Multispecies Model

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1 Introduction and motivation

2 Model of one season detection

2.1 Modelling assumptions

- No seasonality
-

2.2 Data model

For $i = 1, \ldots, n$ sites, $k = 1, \ldots, K$ observed species (out of a total of Ω species), and for $j = 1, \ldots, J$ revisits at each site

$$y_{ik} = \begin{cases} 0 & \text{if } Z_{ik} = 0 \\ \text{Binom}(J, p_k) & \text{if } Z_{ik} = 1 \end{cases}$$

2.3 Process

$$\begin{aligned} Z_{ik} &= \begin{cases} 0 & \text{if } W_k = 0 \\ \text{Bern}(\Psi_k) & \text{if } W_k = 1 \end{cases} \\ p_k &\sim \text{Beta}(\alpha_p, \beta_p) \\ \Psi_k &\sim \text{Beta}(\alpha_\Psi, \beta_\Psi) \\ W_k &\sim \text{Bern}(\lambda) \end{aligned}$$

2.4 Parameter

$$\begin{aligned} \alpha_p &\sim \text{Gamma}(\alpha_{\alpha_p}, \beta_{\alpha_p}) \\ \beta_p &\sim \text{Gamma}(\alpha_{\beta_p}, \beta_{\beta_p}) \\ \alpha_\Psi &\sim \text{Gamma}(\alpha_{\alpha_\Psi}, \beta_{\alpha_\Psi}) \\ \beta_\Psi &\sim \text{Gamma}(\alpha_{\beta_\Psi}, \beta_{\beta_\Psi}) \\ \lambda &\sim \text{Beta}(\alpha_\lambda, \beta_\lambda) \end{aligned}$$

3 Ideas

- model $\Psi_k = f(\Psi_{-k})+$ randomness and species interaction
- $Z_{ik} = \begin{cases} 0 & \text{if } W_k = 0 \\ \text{Bern}(\Psi_k) & \text{if } W_k = 1 \end{cases}$ where W_k represents "ghost species"
- Let Ω represent the total number of possible species, $\Omega \geq K$
- $W_k \sim \text{Bern}(\lambda) \Leftrightarrow k \sim \text{Binom}(\Omega, \lambda) \Leftrightarrow \sum_{k=1}^{\Omega} W_k = K$
- Goal: estimate Ψ_k and predict Ω

4 Posterior

$$
\begin{aligned}
[\boldsymbol{Z}, \boldsymbol{p}, \boldsymbol{\Psi}, \boldsymbol{W}, \alpha_p, \beta_p, \alpha_\Psi, \beta_\Psi, \lambda | \boldsymbol{y}, J] \propto & \prod_{i=1}^{n} \prod_{k=1}^{\Omega} [y_{ik}|J, p_k]^{Z_{ik}} I\{y_{ik} = 0\}^{(1-Z_{ik})} [Z_{ik}|\Psi_k]^{W_k} I\{Z_{ik} = 0\}^{(1-W_k)} \\
& \times [p_k|\alpha_p, \beta_p][\Psi_k|\alpha_\Psi, \beta_\Psi][W_k|\lambda][\alpha_p][\beta_p][\alpha_\Psi][\beta_\Psi][\lambda]
\end{aligned}
$$

5 Full conditionals

5.1 Z_{ik}

For $y_{ik} = 1$, $Z_{ik} = 1$. For $y_{ik} = 0$ and $W_k = 1$

$$
\begin{aligned}
[Z_i k|\cdot] &\propto [y_{ik}|J, p_k]^{Z_{ik}} I\{y_{ik} = 0\}^{(1-Z_{ik})} [Z_{ik}|\Psi]^{W_k} I\{Z_{ik} = 0\}^{(1-W_k)} \\
&\propto (1 - p_k)^{J Z_{ik}} \Psi_k^{Z_{ik} W_k} (1 - \Psi_k)^{(1-Z_{ik}) W_k} \\
&\propto \left((1 - p_k)^J \Psi_k\right)^{Z_{ik}} (1 - \Psi_k)^{(1-Z_{ik})}
\end{aligned}
$$

which is $\text{Bern}(\tilde{\Psi}_k)$ where $\tilde{\Psi}_k = \left((1-p_k)^J \Psi_k\right) / \left((1-p_k)^J \Psi_k + (1 - \Psi_k)\right)$

5.2 p_k

Only sample when $Z_{ik} = 1$

$$
\begin{aligned}
[p_k|\cdot] &\propto \prod_{i=1}^{n} [y_{ik}|J, p_k]^{Z_{ik}} [p_k|\alpha_p, \beta_p] \\
&\propto \prod_{i=1}^{n} \left(p_k^{y_{ik}} (1 - p_k)^{J - y_{ik}}\right)^{Z_{ik}} p_k^{(\alpha_p - 1)} (1 - p_k)^{(\beta_p - 1)} \\
&\propto p_k^{\left(\alpha_p - 1 + \sum_{i=1}^{n} y_{ik} Z_{ik}\right)} (1 - p_k)^{\left(\beta_p - 1 + \sum_{i=1}^{n} Z_{ik}(J - y_{ik})\right)}
\end{aligned}
$$

which is $\text{Beta}(\alpha_p + \sum_{i=1}^{n} y_{ik} Z_{ik}, \beta_p + \sum_{i=1}^{n} Z_{ik}(J - y_{ik}))$

