Banach-Tarski Embeddings and Transformers

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Abstract

We introduce a new construction of embeddings of arbitrary recursive data structures into high dimensional vectors. These embeddings provide an interpretable model for the latent state vectors of transformers. We demonstrate that these embeddings can be decoded to the original data structure when the embedding dimension is sufficiently large. This decoding algorithm has a natural implementation as a transformer. We also show that these embedding vectors can be manipulated directly to perform computations on the underlying data without decoding. As an example we present an algorithm that constructs the embedded parse tree of an embedded token sequence using only vector operations in embedding space.

1 Introduction

Transformer models, as introduced by Vaswani et al. [18], have led to significant advancements in various machine learning domains. One notable feature of these models is that embeddings produced from the internal activations of the models can effectively capture high-level information about the model inputs, where similar model inputs produce correlated embedding vectors [9].

To understand and interpret how transformer models work, we need a theory of how data is represented and manipulated within these models. To work towards such a theory, we consider the following questions:

1. Can we explicitly construct embeddings that represent complex data structures?

2. Is is possible to construct algorithms that use these embeddings as data representations?

In this paper, we describe a family of embeddings of arbitrary recursive data structures to vectors in $\mathbb{R}^d$. These embeddings are designed so that many natural operations on data structures translate to linear operations on the embedding vectors. We call this the "BT Embedding" because it was inspired by the proof of the famous Banach-Tarski paradox.

The BT embedding is constructed using random vectors and matrices. The construction requires only a schema for the data, and does not need training or optimization. It has the property that similarities in data structures can be detected using linear operations (e.g. dot products) to compare embedding vectors.

This embedding enables the design of transformer models that compute directly on the embedding vectors and that do not require custom indexing schemes to represent nested data structures. These constructions demonstrate that transformers can directly process and transform recursive data structures encoded as BT embedding vectors.

The main technical result is that the BT embedding is invertible with high probability when the dimension of the embedding is sufficiently large. At a high level, the method is recursive application of the Johnson-Lindenstrauss lemma and 1-nearest-neighbors with respect to dot product similarity. The embedding dimension required to reversibly encode a data structure in a single vector is approximately linear in the size of structure. We will show that this decoding algorithm has a natural implementation as a transformer model.

To illustrate how BT encodings can be used directly in algorithms on recursive data structures, we construct an algorithm to parse a sequence of BT encoded tokens with a collection of BT encoded production rules. The result is a BT encoded parse tree of the input. Notable, the algorithm operates without decoding any of the vectors, does not need access to the full schema of the input data, and can be implemented as a transformer.

These results can be interpreted as an analogy between the theory of data structures and linear algebra:

Computer Science	Linear Algebra
Symbols, Atomic Data Structures	Random Vectors / JL Embeddings
Attributes, Fields of Data Structures	Random Orthogonal Matrices
Recursive, Tree-like Data Structures	Banach-Tarski Encoding
Paths in Data Structures	Representations of Free Groups on Attributes
Algorithms on Recursive Data Types	Transformers

We have implemented the encoding, decoding, transformer model, and parsing algorithms described in this paper. The code, tests, and experiments are available at `https://github.com/jtmaher/Embedding`.

1.1 Related Work

In the paper “Random Features for Large-Scale Kernel Machines” [16] Rahimi and Recht showed that randomized features can be a powerful tool for various machine learning problems. This work, along with the classic embedding lemma of Johnson and Lindenstrauss [13] led us to consider the construction of embeddings from random vectors.

The Banach-Tarski Theorem [2] [19] showed that there is a decomposition of a ball in $\mathbb{R}^3$ into several subsets that can be isometrically assembled into two copies of the original ball. The proof of this theorem involves constructing a free group F_2 in the orthogonal group O_3 [7]. The natural self-similar structure of the Cayley graph of the free group on two generators is key in the proof. The presence of these groups in all orthogonal groups O_d, $d > 2$ [5], led to the hypothesis that such symmetry groups could be used to create representations of tree structures in high dimensional vector spaces.

In “Attention is Turing Complete” by Perez et al [15] it was shown that transformers with positional embedding are Turing complete. This work has been extended with more practical constructions in [20], [11]. Our construction of transformers in this paper uses similar methods.

In “Neural Turing Machines” by Graves et al [12] it was demonstrated that state machines with memory can be directly optimized to perform computational tasks.

“Universal Transformers” by Dehghani et al [8] introduced the idea that powerful transformer models can be produced by stacking identical (recurrent) transformer blocks. While the transformers in our paper are not recurrent, our constructions are produced by iterating layers in a similar way (this method is also used in [11]).

In the context of tree kernels [4], Zanzotto and Dell’Archiprete [22] constructed features of trees using random embeddings of node labels. Further work [10] shows under certain conditions, parse trees of sentences can be decoded from the original sentence and these features. This approach uses a nonlinear, weighted construction to combine node embeddings.

Shiv and Quirk in [17] have constructed a positional embedding for paths in regular trees that can be traversed by linear operators. Yao et al [21] produced a direct construction of a parser transformer for certain languages.

Hyperbolic geometry of activations has been proposed in [23] as a natural way to embed trees in neural networks.

1.2 Acknowledgments

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2 The BT Embedding

2.1 Definitions

A *schema* is a pair of finite sets (T, A) where T is a set of *tokens* and A is a set of *attributes*.

