Internal MMR

External MMR

Equations of Motion

Internal MMR

$$\lambda \longleftrightarrow \Lambda = \sqrt{a_1/a_p}$$
$$\gamma = -\varpi \longleftrightarrow \Gamma = \Lambda(1-\sqrt{1-e^2}) \approx \Lambda\frac{e^2}{2}$$

$$\theta_{\mathrm{p}} \equiv (j+1)\tau - j\lambda$$
$$\theta \equiv (j+1)\tau - j\lambda + \gamma$$

$$\omega_0 = n_p$$
$$H_0 = \frac{GM}{a_p}$$

$$\frac{\dot{a}}{a} = \frac{1}{T_m} - \frac{2e^2}{T_e}$$
$$\frac{\dot{e}}{e} = -\frac{1}{T_e}$$

$$A = \frac{1}{2}[2(j+1) + \alpha D]b_{1/2}^{(j+1)}(\alpha) \approx 2.0$$
$$B = -\frac{1}{2}[-1 + 2(j+1) + \alpha D]b_{1/2}^{(j)}(\alpha) \approx -2.5$$
$$C = \frac{1}{8}[2\alpha D + \alpha^2 D^2]b_{1/2}^{(0)}(\alpha) \approx 1.15$$
$$D = \frac{1}{4}[2 - 2\alpha D - \alpha^2 D^2]b_{1/2}^{(1)}(\alpha) \approx -2.0$$

External MMR

$$\lambda \longleftrightarrow \Lambda = \sqrt{a_1/a_p}$$
$$\gamma = -\varpi \longleftrightarrow \Gamma = \Lambda(1-\sqrt{1-e^2}) \approx \Lambda\frac{e^2}{2}$$

$$\theta_{\mathrm{p}} \equiv (j+1)\tau - j\lambda$$
$$\theta \equiv (j+1)\tau - j\lambda + \gamma$$

$$\omega_0 = n_p$$
$$H_0 = \frac{GM}{a_p}$$

$$\frac{\dot{a}}{a} = -\frac{1}{T_m} - \frac{2e^2}{T_e}$$
$$\frac{\dot{e}}{e} = -\frac{1}{T_e}$$

$$A = \frac{1}{2}\alpha[-1 + 2(j+1) + \alpha D]b_{1/2}^{(j)}(\alpha) \approx 1.9$$
$$B = -\frac{1}{2}\alpha[2(j+1) + \alpha D]b_{1/2}^{(j+1)}(\alpha) \approx -1.5$$
$$C = \frac{1}{8}\alpha[2\alpha D + \alpha^2 D^2]b_{1/2}^{(0)}(\alpha) \approx 0.9$$
$$D = \frac{1}{4}\alpha[2 - 2\alpha D - \alpha^2 D^2]b_{1/2}^{(1)}(\alpha) \approx -1.5$$

$$H = -\frac{1}{2\Lambda^2} + \mu_p \left(A \left(\frac{2\Gamma}{\Lambda} \right)^{1/2} \cos\theta + B e_p \cos\theta_p \right)$$
$$- \mu_p \left(C \left(\frac{2\Gamma}{\Lambda} + e_p^2 \right) + D e_p \sqrt{\frac{2\Gamma}{\Lambda}} \cos\gamma \right)$$

$$\dot{\lambda} = \frac{1}{\Lambda^3} + \mu_p \left(-A \sqrt{\frac{\Gamma}{2\Lambda^3}} \cos\theta + C \frac{2\Gamma}{\Lambda^2} + D e_p \sqrt{\frac{\Gamma}{2\Lambda^3}} \cos\gamma \right)$$
$$\dot{\Lambda} = -\mu_p \left(A j \sqrt{\frac{2\Gamma}{\Lambda}} \sin\theta + B j e_p \sin\theta_p \right) + \frac{\Lambda}{2} \left(\frac{1}{T_m} - \frac{4\Gamma}{\Lambda T_e} \right)$$
$$\dot{\gamma} = \mu_p \left(\frac{A\cos\theta}{\sqrt{2\Gamma\Lambda}} - \frac{2C}{\Lambda} - D e_p \frac{\cos\gamma}{\sqrt{2\Gamma\Lambda}} \right)$$
$$\dot{\Gamma} = \mu_p \left(A \sqrt{\frac{2\Gamma}{\Lambda}} \sin\theta - D e_p \sqrt{\frac{2\Gamma}{\Lambda}} \sin\gamma \right)$$
$$- \frac{\Gamma}{\Lambda} \frac{\Lambda}{2} \left(\frac{1}{T_m} - \frac{4\Gamma}{\Lambda T_e} \right) - \frac{2\Gamma}{T_e}$$

$$H = -\frac{1}{2\Lambda^2} - \mu_p \left(A \left(\frac{2\Gamma}{\Lambda} \right)^{1/2} \cos\theta + B e_p \cos\theta_p \right)$$
$$- \mu_p \left(C \left(\frac{2\Gamma}{\Lambda} + e_p^2 \right) + D e_p \sqrt{\frac{2\Gamma}{\Lambda}} \cos\gamma \right)$$

$$\dot{\lambda} = \frac{1}{\Lambda^3} + \mu_p \left(A \sqrt{\frac{\Gamma}{2\Lambda^3}} \cos\theta + C \frac{2\Gamma}{\Lambda^2} + D e_p \sqrt{\frac{\Gamma}{2\Lambda^3}} \cos\gamma \right)$$
$$\dot{\Lambda} = \mu_p \left(A j \sqrt{\frac{2\Gamma}{\Lambda}} \sin\theta B j e_p \sin\theta_p \right) + \frac{\Lambda}{2} \left(-\frac{1}{T_m} - \frac{4\Gamma}{\Lambda T_e} \right)$$
$$\dot{\gamma} = \mu_p \left(\frac{-A\cos\theta}{\sqrt{2\Gamma\Lambda}} - \frac{2C}{\Lambda} - D e_p \frac{\cos\gamma}{\sqrt{2\Gamma\Lambda}} \right)$$
$$\dot{\Gamma} = \mu_p \left(-A \sqrt{\frac{2\Gamma}{\Lambda}} \sin\theta - D e_p \sqrt{\frac{2\Gamma}{\Lambda}} \sin\gamma \right)$$
$$- \frac{\Gamma}{\Lambda} \frac{\Lambda}{2} \left(-\frac{1}{T_m} - \frac{4\Gamma}{\Lambda T_e} \right) - \frac{2\Gamma}{T_e}$$

