Barcast Model

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1 Barcast Model

1.1 Data Model

$$\boldsymbol{W}_t = \boldsymbol{H}_t \boldsymbol{T}_t + \boldsymbol{B}_t + \boldsymbol{\eta}_t \qquad \boldsymbol{\eta}_t \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Sigma}_t) \qquad \boldsymbol{\Sigma}_t = \begin{pmatrix} \boldsymbol{\Sigma}_{It} & \mathbf{0} \\ \mathbf{0} & \boldsymbol{\Sigma}_{Pt} \end{pmatrix}$$

where $\boldsymbol{W}_t = \begin{pmatrix} \boldsymbol{W}_{It} \\ \boldsymbol{W}_{Pt} \end{pmatrix}$ is a vector of instrumental observations $\boldsymbol{W}_{It}$ and proxy observations $\boldsymbol{W}_{Pt}$ at time t, $\boldsymbol{T}_t = \begin{pmatrix} \boldsymbol{T}_{It} \\ \boldsymbol{T}_{Pt} \end{pmatrix}$ is a vector of latent climate variables , $\boldsymbol{H}_t = \begin{pmatrix} \boldsymbol{H}_{It} \\ \beta_1 \boldsymbol{H}_{Pt} \end{pmatrix}$, $\boldsymbol{B}_t = \begin{pmatrix} \boldsymbol{H}_{It} \\ \boldsymbol{H}_{Pt} \end{pmatrix} \begin{pmatrix} \mathbf{0}_{N_I} \\ \beta_0 \mathbf{1}_{N_P} \end{pmatrix}$, $\boldsymbol{\Sigma}_{It} = \tau_I^2 \boldsymbol{I}_{N_{It}}$, $\boldsymbol{\Sigma}_{Pt} = \tau_P^2 \boldsymbol{I}_{N_{Pt}}$, N_I is the total number of instrumental observations, N_P is the total number of proxy observations, N_{It} is the number of instrumental observations at time t, N_{Pt} is the number of proxy observations at time t, $\boldsymbol{H}_{It}$ and $\boldsymbol{H}_{Pt}$ are selection matrices indicating whether the instrumental or proxy variable was measured at time t, and β_0 and β_1 are regression coefficients relating the proxy observations and latent field.

1.2 Process Model

$$\boldsymbol{T}_t - \mu \mathbf{1}_N = \alpha \left(\boldsymbol{T}_{t-1} - \mu \mathbf{1}_N \right) + \boldsymbol{\epsilon}_t \qquad \boldsymbol{\epsilon}_t \sim \mathrm{N}(\mathbf{0}, \boldsymbol{\Sigma}_\epsilon) \qquad \boldsymbol{\Sigma}_{\boldsymbol{\epsilon}} = \sigma^2 \boldsymbol{Q}$$

where $\boldsymbol{Q}$ is a covariance matrix modeled by an inverse Wishart distribution.

1.3 Parameter Model

$$
\begin{aligned}
\boldsymbol{T}_0 &\sim \mathrm{N}\left(\tilde{\boldsymbol{\mu}}_0, \tilde{\boldsymbol{\Sigma}}_0\right) & \tilde{\boldsymbol{\mu}}_0 &= \mathbf{0} & \tilde{\boldsymbol{\Sigma}}_0 &= \tilde{\sigma}_0^2 \boldsymbol{I} \\
\alpha &\sim \mathrm{U}\left(0, 1, \right) & & & & \\
\mu &\sim \mathrm{N}\left(\mu_0, \sigma_0^2\right) & \text{For PDSI } \mu_0 &= 0 & \sigma_0^2 &= 1 \\
\sigma^2 &\sim \mathrm{IG}\left(\alpha_{\sigma^2}, \beta_{\sigma^2}\right) & & & & \\
\tau_I &\sim \mathrm{IG}\left(\alpha_I, \beta_I\right) & & & & \\
\tau_P &\sim \mathrm{IG}\left(\alpha_P, \beta_P\right) & & & & \\
\beta_1 &\sim \mathrm{N}\left(\mu_{\beta_1}, \sigma_{\beta_1}^2\right) & \mu_{\beta_1} &= \left(\frac{(1-\tau_P^2)(1-\alpha^2)}{\sigma^2}\right)^{-\frac{1}{2}} & \sigma_{\beta_1}^2 &= 8 \\
\beta_0 &\sim \mathrm{N}\left(\mu_{\beta_0}, \sigma_{\beta_0}^2\right) & \mu_{\beta_0} &= -\mu\beta_1 & \sigma_{\beta_1}^2 &= 8 \\
\boldsymbol{Q} &\sim \mathrm{InvWish}\left(\nu, \boldsymbol{I}_{N\times N}\right) & & & &
\end{aligned}
$$

Note that the prior values are set as the prior modes (e.g. μ_{β_1} is a function of the prior modes for $\tau_P^2, \alpha,$ and σ^2).

