Multispecies Model

Kristin Broms, Viviana Ruiz-Guiterrez, John Tipton

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1 Introduction and motivation

2 Model of one season detection

2.1 Modelling assumptions

- There is an ecological motivation for $\boldsymbol{p}$ and $\boldsymbol{\Psi}$ coming from the same distribution
- There is no heterogeneity in unobserved covariates for $\boldsymbol{p}$ and $\boldsymbol{\Psi}$
- All species are "theoretically" available to be sampled
- No seasonality
- Closed populations within the J visits
- The actual number of species N lies between the observed number of species K and the number of augmented species Ω

2.2 Data model

For $i = 1, \ldots, n$ sites, $k = 1, \ldots, K$ observed species (out of a total of Ω species), and for $j = 1, \ldots, J$ revisits at each site

$$y_{ik} = \begin{cases} 0 & \text{if } Z_{ik} = 0 \\ \text{Binom}(J, p_k) & \text{if } Z_{ik} = 1 \end{cases}$$

where y_{ik} is the number of times that species i is seen at site k out of J visits, Z_{ik} is an indicator of presence or absence of species i at site k, and p_k is the detection probability for species k.

2.3 Process

$$\begin{aligned} Z_{ik} &= \begin{cases} 0 & \text{if } W_k = 0 \\ \text{Bern}(\Psi_k) & \text{if } W_k = 1 \end{cases} \\ p_k &\sim \text{Beta}(\alpha_p, \beta_p) \\ \Psi_k &\sim \text{Beta}(\alpha_\Psi, \beta_\Psi) \\ W_k &\sim \text{Bern}(\lambda) \end{aligned}$$

where W_k is an indicator of whether an augmented species exists or not in the population and $\sum_{k=1}^{\Omega} W_k = N$, the species richness, $\boldsymbol{\Psi}_k$ is the presence probability for species k, and λ is the probability of an augmented species being real

2.4 Parameter

$$
\begin{aligned}
\alpha_p &\sim \text{Gamma}(\alpha_{\alpha_p}, \beta_{\alpha_p}) \\
\beta_p &\sim \text{Gamma}(\alpha_{\beta_p}, \beta_{\beta_p}) \\
\alpha_\Psi &\sim \text{Gamma}(\alpha_{\alpha_\Psi}, \beta_{\alpha_\Psi}) \\
\beta_\Psi &\sim \text{Gamma}(\alpha_{\beta_\Psi}, \beta_{\beta_\Psi}) \\
\lambda &\sim \text{Beta}(\alpha_\lambda, \beta_\lambda)
\end{aligned}
$$

3 Ideas

- model $\Psi_k = f(\Psi_{-k})+$ randomness and species interaction
- $Z_{ik} = \begin{cases} 0 & \text{if } W_k = 0 \\ \text{Bern}(\Psi_k) & \text{if } W_k = 1 \end{cases}$ where W_k represents "ghost species"
- Let Ω represent the total number of possible species, $\Omega \geq N \geq K$
- $W_k \sim \text{Bern}(\lambda) \Leftrightarrow N \sim \text{Binom}(\Omega, \lambda) \Leftrightarrow \sum_{k=1}^{\Omega} W_k = N$
- Goal: estimate Ψ_k and predict N

4 Posterior

$$
\begin{aligned}
[\boldsymbol{Z}, \boldsymbol{p}, \boldsymbol{\Psi}, \boldsymbol{W}, \alpha_p, \beta_p, \alpha_\Psi, \beta_\Psi, \lambda | \boldsymbol{y}, J] \propto & \prod_{i=1}^{n} \prod_{k=1}^{\Omega} [y_{ik}|J, p_k]^{Z_{ik}} I\{y_{ik} = 0\}^{(1-Z_{ik})} [Z_{ik}|\Psi_k]^{W_k} I\{Z_{ik} = 0\}^{(1-W_k)} \\
& \times [p_k|\alpha_p, \beta_p][\Psi_k|\alpha_\Psi, \beta_\Psi][W_k|\lambda][\alpha_p][\beta_p][\alpha_\Psi][\beta_\Psi][\lambda]
\end{aligned}
$$

5 Full conditionals

5.1 Z_{ik}

For $y_{ik} = 1$, $Z_{ik} = 1$.