Shifted Hamiltonian

$$\overline{\Gamma} = \Gamma + \frac{\sqrt{\Lambda_0}B}{A} e_p \sqrt{2\Gamma}\cos\gamma + \frac{\Lambda_0 B^2}{A^2} e_p^2$$
$$\overline{\gamma} = \tan^{-1}\left(\frac{e\sin\gamma}{e\cos\gamma + Be_p/A}\right)$$
$$\overline{e}^2 = e^2 + \frac{2B}{A} e_p e\cos\gamma + \frac{B^2}{A^2} e_p^2$$
$$\overline{\theta} = (j+1)\lambda_p - j\lambda + \overline{\gamma}$$
$$\overline{H} = -\frac{(GM)^2}{2\Lambda^2} + \frac{Gm_p}{a_p} A\sqrt{\frac{2\overline{\Gamma}}{\Lambda}}\cos\overline{\theta}$$

$$\overline{\Gamma} = \Gamma + \frac{\sqrt{\Lambda_0}B}{A} e_p \sqrt{2\Gamma}\cos\gamma + \frac{\Lambda_0 B^2}{A^2} e_p^2$$
$$\overline{\gamma} = \tan^{-1}\left(\frac{e\sin\gamma}{e\cos\gamma + Be_p/A}\right)$$
$$\overline{e}^2 = e^2 + \frac{2B}{A} e_p e\cos\gamma + \frac{B^2}{A^2} e_p^2$$
$$\overline{\theta} = (j+1)\lambda - j\lambda_p + \overline{\gamma}$$
$$\overline{H} = -\frac{(GM)^2}{2\Lambda^2} - \frac{Gm_p}{a_p} \alpha A\sqrt{\frac{2\overline{\Gamma}}{\Lambda}}\cos\overline{\theta}$$

Reduced Hamiltonian

$$H = \eta R - R^2 + \sqrt{R}\cos\overline{\theta}$$
$$\alpha_0 = \frac{a}{a_p}(1 + j\overline{e}^2)$$
$$\eta = \frac{(j+1)\alpha^{3/2} - j}{3^{1/3}(\mu_p j\alpha_0 A)^{2/3}}$$
$$\tau = 3^{1/3}(\mu_p j\alpha_0 A)^{2/3} n_0 t$$
$$R = \overline{e}^2\left(\frac{3j^2}{8\mu_p\alpha_0 A}\right)^{2/3}$$

$$H = \eta R - R^2 - \sqrt{R}\cos\overline{\theta}$$
$$\alpha_0 = \frac{a_p}{a}(1 + (j+1)\overline{e}^2)$$
$$\eta = \frac{(j+1) - j\alpha^{-3/2}}{3^{1/3}(\mu_p (j+1) A)^{2/3}}$$
$$\tau = 3^{1/3}(\mu_p (j+1) A)^{2/3} n_0 t$$
$$R = \overline{e}^2\left(\frac{3(j+1)^2}{8\mu_p A}\right)^{2/3}$$

Capture and Stability (Circular)

$$\frac{\delta a}{a} \sim \frac{\delta n}{n} \sim \mu_p^{2/3}$$

$$\frac{\delta e}{e} \sim \mu_p^{1/3}$$

$$\mu_{\mathrm{cap}}^{4/3} \gg \frac{1}{T_m}$$

$$\mu_{\mathrm{escape}} < \frac{3j^2}{8\alpha_0 A} e_{\mathrm{eq}}^3 < \mu_{\text{finite lib.}} < \frac{3j}{\alpha_0 A} e_{\mathrm{eq}}^3 < \mu_{\mathrm{stable}}$$

