CONCAVITY OF RANDOM SIGN ℓ_p-TYPE SUMS AND THEIR IMPLICATIONS FOR L^p SPACE STRUCTURE

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ABSTRACT. Motivated by a result by G. Schechtman which facilitated the computation of the p-concavity constant of Banach lattices that are the span of 1-unconditional basic sequences in $L^p([0,2])$ for $p > 2$, we consider a probabilistic generalization which in a specific case would resolve the $2 < p < 3$ case which has been open since Schechtman's original paper. Our main novel result is showing that the expected value of the $\ell_{1/p}$ norm where each element is weighted by a uniform random on $[-1,1]$ is concave for all $p > 2$.

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1. INTRODUCTION

This paper has two main aims. The first is to present my research about the p-concavity constant of L_p spaces. The second aim is to present exposition on general Schauder basis theory, general unconditional convergence, and some very basic theory on Banach lattices. In particular, more exposition and background is included than is strictly necessary to explain my results. One such example is the inclusion of the proof of the Riemann summation theorem, and the result by Dvoretsky and Rogers which shows that in every infinite dimensional Banach space, there exists a series which converges unconditionally but not absolutely. With that in mind, we begin with the research problem of interest and then proceed to go into more detailed exposition.

Fix $n \in \mathbb{N}$ where by convention $0 \notin \mathbb{N}$. We write $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_+^n = \{x \in \mathbb{R}^n \mid x_i \in \mathbb{R}_+, \forall i \leqslant n\}$ to be the set of all non-negative real numbers and the set of all vectors in $\mathbb{R}^n$ with non-negative coordinates respectively. Consider $\{\epsilon_i\}_{i=1}^n$ to be an independent identically distributed (abbreviated to i.i.d.) sequence of Rademacher random variables with $\mathbb{P}(\epsilon_i = 1) = \mathbb{P}(\epsilon_i = -1) = \frac{1}{2}$ for each $i \leqslant n$.

Then for each $p \in [1, \infty)$ we define $\varphi : \mathbb{R}_+^n \to \mathbb{R}_+$ by

$$\varphi(x) = \mathbb{E}\left|\sum_{i=1}^n \epsilon_i x_i^{1/p}\right|^p$$

where our expectation is taken with respect to the probability measure generated by $\{\epsilon_i\}_{i=1}^n$.

Remark. *We note that the ℓ_p norm of x is defined as $\|x\|_{\ell_p} = (\sum_{i=1}^n |x_i|^p)^{1/p}$. In particular, φ can be viewed as taking the average of the $\ell_{1/p}$ norm of x over every sign choice for each coordinate. The exact reason for this, specifically why $\ell_{1/p}$ and not ℓ_p, arises from Banach lattice theory and will be explained later in the paper.*

Remark. *The language used in this paper will switch between using measure-theoretic integration and probabilistic expected value whenever one is more natural than the other. For instance, the Rademacher random variables can also be considered as the Rademacher functions, and we switch between these two perspectives frequently. For instance, the Khintchine inequality can be written using either of these two forms, differing only in notation, and hence we switch between them whenever convenient.*

As will be shown, the function φ has connections to both the Khintchine inequality and a structural property of L_p spaces. In G. Schechtman's paper (see [Sch95]), it was shown that $\varphi(x)$ is concave when $p \geqslant 3$, and using this, it was shown that the the p-concavity constant of every 1-unconditional basic sequence in L_p is 1. In the case when $2 < p < 3$, it was conjectured that φ remains concave (and the case where $1 < p < 2$ is conjectured but not proven to be convex). The main novel result my research has shown is a partial result towards the case where $2 < p < 3$, namely if we replace our Rademacher random variables with uniform random variables on $[-1, 1]$, then φ is concave. If we view the concavity of φ as a probabilistic question and ask for what sequences of i.i.d. random variables is φ concave, knowing that it is concave for random signs tells us that is concave for any symmetric random variables. Knowing that φ is concave with uniform random variables on $[-1, 1]$ tells us that φ is concave for all unimodal symmetric random variables.

We outline the paper now. We begin by defining Schauder bases and provide some of the basic conditions that are required to have a Schauder basis. Afterwards we begin to examine unconditional and conditional series in $\mathbb{R}$, $\mathbb{R}^d$, and general Banach spaces: of note is the presentation of the Riemann series theorem and Dvoretsky and Rogers' theorem.

After this, we define unconditional Schauder bases and basic sequences, and define the unconditional constant for unconditional bases. Afterwards we define Banach lattices and define the notion of p-convexity and p-concavity.

Next, we prove a particular version of the Khintchine inequality which gives an explicit bound on the growth of the Khintchine constant which will be an important part of the motivation for Schechtman's paper. The final collection of exposition explains the results contained in Schechtman's paper which is the direct motivation for our research. Finally we present our novel result stated earlier.

1.1. **Schauder Basis Theory Introduction.** For the purposes of this paper, we only consider real Banach spaces, that is Banach spaces whose underlying scalar field is the reals. Let X be a real Banach space.

Definition 1.1. *A sequence $\{x_n\}_{n=1}^\infty$ is a Schauder basis if for every point $x \in X$, there exists a unique sequence $\{a_n\}_{n=1}^\infty \subseteq \mathbb{R}$ with $x = \sum_{n=1}^\infty a_n x_n$ where the infinite sum is understood in the sense that $\sum_{n=1}^\infty a_n x_n = \lim_{M\to\infty} \sum_{n=1}^M a_n x_n$.*

Definition 1.2. *Any sequence $\{x_n\}_{n=1}^{\infty}$ which is a Schauder basis of its closed linear span is called a basic sequence.*

Remark. *We note for posterity why Schauder bases are of particular interest to analysis. The typical definition of a basis for finite dimensional spaces is a finite set of vectors that are both linearly independent and span the underlying space. When moving to infinite dimensional spaces, this definition requires us to consider infinite bases since no finite set can span an infinite dimensional space by definition. The next question is about the specific definition of spanning sets: should we consider infinite linear combinations or only those with finitely many elements? Schauder bases consider the former while Hamel bases only allow finite linear combinations. To be explicit, a Hamel basis is a sequence $\{x_n\}_{n=1}^{\infty} \subseteq X$ such that every element $x \in X$ can be represented uniquely as the finite linear combination of elements of our sequence.*

From an algebraic viewpoint, Hamel bases are much nicer. In particular, in order to allow infinite linear combinations of elements, we require some topology on the underlying space to even define what an infinite sum means. In typical analytic contexts, this presents no challenge since usually we work with topological vector spaces. Additionally, Hamel bases necessarily must be uncountably infinite in infinite dimensional Banach spaces due to the Baire category theorem, whereas Schauder bases are by definition countably infinite. The following theorem shows that Hamel bases must be uncountably infinite in infinite dimensional Banach spaces.

Theorem 1.3. *Suppose X is an infinite dimensional Banach space. Then any Hamel basis of x is necessarily uncountably infinite.*

Proof. This proof follows from the Baire Category theorem. Suppose that this is false, and $\{x_n\}_{n\in\mathbb{N}}$ is a countable Hamel basis for X. Define $X_n = \operatorname{span}\{x_i \mid i \leqslant n\}$. Note that X_n is a finite dimensional subspace of X. In particular X_n is closed and X_n has empty interior. However, because $\{x_n\}_{n\in\mathbb{N}}$ is a Hamel basis, for every $x \in X$, there is some $m \in \mathbb{N}$ with $x = \sum_{n=1}^{m} a_n x_n$. In particular $x \in X_n$, and hence $X = \bigcup_{n=1}^{\infty} X_n$. Thus X is the countable union of nowhere dense sets. However this contradicts the Baire Category theorem, and hence we deduce that X must not have a countable Hamel basis. □

For the rest of the paper we say basis to mean Schauder basis unless specified otherwise. With the obvious choice of an analytic basis being a Schauder basis, there are a few natural questions to ask. These questions will be the main topics of the next section. The first question is what spaces actually have bases. Ideally we would hope that every Banach space has such a basis, but this is false. As we will show, any Banach space that has a basis is separable (and in particular not every Banach space has a basis). In the specific case of a separable Hilbert space however, every such space has a basis that is orthonormal. The second question to ask is whether subspaces of Banach spaces must have a basis, and in fact every infinite dimensional Banach space must have an infinite dimensional subspace with a basis. With these questions stated, we proceed to answering them.

1.2. **Where Existence of a Basis Holds.** In this section, we give a few basic criteria for when a basis exists in a Banach space, and give the easiest necessary condition to have a basis. As will be shown, a Banach space with a basis must be separable, and in fact all separable Hilbert spaces have a basis (which can be chosen to be orthonormal).

Theorem 1.4. *Suppose X is a Banach space and $\{e_n\}_{n=1}^{\infty}$ is a basis. Then X must be separable.*

Proof. The basic idea of the proof is that the set of all finite linear combinations of basis elements with rational coefficients is both countable and dense. Countability is trivial. Density can be shown directly. The proof is very standard and hence omitted. □

As stated earlier, the next obvious question to ask is whether every separable Banach space has a basis. One easy partial result shows that this is indeed true in separable Hilbert spaces. First we need a lemma that ensures that our basis is indeed countable.

Lemma 1.5. *Suppose X is a separable Hilbert space and $H \subseteq X$ is a orthonormal set. Then H is countable.*

Proof. Suppose for the sake of contradiction that H is uncountable and fix $x_1, x_2 \in H$. Then $\|x_1\| = \|x_2\| = 1$ and $\langle x_1, x_2 \rangle = 0$. Then by a simple computation

$$\|x_1 - x_2\|^2 = \langle x_1 - x_2, x_1 - x_2 \rangle = \|x_1\|^2 + \|x_2\|^2 - 2\langle x_1, x_2 \rangle = \|x_1\|^2 + \|x_2\|^2 = 2$$

Thus $\|x_1 - x_2\| = \sqrt{2}$ and thus we have that $\{B(x, \frac{\sqrt{2}}{2}) \mid x \in H\}$ is a disjoint uncountable collection of balls. Since X is separable, we must have a countable dense subset. By density, each such ball contains at least one point in our dense subset, and hence our dense subset must be uncountable. This is a contradiction and hence H must be countable as desired. □

Theorem 1.6. *Suppose X is a separable Hilbert space. Then there exists a basis. Moreover there exists an orthonormal basis.*

Proof. Consider the set of orthonormal subsets of X partially ordered by set inclusion. We show that every chain has an upper bound and apply Zorn's Lemma.

Suppose $Z \subseteq X$ is a chain. We note that $H = \bigcup_{h \in Z} h$ is orthonormal and an upper bound. In particular for every $x, y \in H$, there exists $Z_x, Z_y \in Z$ containing x and y respectively. Either $Z_x \subseteq Z_y$ or $Z_y \subseteq Z_x$ and without loss of generality $Z_x \subseteq Z_y$. Then $x, y \in Z_y$ and Z_y is orthonormal. Thus $\|x\| = \|y\| = 1$ and $\langle x, y \rangle = 0$ as desired. Thus H is indeed orthonormal.

Note that every chain has an upper bound and hence by Zorn's lemma there exists a maximal element of our set, lets say H. By our previous Lemma 1.5 we know that H is countable. We claim further that H is dense.

Enumerate H as $\{e_n\}_{n=1}^{\infty}$. Note that by standard Hilbert theory there exists a unique projection map P that maps X to the closed span of H and furthermore for every $x \in X$ we have that $P(x) = \sum_{n=1}^{\infty} \langle x, e_n \rangle e_n$. In particular, the unique projection map onto H shows exactly that H is dense and that every element in our space can be represented uniquely as the infinite linear combination of elements in H. Thus H is a basis as desired. □

Thus far, we have shown that all separable Hilbert spaces contain a basis and in fact an orthonormal basis. Separability is required to have a basis for Banach (and hence Hilbert) spaces. The next natural question is to ask if every separable Banach space has a basis. This was answered in the negative by Per Enflo in his seminal paper (see [Enf73]). The proof of this fact is beyond the scope of this paper. While every separable Banach space may not have a basis, it turns out that every infinite dimensional Banach space has a closed infinite dimensional subspace with a basis (and of course every finite dimensional subspace has a basis). The language typically used to describe this result relies on the notion of a basic sequence. We recall the definition now.

Definition 1.7. *The sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$, where X is an infinite dimensional Banach space, is a basic sequence if $\{x_n\}_{n \in \mathbb{N}}$ is a basis for its closed linear span.*

The previous question can now be rephrased as the following: does every Banach space admit a basic sequence? Since finite dimensional Banach spaces always have a basis, it suffices to consider the infinite dimensional case. As will be shown, the answer to this question is yes. The proof is due to Mazur and follows along the lines of Theorem 1.a.5 in [LT13]. Our proof relies on a computational lemma and then a technical lemma that will be used to generate the elements of our basic sequence.

Said elements will be generated to maintain linear independence in such a way that the generated series remains linear independent as well.

Lemma 1.8. *Suppose X is an infinite dimensional Banach space and $B \subseteq X$ is a finite dimensional subspace. Let $\epsilon > 0$. Then we have an $x \in X$ with $\|x\| = 1$ and for all $y \in B$ and $\lambda \in \mathbb{R}$, we have that $\|y\| \leqslant (1+\epsilon)\|y + \lambda x\|$.*

Proof. Without loss of generality, we can assume $\epsilon < 1$. Note that since B is finite dimensional, we have that $K = \{y \in B \mid \|y\| = 1\}$ is compact. In particular, we can find $\{y_i\}_{i=1}^m \subseteq K$ such that $K \subseteq \bigcup_{i=1}^m B(y_i, \epsilon/2)$. Using Hahn-Banach, we can create $\{y_i^*\}_{i=1}^m \subseteq X^*$ such that $y_i^*(y_i) = 1$ for all $i \leqslant m$ and $\|y_i^*\|_{X^*} = 1$. We aim to find $x \in X$ with $\|x\| = 1$ and $y_i^*(x) = 0$ for all $i \leqslant n$. This x will be exactly the x stated in the lemma. In order to construct such an x, we show that $\bigcap_{i=1}^m \ker y_i^* \neq \{0\}$.

Suppose for the sake of contradiction that $\bigcap_{i=1}^m \ker y_i^* = \{0\}$. Then, if we define $f : X \to \mathbb{R}^m$ by $f(x) = (y_1^*(x), \ldots, y_m^*(x))$, we have that $\ker f = \bigcap_{i=1}^m \ker y_i^* = \{0\}$. This implies that f is a linear, injective function from $X \to \mathbb{R}^m$. Since X is infinite dimensional, this is a contradiction, and hence $\bigcap_{i=1}^m \ker y_i^*$ is non-trivial.

Fix any non-zero $x \in \bigcap_{i=1}^m \ker y_i^*$ with $\|x\| = 1$. By construction $y_i^*(x) = 0$ for all $i \leqslant m$. Now we show that x satisfies the specified property. Fix $\lambda \in \mathbb{R}$ and $y \in B$. If $y = 0$, then our inequality is trivial, so assume $y \neq 0$. Define $y' = y/\|y\|$, and $\lambda' = \lambda/\|y\|$. Since $\|y'\| = 1$, we have some y_i with $\|y' - y_i\| < \epsilon/2$ (since our previously defined K covers the surface of the unit ball). Then

$$\|y' + \lambda' x\| = \|y_i + \lambda' x + (y' - y_i)\| \geqslant \|y_i + \lambda' x\| - \|y' - y_i\| \geqslant \|y_i + \lambda' x\| - \epsilon/2$$

Since $\|y_i^*\|_{X^*} = 1$, we have then that $\|y_i + \lambda' x\| \geqslant y_i^*(y_i + \lambda' x) = 1$. Accordingly we deduce that

$$\|y' + \lambda' x\| \geqslant 1 - \epsilon/2$$

Multiplying both sides by $\|y\|$, we deduce that

$$\|y + \lambda x\| \geqslant \|y\|(1 - \epsilon/2) \geqslant \frac{\|y\|}{1+\epsilon}$$

where we used the fact that $1 - \frac{\epsilon}{2} > \frac{1}{1+\epsilon}$ for all $0 < \epsilon < 1$. Rearranging we deduce our desired inequality. ☐

The purpose of this lemma is to allow us to inductively build a basic sequence. If you imagine B to be the span of finitely many vectors, then the inequality we just proved tells us that adding one more element to our (finite dimensional) basis keeps it linearly independent. In particular the new element cannot decrease the norm of our element too much, and we have precise control over what "too much" means by changing ϵ. We formalize this remark and use it to construct a basic sequence now.

Theorem 1.9. *Suppose X is an infinite dimensional Banach space. Then there exists a basic sequence $\{x_n\}_{n=1}^\infty \subseteq X$.*

Proof. We begin by picking a sequence of positive real numbers $\{\epsilon_i\}_{i\in\mathbb{N}}$ with $0 < \epsilon_i < 1$ and $\prod_{i=1}^\infty (1+\epsilon_i) < 1 + \epsilon$ for some $\epsilon > 0$.

Pick some $x_1 \in X$ with $\|x_1\| = 1$. Denote $B_1 = \operatorname{span}\{x_i\}_{i=1}^1$. By Lemma 1.8, we can find $x_2 \in X$ with $\|x_2\| = 1$ such that for all $y \in B_1$, $\|y\| \leqslant (1+\epsilon_1)\|y + \lambda x_2\|$. Denote $B_2 = \operatorname{span}\{x_1, x_2\}$.

Continuing this inductively, we build a sequence $\{x_i\}_{i=1}^\infty$ and a sequence of finite dimensional subspaces $B_i = \operatorname{span}\{x_1, \ldots, x_i\}$. These subspaces have the property that for all $y \in B_i$ and all $\lambda \in \mathbb{R}$, $\|y\| \leqslant (1+\epsilon_i)\|y + \lambda x_{i+1}\|$. Denote $B = \overline{\operatorname{span}\{x_i\}_{i\in\mathbb{N}}}$. Clearly $\{x_i\}_{i=1}^\infty$ spans B, so it remains to be shown that $\{x_i\}_{i=1}^\infty$ is actually a basis and namely we need to check linear independence. Suppose we have some element $y = \sum_{i=1}^{\infty} a_i x_i \in B$ with $y = 0$. We show that $a_i = 0$ for all $i \in \mathbb{N}$. First we denote $y_n = \sum_{i=1}^{n} a_i x_i$. By our subspace property we must have that for all $n \in \mathbb{N}$,

$$\|y_n\| \leqslant (1 + \epsilon_n) \|y_n + a_{n+1} x_{n+1}\| = (1 + \epsilon_n) \|y_{n+1}\|$$

Iterating this, we deduce that for all $k \in \mathbb{N}$

$$\|y_n\| \leqslant \prod_{i=n}^{n+k} (1 + \epsilon_i) \|y_{n+k}\| \leqslant (1 + \epsilon) \|y_{n+k}\|$$

Since $0 = \lim_{n\to\infty} y_n$, if we take the limit as $k \to \infty$, we deduce that $\|y_n\| = 0$ for all $n \in \mathbb{N}$. In particular, $y_1 = a_1 x_1 = 0$, so $a_1 = 0$. Then $y_2 = a_2 x_2 = 0$ so $a_2 = 0$. Repeating this inductively, we deduce that $a_i = 0$ for all $i \in \mathbb{N}$ and hence deduce that that $\{x_i\}_{i\in\mathbb{N}}$ is indeed a basis on its closed linear span. □

Having built up some basic theory on bases of Banach spaces, we now switch to studying conditional and unconditional convergence of series and bases.

