Barcast Model

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1 Barcast Model

1.1 Data Model

$$\boldsymbol{W}_t = \boldsymbol{H}_t \boldsymbol{T}_t + \boldsymbol{B}_t + \boldsymbol{\eta}_t \qquad \boldsymbol{\eta}_t \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_t) \qquad \boldsymbol{\Sigma}_t = \left(\begin{array}{cc} \boldsymbol{\Sigma}_{It} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma}_{Pt} \end{array} \right)$$

where $\boldsymbol{W}_t = \left(\begin{array}{c} \boldsymbol{W}_{It} \\ \boldsymbol{W}_{Pt} \end{array} \right)$ is a vector of instrumental observations $\boldsymbol{W}_{It}$ and proxy observations $\boldsymbol{W}_{Pt}$ at time t, $\boldsymbol{T}_t = \left(\begin{array}{c} \boldsymbol{T}_{It} \\ \boldsymbol{T}_{Pt} \end{array} \right)$ is a vector of latent climate variables , $\boldsymbol{H}_t = \left(\begin{array}{c} \boldsymbol{H}_{It} \\ \beta_1 \boldsymbol{H}_{Pt} \end{array} \right)$, $\boldsymbol{B}_t = \left(\begin{array}{c} \boldsymbol{H}_{It} \\ \boldsymbol{H}_{Pt} \end{array} \right) \left(\begin{array}{c} \boldsymbol{0}_{N_I} \\ \beta_0 \boldsymbol{1}_{N_P} \end{array} \right)$, $\boldsymbol{\Sigma}_{It} = \tau_I^2 \boldsymbol{I}_{N_{It}}$, $\boldsymbol{\Sigma}_{Pt} = \tau_P^2 \boldsymbol{I}_{N_{Pt}}$, N_I is the total number of instrumental observations, N_P is the total number of proxy observations, N_{It} is the number of instrumental observations at time t, N_{Pt} is the number of proxy observations at time t, $\boldsymbol{H}_{It}$ and $\boldsymbol{H}_{Pt}$ are selection matrices indicating whether the instrumental or proxy variable was measured at time t, and β_0 and β_1 are regression coefficients relating the proxy observations and latent field.

1.2 Process Model

$$\boldsymbol{T}_t - \mu \boldsymbol{1}_N = \alpha \left(\boldsymbol{T}_{t-1} - \mu \boldsymbol{1}_N \right) + \boldsymbol{\epsilon}_t \quad \boldsymbol{\epsilon}_t \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}}) \quad \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} = \sigma^2 \exp\left(-\phi \boldsymbol{D} \right)$$

where $\boldsymbol{D}$ is the distance matrix between observation points

1.3   Parameter Model

$$\boldsymbol{T}_0 \sim \text{N}\left(\tilde{\boldsymbol{\mu}}_0, \tilde{\boldsymbol{\Sigma}}_0\right) \qquad\qquad \tilde{\boldsymbol{\mu}}_0 = \boldsymbol{0} \qquad\qquad\qquad \tilde{\boldsymbol{\Sigma}}_0 = \tilde{\sigma}_0^2 \boldsymbol{I}$$

$$\alpha \sim \text{U}\left(0,1,\right)$$

$$\mu \sim \text{N}\left(\mu_0, \sigma_0^2\right) \qquad \text{For PDSI } \mu_0 = 0 \qquad\qquad\qquad \sigma_0^2 = 1$$

$$\sigma^2 \sim \text{IG}\left(\alpha_{\sigma^2}, \beta_{\sigma^2}\right)$$

$$\phi \sim \text{IG}\left(\alpha_\phi, \beta_\phi\right)$$

$$\tau_I \sim \text{IG}\left(\alpha_I, \beta_I\right)$$

$$\tau_P \sim \text{IG}\left(\alpha_P, \beta_P\right)$$

$$\beta_1 \sim \text{N}\left(\mu_{\beta_1}, \sigma_{\beta_1}^2\right) \qquad\qquad \mu_{\beta_1} = \left(\frac{(1-\tau_P^2)(1-\alpha^2)}{\sigma^2}\right)^{-\frac{1}{2}} \qquad \sigma_{\beta_1}^2 = 8$$

$$\beta_0 \sim \text{N}\left(\mu_{\beta_0}, \sigma_{\beta_0}^2\right) \qquad\qquad \mu_{\beta_0} = -\mu_{\beta_1} \qquad\qquad\qquad\quad \sigma_{\beta_1}^2 = 8$$

Note that the prior values are set as the prior modes (e.g. μ_{β_1} is a function of the prior modes for τ_P^2, α, and σ^2).