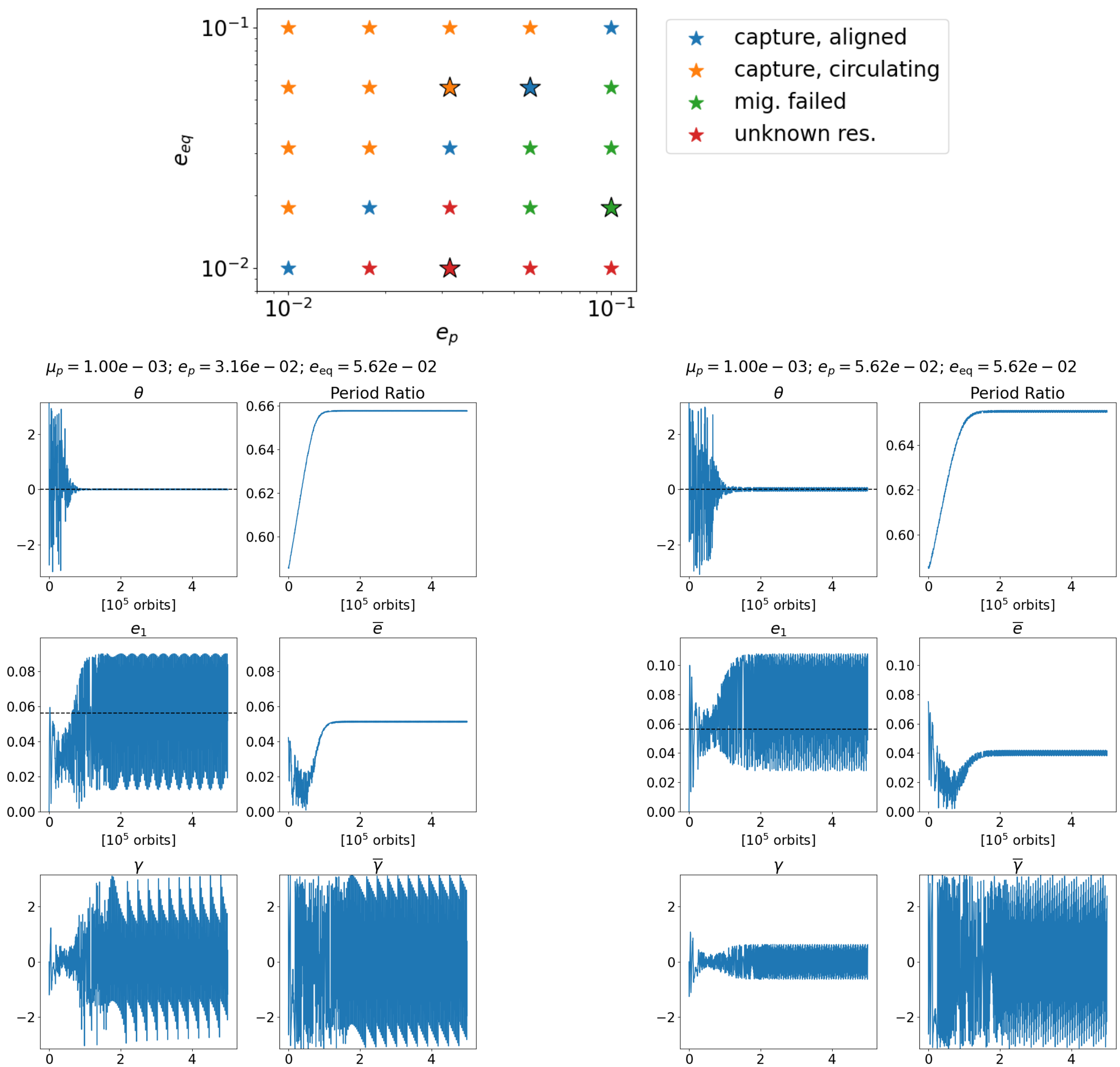
Always stable

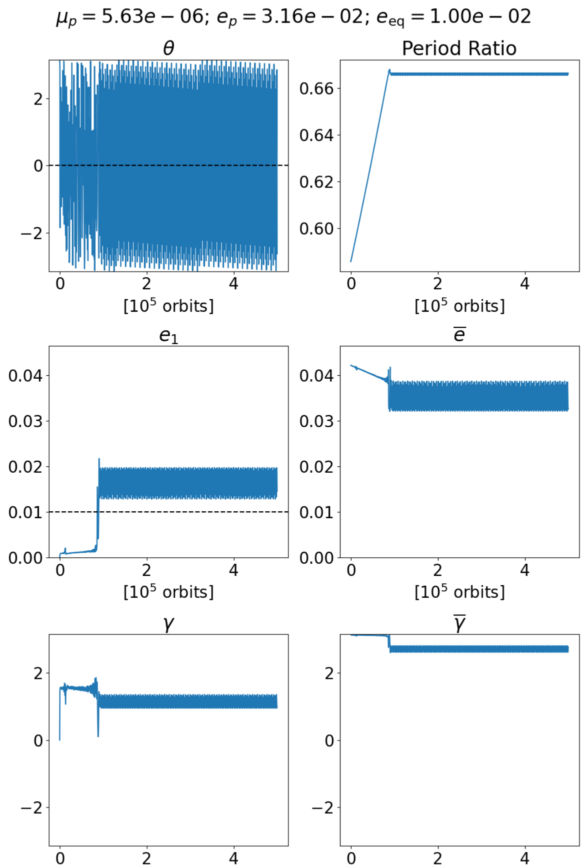
External Perturber Effects

$$H_{\gamma,\mathrm{sec}} = -\Gamma(\omega_{1,\mathrm{ext}} - \omega_{p,\mathrm{ext}}) \equiv -\Gamma\omega_{\mathrm{eff}}$$

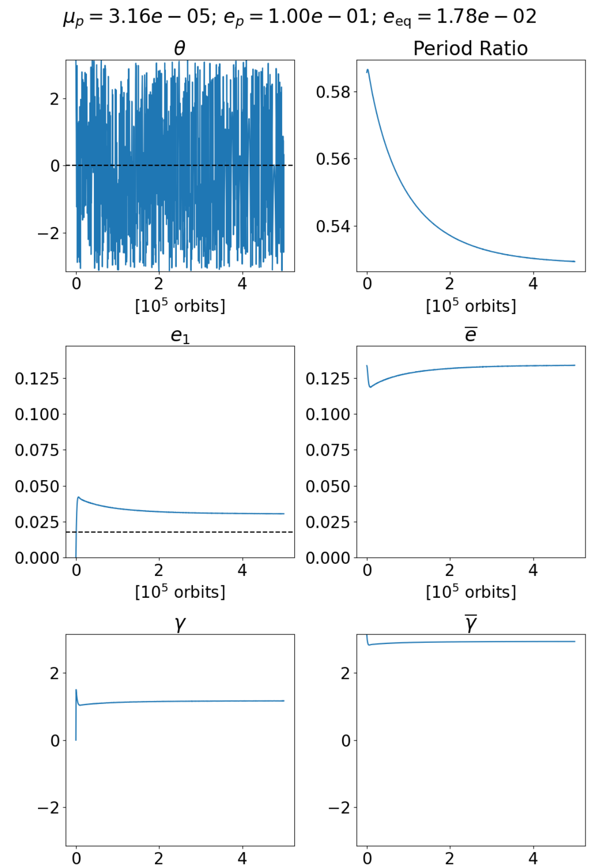
$$\begin{aligned} H_{\lambda,\mathrm{sec}} = &- \frac{GM}{2a_p}\mu_p b_{1/2}^{(0)}\left(\frac{a}{a_p}\right) - \frac{GM}{2a_{\mathrm{ext}}}\mu_{\mathrm{ext}} b_{1/2}^{(0)}\left(\frac{a}{a_{\mathrm{ext}}}\right) \\ &- \frac{GM}{2a_{\mathrm{ext}}}\mu_{\mathrm{ext}} b_{1/2}^{(0)}\left(\frac{a_p}{a_{\mathrm{ext}}}\right) \end{aligned}$$

