Barcast Model

March 25, 2014

1 Barcast Model

1.1 Data Model

$$\boldsymbol{W}_t = \boldsymbol{H}_t \boldsymbol{T}_t + \boldsymbol{B}_t + \boldsymbol{\eta}_t \qquad \boldsymbol{\eta}_t \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_t) \qquad \boldsymbol{\Sigma}_t = \begin{pmatrix} \tau_I^2 \boldsymbol{I}_{N_{It}} & \boldsymbol{0} \\ \boldsymbol{0} & \tau_P^2 \boldsymbol{I}_{N_{Pt}} \end{pmatrix}$$

where $\boldsymbol{H}_t = \begin{pmatrix} \boldsymbol{H}_{It} \\ \beta_1 \boldsymbol{H}_{Pt} \end{pmatrix}$ and $\boldsymbol{B}_t = \begin{pmatrix} \boldsymbol{H}_{It} \\ \boldsymbol{H}_{Pt} \end{pmatrix} \begin{pmatrix} \boldsymbol{0}_{N_I} \\ \beta_0 \boldsymbol{1}_{N_P} \end{pmatrix}$.

1.2 Process Model

$$\boldsymbol{T}_t - \mu \boldsymbol{1} = \alpha \left(\boldsymbol{T}_{t-1} - \mu \boldsymbol{1} \right) + \boldsymbol{\epsilon}_t \qquad \boldsymbol{\epsilon}_t \sim \mathrm{N}(\boldsymbol{0}, \boldsymbol{\Sigma}_\epsilon) \qquad \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} = \sigma^2 \exp\left(-\phi \boldsymbol{D}\right)$$

where $\boldsymbol{D}$ is the distance matrix between observation points

1.3 Parameter Model

$$\begin{aligned}
\boldsymbol{T}_0 &\sim \mathrm{N}\left(\tilde{\boldsymbol{\mu}}_0, \tilde{\boldsymbol{\Sigma}}_0\right) & \tilde{\boldsymbol{\mu}}_0 &= \boldsymbol{0} & \tilde{\boldsymbol{\Sigma}}_0 &= \tilde{\sigma}_0^2 \boldsymbol{I} \\
\alpha &\sim \mathrm{U}\left(0, 1, \right) & & & & \\
\mu &\sim \mathrm{N}\left(\mu_0, \sigma_0^2\right) & \text{For PDSI } \mu_0 &= 0 & \sigma_0^2 &= 1 \\
\sigma^2 &\sim \mathrm{IG}\left(\alpha_{\sigma^2}, \beta_{\sigma^2}\right) & & & & \\
\phi &\sim \mathrm{IG}\left(\alpha_\phi, \beta_\phi\right) & & & & \\
\tau_I &\sim \mathrm{IG}\left(\alpha_I, \beta_I\right) & & & & \\
\tau_P &\sim \mathrm{IG}\left(\alpha_P, \beta_P\right) & & & & \\
\beta_1 &\sim \mathrm{N}\left(\mu_{\beta_1}, \sigma_{\beta_1}^2\right) & \mu_{\beta_1} &= \left(\frac{(1-\tau_P^2)(1-\alpha^2)}{\sigma^2} \right)^{-\frac{1}{2}} & \sigma_{\beta_1}^2 &= 8 \\
\beta_0 &\sim \mathrm{N}\left(\mu_{\beta_0}, \sigma_{\beta_0}^2\right) & \mu_{\beta_0} &= -\mu\beta_1 & \sigma_{\beta_1}^2 &= 8
\end{aligned}$$

Note that the prior values are set as the prior modes (e.g. μ_{β_1} is a function of the prior modes for $\tau_P^2, \alpha,$ and σ^2).

2 Posterior

$$\prod_{t=1}^{T}\left[\boldsymbol{T}_t,\beta_0,\beta_1,\mu,\alpha,\tau_I^2,\tau_P^2,\sigma^2,\phi\middle|\boldsymbol{W}_t,T\right]\propto\prod_{t=1}^{T}\left[\boldsymbol{W}_t\middle|\boldsymbol{T}_t,\beta_0,\beta_1,\tau_I^2,\tau_P^2\right]\propto\left[\boldsymbol{T}_t\middle|\boldsymbol{T}_t,\mu,\alpha,\sigma^2,\phi\right][\boldsymbol{T}_0]$$
$$\times[\beta_0][\beta_1]\left[\tau_I^2\right]\left[\tau_P^2\right][\mu][\alpha]\left[\sigma^2\right][\phi]$$

