

# Barcast Model

March 25, 2014

## 1 Barcast Model

### 1.1 Data Model

$$\mathbf{W}_t = \mathbf{H}_t \mathbf{T}_t + \mathbf{B}_t + \boldsymbol{\eta}_t \quad \boldsymbol{\eta}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_t) \quad \boldsymbol{\Sigma}_t = \begin{pmatrix} \tau_I^2 \mathbf{I}_{N_{It}} & \mathbf{0} \\ \mathbf{0} & \tau_P^2 \mathbf{I}_{N_{Pt}} \end{pmatrix}$$

$$\text{where } \mathbf{H}_t = \begin{pmatrix} \mathbf{H}_{It} \\ \beta_1 \mathbf{H}_{Pt} \end{pmatrix} \text{ and } \mathbf{B}_t = \begin{pmatrix} \mathbf{0}_{N_{It}} \\ \beta_0 \mathbf{1}_{N_{Pt}} \end{pmatrix}.$$

### 1.2 Process Model

$$\mathbf{T}_t - \mu \mathbf{1} = \alpha (\mathbf{T}_{t-1} - \mu \mathbf{1}) + \boldsymbol{\epsilon}_t \quad \boldsymbol{\epsilon}_t \sim \mathcal{N}(\mathbf{0}, \boldsymbol{\Sigma}_\epsilon) \quad \boldsymbol{\Sigma}_\epsilon = \sigma^2 \exp(-\phi \mathbf{D})$$

where  $\mathbf{D}$  is the distance matrix between observation points

### 1.3 Parameter Model

$$\mathbf{T}_0 \sim \mathcal{N}(\tilde{\boldsymbol{\mu}}_0, \tilde{\boldsymbol{\Sigma}}_0) \quad \tilde{\boldsymbol{\mu}}_0 = \mathbf{0} \quad \tilde{\boldsymbol{\Sigma}}_0 = \tilde{\sigma}_0^2 \mathbf{I}$$

$$\alpha \sim \mathcal{U}(0, 1)$$

$$\mu \sim \mathcal{N}(\mu_0, \sigma_0^2) \quad \text{For PDSI } \mu_0 = 0 \quad \sigma_0^2 = 1$$

$$\sigma^2 \sim \text{IG}(\alpha_{\sigma^2}, \beta_{\sigma^2})$$

$$\phi \sim \text{IG}(\alpha_\phi, \beta_\phi)$$

$$\tau_I \sim \text{IG}(\alpha_I, \beta_I)$$

$$\tau_P \sim \text{IG}(\alpha_P, \beta_P)$$

$$\beta_1 \sim \mathcal{N}(\mu_{\beta_1}, \sigma_{\beta_1}^2) \quad \mu_{\beta_1} = \left( \frac{(1 - \tau_P^2)(1 - \alpha^2)}{\sigma^2} \right)^{-\frac{1}{2}} \quad \sigma_{\beta_1}^2 = 8$$

$$\beta_0 \sim \mathcal{N}(\mu_{\beta_0}, \sigma_{\beta_0}^2) \quad \mu_{\beta_0} = -\mu_{\beta_1} \quad \sigma_{\beta_1}^2 = 8$$

Note that the prior values are set as the prior modes (e.g.  $\mu_{\beta_1}$  is a function of the prior modes for  $\tau_P^2$ ,  $\alpha$ , and  $\sigma^2$ ).

## 2 Posterior

$$\prod_{t=1}^T [\mathbf{T}_t, \beta_0, \beta_1, \mu, \alpha, \tau_I^2, \tau_P^2, \sigma^2, \phi | \mathbf{W}_t, T] \propto \prod_{t=1}^T [\mathbf{W}_t | \mathbf{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2] \propto [\mathbf{T}_t | \mathbf{T}_t, \mu, \alpha, \sigma^2, \phi] [\mathbf{T}_0] \\ \times [\beta_0] [\beta_1] [\tau_I^2] [\tau_P^2] [\mu] [\alpha] [\sigma^2] [\phi]$$