Internal Apsidal Alignment



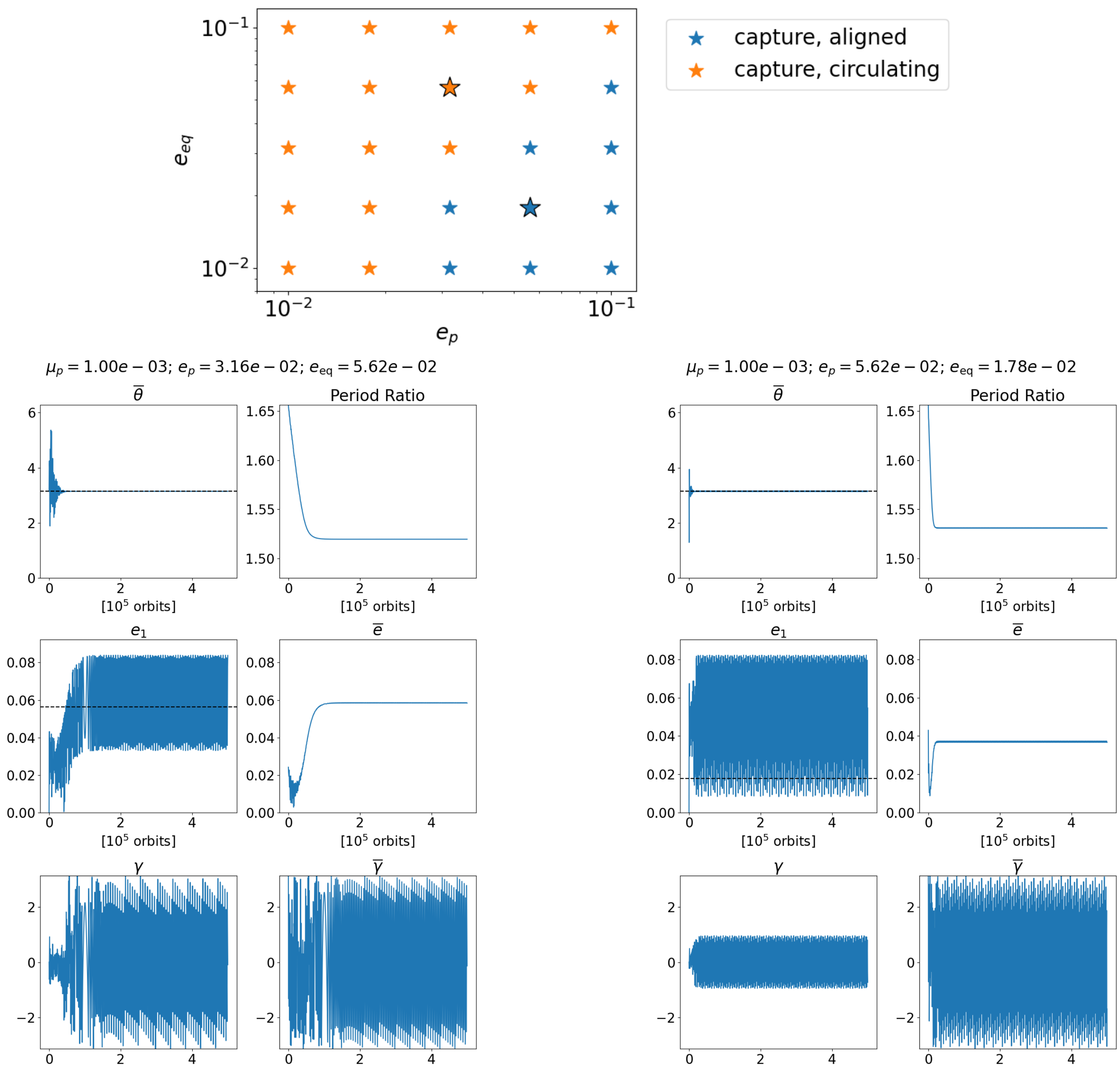


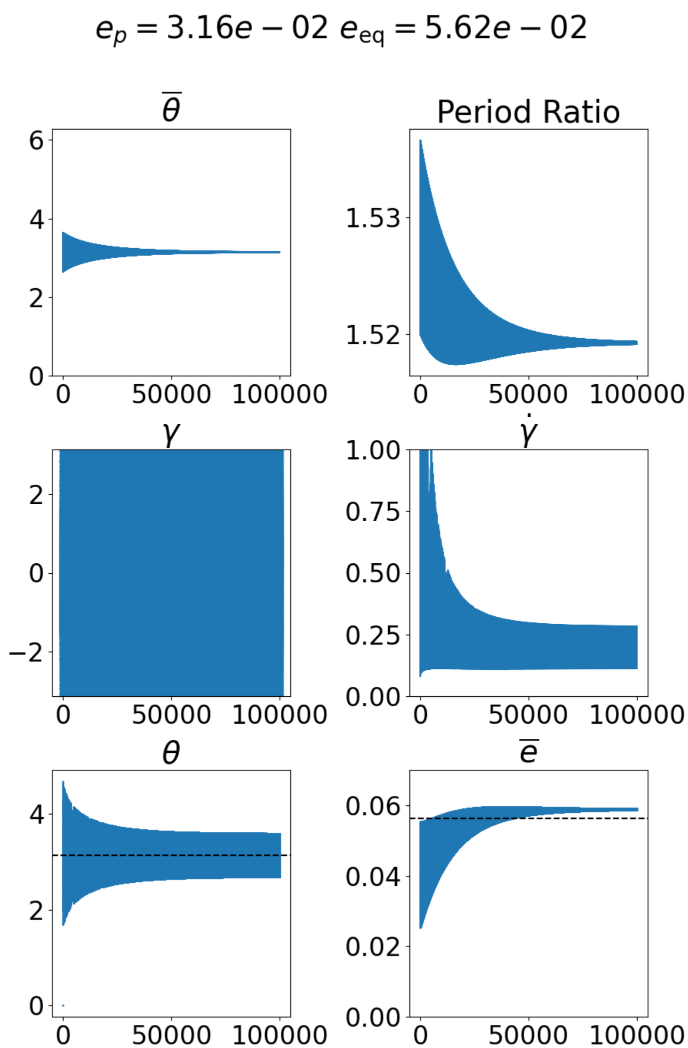
$\mu_p = 5.63e-06$; $e_p = 3.16e-02$; $e_{\rm eq} = 1.00e-02$
θ
Period Ratio
e_1
$\overline{e}$
γ
$\overline{\gamma}$
[10^5 orbits]

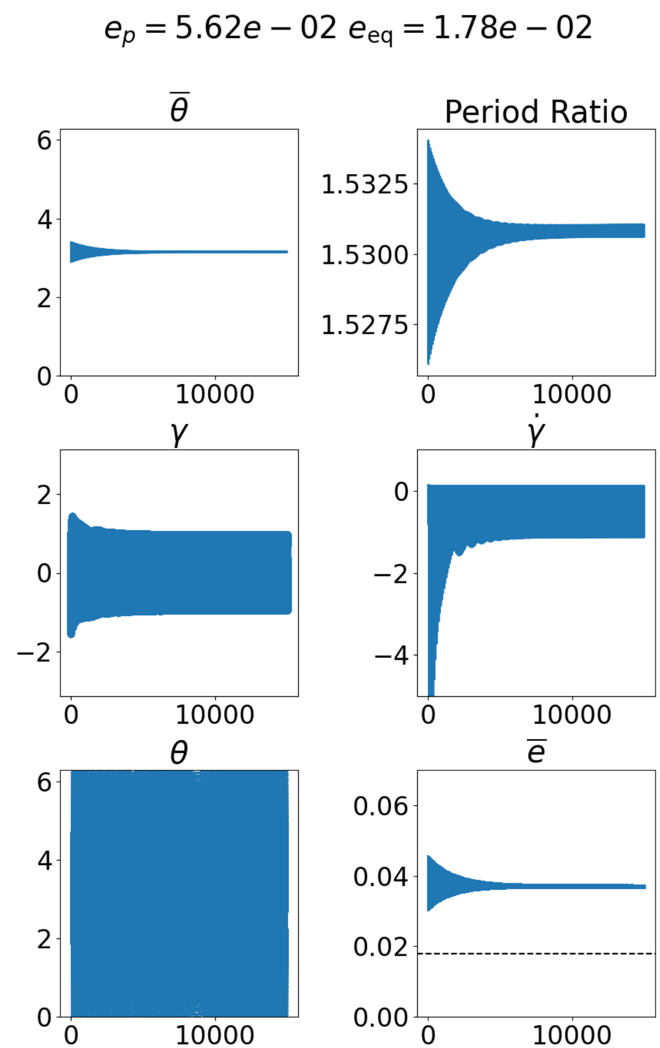


$\mu_p = 3.16e-05$; $e_p = 1.00e-01$; $e_{\rm eq} = 1.78e-02$
θ
Period Ratio
e_1
$\overline{e}$
γ
$\overline{\gamma}$
[10^5 orbits]

External Apsidal Alignment







For $e_{\mathrm{eq}} > e_p$, there exists $\theta_{p,\mathrm{eq}}$

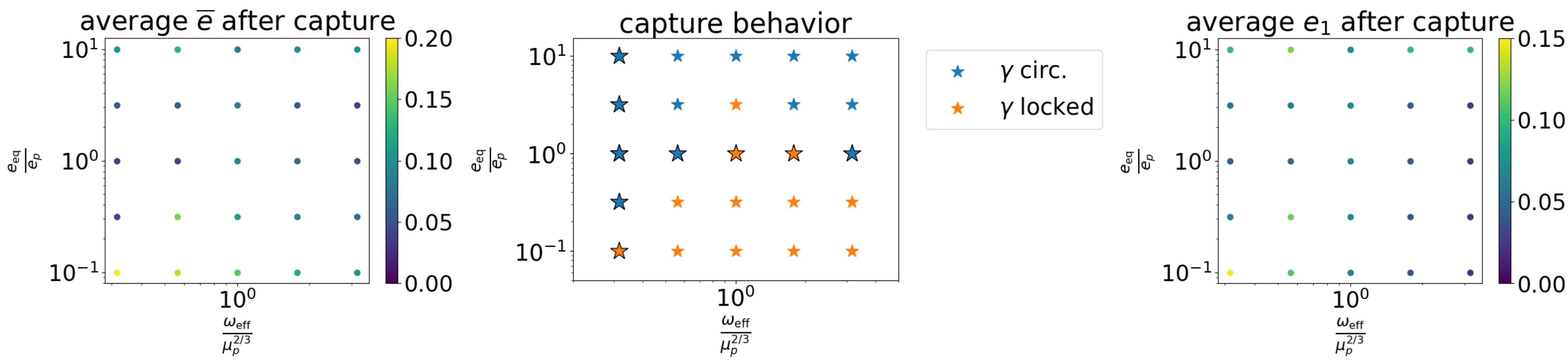
For $e_{\mathrm{eq}} > e_p$, $\theta_{p,\mathrm{eq}}$ does not exist