1.3. **Conditional Convergence.** In this section we present a number of results on conditional and unconditional convergence in both finite and infinite dimensional Banach spaces. Before discussing conditional convergence of bases in generality, we give two classic theorem on conditional series convergence in $\mathbb{R}^n$: the fact that unconditional convergence is equivalent to absolute convergence in finite dimensional spaces and the Riemann Rearrangement Theorem. Afterwards we give some equivalent conditions to unconditional convergence that hold in all Banach spaces, not just finite dimensional ones. Finally, we show a theorem due to Dvoertzky and Rogers which shows that in infinite dimensional Banach spaces, unconditional convergence is a strictly weaker condition than absolute convergence. Some remarks about conditional convergence in infinite dimensional spaces will be included at the end without proof. First we define some standard notions. In this section, we suppose $\{a_n\}_{n=1}^{\infty} \subseteq X$ where X is a Banach space.

Definition 1.10. *We define the series $\sum_{n=1}^{\infty} a_n$ to be the formal limit $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$. This limit may or may not exist in generality. If the limit of partial sums converges, then we say that the series converges, and if the limit does not exist, then we say that the series diverges.*

Definition 1.11. *We say that the series $\sum_{n=1}^{\infty} a_n$ converges conditionally if the following holds:*

(1) $\lim_{N\to\infty} \sum_{n=1}^{N} a_n$ exists
(2) There exists a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ such that $\lim_{N\to\infty} \sum_{n=1}^{N} a_{\sigma(n)}$ does not exist.

Conversely, a series converges unconditionally if the series converges and item (2) *is false, namely every permutation of the series converges as well.*

Definition 1.12. *For each $\{a_n\}_{n=1}^{\infty} \subseteq X$, a Banach space, we say that the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent if $\sum_{n=1}^{\infty} \|a_n\| < \infty$.*

We first prove that unconditional convergence is equivalent to absolute convergence in finite dimensional spaces. The proof, as with all equivalence proofs, has two steps, namely showing that absolute convergence implies unconditional convergence and vice versa. In fact, absolute convergence always implies unconditional convergence in Banach spaces without any requirements on the dimension. We prove this first before proceeding with the theory specific to finite dimensional spaces.

Theorem 1.13. *Fix $\{a_n\}_{n=1}^{\infty} \subseteq X$ where X is a Banach space. If the series $\sum_{n=1}^{\infty} a_n$ is absolutely convergent, then the series is unconditionally convergent.*

Proof. Suppose our series is absolutely convergent. Fix $\sigma : \mathbb{N} \to \mathbb{N}$ a permutation and $\epsilon > 0$. Since our series is absolutely convergent, there exists $N \in \mathbb{N}$ such that $\sum_{n=N}^{\infty} \|a_n\| < \epsilon$. Since σ is a permutation, there exists $M \in \mathbb{N}$ such that $\{1, \ldots, N\} \subseteq \{\sigma(1), \ldots, \sigma(M)\}$. In particular, for all $n > M$ we have that $\sigma(n) > N$. Fix $n_0, n_1 > M$. Without loss of generality $n_0 \geqslant n_1$, and we compute

$$\left\| \sum_{n=1}^{n_0} a_{\sigma(n)} - \sum_{n=1}^{n_1} a_{\sigma(n)} \right\| = \left\| \sum_{n=n_1+1}^{n_0} a_{\sigma(n)} \right\| \leqslant \sum_{n=n_1+1}^{n_0} \left\| a_{\sigma(n)} \right\| \leqslant \sum_{n=N+1}^{\infty} \|a_n\| < \epsilon$$

In particular we have that our permuted series is Cauchy and hence convergent. We deduce that our series is indeed unconditionally convergent. □

Theorem 1.14. *Fix $\{a_n\}_{n=1}^{\infty} \subseteq \mathbb{R}^d$ where $\mathbb{R}^d$ is adorned with the Euclidean norm. The series $\sum_{n=1}^{\infty} a_n$ is unconditionally convergent if and only if the series is absolutely convergent. Since all finite dimensional Banach spaces are isomorphic to $\mathbb{R}^d$ for some d, this implies that unconditional convergence is equivalent to absolute convergence in any finite dimensional Banach space.*

In order to prove this, we prove the special case when $d = 1$ first and use this for the general case:

Proof of Theorem 1.14 when $d = 1$. Suppose $\sum_{n=1}^{\infty} a_n$ is not absolutely convergent. In this case, we have that $\sum_{n=1}^{\infty} a_n^+ = \infty$ or $\sum_{n=1}^{\infty} a_n^- = -\infty$ where

$$a_n^+ = \begin{cases} a_n & a_n \geqslant 0 \\ 0 & a_n < 0 \end{cases} \qquad\qquad a_n^- = \begin{cases} a_n & a_n \leqslant 0 \\ 0 & a_n > 0 \end{cases}$$

Note that if exactly one of these conditions is true, then our series is not convergent at all, so both conditions must be true. We construct a permutation inductively to show that our series is conditionally convergent. In particular we construct a permutation such that the permuted series diverges.

Denote $A = \{n \in \mathbb{N} \mid a_n > 0\}$ and $B = \{n \in \mathbb{N} \mid a_n \leqslant 0\}$ (both of which are infinite). Fix $p : \mathbb{N} \to \mathbb{N}$ an enumeration of A and $q : \mathbb{N} \to \mathbb{N}$ an enumeration of B. To define our permutation, we use induction. Define $n_0 = 1$. Fix $i \in \mathbb{N}$. Our induction hypothesis is the following statements:

(1) $1 = n_0 < n_1 < \cdots < n_i < \infty$

(2) For each $k \leqslant i$ we have that

$$\sum_{n=n_{k-1}}^{n_k} a_{p(n)} > 1 - a_{q(k)}$$

(3) For each $k \leqslant i$ and n with $n_{k-1} < n \leqslant n_k$ we have that

$$\sigma(k + n) = a_{p(n)}$$

(4) For each $k \leqslant i$ we have that

$$\sigma(k + n_{k-1} - 1) = a_{q(n)}$$

For the $i = 1$ base case, denote $\sigma(1) = a_{q(1)}$. Then there exists $n_1 \in \mathbb{N}$ with $n_0 < n_1$ and $\sum_{n=n_0}^{n_1} a_{p(n)} > 1 - a_{q(1)}$. Define $\sigma(1 + n) = a_{p(n)}$ for all $n_0 \leqslant n \leqslant n_1$. These all show items (1) through (4).

Suppose now that the induction hypothesis holds for some $i \in \mathbb{N}$. Then note that $\sum_{n=1}^{n_i} a_{p(n)} < \infty$ while $\sum_{n=1}^{\infty} a_{p(n)} = \infty$. Since and $1 - a_{q(i+1)} < \infty$, there must exist n_{i+1} with $n_i < n_{i+1} \in \mathbb{N}$ and $\sum_{n=n_i}^{n_{i+1}} a_{p(n)} > 1 - a_{q(i+1)}$. This checks items (1) and (2). Define $\sigma(i + n_i) = a_{q(n)}$ and for each n with $n_i < n \leqslant n_{i+1}$ define $\sigma(i + 1 + n) = a_{p(n)}$. This gives us items (3) and (4).

By induction and construction we have an increasing sequence $\{n_k\}_{k=0}^\infty$ and an enumeration of $\mathbb{N}$. Furthermore, for each $k \in \mathbb{N}$ we have that

$$\sum_{n=1}^{k+n_k} a_{\sigma(n)} = \sum_{i=1}^{k} \left(a_{q(i)} + \sum_{n=n_{i-1}}^{n_i} a_{p(n)} \right) > \sum_{i=1}^{k} 1 = k$$

Hence $\lim_{k\to\infty} \sum_{n=1}^{k+n_k} a_{\sigma(n)} = \infty$, and in particular $\sum_{n=1}^\infty a_{\sigma(n)}$ is not a convergent series. Thus unconditionally convergent series are absolutely convergent. $\square$

Proof of Theorem 1.14. Having proved the case when $d = 1$, the general theorem comes from applying the $d = 1$ case to each coordinate separately. For notation, we write $a^{(p)}$ to be the p-th coordinate of a.

Suppose that $\sum_{n=1}^\infty a_n$ is unconditionally convergent. Like the above, the series generated by any one coordinate must be an unconditional series, and in particular $\sum_{n=1}^\infty a_n^{(p)}$ must be unconditionally convergent. Thus $\sum_{n=1}^\infty a_n^{(p)}$ is absolutely convergent. Fixing $N \in \mathbb{N}$ we have then that

$$\sum_{n=1}^{N} \|a_n\| \leqslant \sum_{n=1}^{N} \sum_{p=1}^{d} \left| a_n^{(p)} \right| = \sum_{p=1}^{d} \left(\sum_{n=1}^{N} \left| a_n^{(p)} \right| \right)$$

Of course $\lim_{N\to\infty} \sum_{n=1}^N \left| a_n^{(p)} \right| < \infty$ since this is an absolutely convergent series, and thus $\lim_{N\to\infty} \sum_{n=1}^N \|a_n\| < \infty$ as well. Thus unconditional convergence implies absolute convergence in the finite dimensional case. $\square$

With this proven, we can now state and prove the Riemann Rearrangement Theorem.

Theorem 1.15. *Fix $\{a_n\}_{n=1}^\infty \subseteq \mathbb{R}$. Suppose $\sum_{n=1}^\infty a_n$ is conditionally convergent. Then for every $\alpha \in \mathbb{R}$ or $\alpha = \infty$ or $\alpha = -\infty$, there exists a permutation $\sigma : \mathbb{N} \to \mathbb{N}$ with $\sum_{n=1}^\infty a_{\sigma(n)} = \alpha$.*

Proof. We already showed the $\alpha = \infty$ case in Theorem 1.13, and the $\alpha = -\infty$ case can be done in an analogous way by interchanging the role of the positive and negative terms (and making the few necessary minor modifications). Thus we focus on the $\alpha \in \mathbb{R}$ case. The language of the $\alpha = \infty$ case was more powerful than necessary, and we modify that proof as necessary to show this case. Fix $\alpha \in \mathbb{R}$. The first modification we need is to handle the case when $a_n = 0$ separately. It suffices to show the theorem when $a_n \neq 0$ for all n. Suppose we have shown this result, and suppose $\{a_n\}_{n=1}^\infty \subseteq \mathbb{R}$ gives a conditionally convergent series. Fix $\alpha \in \mathbb{R}$.

If we write $I = \{n \in \mathbb{N} \mid a_n \neq 0\}$, then we can enumerate I as $i : \mathbb{N} \to \mathbb{N}$. If we define $a_n^* = a_{i(n)}$ then $\sum_{n=1}^\infty a_n^*$ is conditionally convergent. Then we have a permutation $\sigma^* : \mathbb{N} \to \mathbb{N}$ with $\sum_{n=1}^\infty a_{\sigma^*(n)}^* = \alpha$. Define $O = \{n \in \mathbb{N} \mid a_n = 0\}$. If O is finite we define $o : \{1, 2, \ldots, |O|\} \to O$ and define

$$\sigma(n) = \begin{cases} 0 & n \leqslant |O| \\ \sigma^*(n - |O|) & n > |O| \end{cases}$$

and then $\sum_{n=1}^\infty a_{\sigma(n)} = \alpha$ and σ is a permutation. If O is infinite then we can enumerate O as $o : \mathbb{N} \to \mathbb{N}$ and define

$$\sigma(n) = \begin{cases} \sigma^*(n/2) & n \text{ even} \\ o\left(\frac{n+1}{2}\right) & n \text{ odd} \end{cases}$$

Similarly $\sum_{n=1}^\infty a_{\sigma(n)} = \alpha$ and σ is a permutation. Thus it suffices to prove the case when $a_n \neq 0$ for all n.

To do this, we copy the setup of the previous proof with a slight modification. Denote $A = \{n \in \mathbb{N} \mid a_n > 0\}$ and $B = \{n \in \mathbb{N} \mid a_n < 0\}$ (both of which are infinite). Note that since $\sum_{n=1}^N a_n$ converges, we have that our series is Cauchy. In particular for any $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if $m > N$, then $\left|\sum_{n=m}^\infty a_n\right| < \epsilon$ and hence $|a_n| < \epsilon$ for all $n > N$. In particular this tells us that $\lim_{n\to\infty} a_n = 0$. Because of this, for each $\epsilon > 0$, there exists finitely many n with $|a_n| \geqslant \epsilon$. As before, fix $p : \mathbb{N} \to \mathbb{N}$ an enumeration of A and $q : \mathbb{N} \to \mathbb{N}$ an enumeration of B. Since $a_n \neq 0$ for all n and $\lim_{n\to\infty} a_n = 0$, we can assume $a_{p(n)}$ is decreasing and that $a_{q(n)}$ is increasing.

We define our permutation inductively. Define $n_0 = 1$. Fix $i \in \mathbb{N}$. This induction hypothesis is similar to our previous one but is more complicated to allow for a more precise convergence.

(1) $1 = n_0 < n_1 < m_1 < n_2 < m_2 < \ldots \cdots < n_i < m_i < \infty$
(2) We have that n_i is the minimal integer greater than m_{i-1} (or 0 if $i = 1$) such that

$$\sum_{n=1}^{n_i} a_{\sigma(n)} > \alpha$$

where $\sigma(n) = a_{p\left(n-m_{i-1}+n_1+\sum_{k=2}^{i-1}(n_i-m_i)\right)}$ when $m_{i-1} < n \leqslant n_i$ and $i > 1$ or $\sigma(n) = a_{p(n)}$ if $i = 1$ for $n \leqslant n_1$.
(3) We have that m_i is the minimal integer greater than n_{i-1} such that

$$\sum_{n=1}^{m_i} a_{\sigma(n)} < \alpha$$

where $\sigma(n) = a_{q\left(n-n_i+\sum_{k=1}^{i-1}(m_i-n_i)\right)}$ when $n_i < n \leqslant m_i$.

The proof proceeds in exactly the same way as before and as such we skip the precise details. The main difference is that we add only as many positive terms as necessary to have our partial sum exceed α, and then add only as many negative as necessary to have our partial sum fall below α.

Now we show that $\sum_{n=1}^N a_{\sigma(n)}$ converges to α. Fix $\epsilon > 0$. Since $a_{p(n)}$ is decreasing to 0 and $a_{q(n)}$ is increasing to 0, there exists $N \in \mathbb{N}$ such that if $m > N$ then $a_{p(m)} < \epsilon$ and $a_{q(m)} > -\epsilon$. Suppose $m > N$.

Then we have exactly one i with $n_i \leqslant m < n_{i+1}$. For j with $n_i \leqslant j \leqslant m_i$ we have that $\sum_{n=1}^j a_{\sigma(n)}$ is decreasing, and hence

$$\left|\sum_{n=1}^{j} a_{\sigma(n)} - \alpha\right| \leqslant \max\left\{\left|\sum_{n=1}^{n_i} a_{\sigma(n)} - \alpha\right|, \left|\sum_{n=1}^{m_i} a_{\sigma(n)} - \alpha\right|\right\} \leqslant \epsilon$$

since if $\left|\sum_{n=1}^{n_i} a_{\sigma(n)} - \alpha\right| > \epsilon$ then n_i is not the minimal index, and similarly with the other case.

For j with $m_i \leqslant j \leqslant n_{i+1}$, we have that $\sum_{n=1}^j a_{\sigma(j)}$ is increasing and we have a similar estimate. In particular, we must have that

$$\left|\sum_{n=1}^{m} a_{\sigma(n)} - \alpha\right| \leqslant \epsilon$$

and hence $\sum_{n=1}^\infty a_{\sigma(n)} = \alpha$ as desired. This completes the proof of the Riemann Rearrangement theorem. □

1.3.1. *Conditional Convergence in Infinite Dimensional Banach Spaces.* In contrast to finite dimensional Banach spaces, unconditional convergence is no longer equivalent to absolute convergence. In generality, we have already shown that absolute convergence implies unconditional convergence, and in fact absolute convergence is a stronger property than unconditional convergence.

For our treatment of infinite dimensional conditional convergence theory, we present a theorem giving some equivalent conditions to unconditional convergence that hold in generality. This theorem is well known, and the statement (excluding item (5)) is mostly verbatim from Proposition 1.c.1. in [LT13], and the proof is an expanded and more detailed version of the proof presented there. Afterwards, we present a remarkable result by Dvoretzky and Rogers which shows that in infinite dimensional Banach spaces, absolute convergence is never equivalent to unconditional convergence. In particular, unconditional convergence is a strictly weaker property than absolute convergence in every infinite dimensional Banach space.

Theorem 1.16. *Let $\{x_n\}_{n=1}^\infty \subseteq X$ where X is a Banach space (either infinite dimensional or finite dimensional). Then the following are equivalent.*

(1) The series $\sum_{n=1}^\infty x_{\sigma(n)}$ converges for every permutation σ.
(2) The series $\sum_{i=1}^\infty x_{n_i}$ converges for every choice of $n_1 < n_2 < \dots$
(3) The series $\sum_{n=1}^\infty \theta_n x_n$ converges for every choice of signs θ_n.
(4) For every $\epsilon > 0$ there exists an integer n so that $\left\|\sum_{i\in\psi} x_i\right\| < \epsilon$ for every finite set of integers ψ which satisfies $\min\{i \in \psi\} > n$.
(5) Item (1) is true and additionally, all permuted series converge to the same value.

Proof. We first note that (2) and (3) are equivalent. If (2) holds, then fix a choice of signs. Suppose $P = \{n \in \mathbb{N} \mid \theta_n = 1\}$ and $N = \{n \in \mathbb{N} \mid \theta_n = -1\}$. Then for any $m \in \mathbb{N}$, we have that

$$\sum_{n=1}^m \theta_n x_n = \sum_{n\in[m]\cap P} x_n - \sum_{n\in[m]\cap N} x_n$$

By (2) both these sums converge as $m \to \infty$, and hence $\sum_{n=1}^\infty \theta_n x_n$ is a convergent series.

Now we suppose that (3) holds. Fix a choice $n_1 < n_2 < \dots$. Write $P = \{n_k \mid k \in \mathbb{N}\}$. We create a sequence of signs as follows. For each $n \in \mathbb{N}$, define

$$\theta_n = \begin{cases} 1 & n \in P \\ -1 & n \notin P \end{cases}$$

Since our series converges for every choice of signs, it also converges when all signs are chosen to be 1, so in particular we have that $\sum_{n=1}^\infty \theta_n x_n$ and $\sum_{n=1}^\infty x_n$ exist. In particular then, the partial sums of the following must converge:

$$\frac{1}{2}\left(\sum_{n=1}^\infty \theta_n x_n + \sum_{n=1}^\infty x_n\right)$$

Fix $m \in \mathbb{N}$, and consider the mth partial sum of the above. This equals

$$\frac{1}{2}\left(\sum_{n=1}^m \theta_n x_n + \sum_{n=1}^m x_n\right) = \frac{1}{2}\left(\sum_{n\in[m]\cap P} 2x_n + \sum_{n\in[m]\setminus P} (x_n - x_n)\right) = \sum_{n\in[m]\cap P} x_n$$

Since the left hand side converges as $m \to \infty$, the right hand side must also converge as well. In particular we deduce that $\sum_{n\in P} x_n = \sum_{k=1}^\infty x_{n_k}$ converges as required. Thus (2) and (3) are equivalent.

We now show that (4) implies (1) and (4) implies (2). Suppose (4) holds, and fix a permutation σ. For $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that $\left\|\sum_{i\in\psi} x_i\right\| < \epsilon$ for all finite sets of integers ψ satisfying $\min\psi > N$. Since σ is a permutation, there exists $M \in \mathbb{N}$ such that $[N] \subseteq \sigma([M])$. In particular for all $n > M$ we have that $\sigma(n) \notin [N]$, and in particular $\sigma(n) > N$.