2 Posterior

$$
\begin{aligned}
\prod_{t=1}^{T}\left[\boldsymbol{T}_t, \beta_0, \beta_1, \mu, \alpha, \tau_I^2, \tau_P^2, \sigma^2, \boldsymbol{Q} \middle| \boldsymbol{W}_t, T\right] \propto \prod_{t=1}^{T}&\left[\boldsymbol{W}_t \middle| \boldsymbol{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \propto \left[\boldsymbol{T}_t \middle| \boldsymbol{T}_t, \mu, \alpha, \sigma^2, \boldsymbol{Q}\right]\left[\boldsymbol{T}_0\right] \\
&\times \left[\beta_0\right]\left[\beta_1\right]\left[\tau_I^2\right]\left[\tau_P^2\right]\left[\mu\right]\left[\alpha\right]\left[\sigma^2\right]\left[\boldsymbol{Q}\right]
\end{aligned}
$$

3 Full Conditionals

3.1 Full Conditional for $\boldsymbol{T}_0$

$$
\begin{aligned}
\left[\boldsymbol{T}_0|\cdot\right] &\propto \left[\boldsymbol{T}_1 \middle| \boldsymbol{T}_0, \mu, \alpha, \sigma^2, \boldsymbol{Q}\right]\left[\boldsymbol{T}_0\right] \\
&\propto \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_1 - \alpha\boldsymbol{T}_0 - (1-\alpha)\mu\mathbf{1}_N\right)^T \boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_1 - \alpha\boldsymbol{T}_0 - (1-\alpha)\mu\mathbf{1}_N\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_0 - \tilde{\boldsymbol{\mu}}_0\right)^T \tilde{\boldsymbol{\Sigma}}_0^{-1}\left(\boldsymbol{T}_0 - \tilde{\boldsymbol{\mu}}_0\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\left[\boldsymbol{T}_0^T\left(\alpha^2\boldsymbol{\Sigma}_\epsilon^{-1} + \tilde{\boldsymbol{\Sigma}}_0^{-1}\right)\boldsymbol{T}_0 - 2\boldsymbol{T}_0^T\left(\alpha\boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_1 - (1-\alpha)\mu\mathbf{1}_N\right) + \tilde{\boldsymbol{\Sigma}}_0^{-1}\tilde{\boldsymbol{\mu}}\right)\right]\right\}
\end{aligned}
$$

which is MVN($\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1}$) with

$$
\begin{aligned}
\boldsymbol{A}^{-1} &= \left(\alpha^2 \boldsymbol{\Sigma}_\epsilon^{-1} + \tilde{\boldsymbol{\Sigma}}_0^{-1}\right)^{-1} \\
\boldsymbol{b} &= \alpha \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_1 - (1-\alpha)\,\mu \boldsymbol{1}_N\right) + \tilde{\boldsymbol{\Sigma}}_0^{-1} \tilde{\boldsymbol{\mu}}.
\end{aligned}
$$

3.2 Full Conditional for $\boldsymbol{T}_t$

For $t = 1, \ldots T-1$,

$$
\begin{aligned}
[\boldsymbol{T}_t|\cdot] &\propto \left[\boldsymbol{W}_t \middle| \boldsymbol{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \left[\boldsymbol{T}_t \middle| \boldsymbol{T}_{t-1}, \mu, \alpha, \sigma^2, \boldsymbol{Q}\right] \left[\boldsymbol{T}_{t+1} \middle| \boldsymbol{T}_t, \mu, \alpha, \sigma^2, \boldsymbol{Q}\right] \\
&\propto \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}_N\right)^T \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\,\mu\boldsymbol{1}_N\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_{t+1} - \alpha\boldsymbol{T}_{t} - (1-\alpha)\,\mu\boldsymbol{1}_N\right)^T \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_{t+1} - \alpha\boldsymbol{T}_{t} - (1-\alpha)\,\mu\boldsymbol{1}_N\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\boldsymbol{T}_t^T \left(\boldsymbol{H}_t^T\boldsymbol{\Sigma}_t^{-1}\boldsymbol{H}_t + \left(\alpha^2+1\right)\boldsymbol{\Sigma}_\epsilon^{-1}\right)\boldsymbol{T}_t\right\} \\
&\quad \times \exp\left\{-\boldsymbol{T}_t^T \left(\boldsymbol{H}_t^T\boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t - \boldsymbol{B}_t\right) + \boldsymbol{\Sigma}_\epsilon^{-1}\left(\alpha\boldsymbol{T}_{t-1} + (1-\alpha)\,\mu\boldsymbol{1}_N\right) + \alpha\boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_{t+1} - (1-\alpha)\,\mu\boldsymbol{1}_N\right)\right)\right\}
\end{aligned}
$$

which is MVN$(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with

$$
\begin{aligned}
\boldsymbol{A}^{-1} &= \left(\boldsymbol{H}_t^T\boldsymbol{\Sigma}_t^{-1}\boldsymbol{H}_t + \left(\alpha^2+1\right)\boldsymbol{\Sigma}_\epsilon^{-1}\right)^{-1} \\
\boldsymbol{b} &= \boldsymbol{H}_t^T\boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t - \boldsymbol{B}_t\right) + \boldsymbol{\Sigma}_\epsilon^{-1}\left(\alpha\left(\boldsymbol{T}_{t+1} + \boldsymbol{T}_{t-1}\right) + (1-\alpha)^2\mu\boldsymbol{1}_N\right).
\end{aligned}
$$

