Simulating a 1D Sod Shock Tube with SPH

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In this report, we solve Euler's equations for a one dimensional shock tube by means of Smoothed Particle Hydrodynamics. We first provide an overview of the theory and model and then present our results. Our results are in agreeance to what is predicted by theory as all major components of the shock are seen. Although they agree, there are slight discrepancies that arise from the particle-based approximation. One being the clumping of particles near the rarefaction wave.

I. INTRODUCTION

Computational Fluid Dynamics (CFD) is a robust field with varying techniques that seeks to solve the Navier-Stokes or Euler equations. There currently seems to be two paradigms of solvers in this field. The first being Eulerian based solvers, in which the mesh used is stationary. The ladder being Lagrangian codes. One such Lagrangian based solver is known as Smoothed Particle Hydrodynamics (SPH).

SPH was first introduced by Gingold and Monaghan to study non-spherical stars that have a polytropic equation of state. [1] This first paper sets the ground work to be able to extend such a formulation to other physical phenomena. To begin, a discretized Euler's equations state that for a fluid with no external forces the equation of motion for a segment i is given as:

$$\frac{d^2 r_i}{dt^2} = -\frac{1}{\rho_i}\nabla P, \tag{1}$$

where ρ_i is the density of the segment and ∇P is the pressure gradient. The fluid segments are commonly referred to as particles.

In SPH, the density is approximated (smoothed) with the use of kernel function, $W(r)$. A kernel smoother is a function that estimates the value of a real valued function as the weighted average of neighboring data points. Such a function must also follow a normalisation condition:

$$\int dr\, W(r) = 1 \tag{2}$$

Our smoothed density, $\rho_s(r)$, can then be expressed as:

$$\rho_s(r) = \int dr'\, W(r - r')\rho(r') \tag{3}$$

$\rho(r')$ is an unknown function. This means that eq. (3) can not be solved in its current form. In order to arrive at a solution, we must make use of Monte Carlo integration. With this, we can write the density for a particle i as:

$$\rho_i = \sum_j m_j W(r_i - r_j), \tag{4}$$

where j notates some other particle in the system. It is important to note that the particles have a self-contribution to their density.

With our particle discretization of our fluid and kernel estimation, we can express eq. (1) as:

$$\frac{d^2 r_i}{dt^2} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2}\right)\nabla W(r_i - r_j),$$

where $\nabla W(r_i - r_j)$ is the gradient of our chosen kernel and P_i is given by a chosen equation of state. The chosen equation of state for our model is that of the Ideal Gas. This equation of state is given as:

$$P = e\rho(\gamma - 1), \tag{5}$$

where e is the internal energy and γ is set to 1.4 to model an adiabatic ideal gas. Euler's equations states that energy conservation is given as:

$$\frac{de}{dt} = -\frac{P}{\rho}\frac{\partial v}{\partial x}, \tag{6}$$

In our particle bases representation, we can express the previous equation as:

$$\frac{de}{dt} = -\sum_j m_j \left(\frac{P_i}{\rho_i^2} + \frac{P_j}{\rho_j^2}\right)(v_i - v_j)\nabla W(r_i - r_j) \tag{7}$$

For our simulation, these are the only two quantities that are necessary for time-evolution as they both relate to energy and momentum conservation. In fact after initial condition are set, these quantities are integrated using the sympletic leap-frog integration method.

The dynamics of the system is not invariant under kernel choice. This makes it important to understand which kernel you are working with. A function that fits all requirements for a good kernel would be the Gaussian distribution, but this is in fact problematic. This is due to the fact that the Gaussian distribution only converges to zero after an infinite distance from the origin making the computation more expensive. Although this is not an issue for the systems we consider, the chosen kernel for our system is the cubic spline kernel:

$$W(r) = \begin{cases} \sigma_3\left[1 - \frac{3}{2}q^2(1 - \frac{q}{2})\right] & \text{if } 0 \le q \le 1 \\ \frac{\sigma_3}{4}(2 - q)^3 & \text{if } 1 < q \le 2 \\ 0 & \text{if } 2 < q \end{cases} \tag{8}$$

In eq. (8), σ_3 is a normalization constant that equals

$(2/3)h$ and $q = \frac{r}{h}$, where h is the smoothing length.

For this report, we consider one system. That system being a classic test for SPH solvers: the Sod shock tube. The Sod shock tube was heavily considered by Gary A. Sod in 1978. This shock tube has an analytical solution for one dimension which makes it a great test for solvers.

II. MODEL AND PARAMETERS

Our parameters were obtained from Liu and Liu in their book for SPH. [2] For our simulation, we consider a one dimensional tube with dimensionless length 1.2. The fluid in this tube is then split into 400 particles. In order to get the desired contact discontinuity, we partition particle i as:

$$x_i = \begin{cases} i(\Delta x/4) - x_0 + \Delta x/4 & \text{if } 0 \leq i \leq 319, \\ (i - 320)\Delta x + 0.5\Delta x & \text{if } 320 \leq i \leq 399, \end{cases} \tag{9}$$

where $x_0 = 0.6$ and $\Delta = x_0/80$. This partition gives the particles on the left side a density of 1 and on the right side a density of 0.25. We then assign initial internal energy as:

$$e_i = \begin{cases} 2.5 & \text{if } 0 \leq i \leq 319, \\ 1.795 & \text{if } 320 \leq i \leq 399, \end{cases} \tag{10}$$

The smoothing length is chosen to be $h = 0.016$ unless stated otherwise. Due to the small timescales we consider, we do not employ any sort of boundary conditions on our system.