A *data structure* with schema (T, A) is a finite tree where each node has a label from T and each edge has an attribute from A.

A schema is called *reflexive* if $A \subset T$. This allows us to use attributes as values, which allows self describing structures. We will assume that all schemas below are reflexive.

Let $\mathcal{T}_{(T,A)}$ denote the set of all finite trees with nodes labeled with elements of T and edges labeled with elements of A, with the restriction that a node can have at most one branch labeled any particular element of A

We think of these labeled trees as a general model of nested data structures, and for this reason we will use *data structure* as a synonym for a tree in $\mathcal{T}_{(T,A)}$.

Each node of such a tree has a *path*, which is the unique sequence of edge labels starting at the root and ending at the node in question.

2.2 Examples

Figure 1 shows a simple schema for a data type that describes a pet.

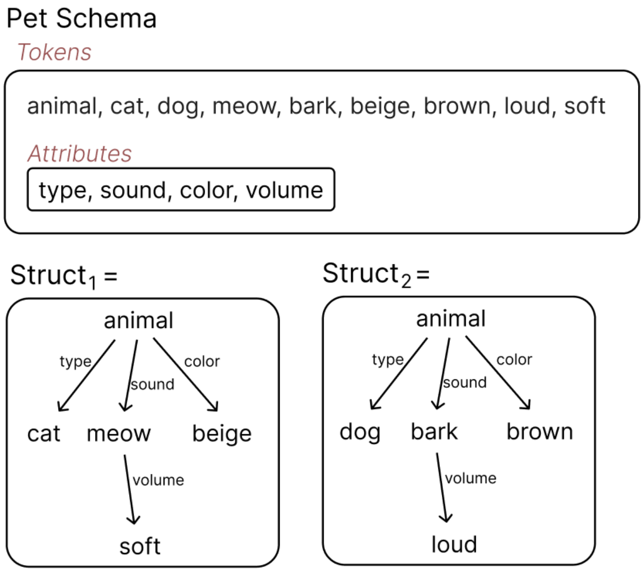
This type of example can be generalized to any type of structured data, such as JSON, with a finite number of atomic values. In practice, there are typically constraints on which attributes can be attached to a particular token, but in the present discussion we will ignore this detail.

An important class of examples is linked lists, which can be modeled with a single “next” attribute (Figure 2). Further specializing to the case of binary lists allows representation of bit strings.

2.3 Random Embeddings

An *embedding* E of a schema (T, A) is a pair of maps:

Figure 1:



$$E_{token} : T \to \mathbb{S}^{d-1} \subset \mathbb{R}^d$$

$$E_{attr} : A \to O_d$$

In other words, each token is mapped to a unit vector, and each attribute is mapped to an orthogonal matrix.

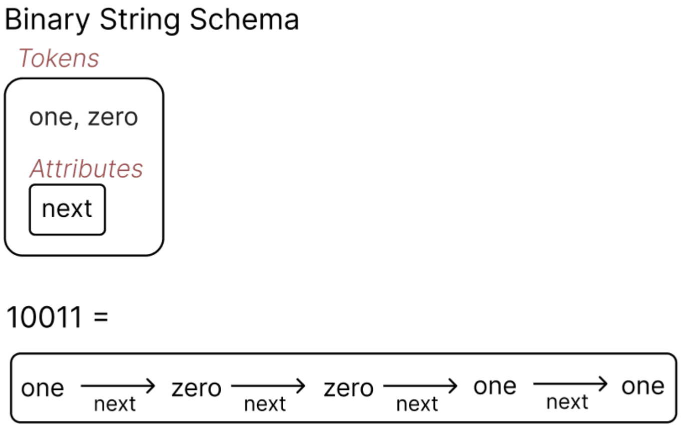
Going forward, we will assume that E is a *random* embedding with respect to the uniform measure on $\mathbb{S}^{d-1}$ and Haar measure on O_d. That said, it is possible that our constructions could be made substantially more efficient using other random embedding schemes, such as [1].

2.4 BT Embedding

The *BT embedding* associated with E is a map $BT_E : \mathcal{T}_{(T,A)} \to \mathbb{R}^d$ which extends E to a map of all data structures to vectors. Let $\tau \in \mathcal{T}_{(T,A)}$ be a tree:

$$BT_E(\tau) = \sum_{x \in nodes(\tau)} \Big(\prod_{i=1}^{\#(path(x))} E_{attr}(path(x)_i) \big) E_{tokens}(label(x)) \Big)$$

Figure 2:

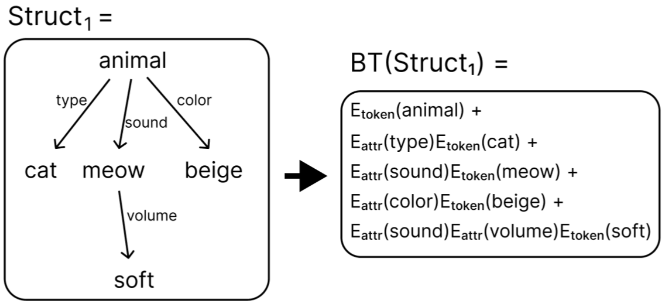


In other words, the BT encoding of a tree is the sum of the embeddings of the tokens at the nodes, transformed by the product of matrices corresponding to the attributes of the path of each node. Note that the order of operations in the matrix product is significant and runs from left to right as we move from the root to the leaves: i.e. $A_1 A_2 \dots A_k$ where A_i is the matrix corresponding to the ith element of the path.

