Multispecies Model

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1 Introduction and motivation

2 Model of one season detection

2.1 Modelling assumptions

- There is an ecological motivation for $\boldsymbol{p}$ and $\boldsymbol{\Psi}$ coming from the same distribution

- There is no heterogeneity in unobserved covariates for $\boldsymbol{p}$ and $\boldsymbol{\Psi}$

- All species are "theoretically" available to be sampled

- No seasonality

- Closed populations within the J visits

- The actual number of species N lies between the observed number of species K and the number of augmented species Ω

2.2 Data model

For $i = 1, \ldots, n$ sites, $k = 1, \ldots, K$ observed species (out of a total of Ω species), and for $j = 1, \ldots, J$ revisits at each site

$$y_{ik} = \begin{cases} 0 & \text{if } Z_{ik} = 0 \\ \text{Binom}(J, p_k) & \text{if } Z_{ik} = 1 \end{cases}$$

where y_{ik} is the number of times that species i is seen at site k out of J visits, Z_{ik} is an indicator of presence or absence of species i at site k, and p_k is the detection probability for species k.

2.3 Process Model

$$Z_{ik} = \begin{cases} 0 & \text{if } W_k = 0 \\ \text{Bern}(\Psi_k) & \text{if } W_k = 1 \end{cases}$$

$$W_k \sim \text{Bern}(\lambda)$$

where W_k is an indicator of whether an augmented species exists or not in the population and $\sum_{k=1}^{\Omega} W_k = N$, the species richness, $\boldsymbol{\Psi}_k$ is the presence probability for species k, and λ is the probability of an augmented species being real

2.4   Parameter Model

$$p_k \sim \text{Beta}(\alpha_p, \beta_p)$$
$$\Psi_k \sim \text{Beta}(\alpha_\Psi, \beta_\Psi)$$
$$\lambda \sim \text{Beta}(\alpha_\lambda, \beta_\lambda)$$

3   Ideas

- model $\Psi_k = f(\Psi_{-k}) +$ randomness and species interaction

- $Z_{ik} = \begin{cases} 0 & \text{if } W_k = 0 \\ \text{Bern}(\Psi_k) & \text{if } W_k = 1 \end{cases}$ where W_k represents "ghost species"

- Let Ω represent the total number of possible species, $\Omega \geq N \geq K$

- $W_k \sim \text{Bern}(\lambda) \Leftrightarrow N \sim \text{Binom}(\Omega, \lambda) \Leftrightarrow \sum_{k=1}^{\Omega} W_k = N$

- Goal: estimate Ψ_k and predict N

4   Posterior

$$[\boldsymbol{Z}, \boldsymbol{p}, \boldsymbol{\Psi}, \boldsymbol{W}, \lambda | \boldsymbol{y}, J] \propto \prod_{i=1}^{n} \prod_{k=1}^{\Omega} [y_{ik}|J, p_k]^{Z_{ik}} I\{y_{ik} = 0\}^{(1-Z_{ik})} [Z_{ik}|\Psi_k]^{W_k} I\{Z_{ik} = 0\}^{(1-W_k)}$$
$$\times [W_k|\lambda][p_k][\Psi_k][\lambda]$$

5   Full conditionals

5.1 Z_{ik}

$$[Z_{ik}|\cdot] \propto [y_{ik}|J, p_k]^{Z_{ik}} I\{y_{ik} = 0\}^{(1-Z_{ik})} [Z_{ik}|\Psi_k]^{W_k} I\{Z_{ik} = 0\}^{(1-W_k)}$$
$$\propto \left(p_k^{y_{ik}} (1-p_k)^{(J-y_{ik})} \Psi_k^{W_k} \right)^{Z_{ik}} (1-\Psi_k)^{W_k^{(1-Z_{ik})}}$$
$$\propto \left(p_k^{y_{ik}} (1-p_k)^{(J-y_{ik})} \Psi_k^{W_k} \right)^{Z_{ik}} \left((1-\Psi_k)^{W_k} \right)^{(1-Z_{ik})}$$

For $y_{ik} > 0$ then $Z_{ik} = 1$.

For $y_{ik} = 0$ and $W_k = 0$ then $Z_{ik} = 0$.

For $y_{ik} = 0$ and $W_k = 1$ then $Z_{ik} \sim \text{Bern}(\tilde{\Psi})$ where $\tilde{\Psi}_k = \left((1-p_k)^J \Psi_k \right) / \left((1-p_k)^J \Psi_k + 1 - \Psi_k \right)$

5.2 p_k

$$[p_k|\cdot] \propto \prod_{i=1}^{n}[y_{ik}|J,p_k]^{Z_{ik}}[p_k|\alpha_p,\beta_p]$$

$$\propto \prod_{i=1}^{n}\left(p_k^{y_{ik}}(1-p_k)^{(J-y_{ik})}\right)^{Z_{ik}}p_k^{(\alpha_p-1)}(1-p_k)^{(\beta_p-1)}$$

$$\propto p_k^{\left(\alpha_p-1+\sum_{i=1}^{n}y_{ik}Z_{ik}\right)}(1-p_k)^{\left(\beta_p-1+\sum_{i=1}^{n}Z_{ik}(J-y_{ik})\right)}$$

which is Beta$(\alpha_p+\sum_{i=1}^{n}y_{ik}Z_{ik},\ \beta_p+\sum_{i=1}^{n}Z_{ik}(J-y_{ik}))$