## 3 Full Conditionals

### 3.1 Full Conditional for $\mathbf{T}_0$

$$[\mathbf{T}_0 | \cdot] \propto [\mathbf{T}_1 | \mathbf{T}_0, \mu, \alpha, \sigma^2, \phi] [\mathbf{T}_0] \\ \propto \exp \left\{ -\frac{1}{2} (\mathbf{T}_1 - \alpha \mathbf{T}_0 - (1 - \alpha) \mu \mathbf{1})^T \boldsymbol{\Sigma}_\epsilon^{-1} (\mathbf{T}_1 - \alpha \mathbf{T}_0 - (1 - \alpha) \mu \mathbf{1}) \right\} \\ \times \exp \left\{ -\frac{1}{2} (\mathbf{T}_0 - \tilde{\mu}_0)^T \tilde{\boldsymbol{\Sigma}}_0^{-1} (\mathbf{T}_0 - \tilde{\mu}_0) \right\} \\ \propto \exp \left\{ -\frac{1}{2} \left[ \mathbf{T}_0^T \left( \alpha \boldsymbol{\Sigma}_\epsilon^{-1} + \tilde{\boldsymbol{\Sigma}}_0^{-1} \right) \mathbf{T}_0 - \mathbf{T}_0 \left( \alpha \boldsymbol{\Sigma}_\epsilon^{-1} (\mathbf{T}_1 - (1 - \alpha) \mu \mathbf{1}) + \tilde{\boldsymbol{\Sigma}}_0^{-1} \tilde{\mu} \right) \right] \right\}$$

which is  $\text{MVN}(\mathbf{A}^{-1} \mathbf{b}, \mathbf{A}^{-1})$  with  $\mathbf{A}^{-1} = \left( \alpha \boldsymbol{\Sigma}_\epsilon^{-1} + \tilde{\boldsymbol{\Sigma}}_0^{-1} \right)^{-1}$  and  $\mathbf{b} = \alpha \boldsymbol{\Sigma}_\epsilon^{-1} (\mathbf{T}_1 - (1 - \alpha) \mu \mathbf{1}) + \tilde{\boldsymbol{\Sigma}}_0^{-1} \tilde{\mu}$ .

### 3.2 Full Conditional for $\mathbf{T}_t$

For  $t = 1, \dots, T - 1$ ,

$$[\mathbf{T}_t | \cdot] \propto [\mathbf{W}_t | \mathbf{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2] [\mathbf{T}_t | \mathbf{T}_{t-1}, \mu, \alpha, \sigma^2, \phi] [\mathbf{T}_{t+1} | \mathbf{T}_t, \mu, \alpha, \sigma^2, \phi] \\ \propto \exp \left\{ -\frac{1}{2} (\mathbf{W}_t - (\mathbf{H}_t \mathbf{T}_t - \mathbf{B}_t))^T \boldsymbol{\Sigma}_t^{-1} (\mathbf{W}_t - (\mathbf{H}_t \mathbf{T}_t - \mathbf{B}_t)) \right\} \\ \times \exp \left\{ -\frac{1}{2} (\mathbf{T}_t - \alpha \mathbf{T}_{t-1} - (1 - \alpha) \mu \mathbf{1})^T \boldsymbol{\Sigma}_\epsilon^{-1} (\mathbf{T}_t - \alpha \mathbf{T}_{t-1} - (1 - \alpha) \mu \mathbf{1}) \right\} \\ \times \exp \left\{ -\frac{1}{2} (\mathbf{T}_{t+1} - \alpha \mathbf{T}_t - (1 - \alpha) \mu \mathbf{1})^T \boldsymbol{\Sigma}_\epsilon^{-1} (\mathbf{T}_{t+1} - \alpha \mathbf{T}_t - (1 - \alpha) \mu \mathbf{1}) \right\} \\ \propto \exp \left\{ -\frac{1}{2} \mathbf{T}_t^T \left( \mathbf{H}_t^T \boldsymbol{\Sigma}_t^{-1} \mathbf{H}_t + (\alpha^2 + 1) \boldsymbol{\Sigma}_\epsilon^{-1} \right) \mathbf{T}_t \right\} \\ \times \exp \left\{ -\mathbf{T}_t^T \left( \mathbf{H}_t^T \boldsymbol{\Sigma}_t^{-1} (\mathbf{W}_t - \mathbf{B}_t) + \boldsymbol{\Sigma}_\epsilon^{-1} (\alpha \mathbf{T}_{t-1} + (1 - \alpha) \mu \mathbf{1}) + \alpha \boldsymbol{\Sigma}_\epsilon^{-1} (\mathbf{T}_{t+1} + (1 - \alpha) \mu \mathbf{1}) \right) \right\}$$