2   Posterior

$$\prod_{t=1}^{T}\left[\boldsymbol{T}_t, \beta_0, \beta_1, \mu, \alpha, \tau_I^2, \tau_P^2, \sigma^2, \phi \big| \boldsymbol{W}_t, T\right] \propto \prod_{t=1}^{T}\left[\boldsymbol{W}_t \big| \boldsymbol{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \propto \left[\boldsymbol{T}_t \big| \boldsymbol{T}_t, \mu, \alpha, \sigma^2, \phi\right]\left[\boldsymbol{T}_0\right]$$

$$\times \left[\beta_0\right]\left[\beta_1\right]\left[\tau_I^2\right]\left[\tau_P^2\right]\left[\mu\right]\left[\alpha\right]\left[\sigma^2\right]\left[\phi\right]$$

3   Full Conditionals

3.1   Full Conditional for $\boldsymbol{T}_0$

$$\left[\boldsymbol{T}_0 \big| \cdot\right] \propto \left[\boldsymbol{T}_1 \big| \boldsymbol{T}_0, \mu, \alpha, \sigma^2, \phi\right]\left[\boldsymbol{T}_0\right]$$

$$\propto \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_1 - \alpha\boldsymbol{T}_0 - (1-\alpha)\mu\boldsymbol{1}_N\right)^T \boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_1 - \alpha\boldsymbol{T}_0 - (1-\alpha)\mu\boldsymbol{1}_N\right)\right\}$$

$$\times \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_0 - \tilde{\boldsymbol{\mu}}_0\right)^T \tilde{\boldsymbol{\Sigma}}_0^{-1}\left(\boldsymbol{T}_0 - \tilde{\boldsymbol{\mu}}_0\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{T}_0^T\left(\alpha^2\boldsymbol{\Sigma}_\epsilon^{-1} + \tilde{\boldsymbol{\Sigma}}_0^{-1}\right)\boldsymbol{T}_0 - 2\boldsymbol{T}_0^T\left(\alpha\boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_1 - (1-\alpha)\mu\boldsymbol{1}_N\right) + \tilde{\boldsymbol{\Sigma}}_0^{-1}\tilde{\boldsymbol{\mu}}\right)\right]\right\}$$

which is MVN$(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with

$$\boldsymbol{A}^{-1} = \left(\alpha^2 \boldsymbol{\Sigma}_\epsilon^{-1} + \tilde{\boldsymbol{\Sigma}}_0^{-1}\right)^{-1}$$

$$\boldsymbol{b} = \alpha \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_1 - (1-\alpha)\mu \boldsymbol{1}_N\right) + \tilde{\boldsymbol{\Sigma}}_0^{-1} \tilde{\boldsymbol{\mu}}.$$

3.2 Full Conditional for $\boldsymbol{T}_t$

For $t = 1, \dots T - 1$,

$$[\boldsymbol{T}_t|\cdot] \propto \left[\boldsymbol{W}_t|\boldsymbol{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \left[\boldsymbol{T}_t|\boldsymbol{T}_{t-1}, \mu, \alpha, \sigma^2, \phi\right] \left[\boldsymbol{T}_{t+1}|\boldsymbol{T}_t, \mu, \alpha, \sigma^2, \phi\right]$$

$$\propto \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t \boldsymbol{T}_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t \boldsymbol{T}_t + \boldsymbol{B}_t)\right)\right\}$$

$$\times \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\mu\boldsymbol{1}_N\right)^T \boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\mu\boldsymbol{1}_N\right)\right\}$$