3 Full Conditionals

3.1 Full Conditional for $\boldsymbol{T}_0$

$$\begin{aligned}
[\boldsymbol{T}_0|\cdot] &\propto \left[\boldsymbol{T}_1\middle|\boldsymbol{T}_0,\mu,\alpha,\sigma^2,\phi\right][\boldsymbol{T}_0]\\
&\propto \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_1-\alpha\boldsymbol{T}_0-(1-\alpha)\,\mu\mathbf{1}\right)^T\boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_1-\alpha\boldsymbol{T}_0-(1-\alpha)\,\mu\mathbf{1}\right)\right\}\\
&\quad\times\exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_0-\tilde{\boldsymbol{\mu}}_0\right)^T\tilde{\boldsymbol{\Sigma}}_0^{-1}\left(\boldsymbol{T}_0-\tilde{\boldsymbol{\mu}}_0\right)\right\}\\
&\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{T}_0^T\left(\alpha\boldsymbol{\Sigma}_\epsilon^{-1}+\tilde{\boldsymbol{\Sigma}}_0^{-1}\right)\boldsymbol{T}_0-\boldsymbol{T}_0\left(\alpha\boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_1-(1-\alpha)\,\mu\mathbf{1}\right)+\tilde{\boldsymbol{\Sigma}}_0^{-1}\tilde{\boldsymbol{\mu}}\right)\right]\right\}
\end{aligned}$$

which is MVN($\boldsymbol{A}^{-1}\boldsymbol{b},\boldsymbol{A}^{-1}$) with $\boldsymbol{A}^{-1}=\left(\alpha\boldsymbol{\Sigma}_\epsilon^{-1}+\tilde{\boldsymbol{\Sigma}}_0^{-1}\right)^{-1}$ and $\boldsymbol{b}=\alpha\boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_1-(1-\alpha)\,\mu\mathbf{1}\right)+\tilde{\boldsymbol{\Sigma}}_0^{-1}\tilde{\boldsymbol{\mu}}$.

3.2 Full Conditional for $\boldsymbol{T}_t$

For $t=1,\ldots T-1$,

$$\begin{aligned}
[\boldsymbol{T}_t|\cdot] &\propto \left[\boldsymbol{W}_t\middle|\boldsymbol{T}_t,\beta_0,\beta_1,\tau_I^2,\tau_P^2\right]\left[\boldsymbol{T}_t\middle|\boldsymbol{T}_{t-1},\mu,\alpha,\sigma^2,\phi\right]\left[\boldsymbol{T}_{t+1}\middle|\boldsymbol{T}_t,\mu,\alpha,\sigma^2,\phi\right]\\
&\propto \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t-(\boldsymbol{H}_t\boldsymbol{T}_t-\boldsymbol{B}_t)\right)^T\boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t-(\boldsymbol{H}_t\boldsymbol{T}_t-\boldsymbol{B}_t)\right)\right\}\\
&\quad\times\exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_t-\alpha\boldsymbol{T}_{t-1}-(1-\alpha)\,\mu\mathbf{1}\right)^T\boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_t-\alpha\boldsymbol{T}_{t-1}-(1-\alpha)\,\mu\mathbf{1}\right)\right\}\\
&\quad\times\exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_{t+1}-\alpha\boldsymbol{T}_t-(1-\alpha)\,\mu\mathbf{1}\right)^T\boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_{t+1}-\alpha\boldsymbol{T}_t-(1-\alpha)\,\mu\mathbf{1}\right)\right\}\\
&\propto \exp\left\{-\frac{1}{2}\boldsymbol{T}_t^T\left(\boldsymbol{H}_t^T\boldsymbol{\Sigma}_t^{-1}\boldsymbol{H}_t+\left(\alpha^2+1\right)\boldsymbol{\Sigma}_\epsilon^{-1}\right)\boldsymbol{T}_t\right\}\\
&\quad\times\exp\left\{-\boldsymbol{T}_t^T\left(\boldsymbol{H}_t^T\boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t-\boldsymbol{B}_t\right)+\boldsymbol{\Sigma}_\epsilon^{-1}\left(\alpha\boldsymbol{T}_{t-1}+(1-\alpha)\,\mu\mathbf{1}\right)+\alpha\boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_{t+1}+(1-\alpha)\,\mu\mathbf{1}\right)\right)\right\}
\end{aligned}$$