which is  $\text{MVN}(\mathbf{A}^{-1} \mathbf{b}, \mathbf{A}^{-1})$  with  $\mathbf{A}^{-1} = \left( \mathbf{H}_t^T \boldsymbol{\Sigma}_t^{-1} \mathbf{H}_t + (\alpha^2 + 1) \boldsymbol{\Sigma}_\epsilon^{-1} \right)^{-1}$  and  $\mathbf{b} = \mathbf{H}_t^T \boldsymbol{\Sigma}_t^{-1} (\mathbf{W}_t - \mathbf{B}_t) + \boldsymbol{\Sigma}_\epsilon^{-1} \left( \alpha (\mathbf{T}_{t+1} + \mathbf{T}_{t-1}) + (1 - \alpha)^2 \mu \mathbf{1} \right)$ .

For  $t = T$ ,

$$\begin{aligned}
[\mathbf{T}_T|\cdot] &\propto [\mathbf{W}_T|\mathbf{T}_T, \beta_0, \beta_1, \tau_I^2, \tau_P^2] [\mathbf{T}_T|\mathbf{T}_{T-1}, \mu, \alpha, \sigma^2, \phi] \\
&\propto \exp \left\{ -\frac{1}{2} (\mathbf{W}_T - (\mathbf{H}_T \mathbf{T}_T - \mathbf{B}_T))^T \boldsymbol{\Sigma}_T^{-1} (\mathbf{W}_T - (\mathbf{H}_T \mathbf{T}_T - \mathbf{B}_T)) \right\} \\
&\quad \times \exp \left\{ -\frac{1}{2} (\mathbf{T}_T - \alpha \mathbf{T}_{T-1} - (1-\alpha) \mu \mathbf{1})^T \boldsymbol{\Sigma}_\epsilon^{-1} (\mathbf{T}_T - \alpha \mathbf{T}_{T-1} - (1-\alpha) \mu \mathbf{1}) \right\} \\
&\propto \exp \left\{ -\frac{1}{2} \mathbf{T}_T^T (\mathbf{H}_T^T \boldsymbol{\Sigma}_T^{-1} \mathbf{H}_T + \boldsymbol{\Sigma}_\epsilon^{-1}) \mathbf{T}_T \right\} \\
&\quad \times \exp \left\{ -\mathbf{T}_T^T (\mathbf{H}_T^T \boldsymbol{\Sigma}_T^{-1} (\mathbf{W}_T - \mathbf{B}_T) + \boldsymbol{\Sigma}_\epsilon^{-1} (\alpha \mathbf{T}_{T-1} + (1-\alpha) \mu \mathbf{1})) \right\}
\end{aligned}$$

which is  $\text{MVN}(\mathbf{A}^{-1} \mathbf{b}, \mathbf{A}^{-1})$  with  $\mathbf{A}^{-1} = (\mathbf{H}_T^T \boldsymbol{\Sigma}_T^{-1} \mathbf{H}_T + \boldsymbol{\Sigma}_\epsilon^{-1})^{-1}$  and  $\mathbf{b} = \mathbf{H}_T^T \boldsymbol{\Sigma}_T^{-1} (\mathbf{W}_T - \mathbf{B}_T) + \boldsymbol{\Sigma}_\epsilon^{-1} (\alpha \mathbf{T}_{T-1} + (1-\alpha) \mu \mathbf{1})$ .

### 3.3 Full conditional for $\beta_0$

$$\begin{aligned}
[\beta_0|\cdot] &\propto \prod_{t=1}^T [\mathbf{W}_t|\mathbf{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2] [\beta_0] \\
&\propto \prod_{t=1}^T \exp \left\{ -\frac{1}{2} (\mathbf{W}_t - (\mathbf{H}_t \mathbf{T}_t + \mathbf{B}_t))^T \boldsymbol{\Sigma}_t^{-1} (\mathbf{W}_t - (\mathbf{H}_t \mathbf{T}_t + \mathbf{B}_t)) \right\} \\
&\quad \times \exp \left\{ -\frac{1}{2} (\beta_0 - \mu_{\beta_0})^T \boldsymbol{\Sigma}_{\beta_0}^{-1} (\beta_0 - \mu_{\beta_0}) \right\} \\
&\propto \exp \left\{ -\frac{1}{2} \beta_0^2 \left( \sum_{t=1}^T \mathbf{1}^T \boldsymbol{\Sigma}_{P_t}^{-1} \mathbf{1} + \boldsymbol{\Sigma}_{\beta_0}^{-1} \right) - \beta_0 \left( \sum_{t=1}^T \mathbf{1}^T \boldsymbol{\Sigma}_{P_t}^{-1} (\mathbf{W}_{P_t} - \beta_1 \mathbf{H}_{P_t} \mathbf{T}_{P_t}) + \boldsymbol{\Sigma}_{\beta_0}^{-1} \mu_{\beta_0} \right) \right\}
\end{aligned}$$