For $t = T$,

$$
\begin{aligned}
[\boldsymbol{T}_T|\cdot] &\propto \left[\boldsymbol{W}_T \middle| \boldsymbol{T}_T, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \left[\boldsymbol{T}_T \middle| \boldsymbol{T}_{T-1}, \mu, \alpha, \sigma^2, \boldsymbol{Q}\right] \\
&\propto \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_T - (\boldsymbol{H}_T\boldsymbol{T}_T - \boldsymbol{B}_T)\right)^T \boldsymbol{\Sigma}_T^{-1} \left(\boldsymbol{W}_T - (\boldsymbol{H}_T\boldsymbol{T}_T - \boldsymbol{B}_T)\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_T - \alpha\boldsymbol{T}_{T-1} - (1-\alpha)\,\mu\boldsymbol{1}_N\right)^T \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_T - \alpha\boldsymbol{T}_{T-1} - (1-\alpha)\,\mu\boldsymbol{1}_N\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\boldsymbol{T}_T^T \left(\boldsymbol{H}_T^T\boldsymbol{\Sigma}_T^{-1}\boldsymbol{H}_T + \boldsymbol{\Sigma}_\epsilon^{-1}\right)\boldsymbol{T}_T\right\} \\
&\quad \times \exp\left\{-\boldsymbol{T}_T^T \left(\boldsymbol{H}_T^T\boldsymbol{\Sigma}_T^{-1}\left(\boldsymbol{W}_T - \boldsymbol{B}_T\right) + \boldsymbol{\Sigma}_\epsilon^{-1}\left(\alpha\boldsymbol{T}_{T-1} + (1-\alpha)\,\mu\boldsymbol{1}_N\right)\right)\right\}
\end{aligned}
$$

which is MVN$(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with

$$\begin{aligned}
\boldsymbol{A}^{-1} &= \left(\boldsymbol{H}_T^T \boldsymbol{\Sigma}_T^{-1} \boldsymbol{H}_T + \boldsymbol{\Sigma}_\epsilon^{-1}\right)^{-1} \\
\boldsymbol{b} &= \boldsymbol{H}_T^T \boldsymbol{\Sigma}_T^{-1} \left(\boldsymbol{W}_T - \boldsymbol{B}_T\right) + \boldsymbol{\Sigma}_\epsilon^{-1} \left(\alpha \boldsymbol{T}_{T-1} + (1-\alpha)\,\mu \boldsymbol{1}_N\right).
\end{aligned}$$

3.3 Full conditional for β_0

$$\begin{aligned}
[\beta_0|\cdot] &\propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| \boldsymbol{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] [\beta_0] \\
&\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\frac{\left(\beta_0 - \mu_{\beta_0}\right)^2}{\sigma_{\beta_0}^2}\right\} \\
&\propto \exp\left\{-\frac{1}{2}\beta_0^2\left(\sum_{t=1}^{T} \boldsymbol{1}_{N_{Pt}}^T \boldsymbol{\Sigma}_{Pt}^{-1} \boldsymbol{1}_{N_{Pt}} + \frac{1}{\sigma_{\beta_0}^2}\right) - \beta_0\left(\sum_{t=1}^{T} \boldsymbol{1}_{N_{Pt}}^T \boldsymbol{\Sigma}_{Pt}^{-1} \left(\boldsymbol{W}_{Pt} - \beta_1 \boldsymbol{H}_{Pt}\boldsymbol{T}_t\right) + \frac{\mu_{\beta_0}}{\sigma_{\beta_0}^2}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\beta_0^2\left(\frac{M_P}{\tau_P^2} + \frac{1}{\sigma_{\beta_0}^2}\right) - \beta_0\left(\frac{1}{\tau_P^2}\sum_{t=1}^{T} \boldsymbol{1}_{N_{Pt}}^T \left(\boldsymbol{W}_{Pt} - \beta_1 \boldsymbol{H}_{Pt}\boldsymbol{T}_t\right) + \frac{\mu_{\beta_0}}{\sigma_{\beta_0}^2}\right)\right\}
\end{aligned}$$

which is MVN$(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with

$$\begin{aligned}
\boldsymbol{A}^{-1} &= \left(\frac{M_P}{\tau_P^2} + \frac{1}{\sigma_{\beta_0}^2}\right)^{-1} \\
\boldsymbol{b} &= \frac{1}{\tau_P^2}\sum_{t=1}^{T} \boldsymbol{1}_{N_{Pt}}^T \left(\boldsymbol{W}_{Pt} - \beta_1 \boldsymbol{H}_{Pt}\boldsymbol{T}_t\right) + \frac{\mu_{\beta_0}}{\sigma_{\beta_0}^2}.
\end{aligned}$$

