Multispecies Model- Dorazio Notation

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February 17, 2014

1 Introduction and motivation

2 Model of one season detection

2.1 Modelling assumptions

- There is an ecological motivation for the detection probabilities, the $\boldsymbol{p}$, coming from a common distribution and the ocucpancy probabilities, the $\boldsymbol{\Psi}$, coming from a common distribution
- There is no heterogeneity in unobserved covariates for $\boldsymbol{p}$ and $\boldsymbol{\Psi}$
- All species are "theoretically" available to be sampled
- No seasonality
- Closed populations within the K visits
- The actual number of species, N, lies between the observed number of species n and the number of augmented species Ω

2.2 Data model

For $j = 1, \ldots, J$ sites, $i = 1, \ldots, N$ 'actual' species (out of a total of Ω species), and for $k = 1, \ldots, K$ revisits at each site

$$y_{ij} = \begin{cases} 0 & \text{if } Z_{ij} = 0 \\ \text{Binom}(K, p_i) & \text{if } Z_{ij} = 1 \end{cases}$$

where y_{ij} is the number of times that species i is seen at site j during K visits, Z_{ij} is an indicator of presence or absence of species i at site j, and p_i is the detection probability for species i.

2.3 Process

$$\begin{aligned} Z_{ij} &= \begin{cases} 0 & \text{if } W_i = 0 \\ \text{Bern}(\Psi_i) & \text{if } W_i = 1 \end{cases} \\ p_i &\sim \text{Beta}(\alpha_p, \beta_p) \\ \Psi_i &\sim \text{Beta}(\alpha_\Psi, \beta_\Psi) \\ W_i &\sim \text{Bern}(\lambda) \end{aligned}$$

where W_i is an indicator of whether a species exists or not in the population; $\sum_{i=1}^{\Omega} W_i = N$ is the species richness; $\boldsymbol{\Psi}_i$ is the presence probability for species i; and λ is the probability of a species being real.

2.4 Parameter

$$\begin{aligned}
\alpha_p &\sim \text{Gamma}(\alpha_{\alpha_p}, \beta_{\alpha_p}) \\
\beta_p &\sim \text{Gamma}(\alpha_{\beta_p}, \beta_{\beta_p}) \\
\alpha_\Psi &\sim \text{Gamma}(\alpha_{\alpha_\Psi}, \beta_{\alpha_\Psi}) \\
\beta_\Psi &\sim \text{Gamma}(\alpha_{\beta_\Psi}, \beta_{\beta_\Psi}) \\
\lambda &\sim \text{Beta}(\alpha_\lambda, \beta_\lambda)
\end{aligned}$$

3 Ideas

- model $\Psi_k = f(\Psi_{-i})+$ randomness and species interaction
- $Z_{ij} = \begin{cases} 0 & \text{if } W_i = 0 \\ \text{Bern}(\Psi_i) & \text{if } W_i = 1 \end{cases}$ where W_i represents "ghost species"
- Let Ω represent the total number of possible species, $\Omega \geq N \geq n$
- $W_i \sim \text{Bern}(\lambda) \Leftrightarrow N \sim \text{Binom}(\Omega, \lambda) \Leftrightarrow \sum_{i=1}^{\Omega} W_i = N$
- Goal: estimate Ψ_i and predict N

4 Posterior

$$\begin{multline}
[\boldsymbol{Z}, \boldsymbol{p}, \boldsymbol{\Psi}, \boldsymbol{W}, \alpha_p, \beta_p, \alpha_\Psi, \beta_\Psi, \lambda | \boldsymbol{y}, K] \propto \\
\left[\prod_{i=1}^{\Omega}\left[\prod_{j=1}^{J}[y_{ij}|K, p_i]^{Z_{ij}} I\{y_{ij} = 0\}^{(1-Z_{ij})}[Z_{ij}|\Psi_i]^{W_i} I\{Z_{ij} = 0\}^{(1-W_i)}\right][p_i|\alpha_p, \beta_p][\Psi_i|\alpha_\Psi, \beta_\Psi][W_i|\lambda]\right] \\
\times [\alpha_p][\beta_p][\alpha_\Psi][\beta_\Psi][\lambda] \quad (1)
\end{multline}$$

5 Full conditionals

5.1 Z_{ij}

For $W_i = 0$, $Z_{ij} = 0$.

For $y_{ij} \geq 1$, $Z_{ij} = 1$.

For $y_{ij} = 0$ and $W_i = 1$:

$$\begin{aligned}
[Z_{ij}|\cdot] &\propto [y_{ij}|K, p_i]^{Z_{ij}} I\{y_{ij} = 0\}^{(1-Z_{ij})}[Z_{ij}|\Psi]^{W_i} I\{Z_{ij} = 0\}^{(1-W_i)} \\
&\propto (1-p_i)^{JZ_{ij}} \Psi_i^{Z_{ij}W_i}(1-\Psi_i)^{(1-Z_{ij})W_i} \\
&\propto \left((1-p_i)^K \Psi_i\right)^{Z_{ij}} (1-\Psi_i)^{(1-Z_{ij})}
\end{aligned}$$

which is Bern($\tilde{\Psi}_i$) where $\tilde{\Psi}_i = \left((1-p_i)^K \Psi_i\right) / \left((1-p_i)^K \Psi_i + (1-\Psi_i)\right)$