Now, fix $n_1 \geqslant n_2 > M$. We test the Cauchy criteria. Define $\psi = \{n \in \mathbb{N} \mid n_2 + 1 \leqslant n \leqslant n_1\}$. Then note that $\min \psi = n_2 + 1 > M$ and hence $\min \sigma(\psi) > N$.

$$\left\| \sum_{i=1}^{n_1} x_{\sigma(i)} - \sum_{i=1}^{n_2} x_{\sigma(i)} \right\| = \left\| \sum_{i=n_2+1}^{n_1} x_{\sigma(i)} \right\| = \left\| \sum_{i\in\psi} x_{\sigma(i)} \right\| = \left\| \sum_{i\in\sigma(\psi)} x_i \right\| < \epsilon$$

Thus $\{\sum_{n=1}^{k} x_{\sigma(n)}\}_{k\in\mathbb{N}}$ is a Cauchy sequence and hence convergent. Thus (4) implies (1).

To show that (4) implies (2), we use essentially the same reasoning. Fix $n_1 < n_2 < \ldots$. By assumption, for each $\epsilon > 0$, there exists $N \in \mathbb{N}$ such that if ψ is a finite set of integers with $\min \psi > N$, then $\left\| \sum_{i\in\psi} x_i \right\| < \epsilon$. Then because $\{n_k\}_{k\in\mathbb{N}}$ is an increasing sequence going to infinity, there exists $M \in \mathbb{N}$ such that if $k > M$ then $n_k > N$. Fix $n_1 \geqslant n_2 > M$, and define $\psi = \{n_k \mid n_2 + 1 \leqslant k \leqslant n_1\}$. Then our Cauchy criteria yields

$$\left\| \sum_{k=1}^{n_1} x_{n_k} - \sum_{k=1}^{n_2} x_{n_k} \right\| = \left\| \sum_{k=n_2+1}^{n_1} x_{n_k} \right\| = \left\| \sum_{i\in\psi} x_i \right\| < \epsilon$$

Thus the partial sums from (2) are Cauchy, and hence (4) implies (2).

It remains to be shown that (1) implies (4), and similarly that (2) implies (4). To use a contrapositive argument, assume (4) is false. Then there exists an $\epsilon > 0$ such that for all $n \in \mathbb{N}$, there exists a finite set of integers ψ with $\min \psi > n$ and $\left\| \sum_{i\in\psi} x_i \right\| \geqslant \epsilon$. This generates a sequence of finite sets of integers, $\{\psi_n\}_{n\in\mathbb{N}}$. Potentially extracting a subsequence and relabelling, we can assume without loss of generality that $q_n := \max \psi_n < \min \psi_{n+1} := p_{n+1}$. To show that (2) is false, we show that the Cauchy criteria does not hold.

We note that by our construction of $\{\psi_n\}_{n\in\mathbb{N}}$, this sequence of sets is pairwise disjoint. In particular, we inductively build an enumeration of $\bigcup_{i\in\mathbb{N}} \psi_i$ as $\{n_k\}_{k\in\mathbb{N}}$, where n_k is increasing. Note that $q_n = \max \psi_n \to \infty$ as $n \to \infty$, and hence for any fixed $N \in \mathbb{N}$, we have some $k \in \mathbb{N}$ such that $q_k > q_{k-1} \in N$. Then by the construction of our enumeration $\{n_k\}_{k\in\mathbb{N}}$, there exists some $\ell_0 < \ell_1 \in \mathbb{N}$ with $n_{\ell_0} = q_{k-1}$ and $n_{\ell_1} = q_k$. Finally we deduce that

$$\left\| \sum_{i=1}^{\ell_1} x_{n_i} - \sum_{i=1}^{\ell_0} x_{n_i} \right\| = \left\| \sum_{i=\ell_0+1}^{\ell_1} x_{n_i} \right\| = \left\| \sum_{i\in\psi_k} x_i \right\| \geqslant \epsilon$$

In particular, we have found a subsequences of indices $n_1 < n_2 < \ldots$ where the series $\sum_{i=1}^{\infty} x_{n_i}$ fails to be Cauchy and hence does not converge. Thus (2) is false. The same argument works for (1) as follows: our subsequence does not define a permutation, but if we insert the missing values in between the blocks like the above, we can build a permutation that similarly fails to be Cauchy. In particular by contrapositive, we deduce that (1) implies (4) and (2) implies (4). Thus we have determined that items (1) through (4) are equivalent, leaving us to just show equivalence to (5).

Note that (5) trivially implies (1), so we only need to show that (1) implies (5). Suppose (4) is true (which is equivalent to (1)). Fix any two permutations σ, σ'. By assumption each permuted series converges. Fix $\epsilon > 0$. By (4), there exists $N_0 \in \mathbb{N}$ such that if $\psi \subseteq \mathbb{N}$ is a finite set with $\min \psi > N_0$, then

$$\left\| \sum_{k\in\psi} x_{\sigma(k)} \right\| < \epsilon/2 \qquad\qquad \left\| \sum_{k\in\psi} x_{\sigma'(k)} \right\| < \epsilon/2$$

Then, since σ and ψ are permutations, there exists $N_1 \in \mathbb{N}$ such that $[N_0] \subseteq \sigma([N_1])$ and $[N_0] \subseteq \psi([N_1])$. Then

$$\left\| \sum_{k=1}^{n} x_{\sigma(k)} - \sum_{k=1}^{n} x_{\sigma'(k)} \right\| \leqslant \left\| \sum_{k \in \sigma([n]) \setminus \sigma'([n])} x_k \right\| + \left\| \sum_{k \in \sigma'([n]) \setminus \sigma([n])} x_k \right\| < \epsilon/2 + \epsilon/2 = \epsilon$$

since $\min \sigma([n]) \setminus \sigma'([n]), \min \sigma'([n]) \setminus \sigma([n]) > N_0$ by construction. In particular this shows that $\sum_{k=1}^{\infty} x_{\sigma(k)} = \sum_{k=1}^{\infty} x_{\sigma'(k)}$ as desired. Thus (1) implies (5) and hence items (1) through (5) are equivalent as stated. $\square$

Remark. *The above theorem makes clear one subtle detail that differs in some treatments of unconditional convergence. Some authors will state that unconditionally convergent series must converge to the same value when permuted, but in fact that is not necessary as shown. The seemingly weaker property that all permuted sequences must converge, not necessarily to the same value, is actually an equivalent condition.*

We now proceed to Dvoretzky and Rogers' theorem. We begin by stating it. This proof and the two lemmas after it correspond to results 1.c.2 to 1.c.4 in [LT13]

Theorem 1.17. *Let X be an infinite-dimensional Banach space. Let $\{\lambda_n\}_{n=1}^{\infty}$ be a sequence of positive real numbers with $\sum_{n=1}^{\infty} \lambda_n^2 < \infty$. Then there exists $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ such that the series $\sum_{n=1}^{\infty} x_n$ is unconditionally convergent and $\|x_n\| = \lambda_n$. In particular, we can choose $\{\lambda_i\}_{i=1}^{\infty} \subseteq \mathbb{R}$ with $\lambda_i > 0$ for all $i \in \mathbb{N}$ such that the series $\sum_{n=1}^{\infty} x_n$ generated is unconditionally convergent but not absolutely convergent.*

The first lemma we need develops the idea of an Auerbach system. Essentially Auerbach systems create an analog of the standard basis in $\mathbb{R}^n$ that generalizes the property that the dual of $\mathbb{R}^n$ is spanned by the functionals that map exactly one standard basis element to 1 and maps the rest to 0.

Lemma 1.18. *Let X be a Banach space of dimension n. Then, there exists a basis $x_1, \ldots, x_n \in X$ and distinct functionals $x_1^*, \ldots, x_n^* \in X'$ with $\|x_i^*\|_{X'} = 1$ and $\|x_i\|_X = 1$, and $x_i^*(x_j) = 1$ if $i = j$ and 0 otherwise. Such a system is called an Auerbach system. Any Auerbach system satisfies $x = \sum_{i=1}^{n} x_i x_i^*(x)$, and $x_1^*, \ldots, x_n^*$ is a basis for X'.*

Proof. We begin by introducing a coordinate system. Fix a basis $z_1, \ldots, z_n \in X$ and without loss of generality assume $\|z_i\| = 1$ for all $i \leqslant n$. Our linear continuous coordinate function $\varphi : X \to \mathbb{R}^n$ is then defined as follows: $\varphi(z_i) = e_i$ where e_i is the ith standard basis vector in $\mathbb{R}^n$, and by linearity φ is defined for all $x \in X$.

Define $V : X^n \to \mathbb{R}$ to be $V(x_1, \ldots, x_n) = \det(\varphi(x_1), \varphi(x_2), \ldots, \varphi(x_n))$. Note $\det : \mathbb{R}^{n \times n} \to \mathbb{R}$ is continuous since it can be expanded as a polynomial in terms of the $n \times n$ inputs, and hence V is continuous. In particular, V must attains it maximum over $\overline{B(0,1)}$ at some point $(x_1, \ldots, x_n)$. We claim that $\|x_i\| = 1$ for all $i \leqslant n$. To see this, note that if $x_i = 0$ then $V(x_1, \ldots, x_n) = 0$. Since $V(z_1, \ldots, z_n) \neq 0$ we obtain that $x_i \neq 0$ for all $i \leqslant n$. Now, fix $i \leqslant n$ and $\alpha \in \mathbb{R}$. Then by construction V is a multilinear map, and hence $V(x_1, \ldots, \alpha x_i, \ldots, x_n) = \alpha V(x_1, \ldots, x_n)$. Since $V(x_1, \ldots, x_n)$ is assumed to be the max that V attains on the closed unit ball, $V(x_1, \ldots, x_n) > 0$. In particular, if $\alpha > 1$ then $V(x_1, \ldots, \alpha x_i, \ldots, x_n) \geqslant \alpha V(x_1, \ldots, x_n)$, so we must have that $\alpha \|x_i\| > 1$. In particular we deduce that $\|x_i\| = 1$ as desired.

Let $x_1, \ldots, x_n \in X$ be defined as the tuple where V attains its max. Note that since $V(x_1, \ldots, x_n) \neq 0$, we must have that $x_1, \ldots, x_n$ are linearly independent. Since our space is n-dimensional, this implies that $x_1, \ldots, x_n$ is a basis. Then for each $i \leqslant n$ define

$$x_i^*(x) = \frac{V(x_1, \ldots, x_{i-1}, x, x_{i+1}, \ldots, x_n)}{V(x_1, \ldots, x_n)}$$

Note that x_i^* is continuous and linear and hence $x_i^* \in X'$. By construction $x_i^*(x_j) = 1$ if $i = j$ and 0 otherwise. Furthermore $\|x_i^*\|_{X'} = 1$. $x_1, \ldots, x_n$ and $x_1^*, \ldots, x_n^*$ are exactly the Auerbach system we aimed to construct and satisfy the specified properties.

For the moreover, fix an Auerbach system. We first show that $x_1^*, \ldots, x_n^*$ spans X'. Fix $\varphi \in X'$. By linearity, φ is defined by its action on the basis vectors $x_1, \ldots, x_n$. In particular for any $x = \sum_{i=1}^n a_i x_i$, we have that

$$\varphi(x) = \sum_{i=1}^n a_i \varphi(x_i) = \sum_{i=1}^n \varphi(x_i)(a_i x_i^*(x_i)) = \sum_{i=1}^n \varphi(x_i) x_i^*(x)$$

Thus φ is a linear combination of $x_1^*, \ldots, x_n^*$, so this spans X'. Furthermore, this system of dual vectors is linearly independent, since for all $a_1, \ldots, a_n \in \mathbb{R}$, we have that $\sum_{i=1}^n a_i x_i^* = 0$ if and only if $\sum_{i=1}^n a_i x_i^*(x_j) = 0$ for all $j \leqslant n$. Since the left hand side simplifies to a_j, we deduce that $\sum_{i=1}^n a_i x_i^* = 0$ if and only if $a_i = 0$ for all $i \leqslant n$, and hence $\{x_i^*\}_{i=1}^n$ is a basis for X'.

Finally we show that for all $x \in X$ we have that $x = \sum_{i=1}^n x_i x_i^*(x)$. Fix $x \in X$. Since $\{x_i\}_{i=1}^n$ is a basis, we must have $a_1, \ldots, a_n \in \mathbb{R}$ with $x = \sum_{i=1}^n a_i x_i$. Apply x_j^* to both sides to deduce that $x_j^*(x) = a_j$, which concludes our proof of this lemma. □

We now prove our final lemma we need to show Dvoretsky and Roger's theorem, which is a technical lemma.

Lemma 1.19. *Let X be a Banach space of dimension n^2 with norm $\|\cdot\|_0$. Then there exists a n-dimensional subspace C and an inner product norm $\|\cdot\|_C$ on C with $\|\cdot\|_0 \leqslant \|\cdot\|_C$ on C. Furthermore, there exists an orthonormal (with respect to $\|\cdot\|_C$) basis $y_1, \ldots, y_n$ of C with $\|y_i\|_0 > 1/8$ for all $i \leqslant n$.*

Proof. We begin by creating an inner product norm on X. Fix an Auerbach system in B, $x_1, \ldots, x_{n^2}$. Define

$$\|x\|_1 = n \left(\sum_{j=1}^{n^2} x_j^*(x)^2 \right)^{1/2}$$

This is an inner product norm generated by the inner product $\langle x, y \rangle_1 = n^2 \left(\sum_{j=1}^{n^2} x_j^*(x) x_j^*(y) \right)$. We aim to show the following inequality for all $x \in X$

$$\frac{\|x\|_1}{n^2} \leqslant \|x\|_0 \leqslant \|x\|_1$$

If we expand the left hand side, we find that

$$\frac{\|x\|_1}{n^2} = \left(\frac{1}{n^2} \sum_{j=1}^{n^2} x_j^*(x)^2 \right)^{1/2} \leqslant \max_j \left| x_j^*(x) \right| \leqslant \|x\|_0$$

where the last step follows from the fact that $\left\| x_j^* \right\|_{X'} = 1$. Then by triangle inequality, we have that

$$\|x\|_0 = \left\| \sum_{i=1}^{n^2} x_i x_i^*(x) \right\|_0 \leqslant \sum_{i=1}^{n^2} |x_i^*(x)|$$

We can view that sum as the L^1 norm with respect to the counting measure, and applying Hölder's inequality (or Cauchy-Schwarz), we obtain that

$$\sum_{i=1}^{n^2} |x_i^*(x)| \leqslant (n^2)^{1/2} \left(\sum_{i=1}^{n^2} x_i^*(x)^2 \right)^{1/2} = n \left(\sum_{i=1}^{n^2} x_i^*(x)^2 \right)^{1/2} = \|x\|_1$$

and hence our inequality $\frac{\|x\|_1}{n^2} \leqslant \|x\|_0 \leqslant \|x\|_1$ holds.

We note now that if $n = 1$ then considering the vector $y_1 \in X$ with $\|y_1\|_1 = 1$, then by our inequality we deduce that $1/8 < \|y_1\|_0 = 1$ as well which gives our desired result. If $n = 2$, then pick any two orthonormal vectors $y_1, y_2 \in X$. Then by the same inequality we deduce that $\|y_i\|_0 \geqslant 1/4$ which gives our desired result. Thus for the rest of the proof assume $n > 2$.

Consider now the statement that every subspace C of X with $\dim C > \dim X/2$ contains at least one element $y \in C$ with $\|y\|_1 = 1$ and $\|y\|_0 > 1/8$.

Suppose this statement is true. Then there is $y_1 \in X$ with $\|y_1\|_1 = 1$ and $\|y_1\|_0 > 1/8$. If we remove all vectors from X that are not orthogonal to y_1, we get a subspace X_1 of dimension $n^2 - 1$. We can then pick $y_2 \in X_1$ with $\|y_2\|_1 = 1$ and $\|y_2\|_0 > 1/8$. We can repeat this process at least $n^2/2 - 1$ times, since each step removes lowers the dimension of our subspace by 1, and $n^2 - (n^2/2 - 1) = n^2/2 + 1 > \dim X/2$. Since $n > 2$, we have that $n^2/2 - 1 > n$ and hence we can select orthonormal $y_1, \ldots, y_n$ with $\|y_i\|_1 = 1$ and $\|y_i\|_0 > 1/8$ for all $i \leqslant n$. Then defining our subspace C to the span of $y_1, \ldots, y_n$, we are finished.

Now, assume the specified statement is false. Then we have some subspace C of X with $\dim C > \dim X/2$, but for all $y \in C$ with $\|y\|_1 = 1$, we have that $\|y\|_0 \leqslant 1/8$. By homogeneity, we deduce that for all $y \in C$, we have that

$$\|y\|_0 = \|y\|_1 \left\| \frac{y}{\|y\|_1} \right\|_0 \leqslant \frac{\|y\|_1}{8}$$

Thus, for all $y \in C$, we can tighten our inequality above. In particular if we define $\|x\|_2 = \|x\|_1/8$, then for all $x \in C$ we have that

$$\frac{8\|x\|_2}{n^2} \leqslant \|x\|_0 \leqslant \frac{\|x\|_1}{8} = \|x\|_2$$

Take our specified statement and replace X with C and $\|\cdot\|_1$ with $\|\cdot\|_2$. Once again, either our specified statement is true or it is false. If it is false, we can iterative the same process of selecting a subspace where the statement fails and defining a new norm that is 1/8 the previous one. After k-many iterations, our inequality will read

$$\frac{8^k}{n^2} \|x\|_{k+1} \leqslant \|x\|_0 \leqslant \|x\|_{k+1}$$

Since this inequality must be true, this process must stop within $\ell - 1$ many steps where ℓ is the largest natural number where $8^\ell \leqslant n^2$. Suppose after $\ell - 1$ many iterations our statement is true, and we have a subspace $X_\ell \subseteq X$ where every subspace of X_ℓ has our desired property.

Note that $\dim X_\ell > \frac{1}{2^\ell} \dim X$. By our earlier argument where we initially assumed that our statement was true, we can find at least $\frac{\dim X_\ell}{2} - 1$ many orthonormal vectors with respect to $\|\cdot\|_\ell$ with our original norm being greater than 1/8. Note that since ℓ satisfies $8^\ell \leqslant n^2$, if we take the square root on both sides we obtain that $2^{\frac{3\ell}{2}} = 8^{\ell/2} \leqslant n$. In particular we have that $2^\ell 2^{\ell/2} \leqslant n$, so $2^{-\ell} \geqslant \frac{2^{\ell/2}}{n}$. Thus

$$\frac{\dim X_\ell}{2} - 1 \geqslant n^2 2^{-\ell} - 1 \geqslant \frac{n^2}{n} 2^{\ell/2} - 1 = n2^{\ell/2} - 1 \geqslant n$$

Thus we can find $y_1, \ldots, y_n$ orthogonal with respect to $\|\cdot\|_C = \|\cdot\|_\ell$, and $\|y_1\|_0 > 1/8$. Our subspace C is then the span of $y_1, \ldots, y_n$. Since $y_1, \ldots, y_n$ are orthonormal and hence linearly independent, we obtain that $y_1, \ldots, y_n$ is a basis of C immediately. This concludes the proof of the lemma. $\square$

Before proving the full Theorem 1.17, we state a direct consequence of the previous lemma.