3.4 Full conditional for β_1

$$
\begin{aligned}
[\beta_1|\cdot] &\propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| \boldsymbol{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] [\beta_1] \\
&\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - \left(\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t\right)\right)^T \boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t - \left(\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t\right)\right)\right\} \\
&\quad \times \exp\left\{-\frac{1}{2}\frac{\left(\beta_1 - \mu_{\beta_1}\right)^2}{\sigma^2_{\beta_1}}\right\} \\
&\propto \exp\left\{-\frac{1}{2}\beta_1^2\left(\sum_{t=1}^{T}\left(\boldsymbol{H}_{Pt}\boldsymbol{T}_t\right)^T \boldsymbol{\Sigma}_{Pt}^{-1}\left(\boldsymbol{H}_{Pt}\boldsymbol{T}_t\right) + \frac{1}{\sigma^2_{\beta_1}}\right)\right\} \\
&\quad \times \exp\left\{-\beta_1\left(\sum_{t=1}^{T}\left(\boldsymbol{H}_{Pt}\boldsymbol{T}_t\right)^T \boldsymbol{\Sigma}_{Pt}^{-1}\left(\boldsymbol{W}_{Pt} - \beta_0\boldsymbol{1}_{N_{Pt}}\right) + \frac{\mu_{\beta_1}}{\sigma^2_{\beta_1}}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\beta_1^2\left(\frac{1}{\tau_P^2}\sum_{t=1}^{T}\left(\boldsymbol{H}_{Pt}\boldsymbol{T}_t\right)^T \boldsymbol{\Sigma}_{Pt}^{-1}\left(\boldsymbol{H}_{Pt}\boldsymbol{T}_t\right) + \frac{1}{\sigma^2_{\beta_1}}\right)\right\} \\
&\quad \times \exp\left\{-\beta_1\left(\frac{1}{\tau_P^2}\sum_{t=1}^{T}\left(\boldsymbol{H}_{Pt}\boldsymbol{T}_t\right)^T \left(\boldsymbol{W}_{Pt} - \beta_0\boldsymbol{1}_{N_{Pt}}\right) + \frac{\mu_{\beta_1}}{\sigma^2_{\beta_1}}\right)\right\}
\end{aligned}
$$

which is MVN$(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$ with

$$
\begin{aligned}
\boldsymbol{A}^{-1} &= \left(\frac{1}{\tau_P^2}\sum_{t=1}^{T}\left(\boldsymbol{H}_{Pt}\boldsymbol{T}_t\right)^T\left(\boldsymbol{H}_{Pt}\boldsymbol{T}_t\right) + \frac{1}{\sigma^2_{\beta_1}}\right)^{-1} \\
\boldsymbol{b} &= \frac{1}{\tau_P^2}\sum_{t=1}^{T}\left(\boldsymbol{H}_{Pt}\boldsymbol{T}_t\right)^T\left(\boldsymbol{W}_{Pt} - \beta_0\boldsymbol{1}_{N_{Pt}}\right) + \frac{\mu_{\beta_1}}{\sigma^2_{\beta_1}}.
\end{aligned}
$$

NOTE: My devivation includes $\boldsymbol{H}_{Pt}$ in the $\boldsymbol{A}^{-1}$ and $\boldsymbol{b}$ terms but is not present in Martin's paper... This is worth pursuing futher. It seems that Martin uses $\boldsymbol{T}_{Pt}$ to denote $\boldsymbol{H}_{Pt}\boldsymbol{T}_t$??

3.5 Full conditional for μ

$$
\begin{aligned}
[\mu|\cdot] &\propto \prod_{t=1}^{T} \left[\boldsymbol{T}_t \middle| \boldsymbol{T}_{t-1}, \mu, \alpha, \sigma^2, \boldsymbol{Q}\right] [\mu] \\
&\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - (1-\alpha)\,\mu \mathbf{1}_N\right)^T \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - (1-\alpha)\,\mu \mathbf{1}_N\right)\right\} \\
&\qquad \times \exp\left\{-\frac{1}{2}\frac{(\mu-\mu_0)^2}{\sigma_0^2}\right\} \\
&\propto \exp\left\{-\frac{1}{2}\mu^2\left(\sum_{t=1}^{T}(1-\alpha)^2\, \mathbf{1}_N^T \boldsymbol{\Sigma}_\epsilon^{-1} \mathbf{1}_N + \frac{1}{\sigma_0^2}\right) - \mu\left((1-\alpha)\, \mathbf{1}_N^T \boldsymbol{\Sigma}_\epsilon^{-1} \sum_{t=1}^{T}\left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1}\right) + \frac{\mu_0}{\sigma_0^2}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\mu^2\left(T\,(1-\alpha)^2\, \mathbf{1}_N^T \boldsymbol{\Sigma}_\epsilon^{-1} \mathbf{1}_N + \frac{1}{\sigma_0^2}\right) - \mu\left((1-\alpha)\, \mathbf{1}_N \boldsymbol{\Sigma}_\epsilon^{-1} \sum_{t=1}^{T}\left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1}\right) + \frac{\mu_0}{\sigma_0^2}\right)\right\}
\end{aligned}
$$

