Barcast Model

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1 Barcast Model

1.1 Data Model

$$\boldsymbol{W}_t = \boldsymbol{H}_t T_t + \boldsymbol{B}_t + \boldsymbol{\eta}_t \qquad \boldsymbol{\eta}_t \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_t) \qquad \boldsymbol{\Sigma}_t = \begin{pmatrix} \boldsymbol{\Sigma}_{It} & \boldsymbol{0} \\ \boldsymbol{0} & \boldsymbol{\Sigma}_{Pt} \end{pmatrix}$$

where $\boldsymbol{W}_t = \begin{pmatrix} \boldsymbol{W}_{It} \\ \boldsymbol{W}_{Pt} \end{pmatrix}$ is a vector of instrumental observations $\boldsymbol{W}_{It}$ and proxy observations $\boldsymbol{W}_{Pt}$ at time t, T_t is a latent climate variable, $\boldsymbol{H}_t = \begin{pmatrix} \boldsymbol{H}_{It} \\ \boldsymbol{\beta}_1 \boldsymbol{H}_{Pt} \end{pmatrix}$, $\boldsymbol{B}_t = \begin{pmatrix} \boldsymbol{H}_{It} \\ \boldsymbol{H}_{Pt} \end{pmatrix} \begin{pmatrix} \boldsymbol{0}_{N_I} \\ \boldsymbol{\beta}_0 \end{pmatrix}$, $\boldsymbol{\Sigma}_{It} = \tau_I^2 \boldsymbol{I}_{N_{It}}$, $\boldsymbol{\Sigma}_{Pt} = \tau_P^2 \boldsymbol{I}_{N_{Pt}}$, N_I is the total number of instrumental observations, N_P is the total number of proxy observations, N_{It} is the number of instrumental observations at time t, N_{Pt} is the number of proxy observations at time t, $\boldsymbol{H}_{It}$ and $\boldsymbol{H}_{Pt}$ are selection matrices indicating whether the instrumental or proxy variable was measured at time t, and $\boldsymbol{\beta}_0$ and $\boldsymbol{\beta}_1$ are regression coefficients relating the proxy observations and latent field.

1.2 Process Model

$$\boldsymbol{T} \sim \mathrm{N}\left(\boldsymbol{0}, \sigma^2 \boldsymbol{Q}\right)$$

where $\boldsymbol{Q} = \boldsymbol{D} - \boldsymbol{W}$ is an intrinsic conditionally autoregressive covariance matrix where $\boldsymbol{D}$ is a matrix that has the counts of the number of neighbors of each location on the diagonal and $\boldsymbol{W}_{ij}$ is 1 if location i is a neighbor of location j and 0 otherwise. The process can be written using a basis representation of the spectral decomposition of $\boldsymbol{Q}$ as

$$\boldsymbol{T} \sim \mathrm{N}\left(\boldsymbol{Z}\boldsymbol{\alpha}, \sigma^2 \boldsymbol{I}\right)$$

where $\boldsymbol{Z}$ is the matrix with the eignevectors of $\boldsymbol{Q}$ as columns that represent the basis functions with random coefficients α given by

$$\boldsymbol{\alpha} \sim \mathrm{N}\left(\mathbf{0}, \boldsymbol{\Lambda}\right)$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix of the eigenvalues of $\boldsymbol{Q}$.

1.3 Parameter Model

$$\begin{aligned}
\boldsymbol{\alpha}|\boldsymbol{Q} &\sim \mathrm{N}\left(\mathbf{0}, \boldsymbol{\Lambda}, \right) \\
\left[\sigma^2\right] &\propto \frac{1}{\sigma^2} \\
\left[\tau_I^2\right] &\propto \frac{1}{\tau_I^2} \\
\left[\tau_P^2\right] &\propto \frac{1}{\tau_P^2} \\
\boldsymbol{\beta}_0|\tau_P^2 &\sim \mathrm{N}\left(\mathbf{0}, \tau_P^2\boldsymbol{\Delta}_0^{-1}\right) \\
\boldsymbol{\beta}_1|\tau_P^2 &\sim \mathrm{N}\left(\mathbf{0}, \tau_P^2\boldsymbol{\Delta}_1^{-1}\right)
\end{aligned}$$