Corollary 1.20. *Let X be an n^2-dimensional Banach space with norm $\|\cdot\|_0$. Let C be a n-dimensional subspace as defined in the previous lemma with inner product norm $\|\cdot\|_C$ with $\|\cdot\|_0 \leqslant \|\cdot\|_C$ on C. Then there exists $u_1, \ldots, u_n \in C$ with $\|u_i\|_0 = 1$ for all $i \leqslant n$ such that for all $a_1, \ldots, a_n \in \mathbb{R}$ we have that*

$$\left\|\sum_{i=1}^n a_i u_i\right\|_0 \leqslant 8 \left(\sum_{i=1}^n a_i^2\right)^{1/2}$$

Proof. Fix $y_1, \ldots, y_n \in C$ from the previous lemma. Define $u_i = \frac{y_i}{\|y_i\|_0}$. Then $\|u_i\|_0 = 1$ by construction. Furthermore since $\|\cdot\|_C$ is an inner product norm, $\|y_i\|_0 > 1/8$, and $\|y_i\|_C = 1$, we have that

$$\left\|\sum_{i=1}^n a_i u_i\right\|_0 \leqslant \left\|\sum_{i=1}^n a_i u_i\right\|_C = \left(\sum_{i=1}^n a_i^2 \|u_i\|_C^2\right)^{1/2} = \left(\sum_{i=1}^n a_i^2 \frac{\|y_i\|_C^2}{\|y_i\|_0^2}\right)^{1/2} \leqslant 8 \left(\sum_{i=1}^n a_i^2\right)^{1/2}$$

This proves the desired result. $\square$

We are now in the position to prove Theorem 1.17.

Proof of Theorem 1.17. Pick $\{\lambda_i\}_{i\in\mathbb{N}}$ to be a sequence of positive real numbers with $\sum_{i=1}^\infty \lambda_i^2 < \infty$. Pick an increasing sequence of natural numbers $n_1 < n_2 < \ldots$ such that $\sum_{i=n_k}^\infty \lambda_i^2 < 2^{-2k}$ for each $k \in \mathbb{N}$.

Fix $k \in \mathbb{N}$ and fix a subspace of X of dimension $(n_{k+1} - n_k)^2$. By the previous lemma and its corollary we can find unit vectors $u_{n_k}, \ldots, u_{n_{k+1}-1}$ such that if we define $x_i = \lambda_i u_i$, we have that for any choice of signs θ_i

$$\left\|\sum_{i=n_k}^{n_{k+1}-1} \theta_i x_i\right\| \leqslant 8 \left(\sum_{i=1}^n \lambda_i^2\right)^{1/2} < 8 \cdot 2^{-k}$$

For any $i < n_1$, define x_i to be any vector of norm λ_i. This defines a sequence of vectors $\{x_i\}_{i=1}^\infty$ with $\|x_i\| = \lambda_i$. Furthermore, for any fixed choice of signs θ_i, we have that $\sum_{i=1}^\infty \theta_i x_i$ is Cauchy and hence convergent. By our equivalence theorem, Theorem 1.16, we have that this series is unconditionally convergent. As a consequence, if we choose λ_i so that $\sum_{i=1}^\infty |\lambda_i| = \infty$, say $\lambda_i = \frac{1}{i}$, we find an example of a series which is unconditionally but not absolutely convergent. $\square$

We now state an easy but important corollary of Dvoretzky and Rogers' result.

Corollary 1.21. *Let X be a Banach space. Then X is finite dimensional if and only the following is true: every series in X is absolutely convergent if and only it is unconditionally convergent.*

Remark. *For posterity, we note that this proof differs from the original proof by Dvoretzky and Rogers (see [DR50] for the original paper). Their paper relied heavily on geometric arguments about convex geometry while this proof is mostly analytic.*

To explain how remarkable this result is requires a bit of history. Stefan Banach originally asked when absolute convergence is equivalent to unconditional convergence in Banach spaces. It was relatively easy to check that this is true for finite dimensional spaces, and prior to the paper by Dvoretzky and Rogers, examples had been found that showed unconditional convergent series that fail to be absolutely convergent in most of the commonly encountered infinite dimensional Banach spaces. This result manages to construct such a sequence in a very large class of spaces.

First, it says that in any infinite dimensional Banach space, there is a series which is unconditionally convergent but not absolutely convergent. In particular (as stated in our corollary), a Banach space is infinite dimensional if and only if there is a series which is conditionally but not absolutely convergent.

Perhaps more remarkably, with some careful choice $\{\lambda_n\}_{n=1}^{\infty}$ (specifically from the original paper, let $\lambda_n = \frac{1}{\sqrt{n}\log(1+n)}$, it produces a sequence which is in ℓ_2 but not in $\ell_{2-\epsilon}$ for any $\epsilon > 0$. In particular our series is not absolute convergent, so our sequence is not in ℓ_1 and moreover is not in $\ell_{2-\epsilon}$ for any $\epsilon > 0$. Such precise control is a remarkable property, and the fact that it happens in every infinite dimensional Banach space, a very large class of spaces, is even more remarkable.

Remark. *As stated in the beginning of this section, we include a brief remark about how the typical results about finite dimensional unconditional and conditional convergence generalize to infinite dimensions. We have already shown that the typical characterization of unconditional convergence through absolute convergence is no longer true in infinite dimensional Banach spaces, and the analog of the Riemann rearrangement theorem is drastically weirder in infinite dimensions.*

While not proven or stated in this paper, in $\mathbb{R}^d$ there is an analog of the Riemann rearrangement theorem. Roughly what it states is that any series in $\mathbb{R}^d$, permuting the series may change what it converges to. The set of values it could possibly converge to is exactly one of the following two cases:

1. *The empty set*
2. *The translate of a vector subspace of $\mathbb{R}^d$*

For instance, this tells us that the set of rearrangements could form just a point in the case of unconditionally convergent series, or it could be a line, a plane, or any higher dimensional subspaces, or even the whole space. Note that in $\mathbb{R}$ and $\mathbb{R}^d$, the set of convergent values was always convex and linear, and either took 1 *value or infinitely many.*

Around the turn of the 20th century, Banach himself asked how this generalized: what could the set of values that a rearrangement could take for a series in an infinite dimensional Banach space? It was quickly realized that in fact, the set of rearrangements no longer needed to be a translated linear subspace or convex. As time went on, people found more and more bizarre examples of series whose rearrangement set was badly behaved. Amazingly, two mathematicians Kadec and Woźniakowski proved in 1988 that the situation was far more unintuitive than expected: in every infinite dimensional Banach space, there exists a series which, when permuted, converges to exactly one of two values (see [KW89]). Later in 2005, Wojtaszczyk proved the following stronger result: in any infinite dimensional Banach space and given any finite set of elements of that Banach space, there exists a series which takes exactly those values when rearranged (see [Woj05]).

With the previous work developed and these results mentioned, it should be clear to the reader that the behavior of series convergence in infinite dimensional Banach spaces is both incredibly complex and incredibly different than in finite dimensions.

2. Unconditional Bases

Continuing with the build up towards Schechtman's paper, we need to define a few more notions. The goal of this section is to first define unconditional bases, and then to introduce a numerical properties of those bases: the unconditionally constant.

Definition 2.1. *Consider X an infinite dimensional Banach space, and suppose $\{x_i\}_{i\in\mathbb{N}}$ is a basis. Then this basis is said to be unconditional if for all $x \in X$, the series expansion of x in terms of the basis converges unconditionally. In particular, for all $x \in X$ where $x = \sum_{i=1}^{\infty} a_i x$, we have that that series converges unconditionally.*

For the sake of concreteness, we give a simple example of an unconditional basis and a basis which fails to be unconditional (or in short, a conditional basis).

Example 1. *Consider $\ell_p = \{f : \mathbb{N} \to \mathbb{R} \mid \left(\sum_{i=1}^{\infty} |f(i)|^p\right)^{1/p} < \infty\}$ where $1 \leqslant p < \infty$. Then the sequence of functions $e_i : \mathbb{N} \to \mathbb{R}$ defined by $e_i(n) = 1$ if $n = i$ and 0 otherwise is an unconditional basis.*

Proof. The fact that this is a basis is trivial. To show that it is unconditional, fix $f \in \ell_p$. Then $f = \sum_{i=1}^{\infty} e_i f(i)$. Fix any choice of signs θ_i and $\epsilon > 0$. Denote $f^n = \sum_{i=1}^{n} \theta_i e_i f(i)$. Since $f \in \ell_p$, there is some $N \in \mathbb{N}$ such that $\left(\sum_{i=N+1}^{\infty} |f(i)|^p\right)^{1/p} < \epsilon^{1/p}$. Then if $n \geqslant m > N$, we have that

$$\|f^n - f^m\|_{\ell_p}^p = \sum_{i=1}^{\infty} |f^n(i) - f^m(i)|^p = \sum_{i=m+1}^{n} |\theta_i f(i)|^p = \sum_{i=m+1}^{n} |f(i)|^p \leqslant \sum_{i=N+1}^{\infty} |f(i)|^p < \epsilon$$

Thus the sequence $\{f^n\}_{n\in\mathbb{N}}$ is Cauchy and hence convergent. In particular the series $\sum_{i=1}^{\infty} \theta_i e_i f(i)$ is convergent, and hence $\sum_{i=1}^{\infty} e_i f(i)$ is unconditionally convergent. Since this holds for all $f \in \ell_p$, we have that our basis is indeed an unconditional basis. □

Example 2. *Consider ℓ_1 as before, and $\{e_i\}$ as the standard basis of ℓ_1 which was defined in the previous example. We define a new basis as follows: define $x_1 = e_1$, and for all $i > 1$ define $x_i = e_{i-1} - e_i$. Then this sequence is a conditional basis.*

Proof. Clearly this is a basis so we just need to show conditionality by finding a series that converges conditionally. Suppose $x \in \ell_p$ and $\{a_i\}_{i\in\mathbb{N}}$ is chosen so that $x = \sum_{i=1}^{\infty} a_i x_i$. Then note that

$$\sum_{i=1}^{n} a_i x_i(j) = \begin{cases} a_1 + a_2 & j = 1 \\ a_{j+1} - a_j & 1 < j < n \\ -a_j & j = n \\ 0 & j > n \end{cases}$$

Accordingly, we have that

$$\left\| \sum_{i=1}^{n} a_i x_i \right\|_{\ell_1} = |a_1 + a_2| + |a_n| + \sum_{i=2}^{n-1} |a_{i+1} - a_i|$$

To test for conditional convergence, we replace a_i with $a_i\theta_i$ for some choice of signs. If we choose $\theta_i = (-1)^i$, then by the above we find that

$$\left\| \sum_{i=1}^{n} a_i \theta_i x_i \right\|_{\ell_1} = |a_1 - a_2| + |a_n| + \sum_{i=2}^{n-1} |a_{i+1} + a_i|$$

This shows how we can find a choice of $a_1, \ldots, a_n$ such that yields a conditional series. Note that in the first expression, our sum involves the difference between $a_{i+1} - a_i$ while the second expresssion involves $a_{i+1} + a_i$. Set $a_i = \frac{1}{i}$. Then $a_{i+1} - a_i = \frac{-1}{i(i+1)}$ which is summable through basic calculus. In particular $\sum_{i=1}^{\infty} a_i x_i \in \ell_1$.

However, if we consider the second term, we have that

$$\left\| \sum_{i=1}^{n} a_i \theta_i x_i \right\|_{\ell_1} \geqslant \sum_{i=1}^{n} \frac{1}{n}$$

The sum on the right diverges and hence $\sum_{i=1}^{n} a_i \theta_i x_i$ is not a convergent series. In particular we deduce that our base is a conditional base. □

With these examples done, we define an unconditional basic sequence.

Definition 2.2. *A basic sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is unconditional if for every $\sum_{n=1}^{\infty} a_n x_n \in \overline{\operatorname{span}\{x_n\}_{n\in\mathbb{N}}}$, we have that the series converges unconditionally. Equivalently, an unconditional basic sequence is a basic sequence which is an unconditional basis for its span.*

Lemma 2.3. *A basic sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is unconditional if and only if any of the following hold:*

(1) For every permutation σ, we have that $\{x_{\sigma(n)}\}_{n\in\mathbb{N}}$ is a basic sequence.
(2) For every subset of the integers ψ, we have that convergence of $\sum_{n=1}^{\infty} a_n x_n$ implies convergence of $\sum_{n\in\psi} a_n x_n$.

Proof. Both of these equivalent conditions are equivalent by Theorem 1.16 where we showed many conditions that are equivalent to unconditional convergence. The equivalence of the basis being unconditional with (1) follows from the fact that permuted unconditional series converge to the original series. The fact that either of these are equivalent to (2) follows from combining items (2) and (1) in Theorem 1.16. □

Before proceeding further into our theory of unconditional bases, we show that projections are continuous in general, as well as give a precise definition of a projection. This will be used to justify continuity of a family of linear operators we will define that relate to unconditional bases.

Definition 2.4. *A linear operator $P : X \to X$ is said to be a projection if it satisfies the following properties:*

(1) $P \circ P = P$
(2) The range of P is closed
(3) $P^{-1}(\{0\})$ is closed.

Lemma 2.5. *All projections $P : X \to X$ are continuous.*

Proof. We aim to use the closed graph theorem. Suppose $\{z_n\}_{n\in\mathbb{N}} \subseteq X$ with $z_n \to z \in X$ and $P(z_n) \to y \in X$. We aim to show that $P(z) = y$.

Since the range of P is closed, we deduce that y is in the range of P. In particular, there is some $z' \in X$ with $P(z') = y$. If we examine the sequence $z_n - P(z_n)$, note that since $P \circ P = P$, we have that $P(z_n - P(z_n)) = P(z_n) - P(z_n) = 0$, so $z_n - P(z_n) \in P^{-1}(\{0\})$. Taking the limit as $n \to \infty$ and using the fact that $P^{-1}(\{0\})$ is closed, we deduce that $z - y \in P^{-1}(\{0\})$. Accordingly we deduce that

$$0 = P(z) - P(y) = P(z) - P \circ P(z') = P(z) - P(z') = P(z - z')$$

and hence $P(z) = P(z') = y$. Thus the graph of P is closed, and by the closed graph theorem we deduce that P is continuous as desired. □

With the general projection theory completed, we define one family of linear operators acting on the closed linear span of a basic sequence. Suppose $\{x_n\}_{n\in\mathbb{N}}$ is a basic sequence for the space X. For every $\psi \subseteq \mathbb{N}$, we define $P_\psi\left(\sum_{i=1}^{\infty} a_i x_i\right) = \sum_{i\in\psi} a_i x_i$.

Lemma 2.6. *For every choice of $\psi \subseteq \mathbb{N}$, we have that P_ψ is a projection. In particular, P_ψ is a continuous operator.*

Proof. Fix a choice $\psi \subseteq \mathbb{N}$. Note that $P_\psi \circ P_\psi = P_\psi$ trivially and P_ψ is linear by definition, so we need to show that $P_\psi^{-1}(\{0\})$ is closed and the range of P is closed. Since $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is a basis, we have that $P_\psi(\sum_{n=1}^{\infty} a_n x_n) = \sum_{n\in\psi} a_n x_n = 0$ if and only if $a_n = 0$ for all $n \in \psi$.

Then $P^{-1}(\{0\})$ is closed by the following simple observation. $P_\psi^{-1}(\{0\}) = \{\sum_{n\notin\psi} a_n x_n \in X\} = \overline{\text{span}\{x_n \mid n \notin \psi\}}$, which is closed by definition. For the same reason the range of P_ψ is closed. In particular by the previous lemma, P_ψ is continuous. □

Using these projection operators, we can define our main bounded linear operator of interest. Fix a choice of signs $\theta = \{\theta_i\}_{i\in\mathbb{N}} \subseteq \{-1, 1\}$ and define $M_\theta\left(\sum_{i=1}^\infty a_i x_i\right) = \sum_{i=1}^\infty \theta_i a_i x_i$ where $\{x_k\}_{k\in\mathbb{N}}$ is an unconditional basis of X. Since our basis is unconditional, the series $\sum_{i=1}^\infty a_i x_i$ converges unconditionally and hence by Theorem 1.16, we must have that $\sum_{i=1}^\infty \theta_i a_i x_i$ converges as well. This justifies why M_θ is well-defined. By definition, M_θ is clearly linear, and showing continuity/boundness follows from the fact that our projections are continuous/bounded. In particular, if we define $\psi_+ = \{n \in \mathbb{N} \mid \theta_i = 1\}$ and $\psi_- = \{n \in \mathbb{N} \mid \theta_i = -1\}$, then we have that

$$M_\theta(\sum_{i=1}^\infty a_i x_i) = P_{\theta_+}\left(\sum_{i=1}^\infty a_i x_i\right) - P_{\theta_-}\left(\sum_{i=1}^\infty a_i x_i\right)$$

Thus M_θ is just the sum of continuous operators and hence M_θ is continuous. One property of M_θ is that taking the supremum of $\|M_\theta(x)\|$ essentially gives us a measure of how changing the sign of our basis can change the norm of our series. Similarly, taking the supremum of the operator norm of M_θ over all θ grants us a quantitative measure for how our basis behaves under sign changes. This motivates the definition of a new constant: the unconditional constant of an unconditional basic sequence.

Definition 2.7. *The unconditional constant of an unconditional basic sequence $\{x_n\}_{n\in\mathbb{N}} \subseteq X$ is defined to be $\kappa = \sup_\theta \|M_\theta\|_{op}$. We say that an unconditional basic sequence is a κ-unconditional basic sequence if $\sup_\theta \|M_\theta\|_{op} = \kappa$.*

This definition of course needs to be shown to be well-defined. Indeed, without proof there is no reason to assume that $M_\theta(x)$ is bounded over all θ for any fixed x, and it is an even bigger claim that M_θ in operator norm over all θ. During the author's review of the literature on this topic, all references seem to leave this as an exercise or otherwise leave this unproven (although most likely this has been written out). The most complete reference found was in Section 10 of [Car04], which provides a high-level sketch of the argument. The proof is sufficiently beautiful to deserve an explicit proof.

At a high level, we will fix an unconditionally convergent series $\sum_{i=1}^\infty x_i \subseteq X$. We then define $T : \{-1, 1\}^\mathbb{N} \to X$ by $T(\theta) = \sum_{i=1}^\infty \theta_i x_i$ so that $T(\theta) = M_\theta(\sum_{i=1}^\infty x_i)$. Adorning $\{-1, 1\}^\mathbb{N}$ with a particular metric, we find that that metric space is a complete compact metric space and T is continuous. As such, the image of $\{-1, 1\}^\mathbb{N}$ under T is compact and hence bounded. Note that this image is exactly the image of $M_\theta(\sum_{i=1}^\infty x_i)$ over all θ. Since M_θ is continuous and linear and for each $x \in X$ we have that $\sup_\theta \|M_\theta(x)\| < \infty$, by Banach-Steinhaus we deduce our desired bound in operator norm. With the full proof sketched out, we begin by defining the metric on $\{-1, 1\}^\mathbb{N}$.

Lemma 2.8. *The metric space $\{-1, 1\}^\mathbb{N}$ adorned with the metric*

$$d(\sigma, \psi) = \sum_{i=1}^\infty \frac{|\sigma_i - \psi_i|}{2^i}$$

is complete and compact.

Proof. The fact that d is a metric is a standard proof and hence omitted. The fact that this metric makes our space complete is also easy to see (we prove compactness explicitly using essentially the same technique, so we keep the details sparse and here give a more detailed proof for compactness). For any Cauchy sequence $\{\theta^{(k)}\}_{k\in\mathbb{N}}$ and any $\ell \in \mathbb{N}$, we have some $N \in \mathbb{N}$ so that if $n, m > N$ then

$d(\theta^{(n)}, \theta^{(m)}) < 2^{\ell}$. By the definition of the metric we deduce then that $\theta_i^{(n)} = \theta_i^{(m)}$ for all $i \leqslant \ell$. In particular the first ℓ elements are eventually constant. Choosing $\theta \in \{-1,1\}^{\mathbb{N}}$ so that θ_i matches the digits of the Cauchy sequence once they become constant, we deduce that $\theta^{(n)} \to \theta$ as $n \to \infty$ and hence this metric space is complete.