For $y_{ik} = 0$ and $W_k = 1$

$$
\begin{aligned}
[Z_{ik}|\cdot] &\propto [y_{ik}|J, p_k]^{Z_{ik}} I\{y_{ik} = 0\}^{(1-Z_{ik})} [Z_{ik}|\Psi]^{W_k} I\{Z_{ik} = 0\}^{(1-W_k)} \\
&\propto (1-p_k)^{JZ_{ik}} \Psi_k^{Z_{ik}W_k} (1-\Psi_k)^{(1-Z_{ik})W_k} \\
&\propto \left((1-p_k)^J \Psi_k\right)^{Z_{ik}} (1-\Psi_k)^{(1-Z_{ik})}
\end{aligned}
$$

which is Bern($\tilde{\Psi}_k$) where $\tilde{\Psi}_k = \left((1-p_k)^J \Psi_k\right) / \left((1-p_k)^J \Psi_k + (1-\Psi_k)\right)$

For $y_{ik} = 0$ and $W_k = 0$

$$
\begin{aligned}
[Z_{ik}|\cdot] &\propto [y_{ik}|J, p_k]^{Z_{ik}} I\{y_{ik} = 0\}^{(1-Z_{ik})} [Z_{ik}|\Psi]^{W_k} I\{Z_{ik} = 0\}^{(1-W_k)} \\
&\propto (1-p_k)^{JZ_{ik}} \Psi_k^{Z_{ik}W_k} (1-\Psi_k)^{(1-Z_{ik})W_k} \\
&\propto ((1-p_k)^J)^{Z_{ik}}
\end{aligned}
$$

which is Bern$((1-p_k)^J)$

5.2 p_k

$$
\begin{aligned}
[p_k|\cdot] &\propto \prod_{i=1}^{n}[y_{ik}|J,p_k]^{Z_{ik}}[p_k|\alpha_p,\beta_p] \\
&\propto \prod_{i=1}^{n}\left(p_k^{y_{ik}}(1-p_k)^{J-y_{ik}}\right)^{Z_{ik}} p_k^{(\alpha_p-1)}(1-p_k)^{(\beta_p-1)} \\
&\propto p_k^{\left(\alpha_p-1+\sum_{i=1}^{n}y_{ik}Z_{ik}\right)}(1-p_k)^{\left(\beta_p-1+\sum_{i=1}^{n}Z_{ik}(J-y_{ik})\right)}
\end{aligned}
$$

which is Beta$(\alpha_p+\sum_{i=1}^{n}y_{ik}Z_{ik}, \beta_p+\sum_{i=1}^{n}Z_{ik}(J-y_{ik}))$

5.3 Ψ_k

When $W_k=1$

$$
\begin{aligned}
[\Psi_k|\cdot] &\propto \prod_{i=1}^{n}[Z_{ik}|\Psi_k]^{W_k}[\Psi_k|\alpha_\Psi,\beta_\Psi] \\
&\propto \prod_{i=1}^{n}\Psi_{ik}^{Z_{ik}}(1-\Psi_k)^{(1-Z_{ik})}\Psi_k^{(\alpha_\Psi-1)}(1-\Psi_k)^{(\beta_\Psi-1)} \\
&\propto \Psi_{ik}^{(\alpha_\Psi-1+\sum_{i=1}^{n}Z_{ik})}(1-\Psi_k)^{(\beta_\Psi-1+\sum_{i=1}^{n}(1-Z_{ik}))}
\end{aligned}
$$

which is Beta$(\alpha_\Psi+\sum_{i=1}^{n}Z_{ik},\ \beta_\Psi+\sum_{i=1}^{n}(1-Z_{ik}))$