which is MVN($\boldsymbol{A}^{-1}\boldsymbol{b},\boldsymbol{A}^{-1}$) with $\boldsymbol{A}^{-1}=\left(\boldsymbol{H}_t^T\boldsymbol{\Sigma}_t^{-1}\boldsymbol{H}_t+\left(\alpha^2+1\right)\boldsymbol{\Sigma}_\epsilon^{-1}\right)^{-1}$ and $\boldsymbol{b}=\boldsymbol{H}_t^T\boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t-\boldsymbol{B}_t\right)+\boldsymbol{\Sigma}_\epsilon^{-1}\left(\alpha\left(\boldsymbol{T}_{t+1}+\boldsymbol{T}_{t-1}\right)+(1-\alpha)^2\mu\mathbf{1}\right)$.

For $t = T$,

$$
\begin{aligned}
[\boldsymbol{T}_T|\cdot] &\propto \left[\boldsymbol{W}_T \middle| \boldsymbol{T}_T, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \left[\boldsymbol{T}_T \middle| \boldsymbol{T}_{T-1}, \mu, \alpha, \sigma^2, \phi\right] \\
&\propto \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_T - \left(\boldsymbol{H}_T\boldsymbol{T}_T - \boldsymbol{B}_T\right)\right)^T \boldsymbol{\Sigma}_T^{-1}\left(\boldsymbol{W}_T - \left(\boldsymbol{H}_T\boldsymbol{T}_T - \boldsymbol{B}_T\right)\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_T - \alpha\boldsymbol{T}_{T-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)^T \boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_T - \alpha\boldsymbol{T}_{T-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\boldsymbol{T}_T^T\left(\boldsymbol{H}_T^T\boldsymbol{\Sigma}_T^{-1}\boldsymbol{H}_T + \boldsymbol{\Sigma}_\epsilon^{-1}\right)\boldsymbol{T}_T\right\} \\
&\quad \times \exp\left\{-\boldsymbol{T}_T^T\left(\boldsymbol{H}_T^T\boldsymbol{\Sigma}_T^{-1}\left(\boldsymbol{W}_T - \boldsymbol{B}_T\right) + \boldsymbol{\Sigma}_\epsilon^{-1}\left(\alpha\boldsymbol{T}_{T-1} + (1-\alpha)\,\mu\boldsymbol{1}\right)\right)\right\}
\end{aligned}
$$

which is MVN$(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with $\boldsymbol{A}^{-1} = \left(\boldsymbol{H}_T^T\boldsymbol{\Sigma}_T^{-1}\boldsymbol{H}_T + \boldsymbol{\Sigma}_\epsilon^{-1}\right)^{-1}$ and $\boldsymbol{b} = \boldsymbol{H}_T^T\boldsymbol{\Sigma}_T^{-1}\left(\boldsymbol{W}_T - \boldsymbol{B}_T\right) + \boldsymbol{\Sigma}_\epsilon^{-1}\left(\alpha\boldsymbol{T}_{T-1} + (1-\alpha)\,\mu\boldsymbol{1}\right)$.

3.3 Full conditional for β_0

$$
\begin{aligned}
[\beta_0|\cdot] &\propto \prod_{t=1}^{T}\left[\boldsymbol{W}_t \middle| \boldsymbol{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right][\beta_0] \\
&\propto \prod_{t=1}^{T}\exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - \left(\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t\right)\right)^T \boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t - \left(\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t\right)\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2}\left(\beta_0 - \mu_{\beta_0}\right)^T \boldsymbol{\Sigma}_{\beta_0}^{-1}\left(\beta_0 - \mu_{\beta_0}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\beta_0^2\left(\sum_{t=1}^{T}\boldsymbol{1}^T\boldsymbol{\Sigma}_{Pt}^{-1}\boldsymbol{1} + \boldsymbol{\Sigma}_{\beta_0}^{-1}\right) - \beta_0\left(\sum_{t=1}^{T}\boldsymbol{1}^T\boldsymbol{\Sigma}_{Pt}^{-1}\left(\boldsymbol{W}_{Pt} - \beta_1\boldsymbol{H}_{Pt}\boldsymbol{T}_{Pt}\right) + \boldsymbol{\Sigma}_{\beta_0}^{-1}\mu_{\beta_0}\right)\right\}
\end{aligned}
$$

which is MVN$(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with $\boldsymbol{A}^{-1} = \left(\sum_{t=1}^{T}\boldsymbol{1}^T\boldsymbol{\Sigma}_{Pt}^{-1}\boldsymbol{1} + \boldsymbol{\Sigma}_{\beta_0}^{-1}\right)^{-1}$ and $\boldsymbol{b} = \sum_{t=1}^{T}\boldsymbol{1}^T\boldsymbol{\Sigma}_{Pt}^{-1}\left(\boldsymbol{W}_{Pt} - \beta_1\boldsymbol{H}_{Pt}\boldsymbol{T}_{Pt}\right) + \boldsymbol{\Sigma}_{\beta_0}^{-1}\mu_{\beta_0}$. Note that β_0 is univariate so this is really a 1-d density.