The proof of compactness follows in much the same way. Since this space is a metric space, sequential compactness and compactness are equivalent properties. Thus it suffices to show sequential compactness. To do this, suppose $\{\theta^{(k)}\}_{k\in\mathbb{N}} \subseteq \{-1,1\}^{\mathbb{N}}$. We inductively construct a subsequence and a $\theta \in \{-1,1\}^{\mathbb{N}}$ so that the subsequence converges to θ.

The inductive proof relies on showing the following statement for each $i \in \mathbb{N}$ which we will denote $P(i)$ for brevity: for at least one element of $\{-1,1\}^i$, there exists infinitely many elements in our sequence whose first i digits match the chosen sequence.

Note that since our sequence is infinite, infinitely many elements must begin with -1 or infinitely many elements must begin with 1. Pick -1 or 1 so that infinitely many elements begin with the chosen number. Then we note that $P(1)$ is thus true.

For induction, suppose for some $i \in \mathbb{N}$ that $P(i)$ is true. Suppose $\theta_1, \ldots, \theta_i \in \{-1,1\}$ are chosen so that infinitely many elements in $\{\theta^{(k)}\}_k$ have $\theta_1, \ldots, \theta_i$ as their first i elements. Then among all $\theta^{(k)}$ that have these specified initial elements, either infinitely many have their $i+1$-th digit as 1, or infinitely many have their $i+1$-th digit as -1. Picking one of the digits that occurs infinitely many times we deduce that $P(i+1)$ is true. Thus $P(i)$ is true for all $i \in \mathbb{N}$, and moreover we can chose the initial strings to grow in the following sense: we can pick each initial string so that every subsequent initial string is an extension of the chosen one. By doing so we define a $\theta \in \{-1,1\}^{\mathbb{N}}$ so that any initial string in θ occurs as the initial string in some element of $\{\theta^{(k)}\}_{k\in\mathbb{N}}$.

All that remains to be shown is that some subsequence converges to θ. This follows by picking one index n_1 so that $\theta^{(n_1)}$ matches θ in the first coordinate. By hypothesis there is infinitely many elements that match θ in both the first and second coordinate, so there is some $n_2 > n_1$ such that $\theta^{(n_2)}$ matches θ in the second coordinate. Repeating this inductively grants our subsequence, and note that

$$d(\theta, \theta^{(n_k)}) = \sum_{\ell=k+1}^{\infty} \frac{\left|\theta_\ell - \theta_\ell^{(n_k)}\right|}{2^\ell} \leqslant 2 \sum_{\ell=k+1}^{\infty} 2^{-\ell}$$

The right hand side goes to 0 as $k \to \infty$ and hence we deduce that $\theta^{(n_k)} \to \theta$ with respect to our metric. Thus our metric space is compact as desired. □

Lemma 2.9. *For any fixed unconditional series $\sum_{i=1}^{\infty} x_i \subseteq X$, we have that $T : \{-1,1\}^{\mathbb{N}} \to X$ defined by $T(\theta) = \sum_{i=1}^{\infty} \theta_i x_i$ is continuous when the domain is adorned with metric $d(\theta,\psi) = \sum_{i=1}^{\infty} \frac{|\theta_i - \psi_i|}{2^i}$.*

Proof. This proof follows from a fact proven earlier in the paper. In Theorem 1.16, we proved a collection of statements that are equivalent to a series being unconditionally convergent. The equivalent condition we use now is that the series $\sum_{i=1}^{\infty} x_i$ converges unconditionally if for every $\epsilon > 0$, there exists an integer n so that any finite set of integers ψ with $\min\{i \in \psi\} > n$ satisfies $\left\|\sum_{i\in\psi} x_i\right\| < \epsilon$. To that end, with $\epsilon > 0$ we aim to find $\delta > 0$ so that if $\theta, \sigma \in \{-1,1\}^{\mathbb{N}}$ satisfy $d(\theta,\sigma) < \delta$ then $\|T(\theta) - T(\sigma)\| < \epsilon$.

By our equivalent condition, we must have some $N \in \mathbb{N}$ so that if ψ is a finite set of integers with $\min\{i \in \psi\} > N$, then $\left\|\sum_{i\in\psi} x_i\right\| < \epsilon/3$. Then pick $\delta > 0$ so that $2^{1-k} > \delta$ for each $k \leqslant N$. Then if $\theta, \sigma \in \{-1,1\}^{\mathbb{N}}$ with $d(\theta,\sigma) < \delta$, we deduce from our definition of the metric that $\theta_k = \sigma_k$ for each

$k \leqslant N$ since $\delta > d(\theta, \sigma) \geqslant |\theta_k - \sigma_k| \, 2^{-k}$. Thus, for each $m \in \mathbb{N}$ with $m > N$, we deduce that

$$\left\| \sum_{i=1}^{m} \theta_i x_i - \sum_{i=1}^{m} \sigma_i x_i \right\| = \left\| 2 \sum_{\substack{N+1 \leqslant i \leqslant m \\ \theta_i \neq \sigma_i}} x_i \right\| \leqslant \frac{2\epsilon}{3}$$

Taking the limit as $m \to \infty$ we deduce that

$$\|T(\theta) - T(\sigma)\| \leqslant \frac{2\epsilon}{3} < \epsilon$$

and hence T is continuous. ☐

The preceding two lemmas are sufficient to prove that the unconditional constant is finite.

Theorem 2.10. *The unconditional constant of an unconditional basic sequence $\{x_n\}_{n \in \mathbb{N}} \subseteq X$ denoted K and defined by $K := \sup_{\theta \in \{-1,1\}^{\mathbb{N}}} \|M_\theta\|_{op}$ is well-defined. In particular $K < \infty$.*

Proof. We briefly recap what we know already before combining everything to prove this. Recall that we know that M_θ is continuous and linear for each $\theta \in \{-1,1\}^{\mathbb{N}}$. As such, if we knew that for each $x \in X$, $\sup_{\theta \in \{-1,1\}^{\mathbb{N}}} \|M_\theta(x)\| < \infty$, then Banach-Steinhaus would then imply that the supremum of M_θ in operator norm was finite, so the main result of this theorem is to show that for each $x \in X$, we have that $\sup_{\theta \in \{-1,1\}^{\mathbb{N}}} \|M_\theta(x)\| < \infty$.

Fix $x = \sum_{i=1}^{\infty} a_i x_i \in X$. Since $\{x_i\}_{i \in \mathbb{N}}$ is an unconditional basic sequence, we have that the series defining x converges unconditionally. As defined in the previous lemma, we have that $T : \{-1,1\}^{\mathbb{N}} \to X$ defined $T(\theta) = \sum_{i=1}^{\infty} \theta_i a_i x_i$ is continuous when $\{-1,1\}^{\mathbb{N}}$ is adorned by the metric $d(\theta, \psi) = \sum_{i=1}^{\infty} \frac{|\theta_i - \psi_i|}{2^i}$. As shown in Lemma 2.8, we know that this metric space is compact, and since the continuous image of a compact set is compact, we deduce that $T(\{-1,1\}^{\mathbb{N}}) \subseteq X$ is compact and hence bounded. In other words, $\sup_{\theta \in \{-1,1\}^{\mathbb{N}}} \|T(\theta)\| < \infty$, and since $T(\theta) = \sum_{i=1}^{\infty} \theta_i a_i x_i = M_\theta(x)$, we deduce as desired that $\sup_{\theta \in \{-1,1\}^{\mathbb{N}}} \|M_\theta(x)\| < \infty$. This holds for all $x \in X$ and by our functional analytic argument using Banach-Steinhaus, we deduce that $K := \sup_{\theta \in \{-1,1\}^{\mathbb{N}}} \|M_\theta\|_{op} < \infty$ as desired. ☐

3. Banach Lattice Theory

Many examples of Banach spaces can be ordered in a way that respects the norm. For instance, if we examine $C([0,1])$, the space of continuous real valued functions defined on $[0,1]$ adorned with the supremum norm, then $C([0,1])$ is a Banach space. One natural partial order on $C([0,1])$ is the pointwise order: for all $f, g \in C([0,1])$, we have that $f \leqslant g$ if and only if $f(x) \leqslant g(x)$ for all $x \in [0,1]$. This order generalizes some of the properties that the order on $\mathbb{R}$ has due to the fact that the norm and this order are both built from the order on $\mathbb{R}$. Some properties we have are as follows:

(1) $f \leqslant g$ if and only if $f + z \leqslant g + z$ for all $z \in C([0,1])$
(2) If $f \geqslant 0$, then $af \geqslant 0$ for all $a > 0$.
(3) For every $f, g \in C([0,1])$, we can define a least upper bound $f \vee g \in C([0,1])$ by $f \vee g(x) = \max\{f(x), g(x)\}$. We can similarly define a greatest lower bound $f \wedge g$ by $f \wedge g(x) = \min\{f(x), g(x)\}$.
(4) Using the above, we can define $|f| = f \vee (-f)$. Then, if $|f| \leqslant |g|$, we can show that $\|f\| \leqslant \|g\|$.

In particular, this order gives an absolute value function that generalizes the absolute value function on $\mathbb{R}$. We can give other examples of spaces that are naturally ordered in a fashion like this, and this motivates defining a special class of Banach spaces which has an order with all the above properties. Said class is known as a Banach lattice.

Definition 3.1. *Suppose X is a Banach space defined over the reals that is partially ordered by $\leqslant$. Then X is Banach lattice if the following properties hold:*

1. *For all $x, y \in X$, we have that $x \leqslant y$ if and only if $x + z \leqslant y + z$ for all $z \in X$.*
2. *If $x \geqslant 0$, then $ax \geqslant 0$ for all $a > 0$.*
3. *For every $x, y \in X$, we can define a least upper bound $x \vee y \in X$. We can similarly define a greatest lower bound $x \wedge y \in X$.*
4. *Using the above, we can define $|x| = x \vee (-x)$. Then, if $|x| \leqslant |y|$ for some $x, y \in X$, we have that $\|x\| \leqslant \|y\|$.*

Banach lattices have a deep theory that has been widely studied. For instance, in spite of the generality seen in the definition, we can develop a functional calculus on a general Banach lattice, and said will exactly model arithmetic of real numbers. Informally, any inequality (or equality) using just addition, scalar multiplication, and the lattice operations (so least upper bound or greatest lower bound) in the Banach lattice will be true if it is in the real line. Making this precise requires the development of a lot of theory (for instance, the backbone of developing this functional calculus is an abstract version of Stone-Weierstrass on Banach lattices). In the case where the Banach lattice is actually a space of functions, the functional calculus follows exactly from arithmetic on the real numbers and the deep theoretical work is unnecessary. For this paper, the only concrete Banach lattice we consider is L^p and hence we skip the development of this functional calculus. To see exactly how the functional calculus is developed, one good reference is Chapter 16 of [DJT95].

Two particularly studied notions, and of importance to this paper, are the notion of p-convex and p-concave linear operators. We define these two notions now (see [LT78]). In broader theory, these notions arise when studying isomorphisms of lattices, with some specific examples being the study of rearrangement invariant function spaces and the study of uniform convexity in Banach lattices.

Definition 3.2. *Let X be a Banach lattice and let V be an arbitrary Banach space. Fix $1 \leqslant p \leqslant \infty$.*

1. *A linear operator $T : V \to X$ is called p-convex if there is some $M < \infty$ such that for all $n \in \mathbb{N}$ and $v_1, \ldots, v_n \in V$, we have that*

$$\left\| \left(\sum_{i=1}^{n} |Tv_i|^p \right)^{1/p} \right\| \leqslant M \left(\sum_{i=1}^{n} \|v_i\|^p \right)^{1/p}$$

   The smallest possible such M is denoted M^p.

2. *A linear operator $T : X \to V$ is p-concave if there is some constant $M < \infty$ such that for all $n \in \mathbb{N}$ and $x_1, \ldots, x_n \in X$, we have that*

$$\left(\sum_{i=1}^{n} \|Tx_i\|^p \right)^{1/p} \leqslant M \left\| \left(\sum_{i=1}^{n} |x_i|^p \right)^{1/p} \right\|$$

   The smallest possible M is denoted M_p.

Definition 3.3. *We say that a Banach lattice X is p-convex (or p-concave) if the identity map is p-convex (or p-concave respectively). The minimum M satisfying the condition above is known as the p-convexity (or p-concavity respectively) constant.*

One very important caveat with the above definitions: $|\cdot|$ in the above is the absolute value with respect to the order on the Banach lattice, not the normal absolute value on $\mathbb{R}$. The exact meaning of $(\sum_{i=1}^n |x_i|^p)^{1/p}$ when the Banach lattice is not L^p, or the order is not the standard pointwise is defined in terms of the previously mentioned functional calculus. We will need the definition for this for one part later in the paper and will be introduced when necessary.

We now switch from general Banach lattice theory to the Khintchine inequality which will be necessary later on.

3.1. **Khintchine's Inequality and its Application.** We begin with two technical lemmas regarding Gaussian random variables and a third technical lemma showing the monotonicity of p norms. The first lemma will be used in the Khintchine Inequality proof to insert Gaussian random variables into the statement, and then the second lemma on Gaussian moments will be used to complete the calculation. The third lemma will be used to make the second lemma applicable. For attribution: the Khintchine inequality is standard material and available in many resources. This specific version of the proof differs from the most direct method by using Gaussian random variables and is from lecture notes from Nelson (see [Nel15]). The presentation here follows the same lines as Nelson's proof but has the details filled in. We state some basic calculation results without proof first.

Lemma 3.4. *Let g be a Gaussian random variable with mean zero and variance one. Then $\mathbb{E}|g| = \sqrt{\frac{2}{\pi}}$.*

Proof. This is well known and follows exactly from computation. ☐

Lemma 3.5. *Let g be a Gaussian random variable with mean 0 and variance σ^2. Then $\mathbb{E}g^p \leqslant \sqrt{2e}(\sigma\sqrt{p})^p$.*

Proof. Through basic calculus we find that $\mathbb{E}\,|g|^p = \sigma^p \frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\sqrt{\pi}}$, using the crude estimate that for all $x \geqslant 1$ we have that $\Gamma(x) \leqslant \sqrt{2\pi}x^{x-1/2}$ (see Theorem 1 in [Jam15] for a proof of this), we deduce that

$$\mathbb{E}\,|g|^p = \sigma^p \frac{2^{p/2}\Gamma(\frac{p+1}{2})}{\sqrt{\pi}} \leqslant \frac{\sigma^p}{\sqrt{\pi}}2^{p/2}\cdot\sqrt{2\pi}\cdot\left(\frac{p+1}{2}\right)^{\frac{p}{2}} = \sigma^p\sqrt{2}(p+1)^{p/2} \leqslant \sqrt{2e}(\sigma\sqrt{p})^p$$

where the final inequality above arises from the fact that $\sup_{p\geqslant 1}\left(\frac{p+1}{p}\right)^{p/2} = \sup_{p\geqslant 1}(1+1/p)^{p/2} = \sqrt{e}$, and hence $(p+1)^{p/2} \leqslant \sqrt{e}p^{p/2}$. ☐

Lemma 3.6. *Suppose $1 \leqslant p \leqslant q < \infty$ and X is any random variable. Then $(\mathbb{E}\,|X|^p)^{1/p} \leqslant (\mathbb{E}\,|X|^q)^{1/q}$.*

Proof. This is standard resulting from Jensen's inequality applied to the function $x \mapsto |x|^{q/p}$. ☐

Theorem 3.7 (Khintchine Inequality)**.** *Fix $0 < p < \infty$. Let $\{\epsilon_n\}_{n=1}^N$ be i.i.d. random signs (Rademacher random variables) with $\mathbb{P}(\epsilon_n = 1) = \mathbb{P}(\epsilon_n = -1) = \frac{1}{2}$. Then there exists $B_p > 0$ such that for all $\{x_n\}_{n=1}^N \subseteq \mathbb{R}$, we have that*

$$\left(\mathbb{E}\left|\sum_{n=1}^N \epsilon_n x_n\right|^p\right)^{1/p} \leqslant B_p\left(\sum_{n=1}^N |x_n|^2\right)^{1/2}$$

Proof. The inequality is trivial if $x_n = 0$ for all n, so suppose that this is not true. We note that if $p = 2$, then we have equality with $B_p = 1$. Suppose $p < 2$. Then by our previous lemma we deduce that

$$\left(\mathbb{E}\left|\sum_{n=1}^{N}\epsilon_n x_n\right|^p\right)^{1/p} \leqslant \left(\mathbb{E}\left|\sum_{n=1}^{N}\epsilon_n x_n\right|^2\right)^{1/2} = \left(\mathbb{E}\sum_{n=1}^{N}|x_n|^2\right)^2$$

where we used the fact that $\mathbb{E}\epsilon_n\epsilon_j = 0$ if $n \neq j$. Thus for $p \leqslant 2$, Khintchine's inequality holds with $B_p = 1$.

Suppose $p > 2$. Consider $\{g_n\}_{n=1}^N$ to be an i.i.d. sequence of mean zero variance one Gaussian random variables. Note that $\mathbb{E}|g_n| = \sqrt{\frac{2}{\pi}}$ by our first technical lemma, and hence

$$\left(\mathbb{E}\left|\sum_{n=1}^{N}\epsilon_n x_n\right|^p\right)^{1/p} = \sqrt{\frac{\pi}{2}}\left(\mathbb{E}\left|\mathbb{E}_g\sum_{n=1}^{N}\epsilon_n|g_n|x_n\right|^p\right)^{1/p}$$

Where $\mathbb{E}_g$ is the expectation with respect to our Gausiann random variables. Applying Jensen's inequality to $\mathbb{E}_g$ and noting that $|\cdot|^{1/p}$ is increasing, we obtain then that

$$\left(\mathbb{E}\left|\sum_{n=1}^{N}\epsilon_n x_n\right|^p\right)^{1/p} \leqslant \sqrt{\frac{\pi}{2}}\left(\mathbb{E}\mathbb{E}_g\left|\sum_{n=1}^{N}\epsilon_n|g_n|x_n\right|^p\right)^{1/p}$$

Then $\epsilon_n\,|g_n|$ has the same distribution as g_n, so removing the absolute value bars we find that

$$\left(\mathbb{E}\left|\sum_{n=1}^{N}\epsilon_n x_n\right|^p\right)^{1/p} \leqslant \sqrt{\frac{\pi}{2}}\left(\mathbb{E}\left|\sum_{n=1}^{N}g_n x_n\right|^p\right)^{1/p}$$

We note now that $\sum_{n=1}^N g_n x_n$ is the sum of Gaussian random variables with mean 0 and variance x_n^2. In particular, $\sum_{n=1}^N g_n x_n$ has the same distribution as G, a Gaussian with mean 0 and variance equal to $\sum_{n=1}^N x_n^2$. Accordingly, we have that the right hand side of the above is just the p-th moment of this Gaussian, raised to the power of $1/p$. By our moment calculating lemma, we deduce then that

$$\sqrt{\frac{\pi}{2}}\left(\mathbb{E}\left|\sum_{n=1}^{N}g_n x_n\right|^p\right)^{1/p} \leqslant \sqrt{\frac{\pi}{2}}(2e)^{\frac{1}{2p}}\left(\sum_{n=1}^{N}x_n^2\right)^{1/2}\sqrt{p}$$

Absorbing all the constants into B_p, we deduce that

$$\left(\mathbb{E}\left|\sum_{n=1}^{N}\epsilon_n x_n\right|^p\right)^{1/p} \leqslant B_p\left(\sum_{n=1}^{N}|x_n|^2\right)^{1/2}$$

Where $B_p = \sqrt{\frac{\pi}{2}}(2e)^{\frac{1}{2p}}\sqrt{p}$. This concludes the proof. □

4. A Summary of Schechtman's Paper

With all the previous developments, we are finally in a position to discuss Schechtman's paper which is the principal topic of research. The main idea is to consider the p-concavity constant for various subspaces of $L^p([0,1])$ with the Lebesgue measure (for brevity we write this as L^p for the rest of this paper). As an example, we can easily show that the p-concavity constant of L^p is 1.

