# Barcast Model

June 19, 2014

## 1 Barcast Model

### 1.1 Data Model

$$\boldsymbol{W}_t = \boldsymbol{H}_t T_t + \boldsymbol{B}_t + \boldsymbol{\eta}_t \qquad \boldsymbol{\eta}_t \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Sigma}_t) \qquad \boldsymbol{\Sigma}_t = \begin{pmatrix} \boldsymbol{\Sigma}_{It} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{Pt} \end{pmatrix}$$

where $\boldsymbol{W}_t = \begin{pmatrix} \boldsymbol{W}_{It} \\ \boldsymbol{W}_{Pt} \end{pmatrix}$ is a vector of instrumental observations $\boldsymbol{W}_{It}$ and proxy observations $\boldsymbol{W}_{Pt}$ at time $t$, $T_t$ is a latent climate variable, $\boldsymbol{H}_t = \begin{pmatrix} \boldsymbol{H}_{It} \\ \beta_1 \boldsymbol{H}_{Pt} \end{pmatrix}$, $\boldsymbol{B}_t = \begin{pmatrix} \boldsymbol{H}_{It} \\ \boldsymbol{H}_{Pt} \end{pmatrix} \begin{pmatrix} \mathbf{0}_{N_I} \\ \beta_0 \mathbf{1}_{N_P} \end{pmatrix}$, $\boldsymbol{\Sigma}_{It} = \tau_I^2 \boldsymbol{I}_{N_{It}}$, $\boldsymbol{\Sigma}_{Pt} = \tau_P^2 \boldsymbol{I}_{N_{Pt}}$, $N_I$ is the total number of instrumental observations, $N_P$ is the total number of proxy observations, $N_{It}$ is the number of instrumental observations at time $t$, $N_{Pt}$ is the number of proxy observations at time $t$, $\boldsymbol{H}_{It}$ and $\boldsymbol{H}_{Pt}$ are selection matrices indicating whether the instrumental or proxy variable was measured at time $t$, and $\beta_0$ and $\beta_1$ are regression coefficients relating the proxy observations and latent field.

### 1.2 Process Model

$$\boldsymbol{T} \sim \mathrm{N}\left(\mathbf{0}, \sigma^2 \boldsymbol{Q}\right)$$

where $\boldsymbol{Q} = \boldsymbol{D} - \boldsymbol{W}$ is an intrinsic conditionally autoregressive covariance matrix where $\boldsymbol{D}$ is a matrix that has the counts of the number of neighbors of each location on the diagonal and $\boldsymbol{W}_{ij}$ is 1 if location $i$ is a neighbor of location $j$ and 0 otherwise. The process can be written using a basis representation of the spectral decomposition of $\boldsymbol{Q}$ as

$$\boldsymbol{T} \sim \mathrm{N}\left(\boldsymbol{Z}\boldsymbol{\alpha}, \sigma^2 \boldsymbol{I}\right)$$

where $\boldsymbol{Z}$ is the matrix with the eignevectors of $\boldsymbol{Q}$ as columns that represent the basis functions with random coefficients $\alpha$ given by

$$\boldsymbol{\alpha} \sim \mathrm{N}\left(\mathbf{0}, \boldsymbol{\Lambda}\right)$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix of the eigenvalues of $\boldsymbol{Q}$.

### 1.3 Parameter Model

$$\begin{aligned}
\boldsymbol{\alpha}|\boldsymbol{Q} &\sim \mathrm{N}\left(\mathbf{0}, \boldsymbol{\Lambda}, \right) \\
\sigma^2 &\sim \mathrm{IG}\left(\alpha_{\sigma^2}, \beta_{\sigma^2}\right) \\
\tau_I^2 &\sim \mathrm{IG}\left(\alpha_I, \beta_I\right) \\
\tau_P^2 &\sim \mathrm{IG}\left(\alpha_P, \beta_P\right) \\
\beta_0 &= 0 \\
\beta_1|\tau_P^2 &\sim \mathrm{N}\left(0, \frac{\tau_P^2}{\delta}\right)
\end{aligned}$$