which is  $\text{MVN}(\mathbf{A}^{-1} \mathbf{b}, \mathbf{A}^{-1})$  with  $\mathbf{A}^{-1} = (\sum_{t=1}^T \mathbf{1}^T \boldsymbol{\Sigma}_{P_t}^{-1} \mathbf{1} + \boldsymbol{\Sigma}_{\beta_0}^{-1})^{-1}$  and  $\mathbf{b} = \sum_{t=1}^T \mathbf{1}^T \boldsymbol{\Sigma}_{P_t}^{-1} (\mathbf{W}_{P_t} - \beta_1 \mathbf{H}_{P_t} \mathbf{T}_{P_t}) + \boldsymbol{\Sigma}_{\beta_0}^{-1} \mu_{\beta_0}$ . Note that  $\beta_0$  is univariate so this is really a 1-d density.

### 3.4 Full conditional for $\beta_1$

$$\begin{aligned}
[\beta_1|\cdot] &\propto \prod_{t=1}^T [\mathbf{W}_t | \mathbf{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2] [\beta_1] \\
&\propto \prod_{t=1}^T \exp \left\{ -\frac{1}{2} (\mathbf{W}_t - (\mathbf{H}_t \mathbf{T}_t + \mathbf{B}_t))^T \boldsymbol{\Sigma}_t^{-1} (\mathbf{W}_t - (\mathbf{H}_t \mathbf{T}_t + \mathbf{B}_t)) \right\} \\
&\quad \times \exp \left\{ -\frac{1}{2} (\beta_1 - \mu_{\beta_1})^T \boldsymbol{\Sigma}_{\beta_1}^{-1} (\beta_1 - \mu_{\beta_1}) \right\} \\
&\propto \exp \left\{ -\frac{1}{2} \beta_1^2 \left( \sum_{t=1}^T (\mathbf{H}_{Pt} \mathbf{T}_{Pt})^T \boldsymbol{\Sigma}_{Pt}^{-1} (\mathbf{H}_{Pt} \mathbf{T}_{Pt}) + \boldsymbol{\Sigma}_{\beta_1}^{-1} \right) \right\} \\
&\quad \times \exp \left\{ -\beta_1 \left( \sum_{t=1}^T (\mathbf{H}_{Pt} \mathbf{T}_{Pt})^T \boldsymbol{\Sigma}_{Pt}^{-1} (\mathbf{W}_{Pt} - \beta_0 \mathbf{1}) + \boldsymbol{\Sigma}_{\beta_1}^{-1} \mu_{\beta_1} \right) \right\}
\end{aligned}$$

which is  $\text{MVN}(\mathbf{A}^{-1} \mathbf{b}, \mathbf{A}^{-1})$  with  $\mathbf{A}^{-1} = \left( \sum_{t=1}^T (\mathbf{H}_{Pt} \mathbf{T}_{Pt})^T \boldsymbol{\Sigma}_{Pt}^{-1} (\mathbf{H}_{Pt} \mathbf{T}_{Pt}) + \boldsymbol{\Sigma}_{\beta_1}^{-1} \right)^{-1}$  and  $\mathbf{b} = \sum_{t=1}^T (\mathbf{H}_{Pt} \mathbf{T}_{Pt})^T \boldsymbol{\Sigma}_{Pt}^{-1} (\mathbf{W}_{Pt} - \beta_0 \mathbf{1}) + \boldsymbol{\Sigma}_{\beta_1}^{-1} \mu_{\beta_1}$ . Note that  $\beta_0$  is univariate so this is really a 1-d density.