2 Posterior

$$\begin{aligned}
\prod_{t=1}^{T}\left[T_t, \boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \boldsymbol{\alpha}, \tau_I^2, \tau_P^2, \sigma^2 \middle| \boldsymbol{W}_t\right] \propto & \prod_{t=1}^{T}\left[\boldsymbol{W}_t \middle| T_t, \boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \tau_I^2, \tau_P^2\right]\left[\boldsymbol{T}|\boldsymbol{\alpha}, \sigma^2\right] \\
& \times \left[\boldsymbol{\beta}_0 \middle| \tau_P^2\right]\left[\boldsymbol{\beta}_1 \middle| \tau_P^2\right]\left[\tau_I^2\right]\left[\tau_P^2\right]\left[\boldsymbol{\alpha}|\boldsymbol{Q}\right]\left[\sigma^2\right]
\end{aligned}$$

3 Appendix

3.1 Full Conditional for $\boldsymbol{T}$

$$
\begin{aligned}
[\boldsymbol{T}|\cdot] &\propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| T_t, \boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \tau_I^2, \tau_P^2\right] \left[\boldsymbol{T} \middle| \boldsymbol{\alpha}, \sigma^2\right] \\
&\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2\sigma^2}\left(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha}\right)^T \left(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\sum_{t=1}^{T}\left[T_t^2\left(\boldsymbol{H}_t^T \boldsymbol{\Sigma}_t^{-1} \boldsymbol{H}_t\right) - T_t\left(\boldsymbol{H}_t^T \boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t - \boldsymbol{B}_t\right)\right)\right]\right\} \\
&\quad \times \exp\left\{-\frac{1}{2\sigma^2}\left[\boldsymbol{T}^T\boldsymbol{T} - 2\boldsymbol{T}^T\boldsymbol{Z}\boldsymbol{\alpha}\right]\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{T}^T\left(\tilde{\boldsymbol{A}} + \frac{1}{\sigma^2}\boldsymbol{I}\right)\boldsymbol{T} - 2\boldsymbol{T}^T\left(\tilde{\boldsymbol{b}} + \frac{\boldsymbol{Z}\boldsymbol{\alpha}}{\sigma^2}\right)\right]\right\}
\end{aligned}
$$

which is MVN($\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1}$) with

$$
\begin{aligned}
\boldsymbol{A}^{-1} &= \left(\tilde{\boldsymbol{A}} + \frac{1}{\sigma^2}\boldsymbol{I}\right)^{-1} \\
\boldsymbol{b} &= \tilde{\boldsymbol{b}} + \frac{\boldsymbol{Z}\boldsymbol{\alpha}}{\sigma^2}.
\end{aligned}
$$

where $\tilde{\boldsymbol{A}} = \operatorname{diag}\left(\boldsymbol{H}_1^T\boldsymbol{\Sigma}_1^{-1}\boldsymbol{H}_1, \ldots, \boldsymbol{H}_T^T\boldsymbol{\Sigma}_T^{-1}\boldsymbol{H}_T\right)$ and $\tilde{\boldsymbol{b}} = \left(\boldsymbol{H}_1^T\boldsymbol{\Sigma}_1^{-1}\left(\boldsymbol{W}_1 - \boldsymbol{B}_1\right), \ldots, \boldsymbol{H}_T^T\boldsymbol{\Sigma}_T^{-1}\left(\boldsymbol{W}_T - \boldsymbol{B}_T\right)\right)^T$

3.2 Full conditional for $\boldsymbol{\beta}_0$

$$
\begin{aligned}
[\boldsymbol{\beta}_0|\cdot] &\propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| T_t, \boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \tau_I^2, \tau_P^2\right] \left[\boldsymbol{\beta}_0 \middle| \tau_P^2\right] \\
&\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2\tau_P^2} \boldsymbol{\beta}_0^T \boldsymbol{\Delta}_0 \boldsymbol{\beta}_0\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{\beta}_0^T \left(\sum_{t=1}^{T} \boldsymbol{H}_{Pt}^T \boldsymbol{\Sigma}_{Pt}^{-1} \boldsymbol{H}_{Pt} + \boldsymbol{\Delta}_0\right) \boldsymbol{\beta}_0 - 2\boldsymbol{\beta}_0^T \left(\sum_{t=1}^{T} \boldsymbol{H}_{Pt}^T \boldsymbol{\Sigma}_{Pt}^{-1} \left(\boldsymbol{W}_{Pt} - \boldsymbol{\beta}_1 \boldsymbol{H}_{Pt} \boldsymbol{T}_t\right)\right)\right]\right\}
\end{aligned}
$$

which is MVN($\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1}$) with

$$
\begin{aligned}
\boldsymbol{A}^{-1} &= \left(\sum_{t=1}^{T} \boldsymbol{H}_{Pt}^T \boldsymbol{\Sigma}_{Pt}^{-1} \boldsymbol{H}_{Pt} + \boldsymbol{\Delta}_0\right)^{-1} \\
\boldsymbol{b} &= \sum_{t=1}^{T} \boldsymbol{H}_{Pt}^T \boldsymbol{\Sigma}_{Pt}^{-1} \left(\boldsymbol{W}_{Pt} - \boldsymbol{\beta}_1 \boldsymbol{H}_{Pt} \boldsymbol{T}_t\right).
\end{aligned}
$$