We can extend this model by accepting trees with a *set* of tokens for each node. In this setting BT is a linear map w.r.t. unions of trees.

Figure 3 shows how the embedding works for our pet schema example:

Figure 3:



2.5 Properties of the BT Embedding

Cardinality

$$\#nodes(x) \approx ||BT_E(x)||^2$$

This becomes arbitrary close to equality as the embedding dimension $d \to \infty$.

Linearity

Suppose that $\tau_1, \tau_2 \in \mathcal{T}_{(T,A)}$ are trees, and $\nu \in Nodes(\tau_1)$ is a leaf node. Given an attribute a, we can append τ_2 to τ_1 with the edge from ν to the root of τ_2 labeled by a. In this case we have:

$$BT(\tau_1 \cup_{\nu,a} \tau_2) = BT(\tau_1) + A \cdot BT(\tau_2)$$

where $A = (E_{attr}(a) \prod_{p \in path(\nu)} E_{attr}(p)) \in O_d$

This implies that if two trees share a common sub-tree, then there is a corresponding linear relationship between the BT embeddings. Thus, the BT embedding gives us a way to transform arbitrary data into vectors that can plausibly be used for linear regression.

Extension

If we extend BT to *tree fragments*, where the node labels are optional (but where we still have a tree with all the edges labeled), then BT is linear on disjoint tree fragments. While this extension can be useful, note that we cannot expect to decode fragments using a direct tree traversal, so decoding of fragments would likely be exponential time with respect to tree size.

Lists

Linked lists, as in figure 2, can be constructed in any schema using any attribute, but for clarity let us assume we have an attribute called “next”. For a list of tokens $t_1, \ldots, t_n$, we can form a BT encoded list as follows, with the matrix A_{next} being the embedding of the “next” attribute:

$$\sum_{i=1,\ldots,n} A_{\text{next}}^{i-1} E(t_i)$$

This has analogous properties to the positional embeddings used in transformers [18], namely, that translations of the original sequence of tokens correspond to multiplication by an orthogonal matrix.

We can push tokens onto an embedded list without decoding the list. If $v = BT(t_1, \ldots, t_n)$ as above, then:

$$\text{push}(v, t) = E(t) + A_{\text{next}} \cdot v$$

On the other hand, popping tokens from the list requires decoding, and this illustrates a general property of BT embeddings: *Write* operations are linear, while *read* operations are typically nonlinear, as we will see in the discussion of decoding.

3 Decoding BT Embeddings

In this section, we describe conditions under which the BT embedding is invertible. We introduce a recursive algorithm to decode these embeddings under these conditions.

Let $\tau \in \mathcal{T}_{(T,A)}$ be a data structure with l nodes, and $v = BT_E(\tau)$ be its BT encoding with respect to randomly chosen $E_{tokens} \in \mathbb{R}^d, E_{attr} \in O_d$. The following algorithm is based on the idea that we can find the most likely token for the root by maximizing the inner product between v and the token vectors E_{tokens}. To check for the presence of tokens associated with each attribute, we transform the vector v by the inverse of the corresponding attribute matrix, and then attempt to decode a token in the same way. Proceeding recursively we get:

Algorithm 1:

1. Set $v = BT_E(\tau)$, output = empty tree, path = root.

2. For each $w_i \in E_{tokens}$, compute $x_i = \langle w_i, v \rangle$.

3. If $\max x_i > 1/2$ then place a token y corresponding to argmax(x_i) at the path on output, else return.

4. Transform v by the inverse of each attribute matrix and recursively call step (2) with $v = A_i^{-1} v$ and path = path + A_i for each A_i.

An implementation of this algorithm can be found at `https://github.com/jtmaher/Embedding/blob/master/embedding/encoder.py`.

This algorithm runs in linear time with respect to the number of nodes in τ, assuming that we can correctly decode all tokens (or lack thereof) with the comparison in step 3. The tree structure is critical here, since if there is a disconnected tree fragment, we would need to search for tokens over an exponentially large set of potential nodes.

To understand how Algorithm 1 works, suppose the y is the transformed embedding of the token at any node in τ and $v = BT(\tau)$. We have:

$$\langle y, v \rangle = 1 + \sum_{i=1}^{l-1} \langle y, v_i \rangle \tag{1}$$

where the v_i range over all other $l-1$ terms in $BT(\tau)$. Note that l is the number of nodes of τ. If we happen to know that all the inner products $\langle y, v_i \rangle < \epsilon$ for $v_i \neq y$ and where $\epsilon \cdot l < \frac{1}{2}$, then Algorithm 1 always picks the correct token (or correctly infers that there is no token present if the maximum inner product is $< \frac{1}{2}$).

Let $\Gamma_l \subset O_d$ denote the set of all l-fold products of the matrices in $E_{attr} \cup I_d$ where I_d is the $d \times d$ identity. Let $V = \Gamma_l E_{tokens}$ be the set of all products of the token vectors with these matrices. V is just the set of all possible terms in BT embeddings of trees with $\leq l$ nodes. Combining this with the above analysis of the algorithm yields:

Lemma 1. *If $|\langle v_1, v_2 \rangle| < \frac{1}{2l}$ for all $v_1 \neq v_2 \in V$, then Algorithm 1 decodes all trees with $\leq l$ nodes.*

Intuitively, we expect that this condition is satisfied in sufficiently high dimensions, because the JL Lemma [13] shows that the inner products of random vectors become increasingly clustered around zero as dimensionality is increased.