## 2 Posterior

$$\begin{aligned}
\prod_{t=1}^{T}\left[T_t, \beta_0, \beta_1, \boldsymbol{\alpha}, \tau_I^2, \tau_P^2, \sigma^2 \middle| \boldsymbol{W}_t\right] \propto & \prod_{t=1}^{T}\left[\boldsymbol{W}_t \middle| T_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right]\left[\boldsymbol{T} \middle| \boldsymbol{\alpha}, \sigma^2\right] \\
& \times \left[\beta_0 \middle| \tau_P^2\right]\left[\beta_1 \middle| \tau_P^2\right]\left[\tau_I^2\right]\left[\tau_P^2\right]\left[\boldsymbol{\alpha}|\boldsymbol{Q}\right]\left[\sigma^2\right]
\end{aligned}$$

# 3 Appendix

## 3.1 Full Conditional for $\boldsymbol{T}$

$$
\begin{aligned}
[\boldsymbol{T}|\cdot] \propto & \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| T_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \left[\boldsymbol{T} \middle| \boldsymbol{\alpha}, \sigma^2\right] \\
& \propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - \left(\boldsymbol{H}_t T_t + \boldsymbol{B}_t\right)\right)^T \boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t - \left(\boldsymbol{H}_t T_t + \boldsymbol{B}_t\right)\right)\right\} \\
& \quad \times \exp\left\{-\frac{1}{2\sigma^2}\left(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha}\right)^T\left(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha}\right)\right\} \\
& \propto \exp\left\{-\frac{1}{2}\sum_{t=1}^{T}\left[T_t^2\left(\boldsymbol{H}_t^T\boldsymbol{\Sigma}_t^{-1}\boldsymbol{H}_t\right) - T_t\left(\boldsymbol{H}_t^T\boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t - \boldsymbol{B}_t\right)\right)\right]\right\} \\
& \quad \times \exp\left\{-\frac{1}{2\sigma^2}\left[\boldsymbol{T}^T\boldsymbol{T} - 2\boldsymbol{T}^T\boldsymbol{Z}\boldsymbol{\alpha}\right]\right\} \\
& \propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{T}^T\left(\tilde{\boldsymbol{A}} + \frac{1}{\sigma^2}\boldsymbol{I}\right)\boldsymbol{T} - 2\boldsymbol{T}^T\left(\tilde{\boldsymbol{b}} + \frac{\boldsymbol{Z}\boldsymbol{\alpha}}{\sigma^2}\right)\right]\right\}
\end{aligned}
$$

which is MVN$(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with

$$
\begin{aligned}
\boldsymbol{A}^{-1} &= \left(\tilde{\boldsymbol{A}} + \frac{1}{\sigma^2}\boldsymbol{I}\right)^{-1} \\
\boldsymbol{b} &= \tilde{\boldsymbol{b}} + \frac{\boldsymbol{Z}\boldsymbol{\alpha}}{\sigma^2}.
\end{aligned}
$$

where $\tilde{\boldsymbol{A}} = \mathrm{diag}\left(\boldsymbol{H}_1^T\boldsymbol{\Sigma}_1^{-1}\boldsymbol{H}_1, \ldots, \boldsymbol{H}_T^T\boldsymbol{\Sigma}_T^{-1}\boldsymbol{H}_T\right)$ and $\tilde{\boldsymbol{b}} = \left(\boldsymbol{H}_1^T\boldsymbol{\Sigma}_1^{-1}\left(\boldsymbol{W}_1 - \boldsymbol{B}_1\right), \ldots, \boldsymbol{H}_T^T\boldsymbol{\Sigma}_T^{-1}\left(\boldsymbol{W}_T - \boldsymbol{B}_T\right)\right)^T$

## 3.2 Full conditional for $\beta_0$

$$
\begin{aligned}
[\beta_0|\cdot] &\propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| T_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \left[\beta_0 \middle| \tau_P^2\right] \\
&\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2\tau_P^2}\beta_0^T \boldsymbol{\Delta}_0 \beta_0\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\beta_0^T \left(\sum_{t=1}^{T} \boldsymbol{H}_{Pt}^T \boldsymbol{\Sigma}_{Pt}^{-1} \boldsymbol{H}_{Pt} + \boldsymbol{\Delta}_0\right)\beta_0 - 2\beta_0^T \left(\sum_{t=1}^{T} \boldsymbol{H}_{Pt}^T \boldsymbol{\Sigma}_{Pt}^{-1} \left(\boldsymbol{W}_{Pt} - \beta_1 \boldsymbol{H}_{Pt} \boldsymbol{T}_t\right)\right)\right]\right\}
\end{aligned}
$$

which is MVN($\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1}$) with

$$
\begin{aligned}
\boldsymbol{A}^{-1} &= \left(\sum_{t=1}^{T} \boldsymbol{H}_{Pt}^T \boldsymbol{\Sigma}_{Pt}^{-1} \boldsymbol{H}_{Pt} + \boldsymbol{\Delta}_0\right)^{-1} \\
\boldsymbol{b} &= \sum_{t=1}^{T} \boldsymbol{H}_{Pt}^T \boldsymbol{\Sigma}_{Pt}^{-1} \left(\boldsymbol{W}_{Pt} - \beta_1 \boldsymbol{H}_{Pt} \boldsymbol{T}_t\right).
\end{aligned}
$$