3.4 Full conditional for β_1

$$
\begin{aligned}
[\beta_1|\cdot] &\propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| \boldsymbol{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] [\beta_1] \\
&\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)\right\} \\
&\quad\times \exp\left\{-\frac{1}{2}\left(\beta_1 - \mu_{\beta_1}\right)^T \boldsymbol{\Sigma}_{\beta_1}^{-1} \left(\beta_1 - \mu_{\beta_1}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\beta_1^2 \left(\sum_{t=1}^{T} \left(\boldsymbol{H}_{Pt}\boldsymbol{T}_{Pt}\right)^T \boldsymbol{\Sigma}_{Pt}^{-1} \left(\boldsymbol{H}_{Pt}\boldsymbol{T}_{Pt}\right) + \boldsymbol{\Sigma}_{\beta_1}^{-1}\right)\right\} \\
&\quad\times \exp\left\{-\beta_1 \left(\sum_{t=1}^{T} \left(\boldsymbol{H}_{Pt}\boldsymbol{T}_{Pt}\right)^T \boldsymbol{\Sigma}_{Pt}^{-1} \left(\boldsymbol{W}_{Pt} - \beta_0\boldsymbol{1}\right) + \boldsymbol{\Sigma}_{\beta_1}^{-1}\mu_{\beta_1}\right)\right\}
\end{aligned}
$$

which is MVN$(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with $\boldsymbol{A}^{-1} = \left(\sum_{t=1}^{T} \left(\boldsymbol{H}_{Pt}\boldsymbol{T}_{Pt}\right)^T \boldsymbol{\Sigma}_{Pt}^{-1} \left(\boldsymbol{H}_{Pt}\boldsymbol{T}_{Pt}\right) + \boldsymbol{\Sigma}_{\beta_1}^{-1}\right)^{-1}$ and $\boldsymbol{b} = \sum_{t=1}^{T} \left(\boldsymbol{H}_{Pt}\boldsymbol{T}_{Pt}\right)^T \boldsymbol{\Sigma}_{Pt}^{-1} \left(\boldsymbol{W}_{Pt} - \beta_0\boldsymbol{1}\right) + \boldsymbol{\Sigma}_{\beta_1}^{-1}\mu_{\beta_1}$. Note that β_0 is univariate so this is really a 1-d density.

3.5 Full conditional for μ

$$
\begin{aligned}
[\mu|\cdot] &\propto \prod_{t=1}^{T} \left[\boldsymbol{T}_t \middle| \boldsymbol{T}_{t-1}, \mu, \alpha, \sigma^2, \phi\right] [\mu] \\
&\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)^T \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)\right\} \\
&\quad\times \exp\left\{-\frac{1}{2}\frac{\left(\mu - \mu_0\right)^2}{\sigma_0^2}\right\} \\
&\propto \exp\left\{-\frac{1}{2}\mu^2 \left(\sum_{t=1}^{T} (1-\alpha)^2 \boldsymbol{1}^T\boldsymbol{\Sigma}_\epsilon^{-1}\boldsymbol{1} + \frac{1}{\sigma_0^2}\right) - \mu\left((1-\alpha)\,\boldsymbol{1}^T\boldsymbol{\Sigma}_\epsilon^{-1} \sum_{t=1}^{T} \left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1}\right) + \frac{\mu_0}{\sigma_0^2}\right)\right\}
\end{aligned}
$$

which is MVN$(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with $\boldsymbol{A}^{-1} = \left(\sum_{t=1}^{T} (1-\alpha)^2 \boldsymbol{1}^T\boldsymbol{\Sigma}_\epsilon^{-1}\boldsymbol{1} + \frac{1}{\sigma_0^2}\right)^{-1}$ and $\boldsymbol{b} = (1-\alpha)\,\boldsymbol{1}^T\boldsymbol{\Sigma}_\epsilon^{-1} \sum_{t=1}^{T} \left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1}\right) + \frac{\mu_0}{\sigma_0^2}$.

