# Proofs as Programs

## Exercises in Type Level Programming

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Kosice, 16 Nov. 2023

## About the project

The **Modern Approaches and Tools for Teaching Classes at the University Level in Theoretical Computer Science Courses of Logic, Types, and Semantics** initiative no: **FBR-PDI-025** is funded by Iceland, Liechtenstein and Norway through the EEA Grants and Norway Grants.

Iceland Liechtenstein Norway grants

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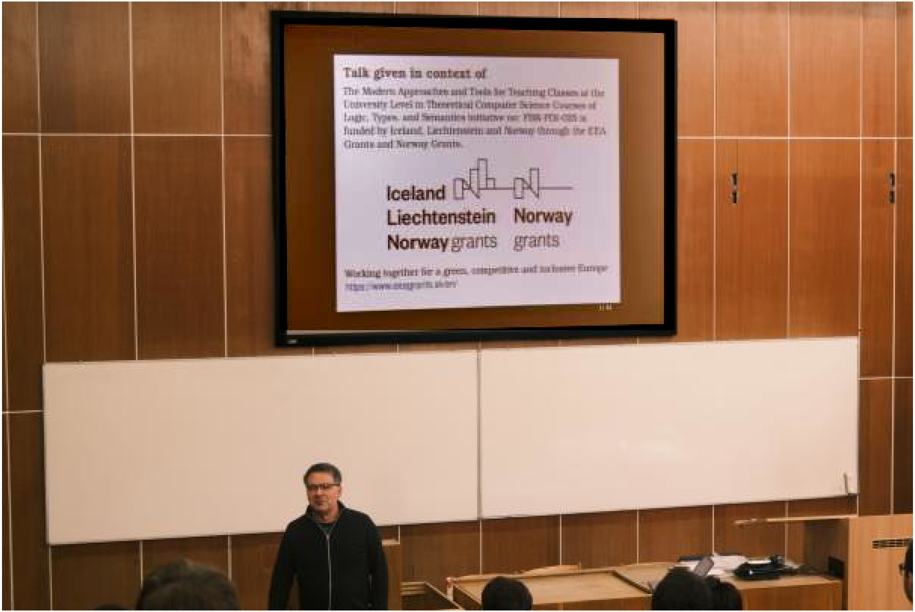
Talk given in context of
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## Agenda

- Propositions as types, proofs as programs.
- Exercises.
- What are dependent types.
- How can we simulate dependent types in Haskell.
- Exercises.
- Discussion.

# The Curry-Howard-Lambek Correspondence

## From Wikipedia

- Curry 1934 : Types look like axiom schemes for intuitionistic logic.
- Curry 1958 : Hilbert style deduction systems coincide with the typed fragment of combinatory logic.
- Howard 1969 : Natural deduction can be directly interpreted in a typed lambda calculus.
- Lambek 2005 : Lambda calculi are structurally equivalent to Cartesian closed categories.

# Propositions as types, Proofs as Programs (Exercises)

`https://github.com/jtkristensen/exercises-in-type-level-programming/tree/main/tuke-2023/ProofsAsPrograms`

# Lambda Cube ($\lambda_{\to}$)

$$t ::= x \mid \lambda x : \tau.\ t \mid t_1\ t_2$$
$$\tau ::= \tau_1 \to \tau_2$$

$\boxed{\Gamma \vdash t : \tau}$, for closed terms $t$:

$$\text{Var}_t : \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \text{Abs}_{t \to t} : \frac{\Gamma[x \mapsto \tau_1] \vdash t : \tau_2}{\Gamma \vdash \lambda x : \tau_1.\ t : \tau_1 \to \tau_2}$$

$$\text{App}_{tt} : \frac{\Gamma \vdash t_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash t_2 : \tau_1}{\Gamma \vdash t_1\ t_2 : \tau_2}$$

# Lambda Cube ($\lambda_2$)

$$t ::= x \mid \lambda x : \tau.\ t \mid \Lambda X.t \mid t_1\ t_2 \mid t[\tau]$$
$$\tau ::= X \mid \forall X.\tau \mid \tau_1 \to \tau_2$$