5.3 Ψ_k

$$[\Psi_k|\cdot] \propto \prod_{i=1}^{n}[Z_{ik}|\Psi_k]^{W_k}[\Psi_k|\alpha_\Psi,\beta_\Psi]$$

$$\propto \prod_{i=1}^{n}(\Psi_k^{Z_{ik}})^{W_k}\left((1-\Psi_k)^{(1-Z_{ik})}\right)^{W_k}\Psi_k^{(\alpha_\Psi-1)}(1-\Psi_k)^{(\beta_\Psi-1)}$$

$$\propto \Psi_{ik}^{\left(\alpha_\Psi-1+W_k\sum_{i=1}^{n}Z_{ik}\right)}(1-\Psi_k)^{\left(\beta_\Psi-1+W_k\sum_{i=1}^{n}(1-Z_{ik})\right)}$$

which is Beta $\left(\alpha_\Psi+W_k\sum_{i=1}^{n}Z_{ik},\ \beta_\Psi+W_k\sum_{i=1}^{n}(1-Z_{ik})\right)$

5.4 W_k

$$[W_k|\cdot] \propto \prod_{i=1}^{n}[Z_{ik}|\Psi_k]^{W_k}I\{Z_{ik}=0\}^{(1-W_k)}[W_k|\lambda]$$

$$\propto \prod_{i=1}^{n}\Psi_k^{(Z_{ik}W_k)}(1-\Psi_k)^{((1-Z_{ik})W_k)}I\{Z_{ik}=0\}^{(1-W_k)}\lambda^{W_k}(1-\lambda)^{(1-W_k)}$$

$$\propto \prod_{i=1}^{n}\left(\Psi_k^{Z_{ik}}(1-\Psi_k)^{(1-Z_{ik})}\lambda\right)^{W_k}\left(I\{Z_{ik}=0\}*(1-\lambda)\right)^{(1-W_k)}$$

$$\propto \left(\Psi_k^{\sum_{i=1}^{n}Z_{ik}}(1-\Psi_k)^{\sum_{i=1}^{n}(1-Z_{ik})}\lambda\right)^{W_k}\left(I\{\sum_{i=1}^{n}Z_{ik}=0\}*(1-\lambda)\right)^{(1-W_k)}$$

If $\sum_{i=1}^{n}Z_{ik}>0$ then $W_k=1$.

If $\sum_{i=1}^{n}Z_{ik}=0$ then $W_k \sim \text{Bern}(\tilde\lambda)$ where $\tilde\lambda=((1-\Psi_k)^n\lambda)/((1-\Psi_k)^n\lambda+(1-\lambda))$

5.5 α_p

$$[\alpha_p|\cdot] \propto \prod_{k=1}^{\Omega}[p_k|\alpha_p,\beta_p][\alpha_p]I\{W_k=1\}$$

where $[p_k|\alpha_p,\beta_p]$ is Beta(α_p,β_p) and $[\alpha_p]$ is Gamma($\alpha_{\alpha_p},\beta_{\alpha_p}$). This can be sampled using Metropolis-Hastings

5.6 β_p

$$[\beta_p|\cdot] \propto \prod_{k=1}^{\Omega}[p_k|\alpha_p,\beta_p][\beta_p]I\{W_k=1\}$$

where $[p_k|\alpha_p,\beta_p]$ is Beta(α_p,β_p) and $[\beta_p]$ is Gamma($\alpha_{\beta_p},\beta_{\beta_p}$). This can be sampled using Metropolis-Hastings

5.7 α_Ψ

$$[\alpha_\Psi|\cdot] \propto \prod_{k=1}^{\Omega}[\Psi_k|\alpha_\Psi,\beta_\Psi][\alpha_\Psi]I\{W_k=1\}$$

where $[\Psi_k|\alpha_\Psi,\beta_\Psi]$ is Beta(α_Ψ,β_Ψ) and $[\alpha_\Psi]$ is Gamma($\alpha_{\alpha_\Psi},\beta_{\alpha_\Psi}$). This can be sampled using Metropolis-Hastings

5.8 β_Ψ

$$[\beta_\Psi|\cdot] \propto \prod_{k=1}^{\Omega}[\Psi_k|\alpha_\Psi,\beta_\Psi][\beta_\Psi]I\{W_k=1\}$$

where $[\Psi_k|\alpha_\Psi,\beta_\Psi]$ is Beta(α_Ψ,β_Ψ) and $[\beta_\Psi]$ is Gamma($\alpha_{\beta_\Psi},\beta_{\beta_\Psi}$). This can be sampled using Metropolis-Hastings

5.9 λ

$$[\lambda|\cdot] \propto \prod_{k=1}^{\Omega}[W_k|\lambda][\lambda]$$
$$\propto \prod_{k=1}^{\Omega}\lambda^{W_k}(1-\lambda)^{(1-W_k)}\lambda^{(\alpha_\lambda-1)}(1-\lambda)^{(\beta_\lambda-1)}$$

which is Beta($\alpha_\lambda + \sum_{k=K+1}^{\Omega}W_k, \beta_\lambda + \sum_{k=K+1}^{\Omega}(1-W_k)$)

6 Next Steps

- Include a probit link function