Proposition 4.1. *The p-concavity constant of L^p is 1 when viewed as Banach lattice under the point-wise order.*

Proof. To be explicit, L^p is a Banach lattice with the point-wise order defined by $f \geqslant 0$ if $f(x) \geqslant 0$ for almost every x. Then $|f| = \max\{f, -f\}$ pointwise. Then by definition, we have that this space is p-concave with constant 1 if and only if for all $x_1, \ldots, x_n \in L^p$, we have that

$$\left(\sum_{i=1}^{n} \|x_i\|_p^p\right)^{1/p} \leqslant \left\|\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}\right\|_p$$

It is easy to see that in fact we have equality:

$$\left(\sum_{i=1}^{n} \|x_i\|_p^p\right)^{1/p} = \left(\sum_{i=1}^{n} \int |x_i|^p\right)^{1/p} = \left(\int \left(\sum_{i=1}^{n} |x_i|^p\right)^{p/p}\right)^{1/p} = \left\|\left(\sum_{i=1}^{n} |x_i|^p\right)^{1/p}\right\|_p$$

Thus the p-concavity constant of L^p is 1. □

For other subspaces with a defined lattice structure, the exact value of p-concavity constant does not follow trivially from the above since they typically have a different order and hence a different lattice structure. The specific subspaces we study are those generated by 1-unconditional basic sequences. We now define the lattice structure on the span of such a sequence.

Definition 4.2. *Take any* 1*-unconditional basic sequence in a Banach space, say* $\{e_i\}_{i=1}^{\infty}$*. We adorn the closed linear span with the following order: assuming* $\{a_i\}_i$ *is a sequence of real scalars and we have that* $\sum_{i=1}^{\infty} a_i e_i$ *converges, we have that* $\sum_{i=1}^{\infty} a_i e_i \geqslant 0$ *if and only if* $a_i \geqslant 0$ *for all* $i \in \mathbb{N}$*.*

Under this ordering, we have that the closed linear span of any 1-unconditional basic sequence is a Banach lattice, and the question of p-concavity is now applicable. For the rest of this paper we restrict ourselves to studying 1-unconditional basic sequences in $L^p([0,1])$. One sticking point is that the terms occurring in the definition of p-concavity are no longer understood in the simple pointwise fashion and are instead computed using the functional calculus theory. For these very special types of lattices, we can write out the translated version of p-concavity which we state without proof.

Proposition 4.3. *The closed linear span of a* 1*-unconditional basic sequence* $\{e_i\}_{i\in\mathbb{N}}$ *in* L^p *is* p*-concave with constant* C *if for all finite sequences of elements in the span* $y_1, \ldots, y_n$ *with* $y_i = \sum_{k=1}^{\infty} a_{k,i} e_k$*, we have that*

$$\left(\sum_{i=1}^{n} \left\|\sum_{k=1}^{\infty} a_{k,i} e_k\right\|_p^p\right)^{1/p} \leqslant C \left\|\sum_{k=1}^{\infty} \left(\sum_{i=1}^{n} |a_{k,i}|^p\right)^{1/p} e_k\right\|_p$$

Using Khintchine's inequality, we can actually show that for all $p \geqslant 2$, we have that the closed linear span of a 1-unconditional basic sequence in L^p is p-concave with a constant no more than the Khintchine constant B_p. The proof is not especially long or hard, but it is non-trivial, so it is worth writing out in its entirety.

Proposition 4.4. *Fix any* 1*-unconditional basic sequence* $\{e_i\}_{i=1}^{\infty} \subseteq L^p([0,1])$ *where* $p \geqslant 2$*. Then the Banach lattice defined on the closed span of the sequence is* p*-concave with constant at most* B_p*, the constant arising in the Khintchine inequality.*

Proof. We begin by recalling Minkowski's integral inequality, namely for any two σ-finite measure spaces (S_1, μ_1) and (S_2, μ_2) and $F : S_1 \times S_2 \to \mathbb{R}$ a measurable function, we have that

$$\left(\int_{S_2} \left|\int_{S_1} F(x,y) d\mu_1(x)\right|^p d\mu_2(y)\right)^{1/p} \leqslant \int_{S_1} \left(\int_{S_2} |F(x,y)|^p \, d\mu_2(y)\right)^{1/p} d\mu_1(x)$$

We aim to prove the inequality in Proposition 4.3, so assume we have $y_1, \ldots, y_n$ in the span of $\{e_i\}_{i\in\mathbb{N}}$ so that $\sum_{k=1}^{\infty} a_{k,i}e_k = y_i$. Then starting from the left side of the inequality in that proposition:

$$\left(\sum_{i=1}^{n}\left\|\sum_{k=1}^{\infty} a_{k,i}e_k\right\|_p^p\right)^{1/p} = \left(\sum_{i=1}^{n}\int_0^1 \left|\sum_{k=1}^{\infty} a_{k,i}e_k(r)\right|^p dr\right)^{1/p} = \left(\int_0^1 \left(\sum_{i=1}^{n}\left|\sum_{k=1}^{\infty} a_{k,i}e_k(r)\right|^p\right)^{p/p} dr\right)^{1/p}$$

$$= \left\|\left(\sum_{i=1}^{n}\left|\sum_{k=1}^{\infty} a_{k,i}e_k\right|^p\right)^{1/p}\right\|_p$$

The main idea is that we can insert Rademacher functions $\{\epsilon_k\}_{k\in\mathbb{N}}$ into the above to apply Khintchine's inequality. Since $\|\epsilon_k\|_p = 1$ for every $k \in \mathbb{N}$, we have that

$$\left\|\left(\sum_{i=1}^{n}\left|\sum_{k=1}^{\infty} a_{k,i}e_k\right|^p\right)^{1/p}\right\|_p = \left\|\left(\sum_{i=1}^{n}\left(\int_0^1 \left|\sum_{k=1}^{\infty} a_{k,i}e_k\epsilon_k(t)\right|^p dt\right)^{p/p}\right)^{1/p}\right\|_p$$

The inner most integral is then exactly in the setup for Khintchine's inequality and hence we find that

$$\left\|\left(\sum_{i=1}^{n}\left|\sum_{k=1}^{\infty} a_{k,i}e_k\right|^p\right)^{1/p}\right\|_p \leqslant \left\|\left(\sum_{i=1}^{n}\left(B_p^p\sum_{k=1}^{\infty} |a_{k,i}e_k|^2\right)^{p/2}\right)^{1/p}\right\|_p$$

$$= B_p\left\|\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{\infty} |a_{k,i}e_k|^2\right)^{p/2}\right)^{1/p}\right\|_p$$

where B_p is the Khintchine constant. Now using triangle inequality we deduce that

$$B_p\left\|\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{\infty} |a_{k,i}e_k|^2\right)^{p/2}\right)^{1/p}\right\|_p \leqslant B_p\left\|\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{\infty} |a_{k,i}|\,|e_k|\right)^{p}\right)^{1/p}\right\|_p$$

Using Minkowski's inequality now, we deduce that

$$B_p\left\|\left(\sum_{i=1}^{n}\left(\sum_{k=1}^{\infty} |a_{k,i}|\,|e_k|\right)^{p}\right)^{1/p}\right\|_p \leqslant B_p\left\|\sum_{k=1}^{\infty}\left(\sum_{i=1}^{n} |a_{k,i}|^p\,|e_k|^p\right)^{1/p}\right\|_p$$

$$= B_p\left\|\sum_{k=1}^{\infty}\left(\sum_{i=1}^{n} |a_{k,i}|^p\right)^{1/p} |e_k|\right\|_p$$

Finally using 1-unconditionality we have that

$$\left(\sum_{i=1}^{n}\left\|\sum_{k=1}^{\infty} a_{k,i}e_k\right\|_p^p\right)^{1/p} \leqslant B_p\left\|\sum_{k=1}^{\infty}\left(\sum_{i=1}^{n} |a_{k,i}|^p\right)^{1/p} |e_k|\right\|_p \leqslant B_p\left\|\sum_{k=1}^{\infty}\left(\sum_{i=1}^{n} |a_{k,i}|^p\right)^{1/p} e_k\right\|_p$$

Accordingly, we deduce that the closed linear span of our 1-unconditional basic sequence is p-concave in L^p with constant at most the Khintchine inequality as desired. $\square$

As shown in our section about Khintchine's inequality, $B_p \leqslant C\sqrt{p}$ for some $C > 0$ independent of p, so the above proof shows a bound on the p-concavity constant that grows like $\sqrt{p}$ as p increases. In particular, it seems reasonable that this constant must grow as p does, but in [Sch95], Schechtman had a key insight that provided a way of showing that this constant is actually identically 1. The manner in which he did that was by showing a lemma that showed that the Rademacher functions form a very special 1-unconditional basic sequence in L^p, in the sense that they are the most extremal possible such basic sequence. We state the lemma now.

Lemma 4.5 (Lemma 1 from [Sch95])**.** *Let $2 \leqslant p < \infty$. Then the following conditions are equivalent.*

(1) The p-concavity constant of every 1-unconditional basic sequence in L^p is 1.
(2) The p-concavity constant of the Rademacher sequence in L^p (as a 1-unconditional basic sequence) is 1.
(3) Every normalized (so L^p norm is 1) 1-unconditional basic sequence $\{e_n\}_{n=1}^{\infty}$ satisfies for all $N \in \mathbb{N}$ and sequences of real numbers $\{a_n\}_{n=1}^{\infty}$

$$\left\| \sum_{k=1}^{N} a_k e_k \right\|_p \leqslant \left\| \sum_{k=1}^{N} a_k r_k \right\|_p$$

where $\{r_k\}_{k=1}^{\infty}$ are the Rademacher functions.

Proof. The proof of this can be seen in [Sch95]. It follows in a similar vein of calculations as the previous lemma (and the lemma that comes after this). ☐

This lemma motivates a deeper investigation of the p-concavity constant of the Rademacher functions in L^p, because showing that it is 1 would tell us a relatively strong structural property for all 1-unconditional basic sequences in L^p, namely that their p-concavity constant is 1. Moreover, we additionally would know that the Rademacher functions are in some sense the most extreme normalized 1-unconditional basic sequence, made precise by (3) of the above lemma.

Through one final technical computation, we can give the easiest way of computing the p-concavity constant of the Rademacher functions.

Lemma 4.6. *The p-concavity constant of the Rademacher functions is 1 if and only if for all $n \in \mathbb{N}$ we have that $\varphi : \mathbb{R}_+^n \to \mathbb{R}$ is concave where*

$$\varphi(x_1, \ldots, x_n) = \mathbb{E}\left| \sum_{i=1}^{n} x_i^{1/p} X_i \right|^p$$

Proof. Note that an equivalent condition to the Rademachers having a p-concavity constant of 1 is given by 4.3. In particular the Rademacher functions $\{r_k\}_{k\in\mathbb{N}}$ have a p-concavity constant of 1 (as a 1-unconditional basic sequence) if and only if for all $n \in \mathbb{N}$ and $y_1, \ldots, y_n$ in the span of the Rademacher functions with $y_i = \sum_{k=1}^{\infty} a_{k,i} e_k$, we have that

$$\left(\sum_{i=1}^{n} \left\| \sum_{k=1}^{\infty} a_{k,i} r_k \right\|_p^p \right)^{1/p} \leqslant \left\| \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n} |a_{k,i}|^p \right)^{1/p} r_k \right\|_p$$

Raising both sides to the pth power we deduce that

$$\sum_{i=1}^{n} \mathbb{E} \left| \sum_{k=1}^{\infty} a_{k,i} r_k \right|^p \leqslant \mathbb{E} \left| \sum_{k=1}^{\infty} \left(\sum_{i=1}^{n} |a_{k,i}|^p \right)^{1/p} r_k \right|^p$$

Of course, the Rademacher random variables are symmetric (or alternatively by 1-unconditionality) and hence with loss of generality we can replace $a_{k,i}$ on the left by $|a_{k,i}|$. Additionally, we truncate the infinite sums to $N \in \mathbb{N}$.

$$\sum_{i=1}^{n} \mathbb{E}\left|\sum_{k=1}^{N} |a_{k,i}|\, r_k\right|^p \leqslant \mathbb{E}\left|\sum_{k=1}^{N}\left(\sum_{i=1}^{n} |a_{k,i}|^p\right)^{1/p} r_k\right|^p$$

If the above inequality holds for all $N \in \mathbb{N}$ we can pass to the infinite sum case by taking the limit. Moreover, if the inequality holds for the infinite sum case, then replacing $a_{k,i}$ by 0 for all $i \leqslant n$ and $k > N$ yields the finite case. In particular it suffices to consider the finite case.

At this stage we define a function that will be the main focus for the rest of the paper. Define $\varphi : \mathbb{R}_+^n \to \mathbb{R}$ by

$$\varphi(x_1, \ldots, x_n) = \mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} r_k\right|^p$$

Then the inequality above becomes

$$\sum_{i=1}^{n} \varphi(|a_{1,i}|^p, |a_{2,i}|^p, \ldots, |a_{N,i}|^p) \leqslant \varphi\left(\sum_{i=1}^{n} |a_{1,i}|^p, \ldots, \sum_{i=1}^{n} |a_{N,i}|^p\right)$$

The above inequality holds if and only if the following holds for all $\{x_{k,i}\}_{k\leqslant N, i\leqslant n} \subseteq \mathbb{R}_+$:

$$\sum_{i=1}^{n} \varphi(x_{1,i}, x_{2,i}, \ldots, x_{N,i}) \leqslant \varphi\left(\sum_{i=1}^{n} x_{1,i}, \ldots, \sum_{i=1}^{n} x_{N,i}\right)$$

We claim that the above inequality holds if and only if φ is concave. Suppose the above inequality holds for all choices x. Then note that φ is homogeneous of degree 1 and hence if we pick $\lambda_1, \ldots, \lambda_n$ with $\lambda_i \in (0,1)$ and $\sum_{i=1}^{N} \lambda_i = 1$, scaling $x_{k,i}$ by λ_i, we deduce that

$$\sum_{i=1}^{n} \lambda_i \varphi(x_{1,i}, \ldots, x_{N,i}) \leqslant \varphi(\sum_{i=1}^{n} \lambda_i x_{1,i}, \ldots, \sum_{i=1}^{n} \lambda_i x_{N,i})$$

Of course if we write $y_i = (x_{1,i}, \ldots, x_{N,i})$ then the above reduces to

$$\sum_{i=1}^{n} \lambda_i \varphi(y_i) \leqslant \varphi(\sum_{i=1}^{n} \lambda_i y_i)$$

This is exactly the condition that φ is concave, and hence we deduce that the p-concavity constant of the Banach lattice generated by the Rademacher functions is 1 if and only if φ is concave. □

As a consequence of the previous lemma, our main goal of research is thus to show that φ is concave. While we have already defined φ, we will later modify it slightly so it seems appropriate to give the general definition.

Definition 4.7. *Let $p \geqslant 2$ and let $X_1, \ldots, X_n$ be i.i.d. symmetric random variables. We define $\varphi : \mathbb{R}_+^n \to \mathbb{R}$ by*

$$\varphi(x_1, \ldots, x_n) = \mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^p$$

Remark. *In order to have the concavity of φ imply the p-concavity constant of the Rademacher random variables, of course we need that X_i be Rademacher random variables. However, later in the paper, we will change X_i from a random sign to a uniform random on $[-1, 1]$, so for now we assume* *that X_i being a generic sequence of i.i.d. symmetric random variable and do our basic calculations in this setup before specifying X_i.*

Remark. *The motivation for requiring $p \geqslant 2$ in our concavity calculations is as follows. Suppose $p = 2$ and $\{X_i\}_{i=1}^n$ is any i.i.d. sequence of mean zero random variables. If we simply expand the definition of φ using independence and the fact that our random variables are mean zero, we obtain that*

$$\varphi(x_1, \dots, x_n) = \mathbb{E}\left|\sum_{i=1}^n x_i^{1/2} X_i\right|^2 = \mathbb{E}\left(\sum_{i=1}^n x_i X_i^2 + \sum_{i\neq j} \sqrt{x_i x_j} X_i X_j\right) = \sum_{i=1}^n x_i \operatorname{var} X_i$$

Thus φ is actually linear when $p = 2$. It remains open whether φ is concave when $2 < p < 3$, and it is conjectured that φ is convex when $1 < p < 2$ (and moreover the lattice theory has an analogous version for p-convexity which is mentioned in Schechtman's original paper, see [Sch95]). It will be shown that φ is concave for $p \geqslant 3$, so it appears that $p = 2$ is a turning point where φ switches from convex to concave. Unfortunately, this remains an open problem.

In Schectman's paper, he managed to prove that φ is indeed concave for $p \geqslant 3$, beginning with the following Hessian computation lemma.

Theorem 4.8. *Suppose $p \geqslant 2$. Then $\varphi : \mathbb{R}_+^n \to \mathbb{R}$ is concave if for all $a \in \mathbb{R}^n$, $x \in \mathbb{R}_+^n$ and $j \neq k \leqslant n$, we have that*

$$0 \leqslant \mathbb{E}\left|\sum_{i=1}^n x_i^{1/p} X_i\right|^{p-2} X_j X_k$$

Proof. This theorem is the main content of Lemma 2 in [Sch95]. The calculations here are done in slightly more generality than in Schechtman's work since we make no assumptions on our random variables, whereas Schechtman assumes they are Rademacher random variables. The proof follows entirely from checking the condition that the Hessian of φ is negative semi-definite.