which is MVN($\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1}$) with

$$
\begin{aligned}
\boldsymbol{A}^{-1} &= \left(T\,(1-\alpha)^2\, \mathbf{1}_N^T \boldsymbol{\Sigma}_\epsilon^{-1} \mathbf{1}_N + \frac{1}{\sigma_0^2}\right)^{-1} \\
\boldsymbol{b} &= (1-\alpha)\, \mathbf{1}_N \boldsymbol{\Sigma}_\epsilon^{-1} \sum_{t=1}^{T}\left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1}\right) + \frac{\mu_0}{\sigma_0^2}
\end{aligned}
$$

where $N = N_I + N_P = 33$ for the Hudson Valley PDSI

3.6 Full conditional for α

$$
\begin{aligned}
[\alpha|\cdot] &\propto \prod_{t=1}^{T} \left[\boldsymbol{T}_t \middle| \boldsymbol{T}_{t-1}, \mu, \alpha, \sigma^2, \boldsymbol{Q}\right] [\alpha] \\
&\propto \prod_{t=1}^{T} \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - (1-\alpha)\,\mu \mathbf{1}_N\right)^T \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - (1-\alpha)\,\mu \mathbf{1}_N\right)\right\} I\left\{\alpha \in (0,1)\right\} \\
&\propto \exp\left\{-\frac{1}{2}\alpha^2 \sum_{t=1}^{T}\left(\boldsymbol{T}_{t-1} - \mu \mathbf{1}_N\right)^T \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_{t-1} - \mu \mathbf{1}_N\right) - \alpha \sum_{t=1}^{T}\left(\boldsymbol{T}_{t-1} - \mu \mathbf{1}_N\right)^T \boldsymbol{\Sigma}_\epsilon^{-1} \left(\boldsymbol{T}_t - \mu \mathbf{1}_N\right)\right\} \\
&\qquad \times I\left\{\alpha \in (0,1)\right\}
\end{aligned}
$$

which is truncated $\mathrm{N}(\boldsymbol{A}^{-1}\boldsymbol{b}, \boldsymbol{A}^{-1})$, , restricted to $\alpha \in (0,1)$, with

$$\boldsymbol{A}^{-1} = \left(\sum_{t=1}^{T} \left(\boldsymbol{T}_{t-1} - \mu \mathbf{1}_N\right)^T \boldsymbol{\Sigma}_{\epsilon}^{-1} \left(\boldsymbol{T}_{t-1} - \mu \mathbf{1}_N\right) \right)^{-1}$$

$$\boldsymbol{b} = \sum_{t=1}^{T} \left(\boldsymbol{T}_{t-1} - \mu \mathbf{1}_N\right)^T \boldsymbol{\Sigma}_{\epsilon}^{-1} \left(\boldsymbol{T}_{t} - \mu \mathbf{1}_N\right).$$

3.7 Full conditional for τ_I^2

$$\begin{aligned}
\left[\tau_I^2 \middle| \cdot\right] &\propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| \boldsymbol{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \left[\tau_I^2\right] \\
&\propto \prod_{t=1}^{T} \left|\boldsymbol{\Sigma}_t\right|^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2} \left(\boldsymbol{W}_t - \left(\boldsymbol{H}_t \boldsymbol{T}_t + \boldsymbol{B}_t\right)\right)^T \boldsymbol{\Sigma}_t^{-1} \left(\boldsymbol{W}_t - \left(\boldsymbol{H}_t \boldsymbol{T}_t + \boldsymbol{B}_t\right)\right) \right\} \\
&\quad \times \left(\tau_I^2\right)^{-\alpha_I - 1} \exp\left\{ \frac{\beta_I}{\tau_I^2} \right\} \\
&\propto \left(\tau_I^2\right)^{-\frac{1}{2}\sum_{t=1}^{T} N_{It}} \exp\left\{ -\frac{1}{2} \sum_{t=1}^{T} \left(\boldsymbol{W}_{It} - \left(\boldsymbol{H}_{It} \boldsymbol{T}_t + \boldsymbol{B}_{It}\right)\right)^T \boldsymbol{\Sigma}_{It}^{-1} \left(\boldsymbol{W}_{It} - \left(\boldsymbol{H}_{It} \boldsymbol{T}_t + \boldsymbol{B}_{It}\right)\right) \right\} \\
&\quad \times \left(\tau_I^2\right)^{-\alpha_I - 1} \exp\left\{ \frac{\beta_I}{\tau_I^2} \right\} \\
&\propto \left(\tau_I^2\right)^{-\alpha_I - \frac{M_I}{2} - 1} \exp\left\{ -\frac{1}{\tau_I^2} \left(\beta_I + \frac{1}{2} \sum_{t=1}^{T} \left(\boldsymbol{W}_{It} - \left(\boldsymbol{H}_{It} \boldsymbol{T}_t\right)\right)^T \left(\boldsymbol{W}_{It} - \left(\boldsymbol{H}_{It} \boldsymbol{T}_t\right)\right) \right) \right\}
\end{aligned}$$

