Statistical Ecology HW#3

1.

(a) Write the 5 plausible models

(b) For each Model, Propose a discrete Probability model of how the data Arose

Model 0: $\mu_1 = \mu_2 = \mu_3 = \mu$ $\longrightarrow X_{i,j} \sim Poisson(\mu)$

Model 1: $\mu_1 \neq \mu_2 \neq \mu_3$ $\longrightarrow X_{i,j} \sim Poisson(\mu_i)$

Model 2: $(\mu_1 = \mu_2) \neq \mu_3$ $\longrightarrow X_{i,j} \sim Poisson(\mu_{12})$ for $i=1,2$ & $X_{3,j} \sim Poisson(\mu_1)$

Model 3: $(\mu_1 = \mu_3) \neq \mu_2$ $\longrightarrow X_{i,j} \sim Poisson(\mu_{13})$ for $i=1,3$ & $X_{2,j} \sim Poisson(\mu_2)$

Model 4: $(\mu_2 = \mu_3) \neq \mu_1$ $\longrightarrow X_{i,j} \sim Poisson(\mu_{23})$ for $i=2,3$ & $X_{1,j} \sim Poisson(\mu_1$

(b) Write down the likelihood function for each model, then Solve for the Maximum Likelihood Estimate (MLEs) of the model Parameters

Model 1: $\mathcal{L}(\mu_1, \mu_2, \mu_3)$ = the joint Prob of obs $\rightarrow P(A)\cdot P(B) \ldots P(Z)$

$$= P(\overset{A}{X_{11} = x_{11}}, \overset{B}{X_{12} = x_{12}}, \ldots, \overset{Z}{X_{3n_3} = x_{3j}})$$

$$= P(X_{11} = x_{11}) \cdot P(X_{12} = x_{12}) \ldots P(X_{nj} = x_{3j})$$

$$= \prod_{i=1}^{3} \prod_{j=1}^{n_i} P(X_{i,j} = x_{i,j}) = \prod_{j=1}^{n_1} P(X_{1j} = x_{1j}) \prod_{j=2}^{n_2} P(X_{2j} = x_{2j}) \prod_{j=3}^{n_3} P(X_{3j} = x_{3j})$$

$$\mathcal{L}(\mu_1, \mu_2, \mu_3) = \left(\prod_{j=1}^{n_1} \frac{e^{-\mu_1} \mu_1^{x_{1j}}}{x_{1j}!}\right) \left(\prod_{j=2}^{n_2} \frac{e^{-\mu_2} \mu_2^{x_{2j}}}{x_{2j}!}\right) \left(\prod_{j=3}^{n_3} \frac{e^{-\mu_3} \mu_3^{x_{3j}}}{x_{3j}!}\right)$$

Take the Log likelihood

$$\ln \mathcal{L}(\mu_1, \mu_2, \mu_3) = \sum_{j=1}^{n_1} \left[-\mu_1 + x_{1j} \ln \mu_1 - \ln(x_{1j}!)\right] + \sum_{j=2}^{n_2} \left[-\mu_2 + x_{2j} \ln \mu_2 - \ln(x_{2j}!)\right]$$

$$+ \sum_{j=3}^{n_3} \left[-\mu_3 + x_{3j} \ln \mu_3 - \ln(x_{3j}!)\right]$$

Take the first (for Param 1) derivative

$$\frac{d \ln \mathcal{L}(\mu_1, \mu_2, \mu_3)}{d\mu_1} = \frac{d}{d\mu_1}\left[-n_1\mu_1 + (\ln \mu_1)\left(\sum_{j=1}^{n_1} x_{ij}\right) - \sum_{j=1}^{n_1} \ln(x_{1,j}!)\right]$$

$$= -n_1 + \frac{1}{\mu_1} \cdot \sum_{j=1}^{n_1} x_{i,j}$$

$$0 = -n_1 + \frac{1}{\mu_1} \sum_{j=1}^{n_1} x_{ij}$$

$$n_1 = \frac{\sum_{j=1}^{n_1} x_{i,j}}{\mu}$$

$$\frac{\sum_{j=1}^{n_1} x_{1j}}{n_1} = \bar{x}_1$$

or Rewritten

$\hat{\mu}_1 = \bar{X}_1$

MLE for group 1

& by Same logic for Other Params

$\vdots$

$\hat{\mu}_2 = \bar{X}_2$ MLE for G2

$\hat{\mu}_3 = \bar{X}_3$ MLE for G3

Model 2

Likelihood Fxn: $\mathcal{L}(\mu_{12}, \mu_3)$ MLE's: $\hat{\mu}_{12} = \bar{x}_{12}$, $\hat{\mu}_3 = \bar{x}_3$

Model 3

Likelihood Fxn: $\mathcal{L}(\mu_{13}, \mu_2)$ MLE's: $\hat{\mu}_{13} = \bar{x}_{13}$, $\hat{\mu}_2 = \bar{x}_2$

Model 4

Likelihood Fxn: $\mathcal{L}(\mu_{23}, \mu_1)$ MLE's: $\hat{\mu}_{23} = \bar{x}_{23}$, $\hat{\mu}_1 = \bar{x}_1$

Model Ø

Likelihood Fxn: $\mathcal{L}(\mu)$ MLE: $\hat{\mu} = \bar{x}$

c) For each Model, evaluate the likelihood Fxn at the MLEs & the data at hand. That quantity is the Maximized likelihood Fxn.

~ See Code ~

d) Do a Likelihood Ratio Test Using Samuel Wilk's χ^2 approximation to test the Null Hypothesis Vs each one of the Alternatives Models (Including Model 1). Because We are doing mult. hyp. tests, we need to do a Bonferroni type I error Correction.

If the # of tests you are doing is m and you want to do an overall test with Type I error rate α, then the Bonferroni-Corrected α_B to be used for each of the tests is equal to α/m

~ See Code ~

e) Parametric Bootstrap Likelihood Ratio Tests (PBLRT)

Step 1: Compute the Maximum Likelihood model Parameters under the null for a given data set.

$\hat{\theta}_0$ for M0 & $\hat{\theta}_1$ for M1

$\hat{\theta}_0 = [\hat{\mu}, \hat{\mu}, \hat{\mu}]$ $\quad\quad L(\theta_0) =$

$\hat{\theta}_1 = [\hat{\mu}_1, \hat{\mu}_2, \hat{\mu}_3]$