

# Multispecies Model

Kristin Broms, Viviana Ruiz-Gutierrez, John Tipton

January 31, 2014

## 1 Introduction and motivation

## 2 Model of one season detection

### 2.1 Modelling assumptions

- There is an ecological motivation for  $\mathbf{p}$  and  $\Psi$  coming from the same distribution
- There is no heterogeneity in unobserved covariates for  $\mathbf{p}$  and  $\Psi$
- All species are "theoretically" available to be sampled
- No seasonality
- Closed populations within the  $J$  visits
- The actual number of species  $N$  lies between the observed number of species  $K$  and the number of augmented species  $\Omega$

### 2.2 Data model

For  $i = 1, \dots, n$  sites,  $k = 1, \dots, K$  observed species (out of a total of  $\Omega$  species), and for  $j = 1, \dots, J$  revisits at each site

$$y_{ik} = \begin{cases} 0 & \text{if } Z_{ik} = 0 \\ \text{Binom}(J, p_k) & \text{if } Z_{ik} = 1 \end{cases}$$

where  $y_{ik}$  is the number of times that species  $i$  is seen at site  $k$  out of  $J$  visits,  $Z_{ik}$  is an indicator of presence or absence of species  $i$  at site  $k$ , and  $p_k$  is the detection probability for species  $k$ .

### 2.3 Process

$$Z_{ik} = \begin{cases} 0 & \text{if } W_k = 0 \\ \text{Bern}(\Psi_k) & \text{if } W_k = 1 \end{cases}$$

$$p_k \sim \text{Beta}(\alpha_p, \beta_p)$$

$$\Psi_k \sim \text{Beta}(\alpha_\Psi, \beta_\Psi)$$

$$W_k \sim \text{Bern}(\lambda)$$

where  $W_k$  is an indicator of whether an augmented species exists or not in the population and  $\sum_{k=1}^{\Omega} W_k = N$ , the species richness,  $\Psi_k$  is the presence probability for species  $k$ , and  $\lambda$  is the probability of an augmented species being real

## 2.4 Parameter

$$\begin{aligned}
\alpha_p &\sim \text{Gamma}(\alpha_{\alpha_p}, \beta_{\alpha_p}) \\
\beta_p &\sim \text{Gamma}(\alpha_{\beta_p}, \beta_{\beta_p}) \\
\alpha_\Psi &\sim \text{Gamma}(\alpha_{\alpha_\Psi}, \beta_{\alpha_\Psi}) \\
\beta_\Psi &\sim \text{Gamma}(\alpha_{\beta_\Psi}, \beta_{\beta_\Psi}) \\
\lambda &\sim \text{Beta}(\alpha_\lambda, \beta_\lambda)
\end{aligned}$$

## 3 Ideas

- model  $\Psi_k = f(\Psi_{-k}) + \text{randomness and species interaction}$
- $Z_{ik} = \begin{cases} 0 & \text{if } W_k = 0 \\ \text{Bern}(\Psi_k) & \text{if } W_k = 1 \end{cases}$  where  $W_k$  represents "ghost species"
- Let  $\Omega$  represent the total number of possible species,  $\Omega \geq N \geq K$
- $W_k \sim \text{Bern}(\lambda) \Leftrightarrow N \sim \text{Binom}(\Omega, \lambda) \Leftrightarrow \sum_{k=1}^{\Omega} W_k = N$
- Goal: estimate  $\Psi_k$  and predict  $N$

## 4 Posterior

$$\begin{aligned}
[\mathbf{Z}, \mathbf{p}, \boldsymbol{\Psi}, \mathbf{W}, \alpha_p, \beta_p, \alpha_\Psi, \beta_\Psi, \lambda | \mathbf{y}, J] &\propto \prod_{i=1}^n \prod_{k=1}^{\Omega} [y_{ik} | J, p_k]^{Z_{ik}} I\{y_{ik} = 0\}^{(1-Z_{ik})} [Z_{ik} | \Psi_k]^{W_k} I\{Z_{ik} = 0\}^{(1-W_k)} \\
&\quad \times [p_k | \alpha_p, \beta_p] [\Psi_k | \alpha_\Psi, \beta_\Psi] [W_k | \lambda] [\alpha_p] [\beta_p] [\alpha_\Psi] [\beta_\Psi] [\lambda]
\end{aligned}$$

## 5 Full conditionals

### 5.1 $Z_{ik}$

For  $y_{ik} = 1, Z_{ik} = 1$ .