3.6 Full conditional for α

$$
\begin{aligned}
[\alpha|\cdot] &\propto \prod_{t=1}^{T} \left[\boldsymbol{T}_t \middle| \boldsymbol{T}_{t-1}, \mu, \alpha, \sigma^2, \phi\right] [\alpha] \\
&\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)^T \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)\right\} I\left\{\alpha \in (0,1)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\alpha^2 \sum_{t=1}^{T} \left(\boldsymbol{T}_{t-1} - \mu\boldsymbol{1}\right)^T \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_{t-1} - \mu\boldsymbol{1}\right) - \alpha \sum_{t=1}^{T} \left(\boldsymbol{T}_{t-1} - \mu\boldsymbol{1}\right)^T \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_t - \mu\boldsymbol{1}\right)\right\} I\left\{\alpha \in (0,1)\right\}
\end{aligned}
$$

which is truncated $\mathrm{N}(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with $\boldsymbol{A}^{-1} = \left(\sum_{t=1}^{T} \left(\boldsymbol{T}_{t-1} - \mu\boldsymbol{1}\right)^T \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_{t-1} - \mu\boldsymbol{1}\right)\right)^{-1}$ and $\boldsymbol{b} = \sum_{t=1}^{T} \left(\boldsymbol{T}_{t-1} - \mu\boldsymbol{1}\right)^T \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_t - \mu\boldsymbol{1}\right)$, restricted to $\alpha \in (0,1)$.

3.7 Full conditional for τ_I^2

$$
\begin{aligned}
\left[\tau_I^2 \middle| \cdot\right] &\propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| \boldsymbol{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \left[\tau_I^2\right] \\
&\propto \prod_{t=1}^{T} \left|\boldsymbol{\Sigma}_t\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)\right\} \\
&\qquad \times \left(\tau_I^2\right)^{-\alpha_I - 1} \exp\left\{\frac{\beta_I}{\tau_I^2}\right\} \\
&\propto \left(\tau_I^2\right)^{-\frac{1}{2}\sum_{t=1}^{T} N_{It}} \exp\left\{-\frac{1}{2}\sum_{t=1}^{T} \left(\boldsymbol{W}_{It} - (\boldsymbol{H}_{It}\boldsymbol{T}_{It} + \boldsymbol{B}_{It})\right)^T \left(\frac{1}{\tau_I^2}\boldsymbol{I}_{N_{It}}\right)^{-1} \left(\boldsymbol{W}_{It} - (\boldsymbol{H}_{It}\boldsymbol{T}_{It} + \boldsymbol{B}_{It})\right)\right\} \\
&\qquad \times \left(\tau_I^2\right)^{-\alpha_I - 1} \exp\left\{\frac{\beta_I}{\tau_I^2}\right\} \\
&\propto \left(\tau_I^2\right)^{-\alpha_I - \frac{1}{2}\sum_{t=1}^{T} N_{It} - 1} \exp\left\{-\frac{1}{\tau_I^2}\left(\beta_I + \frac{1}{2}\sum_{t=1}^{T} \left(\boldsymbol{W}_{It} - (\boldsymbol{H}_{It}\boldsymbol{T}_{It})\right)^T \left(\boldsymbol{W}_{It} - (\boldsymbol{H}_{It}\boldsymbol{T}_{It})\right)\right)\right\}
\end{aligned}
$$

which is $\mathrm{IG}\Big(\alpha_I + \frac{1}{2}\sum_{t=1}^{T} N_{It}, \beta_I + \frac{1}{2}\sum_{t=1}^{T} \left(\boldsymbol{W}_{It} - (\boldsymbol{H}_{It}\boldsymbol{T}_{It})\right)^T \left(\boldsymbol{W}_{It} - (\boldsymbol{H}_{It}\boldsymbol{T}_{It})\right)\Big)$