5.2 p_i

$$
\begin{aligned}
[p_i|\cdot] &\propto \prod_{j=1}^{J}[y_{ij}|K,p_i]^{Z_{ij}}[p_i|\alpha_p,\beta_p] \\
&\propto \prod_{j=1}^{J}\left(p_i^{y_{ij}}(1-p_i)^{K-y_{ij}}\right)^{Z_{ij}} p_i^{(\alpha_p-1)}(1-p_i)^{(\beta_p-1)} \\
&\propto p_i^{\left(\alpha_p-1+\sum_{j=1}^{J} y_{ij}Z_{ij}\right)}(1-p_i)^{\left(\beta_p-1+\sum_{j=1}^{J} Z_{ij}(K-y_{ij})\right)}
\end{aligned}
$$

which is Beta($\alpha_p+\sum_{j=1}^{J} y_{ij}Z_{ij},\beta_p+\sum_{j=1}^{J} Z_{ij}(K-y_{ij})$)

5.3 Ψ_i

$$
\begin{aligned}
[\Psi_i|\cdot] &\propto \prod_{j=1}^{J}[Z_{ij}|\Psi_i]^{W_i}[\Psi_i|\alpha_\Psi,\beta_\Psi] \\
&\propto \left(\prod_{j=1}^{J}\Psi_i^{Z_{ij}}(1-\Psi_i)^{(1-Z_{ij})}\right)^{W_i} \Psi_i^{(\alpha_\Psi-1)}(1-\Psi_i)^{(\beta_\Psi-1)} \\
&\propto \Psi_{ij}^{(\alpha_\Psi-1+W_i\sum_{j=1}^{J} Z_{ij})}(1-\Psi_i)^{(\beta_\Psi-1+W_i\sum_{j=1}^{J}(1-Z_{ij}))}
\end{aligned}
$$

which is Beta($\alpha_\Psi+W_i\sum_{j=1}^{J} Z_{ij},\ \beta_\Psi+W_i\sum_{j=1}^{J}(1-Z_{ij})$)

5.4 W_i

If $\sum_{j=1}^{J} Z_{ij}>0$, then $W_k=1$. So only sample for $\sum_{j=1}^{J} Z_{ij}=0$.

$$
\begin{aligned}
[W_i|\cdot] &\propto \prod_{j=1}^{J}[Z_{ij}|\Psi_i]^{W_i}[W_i|\lambda] \\
&\propto \left(\prod_{j=1}^{J}\Psi_i^{(Z_{ij}W_i)}(1-\Psi_i)^{((1-Z_{ij})W_i)}\right)\lambda^{W_i}(1-\lambda)^{(1-W_i)} \\
&\propto \left(\Psi_i^{\sum_{j=1}^{J} Z_{ij}}(1-\Psi_i)^{\sum_{j=1}^{J}(1-Z_{ij})}\lambda\right)^{W_i}(1-\lambda)^{(1-W_i)}
\end{aligned}
$$

which is Bern($\tilde{\lambda}$) where $\tilde{\lambda}=\left((1-\Psi_i)^J\lambda\right)/\left((1-\Psi_i)^J\lambda+(1-\lambda)\right)$

5.5 α_p

$$
[\alpha_p|\cdot] \propto \prod_{i=1}^{\Omega}[p_i|\alpha_p,\beta_p][\alpha_p]
$$

where $[p_i|\alpha_p,\beta_p]$ is Beta(α_p,β_p) and $[\alpha_p]$ is Gamma($\alpha_{\alpha_p},\beta_{\alpha_p}$). This can be sampled using Metropolis-Hastings

5.6 β_p

$$[\beta_p|\cdot] \propto \prod_{i=1}^{\Omega}[p_i|\alpha_p, \beta_p][\beta_p]$$

where $[p_i|\alpha_p, \beta_p]$ is Beta(α_p, β_p) and $[\beta_p]$ is Gamma$(\alpha_{\beta_p}, \beta_{\beta_p})$. This can be sampled using Metropolis-Hastings

5.7 α_Ψ

$$[\alpha_\Psi|\cdot] \propto \prod_{i=1}^{\Omega}[\Psi_i|\alpha_\Psi, \beta_\Psi][\alpha_\Psi]$$

where $[\Psi_i|\alpha_\Psi, \beta_\Psi]$ is Beta$(\alpha_\Psi, \beta_\Psi)$ and $[\alpha_\Psi]$ is Gamma$(\alpha_{\alpha_\Psi}, \beta_{\alpha_\Psi})$. This can be sampled using Metropolis-Hastings

5.8 β_Ψ

$$[\beta_\Psi|\cdot] \propto \prod_{i=1}^{\Omega}[\Psi_i|\alpha_\Psi, \beta_\Psi][\beta_\Psi]$$

where $[\Psi_i|\alpha_\Psi, \beta_\Psi]$ is Beta$(\alpha_\Psi, \beta_\Psi)$ and $[\beta_\Psi]$ is Gamma$(\alpha_{\beta_\Psi}, \beta_{\beta_\Psi})$. This can be sampled using Metropolis-Hastings

5.9 λ

$$\begin{aligned}[\lambda|\cdot] &\propto \prod_{i=1}^{\Omega}[W_i|\lambda][\lambda] \\ &\propto \prod_{i=1}^{\Omega}\lambda^{W_i}(1-\lambda)^{(1-W_i)}\lambda^{(\alpha_\lambda-1)}(1-\lambda)^{(\beta_\lambda-1)}\end{aligned}$$

which is Beta$(\alpha_\lambda + \sum_{i=1}^{\Omega} W_i, \beta_\lambda + \sum_{i=1}^{\Omega}(1-W_i))$

6 Next Steps

- Include a probit link function.