4 dof Hamiltonian differential equations

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1 Hamiltonian & Scaling

Let m_1 denote the inner planet and m_2 denote the outer planet.

$$H = -\frac{GMm_1}{2a_1} - \frac{GMm_2}{2a_2} - \frac{Gm_1m_2}{a_2}\left(f_1e_1\cos\theta_1 + f_2e_2\cos\theta_2\right) \tag{1}$$

$$\begin{aligned}
\theta &= (j+1)\lambda_2 - j\lambda_1 \\
\theta_1 &= (j+1)\lambda_2 - j\lambda_1 + \gamma_1 \\
\theta_2 &= (j+1)\lambda_2 - j\lambda_1 + \gamma_2 \\
q &= \frac{m_1}{m_2} \\
a_0 &= a_{2,\mathrm{init}} \\
H_0 &= \frac{GMm_2}{a_0} \\
\omega_0 &= n_{2,\mathrm{init}} = \sqrt{\frac{GM}{a_0^3}} \\
\Lambda_0 &= m_2\sqrt{GMa_0}
\end{aligned}$$

$$\alpha_1 = \frac{a_1}{a_0}$$
$$\alpha_2 = \frac{a_2}{a_0}$$
$$\Lambda_1 = q\sqrt{\alpha_1}$$
$$\Lambda_2 = \sqrt{\alpha_2}$$
$$x_1 = \Gamma_1 \cos(\gamma_1)$$
$$y_1 = \Gamma_1 \sin(\gamma_1)$$
$$x_2 = \Gamma_2 \cos(\gamma_2)$$
$$y_2 = \Gamma_2 \sin(\gamma_2)$$

2 Resonant Equations of Motion

$$\overline{H} = -\frac{q^3}{2\Lambda_1^2} - \frac{1}{2\Lambda_2^2} - \frac{q\mu_2}{\Lambda_2^2}\left(f_1\sqrt{\frac{2\Gamma_1}{\Lambda_1}}\cos\theta_1 + f_2\sqrt{\frac{2\Gamma_2}{\Lambda_2}}\cos\theta_2\right) \tag{2}$$

Note: $f_1 = -A, f_2 = -B$

Integration variables: $(\theta, \Lambda_1, \Lambda_2, x_1, y_1, x_2, y_2)$

$$\dot{\lambda}_1 = \frac{1}{\alpha_1^{3/2}} + \frac{\mu_2 f_1 e_1}{2\alpha_2\alpha_1^{1/2}}\cos\theta_1 \tag{3}$$

$$\dot{\lambda}_2 = \frac{1}{\alpha_2^{3/2}} + \frac{q\mu_2}{\alpha_2^{3/2}}\left(2f_1e_1\cos\theta_1 + \frac{5}{2}f_2e_2\cos\theta_2\right) \tag{4}$$

$$\dot{\Lambda}_1 = \frac{jq\mu_2}{\alpha_2}\left(f_1e_1\sin\theta_1 + f_2e_2\sin\theta_2\right) \tag{5}$$

$$\dot{\Lambda}_2 = -\frac{(j+1)q\mu_2}{\alpha_2}\left(f_1e_1\sin\theta_1 + f_2e_2\sin\theta_2\right) \tag{6}$$

$$\dot{\gamma}_1 = \frac{-\mu_2}{\sqrt{\alpha}_1\alpha_2e_1}f_1\cos\theta_1 \tag{7}$$

$$\dot{\gamma}_2 = -\frac{q\mu_2}{\alpha_2^{3/2}e_2}f_2\cos\theta_2 \tag{8}$$

$$\dot{\Gamma}_1 = -\frac{q\mu_2}{\alpha_2} f_1 e_1 \sin\theta_1 \tag{9}$$

$$\dot{\Gamma}_2 = -\frac{q\mu_2}{\alpha_2} f_2 e_2 \sin\theta_2 \tag{10}$$

$$\dot{x}_1 = \dot{\Gamma}_1 \cos(\gamma_1) - \Gamma_1 \sin(\gamma_1)\dot{\gamma}_1 \tag{11}$$

$$\dot{y}_1 = \dot{\Gamma}_1 \sin(\gamma_1) + \Gamma_1 \cos(\gamma_1)\dot{\gamma}_1 \tag{12}$$

$$\dot{x}_2 = \dot{\Gamma}_2 \cos(\gamma_2) - \Gamma_2 \sin(\gamma_2)\dot{\gamma}_2 \tag{13}$$

$$\dot{y}_2 = \dot{\Gamma}_2 \sin(\gamma_2) + \Gamma_2 \cos(\gamma_2)\dot{\gamma}_2 \tag{14}$$