In fact, a looser bound than Lemma 1 will suffice due to the fact that the terms in the sum of equation 1 are essentially random and therefore combine sub-additively. For instance, if we assume that the inner products $\langle v_i, v_j \rangle$ are I.I.D., bounded in absolute value $|\langle v_i, v_j \rangle| < \epsilon$, and mean zero for $i \neq j$, then we can conclude, by the central limit theorem, that we can pick C such that the sums $S_l = \sum_{i=1}^{l-1} \langle y, v_i \rangle$ in equation 1 satisfy:

$$|S_l| < C\sqrt{l} \cdot \epsilon$$

with high probability as $l \to \infty$. This yields a bound on the inner products that scales like $\frac{1}{\sqrt{l}}$ w.r.t. the tree size l:

Lemma 2. *There is a constant C such that if $\langle v_i, v_j \rangle$ are I.I.D., $E(\langle v_i, v_j \rangle) = 0$ and $|\langle v_i, v_j \rangle| < \frac{1}{2C\sqrt{l}}$, then $S_l < \frac{1}{2}$ with high probability as $l \to \infty$. Therefore Algorithm 1 succeeds with high probability under these assumptions.*

Note that we can control the bound on $|\langle v_i, v_j \rangle|$ by increasing the embedding dimension d: According to the JL Lemma, the size of these inner products is proportional to $\frac{1}{\sqrt{d}}$. This implies a linear relation between the embedding dimension and the size of trees that can be successfully decoded. For a more detailed discussion, see Section 6.

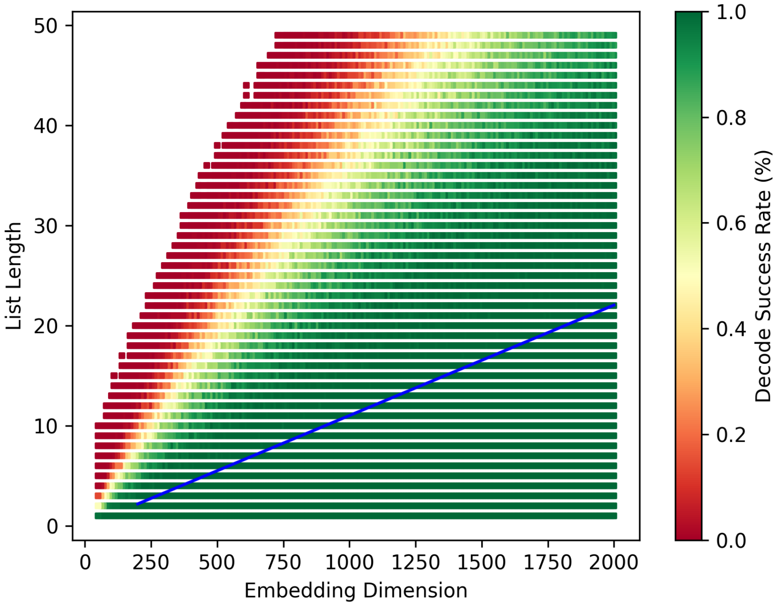
3.1 Empirical Decoding Results

Random Lists

Figure 4 illustrates the results of an experiment on a schema for linked lists of 100 tokens with one attribute "next". For a selection of embedding dimensions and list lengths, we constructed 829,990 random lists and BT encoded them into vectors of various embedding dimensions. The points on the graph indicate the success rate of decoding these vectors to the original lists.

The code for this experiment can be found at `https://github.com/jtmaher/Embedding/blob/master/Arrays.ipynb`.

Figure 4: Algorithm 1 Performance for Lists

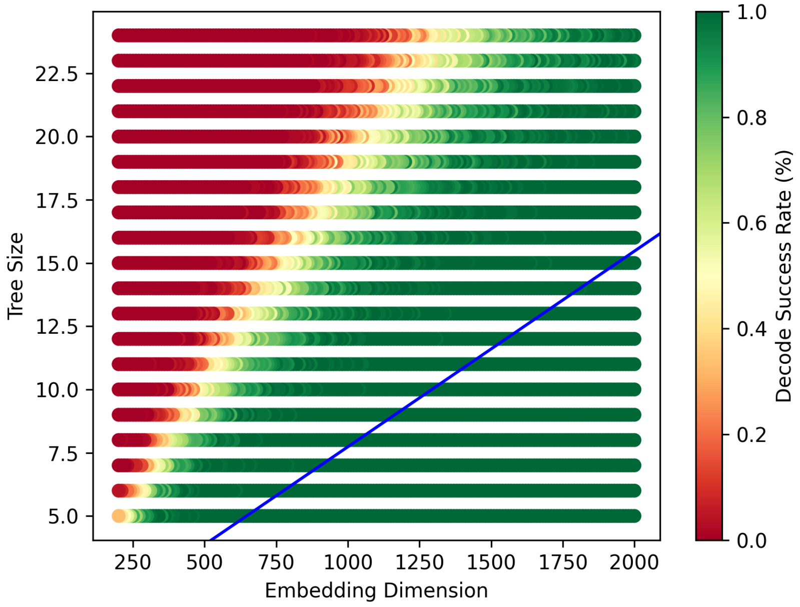


All points below the blue line are at 100% success rate

Random Trees

Figure 5 shows the results of an experiment on a schema of 100 tokens and 4 attributes. We randomly produced 196,000 embeddings of trees of size between 5 and 25 nodes, for embedding dimension $d \in [50, 2000]$, and recorded the success rate by size and dimension.