$\overline{e} \sim e_{\mathrm{eq}} \Rightarrow \gamma$ unconstrained

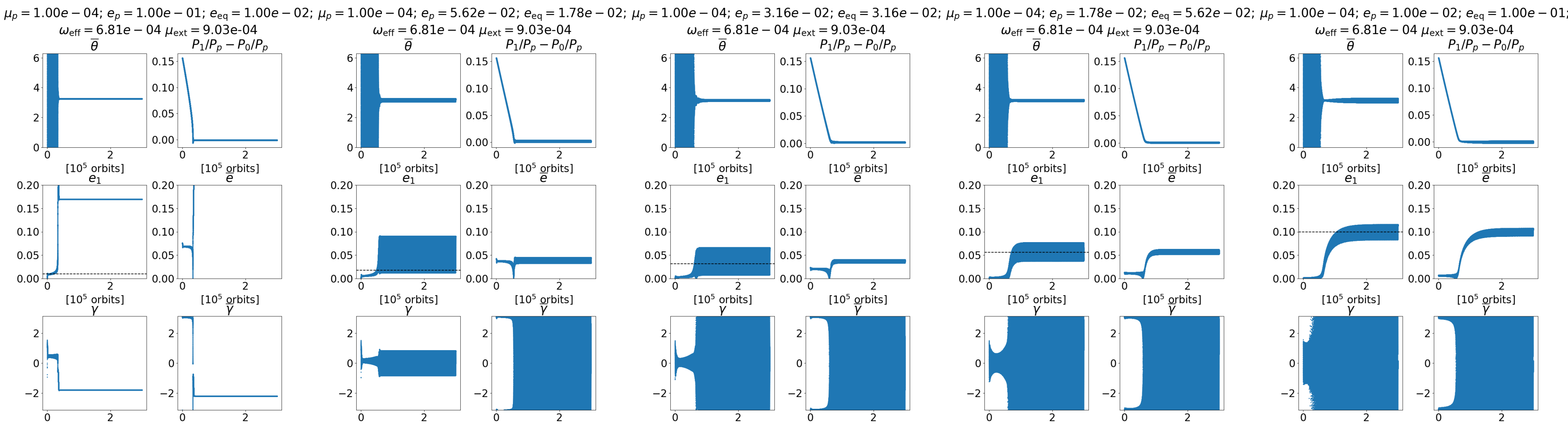
$\overline{e} > e_{\mathrm{eq}} \Rightarrow \gamma \to 0$ to minimize $\overline{e}$