which is $\mathrm{IG}\Big(\alpha_I + \frac{M_I}{2}, \beta_I + \frac{1}{2}\sum_{t=1}^{T} \left(\boldsymbol{W}_{It} - \left(\boldsymbol{H}_{It} \boldsymbol{T}_t\right)\right)^T \left(\boldsymbol{W}_{It} - \left(\boldsymbol{H}_{It} \boldsymbol{T}_t\right)\right)\Big)$

3.8 Full conditional for τ_P^2

$$
\begin{aligned}
\left[\tau_P^2 \middle| \cdot\right] &\propto \prod_{t=1}^{T} \left[\boldsymbol{W}_t \middle| \boldsymbol{T}_t, \beta_0, \beta_1, \tau_I^2, \tau_P^2\right] \left[\tau_P^2\right] \\
&\propto \prod_{t=1}^{T} |\boldsymbol{\Sigma}_t|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)^T \boldsymbol{\Sigma}_t^{-1}\left(\boldsymbol{W}_t - (\boldsymbol{H}_t\boldsymbol{T}_t + \boldsymbol{B}_t)\right)\right\} \\
&\quad \times \left(\tau_P^2\right)^{-\alpha_P - 1} \exp\left\{\frac{\beta_P}{\tau_P^2}\right\} \\
&\propto \left(\tau_P^2\right)^{-\frac{1}{2}\sum_{t=1}^{T} N_{Pt}} \\
&\quad \times \exp\left\{-\frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{W}_{Pt} - (\beta_1\boldsymbol{H}_{Pt}\boldsymbol{T}_t + \boldsymbol{B}_{Pt})\right)^T \boldsymbol{\Sigma}_{Pt}^{-1}\left(\boldsymbol{W}_{Pt} - (\beta_1\boldsymbol{H}_{Pt}\boldsymbol{T}_t + \boldsymbol{B}_{Pt})\right)\right\} \\
&\quad \times \left(\tau_P^2\right)^{-\alpha_P - 1} \exp\left\{\frac{\beta_P}{\tau_P^2}\right\} \\
&\propto \left(\tau_P^2\right)^{-\alpha_P - \frac{M_P}{2} - 1} \\
&\quad \times \exp\left\{-\frac{1}{\tau_P^2}\left(\beta_P + \frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{W}_{Pt} - (\beta_1\boldsymbol{H}_{Pt}\boldsymbol{T}_t + \beta_0\boldsymbol{1}_{N_{Pt}})\right)^T\left(\boldsymbol{W}_{Pt} - (\beta_1\boldsymbol{H}_{Pt}\boldsymbol{T}_t + \beta_0\boldsymbol{1}_{N_{Pt}})\right)\right)\right\}
\end{aligned}
$$

which is IG$\Big(\alpha_P + \frac{M_P}{2}, \beta_P + \frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{W}_{Pt} - (\beta_1\boldsymbol{H}_{Pt}\boldsymbol{T}_t + \beta_0\boldsymbol{1}_{N_{Pt}})\right)^T\left(\boldsymbol{W}_{Pt} - (\beta_1\boldsymbol{H}_{Pt}\boldsymbol{T}_t + \beta_0\boldsymbol{1}_{N_{Pt}})\right)\Big)$