When $W_k=0$

$$
\begin{aligned}
[\Psi_k|\cdot] &\propto \prod_{i=1}^{n}[Z_{ik}|\Psi_k]^{W_k}[\Psi_k|\alpha_\Psi,\beta_\Psi] \\
&\propto \prod_{i=1}^{n}\Psi_{ik}^{Z_{ik}}(1-\Psi_k)^{(1-Z_{ik})}\Psi_k^{(\alpha_\Psi-1)}(1-\Psi_k)^{(\beta_\Psi-1)} \\
&\propto \Psi_{ik}^{(\alpha_\Psi-1)}(1-\Psi_k)^{(\beta_\Psi-1))}
\end{aligned}
$$

which is Beta$(\alpha_\Psi,\ \beta_\Psi)$

5.4 W_k

Only sample for $\sum_{i=1}^{n}y_{ik}=0$

$$
\begin{aligned}
[W_k|\cdot] &\propto \prod_{i=1}^{n}[Z_{ik}|\Psi_k]^{W_k}[W_k|\lambda] \\
&\propto \prod_{i=1}^{n}\Psi_k^{(Z_{ik}W_k)}(1-\Psi_k)^{((1-Z_{ik})W_k)}\lambda^{W_k}(1-\lambda)^{(1-W_k)} \\
&\propto \prod_{i=1}^{n}\left(\Psi_k^{Z_{ik}}(1-\Psi_k)^{(1-Z_{ik})}\lambda\right)^{W_k}(1-\lambda)^{(1-W_k)}
\end{aligned}
$$

which is Bern$(\tilde{\lambda})$ where $\tilde{\lambda}=\left(\Psi_k^{\sum_{i=1}^{n}Z_{ik}}(1-\Psi_k)^{\sum_{i=1}^{n}(1-Z_{ik})}\lambda\right)/\left(\Psi_k^{\sum_{i=1}^{n}Z_{ik}}(1-\Psi_k)^{\sum_{i=1}^{n}(1-Z_{ik})}\lambda+(1-\lambda)\right)$

5.5 α_p

$$[\alpha_p|\cdot] \propto \prod_{k=1}^{\Omega} [p_k|\alpha_p, \beta_p][\alpha_p] I\{W_k = 1\}$$

where $[p_k|\alpha_p, \beta_p]$ is Beta(α_p, β_p) and $[\alpha_p]$ is Gamma$(\alpha_{\alpha_p}, \beta_{\alpha_p})$. This can be sampled using Metropolis-Hastings

5.6 β_p

$$[\beta_p|\cdot] \propto \prod_{k=1}^{\Omega} [p_k|\alpha_p, \beta_p][\beta_p] I\{W_k = 1\}$$

where $[p_k|\alpha_p, \beta_p]$ is Beta(α_p, β_p) and $[\beta_p]$ is Gamma$(\alpha_{\beta_p}, \beta_{\beta_p})$. This can be sampled using Metropolis-Hastings

5.7 α_Ψ

$$[\alpha_\Psi|\cdot] \propto \prod_{k=1}^{\Omega} [\Psi_k|\alpha_\Psi, \beta_\Psi][\alpha_\Psi] I\{W_k = 1\}$$

where $[\Psi_k|\alpha_\Psi, \beta_\Psi]$ is Beta$(\alpha_\Psi, \beta_\Psi)$ and $[\alpha_\Psi]$ is Gamma$(\alpha_{\alpha_\Psi}, \beta_{\alpha_\Psi})$. This can be sampled using Metropolis-Hastings

5.8 β_Ψ

$$[\beta_\Psi|\cdot] \propto \prod_{k=1}^{\Omega} [\Psi_k|\alpha_\Psi, \beta_\Psi][\beta_\Psi] I\{W_k = 1\}$$

where $[\Psi_k|\alpha_\Psi, \beta_\Psi]$ is Beta$(\alpha_\Psi, \beta_\Psi)$ and $[\beta_\Psi]$ is Gamma$(\alpha_{\beta_\Psi}, \beta_{\beta_\Psi})$. This can be sampled using Metropolis-Hastings

5.9 λ

$$\begin{aligned}[\lambda|\cdot] &\propto \prod_{k=1}^{\Omega} [W_k|\lambda][\lambda] \\ &\propto \prod_{k=1}^{\Omega} \lambda^{W_k}(1-\lambda)^{(1-W_k)} \lambda^{(\alpha_\lambda - 1)}(1-\lambda)^{(\beta_\lambda - 1)}\end{aligned}$$

which is Beta$(\alpha_\lambda + \sum_{k=K+1}^{\Omega} W_k, \beta_\lambda + \sum_{k=K+1}^{\Omega}(1 - W_k))$

6 Next Steps

- Include a probit link function