$$\times \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_{t+1} - \alpha\boldsymbol{T}_t - (1-\alpha)\mu\boldsymbol{1}_N\right)^T \boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_{t+1} - \alpha\boldsymbol{T}_t - (1-\alpha)\mu\boldsymbol{1}_N\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\boldsymbol{T}_t^T \left(\boldsymbol{H}_t^T \boldsymbol{\Sigma}_t^{-1}\boldsymbol{H}_t + (\alpha^2+1)\boldsymbol{\Sigma}_\epsilon^{-1}\right)\boldsymbol{T}_t\right\}$$

$$\times \exp\left\{-\boldsymbol{T}_t^T \left(\boldsymbol{H}_t^T \boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t - \boldsymbol{B}_t\right) + \boldsymbol{\Sigma}_\epsilon^{-1}\left(\alpha\boldsymbol{T}_{t-1} + (1-\alpha)\mu\boldsymbol{1}_N\right) + \alpha\boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_{t+1} - (1-\alpha)\mu\boldsymbol{1}_N\right)\right)\right\}$$

which is $\mathrm{MVN}(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with

$$\boldsymbol{A}^{-1} = \left(\boldsymbol{H}_t^T \boldsymbol{\Sigma}_t^{-1}\boldsymbol{H}_t + (\alpha^2+1)\boldsymbol{\Sigma}_\epsilon^{-1}\right)^{-1}$$

$$\boldsymbol{b} = \boldsymbol{H}_t^T \boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t - \boldsymbol{B}_t\right) + \boldsymbol{\Sigma}_\epsilon^{-1}\left(\alpha\left(\boldsymbol{T}_{t+1} + \boldsymbol{T}_{t-1}\right) + (1-\alpha)^2 \mu\boldsymbol{1}_N\right).$$

For $t = T$,

$$[\boldsymbol{T}_T|\cdot] \propto \left[\boldsymbol{W}_T|\boldsymbol{T}_T, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \left[\boldsymbol{T}_T|\boldsymbol{T}_{T-1}, \mu, \alpha, \sigma^2, \phi\right]$$

$$\propto \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_T - (\boldsymbol{H}_T \boldsymbol{T}_T - \boldsymbol{B}_T)\right)^T \boldsymbol{\Sigma}_T^{-1}\left(\boldsymbol{W}_T - (\boldsymbol{H}_T \boldsymbol{T}_T - \boldsymbol{B}_T)\right)\right\}$$

$$\times \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_T - \alpha\boldsymbol{T}_{T-1} - (1-\alpha)\mu\boldsymbol{1}_N\right)^T \boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_T - \alpha\boldsymbol{T}_{T-1} - (1-\alpha)\mu\boldsymbol{1}_N\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\boldsymbol{T}_T^T \left(\boldsymbol{H}_T^T \boldsymbol{\Sigma}_T^{-1}\boldsymbol{H}_T + \boldsymbol{\Sigma}_\epsilon^{-1}\right)\boldsymbol{T}_T\right\}$$

$$\times \exp\left\{-\boldsymbol{T}_T^T \left(\boldsymbol{H}_T^T \boldsymbol{\Sigma}_T^{-1}\left(\boldsymbol{W}_T - \boldsymbol{B}_T\right) + \boldsymbol{\Sigma}_\epsilon^{-1}\left(\alpha\boldsymbol{T}_{T-1} + (1-\alpha)\mu\boldsymbol{1}_N\right)\right)\right\}$$

which is MVN$(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with

$$\boldsymbol{A}^{-1} = \left(\boldsymbol{H}_T^T \boldsymbol{\Sigma}_T^{-1} \boldsymbol{H}_T + \boldsymbol{\Sigma}_\epsilon^{-1}\right)^{-1}$$
$$\boldsymbol{b} = \boldsymbol{H}_T^T \boldsymbol{\Sigma}_T^{-1} \left(\boldsymbol{W}_T - \boldsymbol{B}_T\right) + \boldsymbol{\Sigma}_\epsilon^{-1} \left(\alpha \boldsymbol{T}_{T-1} + (1-\alpha)\,\mu \boldsymbol{1}_N\right).$$

