4. If $X_1, X_2, \dots X_n$ are iid random samples from $f(x)$ (where $f(x)$ is either the pdf or pmf). Find Fishers Information for

a. $f(x) = \lambda \exp -\lambda x$, $(0 < x < \infty)$ Exponential Observations

b. $f(x) = p^x(1-p)^{1-x}$, $x = 0, 1$ Bernoulli

6. $X \sim Bin(N, P) \rightarrow \mathcal{L}(p) = \binom{N}{x} p^x (1-p)^{N-x}$

$\hat{p} = \frac{x}{n}$

$$I(p) = -E\left[\frac{\partial^2 \ln \mathcal{L}(p)}{\partial p^2}\right]$$

$$= -E\left[\frac{\partial^2}{\partial p^2}\left[\ln\left[\binom{n}{x}\right] + x \ln p + (N-x)\ln(1-p)\right]\right]$$

$$= \frac{\partial}{\partial p}\left[x \ln p + (n-x)\ln(1-p)\right]$$

$$0 = \frac{x}{p} - \frac{(n-x)}{(1-p)}$$

Find MLE $(\hat{p})$

$$\boxed{\hat{p} = \frac{x}{n}}$$

Take 2nd Derivative

$$\frac{\partial}{\partial p}\left[\frac{x}{p} - \frac{n}{1-p} + \frac{x}{1-p}\right]$$

$$= \frac{-x}{p^2} - \frac{n}{(1-p)^2} + \frac{x}{(1-p)^2}$$

Noting

$$\frac{\partial(1/y)}{\partial y} = 1/y^2$$

Fisher's Info

$$I(p) = -E_X\left[-\frac{X}{p^2} - \frac{n}{(1-p)^2} + \frac{X}{(1-p)^2}\right]$$

takes into Account all Values of x

$$= -\left[\left(-\frac{1}{p^2}\right)E[X] - \frac{n}{(1-p)^2} + \frac{1}{(1-p)^2}E[X]\right]$$

$$= \frac{E[X]}{p^2} + \frac{n}{(1-p)^2} - \frac{E[X]}{(1-p)^2}$$

$E[X] = np$ for $X \sim Bin(n,p)$

$$= \frac{n\cancel{p}}{p^{\cancel{2}}} + \frac{n}{(1-p)^2} - \frac{np}{(1-p)^2}$$

$$= \frac{n}{p} + \frac{n(1-p)}{(1-p)^2}$$

$$= \frac{(1-p)^{\cancel{2}}n + np\cancel{(1-p)}}{p(1-p)^{\cancel{2}}}$$

$$= \frac{n\cancel{(1-p)}[(1-p)+p)]}{p(1-p)^{\cancel{2}}}$$

$$= \frac{n(1)}{p(1-p)} = E[I(p)] \quad \& \quad I(\hat{p})^{-1} = \frac{\hat{p}(1-\hat{p})}{n}$$

$$\hat{p} \pm z_{\alpha/2} \cdot \sqrt{I(\hat{p})^{-1}}$$

$$= \frac{\left(\frac{x}{n}\right)\left(1-\left(\frac{x}{n}\right)\right)}{n} \quad \&$$

CI $\frac{x}{n} \pm z_{\alpha/2}\sqrt{\frac{\frac{x}{n}(1-\frac{x}{n})}{n}}$