Figure 5: Algorithm 1 Performance for Trees



All points below the blue line are at 100% success rate

The code for this experiment can be found at `https://github.com/jtmaher/Embedding/blob/master/RandomTrees.ipynb`.

Approximate Linear Scaling

In both of these experiments, we observe that to successfully decode with probability approaching 1, we require approximately $d \gtrsim 125 \cdot l$ for embedding dimension d and tree size l.

While this seems to require a very large number of dimensions, note that the set of structures of size l is exponential in l in both of these experiments, so a very large set of structures is being distinguished by this method. In modern transformer models, the embedding space is indeed quite large, e.g. GPT-3 uses 12,288 dimensional embeddings [3], which is sufficient to embed structures of size ≈ 100 using BT embeddings.

4 Decoding with Transformers

In this section, we describe how to implement the BT decoding algorithm with an explicit transformer model. Our implementation uses similar techniques to [11], [15], and [21].

Rather than implement the full recursive form of Algorithm 1, we will focus on the special case of retrieving the label of a node at a particular path from the BT encoding vector. An implementation of this model can be found at `https://github.com/jtmaher/Embedding/blob/master/Transformers2.ipynb`.

Inputs and Outputs

Let τ be a data structure. The inputs to the transformer are:

1. A BT encoded vector $v = BT(\tau)$
2. A BT encoded path $r = BT([r_1, \ldots, r_k])$, where r_i are the tokens of the attributes in the desired path, and where the list is formed as a linked list using a distinguished "next" attribute.

The transformer is a decoder-only (autoregressive) architecture and operates on sequences of k vectors, where k is the length of the path. The input vector and path are placed in the first position of the sequence.

Recall that our schema is assumed to be reflexive, meaning that the attributes all have corresponding tokens, which is necessary to express paths in this way. We also assume that we have a "next" attribute in the schema for formation of the path list.

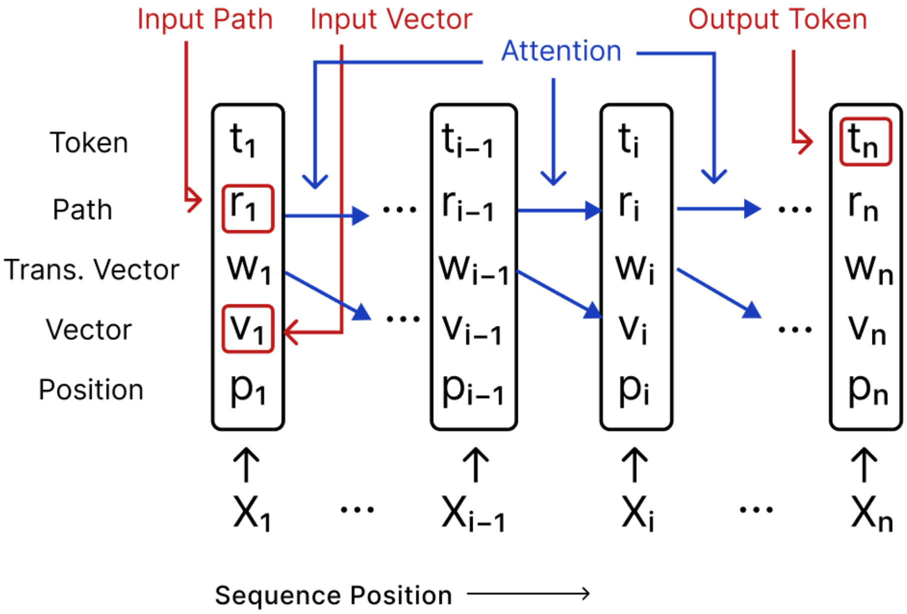
The output of our model is a sequence of k BT encoded tokens corresponding to the labels at each node along the path p, starting at the root. The desired label at the input path will be found in the k-th position. Although this is a "decoder" model, we note that the output and internal state will all consist of BT encoded data. This feature was quite useful for debugging our implementation, since we could decode structures from the intermediate vectors to inspect the internal state of the model during each step of evaluation.

State Vectors

We use position encoding vectors, which we define as follows: Let $p_1 \in S^{k-1} \subset \mathbb{R}^k$ be a random unit vector and $Z \in O_k$ be random orthogonal matrix. Then let $p_i = Z^{i-1}t_1$. We assume that the dimension k is selected so that $\langle p_i, p_j \rangle$ is small for $i \neq j \in [1, \ldots, n]$, where n is our sequence length.

Figure 6 shows the data layout for our model.