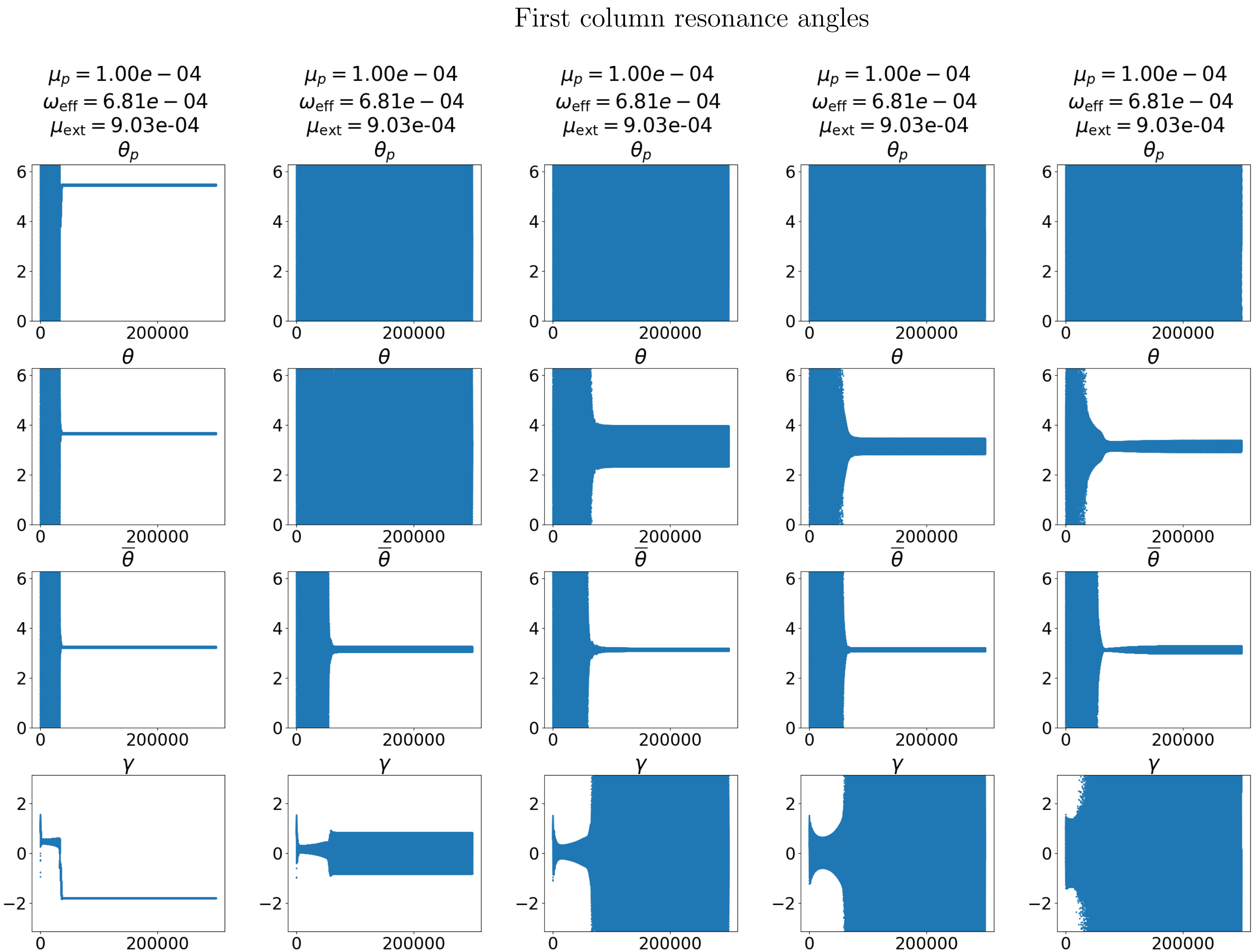
Perturber for outer resonance

$$\mu_p = 1e-4;\ \delta a \sim 3e-3$$



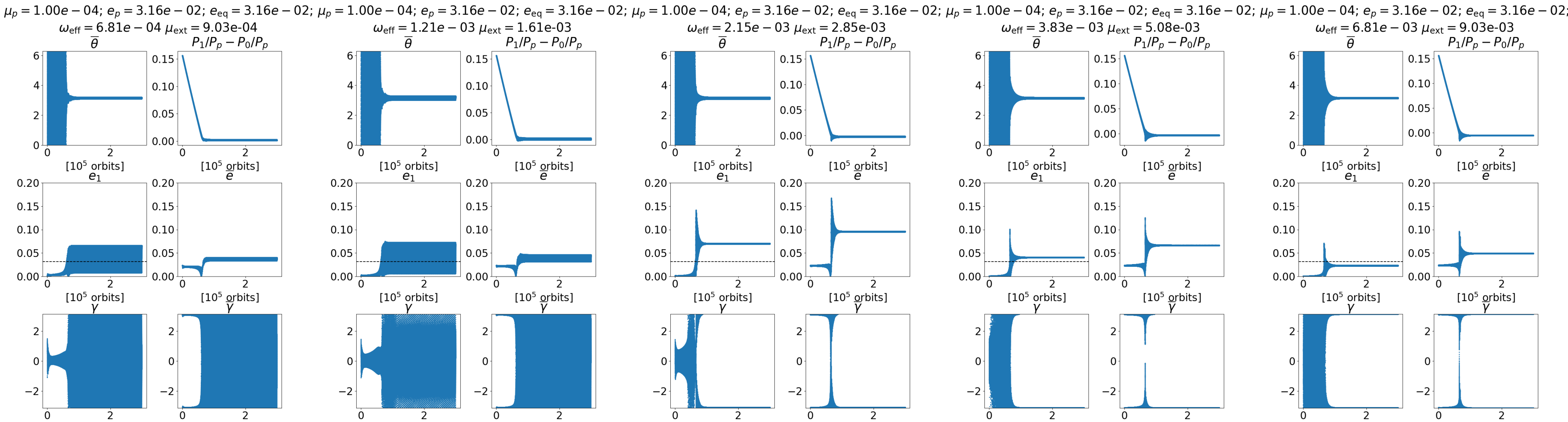
First column simulations



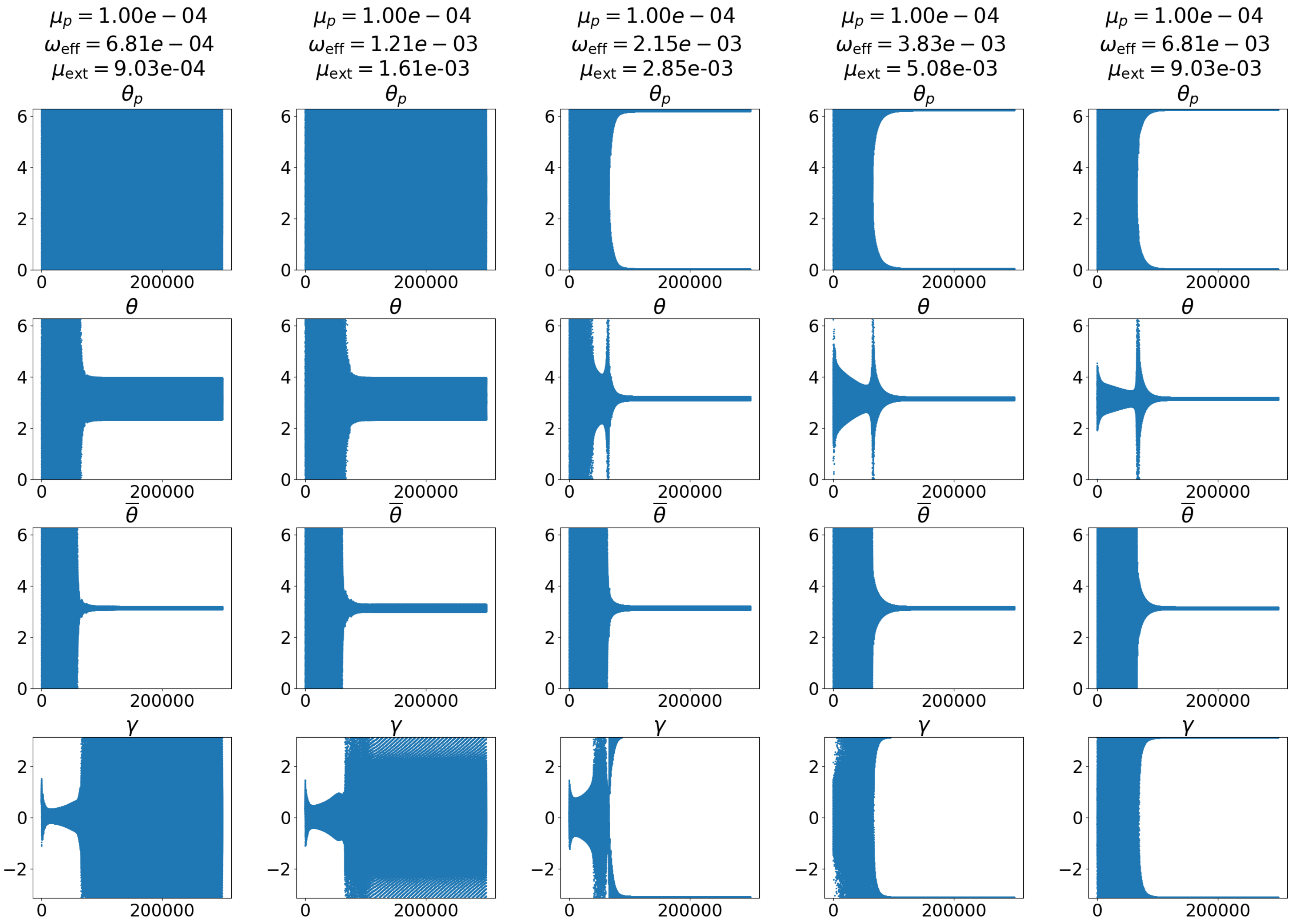


First column resonance angles
$\mu_p = 1.00e-04$
$\omega_{\rm eff} = 6.81e-04$
$\mu_{\rm ext} = 9.03$e-04
θ_p
θ
$\overline{\theta}$
γ
0
2
4
6
−2
200000