3.3   Full conditional for β_0

$$[\beta_0|\cdot] \propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t|\boldsymbol{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right][\beta_0]$$

$$\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)\right\}$$

$$\times \exp\left\{-\frac{1}{2}\frac{(\beta_0 - \mu_{\beta_0})^2}{\sigma_{\beta_0}^2}\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\beta_0^2 \left(\sum_{t=1}^{T} \boldsymbol{1}_{N_{Pt}}^T \boldsymbol{\Sigma}_{Pt}^{-1} \boldsymbol{1}_{N_{Pt}} + \frac{1}{\sigma_{\beta_0}^2}\right) - \beta_0 \left(\sum_{t=1}^{T} \boldsymbol{1}_{N_{Pt}}^T \boldsymbol{\Sigma}_{Pt}^{-1}\left(\boldsymbol{W}_{Pt} - \beta_1 \boldsymbol{H}_{Pt}\boldsymbol{T}_t\right) + \frac{\mu_{\beta_0}}{\sigma_{\beta_0}^2}\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\beta_0^2 \left(\frac{M_P}{\tau_P^2} + \frac{1}{\sigma_{\beta_0}^2}\right) - \beta_0 \left(\frac{1}{\tau_P^2}\sum_{t=1}^{T} \boldsymbol{1}_{N_{Pt}}^T\left(\boldsymbol{W}_{Pt} - \beta_1 \boldsymbol{H}_{Pt}\boldsymbol{T}_t\right) + \frac{\mu_{\beta_0}}{\sigma_{\beta_0}^2}\right)\right\}$$

which is MVN$(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with

$$\boldsymbol{A}^{-1} = \left(\frac{M_P}{\tau_P^2} + \frac{1}{\sigma_{\beta_0}^2}\right)^{-1}$$

$$\boldsymbol{b} = \frac{1}{\tau_P^2}\sum_{t=1}^{T} \boldsymbol{1}_{N_{Pt}}^T\left(\boldsymbol{W}_{Pt} - \beta_1 \boldsymbol{H}_{Pt}\boldsymbol{T}_t\right) + \frac{\mu_{\beta_0}}{\sigma_{\beta_0}^2}.$$

3.4 Full conditional for β_1

$$[\beta_1|\cdot] \propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t|\boldsymbol{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] [\beta_1]$$

$$\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)\right\}$$

$$\times \exp\left\{-\frac{1}{2}\frac{(\beta_1 - \mu_{\beta_1})^2}{\sigma_{\beta_1}^2}\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\beta_1^2\left(\sum_{t=1}^{T}(\boldsymbol{H}_{Pt}\boldsymbol{T}_t)^T \boldsymbol{\Sigma}_{Pt}^{-1}(\boldsymbol{H}_{Pt}\boldsymbol{T}_t) + \frac{1}{\sigma_{\beta_1}^2}\right)\right\}$$

$$\times \exp\left\{-\beta_1\left(\sum_{t=1}^{T}(\boldsymbol{H}_{Pt}\boldsymbol{T}_t)^T \boldsymbol{\Sigma}_{Pt}^{-1}(\boldsymbol{W}_{Pt} - \beta_0\boldsymbol{1}_{N_{Pt}}) + \frac{\mu_{\beta_1}}{\sigma_{\beta_1}^2}\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\beta_1^2\left(\frac{1}{\tau_P^2}\sum_{t=1}^{T}(\boldsymbol{H}_{Pt}\boldsymbol{T}_t)^T \boldsymbol{\Sigma}_{Pt}^{-1}(\boldsymbol{H}_{Pt}\boldsymbol{T}_t) + \frac{1}{\sigma_{\beta_1}^2}\right)\right\}$$