3.3 Full conditional for $\boldsymbol{\beta}_1$

$$
\begin{aligned}
[\boldsymbol{\beta}_1|\cdot] &\propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| T_t, \boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \tau_I^2, \tau_P^2\right] \left[\boldsymbol{\beta}_1 \middle| \tau_P^2\right] \\
&\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2\tau_P^2} \boldsymbol{\beta_1}^T \boldsymbol{\Delta}_1 \boldsymbol{\beta}_1\right\} \\
&\propto \exp\left\{-\frac{1}{2}\boldsymbol{\beta}_1^T \left(\sum_{t=1}^{T} \left(\boldsymbol{H}_{Pt} T_t\right)^T \boldsymbol{\Sigma}_{Pt}^{-1} \left(\boldsymbol{H}_{Pt} T_t\right) + \boldsymbol{\Delta}_1\right) \boldsymbol{\beta}_1\right\} \\
&\quad \times \exp\left\{-\boldsymbol{\beta}_1^T \left(\sum_{t=1}^{T} \left(\boldsymbol{H}_{Pt} T_t\right)^T \boldsymbol{\Sigma}_{Pt}^{-1} \left(\boldsymbol{W}_{Pt} - \boldsymbol{H}_{Pt} \boldsymbol{\beta}_0\right)\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2\tau_P^2}\boldsymbol{\beta}_1^T \left(\sum_{t=1}^{T} \left(\boldsymbol{H}_{Pt} T_t\right)^T \left(\boldsymbol{H}_{Pt} T_t\right) + \boldsymbol{\Delta_1}\right) \boldsymbol{\beta}_1\right\} \\
&\quad \times \exp\left\{-\frac{1}{\tau_P^2}\boldsymbol{\beta}_1^T \left(\sum_{t=1}^{T} \left(\boldsymbol{H}_{Pt} T_t\right)^T \left(\boldsymbol{W}_{Pt} - \boldsymbol{H}_{Pt} \boldsymbol{\beta}_0\right)\right)\right\}
\end{aligned}
$$

which is MVN($\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1}$) with

$$
\begin{aligned}
\boldsymbol{A}^{-1} &= \left(\frac{1}{\tau_P^2} \sum_{t=1}^{T} \left(\boldsymbol{H}_{Pt} T_t\right)^T \left(\boldsymbol{H}_{Pt} T_t\right) + \boldsymbol{\Delta}_1\right)^{-1} \\
\boldsymbol{b} &= \frac{1}{\tau_P^2} \sum_{t=1}^{T} \left(\boldsymbol{H}_{Pt} \boldsymbol{T}_t\right)^T \left(\boldsymbol{W}_{Pt} - \boldsymbol{H}_{Pt} \boldsymbol{\beta}_0\right).
\end{aligned}
$$

3.4 Full conditional for τ_I^2

$$
\begin{aligned}
\left[\tau_I^2 \middle| \cdot\right] &\propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| T_t, \boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \tau_I^2, \tau_P^2\right] \left[\tau_I^2\right] \\
&\propto \prod_{t=1}^{T} |\boldsymbol{\Sigma}_t|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)\right\} \frac{1}{\tau_I^2} \\
&\propto \left(\tau_I^2\right)^{-\frac{1}{2}\sum_{t=1}^{T} N_{It} - 1} \exp\left\{-\frac{1}{2}\sum_{t=1}^{T} \frac{\left(W_{It} - (H_{It} T_t + B_{It})\right)^2}{\tau_I^2}\right\} \\
&\propto \left(\tau_I^2\right)^{-\frac{1}{2}\sum_{t=1}^{T} N_{It} - 1} \exp\left\{-\frac{1}{\tau_I^2}\left(\frac{1}{2}\sum_{t=1}^{T} \left(W_{It} - H_{It} T_t\right)^2\right)\right\}
\end{aligned}
$$

which is $\text{IG}\left(\frac{1}{2}\sum_{t=1}^{T} N_{It}, \frac{1}{2}\sum_{t=1}^{T} \left(W_{It} - H_{It} T_t\right)^2\right)$