3.9 Full conditional for σ^2

$$
\begin{aligned}
\left[\sigma^2|\cdot\right] &\propto \prod_{t=1}^{T} \left[\boldsymbol{T}_t|\boldsymbol{T}_{t-1}, \mu, \alpha, \sigma^2, \boldsymbol{Q}\right]\left[\sigma^2\right] \\
&\propto \prod_{t=1}^{T} |\boldsymbol{\Sigma}_\epsilon|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\mu\mathbf{1}\right)^T \boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\mu\mathbf{1}\right)\right\} \\
&\quad \times \left(\sigma^2\right)^{-\alpha_{\sigma^2}-1} \exp\left\{-\frac{\beta_{\sigma^2}}{\sigma^2}\right\} \\
&\propto \left(\sigma^2\right)^{-\frac{NT}{2}} \exp\left\{-\frac{1}{\sigma^2}\left(\frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\mu\mathbf{1}\right)^T \boldsymbol{Q}^{-1}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\mu\mathbf{1}\right)\right)\right\} \\
&\quad \times \left(\sigma^2\right)^{-\alpha_{\sigma^2}-1} \exp\left\{-\frac{\beta_{\sigma^2}}{\sigma^2}\right\} \\
&\propto \left(\sigma^2\right)^{-\alpha_{\sigma^2}-\frac{NT}{2}-1} \\
&\quad \times \exp\left\{-\frac{1}{\sigma^2}\left(\beta_{\sigma^2} + \frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\mu\mathbf{1}\right)^T \boldsymbol{Q}^{-1}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\mu\mathbf{1}\right)\right)\right\}
\end{aligned}
$$

which is $\text{IG}\Big(\alpha_{\sigma^2} + \frac{NT}{2}, \beta_{\sigma^2} + \frac{1}{2}\sum_{t=1}^{T}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\mu\mathbf{1}\right)^T \boldsymbol{Q}^{-1}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\mu\mathbf{1}\right)\Big)$

3.10 Full conditional for $\boldsymbol{Q}$

$$
\begin{aligned}
\left[\boldsymbol{Q}|\cdot\right] &\propto \prod_{t=1}^{T} \left[\boldsymbol{T}_t|\boldsymbol{T}_{t-1}, \mu, \alpha, \sigma^2, \boldsymbol{Q}\right]\left[\boldsymbol{Q}\right] \\
&\propto \prod_{t=1}^{T} |\boldsymbol{\Sigma}_\epsilon|^{-\frac{1}{2}} \exp\left\{-\frac{1}{2}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\mu\mathbf{1}\right)^T \boldsymbol{\Sigma}_\epsilon^{-1}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\mu\mathbf{1}\right)\right\} \\
&\quad \times |\boldsymbol{Q}|^{-\frac{\nu+N+1}{2}} \exp\left\{-\frac{1}{2}\text{tr}\left(\boldsymbol{I}\boldsymbol{Q}^{-1}\right)\right\} \\
&\propto \exp\left\{-\frac{1}{2\sigma^2}\sum_{t=1}^{T}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\mu\mathbf{1}\right)^T \boldsymbol{Q}^{-1}\left(\boldsymbol{T}_t - \alpha\boldsymbol{T}_{t-1} - (1-\alpha)\mu\mathbf{1}\right)\right\} \qquad (1) \\
&\quad |\boldsymbol{\Sigma}_\epsilon|^{-\frac{\nu+T+N+1}{2}} \times \exp\left\{-\frac{1}{2}\text{tr}\left(\boldsymbol{I}\boldsymbol{Q}^{-1}\right)\right\}
\end{aligned}
$$

where $\text{tr}(\cdot)$ is the trace of a matrix. Now consider the sum in (1) and define $\boldsymbol{T}_{t,i}$ as the i^{th} element of the vector $\boldsymbol{T}_t$ and $\boldsymbol{Q}_{i,j}^{-1}$ as the element in the i^{th} row and j^{th} column of $\boldsymbol{Q}$. Then

$$
\begin{aligned}
&\sum_{t=1}^{T} \left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - (1-\alpha)\,\mu \mathbf{1}\right)^T \boldsymbol{Q}^{-1} \left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - (1-\alpha)\,\mu \mathbf{1}\right) \\
&= \sum_{t=1}^{T} \sum_{i=1}^{N} \sum_{j=1}^{N} \left(\boldsymbol{T}_{t,i} - \alpha \boldsymbol{T}_{t-1,i} - (1-\alpha)\,\mu\right) \boldsymbol{Q}_{i,j}^{-1} \left(\boldsymbol{T}_{t,j} - \alpha \boldsymbol{T}_{t-1,j} - (1-\alpha)\,\mu\right) \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \boldsymbol{Q}_{i,j}^{-1} \sum_{t=1}^{T} \left(\boldsymbol{T}_{t,i} - \alpha \boldsymbol{T}_{t-1,i} - (1-\alpha)\,\mu\right) \left(\boldsymbol{T}_{t,j} - \alpha \boldsymbol{T}_{t-1,j} - (1-\alpha)\,\mu\right) \\
&= \sum_{i=1}^{N} \sum_{j=1}^{N} \boldsymbol{Q}_{i,j}^{-1} \boldsymbol{Q}\left(\boldsymbol{T}, \alpha, \mu\right) \\
&= \operatorname{tr}\left(\boldsymbol{Q}_{i,j}^{-1} \boldsymbol{Q}\left(\boldsymbol{T}, \alpha, \mu\right)\right)
\end{aligned}
$$