### 3.5 Full conditional for $\mu$

$$\begin{aligned}
[\mu|\cdot] &\propto \prod_{t=1}^T [\mathbf{T}_t | \mathbf{T}_{t-1}, \mu, \alpha, \sigma^2, \phi] [\mu] \\
&\propto \prod_{t=1}^T \exp \left\{ -\frac{1}{2} (\mathbf{T}_t - \alpha \mathbf{T}_{t-1} - (1 - \alpha) \mu \mathbf{1})^T \boldsymbol{\Sigma}_\epsilon^{-1} (\mathbf{T}_t - \alpha \mathbf{T}_{t-1} - (1 - \alpha) \mu \mathbf{1}) \right\} \\
&\quad \times \exp \left\{ -\frac{1}{2} \frac{(\mu - \mu_0)^2}{\sigma_0^2} \right\} \\
&\propto \exp \left\{ -\frac{1}{2} \mu^2 \left( \sum_{t=1}^T (1 - \alpha)^2 \mathbf{1}^T \boldsymbol{\Sigma}_\epsilon^{-1} \mathbf{1} + \frac{1}{\sigma_0^2} \right) - \mu \left( (1 - \alpha) \mathbf{1}^T \boldsymbol{\Sigma}_\epsilon^{-1} \sum_{t=1}^T (\mathbf{T}_t - \alpha \mathbf{T}_{t-1}) + \frac{\mu_0}{\sigma_0^2} \right) \right\}
\end{aligned}$$

which is  $\text{MVN}(\mathbf{A}^{-1} \mathbf{b}, \mathbf{A}^{-1})$  with  $\mathbf{A}^{-1} = \left( \sum_{t=1}^T (1 - \alpha)^2 \mathbf{1}^T \boldsymbol{\Sigma}_\epsilon^{-1} \mathbf{1} + \frac{1}{\sigma_0^2} \right)^{-1}$  and  $\mathbf{b} = (1 - \alpha) \mathbf{1}^T \boldsymbol{\Sigma}_\epsilon^{-1} \sum_{t=1}^T (\mathbf{T}_t - \alpha \mathbf{T}_{t-1}) + \frac{\mu_0}{\sigma_0^2}$ .

### 3.6 Full conditional for $\alpha$

$$\begin{aligned}
[\alpha|\cdot] &\propto \prod_{t=1}^T [\mathbf{T}_t | \mathbf{T}_{t-1}, \mu, \alpha, \sigma^2, \phi] [\alpha] \\
&\propto \prod_{t=1}^T \exp \left\{ -\frac{1}{2} (\mathbf{T}_t - \alpha \mathbf{T}_{t-1} - (1-\alpha) \mu \mathbf{1})^T \boldsymbol{\Sigma}_\epsilon^{-1} (\mathbf{T}_t - \alpha \mathbf{T}_{t-1} - (1-\alpha) \mu \mathbf{1}) \right\} I\{\alpha \in (0, 1)\} \\
&\propto \exp \left\{ -\frac{1}{2} \alpha^2 \sum_{t=1}^T (\mathbf{T}_{t-1} - \mu \mathbf{1})^T \boldsymbol{\Sigma}_\epsilon^{-1} (\mathbf{T}_{t-1} - \mu \mathbf{1}) - \alpha \sum_{t=1}^T (\mathbf{T}_{t-1} - \mu \mathbf{1})^T \boldsymbol{\Sigma}_\epsilon^{-1} (\mathbf{T}_t - \mu \mathbf{1}) \right\} I\{\alpha \in (0, 1)\}
\end{aligned}$$

which is truncated  $N(\mathbf{A}^{-1} \mathbf{b}, \mathbf{A}^{-1})$  with  $\mathbf{A}^{-1} = \left( \sum_{t=1}^T (\mathbf{T}_{t-1} - \mu \mathbf{1})^T \boldsymbol{\Sigma}_\epsilon^{-1} (\mathbf{T}_{t-1} - \mu \mathbf{1}) \right)^{-1}$  and  $\mathbf{b} = \sum_{t=1}^T (\mathbf{T}_{t-1} - \mu \mathbf{1})^T \boldsymbol{\Sigma}_\epsilon^{-1} (\mathbf{T}_t - \mu \mathbf{1})$ , restricted to  $\alpha \in (0, 1)$ .

