Time Series

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August 24, 2015

First, we start with the canonical difference equation for the time series autoregressive model of order 1 (AR(1))

$$y_t = \phi y_{t-1} + \epsilon_t \tag{1}$$

where the time series observations for times $t = 1, \ldots, T$ are given by the vector $\mathbf{y} = (y_1, \ldots, y_T)$ where y_t is the observation of the time serise at time t. The autoregressive parameter ϕ controls the strength of autocorrelation in the time series with $-1 < \phi < 1$ and the random error $\epsilon_t \sim N(0, \sigma^2)$ is independent for different times (i.e. the covariance $\text{Cov}(\epsilon_t, \epsilon_{t+k}) = 0$ for $k \neq 0$).

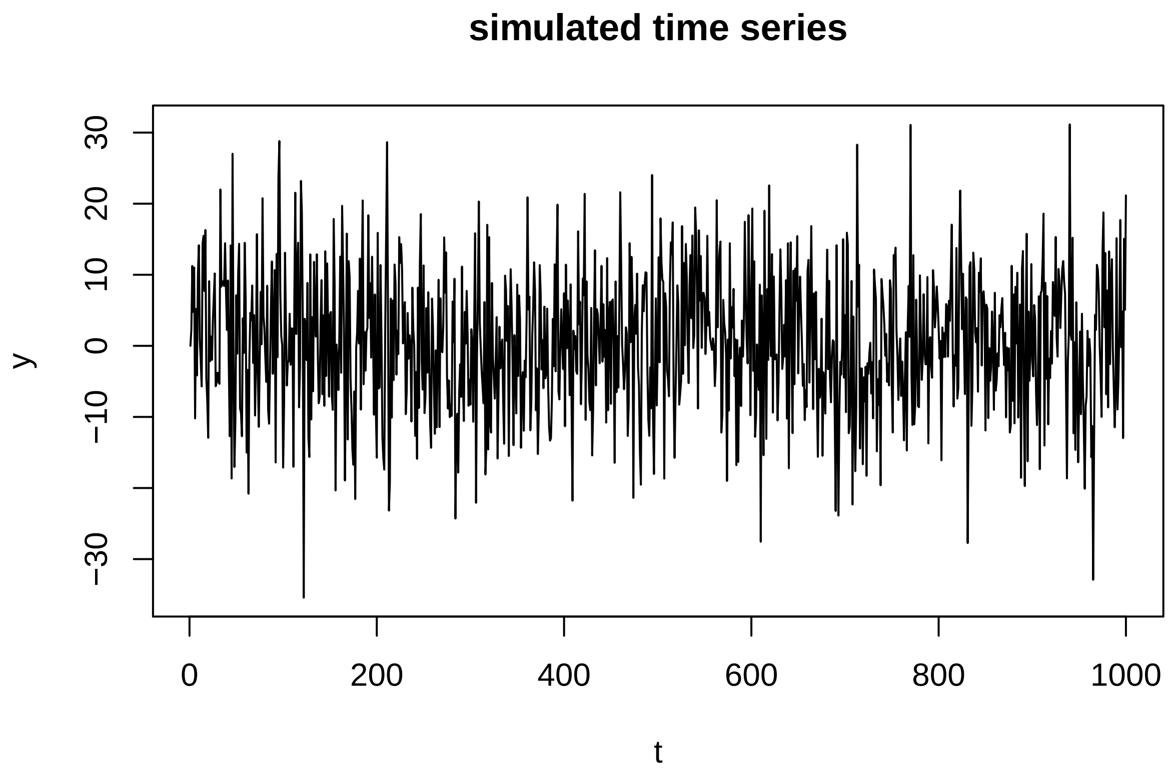
Lets simulate some data here

```r
N <- 1                              ## pick the number of time series
t <- 1000                           ## pick a time series length
mu <- sin(2 * pi * 1:t)
s <- 10                             ## pick standard deviation
phi <- 0.75                         ## pick autocorrelation parameter

##
## function to simulate time series
##

simTimeSeries <- function(t, N, mu, s, phi){
  y <- matrix(mu[1], t, N)                             ## initialize container
  epsilon <- rnorm(N*(t-1), 0, s)                      ## independent random error
  y[2:t, ] <- mu[2:t] + phi * y[1:(t-1), ] + epsilon  ## autoregressive model
  return(y)
}

y <- simTimeSeries(t, N, mu, s, phi)
matplot(y, type="l", main="simulated time series", xlab="t")
```