### 3.3 Full conditional for $\beta_1$

$$
\begin{aligned}
[\beta_1|\cdot] &\propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| T_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \left[\beta_1 \middle| \tau_P^2\right] \\
&\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2\tau_P^2} \boldsymbol{\beta_1}^T \boldsymbol{\Delta}_1 \beta_1\right\} \\
&\propto \exp\left\{-\frac{1}{2}\beta_1^T \left(\sum_{t=1}^{T} \left(\boldsymbol{H}_{Pt} T_t\right)^T \boldsymbol{\Sigma}_{Pt}^{-1} \left(\boldsymbol{H}_{Pt} T_t\right) + \boldsymbol{\Delta}_1\right) \beta_1\right\} \\
&\quad \times \exp\left\{-\beta_1^T \left(\sum_{t=1}^{T} \left(\boldsymbol{H}_{Pt} T_t\right)^T \boldsymbol{\Sigma}_{Pt}^{-1} \left(\boldsymbol{W}_{Pt} - \boldsymbol{H}_{Pt} \beta_0\right)\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2\tau_P^2}\beta_1^T \left(\sum_{t=1}^{T} \left(\boldsymbol{H}_{Pt} T_t\right)^T \left(\boldsymbol{H}_{Pt} T_t\right) + \boldsymbol{\Delta}_1\right) \beta_1\right\} \\
&\quad \times \exp\left\{-\frac{1}{\tau_P^2}\beta_1^T \left(\sum_{t=1}^{T} \left(\boldsymbol{H}_{Pt} T_t\right)^T \left(\boldsymbol{W}_{Pt} - \boldsymbol{H}_{Pt} \beta_0\right)\right)\right\}
\end{aligned}
$$

which is MVN($\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1}$) with

$$
\begin{aligned}
\boldsymbol{A}^{-1} &= \left(\frac{1}{\tau_P^2} \sum_{t=1}^{T} \left(\boldsymbol{H}_{Pt} T_t\right)^T \left(\boldsymbol{H}_{Pt} T_t\right) + \boldsymbol{\Delta}_1\right)^{-1} \\
\boldsymbol{b} &= \frac{1}{\tau_P^2} \sum_{t=1}^{T} \left(\boldsymbol{H}_{Pt} \boldsymbol{T}_t\right)^T \left(\boldsymbol{W}_{Pt} - \boldsymbol{H}_{Pt} \beta_0\right).
\end{aligned}
$$

### 3.4 Full conditional for $\tau_I^2$

$$
\begin{aligned}
\left[\tau_I^2 \middle| \cdot\right] &\propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| T_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \left[\tau_I^2\right] \\
&\propto \prod_{t=1}^{T} |\boldsymbol{\Sigma}_t|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)\right\} \frac{1}{\tau_I^2} \\
&\propto \left(\tau_I^2\right)^{-\frac{1}{2}\sum_{t=1}^{T} N_{It} - 1} \exp\left\{-\frac{1}{2}\sum_{t=1}^{T} \frac{\left(W_{It} - (H_{It} T_t + B_{It})\right)^2}{\tau_I^2}\right\} \\
&\propto \left(\tau_I^2\right)^{-\frac{1}{2}\sum_{t=1}^{T} N_{It} - 1} \exp\left\{-\frac{1}{\tau_I^2}\left(\frac{1}{2}\sum_{t=1}^{T} \left(W_{It} - H_{It} T_t\right)^2\right)\right\}
\end{aligned}
$$

which is $\mathrm{IG}\left(\frac{1}{2}\sum_{t=1}^{T} N_{It}, \frac{1}{2}\sum_{t=1}^{T} \left(W_{It} - H_{It} T_t\right)^2\right)$