### 3.7 Full conditional for $\tau_I^2$

$$\begin{aligned}
[\tau_I^2|\cdot] &\propto \prod_{t=1}^T [\mathbf{W}_t | \mathbf{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2] [\tau_I^2] \\
&\propto \prod_{t=1}^T |\boldsymbol{\Sigma}_t|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (\mathbf{W}_t - (\mathbf{H}_t \mathbf{T}_t + \mathbf{B}_t))^T \boldsymbol{\Sigma}_t^{-1} (\mathbf{W}_t - (\mathbf{H}_t \mathbf{T}_t + \mathbf{B}_t)) \right\} \\
&\quad \times (\tau_I^2)^{-\alpha_I - 1} \exp \left\{ \frac{\beta_I}{\tau_I^2} \right\} \\
&\propto (\tau_I^2)^{-\frac{1}{2} \sum_{t=1}^T N_{It}} \exp \left\{ -\frac{1}{2} \sum_{t=1}^T (\mathbf{W}_{It} - (\mathbf{H}_{It} \mathbf{T}_{It} + \mathbf{B}_{It}))^T \left( \frac{1}{\tau_I^2} \mathbf{I}_{N_{It}} \right)^{-1} (\mathbf{W}_{It} - (\mathbf{H}_{It} \mathbf{T}_{It} + \mathbf{B}_{It})) \right\} \\
&\quad \times (\tau_I^2)^{-\alpha_I - 1} \exp \left\{ \frac{\beta_I}{\tau_I^2} \right\} \\
&\propto (\tau_I^2)^{-\alpha_I - \frac{1}{2} \sum_{t=1}^T N_{It} - 1} \exp \left\{ -\frac{1}{\tau_I^2} \left( \beta_I + \frac{1}{2} \sum_{t=1}^T (\mathbf{W}_{It} - (\mathbf{H}_{It} \mathbf{T}_{It}))^T (\mathbf{W}_{It} - (\mathbf{H}_{It} \mathbf{T}_{It})) \right) \right\}
\end{aligned}$$

which is  $\text{IG} \left( \alpha_I + \frac{1}{2} \sum_{t=1}^T N_{It}, \beta_I + \frac{1}{2} \sum_{t=1}^T (\mathbf{W}_{It} - (\mathbf{H}_{It} \mathbf{T}_{It}))^T (\mathbf{W}_{It} - (\mathbf{H}_{It} \mathbf{T}_{It})) \right)$

### 3.8 Full conditional for $\tau_P^2$

$$\begin{aligned}
[\tau_P^2 | \cdot] &\propto \prod_{t=1}^T [\mathbf{W}_t | \mathbf{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2] [\tau_P^2] \\
&\propto \prod_{t=1}^T |\Sigma_t|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (\mathbf{W}_t - (\mathbf{H}_t \mathbf{T}_t + \mathbf{B}_t))^T \Sigma_t^{-1} (\mathbf{W}_t - (\mathbf{H}_t \mathbf{T}_t + \mathbf{B}_t)) \right\} \\
&\quad \times (\tau_P^2)^{-\alpha_P - 1} \exp \left\{ \frac{\beta_P}{\tau_P^2} \right\} \\
&\propto (\tau_P^2)^{-\frac{1}{2} \sum_{t=1}^T N_{Pt}} \\
&\quad \times \exp \left\{ -\frac{1}{2} \sum_{t=1}^T (\mathbf{W}_{Pt} - (\mathbf{H}_{Pt} \mathbf{T}_{Pt} + \mathbf{B}_{Pt}))^T \left( \frac{1}{\tau_P^2} \mathbf{I}_{N_{Pt}} \right)^{-1} (\mathbf{W}_{Pt} - (\mathbf{H}_{Pt} \mathbf{T}_{Pt} + \mathbf{B}_{Pt})) \right\} \\
&\quad \times (\tau_P^2)^{-\alpha_P - 1} \exp \left\{ \frac{\beta_P}{\tau_P^2} \right\} \\
&\propto (\tau_P^2)^{-\alpha_P - \frac{1}{2} \sum_{t=1}^T N_{Pt} - 1} \\
&\quad \times \exp \left\{ -\frac{1}{\tau_I^2} \left( \beta_I + \frac{1}{2} \sum_{t=1}^T (\mathbf{W}_{Pt} - (\mathbf{H}_{Pt} \mathbf{T}_{Pt} + \mathbf{B}_{Pt}))^T (\mathbf{W}_{Pt} - (\mathbf{H}_{Pt} \mathbf{T}_{Pt} + \mathbf{B}_{Pt})) \right) \right\}
\end{aligned}$$

which is  $\text{IG} \left( \alpha_P + \frac{1}{2} \sum_{t=1}^T N_{Pt}, \beta_P + \frac{1}{2} \sum_{t=1}^T (\mathbf{W}_{Pt} - (\mathbf{H}_{Pt} \mathbf{T}_{Pt} + \mathbf{B}_{Pt}))^T (\mathbf{W}_{Pt} - (\mathbf{H}_{Pt} \mathbf{T}_{Pt} + \mathbf{B}_{Pt})) \right)$