Third row simulations

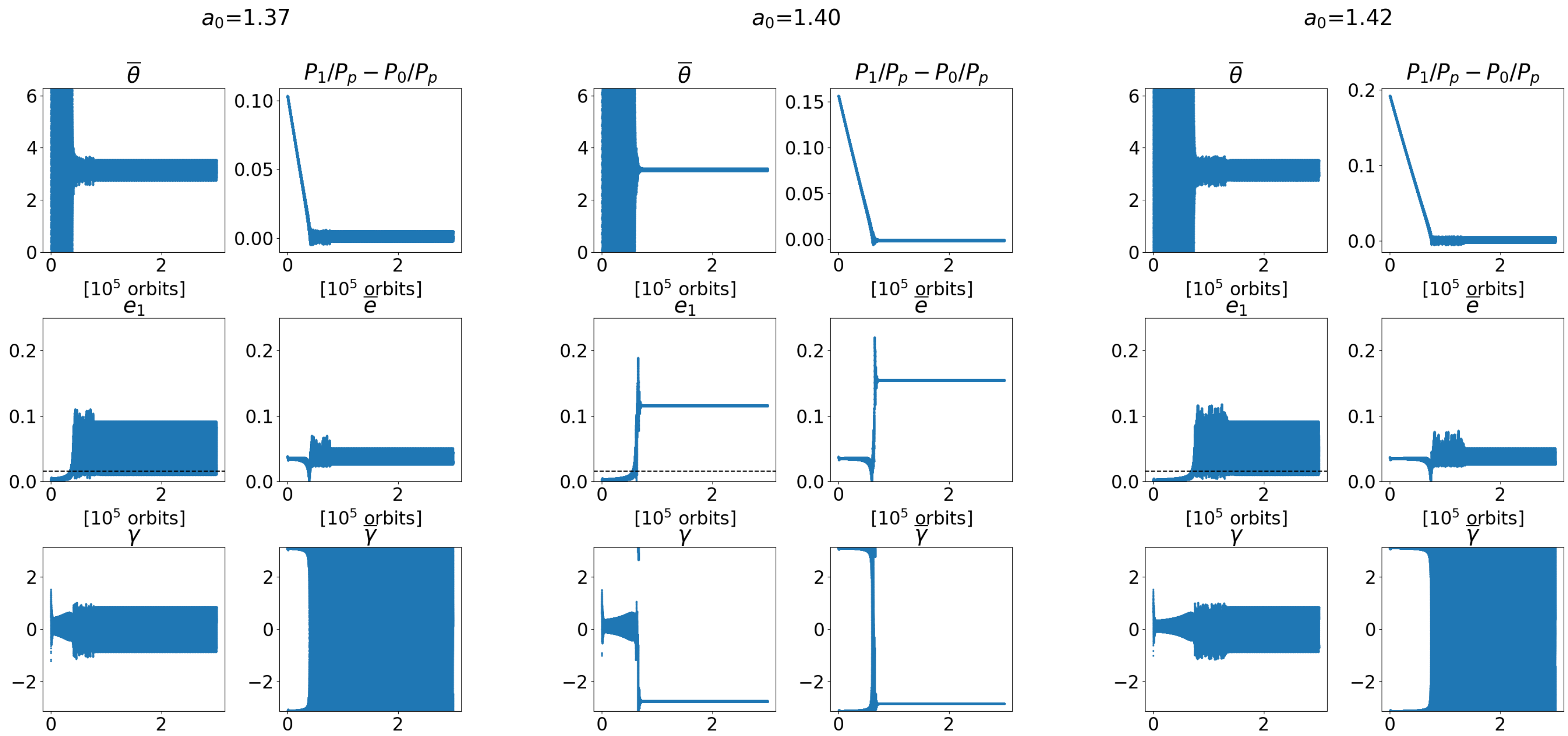


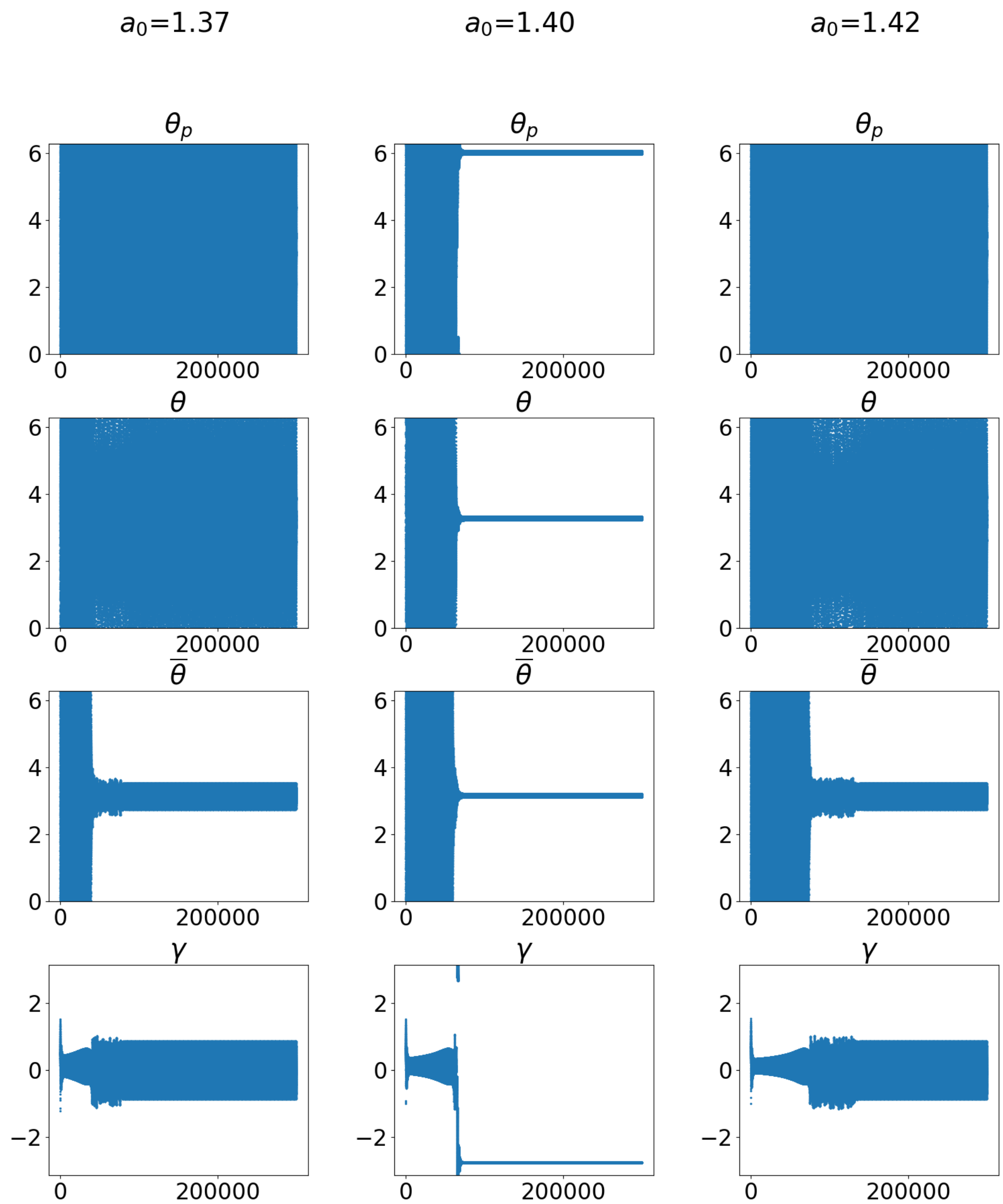
Third row resonance angles



Chaos/Numerical issues

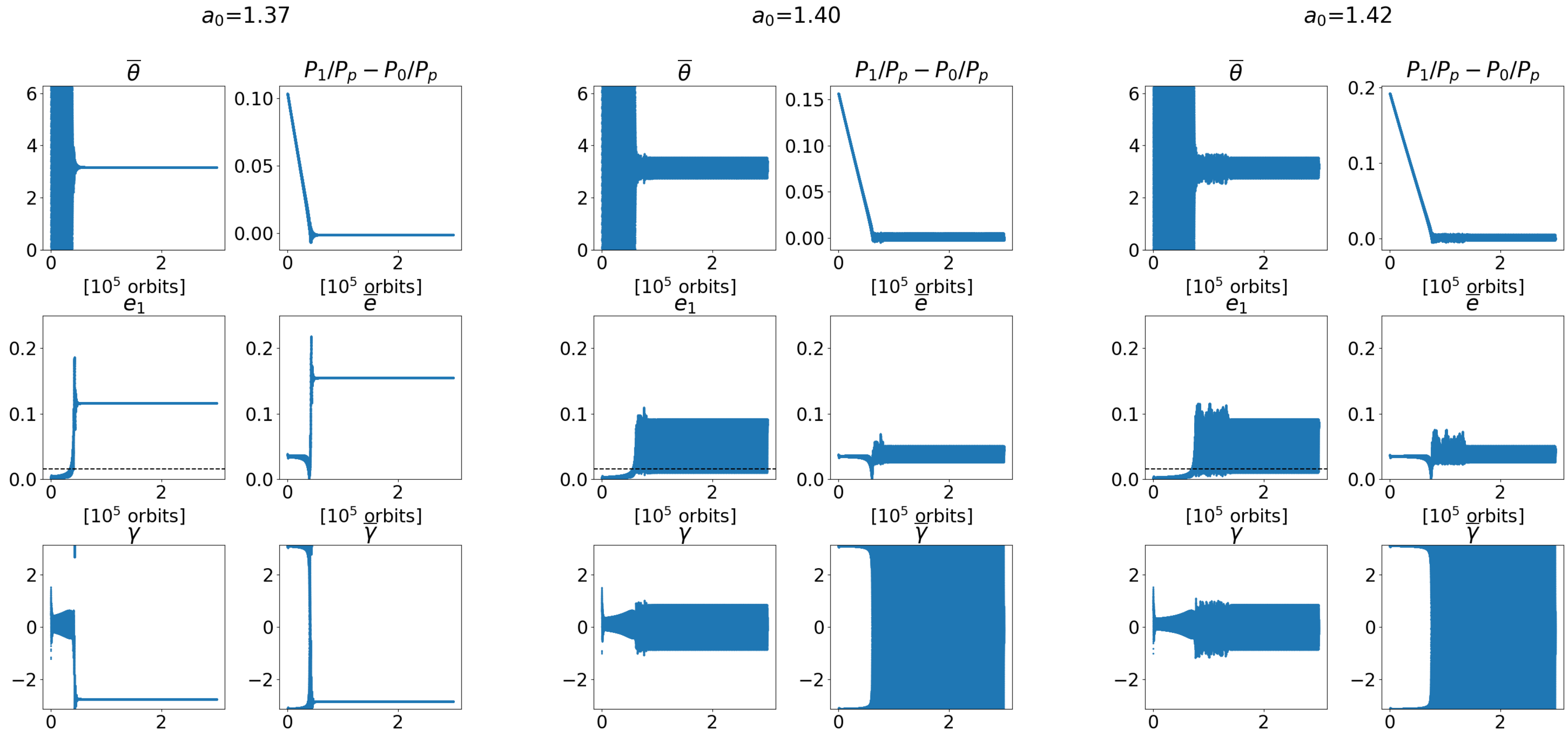
$$\text{Tol}{=}10^{-6},\ e_p = 0.05,\ e_{\text{eq}} = 0.0158,\ \omega_{\text{eff}} = 0.0012$$