Define $\bar{\boldsymbol{T}} = \bar{\boldsymbol{T}}_t - \alpha \bar{\boldsymbol{T}}_{t-1} = \frac{1}{T}\sum_{t=1}^{T} \boldsymbol{T}_t - \frac{\alpha}{T}\sum_{t=1}^{T} \boldsymbol{T}_{t-1}$. Now, examine $\boldsymbol{Q}(\boldsymbol{T}, \alpha, \mu)$ further.

$$
\begin{aligned}
\boldsymbol{Q}(\boldsymbol{T}, \alpha, \mu) &= \sum_{t=1}^{T} (\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - (1-\alpha)\mu \mathbf{1})(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - (1-\alpha)\mu \mathbf{1})^T \\
&= \sum_{t=1}^{T} \left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - \bar{\boldsymbol{T}} + \bar{\boldsymbol{T}} - (1-\alpha)\mu \mathbf{1}\right)\left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - \bar{\boldsymbol{T}} + \bar{\boldsymbol{T}} - (1-\alpha)\mu \mathbf{1}\right)^T \\
&= \sum_{t=1}^{T} \left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - \bar{\boldsymbol{T}}\right)\left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - \bar{\boldsymbol{T}}\right)^T \\
&\quad + \sum_{t=1}^{T} \left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - \bar{\boldsymbol{T}}\right)\left(\bar{\boldsymbol{T}} - (1-\alpha)\mu \mathbf{1}\right)^T \\
&\quad + \sum_{t=1}^{T} \left(\bar{\boldsymbol{T}} - (1-\alpha)\mu \mathbf{1}\right)\left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - \bar{\boldsymbol{T}}\right)^T \\
&\quad + \sum_{t=1}^{T} \left(\bar{\boldsymbol{T}} - (1-\alpha)\mu \mathbf{1}\right)\left(\bar{\boldsymbol{T}} - (1-\alpha)\mu \mathbf{1}\right)^T \\
&= \sum_{t=1}^{T} \left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - \bar{\boldsymbol{T}}\right)\left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - \bar{\boldsymbol{T}}\right)^T \\
&\quad + \left(\sum_{t=1}^{T} (\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1}) - \bar{\boldsymbol{T}}\right)\left(\bar{\boldsymbol{T}} - (1-\alpha)\mu \mathbf{1}\right)^T \\
&\quad + \left(\bar{\boldsymbol{T}} - (1-\alpha)\mu \mathbf{1}\right)\left(\sum_{t=1}^{T} (\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1}) - \bar{\boldsymbol{T}}\right)^T \\
&\quad + \sum_{t=1}^{T} \left(\bar{\boldsymbol{T}} - (1-\alpha)\mu \mathbf{1}\right)\left(\bar{\boldsymbol{T}} - (1-\alpha)\mu \mathbf{1}\right)^T \\
&= \sum_{t=1}^{T} \left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - \bar{\boldsymbol{T}}\right)\left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - \bar{\boldsymbol{T}}\right)^T + T\left(\bar{\boldsymbol{T}} - (1-\alpha)\mu \mathbf{1}\right)\left(\bar{\boldsymbol{T}} - (1-\alpha)\mu \mathbf{1}\right)^T \\
&= \boldsymbol{Q}_0 + T\boldsymbol{q}\boldsymbol{q}^T
\end{aligned}
$$

where $\boldsymbol{Q}_0 = \sum_{t=1}^{T} \left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - \bar{\boldsymbol{T}}\right)\left(\boldsymbol{T}_t - \alpha \boldsymbol{T}_{t-1} - \bar{\boldsymbol{T}}\right)^T$ and $\boldsymbol{q} = \left(\bar{\boldsymbol{T}} - (1-\alpha)\mu \mathbf{1}\right)$. Then, substituting the above derivation into (1) gives

$$
\begin{aligned}
&\propto |\boldsymbol{Q}|^{-\frac{\nu+T+N+1}{2}} \exp\left\{-\frac{1}{2\sigma^2}\mathrm{tr}\left(\boldsymbol{Q}^{-1}\left(\boldsymbol{Q}_0 + T\boldsymbol{q}\boldsymbol{q}^T\right)\right)\right\} \exp\left\{-\frac{1}{2}\mathrm{tr}\left(\boldsymbol{I}\boldsymbol{Q}^{-1}\right)\right\} \\
&\propto |\boldsymbol{Q}|^{-\frac{\nu+T+N+1}{2}} \exp\left\{-\frac{1}{2}\mathrm{tr}\left(\boldsymbol{Q}^{-1}\frac{\left(\boldsymbol{Q}_0 + T\boldsymbol{q}\boldsymbol{q}^T + \boldsymbol{I}\right)}{\sigma^2}\right)\right\}
\end{aligned}
$$

4 Extensions to the model

- Predictive Process for Σ_ϵ in the spatial model
- Include the number of trees used to create a chronology in the model
- ???

5 References

- Martin Tingley's BARCAST papers
- Sayer's discussion on Wishart priors http://www.math.wustl.edu/ sawyer/hmhandouts/Wishart.pdf