$$\times \exp\left\{-\beta_1\left(\frac{1}{\tau_P^2}\sum_{t=1}^{T}(\boldsymbol{H}_{Pt}\boldsymbol{T}_t)^T(\boldsymbol{W}_{Pt} - \beta_0\boldsymbol{1}_{N_{Pt}}) + \frac{\mu_{\beta_1}}{\sigma_{\beta_1}^2}\right)\right\}$$

which is $\mathrm{MVN}(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with

$$\boldsymbol{A}^{-1} = \left(\frac{1}{\tau_P^2}\sum_{t=1}^{T}(\boldsymbol{H}_{Pt}\boldsymbol{T}_t)^T(\boldsymbol{H}_{Pt}\boldsymbol{T}_t) + \frac{1}{\sigma_{\beta_1}^2}\right)^{-1}$$

$$\boldsymbol{b} = \frac{1}{\tau_P^2}\sum_{t=1}^{T}(\boldsymbol{H}_{Pt}\boldsymbol{T}_t)^T(\boldsymbol{W}_{Pt} - \beta_0\boldsymbol{1}_{N_{Pt}}) + \frac{\mu_{\beta_1}}{\sigma_{\beta_1}^2}.$$

NOTE: My devivation includes $\boldsymbol{H}_{Pt}$ in the $\boldsymbol{A}^{-1}$ and $\boldsymbol{b}$ terms but is not present in Martin's paper... This is worth pursuing futher. It seems that Martin uses $\boldsymbol{T}_{Pt}$ to denote $\boldsymbol{H}_{Pt}\boldsymbol{T}_t$??

3.5 Full conditional for μ

$$[\mu|\cdot] \propto \prod_{t=1}^{T} \left[\boldsymbol{T}_t|\boldsymbol{T}_{t-1}, \mu, \alpha, \sigma^2, \phi\right] [\mu]$$

$$\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\mathbf{1}_N\right)^T \boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\mathbf{1}_N\right)\right\}$$

$$\times \exp\left\{-\frac{1}{2}\frac{(\mu-\mu_0)^2}{\sigma_0^2}\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\mu^2\left(\sum_{t=1}^{T}(1-\alpha)^2\,\mathbf{1}_N^T\boldsymbol{\Sigma}_\epsilon^{-1}\mathbf{1}_N + \frac{1}{\sigma_0^2}\right) - \mu\left((1-\alpha)\,\mathbf{1}_N^T\boldsymbol{\Sigma}_\epsilon^{-1}\sum_{t=1}^{T}(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1}) + \frac{\mu_0}{\sigma_0^2}\right)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\mu^2\left(T(1-\alpha)^2\,\frac{N}{\sigma^2} + \frac{1}{\sigma_0^2}\right) - \mu\left(\frac{(1-\alpha)}{\sigma^2}\mathbf{1}_N\sum_{t=1}^{T}(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1}) + \frac{\mu_0}{\sigma_0^2}\right)\right\}$$

which is MVN$(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with

$$\boldsymbol{A}^{-1} = \left(T(1-\alpha)^2\,\frac{N}{\sigma^2} + \frac{1}{\sigma_0^2}\right)^{-1}$$

$$\boldsymbol{b} = \frac{(1-\alpha)}{\sigma^2}\mathbf{1}_N^T\sum_{t=1}^{T}(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1}) + \frac{\mu_0}{\sigma_0^2}$$

where $N = N_I + N_P = 33$ for the Hudson Valley PDSI

3.6 Full conditional for α

$$[\alpha|\cdot] \propto \prod_{t=1}^{T} \left[\boldsymbol{T}_t|\boldsymbol{T}_{t-1}, \mu, \alpha, \sigma^2, \phi\right] [\alpha]$$

$$\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\mathbf{1}_N\right)^T \boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\mathbf{1}_N\right)\right\}I\left\{\alpha \in (0,1)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\alpha^2\sum_{t=1}^{T}(\boldsymbol{T}_{t-1} - \mu\mathbf{1}_N)^T\boldsymbol{\Sigma}_\epsilon^{-1}(\boldsymbol{T}_{t-1} - \mu\mathbf{1}_N) - \alpha\sum_{t=1}^{T}(\boldsymbol{T}_{t-1} - \mu\mathbf{1}_N)^T\boldsymbol{\Sigma}_\epsilon^{-1}(\boldsymbol{T}_t - \mu\mathbf{1}_N)\right\}$$