5.3 Ψ_k

Only sample when $W_k = 1$

$$
\begin{aligned}
[\Psi_k|\cdot] &\propto \prod_{i=1}^{n} [Z_{ik}|\Psi_k]^{W_k} [\Psi_k|\alpha_\Psi, \beta_\Psi] \\
&\propto \prod_{i=1}^{n} \Psi_{ik}^{Z_{ik}} (1-\Psi_k)^{(1-Z_{ik})} \Psi_k^{(\alpha_\Psi - 1)} (1-\Psi_k)^{(\beta_\Psi - 1)} \\
&\propto \Psi_{ik}^{(\alpha_\Psi - 1 + \sum_{i=1}^{n} Z_{ik})} (1-\Psi_k)^{(\beta_\Psi - 1 + \sum_{k=1} (1-Z_{ik}))}
\end{aligned}
$$

which is Beta$(\alpha_\Psi + \sum_{i=1}^{n} Z_{ik},\ \beta_\Psi + \sum_{k=1}(1-Z_{ik}))$

5.4 W_k

Only sample for $y_i = 0$

$$
\begin{aligned}
[W_k|\cdot] &\propto \prod_{i=1}^{n} [Z_{ik}|\Psi_k]^{W_k} [W_k|\lambda] \\
&\propto \prod_{i=1}^{n} \Psi_k^{(Z_{ik} W_k)} (1-\Psi_k)^{((1-Z_{ik})W_k)} \lambda^{W_k} (1-\lambda)^{(1-W_k)} \\
&\propto \prod_{i=1}^{n} \left(\Psi_k^{Z_{ik}} (1-\Psi_k)^{(1-Z_{ik})} \lambda \right)^{W_k} (1-\lambda)^{(1-W_k)}
\end{aligned}
$$

which is Bern$(\tilde{\lambda})$ where $\tilde{\lambda} = \left(\Psi_k^{\sum_{i=1}^{n} Z_{ik}} (1-\Psi_k)^{\sum_{i=1}^{n}(1-Z_{ik})} \lambda \right) / \left(\Psi_k^{\sum_{i=1}^{n} Z_{ik}} (1-\Psi_k)^{\sum_{i=1}^{n}(1-Z_{ik})} \lambda) + (1-\lambda) \right)$

5.5 α_p

Only sample for $W_k = 1$

$$
[\alpha_p|\cdot] \propto \prod_{k=1}^{\Omega} [p_k|\alpha_p, \beta_p][\alpha_p]
$$

where $[p_k|\alpha_p, \beta_p]$ is Beta(α_p, β_p) and $[\alpha_p]$ is Gamma$(\alpha_{\alpha_p}, \beta_{\alpha_p})$. This can be sampled using Metropolis-Hastings

5.6 β_p

Only sample for $W_k = 1$

$$
[\beta_p|\cdot] \propto \prod_{k=1}^{\Omega} [p_k|\alpha_p, \beta_p][\beta_p]
$$

where $[p_k|\alpha_p, \beta_p]$ is Beta(α_p, β_p) and $[\beta_p]$ is Gamma$(\alpha_{\beta_p}, \beta_{\beta_p})$. This can be sampled using Metropolis-Hastings

5.7 α_Ψ

Only sample for $W_k = 1$

$$
[\alpha_\Psi|\cdot] \propto \prod_{k=1}^{\Omega} [\Psi_k|\alpha_\Psi, \beta_\Psi][\alpha_\Psi]
$$

where $[\Psi_k|\alpha_\Psi, \beta_\Psi]$ is Beta$(\alpha_\Psi, \beta_\Psi)$ and $[\alpha_\Psi]$ is Gamma$(\alpha_{\alpha_\Psi}, \beta_{\alpha_\Psi})$. This can be sampled using Metropolis-Hastings

5.8 β_Ψ

Only sample for $W_k = 1$

$$[\beta_\Psi|\cdot] \propto \prod_{k=1}^{\Omega} [\Psi_k|\alpha_\Psi, \beta_\Psi][\beta_\Psi]$$

where $[\Psi_k|\alpha_\Psi, \beta_\Psi]$ is Beta$(\alpha_\Psi, \beta_\Psi)$ and $[\beta_\Psi]$ is Gamma$(\alpha_{\beta_\Psi}, \beta_{\beta_\Psi})$. This can be sampled using Metropolis-Hastings

5.9 λ

$$\begin{aligned}
[\lambda|\cdot] &\propto \prod_{k=1}^{\Omega} [W_k|\lambda][\lambda] \\
&\propto \prod_{k=1}^{\Omega} \lambda^{W_k}(1-\lambda)^{(1-W_k)}\lambda^{(\alpha_\lambda - 1)}(1-\lambda)^{(\beta_\lambda - 1)}
\end{aligned}$$

which is Beta$(\alpha_\lambda + \sum_{k=1}^{\Omega} W_k, \beta_\lambda + \sum_{k=1}^{\Omega}(1 - W_k))$