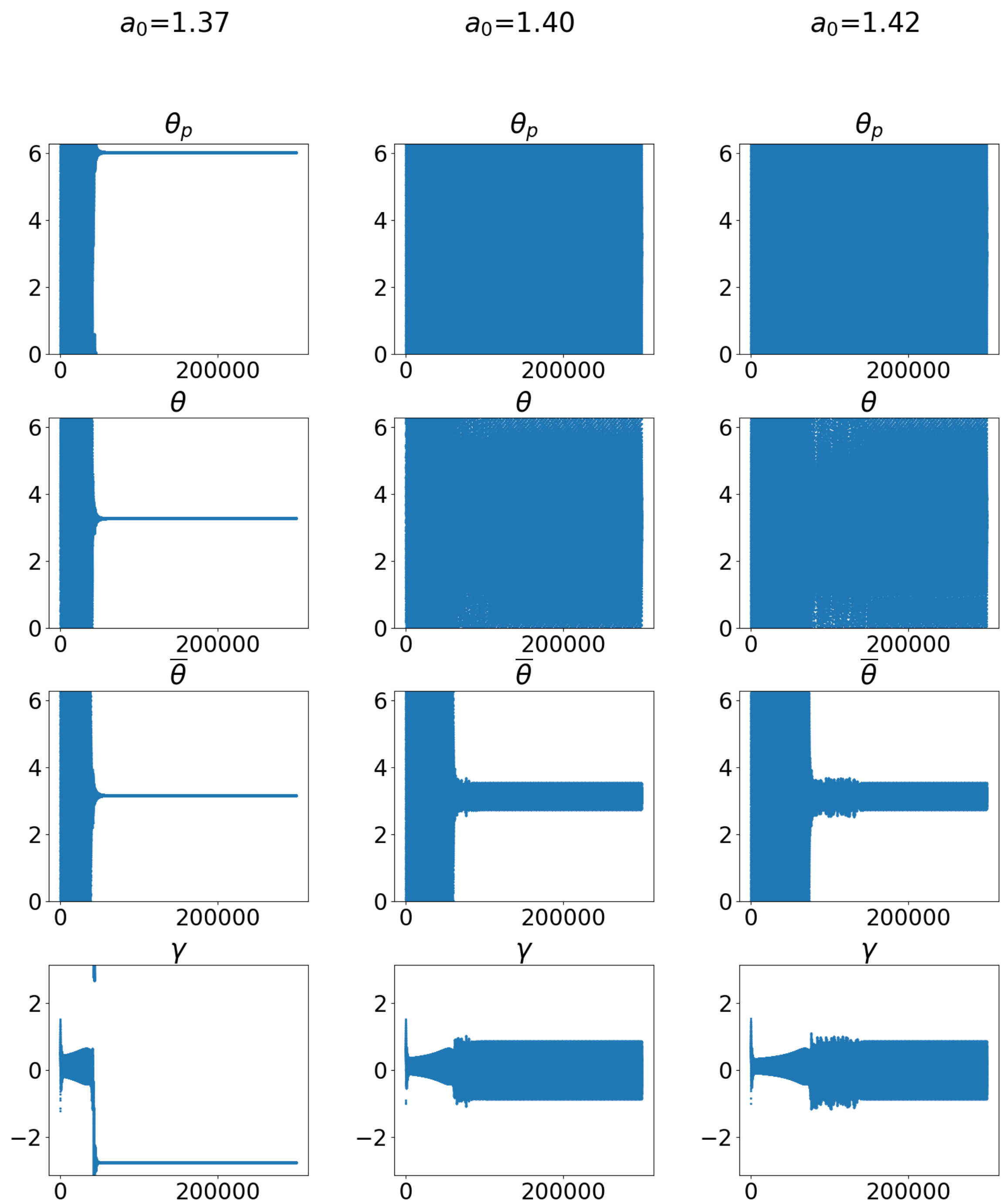




a_0=1.37
a_0=1.40
a_0=1.42
θ_p
θ
$\overline{\theta}$
γ
0
2
4
6
−2
200000

Tol=10^{-9}, $e_p = 0.05$, $e_{\rm eq} = 0.0158$, $\omega_{\rm eff} = 0.0012$





a_0=1.37
a_0=1.40
a_0=1.42
θ_p
θ
$\overline{\theta}$
γ
0
2
4
6
−2
200000