### 3.5 Full conditional for $\tau_P^2$

$$
\begin{aligned}
\left[\tau_P^2 \middle| \cdot\right] &\propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| T_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \left[\tau_P^2\right] \\
&\propto \prod_{t=1}^{T} |\boldsymbol{\Sigma}_t|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{W}_t - (\boldsymbol{H}_t T_t + \boldsymbol{B}_t)\right)\right\} \frac{1}{\tau_P^2} \\
&\propto \left(\tau_P^2\right)^{-\frac{1}{2}\sum_{t=1}^{T} N_{Pt} - 1} \\
&\qquad \times \exp\left\{-\frac{1}{\tau_P^2}\left(\frac{1}{2}\sum_{t=1}^{T} \left(\boldsymbol{W}_{Pt} - (\beta_1 \boldsymbol{H}_{Pt} T_t + \boldsymbol{H}_{Pt}\beta_0)\right)^T \left(\boldsymbol{W}_{Pt} - (\beta_1 \boldsymbol{H}_{Pt} T_t + \boldsymbol{H}_{Pt}\beta_0)\right)\right)\right\}
\end{aligned}
$$

which is $\mathrm{IG}\left(\frac{1}{2}\sum_{t=1}^{T} N_{Pt}, \frac{1}{2}\sum_{t=1}^{T} \left(\boldsymbol{W}_{Pt} - (\beta_1 \boldsymbol{H}_{Pt} T_t + \boldsymbol{H}_{Pt}\beta_0)\right)^T \left(\boldsymbol{W}_{Pt} - (\beta_1 \boldsymbol{H}_{Pt} T_t + \boldsymbol{H}_{Pt}\beta_0)\right)\right)$.

### 3.6 Full conditional for $\boldsymbol{\alpha}$

$$
\begin{aligned}
[\boldsymbol{\alpha}|\cdot] &\propto \left[\boldsymbol{T} \middle| \boldsymbol{\alpha}, \sigma^2\right] [\boldsymbol{\alpha}|\boldsymbol{Q}] \\
&\propto \exp\left\{-\frac{1}{2\sigma^2}\left(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha}\right)^T \left(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha}\right)\right\} \exp\left\{-\frac{1}{2}\boldsymbol{\alpha}^T \boldsymbol{\Lambda}^{-1} \boldsymbol{\alpha}\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{\alpha}^T \left(\frac{\boldsymbol{Z}^T \boldsymbol{Z}}{\sigma^2} + \boldsymbol{\Delta}^{-1}\right) \boldsymbol{\alpha} - 2\boldsymbol{\alpha}^T \frac{\boldsymbol{Z}^T \boldsymbol{T}}{\sigma^2}\right]\right\}
\end{aligned}
$$

which is $\mathrm{N}(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with

$$\boldsymbol{A}^{-1} = \left(\frac{\boldsymbol{Z}^T\boldsymbol{Z}}{\sigma^2} + \boldsymbol{\Delta}^{-1}\right)^{-1}$$
$$\boldsymbol{b} = \frac{\boldsymbol{Z}^T\boldsymbol{T}}{\sigma^2}.$$

### 3.7 Full conditional for $\sigma^2$

$$\begin{aligned}
\left[\sigma^2 \middle| \cdot\right] &\propto \left[\boldsymbol{T}\middle|\boldsymbol{\alpha}, \sigma^2\right]\left[\sigma^2\right] \\
&\propto \left(\sigma^2\right)^{-\frac{T}{2}} \exp\left\{-\frac{1}{2\sigma^2}\left(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha}\right)^T\left(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha}\right)\right\}\frac{1}{\sigma^2} \\
&\propto \left(\sigma^2\right)^{-\frac{T}{2}-1} \exp\left\{-\frac{1}{\sigma^2}\frac{\left(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha}\right)^T\left(\boldsymbol{T} - \boldsymbol{Z}\boldsymbol{\alpha}\right)}{2}\right\}
\end{aligned}$$

which is $\mathrm{IG}\left(\frac{T}{2}, \frac{(\boldsymbol{T}-\boldsymbol{Z}\boldsymbol{\alpha})^T(\boldsymbol{T}-\boldsymbol{Z}\boldsymbol{\alpha})}{2}\right)$

# 4 Extensions to the model

- Predictive Process for $\Sigma_\epsilon$ in the spatial model
- Include the number of trees used to create a chronology in the model
- ???

# 5 References

- Martin Tingley's BARCAST papers
- Sayer's discussion on Wishart priors http://www.math.wustl.edu/ sawyer/hmhandouts/Wishart.pdf