Figure 6: Transformer Data Layout



For each position $i \in [1, \ldots, n]$ in our sequence, we operate on a vector:

$$x_i = (p_i, v_i, w_i, r_i, t_i)$$

where p_i is the position embedding, and $v_i, w_i, u_i, t_i \in \mathbb{R}^d$ are vectors in the d dimensional BT embedding space. Intuitively, v_i is an "input vector", w_i is a "transformed vector", r_i is a "path", and t_i is a "token". The transformed vector will end up being the product of the input vector with the inverse of the attribute matrix corresponding to the current path entry. The token will be decoded from the transformed matrix.

To initialize these vectors, we set:

$$x_1 = (p_1, v, 0, M_{\text{next}}r, 0)$$

where v is the input vector, and r is the input path, and M_{next} is the BT attribute matrix corresponding to "next". Multiplication by M_{next} "shifts" the path vector to skip the first position so that the root token can be decoded.

For $i > 1$ set:

$$x_i = (p_i, 0, 0, 0, 0)$$

Attention Head

We need to copy both the transformed vector and the path from each position in the sequence to the next. This is accomplished with a single attention head.

Define A to be the causal attention block on our sequence of vectors x_i, where the query, key and value are:

$$Q(x_i) = Z^{-1} p_i$$

$$K(x_i) = p_i$$

$$V(x_i) = (0, w_i, 0, M_{\text{next}}^{-1} r_i, 0)$$

For sufficiently low softmax temperature, A mostly attends to the $i-1$th position and produces the transformed token w_{i-1} and a shifted copy of the path vector (except for the degenerate case of $i = 1$, where we retrieve the zero vector by definition).

In other words, if $i > 1$ we have:

$$x_i + A(x_i) = x_i + \sum_{j<i} \text{softmax}_j(\langle Z^{-1} p_i, p_j \rangle)(0, w_j, 0, M_{\text{next}}^{-1} r_j, 0)$$

$$\approx (p_i, w_{i-1}, w_i, M_{\text{next}}^{-1} r_{i-1}, t_i)$$

Note that the transformed vector w_i is being moved to the v_i position in the output.

Feed Forward Layers

We construct two feed forward layers of the form $F(x_i) = M_1 \text{relu}(M_2 x_i) + x_i$ where M_1, M_2 are affine transformations (i.e. linear with a constant bias).

First, we need to transform the input vector v_i by the inverse of the attribute matrix corresponding to the attribute token at the front of the path r_i. Let $E_{attr} \subset E$ denote the token embeddings of the attributes and all tokens, respectively.

Let:

$$y = C(E_{attr} r_i - \frac{1}{2}) \in \mathbb{R}^k$$

where C is a large constant and k is the number of attributes in the schema. y is intended to be very positive on the dimension corresponding to the path attribute encoded at the head of r, and very negative otherwise.

We can use y to form a conditionally transformed version of v using a technique from [11]:

$$f_1(x_i) = \text{relu}(v_i) - \text{relu}(-v_i) + \sum_j \text{relu}(y_i + M_j^{-1}v_i - v_i) - \text{relu}(y_i)$$

where M_j is the jth attribute matrix. It is easy to see that $f_1(x_i) = M^{-1}v_i$ where M is the attribute matrix corresponding to the head of the path vector, or if there is no token decoded from the path vector (i.e. $i = 0$), $f_1(x_i) = v_i$.

Now we can define the first feed forward layer:

$$F_1(x_i) = (0, 0, f_1(x_i) - \text{relu}(w_i) + \text{relu}(-w_i), 0, 0) + x_i$$

The additional relu terms here remove the residual contribution of w_i.

At this point, we need to decode the token at the root of the transformed vector. Let:

$$z = C(Ew_i - \frac{1}{2})$$

$$f_2(x_i) = E^T(\text{relu}(z+1) - \text{relu}(z))$$

where C is a large constant. The purpose of z is to pick out the token with the largest dot product with w_i, which is then converted to the relevant token embedding by the relu expression. Now let:

$$F_2(x_i) = (0, -\text{relu}(v_i) + \text{relu}(-v_i), 0, 0, f_2(x_i)) + x_i$$

The additional relu terms in the v_i component are to zero out this term when combined with the residual state vector x_i.

Decoding Block

To complete the construction, we put everything together into a decoding "block":

$$D(x_i) = F_2(F_1(A(x_i) + x_i))$$

We can think of D as propagating the transformation of the input and path one step forward in the sequence, while simultaneously decoding tokens.

The full transformer decoding model is simply the $n-1$-fold iteration of the D block:

$$x^{out} = (p_i^{out}, v_i^{out}, w_i^{out}, r_i^{out}, t_i^{out}) = D^{n-1}(x_i)$$

Assuming that the input path has length $n-1$, the output token corresponding to the input path is t_n^{out}. The intermediate tokens of v are decoded to t_i^{out} for $i \in [1, \ldots, n]$.

Because this decoding algorithm uses the same inner product logic as Algorithm 1, it will work under identical conditions.

Note that we have two feed forward layers per attention layer, whereas the usual transformers have only one - this is easily remedied by adding a zero-valued attention layer between F_1 and F_2 if desired.

5 Parsing with BT Embeddings

In this section we present an algorithm to parse a sequence of BT encoded tokens according to a finite set of BT encoded production rules. This algorithm does not require decoding of any of the structures and produces the parse tree as a BT encoded vector. In fact, it does not even need access to the schema of the input, except for a small number of attribute matrices required to build up the output tree.

This algorithm can be implemented as a transformer in a similar way as the decoder transformer in Section 4.