III. RESULTS

By plotting the density distribution as in fig. 1, we can get a nice qualitative sense that our code is working. At the beginning of the simulation, there is a contact discontinuity between the left and right sides of our tube. As the system is evolved, the shock begins to move throughout. One can see the rarefaction wave progresses quite nicely. In fig. 2, the velocity distribution is plotted. This plot also agrees with the analytical solution. It is noticeable that the shockwave not only progress towards the right side of the tube, but also perturbs the pre-shock region on the left. The shock appears to progress slower in this region. In both plots, the effects near the precipice of the rarefaction wave are striking. The density in that region grows much higher than that predicted by analytical theory. It is also seen that the velocity distribution becomes highly noisy in this region. This noise is also seen in the density distribution. It is possible to see the ringing near the rarefaction wave which is characteristic of many SPH solvers.

One method to curb this effect is to increase the smoothing length. In fig. 3, the smoothing length is doubled in the blue curve. It is noticeable that the clumping of particles near the rarefaction wave is less than before, but not entirely eliminated.

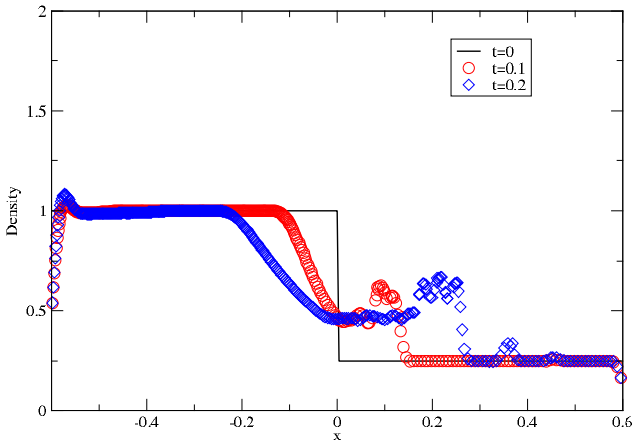

FIG. 1. A plot of the density distribution at varying times in the simulation.

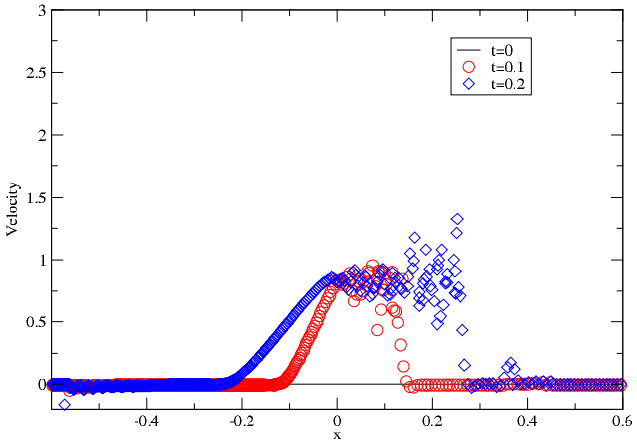

FIG. 2. A plot of the velocity distribution at varying times in the simulation.

IV. SUMMARY AND CONCLUSION

Overall, our SPH solver agrees with theory with some discrepancies. It appears that our simulation is disproportionately clumping particles at the front of out rarefaction wave, which is seen by the peaks near the contact region in fig. 1. There is also ringing in both the veloc-

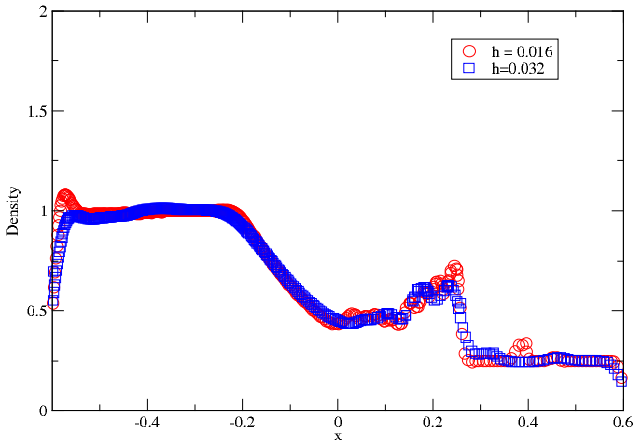

FIG. 3. A plot of the density distribution at $t = 0$ for varying smoothing lengths.

ity and density distributions, although it is much more prevalent in the former. Our implementation of SPH was naive, but this ringing can also be prevalent in even the most robust solvers. We try to curb the peak and ringing by increasing the smoothing length. Although this led to the peak and ringing being lowered, they were not entirely eliminated. Increasing the smoothing distance also has the effect of increased computational complexity as you have to sum over more particle pairs. Our solver could be easily enhanced in the future with the addition of extra kernels and artificial viscosity.

[1] R. A. Gingold and J. J. Monaghan, Monthly Notices of the Royal Astronomical Society **181**, 375 (1977), https://academic.oup.com/mnras/article-pdf/181/3/375/3104055/mnras181-0375.pdf.

[2] G. R. Liu and M. B. Liu, *Smoothed Particle Hydrodynamics* (WORLD SCIENTIFIC, 2003) https://www.worldscientific.com/doi/pdf/10.1142/5340.