Note that since $p > 2$, φ is twice differentiable. First we compute the first partial:

$$\frac{\partial \varphi}{\partial x_i}(x) = x_i^{\frac{1-p}{p}} \mathbb{E}\left|\sum_{k=1}^n x_k^{1/p} X_k\right|^{p-1} \operatorname{Sign}\left(\sum_{k=1}^n x_k^{1/p} X_k\right) X_i$$

If we take another partial with respect to x_j where $i \neq j$, we obtain that

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j}(x) = \frac{p-1}{p} x_i^{\frac{1-p}{p}} x_j^{\frac{1-p}{p}} \mathbb{E}\left|\sum_{k=1}^n x_k^{1/p} X_k\right|^{p-2} X_i X_j$$

We then can compute the final second degree partial as

$$\frac{\partial^2 \varphi}{\partial x_i^2}(x) = \frac{p-1}{p}\left(x_i^{\frac{2(1-p)}{p}} \mathbb{E}\left|\sum_{k=1}^n x_k^{1/p} r_k\right|^{p-2} X_i^2 - x_i^{\frac{1-2p}{p}} \mathbb{E}\left|\sum_{k=1}^n x_k^{1/p} X_k\right|^{p-2}\left(\sum_{k=1}^n x_k^{1/p} X_k\right) X_i\right)$$

To show that the Hessian of φ is negative semi-definite, we need to show for all $x \in \mathbb{R}_+^n$ and $a \in \mathbb{R}^n$ that

$$\sum_{j,k=1}^n a_j a_k \frac{\partial^2 \varphi}{\partial x_j \partial x_k}(x) \leqslant 0$$

In order to do this, we compute as follows:

$$\begin{aligned}
&\frac{p}{p-1}\sum_{j,k=1}^{n} a_j a_k \frac{\partial^2\varphi}{\partial x_j \partial x_k}(x)\\
&\quad= \sum_{j,k=1}^{n} a_j a_k x_j^{\frac{1-p}{p}} x_k^{\frac{1-p}{p}} \mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2} X_j X_k - \sum_{k=1}^{n} a_k^2 x_k^{\frac{1-2p}{p}} \mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2}\left(\sum_{i=1}^{n} x_i^{1/p} X_i\right) X_k\\
&\quad= \sum_{j\neq k} a_j a_k x_j^{\frac{1-p}{p}} x_k^{\frac{1-p}{p}} \mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2} X_j X_k + \sum_{k=1}^{n} a_k^2 x_k^{\frac{2(1-p)}{p}} \mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2} X_k^2\\
&\qquad\qquad - \sum_{k=1}^{n} a_k^2 x_k^{\frac{1-2p}{p}} \mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2}\left(\sum_{i=1}^{n} x_i^{1/p} X_i\right) X_k
\end{aligned}$$

We now take the last term and expand it using linearity of expectation as follows.

$$\begin{aligned}
&- \sum_{k=1}^{n} a_k^2 x_k^{\frac{1-2p}{p}} \mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2}\left(\sum_{i=1}^{n} x_i^{1/p} X_i\right) X_k\\
&= -\sum_{k=1}^{n}\sum_{j=1}^{n} a_k^2 x_j^{1/p} x_k^{\frac{1-2p}{p}} \mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2} X_j X_k\\
&= -\sum_{j\neq k} a_k^2 x_j^{1/p} x_k^{\frac{1-2p}{p}} \mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2} X_j X_k - \sum_{k=1}^{n} a_k^2 x_k^{\frac{2(1-p)}{p}} \mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2} X_k^2
\end{aligned}$$

The right most term cancels in our original sum and hence we find that

$$\begin{aligned}
\frac{p}{p-1}\sum_{j,k=1}^{n} a_j a_k \frac{\partial^2\varphi}{\partial x_j \partial x_k}(x) &= \sum_{j\neq k}\left(a_j a_k x_j^{\frac{1-p}{p}} x_k^{\frac{1-p}{p}} - a_k^2 x_j^{1/p} x_k^{\frac{1-2p}{p}}\right) \mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2} X_j X_k\\
&= -\sum_{j<k}\left(a_j x_j^{\frac{1-2p}{2p}} x_k^{\frac{1}{2p}} - a_k x_k^{\frac{1-2p}{2p}} x_j^{\frac{1}{2p}}\right)^2 \mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2} X_j X_k
\end{aligned}$$

Note that for all $j < k$, the squared coefficient is positive and hence $\mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2} X_j X_k$ being non-negative for all $j < k$ and $x \in \mathbb{R}_+^n$ and would immediately imply that our Hessian is negative semi-definite as desired. □

Through Theorem 4.8, we have reduced a question about concavity to a simpler one about the sign of a function. While this is a large simplification, the new condition is still quite difficult to check. The obvious simplification is to rewrite our expectation in terms of a sequence of conditional expectations, so

$$\mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2} X_j X_k = \mathbb{E}'\mathbb{E}_j\mathbb{E}_k\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2} X_j X_k$$

where $\mathbb{E}'$ means to take the expectation over all X_i where $i \neq j, k$, and $\mathbb{E}_j, \mathbb{E}_k$ mean to take the expectation over X_j and X_k respectively.

Remark. *In the case where $\{X_i\}_{i=1}^n$ are i.i.d. symmetric random signs and $2 < p < 3$, it turns out that this theorem is not applicable. In particular, the condition that* $\mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2} X_j X_k \geqslant 0$

need not hold any longer. Proving the symmetric sign case would require one to consider the entire summand. In particular, the previous theorem allows us to take the Hessian test and remove all dependence on the vector we are testing the Hessian against.

Schechtman's work followed this reasoning, and proved this new condition for the case when our random variables, $\{X_i\}_{i=1}^n$, are symmetric random signs and $p \geqslant 3$. Specifically, he showed the following. We restate his proof for convenience.

Lemma 4.9 ([Sch95])**.** *If $a \in \mathbb{R}$, $b, c \geqslant 0$, and $q \geqslant 1$, $j \neq k \leqslant n$, and $\{X_i\}_{i=1}^n$ are i.i.d. symmetric random signs, then*

$$\mathbb{E}_j \mathbb{E}_k |a + bX_j + cX_k|^q X_j X_k \geqslant 0$$

Proof. Note that

$$\begin{aligned} a + b - c &= \frac{b}{b+c}(a+b+c) + \frac{c}{b+c}(a-b-c) \\ a - b + c &= \frac{c}{b+c}(a+b+c) + \frac{b}{b+c}(a-b-c) \end{aligned}$$

Since $q \geqslant 1$, we have that $|\cdot|^q$ is convex. Thus with the above, we have that

$$\begin{aligned} |a+b-c|^q + |a-b+c|^q &\leqslant \frac{b}{b+c}|a+b+c|^q + \frac{c}{b+c}|a-b-c|^q + \frac{c}{b+c}|a+b+c|^q + \frac{b}{b+c}|a-b-c|^q \\ &= |a+b+c|^q + |a-b-c|^q \end{aligned}$$

Thus

$$0 \leqslant |a+b+c|^q + |a-b-c|^q - |a+b-c|^q - |a-b+c|^q$$

Thus by one final computation, we find that

$$\mathbb{E}_j \mathbb{E}_k |a + bX_j + cX_k|^q X_j X_k = \frac{1}{4}\left(|a+b+c|^q - |a-b+c|^q - |a+b-c|^q + |a-b-c|^q\right) \geqslant 0$$

as desired. ☐

Remark. *Note that we required q, the exponent on our absolute value term, to be at least 1. Since $p \geqslant 3$, and the exponent actually used is $p - 2 \geqslant 1$, this lemma is applicable for Schechtman's case but not when $2 < p < 3$. In fact, this lemma is false for $0 < q < 1$. If we compute the expression in the lemma with $a = 2$, $b = c = 1$ and $q = 0.5$, we obtain*

$$\sqrt{2+1+1} - 2 \cdot \sqrt{2+1-1} + \sqrt{2-1-1} = 2 - 2\sqrt{2} < 0$$

Thus a proof for the random sign case when $2 < p < 3$ would require a fundamentally different technique that cannot use a lemma of this form.

With this and Theorem 4.8, we show the ending of Schechtman's proof.

Corollary 4.10 ([Sch95])**.** *$\varphi : \mathbb{R}_+^n \to \mathbb{R}$ defined by $\varphi(x) = \mathbb{E}\left|\sum_{i=1}^n x_i^{1/p} X_i\right|^p$ is concave when $\{X_i\}_{i=1}^n$ are i.i.d. symmetric random signs and $p \geqslant 3$.*

Proof. By Theorem 4.8 we just need to show that for all $a \in \mathbb{R}^n$, $x \in \mathbb{R}_+^n$ and $j \neq k \leqslant n$, we have that $0 \leqslant \mathbb{E}\left|\sum_{i=1}^n x_i^{1/p} X_i\right|^{p-2} X_j X_k$. Note that $p - 2 \geqslant 1$, and hence by applying Lemma 4.9 with $a = \sum_{i \neq j,k} x_i^{1/p} X_i$, $b = x_j^{1/p}$ and $c = x_k^{1/p}$, we find that

$$\mathbb{E}_j\mathbb{E}_k\left|\sum_{i\neq j,k} x_i^{1/p}X_i + x_j^{1/p}X_j + x_k^{1/p}X_k\right|^{p-2} X_jX_k \geqslant 0$$

and hence

$$\mathbb{E}\left|\sum_{i=1}^n x_i^{1/p}X_i\right|^{p-2} X_jX_k = \mathbb{E}'\mathbb{E}_j\mathbb{E}_k\left|\sum_{i\neq j,k} x_i^{1/p}X_i + x_j^{1/p}X_j + x_k^{1/p}X_k\right|^{p-2} X_jX_k \geqslant 0$$

Thus the corollary is proven. □

In particular, this implies that all the conditions in Lemma 4.5 are true, namely the p-concavity constant of the Rademacher functions is 1, the p-concavity constant of all 1-unconditional basic sequences in L^p are 1, and the Rademacher functions dominate all other normalized 1-unconditional sequences in L^p as made precise by that lemma.

As stated, this avenue of proof fails when $2 < p < 3$ since Lemma 4.9 is false in this case. Furthermore, $|\cdot|^{p-2}$ fails to be convex any longer which takes away the only obvious tool for proving the desired result. The case when $2 < p < 3$ and our random variables are symmetric signs remains open.

5. A Novel Result For Uniform Random Variables where $p > 2$

With the uniform random sign case out of reach for now, the next obvious case to consider is i.i.d. symmetric random variables on $[-1, 1]$. This topic was the main goal of the author's research, and in fact φ is concave under these hypotheses. Proving this requires no advanced techniques but the proof is long and involves many detailed computations. As such, this entire section will be devoted to it. The main work is devoted to devising a new version of Lemma 4.9 that is actually true, which we will state now and postpone the proof.

Lemma 5.1. *Define $f : \mathbb{R}_+^3 \to \mathbb{R}$ by*

$$f(a, b, c) = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} |ax + by + cz|^q yzdydzdx$$

where $q \geqslant 0$. Then $f \geqslant 0$.

Using this, we can immediately prove that φ is concave in the i.i.d. uniform random variable on $[-1, 1]$ case when $p > 2$.

Theorem 5.2. *Recall $\varphi(x) = \mathbb{E}\left|\sum_{i=1}^n x_i^{1/p}X_i\right|^p$ where $\varphi : \mathbb{R}_+^n \to \mathbb{R}$. When $p > 2$ and $\{X_i\}_{i=1}^n$ are uniform random variables on $[-1, 1]$, then φ is concave.*

Proof. By Theorem 4.8, it suffices to show that for all $a \in \mathbb{R}^n$, $x \in \mathbb{R}_+^n$ and $j \neq k \leqslant n$, we have that

$$0 \leqslant \mathbb{E}\left|\sum_{i=1}^n x_i^{1/p}X_i\right|^{p-2} X_jX_k$$

We break up the inside sum as

$$\sum_{i=1}^n x_i^{1/p}X_i = \sum_{i\neq j,k} x_i^{1/p}X_i + x_j^{1/p}X_j + x_k^{1/p}X_k$$

Note that $\sum_{i\neq j,k} x_i^{1/p}X_i$ is just a weighted sum of uniform random variables, so it is a symmetric unimodal (this will be defined and made precise after this theorem). As such, we can find random variables R, U with R being non-negative and U being a uniform random variable on $[-1,1]$ (see the next lemma). As such, we can rewrite our expectation as

$$\mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2} X_j X_k = \mathbb{E}_R \mathbb{E}_U \mathbb{E}_j \mathbb{E}_k \left|RU + x_j^{1/p} X_j + x_k^{1/p} X_k\right|^{p-2}$$

By Lemma 5.1 with $a = R$, $b = x_j^{1/p}$ and $c = x_k^{1/p}$, we have that

$$\mathbb{E}_U \mathbb{E}_j \mathbb{E}_k \left|RU + x_j^{1/p} X_j + x_k^{1/p} X_k\right|^{p-2} \geqslant 0$$

and hence

$$\mathbb{E}\left|\sum_{i=1}^{n} x_i^{1/p} X_i\right|^{p-2} X_j X_k \geqslant 0$$

Finally this implies that φ is concave on $\mathbb{R}_+^n$ as desired. $\square$

Definition 5.3. *Let X be a real random variable. We say that X is a symmetric unimodal random variable if X is continuous (so has a probability density) and its density is non-increasing on $[0, \infty)$.*

We collect some basic facts about symmetric unimodal random variables here. See [LO95] for proofs.

Lemma 5.4. *Suppose X is a symmetric unimodal random variable. Then there exists a non-negative random variable R such that X has the same distribution as RU where U is a uniform random variable and R and U are independent from each other.*

Remark. *One interesting note: if φ is concave for some fixed p when X_i are all symmetric random signs, then for any sequence of i.i.d. symmetric random variables Y_i, we have that $X_i |Y_i|$ has the same distribution as Y_i where X_i is a random sign. In particular, if the random variables inside φ are Y_i, then we can rewrite φ like follows:*

$$\varphi(x) = \mathbb{E}_Y \mathbb{E}_X \left|\sum_{i=1}^{n} (x_i |Y_i|^p)^{1/p} X_i\right|^p$$

In particular: φ is the expectation of concave functions and hence is concave. Thus if we view the concavity of φ as a probabilistic question and ask for what class of random variables is φ concave, knowing that φ is concave for random signs implies it is concave for all symmetric random variables.

Since φ has not yet been shown to be concave for symmetric random signs when $2 < p < 3$, one may ask what the next biggest class of random variables it is concave for. As motivated by the previous lemma, for any symmetric unimodal random variable, we can decompose it as the product of a non-negative random variable multiplied by a uniform random. Repeating the previous computation trick, we can deduce the concavity of φ when the random variables are symmetric unimodal to just the case when they are uniform random variables.

Now we begin the proof of Lemma 5.1. First we handle the easiest case when $a = 0$.

Lemma 5.5. *$f(0, b, c) \geqslant 0$ for all $b, c \geqslant 0$.*

Proof. This is a simple computation. Note that

$$\begin{aligned}\frac{1}{2}f(0,b,c) &= \frac{1}{2}\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}|by+cz|^q\,yzdydzdx \\ &= \int_{-1}^{1}\int_{-1}^{1}|by+cz|^q\,yzdydz \\ &= \int_{0}^{1}\int_{0}^{1}|by+cz|^q\,yzdydz + \int_{-1}^{0}\int_{0}^{1}|by+cz|^q\,yzdydz \\ &+ \int_{0}^{1}\int_{-1}^{0}|by+cz|^q\,yzdydz + \int_{-1}^{0}\int_{-1}^{0}|by+cz|^q\,yzdydz \\ &:= I_1 + I_2\end{aligned}$$

We claim that I_1, I_2 are both non-negative. We compute

$$\begin{aligned}I_1 &= \int_{0}^{1}\int_{0}^{1}|by+cz|^q\,yzdydz + \int_{-1}^{0}\int_{0}^{1}|by+cz|^q\,yzdydz \\ &= \int_{0}^{1}\int_{0}^{1}|by+cz|^q\,yzdydz - \int_{0}^{1}\int_{0}^{1}|by-cz|^q\,yzdydz = \int_{0}^{1}\int_{0}^{1}yz\left(|by+cz|^q - |by-cz|^q\right)dydz \\ &= \int_{0}^{1}\int_{0}^{1}yz\left((by+cz)^q - |by-cz|^q\right)dydz\end{aligned}$$

Since $b, c \geqslant$ and $y, z \geqslant 0$, $by + cz \geqslant |by - cz|$. Also since $(\cdot)^q$ is increasing we have that $(by+cz)^q - |by-cz|^q \geqslant 0$. Since $yz \geqslant 0$ we deduce that $I_1 \geqslant 0$ as desired.

The case for I_2 is similar. Indeed

$$I_2 = \int_{0}^{1}\int_{-1}^{0}|by+cz|^q\,yzdydz + \int_{-1}^{0}\int_{-1}^{0}|by+cz|^q\,yzdydz = \int_{-1}^{0}\int_{-1}^{0}yz\left((-by-cz)^q - |-by+cz|^q)\right)dydz$$

Similarly $yz\left((-by-cz)^q - |-by+cz|^q)\right) \geqslant 0$, and hence $I_2 \geqslant 0$. Since $f(0,b,c) = I_1 + I_2$, we have that $f(0,b,c) \geqslant 0$ for all $b, c \geqslant 0$. $\square$

We now suppose that $a > 0$ and prove a new lemma that implies Lemma 5.1. Define $H : \mathbb{R}_+^2 \to \mathbb{R}$ by

$$H(\alpha,\beta) = \frac{1}{2}\int_{-1}^{1}\int_{-\alpha}^{\alpha}\int_{-\beta}^{\beta}|x+y+z|^q yzdydzdx$$

Lemma 5.6. *For all $a > 0$ and $b, c \geqslant 0$, we have that $f(a,b,c)$ and $H\left(\frac{b}{a},\frac{c}{a}\right)$ have the same sign. In particular, if $H \geqslant 0$ then $f \geqslant 0$.*

Proof. Note that we already know that $f(0,b,c) \geqslant 0$ for all $b, c \geqslant 0$, so we can assume that $a > 0$. Then, we can do the following:

$$\begin{aligned}f(a,b,c) &= \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}|ax+by+cz|^q yzdydzdx \\ &= |a|^q\int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1}|x+\frac{b}{a}y+\frac{c}{a}z|^q yzdydzdx\end{aligned}$$

Performing a change of variables, we find that

$$f(a,b,c) = \frac{|a|^{q+2}}{bc}\int_{-1}^{1}\int_{-c/a}^{c/a}\int_{-b/a}^{b/a}|x+y+z|^q yzdydzdx$$

In particular, we have that

$$f(a,b,c) = \frac{|a|^{q+2}}{bc} H\left(\frac{c}{a}, \frac{b}{a}\right)$$

Since $\frac{|a|^{q+2}}{bc} \geqslant 0$ we deduce that $f(a,b,c)$ and $H\left(\frac{c}{a}, \frac{b}{a}\right)$ have the same sign. Thus if $H \geqslant 0$, then $f \geqslant 0$. □

At the moment, H is clearly symmetric and $H(\alpha,\beta) = H(\beta,\alpha)$. The following lemma is not strictly needed yet, but after this lemma we will put H into a form that appears asymmetric. As such, we prove a crucial (and obvious) symmetry property first that will come back at the end of this section.