We consider the problem of producing the parse tree of a sequence of n tokens, according to k m-ary production rules (i.e. the rules match up to m tokens).

We need a distinguished attribute next to build lists, and m distinguished attributes $\text{arg}_1, \ldots, \text{arg}_m$ that will be used to store the children of replacement node to form the parse tree. Assume we are using a schema (T, A) that includes these attributes.

Each production rule consists of a *pattern* P_i and a *replacement* R_i. Both of these are BT-embedding vectors in $\mathbb{R}^d$.

The P_i will be the encoding of a sequence of $\leq m$ tokens, using the next attribute linked list construction. We will assume here that R_i is a single token indicating the expression type of the replacement value of the production rule.

Example 1. The language of balanced parentheses, where we set $L =$ '(' and $R =$ ')'. The tokens are $\{L, R, E\}$, where E is a placeholder for "expression".

Pattern	Replacement	$P_i \in \mathbb{R}^d$	$R_i \in \mathbb{R}^d$
LR	E	$BT(L) + A_{\text{next}} BT(R)$	$BT(E)$
LER	E	$BT(L) + A_{\text{next}} BT(E) + A_{\text{next}}^2 BT(R)$	$BT(E)$
$E\ E$	E	$BT(E) + A_{\text{next}} BT(E)$	$BT(E)$

Since this language uses ternary productions, the attributes required for parsing are $\{\text{next}, \text{arg}_1, \text{arg}_2, \text{arg}_3\}$.

5.1 Algorithm 2

Assume that we have a set of production rules expressed as strings of tokens in a schema (T, A). We will denote the BT encoded attribute matrices for our distinguished attributes as $A_{\text{next}}, A_{\text{arg}_1}, \ldots$.

1. BT encode each pattern P_i as a list $p_i = \sum_{0<j\leq len(p_i)} A_{\text{next}}^{j-1} P_{i,j}$.
2. Let $r_i = BT(R_i)$ be the encoding of the replacements.
3. Let $x_i = BT(X_i)$ be the encoding of the input sequence.
4. For each p_i
   (a) Let $m = len(p_i)$. Note that this can be computed as $\text{round}(||p_i||^2)$ in sufficient large embedding dimension.
   (b) For each consecutive m-tuple of inputs $x_j, \ldots, x_{j+m}$, form a BT encoded list $x = \sum_{0<k\leq m} A_{\text{next}}^{k-1} x_{j+k}$
      i. Test for a match by comparing $\langle p_i, x\rangle > m - 1/2$. If false, continue on the next m-tuple $x_{j+1}, \ldots$.
      ii. If true, we replace the entire input $m-$tuple with the result r_i corresponding to pattern i, adding the matched tokens as attributes: $r_i + \sum_{0<k\leq m} A_{\text{arg}_k} x_{j+k}$
5. Repeat step (4) until no matches are found.

Note that this algorithm does not require access to the full embedding of (A, T), only the matrices $A_{\text{next}}, A_{\text{arg}_i}$.

An implementation of this algorithm can be found at `https://github.com/jtmaher/Embedding/blob/master/embedding/parser.py`.

The key point is to determine when we can successfully test for a match with the comparison $\langle p_i, x\rangle > m - 1/2$. Assuming that p_i encodes a tree of size m, and x is size l, expanding both arguments of $\langle p_i, x\rangle$ into their BT encoding sums yields $m \cdot l$ terms.

Assuming that the non-matching terms have inner products $< \frac{1}{2C\sqrt{ml}}$, summing up to ml terms, the result will be $< 1/2$ with high probability for an appropriate choice of the constant C. So, by the same logic as Lemma 2, we get:

Lemma 3. *There is a constant C such that if:*

1. *$\langle v_i, v_j\rangle$ are I.I.D.*

2. $E(\langle v_i, v_j \rangle) = 0$

3. $|\langle v_i, v_j \rangle| < \frac{1}{2C\sqrt{ml}}$ *for* $i < j$

4. $x_1, \ldots, x_n$ *is an input sequence of vectors, whose parse tree has* $\leq l$ *nodes, and the maximum arity of the patterns is* m

then Algorithm 2 produces the correct parse tree with high probability for large $m \cdot l$*.*

Note that the intermediate x_i terms are parse trees, and the size of these terms is not necessarily bounded in terms of the input length (for general production rules), which is why we bound in terms of the parse tree size, not the input length.

Observe that all the data at every step of the algorithm is BT encoded, which means that we can inspect or otherwise make use of the intermediate results.

5.2 Empirical Parsing Results

We implemented Algorithm 2 and ran it on an assortment of 17,185 randomly selected sequences of balanced parentheses of length ≤ 34, with embedding dimensions from 200 to 2000. The desired output is a BT encoded parse tree in the first (and only) slot of the output sequence.

In this test, we considered an output to be successful if and only if the output decoded to a correct parse tree of the original input using Algorithm 1.

As in the tests of decoding, we see an approximately linear scaling of the maximum successful sequence length with respect to embedding dimension.

The code for this experiment can be found at `https://github.com/jtmaher/Embedding/blob/master/Parser.ipynb`.