The expected value $\mathrm{E}(y_t)$ of the time series at time t is

$$\begin{aligned}\mathrm{E}(y_t) &= \mathrm{E}(\phi y_{t-1}) + \mathrm{E}(\epsilon_t) \\ &= \phi \mathrm{E}(y_{t-1}) + 0\end{aligned}$$

where, assuming a constant mean μ accross time we have

$$\begin{aligned}\mathrm{E}(y_t) &= \phi \mathrm{E}(y_{t-1}) \\ \rightarrow \mathrm{E}(y_t)(1-\phi) &= 0 \\ \rightarrow \mathrm{E}(y_t) &= 0\end{aligned}$$

and assuming constant variance through time, the variance is

$$\begin{aligned}\mathrm{Var}(y_t) &= \mathrm{Var}(\phi y_{t-1} + \epsilon_t) \\ &= \mathrm{Var}(\phi y_{t-1}) + \mathrm{Var}(\epsilon_t) + 2\mathrm{Cov}(\phi y_{t-1}, \epsilon_t) \\ &= \phi^2 \mathrm{Var}(y_{t-1}) + \sigma^2 + 0\end{aligned}$$

Then using our modeling assumption $\mathrm{Var}(y_t) = \mathrm{Var}(y_{t-1})$,

$$\mathrm{Var}(y_t) - \phi^2 \mathrm{Var}(y_t) = \sigma^2$$

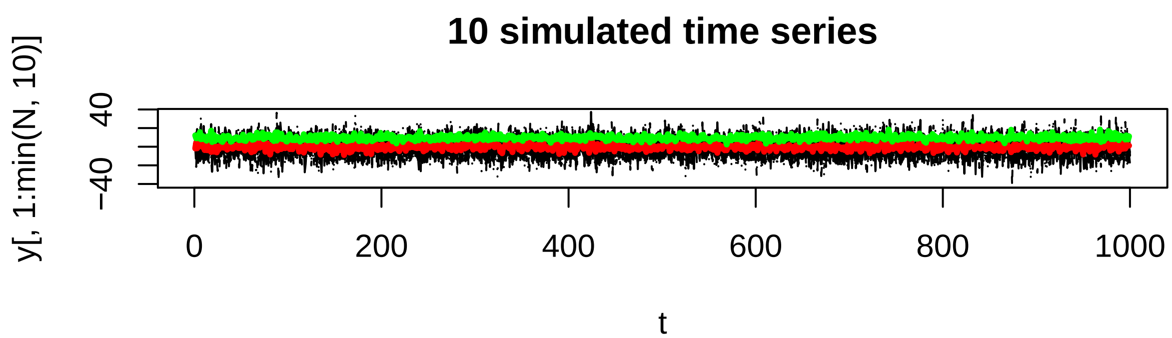
gives the solution $\mathrm{Var}(y_t) = \frac{\sigma^2}{1-\phi^2}$.

```
N <- 10                                    ## replicates N
y <- simTimeSeries(t, N, mu, s, phi)  ## simulate time series
layout(matrix(1:2, 2, 1))
matplot(y[, 1:min(N, 10)], type="l", col="black",
        main=paste(min(N, 10), " simulated time series", sep="") , xlab="t")
## legend() add in legend for mean and y label
## Notice that the y axis is shrunk to 0

##

mean_y <- apply(y[2:t, ], 1, mean)    ##  calculate the mean
var_y <-apply(y[2:t, ], 1, var)       ##  calculate the variance
sd_y <- apply(y[2:t, ], 1, sd)        ##  calculate the standard deviation

matplot(mean_y, type="l", col="red", lwd = 3, add = TRUE)
matplot(sd_y, type = 'l', add = TRUE, col = 'green', lwd = 3)
```