$$\times I\left\{\alpha \in (0,1)\right\}$$

$$\propto \exp\left\{-\frac{1}{2}\alpha^2\frac{1}{\sigma^2}\sum_{t=1}^{T}(\boldsymbol{T}_{t-1} - \mu\mathbf{1}_N)^T(\boldsymbol{T}_{t-1} - \mu\mathbf{1}_N) - \alpha\frac{1}{\sigma^2}\sum_{t=1}^{T}(\boldsymbol{T}_{t-1} - \mu\mathbf{1}_N)^T(\boldsymbol{T}_t - \mu\mathbf{1}_N)\right\}$$

$$\times I\left\{\alpha \in (0,1)\right\}$$

which is truncated $N(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$, , restricted to $\alpha \in (0,1)$, with

$$\boldsymbol{A}^{-1} = \left(\frac{1}{\sigma^2} \sum_{t=1}^{T} (\boldsymbol{T}_{t-1} - \mu \boldsymbol{1}_N)^T (\boldsymbol{T}_{t-1} - \mu \boldsymbol{1}_N) \right)^{-1}$$

$$\boldsymbol{b} = \frac{1}{\sigma^2} \sum_{t=1}^{T} (\boldsymbol{T}_{t-1} - \mu \boldsymbol{1}_N)^T (\boldsymbol{T}_t - \mu \boldsymbol{1}_N).$$

3.7 Full conditional for τ_I^2

$$[\tau_I^2|\cdot] \propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t | \boldsymbol{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2 \right] [\tau_I^2]$$

$$\propto \prod_{t=1}^{T} |\boldsymbol{\Sigma}_t|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} (\boldsymbol{W}_t - (\boldsymbol{H}_t \boldsymbol{T}_t + \boldsymbol{B}_t))^T \boldsymbol{\Sigma}_t^{-1} (\boldsymbol{W}_t - (\boldsymbol{H}_t \boldsymbol{T}_t + \boldsymbol{B}_t)) \right\}$$

$$\times (\tau_I^2)^{-\alpha_I - 1} \exp\left\{ \frac{\beta_I}{\tau_I^2} \right\}$$

$$\propto (\tau_I^2)^{-\frac{1}{2}\sum_{t=1}^{T} N_{It}} \exp\left\{ -\frac{1}{2} \sum_{t=1}^{T} (\boldsymbol{W}_{It} - (\boldsymbol{H}_{It} \boldsymbol{T}_t + \boldsymbol{B}_{It}))^T \boldsymbol{\Sigma}_{It}^{-1} (\boldsymbol{W}_{It} - (\boldsymbol{H}_{It} \boldsymbol{T}_t + \boldsymbol{B}_{It})) \right\}$$

$$\times (\tau_I^2)^{-\alpha_I - 1} \exp\left\{ \frac{\beta_I}{\tau_I^2} \right\}$$

$$\propto (\tau_I^2)^{-\alpha_I - \frac{M_I}{2} - 1} \exp\left\{ -\frac{1}{\tau_I^2} \left(\beta_I + \frac{1}{2} \sum_{t=1}^{T} (\boldsymbol{W}_{It} - (\boldsymbol{H}_{It} \boldsymbol{T}_t))^T (\boldsymbol{W}_{It} - (\boldsymbol{H}_{It} \boldsymbol{T}_t)) \right) \right\}$$

which is $IG\left(\alpha_I + \frac{M_I}{2}, \beta_I + \frac{1}{2} \sum_{t=1}^{T} (\boldsymbol{W}_{It} - (\boldsymbol{H}_{It} \boldsymbol{T}_t))^T (\boldsymbol{W}_{It} - (\boldsymbol{H}_{It} \boldsymbol{T}_t)) \right)$