### 3.9 Full conditional for $\sigma^2$

$$\begin{aligned}
[\sigma^2 | \cdot] &\propto \prod_{t=1}^T [\mathbf{T}_t | \mathbf{T}_{t-1}, \mu, \alpha, \sigma^2, \phi] [\sigma^2] \\
&\propto \prod_{t=1}^T |\Sigma_\epsilon|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (\mathbf{T}_t - \alpha \mathbf{T}_{t-1} - (1-\alpha) \mu \mathbf{1})^T \Sigma_\epsilon^{-1} (\mathbf{T}_t - \alpha \mathbf{T}_{t-1} - (1-\alpha) \mu \mathbf{1}) \right\} \\
&\quad \times (\sigma^2)^{-\alpha_{\sigma^2}-1} \exp \left\{ -\frac{\beta_{\sigma^2}}{\sigma^2} \right\} \\
&\propto (\sigma^2)^{-\frac{nT}{2}} \exp \left\{ -\frac{1}{\sigma^2} \frac{1}{2} \sum_{t=1}^T (\mathbf{T}_t - \alpha \mathbf{T}_{t-1} - (1-\alpha) \mu \mathbf{1})^T \mathbf{R}(\phi)^{-1} (\mathbf{T}_t - \alpha \mathbf{T}_{t-1} - (1-\alpha) \mu \mathbf{1}) \right\} \\
&\quad \times (\sigma^2)^{-\alpha_{\sigma^2}-1} \exp \left\{ -\frac{\beta_{\sigma^2}}{\sigma^2} \right\} \\
&\propto (\sigma^2)^{-\alpha_{\sigma^2} - \frac{nT}{2} - 1} \\
&\quad \times \exp \left\{ -\frac{1}{\sigma^2} \left( \beta_{\sigma^2} + \frac{1}{2} \sum_{t=1}^T (\mathbf{T}_t - \alpha \mathbf{T}_{t-1} - (1-\alpha) \mu \mathbf{1})^T \mathbf{R}(\phi)^{-1} (\mathbf{T}_t - \alpha \mathbf{T}_{t-1} - (1-\alpha) \mu \mathbf{1}) \right) \right\}
\end{aligned}$$

which is  $\text{IG}(\alpha_{\sigma^2} + \frac{nT}{2}, \beta_{\sigma^2} + \frac{1}{2} \sum_{t=1}^T (\mathbf{T}_t - \alpha \mathbf{T}_{t-1} - (1-\alpha) \mu \mathbf{1})^T \mathbf{R}(\phi)^{-1} (\mathbf{T}_t - \alpha \mathbf{T}_{t-1} - (1-\alpha) \mu \mathbf{1}))$

### 3.10 Full conditional for $\phi$

$$\begin{aligned}
[\phi | \cdot] &\propto \prod_{t=1}^T [\mathbf{T}_t | \mathbf{T}_{t-1}, \mu, \alpha, \sigma^2, \phi] [\phi] \\
&\propto \prod_{t=1}^T |\Sigma_\epsilon|^{-\frac{1}{2}} \exp \left\{ -\frac{1}{2} (\mathbf{T}_t - \alpha \mathbf{T}_{t-1} - (1-\alpha) \mu \mathbf{1})^T \Sigma_\epsilon^{-1} (\mathbf{T}_t - \alpha \mathbf{T}_{t-1} - (1-\alpha) \mu \mathbf{1}) \right\} \\
&\quad \times (\phi)^{-\alpha_\phi-1} \exp \left\{ -\frac{\beta_\phi}{\phi} \right\}
\end{aligned}$$

which can be sampled using a Metropolis-Hastings algorithm using a properly tuned normal proposal distribution.

## 4 Extensions to the model

- Predictive Process for  $\Sigma_\epsilon$
- ???