\phi) & \boldsymbol{R}_t(\phi) &= \exp\left(-\boldsymbol{D}_t / \phi\right) \\
\boldsymbol{\epsilon}_t &\sim N(0, \boldsymbol{\Sigma}_{\epsilon_t}) & \boldsymbol{\Sigma}_{\epsilon_t} &= \sigma^2_\epsilon \boldsymbol{I}_{n_t} & & \\
\boldsymbol{\gamma_i^2} &\sim Exp(\lambda^2/2) \text{ Equivalently} & \frac{1}{\gamma_i^2} &\sim IG(1, \lambda^2/2) & & \text{for } i = 1 \ldots, \tau
\end{aligned}$$

1.3 Parameter Model

$$\begin{aligned}
\sigma^2_\eta &\sim IG(\alpha_\eta, \beta_\eta) \\
\sigma^2_\epsilon &\sim IG(\alpha_\epsilon, \beta_\epsilon) \\
\phi &\sim IG(\alpha_\phi, \beta_\phi) \\
\lambda^2 &\sim \Gamma(\alpha_\lambda, \beta_\lambda)
\end{aligned}$$

where $\boldsymbol{I}_{n_t}$ is the identity matrix of size $n_t \times n_t$ where τ is the number of parameters in $\boldsymbol{\beta}_t$ and n_t is the number of samples of y_t at time t and $\boldsymbol{D}_t$ is the distance matrix between locations observed at time t. Define $\boldsymbol{\Sigma}_t = \boldsymbol{\Sigma}_{\eta_t} + \boldsymbol{\Sigma}_{\epsilon_t}$

2 Posterior

$$\prod_{t=1}^T [\boldsymbol{\beta}_t, \boldsymbol{\gamma}^2, \sigma^2_\eta, \sigma^2_\epsilon, \phi, \lambda^2 | \boldsymbol{y}_t] \propto \prod_{t=1}^T [\boldsymbol{y}_t | \boldsymbol{\beta}_t, \sigma^2_\eta, \phi, \sigma^2_\epsilon][\boldsymbol{\beta}_t | \sigma^2_\epsilon, \boldsymbol{\gamma^2}][\boldsymbol{\gamma} | \lambda^2][\sigma^2_\eta][\sigma^2_\epsilon][\phi][\lambda^2]$$

3 Full Conditionals

3.1 Full Conditional for $\boldsymbol{\beta}_t$

For $t = 1, \ldots, T$,

$$\begin{aligned}
[\boldsymbol{\beta}_t|\cdot] &\propto [\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \sigma_\epsilon^2, \phi][\boldsymbol{\beta}_t|\boldsymbol{\mu}_\beta, \sigma_\beta^2] \\
&\propto e^{-\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{H}_t \boldsymbol{X}\boldsymbol{\beta}_t)^T \boldsymbol{\Sigma}_t^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t \boldsymbol{X}\boldsymbol{\beta}_t)} e^{-\frac{1}{2}\boldsymbol{\beta}_t^T \boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\beta}_t} \\
&\propto e^{-\frac{1}{2}\{\boldsymbol{\beta}_t^T(\boldsymbol{X}^T \boldsymbol{H}_t^T \boldsymbol{\Sigma}_t^{-1}\boldsymbol{H}_t \boldsymbol{X} + \boldsymbol{\Sigma}_\beta^{-1})\boldsymbol{\beta}_t - 2\boldsymbol{\beta}_t^T(\boldsymbol{X}^T \boldsymbol{H}_t^T \boldsymbol{\Sigma}_t^{-1}\boldsymbol{y}_t)\}}
\end{aligned}$$

which is Normal with mean $\boldsymbol{A}^{-1}\boldsymbol{b}$ and variance $\boldsymbol{A}^{-1}$ where

$$\begin{aligned}
\boldsymbol{A}^{-1} &= (\boldsymbol{X}^T \boldsymbol{H}_t^T \boldsymbol{\Sigma}_t^{-1}\boldsymbol{H}_t \boldsymbol{X} + \boldsymbol{\Sigma}_\beta^{-1})^{-1} \\
\boldsymbol{b} &= (\boldsymbol{X}^T \boldsymbol{H}_t^T \boldsymbol{\Sigma}_t^{-1}\boldsymbol{y}_t)
\end{aligned}$$

3.2 Full Conditional for γ_i^2

For $i = 1, \ldots, \tau$

$$\begin{aligned}
[\gamma_i^2|\cdot] &\propto \prod_{t=1}^T [\beta_t|\sigma_\epsilon^2, \gamma_i^2][\gamma_i^2|\lambda^2] \\
&\propto \prod_{t=1}^T |\sigma_\epsilon^2 \gamma_i^2|^{-\frac{1}{2}} e^{-\frac{1}{2}\beta_t^T(\sigma_\epsilon^2\gamma_i^2)^{-1}\beta_t} e^{-\frac{\lambda^2}{2}\gamma_i^2} \\
&\propto (\gamma_i^2)^{-\frac{T}{2}} e^{-\frac{1}{\gamma_i^2}(\frac{\sum_{t=1}^T \beta_t^T\beta_t}{2\sigma_\epsilon^2})} e^{-\frac{\lambda^2}{2}\gamma_i^2} \\
&\propto (\frac{1}{\gamma_i^2})^{2-\frac{T}{2}} e^{-\frac{1}{\gamma_i^2}(\frac{\sum_{t=1}^T \beta_t^T\beta_t}{2\sigma_\epsilon^2})} (\frac{1}{\gamma_i^2})^{-2} e^{-\frac{\lambda^2/2}{\frac{1}{\gamma_i^2}}}
\end{aligned}$$