3.8 Full conditional for τ_P^2

$$\left[\tau_P^2|\cdot\right] \propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t|\boldsymbol{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \left[\tau_P^2\right]$$

$$\propto \prod_{t=1}^{T} |\boldsymbol{\Sigma}_t|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)\right\}$$

$$\times \left(\tau_P^2\right)^{-\alpha_P-1} \exp\left\{\frac{\beta_P}{\tau_P^2}\right\}$$

$$\propto \left(\tau_P^2\right)^{-\frac{1}{2}\sum_{t=1}^{T} N_{Pt}}$$

$$\times \exp\left\{-\frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{W}_{Pt} - (\beta_1\boldsymbol{H}_{Pt}\boldsymbol{T}_t + \boldsymbol{B}_{Pt})\right)^T \boldsymbol{\Sigma}_{Pt}^{-1} \left(\boldsymbol{W}_{Pt} - (\beta_1\boldsymbol{H}_{Pt}\boldsymbol{T}_t + \boldsymbol{B}_{Pt})\right)\right\}$$

$$\times \left(\tau_P^2\right)^{-\alpha_P-1} \exp\left\{\frac{\beta_P}{\tau_P^2}\right\}$$

$$\propto \left(\tau_P^2\right)^{-\alpha_P - \frac{M_P}{2} - 1}$$

$$\times \exp\left\{-\frac{1}{\tau_P^2}\left(\beta_P + \frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{W}_{Pt} - (\beta_1\boldsymbol{H}_{Pt}\boldsymbol{T}_t + \beta_0\boldsymbol{1}_{N_{Pt}})\right)^T \left(\boldsymbol{W}_{Pt} - (\beta_1\boldsymbol{H}_{Pt}\boldsymbol{T}_t + \beta_0\boldsymbol{1}_{N_{Pt}})\right)\right)\right\}$$

which is $\text{IG}\left(\alpha_P + \frac{M_P}{2}, \beta_P + \frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{W}_{Pt} - (\beta_1\boldsymbol{H}_{Pt}\boldsymbol{T}_t + \beta_0\boldsymbol{1}_{N_{Pt}})\right)^T \left(\boldsymbol{W}_{Pt} - (\beta_1\boldsymbol{H}_{Pt}\boldsymbol{T}_t + \beta_0\boldsymbol{1}_{N_{Pt}})\right)\right)$

3.9 Full conditional for σ^2

$$[\sigma^2|\cdot] \propto \prod_{t=1}^{T} [\boldsymbol{T}_t|\boldsymbol{T}_{t-1},\mu,\alpha,\sigma^2,\phi]\,[\sigma^2]$$

$$\propto \prod_{t=1}^{T} |\boldsymbol{\Sigma}_\epsilon|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)^T \boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)\right\}$$

$$\times \left(\sigma^2\right)^{-\alpha_{\sigma^2}-1} \exp\left\{-\frac{\beta_{\sigma^2}}{\sigma^2}\right\}$$

$$\propto \left(\sigma^2\right)^{-\frac{NT}{2}} \exp\left\{-\frac{1}{\sigma^2}\left(\frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)^T\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)\right)\right\}$$

$$\times \left(\sigma^2\right)^{-\alpha_{\sigma^2}-1} \exp\left\{-\frac{\beta_{\sigma^2}}{\sigma^2}\right\}$$

$$\propto \left(\sigma^2\right)^{-\alpha_{\sigma^2}-\frac{NT}{2}-1}$$

$$\times \exp\left\{-\frac{1}{\sigma^2}\left(\beta_{\sigma^2} + \frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)^T\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)\right)\right\}$$

which is $\mathrm{IG}\left(\alpha_{\sigma^2} + \frac{NT}{2}, \beta_{\sigma^2} + \frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)^T\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)\right)$

4 Extensions to the model

- Predictive Process for Σ_ϵ in the spatial model

- Include the number of trees used to create a chronology in the model

- Maybe we need to model a covariance between the proxy chronolgies instead of a spatial covariance? e.g. $\boldsymbol{R}(\phi)$ is not a spatial covariance but a covariance between proxy records at a given site

- ???