3.5 Full conditional for τ_P^2

$$
\begin{aligned}
\left[\tau_P^2 \middle| \cdot\right] &\propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| T_t, \boldsymbol{\beta}_0, \boldsymbol{\beta}_1, \tau_I^2, \tau_P^2\right] \left[\tau_P^2\right] \\
&\propto \prod_{t=1}^{T} |\boldsymbol{\Sigma}_t|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)\right\} \frac{1}{\tau_P^2} \\
&\propto \left(\tau_P^2\right)^{-\frac{1}{2}\sum_{t=1}^{T} N_{Pt} - 1} \\
&\qquad \times \exp\left\{-\frac{1}{\tau_P^2}\left(\frac{1}{2}\sum_{t=1}^{T} \left(\boldsymbol{W}_{Pt} - (\boldsymbol{\beta}_1 \boldsymbol{H}_{Pt} T_t + \boldsymbol{H}_{Pt}\boldsymbol{\beta}_0)\right)^T \left(\boldsymbol{W}_{Pt} - (\boldsymbol{\beta}_1 \boldsymbol{H}_{Pt} T_t + \boldsymbol{H}_{Pt}\boldsymbol{\beta}_0)\right)\right)\right\}
\end{aligned}
$$

which is $\text{IG}\left(\frac{1}{2}\sum_{t=1}^{T} N_{Pt}, \frac{1}{2}\sum_{t=1}^{T} \left(\boldsymbol{W}_{Pt} - (\boldsymbol{\beta}_1 \boldsymbol{H}_{Pt} T_t + \boldsymbol{H}_{Pt}\boldsymbol{\beta}_0)\right)^T \left(\boldsymbol{W}_{Pt} - (\boldsymbol{\beta}_1 \boldsymbol{H}_{Pt} T_t + \boldsymbol{H}_{Pt}\boldsymbol{\beta}_0)\right)\right)$.

3.6 Full conditional for $\boldsymbol{\alpha}$

$$
\begin{aligned}
[\boldsymbol{\alpha}|\cdot] &\propto \left[\boldsymbol{T} \middle| \boldsymbol{\alpha}, \sigma^2\right] [\boldsymbol{\alpha}|\boldsymbol{Q}] \\
&\propto \exp\left\{-\frac{1}{2\sigma^2}\left(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha}\right)^T \left(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha}\right)\right\} \exp\left\{-\frac{1}{2}\boldsymbol{\alpha}^T \boldsymbol{\Lambda}^{-1}\boldsymbol{\alpha}\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{\alpha}^T \left(\frac{\boldsymbol{Z}^T\boldsymbol{Z}}{\sigma^2} + \boldsymbol{\Delta}^{-1}\right)\boldsymbol{\alpha} - 2\boldsymbol{\alpha}^T \frac{\boldsymbol{Z}^T\boldsymbol{T}}{\sigma^2}\right]\right\}
\end{aligned}
$$

which is $\mathrm{N}(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with

$$\begin{aligned}
\boldsymbol{A}^{-1} &= \left(\frac{\boldsymbol{Z}^T\boldsymbol{Z}}{\sigma^2} + \boldsymbol{\Delta}^{-1} \right)^{-1} \\
\boldsymbol{b} &= \frac{\boldsymbol{Z}^T\boldsymbol{T}}{\sigma^2}.
\end{aligned}$$

3.7 Full conditional for σ^2

$$\begin{aligned}
\left[\sigma^2 \middle| \cdot\right] &\propto \left[\boldsymbol{T} \middle| \boldsymbol{\alpha}, \sigma^2\right] \left[\sigma^2\right] \\
&\propto \left(\sigma^2\right)^{-\frac{T}{2}} \exp\left\{ -\frac{1}{2\sigma^2} \left(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha}\right)^T \left(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha}\right) \right\} \frac{1}{\sigma^2} \\
&\propto \left(\sigma^2\right)^{-\frac{T}{2}-1} \exp\left\{ -\frac{1}{\sigma^2} \frac{\left(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha}\right)^T \left(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha}\right)}{2} \right\}
\end{aligned}$$

which is $\mathrm{IG}\left(\frac{T}{2}, \frac{(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha})^T (\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha})}{2} \right)$

4 Extensions to the model

- Predictive Process for Σ_ϵ in the spatial model
- Include the number of trees used to create a chronology in the model
- ???

5 References

- Martin Tingley's BARCAST papers
- Sayer's discussion on Wishart priors http://www.math.wustl.edu/ sawyer/hmhandouts/Wishart.pdf