3 Secular Equations of Motion

$$\overline{H}_{\text{sec}} = -\frac{q\mu_2}{\Lambda_2^2}\left(\frac{2C\Gamma_1}{\Lambda_1} + \frac{2C\Gamma_2}{\Lambda_2} + D\sqrt{\frac{2\Gamma_1}{\Lambda_1}}\sqrt{\frac{2\Gamma_2}{\Lambda_2}}\cos(\gamma_1 - \gamma_2)\right) \tag{15}$$

$$\dot{\lambda}_1 = \frac{\mu_2}{\alpha_2\sqrt{\alpha_1}}\left(Ce_1^2 + \frac{De_1e_2}{2}\cos\left(\gamma_1 - \gamma_2\right)\right) \tag{16}$$

$$\dot{\lambda}_2 = \frac{q\mu_2}{\alpha_2^{3/2}}\left(2Ce_1^2 + 3Ce_2^2 + \frac{5}{2}De_1e_2\cos\left(\gamma_1 - \gamma_2\right)\right) \tag{17}$$

$$\dot{\Gamma}_1 = \frac{q\mu_2 De_1e_2}{\alpha_2}\sin\left(\gamma_2 - \gamma_1\right) \tag{18}$$

$$\dot{\Gamma}_2 = \frac{q\mu_2 De_1e_2}{\alpha_2}\sin\left(\gamma_1 - \gamma_2\right) \tag{19}$$

$$\dot{\gamma}_1 = -\frac{\mu_2}{\alpha_2\sqrt{\alpha_1}}\left(2C + D\frac{e_2}{e_1}\right) \tag{20}$$

$$\dot{\gamma}_2 = -\frac{q\mu_2}{\alpha_2^{3/2}}\left(2C + D\frac{e_1}{e_2}\right) \tag{21}$$

4 Dissipation

Parameters: $(T_{m,2}, T_{e,1}, T_{e,2})$; i.e. $T_{m,1} = \infty$; $T_e > 0$; $T_m < 0$

$$\dot{\Lambda}_{1,\mathrm{dis}} = -\frac{\Lambda_1}{2}\left(\frac{4\Gamma_1}{\Lambda_1 T_{e,1}}\right) \tag{22}$$

$$\dot{\Lambda}_{2,\mathrm{dis}} = \frac{\Lambda_1}{2}\left(\frac{1}{T_{m,2}} - \frac{4\Gamma_2}{\Lambda_2 T_{e,2}}\right) \tag{23}$$

$$\dot{x}_{1,\mathrm{dis}} = -\sqrt{\Gamma}_2 \cos(\gamma_1)\left(\frac{1}{T_{e,1}} + \frac{\Gamma_1}{T_{e,1}\Lambda_1}\right) \tag{24}$$

$$\dot{y}_{1,\mathrm{dis}} = -\sqrt{\Gamma}_2 \sin(\gamma_1)\left(\frac{1}{T_{e,1}} + \frac{\Gamma_1}{T_{e,1}\Lambda_1}\right) \tag{25}$$

$$\dot{x}_{2,\mathrm{dis}} = \sqrt{\Gamma}_2 \cos(\gamma_2)\left(-\frac{1}{T_{e,2}} + \frac{1}{4}\left(\frac{1}{T_{m,2}} - \frac{4\Gamma_2}{T_{e,2}\Lambda_2}\right)\right) \tag{26}$$

$$\dot{y}_{2,\mathrm{dis}} = \sqrt{\Gamma}_2 \sin(\gamma_2)\left(-\frac{1}{T_{e,2}} + \frac{1}{4}\left(\frac{1}{T_{m,2}} - \frac{4\Gamma_2}{T_{e,2}\Lambda_2}\right)\right) \tag{27}$$

5 Equilibrium

$$\dot{e}_1 = \frac{\mu_2}{\alpha_2}[f_1 \sin(\theta_1) - De_2 \sin(\gamma_2 - \gamma_1)] - \frac{(e_1 - e_{1,d})}{T_{e,1}} \tag{28}$$

$$\dot{e}_2 = \frac{q\mu_2}{\alpha_2}[f_2 \sin(\theta_2) - De_1 \sin(\gamma_1 - \gamma_2)] - \frac{(e_2 - e_{2,d})}{T_{e,2}} \tag{29}$$

$$\dot{\varpi}_1 - \dot{\varpi}_2 = \frac{\mu_2}{\alpha_2}\left(\frac{f_1\cos\theta_1}{\alpha_1^{1/2}e_1} - \frac{qf_2\cos\theta_2}{\alpha_2^{1/2}e_2} + \frac{2C}{\alpha_1^{1/2}} + \frac{De_2}{\alpha_1^{1/2}e_1} - \frac{2qC}{\alpha_2^{1/2}} - \frac{qDe_1}{\alpha_2^{1/2}e_2}\right) \tag{30}$$

$$\begin{aligned}\dot{\eta} = {}& \frac{q\alpha_0^{1/2}}{j(q\alpha_0^{-1}+1)}\left[\frac{1}{T_{m2}} - \frac{1}{T_{m1}} + \frac{2e_1^2}{T_{e1}} - \frac{2e_2^2}{T_{e2}}\right] - q\alpha_0^{1/2}\frac{2e_1^2}{T_{e1}} - \frac{2e_2^2}{T_{e2}} \\ & + \underbrace{\cancel{\left(q\frac{\mu_2\sin(\varpi_1 - \varpi_2)}{\alpha_2^{3/2}}De_1e_2 + q\frac{\mu_2\sin(\varpi_2 - \varpi_1)}{\alpha_2^{3/2}}De_1e_2\right)}}_{\text{secular contributions}}\end{aligned}$$