3.8 Full conditional for τ_P^2

$$
\begin{aligned}
\left[\tau_P^2 \middle| \cdot\right] &\propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| \boldsymbol{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \left[\tau_P^2\right] \\
&\propto \prod_{t=1}^{T} |\boldsymbol{\Sigma}_t|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)\right\} \\
&\quad\times \left(\tau_P^2\right)^{-\alpha_P - 1} \exp\left\{\frac{\beta_P}{\tau_P^2}\right\} \\
&\propto \left(\tau_P^2\right)^{-\frac{1}{2}\sum_{t=1}^{T} N_{Pt}} \\
&\quad\times \exp\left\{-\frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{W}_{Pt} - (\boldsymbol{H}_{Pt}\boldsymbol{T}_{Pt} + \boldsymbol{B}_{Pt})\right)^T \left(\frac{1}{\tau_P^2}\boldsymbol{I}_{N_{Pt}}\right)^{-1}\left(\boldsymbol{W}_{Pt} - (\boldsymbol{H}_{Pt}\boldsymbol{T}_{Pt} + \boldsymbol{B}_{Pt})\right)\right\} \\
&\quad\times \left(\tau_P^2\right)^{-\alpha_P - 1} \exp\left\{\frac{\beta_P}{\tau_P^2}\right\} \\
&\propto \left(\tau_P^2\right)^{-\alpha_P - \frac{1}{2}\sum_{t=1}^{T} N_{Pt} - 1} \\
&\quad\times \exp\left\{-\frac{1}{\tau_I^2}\left(\beta_I + \frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{W}_{Pt} - (\boldsymbol{H}_{Pt}\boldsymbol{T}_{Pt} + \boldsymbol{B}_{Pt})\right)^T\left(\boldsymbol{W}_{Pt} - (\boldsymbol{H}_{Pt}\boldsymbol{T}_{Pt} + \boldsymbol{B}_{Pt})\right)\right)\right\}
\end{aligned}
$$

which is $\mathrm{IG}\Big(\alpha_P + \frac{1}{2}\sum_{t=1}^{T} N_{Pt}, \beta_P + \frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{W}_{Pt} - (\boldsymbol{H}_{Pt}\boldsymbol{T}_{Pt} + \boldsymbol{B}_{Pt})\right)^T\left(\boldsymbol{W}_{Pt} - (\boldsymbol{H}_{Pt}\boldsymbol{T}_{Pt} + \boldsymbol{B}_{Pt})\right)\Big)$

3.9 Full conditional for σ^2

$$
\begin{aligned}
\left[\sigma^2|\cdot\right] &\propto \prod_{t=1}^{T} \left[\boldsymbol{T}_t \middle| \boldsymbol{T}_{t-1}, \mu, \alpha, \sigma^2, \phi\right] \left[\sigma^2\right] \\
&\propto \prod_{t=1}^{T} \left|\boldsymbol{\Sigma}_\epsilon\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)^T \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)\right\} \\
&\quad \times \left(\sigma^2\right)^{-\alpha_{\sigma^2}-1} \exp\left\{-\frac{\beta_{\sigma^2}}{\sigma^2}\right\} \\
&\propto \left(\sigma^2\right)^{-\frac{nT}{2}} \exp\left\{-\frac{1}{\sigma^2}\frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)^T \boldsymbol{R}(\phi)^{-1} \left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)\right\} \\
&\quad \times \left(\sigma^2\right)^{-\alpha_{\sigma^2}-1} \exp\left\{-\frac{\beta_{\sigma^2}}{\sigma^2}\right\} \\
&\propto \left(\sigma^2\right)^{-\alpha_{\sigma^2}-\frac{nT}{2}-1} \\
&\quad \times \exp\left\{-\frac{1}{\sigma^2}\left(\beta_{\sigma^2} + \frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)^T \boldsymbol{R}(\phi)^{-1} \left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)\right)\right\}
\end{aligned}
$$

which is $\mathrm{IG}\left(\alpha_{\sigma^2} + \frac{nT}{2}, \beta_{\sigma^2} + \frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)^T \boldsymbol{R}(\phi)^{-1} \left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)\right)$

3.10 Full conditional for ϕ

$$
\begin{aligned}
\left[\phi|\cdot\right] &\propto \prod_{t=1}^{T} \left[\boldsymbol{T}_t \middle| \boldsymbol{T}_{t-1}, \mu, \alpha, \sigma^2, \phi\right] \left[\phi\right] \\
&\propto \prod_{t=1}^{T} \left|\boldsymbol{\Sigma}_\epsilon\right|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)^T \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}\right)\right\} \\
&\quad \times \left(\phi\right)^{-\alpha_\phi-1} \exp\left\{-\frac{\beta_\phi}{\phi}\right\}
\end{aligned}
$$

which can be sampled using a Metropolis-Hastings algorithm using a properly tuned normal proposal distribution.

4 Extensions to the model

- Predictive Process for Σ_ϵ
- Include the number of trees used to create a chronology in the model

- ???