For  $y_{ik} = 0$

$$\begin{aligned}
[Z_{ik} | \cdot] &\propto [y_{ik} | J, p_k]^{Z_{ik}} I\{y_{ik} = 0\}^{(1-Z_{ik})} [Z_{ik} | \Psi_k]^{W_k} I\{Z_{ik} = 0\}^{(1-W_k)} \\
&\propto (1 - p_k)^{J Z_{ik}} \Psi_k^{Z_{ik} W_k} (1 - \Psi_k)^{(1-Z_{ik}) W_k} \\
&\propto \left( (1 - p_k)^J \Psi_k^{W_k} \right)^{Z_{ik}} (1 - \Psi_k)^{W_k (1-Z_{ik})}
\end{aligned}$$

which is  $\text{Bern}(\tilde{\Psi}_k)$  where  $\tilde{\Psi}_k = \left( (1 - p_k)^J \Psi_k^{W_k} \right) / \left( (1 - p_k)^J \Psi_k^{W_k} + (1 - \Psi_k)^{W_k} \right)$

## 5.2 $p_k$

$$\begin{aligned}
[p_k|\cdot] &\propto \prod_{i=1}^n [y_{ik}|J, p_k]^{Z_{ik}} [p_k|\alpha_p, \beta_p] \\
&\propto \prod_{i=1}^n (p_k^{y_{ik}} (1-p_k)^{J-y_{ik}})^{Z_{ik}} p_k^{(\alpha_p-1)} (1-p_k)^{(\beta_p-1)} \\
&\propto p_k^{(\alpha_p-1+\sum_{i=1}^n y_{ik} Z_{ik})} (1-p_k)^{(\beta_p-1+\sum_{i=1}^n Z_{ik}(J-y_{ik}))}
\end{aligned}$$

which is  $\text{Beta}(\alpha_p + \sum_{i=1}^n y_{ik} Z_{ik}, \beta_p + \sum_{i=1}^n Z_{ik}(J - y_{ik}))$

## 5.3 $\Psi_k$

When  $W_k = 1$

$$\begin{aligned}
[\Psi_k|\cdot] &\propto \prod_{i=1}^n [Z_{ik}|\Psi_k]^{W_k} [\Psi_k|\alpha_\Psi, \beta_\Psi] \\
&\propto \prod_{i=1}^n \Psi_k^{Z_{ik}} (1-\Psi_k)^{(1-Z_{ik})} \Psi_k^{(\alpha_\Psi-1)} (1-\Psi_k)^{(\beta_\Psi-1)} \\
&\propto \Psi_k^{(\alpha_\Psi-1+\sum_{i=1}^n Z_{ik})} (1-\Psi_k)^{(\beta_\Psi-1+\sum_{i=1}^n (1-Z_{ik}))}
\end{aligned}$$

which is  $\text{Beta}(\alpha_\Psi + \sum_{i=1}^n Z_{ik}, \beta_\Psi + \sum_{i=1}^n (1 - Z_{ik}))$

When  $W_k = 0$

$$\begin{aligned}
[\Psi_k|\cdot] &\propto \prod_{i=1}^n [Z_{ik}|\Psi_k]^{W_k} [\Psi_k|\alpha_\Psi, \beta_\Psi] \\
&\propto \prod_{i=1}^n \Psi_k^{Z_{ik}} (1-\Psi_k)^{(1-Z_{ik})} \Psi_k^{(\alpha_\Psi-1)} (1-\Psi_k)^{(\beta_\Psi-1)} \\
&\propto \Psi_k^{(\alpha_\Psi-1)} (1-\Psi_k)^{(\beta_\Psi-1)}
\end{aligned}$$