3.3 Full Conditional for σ_η^2

$$\begin{aligned}
[\sigma_\eta^2|\cdot] &\propto \prod_{t=1}^T [\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\sigma_\eta^2] \\
&\propto (\prod_{t=1}^T |\boldsymbol{\Sigma}_t|^{-\frac{1}{2}}) e^{-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{y}_t - \boldsymbol{H}_t \boldsymbol{X}\boldsymbol{\beta}_t)^T \boldsymbol{\Sigma}_t^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t \boldsymbol{X}\boldsymbol{\beta}_t)} (\sigma_\eta^2)^{-\alpha_\eta - 1} e^{-\frac{\beta_\eta}{\sigma_\eta^2}} \\
&\propto |\boldsymbol{\Sigma}_t|^{-\frac{T}{2}} e^{-\frac{1}{2}\sum_{t=1}^T(\boldsymbol{y}_t - \boldsymbol{H}_t \boldsymbol{X}\boldsymbol{\beta}_t)^T \boldsymbol{\Sigma}_t^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t \boldsymbol{X}\boldsymbol{\beta}_t)} (\sigma_\eta^2)^{-\alpha_\eta - 1} e^{-\frac{\beta_\eta}{\sigma_\eta^2}}
\end{aligned}$$

which can be sampled using a Metropolis-Hastings step

3.4 Full Conditional for σ_ϵ^2

$$
\begin{aligned}
[\sigma_\epsilon^2|\cdot] &\propto \prod_{t=1}^{T}[\boldsymbol{y}_t|\boldsymbol{\beta}_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\boldsymbol{\beta}_t|\sigma_\eta^2, \boldsymbol{\gamma}][\sigma_\epsilon^2] \\
&\propto (\prod_{t=1}^{T}|\boldsymbol{\Sigma}_t|^{-\frac{1}{2}})e^{-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)}|\boldsymbol{\Sigma}_\beta|^{-\frac{1}{2}}e^{-\frac{1}{2}\sum_{t=1}^{T}\boldsymbol{\beta}_t^T\boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\beta}_t}(\sigma_\epsilon^2)^{-\alpha_\epsilon - 1}e^{-\frac{\beta_\epsilon}{\sigma_\epsilon^2}} \\
&\propto |\boldsymbol{\Sigma}_t|^{-\frac{T}{2}}e^{-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)}|\boldsymbol{\Sigma}_\beta|^{-\frac{1}{2}}e^{-\frac{1}{2}\sum_{t=1}^{T}\boldsymbol{\beta}_t^T\boldsymbol{\Sigma}_\beta^{-1}\boldsymbol{\beta}_t}(\sigma_\epsilon^2)^{-\alpha_\epsilon - 1}e^{-\frac{\beta_\epsilon}{\sigma_\epsilon^2}}
\end{aligned}
$$

which can be sampled using a Metropolis-Hastings step

3.5 Full Conditional for ϕ

$$
\begin{aligned}
[\phi|\cdot] &\propto \prod_{t=1}^{T}[y_t|\beta_t, \sigma_\eta^2, \phi, \sigma_\epsilon^2][\phi] \\
&\propto \prod_{t=1}^{T}|\boldsymbol{\Sigma}_t|^{-\frac{1}{2}}e^{-\frac{1}{2}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)}\phi^{-\alpha_\phi - 1}e^{-\frac{\beta_\phi}{\phi}} \\
&\propto |\boldsymbol{\Sigma}_t|^{-\frac{T}{2}}e^{-\frac{1}{2}\sum_{t=1}^{T}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)^T\boldsymbol{\Sigma}_t^{-1}(\boldsymbol{y}_t - \boldsymbol{H}_t\boldsymbol{X}\boldsymbol{\beta}_t)}\phi^{-\alpha_\phi - 1}e^{-\frac{\beta_\phi}{\phi}}
\end{aligned}
$$

which can be sampled using a Metropolis-Hastings step

Full Conditional for λ^2

$$
\begin{aligned}
[\lambda^2|\cdot] &\propto \prod_{t=1}^{T}\prod_{j=1}^{\tau}(\lambda^2)e^{-\frac{\lambda^2/2}{1/\gamma_j^2}}(\frac{\lambda^2}{2})^{\alpha_\lambda - 1}e^{-\beta_\lambda(\lambda^2/2)} \\
&\propto (\frac{\lambda^2}{2})^{\alpha_\lambda + \tau - 1}e^{-\lambda^2(\beta_\lambda + \sum_{j=1}^{\tau}\gamma_j^2/2)}
\end{aligned}
$$

which is Gamma$(\alpha_\lambda + \tau, \beta_\lambda + \sum_{j=1}^{\tau}\gamma_j^2/2)$

4 Posterior Predictive Distribution

The posterior predictive distribution for $\boldsymbol{y}_t$ is sampled a each MCMC iteration k by

$$
\boldsymbol{y}_t^{(k)} \sim N(\boldsymbol{H_t}\boldsymbol{X}\boldsymbol{\beta}_t^{(k)}, \boldsymbol{\Sigma}^{(k)})
$$