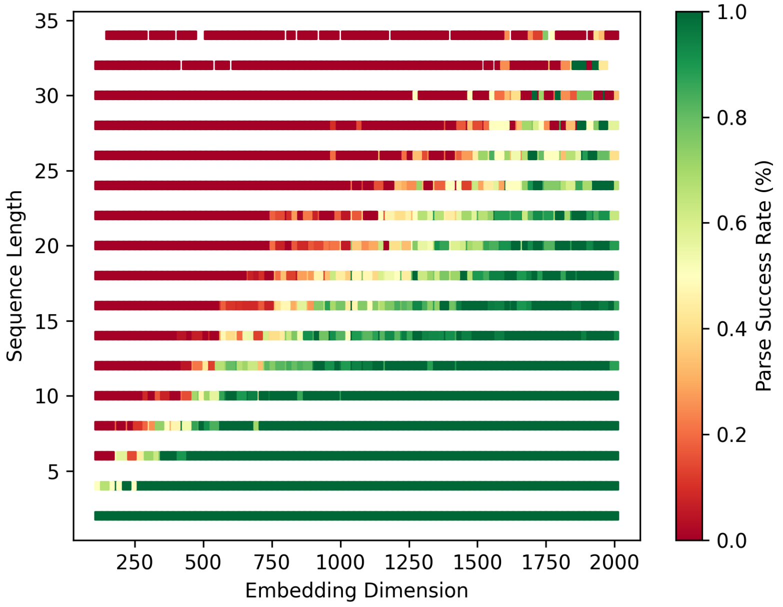
6 The JL Lemma and Decoding

A natural question is whether the conditions of Lemma 2 hold for sufficiently high embedding dimension. We will sketch an argument that the conditions do hold in the special case where our schema contains only one attribute. The general case with > 1 attributes appears to work in practice, but we do not have a proof.

First, we recall the JL Lemma [13], [6], [14]:

Lemma 4. *If we have a collection of* n *independent uniformly distributed random unit vectors* $V \subset S^{d-1} \subset \mathbb{R}^d$*, then if* $d > 16\frac{log(n)}{\epsilon^2}$*, then* $|\langle v_i, v_j \rangle| < \epsilon$ *for any pair of vectors* $v_i, v_j \in V$*, with high probability.*

Figure 7: Algorithm 2 Performance



To show that we can decode BT embeddings with Algorithm 1 in sufficiently high dimensions, we require a similar bound on inner products when V is augmented with orthogonally transformed copies of itself, so that we can apply Lemma 1 or Lemma 2.

Scaling Law

Note that if the conclusion of JL did hold on this augmented set of vectors, then we could substitute ϵ with the bound from Lemma 2 and get:

$$d > 16\frac{log(n)}{\epsilon^2}$$

$$d > 64C^2 \cdot l \cdot log(n)$$

In Lemma 2 we only need the conclusion of JL for sub-collections of $n = l \cdot T$ vectors, where T is the number of tokens in the schema. Therefore if $d > 64C^2 l \cdot log(lT)$, we would expect to decode trees of size $\leq l$ with high probability. Since this is approximately linear in l for small l, it would agree with the empirical scaling results above.

Random Matrices

Let $A_1, \ldots, A_k \in O_d$ represent independent Haar uniform random orthogonal matrices. Let $\Gamma < O_d$ denote the subgroup generated by the A_i. Let ΓV denote the set of all unit vectors of the form gv for $g \in \Gamma, v \in V$.

Definition 5. *We say that ΓV m-ϵ* separated *if for all $v_1, v_2 \in W \subset \Gamma V$ with $v_1 \neq v_2$, we have $| \langle v_1, v_2 \rangle | < \epsilon$ with high probability, for all subsets $W \subset \Gamma V$ of cardinality $\leq m$.*

In other words, the condition is asserting that the conclusion of the JL Lemma continues to hold for subsets of ΓV.

We would like to have a result that implies m-ϵ separation with high probability given $d > C\frac{log(m)}{\epsilon^2}$ for some constant C. Some intuition comes from [5] which proved that the subgroup of SO_d generated by two or more random elements is a free and dense subgroup with probability one, which makes it plausible that the vectors in ΓV are as evenly dispersed as if they were random. This is not true, as can be seen by examining the case of the powers of a single random orthogonal matrix A:

For a full measure set of $A \in O_d$, $\cup_{n=1,\ldots,\infty} A^n$ is dense in a maximal torus of O_d, which has dimension $\lfloor d/2 \rfloor$. Thus, it is easy to see that the orbit of a vector $v \in S^{d-1}$ must be dense in a $\lfloor d/2 \rfloor - 1$ dimensional torus in S^{d-1}. This can be seen explicitly by writing A in block diagonal form. Since Γ acts ergodically on this torus, the distribution of the points in ΓV are uniformly distributed (with respect to an appropriate limit). Therefore we can conclude:

Lemma 6. *If $\Gamma = \{A^i, i \in \mathbb{Z}\}$ and $V \subset S^{d-1}$ is an independent uniformly distributed set of unit vectors, then ΓV is m-ϵ separated when $d > 32\frac{log(m)}{\epsilon^2}$. (Note that the constant is twice that in the JL Lemma.)*

From this and Lemma 2 we can conclude that there is a constant C such that the BT embedding is invertible with high probability for schemas with one attribute and T tokens, when the embedding dimension satisfies:

$$d > 128C^2 \cdot l \cdot log(l \cdot T)$$

Note that this bound on embedding dimension is nearly linear in the tree size l.

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