Time Series

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First, we start with the canonical difference equation for the time series autoregressive model of order 1 (AR(1))

$$y_t = \phi y_{t-1} + \epsilon_t$$

where the time series observations for times $t = 1, \ldots, T$ are given by the vector $\mathbf{y} = (y_1, \ldots, y_T)$ where y_t is the observation of the time serise at time t. The autoregressive parameter ϕ controls the strength of autocorrelation in the time series with $-1 < \phi < 1$ and the random error $\epsilon_t \sim N(0, \sigma^2)$ is independent for different times (i.e. the covariance $\mathrm{Cov}(\epsilon_t, \epsilon_{t+k}) = 0$ for $k \neq 0$).

The expected value $\mathrm{E}(y_t)$ of the time series at time t is

$$\begin{aligned}\mathrm{E}(y_t) &= \phi y_{t-1} + \mathrm{E}(\epsilon_t)\\ &= \phi y_{t-1}\end{aligned}$$

and assuming constant variance through time, the variance is

$$\begin{aligned}\mathrm{Var}(y_t) &= \mathrm{Var}(\phi y_{t-1} + \epsilon_t)\\ &= \mathrm{Var}(\phi y_{t-1}) + \mathrm{Var}(\epsilon_t) + 2\mathrm{Cov}(\phi y_{t-1}\epsilon_t)\\ &= \phi^2 \mathrm{Var}(y_{t-1}) + \sigma^2 + 0\end{aligned}$$

Then using our modeling assumption $\mathrm{Var}(y_t) = \mathrm{Var}(y_{t-1})$,

$$\mathrm{Var}(y_t) - \phi^2 \mathrm{Var}(y_t) = \sigma^2$$

gives the solution $\mathrm{Var}(y_t) = \frac{\sigma^2}{1-\phi^2}$.

The covariance between observations $\mathrm{Cov}(y_t, y_{t+k})$ at times k lags apart (assuming without loss of generality that $k > 0$) is

$$\mathrm{Cov}(y_t, y_{t+k}) = \mathrm{E}(y_t y_{t+k}) - \mathrm{E}(y_t)\mathrm{E}(y_{t+k})$$