which is  $\text{Beta}(\alpha_\Psi, \beta_\Psi)$

## 5.4 $W_k$

Only sample for  $\sum_{i=1}^n y_{ik} = 0$

$$\begin{aligned}
[W_k|\cdot] &\propto \prod_{i=1}^n [Z_{ik}|\Psi_k]^{W_k} [W_k|\lambda] \\
&\propto \prod_{i=1}^n \Psi_k^{(Z_{ik} W_k)} (1-\Psi_k)^{((1-Z_{ik}) W_k)} \lambda^{W_k} (1-\lambda)^{(1-W_k)} \\
&\propto \prod_{i=1}^n \left( \Psi_k^{Z_{ik}} (1-\Psi_k)^{(1-Z_{ik})} \lambda \right)^{W_k} (1-\lambda)^{(1-W_k)}
\end{aligned}$$

which is  $\text{Bern}(\tilde{\lambda})$  where  $\tilde{\lambda} = \left( \Psi_k^{\sum_{i=1}^n Z_{ik}} (1-\Psi_k)^{\sum_{i=1}^n (1-Z_{ik})} \lambda \right) / \left( \Psi_k^{\sum_{i=1}^n Z_{ik}} (1-\Psi_k)^{\sum_{i=1}^n (1-Z_{ik})} \lambda + (1-\lambda) \right)$

### 5.5 $\alpha_p$

$$[\alpha_p | \cdot] \propto \prod_{k=1}^{\Omega} [p_k | \alpha_p, \beta_p] [\alpha_p] I\{W_k = 1\}$$

where  $[p_k | \alpha_p, \beta_p]$  is Beta( $\alpha_p, \beta_p$ ) and  $[\alpha_p]$  is Gamma( $\alpha_{\alpha_p}, \beta_{\alpha_p}$ ). This can be sampled using Metropolis-Hastings

### 5.6 $\beta_p$

$$[\beta_p | \cdot] \propto \prod_{k=1}^{\Omega} [p_k | \alpha_p, \beta_p] [\beta_p] I\{W_k = 1\}$$

where  $[p_k | \alpha_p, \beta_p]$  is Beta( $\alpha_p, \beta_p$ ) and  $[\beta_p]$  is Gamma( $\alpha_{\beta_p}, \beta_{\beta_p}$ ). This can be sampled using Metropolis-Hastings

### 5.7 $\alpha_{\Psi}$

$$[\alpha_{\Psi} | \cdot] \propto \prod_{k=1}^{\Omega} [\Psi_k | \alpha_{\Psi}, \beta_{\Psi}] [\alpha_{\Psi}] I\{W_k = 1\}$$

where  $[\Psi_k | \alpha_{\Psi}, \beta_{\Psi}]$  is Beta( $\alpha_{\Psi}, \beta_{\Psi}$ ) and  $[\alpha_{\Psi}]$  is Gamma( $\alpha_{\alpha_{\Psi}}, \beta_{\alpha_{\Psi}}$ ). This can be sampled using Metropolis-Hastings

### 5.8 $\beta_{\Psi}$

$$[\beta_{\Psi} | \cdot] \propto \prod_{k=1}^{\Omega} [\Psi_k | \alpha_{\Psi}, \beta_{\Psi}] [\beta_{\Psi}] I\{W_k = 1\}$$

where  $[\Psi_k | \alpha_{\Psi}, \beta_{\Psi}]$  is Beta( $\alpha_{\Psi}, \beta_{\Psi}$ ) and  $[\beta_{\Psi}]$  is Gamma( $\alpha_{\beta_{\Psi}}, \beta_{\beta_{\Psi}}$ ). This can be sampled using Metropolis-Hastings

### 5.9 $\lambda$

$$\begin{aligned} [\lambda | \cdot] &\propto \prod_{k=1}^{\Omega} [W_k | \lambda] [\lambda] \\ &\propto \prod_{k=1}^{\Omega} \lambda^{W_k} (1 - \lambda)^{(1 - W_k)} \lambda^{(\alpha_{\lambda} - 1)} (1 - \lambda)^{(\beta_{\lambda} - 1)} \end{aligned}$$

which is Beta( $\alpha_{\lambda} + \sum_{k=K+1}^{\Omega} W_k, \beta_{\lambda} + \sum_{k=K+1}^{\Omega} (1 - W_k)$ )

## 6 Next Steps

- Include a probit link function