Lemma 5.7. *$\partial_b H \geqslant 0$ and $\partial_c H \geqslant 0$ if $\partial_b H(b,c) \geqslant 0$ and $\partial_c H(b,c) \geqslant 0$ for all $0 \leqslant b \leqslant c$.*

Proof. Since H is symmetric, $H(b,c) = H(c,b)$. Then note

$$\partial_b H(b,c) = \lim_{h\to 0} \frac{H(b+h,c) - H(b,c)}{h} = \lim_{h\to 0} \frac{H(c,b+h) - H(c,b)}{h} = \partial_c H(c,b)$$

Thus if $\partial_c H(b,c) \geqslant 0$ for all $0 \leqslant b \leqslant c$, then $\partial_b H(c,b) \geqslant 0$ for all $0 \leqslant b \leqslant c$. In particular we obtain that $\partial_b H \geqslant 0$, and also by symmetry we obtain that $\partial_c H \geqslant 0$. □

Remark. *This lemma suggests heavily how to show that $H \geqslant 0$. By showing that $\partial_b H(b,c) \geqslant 0$ and $\partial_c H(b,c)$ when $0 \leqslant b \leqslant c$, we obtain by this symmetry lemma that $\partial_b H$ and $\partial_c H$ are both non-negative. In particular, we can use the fundamental theorem of calculus to then conclude that $H \geqslant 0$.*

Claim. *We have the following form of H:*

$$H(b,c) = \int_{-1}^{1}\int_{-c}^{c}\int_{0}^{b} |x+y+z|^q yz\,dy\,dz\,dx$$

Proof. Note:

$$H(b,c) = \frac{1}{2}\int_{-1}^{1}\int_{-c}^{c}\int_{-b}^{b} |x+y+z|^q yz\,dy\,dz\,dx$$

We split the b integral into two.

$$\int_{-1}^{1}\int_{-c}^{c}\int_{0}^{b} |x+y+z|^q yz\,dy\,dz\,dx + \int_{-1}^{1}\int_{-c}^{c}\int_{-b}^{0} |x+y+z|^q yz\,dy\,dz\,dx$$

We aim to show that the right integral equals the left integral. If this is proven, then we deduce our desired result immediately. Examine the right integral above. Perform the change of variable $y' = -y$ to obtain that

$$\int_{-1}^{1}\int_{-c}^{c}\int_{-b}^{0} |x+y+z|^q yz\,dy\,dz\,dx = \int_{-1}^{1}\int_{-c}^{c}\int_{b}^{0} |x-y+z|^q (-y)z(-dy)\,dz\,dx$$

Flip the order of integration on the b integral to find that this is

$$= -\int_{-1}^{1}\int_{-c}^{c}\int_{0}^{b} |x-y+z|^q yz\,dy\,dz\,dx$$

Flip the order of integration twice to find that this equals

$$= -\int_{1}^{-1}\int_{c}^{-c}\int_{0}^{b} |x-y+z|^q yz\,dy\,dz\,dx$$

We now perform the change of variables $x \to -x$ and $z \to -z$ to find that

$$= -\int_{-1}^{1}\int_{-c}^{c}\int_{0}^{b} |-x-y-z|^q y(-z)dydzdx = \int_{-1}^{1}\int_{-c}^{c}\int_{0}^{b} |-x-y-z|^q yzdydzdx$$

This is exactly what we wanted to show and hence H is of our desired form. $\square$

Remark. *By the same argument with b and c reversed (and Fubini's theorem) we have that*

$$H(b,c) = \int_{-1}^{1}\int_{-b}^{b}\int_{0}^{c} |x+y+z|^q yzdydzdx$$

As stated earlier, we aim to show that $\partial_b H(b,c) \geqslant 0$ and $\partial_c H(b,c) \geqslant 0$ when $0 \leqslant b \leqslant c$. We now compute $\partial_b H(b,c)$ and $\partial_c H(b,c)$. Note that by the Fundamental Theorem of Calculus, we have that

$$\partial_b H(b,c) = b\int_{-1}^{1}\int_{-c}^{c} |x+b+z|^q zdzdx$$

and

$$\partial_c H(b,c) = c\int_{-1}^{1}\int_{-b}^{b} |x+c+z|^q zdzdx$$

Since $b \geqslant 0$, we have that $\partial_b H(b,c)$ has the same sign as $P_b(b,c) := \int_{-1}^{1}\int_{-c}^{c} |x+b+z|^q zdzdx$ and similarly $P_c(b,c) := \int_{-1}^{1}\int_{-b}^{b} |x+c+z|^q\, zdzdx$ has the same sign as $\partial_c H(b,c)$. We will show $\partial_b H(b,c) \geqslant 0$ by showing that $P_b(b,c) \geqslant 0$, and we will show that $\partial_c H(b,c) \geqslant 0$ after with a different argument. We begin by giving a closed form for P_b and P_c. Integrating with respect to x first, we find that

$$P_b(b,c) = \int_{-c}^{c} |1+b+z|^{q+1}\mathrm{Sign}\,(1+b+z)\,z - |-1+b+z|^{q+1}\mathrm{Sign}\,(-1+b+z)\,zdz \tag{5.1}$$

If we rewrite $z = (1+b+z)-(1+b)$, note that since $\mathrm{Sign}\,(x)\,x = |x|$, we have that

$$\begin{aligned}
&|1+b+z|^{q+1}\mathrm{Sign}\,(1+b+z)\,z\\
&= |1+b+z|^{q+1}\mathrm{Sign}\,(1+b+z)\,(1+b+z) - |1+b+z|^{q+1}\mathrm{Sign}\,(1+b+z)\,(1+b)\\
&= |1+b+z|^{q+1}|1+b+z| - |1+b+z|^{q+1}\mathrm{Sign}\,(1+b+z)\,(1+b)\\
&= |1+b+z|^{q+2} - |1+b+z|^{q+1}\mathrm{Sign}\,(1+b+z)\,(1+b)
\end{aligned}$$

We can also note that $|1+b+z|^{p+1}\mathrm{Sign}\,(1+b+z) = |1+b+z|^p(1+b+z)$. This suggests that we write our integral above in Equation 5.1 as

$$\int_{-c}^{c} |1+b+z|^{q+2} - |1+b+z|^q(1+b+z)(1+b) - |-1+b+z|^{q+2} \\ + |-1+b+z|^q(-1+b+z)(-1+b)dz$$

Each of these integrals can be computed. Doing so, we obtain the following. There are four terms to be integrated and the end result for each is written in order below.

$$\frac{|1+b+c|^{q+3}\mathrm{Sign}\,(1+b+c) - |1+b-c|^{q+3}\mathrm{Sign}\,(1+b-c)}{q+3}$$

$$-\frac{|1+b+c|^{q+2} - |-c+1+b|^{q+2}}{q+2}(1+b)$$

$$-\frac{|-1+b+c|^{q+3}\mathrm{Sign}\,(-1+b+c) - |-1+b-c|^{q+3}\mathrm{Sign}\,(-1+b-c)}{q+3}$$

$$\frac{|-1+b+c|^{q+2}-|c+1-b|^{q+2}}{q+2}(-1+b)$$

Summing these, we obtain a closed form expression for $P_b(b,c)$. Interchanging b and c in the 4 terms above gives us $P_cH(b,c)$. We now perform a change of variables. Denote $k=b+c$ and $s=b-c$. By assumption, $-k \leqslant s \leqslant 0 \leqslant k$. Then our terms for $P_b(b,c)$ becomes

$$\begin{aligned} Q(k,s) := {} & \frac{|1+k|^{q+3}\mathrm{Sign}\,(1+k)-|1+s|^{q+3}\mathrm{Sign}\,(1+s)}{q+3} \\ & - \frac{|1+k|^{q+2}-|1+s|^{q+2}}{q+2}(1+(k+s)/2) \\ & - \frac{|-1+k|^{q+3}\mathrm{Sign}\,(-1+k)-|-1+s|^{q+3}\mathrm{Sign}\,(-1+s)}{q+3} \\ & + \frac{|-1+k|^{q+2}-|1-s|^{q+2}}{q+2}(-1+(k+s)/2) \end{aligned}$$

Then $Q(b+c,b-c)=P_b(b,c)$. In particular, to show that $\partial_b H(b,c) \geqslant 0$ when $0 \leqslant b \leqslant c$, we need to show that $P_b(b,c) \geqslant 0$, and hence $Q(k,s) \geqslant 0$ when $k \geqslant 0$ and $0 \leqslant s \leqslant k$.

Repeating the same computation for $P_c(b,c)$, we find that $P_c(b,c)=Q(b+c,c-b)$. In particular, to show that $\partial_c H(b,c) \geqslant 0$ when $0 \leqslant b \leqslant c$, we need that $P_c(b,c) \geqslant 0$ and hence $Q(k,-s) \geqslant 0$ when $k \geqslant 0$ and $-k \leqslant s \leqslant 0 \leqslant k$.

For fixed s and k, it turns out that we only need to show that $Q(k,|s|) \geqslant 0$ because $Q(k,s) \geqslant Q(k,|s|)$. We first show this, and then show that $Q(k,s) \geqslant 0$ for all $0 \leqslant s \leqslant k$ and conclude that $\partial_b H(b,c)$ and $\partial_c H(b,c)$ are each non-negative when $0 \leqslant b \leqslant c$. Then by our symmetry lemma, Lemma 5.7, we conclude that $\partial_b H$ and $\partial_c H$ are non-negative.

Claim. *$Q(k,-s) \geqslant Q(k,s)$ when $0 \leqslant s \leqslant k$.*

Proof. Define $Z(k,s) := (q+2)\left(Q(k,-s)-Q(k,s)\right)$. We show that $Z(k,s) \geqslant 0$. Note that $Z(0,0)=0$ so we assume that $k>0$. By computation we have that

$$\begin{aligned} Z(k,s) &= s\left(|k+1|^{q+2}-|1-k|^{q+2}\right)-k\left(|s+1|^{q+2}-|1-s|^{q+2}\right) \\ &= s\,|k+1|^{q+2}-k\,|s+1|^{q+2}+k\,|1-s|^{q+2}-s\,|1-k|^{q+2} \end{aligned}$$

Note that $Z(k,0)=Z(k,k)=0$. Computing we also find that

$$\partial_s Z(k,s) = |k+1|^{q+2}-|1-k|^{q+2}-(q+2)k\left(|s+1|^{q+1}+|1-s|^{q+1}\mathrm{Sign}\,(1-s)\right)$$

Note that $\partial_s Z(k,0)=|k+1|^{q+2}-|1-k|^{q+2}>0$. In particular there must exist some $0<c<k$ with $Z(k,c)>0$.

$$\partial_{ss} Z(k,s) = -(q+2)(q+1)k(|s+1|^q-|1-s|^q)$$

Since $|\cdot|^q$ is an increasing function, we have that $|s+1|^q-|1-s|^q \geqslant 0$ and hence $\partial_{ss}Z(k,s) \leqslant 0$. In particular, $\partial_s Z(k,s)$ as a function of s is decreasing.

Since $Z(k,0)=Z(k,k)=0$, there exists $0<c<k$ with $Z(k,c)>0$, and $\partial_s Z(k,s)$ is decreasing, we must have that $Z(k,s) \geqslant 0$ for all $k \geqslant 0$ and $0 \leqslant s \leqslant k$.

In particular, if $Q(k,s) \geqslant 0$ then $Q(k,-s) \geqslant 0$. □

Thus we simply need to show that $Q(k,s) \geqslant 0$ when $0 \leqslant s \leqslant k$.

Lemma 5.8. *$Q(k,s) \geqslant 0$ when $0 \leqslant s \leqslant k$.*

Proof. Note that $Q(0,0)=0$, so we can assume $k>0$. We start be rearranging terms in $Q(k,s)$ as follows. Define the following functions

$$G(k)=\begin{cases}\frac{(1+k)^{q+3}+(1-k)^{q+3}}{q+3}-\frac{(1+k)^{q+2}+(1-k)^{q+2}}{q+2} & 0<k<1\\ \frac{(1+k)^{q+3}-(-1+k)^{q+3}}{q+3}-\frac{(1+k)^{q+2}+(-1+k)^{q+2}}{q+2} & k\geqslant 1\end{cases}$$

$$F(k)=\begin{cases}\frac{(1-k)^{q+2}-(1+k)^{q+2}}{q+2} & 0<k<1\\ \frac{(k-1)^{q+2}-(1+k)^{q+2}}{q+2} & k\geqslant 1\end{cases}$$

Then we have that

$$Q(k,s)=G(k)-G(s)+\frac{k+s}{2}(F(k)-F(s))$$

Now we need to show that $G(k)-G(s)+\frac{k+s}{2}(F(k)-F(s))$ is non-negative. The technique is essentially the same as in the claim preceding this lemma, using computations when $s=0$ and $s=k$ and derivative arguments to conclude. However, the terms involved are much longer. For posterity we compute the first and second derivatives of F and G now.

$$G'(k)=\begin{cases}(1+k)^{q+2}-(1-k)^{q+2}-(1+k)^{q+1}+(1-k)^{q+1} & 0<k<1\\ (1+k)^{q+2}-(-1+k)^{q+2}-(1+k)^{q+1}-(k-1)^{q+1} & k\geqslant 1\end{cases}$$

$$G''(k)=\begin{cases}(q+2)((1+k)^{q+1}+(1-k)^{q+1})-(q+1)((1+k)^{q}+(1-k)^{q}) & 0<k<1\\ (q+2)((1+k)^{q+1}-(-1+k)^{q+1})-(q+1)((1+k)^{q}+(-1+k)^{q}) & k\geqslant 1\end{cases}$$

$$F'(k)=\begin{cases}-(1-k)^{q+1}-(1+k)^{q+1} & 0<k<1\\ (k-1)^{q+1}-(1+k)^{q+1} & k\geqslant 1\end{cases}$$

$$F''(k)=\begin{cases}(q+1)((1-k)^{q}-(1+k)^{q}) & 0<k<1\\ (q+1)((k-1)^{q}-(1+k)^{q}) & k\geqslant 1\end{cases}$$

Claim. *$Q(k,k)=0$ and $Q(k,0)\geqslant 0$.*

Proof. If $k=0$ this is trivial so suppose $k>0$. Then we compute

$$Q(k,k)=G(k)-G(k)+\frac{k+k}{2}(F(k)-F(k))=0$$

Then, computing we have that

$$G(0)=\frac{2}{q+3}-\frac{2}{q+2}$$
$$F(0)=0$$

and hence

$$\begin{aligned}Q(k,0)&=G(k)-G(0)+\frac{k}{2}(F(k)-F(0))\\ &=G(k)+\frac{k}{2}F(k)-\frac{2}{q+3}+\frac{2}{q+2}\end{aligned}$$

Computing the derivative, we find that

$$\partial_k Q(k,0)=G'(k)+\frac{F(k)}{2}+\frac{k}{2}F'(k)$$

$$= \begin{cases} \frac{(kq+k+1)(1-k)^{q+1}+(kq+k-1)(k+1)^{q+1}}{2(q+2)} & k < 1 \\ \frac{-(kq+k+1)(k-1)^{q+1}+(kq+k-1)(k+1)^{q+1}}{2(q+2)} & k \geqslant 1 \end{cases}$$

Plugging in $k = 0$, we have that

$$\partial_k Q(0,0) = 0$$

Computing the second derivative, we have that

$$\partial_{kk} Q(k,0) = G''(k) + \frac{F'(k)}{2} + \frac{k}{2}F''(k) + \frac{F'(k)}{2} = G''(k) + F'(k) + \frac{k}{2}F''(k)$$
$$= \begin{cases} -\frac{1}{2}k(q+1)\left((1-k)^q - (k+1)^q\right) & k < 1 \\ -\frac{1}{2}k(q+1)\left((k-1)^q - (k+1)^q\right) & k \geqslant 1 \end{cases}$$

In both of these cases, $\partial_{kk}Q(k,0) \geqslant 0$. Since $\partial_k Q(0,0) = 0$, we obtain that $\partial_k Q(k,0) \geqslant 0$ for all k. Finally, since $Q(0,0) = 0$, we have that $Q(k,0) \geqslant 0$ for all $k \geqslant 0$ by the Fundamental Theorem of Calculus. $\square$

With this claim proven, we have established our necessary boundary conditions. To show that $Q(k,s) \geqslant 0$ for all $0 \leqslant s \leqslant k$, we need to show that $\partial_s Q(k,s) \leqslant 0$ for all $0 \leqslant s \leqslant k$.

In particular if we can show that the derivative of this function with respect to s is non-positive, then we immediately obtain that our original function is non-negative. Since $Q(k,s) = G(k) - G(s) + \frac{k+s}{2}(F(k) - F(s))$, we have that

$$\partial_s Q(k,s) = -G'(s) + \frac{F(k)}{2} - \frac{k+s}{2}F'(s) - \frac{F(s)}{2} = -G'(s) - \frac{k+s}{2}F'(s) + \frac{1}{2}(F(k) - F(s))$$

We now compute some boundary values for $\partial_s Q(k,s)$.

Claim. $\partial_s Q(k,0) \leqslant \partial_s Q(k,k) = 0$

Proof. By an easy computation we have that

$$\begin{aligned} \partial_s Q(k,k) &= -G'(k) - kF'(k) + \frac{1}{2}(F(k) - F(k)) \\ &= -G'(k) - kF'(k) \\ &= 0 \end{aligned}$$

Note that $G'(0) = 0$, $F'(0) = -2$ and $F(0) = 0$. Thus

$$\partial_s Q(k,0) = k + \frac{1}{2}F(k)$$

When $k = 0$ then we obtain that $\partial_s Q(0,0) = 0$. Taking a derivative with respect to k we get that

$$\partial_k \partial_s Q(k,0) = 1 + \frac{1}{2}F'(k)$$

Similarly we have that $\partial_k \partial_s Q(0,0) = 0$ Taking one final partial we have that

$$\partial_{kk}\partial_s Q(k,0) = \frac{1}{2}F''(k)$$

Recall that

$$F''(k) = \begin{cases} (q+1)((1-k)^q - (1+k)^q) & 0 < k < 1 \\ (q+1)((k-1)^q - (1+k)^q) & k \geqslant 1 \end{cases}$$

Since $(\cdot)^q$ is increasing we have that $F'' \leqslant 0$ as desired. In particular, we deduce that $\partial_{kk}\partial_s Q(k,0) \leqslant 0$. Since $\partial_k \partial_s Q(0,0) = 0$ we have that $\partial_k \partial_s Q(k,0) \leqslant 0$ then. This proves the desired result. $\square$

Since our boundary values for $\partial_s Q$ are non-positive, to show that $\partial_s Q$ is non-positive, it suffices to show that $\partial_{ss} Q$ is non-negative.

Claim. $\partial_{ss}Q(k,s) \geqslant 0$ *for all* $0 \leqslant s \leqslant k$.

Proof. Recall that

$$\partial_s Q(k,s) = -G'(s) - \frac{k+s}{2}F'(s) + \frac{1}{2}(F(k) - F(s))$$

Computing another partial derivative we find that

$$\partial_{ss}Q(k,s) = -G''(s) - \frac{k+s}{2}F''(s) - \frac{1}{2}F'(s) - \frac{1}{2}F'(s) = -G''(s) - \frac{k+s}{2}F''(s) - F'(s)$$

As shown in the previous claim, $F'' \leqslant 0$. Since $s \leqslant k$ we have then that

$$\partial_{ss}Q(k,s) \geqslant -G''(s) - sF''(s) - F'(s)$$

Recall that

$$G''(k) = \begin{cases} (q+2)((1+k)^{q+1} + (1-k)^{q+1}) - (q+1)((1+k)^q + (1-k)^q) & 0 < k < 1 \\ (q+2)((1+k)^{q+1} - (-1+k)^{q+1}) - (q+1)((1+k)^q + (-1+k)^q) & k \geqslant 1 \end{cases}$$

$$F'(k) = \begin{cases} -(1-k)^{q+1} - (1+k)^{q+1} & 0 < k < 1 \\ (k-1)^{q+1} - (1+k)^{q+1} & k \geqslant 1 \end{cases}$$

$$F''(k) = \begin{cases} (q+1)((1-k)^q - (1+k)^q) & 0 < k < 1 \\ (q+1)((k-1)^q - (1+k)^q) & k \geqslant 1 \end{cases}$$

Computing we find that

$$-G''(s) - sF''(s) - F'(s) = 0$$

In particular we deduce that $\partial_{ss}Q(k,s) \geqslant 0$ as desired. $\square$

Since $\partial_{ss}Q(k,s) \geqslant 0$ for all $0 \leqslant s \leqslant k$ and $\partial_s Q(k,s) \leqslant 0$ when $s = 0$ or $s = k$, we find that $\partial_s Q(k,s) \leqslant 0$ for all $0 \leqslant s \leqslant k$. Since $Q(k,k) = 0$ and $Q(k,0) \geqslant 0$ we deduce that $Q(k,s) \geqslant 0$ for all $0 \leqslant s \leqslant k$ as desired, completing the proof. $\square$

By reversing all of our steps, we find that

$$f(a,b,c) = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} |ax+by+cz|^p yz\,dy\,dz\,dx$$

satisfies $f(a,b,c) \geqslant 0$ when $a,b,c \geqslant 0$. In particular, we have shown that Lemma 5.1 holds, and hence our main Theorem 5.2 is true.

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