$\boxed{\Gamma \vdash t : \tau}$ , for closed terms $t$:

$$\text{Var}_t : \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \text{Abs}_{t\to t} : \frac{\Gamma[x \mapsto \tau_1] \vdash t : \tau_2}{\Gamma \vdash \lambda x : \tau_1.\ t : \tau_1 \to \tau_2}$$

$$\text{App}_{tt} : \frac{\Gamma \vdash t_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash t_2 : \tau_1}{\Gamma \vdash t_1\ t_2 : \tau_2}$$

$\boxed{\Gamma \vdash t : \tau}$ , for closed terms $t$:

$$\text{Abs}_{\tau\to t} : \frac{\Gamma \vdash t : \tau}{\Gamma \vdash \Lambda X.t : \forall X.\tau} \qquad \text{App}_{t\tau} : \frac{\Gamma \vdash t : \forall X.\tau_1}{\Gamma \vdash t[\tau_2] : \tau_1[\tau_2/X]}$$

# Lambda Cube ($\lambda_\omega$)

$$t ::= x \mid \lambda x : \tau.\ t \mid t_1 t_2$$

$$\tau ::= X \mid \lambda X : \kappa.\ \tau \mid \tau_1 \to \tau_2 \mid \tau_1 \tau_2$$

$$\kappa ::= * \mid \kappa_1 \to \kappa_2$$

$\boxed{\Gamma \vdash t : \tau}$, for closed terms $t$:

$$\text{Var}_t : \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \text{Abs}_{t \to t} : \frac{\Gamma[x \mapsto \tau_1] \vdash t : \tau_2}{\Gamma \vdash \lambda x : \tau_1.\ t : \tau_1 \to \tau_2}$$

$$\text{App}_{tt} : \frac{\Gamma \vdash t_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash t_2 : \tau_1}{\Gamma \vdash t_1\ t_2 : \tau_2}$$

repeat $\text{Var}_t$, $\text{Abs}_{t \to t}$ and $\text{App}_{tt}$ to get $\text{Var}_\tau$, $\text{Abs}_{\tau \to \tau}$ and $\text{App}_{\tau\tau}$.

# Lambda Cube ($\lambda_\Pi$)

$$t ::= x \mid \lambda x : \tau.\ B \mid t_1 t_2$$

$$\tau ::= \Pi x : A.\ B \mid \tau\ t$$

$$\kappa ::= * \mid \Pi x : \tau.\ \kappa$$

$\boxed{\Gamma \vdash t : \tau}$, for closed terms $t$:

$$\mathrm{Var}_t : \frac{\Gamma(x) = \tau}{\Gamma \vdash x : \tau} \qquad \mathrm{Abs}_{t \to t} : \frac{\Gamma[x \mapsto \tau_1] \vdash t : \tau_2}{\Gamma \vdash \lambda x : \tau_1.\ t : \tau_1 \to \tau_2}$$

$$\mathrm{App}_{tt} : \frac{\Gamma \vdash t_1 : \tau_1 \to \tau_2 \quad \Gamma \vdash t_2 : \tau_1}{\Gamma \vdash t_1\ t_2 : \tau_2}$$

$\boxed{\Gamma \vdash t : \tau}$, for closed terms $t$:

$$\mathrm{Abs}_{t \to \tau} : \frac{\Gamma \vdash A : \tau \quad \Gamma[x \mapsto A] \vdash B : \sigma \quad \Gamma[x \mapsto A] \vdash B' : B}{\Gamma \vdash \lambda x : A.\ B' \ : \Pi x : A.\ B} \quad (\sigma \in \{\kappa, \tau\})$$

$$\mathrm{App}_{\tau t} : \frac{\Gamma \vdash t_1 : \Pi x : \tau.\ B \quad \Gamma \vdash t_2 : \tau}{\Gamma \vdash t_1\ t_2 : B[t_2/x]}$$

# Simulating Dependent types in Haskell (Exercises)

`https://github.com/jtkristensen/exercises-in-type-level-programming/tree/main/tuke-2023/NatProperties`

Discussion