The covariance between observations $\mathrm{Cov}(y_t, y_{t+k})$ at times k lags apart (assuming without loss of generality that $k > 0$) is

$$
\begin{aligned}
\mathrm{Cov}(y_t, y_{t+k}) &= \mathrm{E}(y_t y_{t+k}) - \mathrm{E}(y_t)\mathrm{E}(y_{t+k}) \\
&= \mathrm{E}(y_t(\phi y_{t+k-1} + \epsilon_{t+k})) - 0 \\
&= \mathrm{E}(\phi y_t y_{t+k-1}) + \mathrm{E}(y_t \epsilon_{t+k}) \\
&= \mathrm{E}(\phi y_t y_{t+k-1}) + \mathrm{E}(y_t)\mathrm{E}(\epsilon_{t+k}) \\
&= \mathrm{E}(\phi y_t y_{t+k-1}) + 0 \\
&= \mathrm{E}(y_t(\phi y_{t+k-2} + \epsilon_{t+k-1})) \\
&= \vdots \\
&= \phi^k \mathrm{E}(y_t^2) \\
&= \phi^k \frac{\sigma^2}{1-\phi^2}.
\end{aligned}
$$

Thus, knowing the mean, variance, and covariance at each time t and each lag k, we can write the autoregressive model (**??**) as

$$
\mathbf{y} = \boldsymbol{\mu} + \boldsymbol{\eta} \tag{2}
$$

where $\mu = (0, \ldots, 0)$ and η N $(\mathbf{0}, \boldsymbol{\Sigma})$ where

$$
\boldsymbol{\Sigma} = \frac{\sigma^2}{1-\phi^2} \left(\begin{pmatrix}
1 & \phi & \phi^2 & \phi^3 & \cdots & \phi^{T-1} \\
\phi & 1 & \phi & \phi^2 & \cdots & \phi^{T-2} \\
\phi^2 & \phi & 1 & \phi & \cdots & \phi^{T-3} \\
\phi^3 & \phi^2 & \phi & 1 & \cdots & \phi^{T-4} \\
\vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\
\phi^{T-1} & \phi^{T-2} & \phi^{T-3} & \phi^{T-4} & \cdots & 1
\end{pmatrix} \right)
$$

Typically, the distributions of interest in a time series model are the forecast distribution (used for prediction) and the smoothing distribution (used for estimation of parameters). The forecast distribution at time $\tau + 1$ consists of knowledge of all of the observations of the time series up to the time τ $y_{1:\tau} = (y_1, \ldots, y_\tau)$ given by

$$
[y_{\tau+1} | y_{1:\tau}] = [y_{\tau+1} | y_\tau]
$$

by the Markov assumption in the autoregressive model. Then, the one step ahead expected forecast is

$$
\begin{aligned}
\mathrm{E}(y_{\tau+1} | y_{1:\tau}) &= \mathrm{E}(y_{\tau+1} | y_\tau) \\
&= \mathrm{E}(\phi y_\tau + \epsilon_{\tau+1} | y_\tau) \\
&= \mathrm{E}(\phi y_\tau | y_\tau) + \mathrm{E}(\epsilon_{\tau+1} | y_\tau) \\
& \phi y_\tau + 0.
\end{aligned}
$$

The k step ahead expected forecast is calculated by using a recursive formula of the equation above where $\mathrm{E}(y_{\tau+k} | y_{1:\tau}) = \phi^k y_\tau$.

Likewise, the one step ahead forecast variance is

$$
\begin{aligned}
\mathrm{Var}(y_{\tau+1}|y_{1:\tau}) &= \mathrm{Var}(y_{\tau+1}|y_{\tau}) \\
&= \mathrm{Var}(\phi y_{\tau} + \epsilon_{\tau+1}|y_{\tau}) \\
&= \phi^2 \mathrm{Var}(y_{\tau}|y_{\tau}) + 2\mathrm{Cov}(\phi y_{\tau}, \epsilon_{\tau+1}|y_{\tau}) + \mathrm{Var}(\epsilon_{\tau+1}|y_{\tau}) \\
&= 0 + 0 + \sigma^2.
\end{aligned}
$$

The k step ahead forecast variance can also be calcuated recursively giving $\mathrm{Var}(y_{\tau+k}|y_{1:\tau-1}) = \sum_{i=1}^{k} \phi^{2(i-1)}\sigma^2$.